SOLVING CONTINUOUS SPACE LOCATION PROBLEMS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for

the Degree Of Doctor of Philosophy in the

Graduate School of the Ohio State University

By

Hu Wei, M.A., M.A.S.

The Ohio State University

2008

Dissertation Committee:

Professor Alan T. Murray, Adviser

Professor Morton E. O'Kelly

Professor Ningchuan Xiao

Approved by:

____

Advisor

The Graduate Program in Geography

ABSTRACT

The focus of this research is on location problems where potential facility sites are to be located in continuous space and demand is assumed continuously distributed, and includes the continuous *p*-center and coverage maximization problems. Relevant discrete location models are reviewed for the purpose of comparative analysis, including the vertex *p*-center problem and the maximal covering locational problem.

First, this dissertation explores a simple but effective approach for solving large vertex *p*-center problems, the results of which are to be used as a benchmark for its continuous space counterpart. By introducing a neighborhood facility set, the *p*-center problem can be reformulated such that many redundant variables and constraints are removed but characteristics, including optimality, of the problem are preserved. The problem size of the reformulated model can be substantially smaller than in the original form. This enables the use of general-purpose optimization software to solve large vertex *p*-center instances. Application results are provided and discussed.

The dissertation then studies the continuous space *p*-center problem. A Voronoi diagram heuristic has been proposed for solving the *p*-center problem in continuous space. However, important assumptions underlie this heuristic and may be problematic for practical applications. These simplifying assumptions include uniformly distributed

demand, representing a region as a rectangle, analysis of a simple Voronoi polygon in solving associated one-center problems and no restrictions on potential facility locations. In this dissertation, the complexity of solving the continuous space *p*-center problem in location planning is explored. Considering the issue of solution space feasibility, this research presents a spatially restricted version of this problem and proposes methods for solving it heuristically. The performance of the heuristic is evaluated by comparison with the discrete *p*-center problem. Theoretical and empirical results are provided.

Finally, this dissertation explores approaches for solving the problem of siting service facilities to maximize regional coverage when both facility sites and regional demand are assumed continuous. Traditionally, coverage maximization has been approached using discrete representations of potential facility sites and service demand locations. However, such discretizations of space can lead to significant measurement and coverage errors. Representing candidate facility sites and service demand locations as continuously distributed is more reasonable in many cases. Research on coverage maximization in this context has been limited to siting a single facility in a region. This dissertation addresses multiple facility siting in continuous space. A Voronoi diagram heuristic is proposed to decompose the multiple facility problem into a set of single facility problems. The developed approach is applied to emergency warning siren siting in a region. The results are compared with those obtained from a discrete approach.

ACKNOWLEDGMENTS

Though this dissertation is an individual work, I could never have successfully completed it without the help and support of many people. First and foremost I want to thank my advisor, Professor Alan Murray, for his consistent support and encouragement. Throughout my doctoral work at OSU he continually encouraged me to develop independent thinking, stimulated my analytical skills, and instilled in me the qualities of being a good researcher. This has been invaluable driving forces through my career.

I also wish to thank my doctoral committee members, Professor Morton O'Kelly and Professor Ningchuan Xiao, for their inputs to my education and research. I am grateful to Tim Matisziw for the assistance and valuable suggestions.

I would like to take this opportunity to extend my great gratitude to many friends over the past years. Thanks so much for being friends, for sharing with me their experiences and memories, and for the help and support they have lent me.

Finally, and the most importantly, I wish to thank my family. My wife has supported me with unlimited love, support, understanding, and encouragement, which have kept my life in proper perspective and balance. My son Benjamin has also cheered me in his special way. My mother has unwavering faith and confidence in me. My father passed away several years before I started my doctoral studies. I owe the credit to him for inspiring me a belief in the value of education. I miss him and dedicate this dissertation to him.

VITA

1995…………………Bachelor of Architecture, Tsinghua University, China

1998…………………Master of Engineering in Urban Planning and Design, Tsinghua University, China

2003…………………Master of Arts in Geography, University of Cincinnati, USA

2006…………………Master of Applied Statistics, The Ohio State University, USA

PUBLICATIONS

H. Wei, A.T. Murray, and N. Xiao, 2006. "Solving the continuous space *p*-centre problem: planning application issues," *IMA Journal of Management Mathematics*, 17: 413-425.

T.C. Matisziw, T.H. Grubesic, and H. Wei, 2008. "Downscaling spatial structure for the analysis of epidemiological data," *Computers, Environment and Urban Systems*, 32: 81-93.

FIELDS OF STUDY

Major Field: Geography

Minor Field: Statistics

TABLE OF CONTENTS

Chapters:

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Background

Whenever a question of where to locate something is raised, a location problem arises. One then wonders where places are available and which of these are the best locations based on particular criteria. In the public sector, there are often concerns about the location of facilities or services to be established or extended, such as emergency services (ambulances, warning sirens, fire stations, and police units), school systems and postal facilities. The goals of decision making in the public sector usually include social cost minimization, universality of service, efficiency and equity (Marianov and Serra, 2002), while private sector concerns are generally based on maximizing profit or increasing market share from competitors.

1.1.1 Location models

In support of decision making processes that involve facility siting, location models are generally applied. The field has seen a rapid growth in past decades, mainly due to the evolution of location theory and the advent of computer technology (Church, 1999). Given locations of demand and existing facilities (if relevant), models are structured to locate new facilities to optimize some objective. The objectives are often expressed in measurable terms, such as minimizing the locational and operational costs (e.g., time or distance) needed for service coverage. Many location models have been proposed and applied to solve various location problems. Location models can be classified into three broad categories (Daskin, 1995): *p*-median models, *p*-center models and covering models. The *p*-median models aim to minimize the average weighted distance or time of the system (Hakimi, 1964; ReVelle and Swain, 1970). The *p*-center problem seeks to locate *p* facilities such that the maximum distance from any demand site to its closest service facility is minimized (Hakimi, 1964, 1965). This minimax model has been proposed for public facility planning and emergency services management, such as EMS and fire protection, where the distance from sited facilities to their farthest client is minimized (Love *et al.* 1988). The third category of location models classified in Daskin (1995) is covering models, and are used to serve/cover demand (e.g., population) within a given time or distance standard. A fundamental goal of covering models is to provide a particular level of service coverage to demand. Generally, customers are assumed to use the nearest available facility. A customer is considered covered only if the distance between the customer and the facility to which the customer is served is within a given effective range of the facility. Covering problems can be approached in two basic ways. One is the Location Set Covering Problem (LSCP), which seeks to minimize the number of facilities needed to ensure complete regional coverage of demand (Toregas, 1971). The other is the Maximal Covering Location Problem (MCLP), which aims to maximize service coverage given a fixed number of facilities (Church and ReVelle, 1974). A review of these location models and applications can be found in ReVelle *et al. (*2002).

Other taxonomies of location models are also seen in the location literature, e.g. discrete versus continuous demand, discrete versus continuous facility location, median versus center, location and routing problems, single versus multiple objectives, stochastic versus deterministic problems, etc. (Drezner, 1995). The focus of this dissertation is on location models in which both demand locations and facility sites are assumed continuous, and is limited to *p*-center and coverage maximization problems.

1.1.2 Spatial representation of location

In order to formulate a location model, it is necessary to identify where demand is located and where facilities can be sited. This is related to the representation of geographic space, either discrete or continuous. A discrete approach is based on the assumption that available facility sites and the demand region are represented as a finite set of objects (e.g. points, lines, polygons), while a continuous approach treats demand and candidate facility space as continuous.

Traditionally location models are approached through discrete representation of facility sites and service demand (Bennett and Mirakhor, 1973; ReVelle, 1991; Miller, 1996; Church, 1999). This is mainly due to limited geometric capabilities, data availability, and simplification from a modeling perspective. A continuous demand area, such as an administrative or census unit, is often simplified and/or aggregated into a discrete point, such as the central location of the area. The service facility sites are also considered as point-based. In this context, facilities are to be sited among a finite set of available locations to serve or cover discrete demand points so that particular criteria are satisfied. However, there may be problems with the use of discrete points to represent potential facility sites and demand locations since the point-based assumption about facility location may be too simplistic to represent vector or areal objects as discussed in Miller (1996). On the facility side, the discretization of facility location space limits the search for an optimal solution to the finite and discrete locations. On the demand or service consumption side, representation of space as discrete could lead to pronounced measurement and coverage errors, including imprecise distance measurement and uncertainty in assessing coverage (Daskin *et al.* 1989; Current and Schilling, 1990; Drezner and Drezner, 1997; Murray *et al.* 2002, 2008). Murray and O'Kelly (2002) found that a point-based representation of regional demand in a coverage optimization model generally results in an over-estimation of the actual coverage provided to a region by a service facility configuration. These inaccuracies are the results of inappropriate spatial representation of regional demand in coverage modeling, supporting the representational issues raised in Miller (1996). Thus, there is a need for coverage modeling approaches in which potential facility sites and demand locations are represented as more realistic spatial entities, e.g. lines, polygons or other areal objects.

Another approach for representing geographic space in location modeling is to treat demand locations and/or facility sites as continuous. This assumes that demand is distributed throughout an area and potential facilities can be located anywhere in the analysis region or network. For example, emergency warning sirens need to be audible at all locations of human activity and they can be sited anywhere given their size. This type of location problem has been of interest for a long time in the literature. Continuous space siting problem can be traced back to the Weber problem (Weber, 1909), which is also known in the literature as the minisum problem (Wesolowsky, 1993). The problem aims to find the "minisum" facility location in the continuous plane that minimizes the sum of weighted Euclidean distance from itself to a number of fixed demand points. The Weber problem has given rise to a large number of spatial optimization approaches for formulating and solving continuous space siting problems.

Although representing space as continuous seems to be more reasonable in many situations, it is difficult to extend traditional discrete methods for solving continuous problems. In the discrete case, both demand locations and candidate facilities sites are finite so that standard optimization techniques can be applied. When location space is represented as continuous, it is impossible to enumerate all demand locations and potential places for siting facilities. This makes the continuous location problem challenging to solve. The ability to address this problem has been limited by analytical capabilities associated with geometric computation (Suzuki and Okabe, 1995; Plastria, 2002). A number of solution approaches have been developed in an attempt to solve continuous location problems. Many utilize mathematical properties of the objective function (i.e., the first order conditions for optimality) and formulate the solution as an iterative procedure. For instance, developed by Weiszfeld (1937), the so-called "Weiszfeld procedure" is the most commonly used technique to solve the Weber problem, and falls into this class. Other approaches make use of geometric properties of the region to attack continuous space siting problems. Examples include the solution approach to the 1-center problem (Elzinga and Hearn, 1972; Plastria, 2002), a Voronoi diagram heuristic developed to solve the continuous space *p*-center problem (Suzuki and Okabe, 1995; Okabe and Suzuki, 1997), and the use of the medial axis to search for the location of a facility in a continuous plane that maximizes the coverage of a facility (Matisziw and Murray, 2008). Other geometric solution approaches for solving continuous location problems can be found in the work of Hershberger (1993), Hochbaum and Shmoys (1985), Khuller and Sussmann (2000) and Sharir and Welzl (1996).

Much of the above continuous space work focus on single-facility siting and/or assumes generally idealized conditions. This limits the application of the proposed methods in practice. This dissertation explores approaches for solving general continuous space multi-facility location problems in which some simplifying assumptions are relaxed.

1.2 Research objectives

The focus of this research is on location problems where potential facility sites are to be located in continuous space and demand is assumed continuously distributed. There are two main objectives in the research. The first objective is to develop an efficient and effective heuristic approach for solving the continuous space *p*-center problem. The Voronoi diagram heuristic has been proposed for solving the *p*-center problem in continuous space (Suzuki and Okabe, 1995). However, important assumptions underlying this heuristic may be problematic for practical application, e.g. uniformly distributed demand, representing a region as a rectangle, and no restrictions on potential facility sites. In this research the complexity of solving the continuous space *p*-center problem in location planning is explored. Considering the issue of solution space feasibility, a spatially restricted version of the *p*-center problem and methods for solving it heuristically are developed. The second objective of this research is to develop a heuristic approach for solving a continuous space maximal coverage problem. Even in the simplest case where a single facility is considered, the problem is highly non-linear and cannot be solved analytically. Recently, Matisziw and Murray (2008) propose a method to solve a one-facility problem, and rely on exploiting the geometric properties of a region. In a general case where multiple facilities are to be located, the continuous maximal coverage problem is more challenging and few methods for solving it have been developed in the literature. In this research, a geocomputational approach for solving the general case of the continuous maximal coverage problem is proposed based on Voronoi diagrams and geometric properties of a region. This can be considered an extension of the Voronoi diagram heuristic approach for solving the continuous space complete covering problem as well as an extension to the single facility work of Matisziw and Murray (2008).

1.3 Organization of research

This research explores heuristic approaches for solving continuous space location problems, including the *p*-center problem, complete and partial coverage problems. It is organized as follows.

Following the introduction, Chapter 2 develops a simple and efficient approach for solving large vertex *p*-center problems. The issues of variable reduction and optimality are discussed. A reformulated model is applied to warning siren siting in the City of Dublin, Ohio. This is carried out by experiments of several point representations of demand locations and potential facility sites.

Following the discussion of the vertex *p*-center problem, the dissertation then explores approaches to solve the *p*-center problem in continuous space in Chapter 3. Simplified assumptions in existing approaches and several practical issues are addressed. The complexity of solving the problem is explored. A spatially restricted version of the *p*-center problem is presented, and methods for solving it are proposed. This is followed by an application of the developed model to warning siren siting in the City of Dublin, Ohio.

Application results are then provided and discussed, with comparison to the results obtained for the vertex *p*-center model. The extension of the *p*-center model to the continuous set covering problem and its application are also presented.

Another continuous location problem investigated in the dissertation is a general continuous space maximal covering problem with focus on multiple facility siting. Chapter 4 starts with presenting the rational to approach the problem with the use of a Voronoi diagram. Relevant computational geometry based techniques are then discussed. Next, the proposed solution approach is detailed. The developed method is then applied to warning siren siting in the City of Dublin, Ohio. Following this, application results and discussion are provided.

Finally, Chapter 5 concludes the dissertation by summarizing research results and contributions to the literature. Future research work is also discussed.

CHAPTER 2

THE VERTEX *P*-CENTER PROBLEM

2.1 Introduction

This chapter reviews the vertex *p*-center problem and proposes a simple and efficient approach for solving it. Several point representations of location space are studied and extensive computational results are reported. This discrete model provides a comparative basis for evaluating the continuous space counterpart of the *p*-center problem.

The *p*-center problem was originally defined in Hakimi (1964, 1965). Given the locations of demand points, the objective of the *p*-center problem is to site a given number of facilities so that the maximum distance a demand is from its closest facility is minimized. Thus, it is also known as a minmax location-allocation problem. It is usually assumed that all the facilities provide the same kind of service, and that the amount of demand that can be served by a given facility is unlimited. The problem can be formulated in several model structures according to the treatment of facility siting and demand. In terms of spatial representation, the vertex *p*-center problem restricts candidate facility sites to a set of predefined discrete points, while the absolute or continuous space *p*-center problem allows facilities to be located anywhere in the region (Daskin, 1995). Other variants of the *p*-center problem are either weighted or unweighted demand, depending on whether service points are treated equally (Current *et al.* 2002).

The *p*-center problem can also be used for regional service coverage modeling. For instance, facility service planning often seeks to identify the minimum number of facilities to provide complete service coverage to a region. Murray *et al. (*2008) approach this problem using the *p*-center problem and the Location Set Covering Problem (LSCP), indicating there is a strong relationship between these two models (see also Daskin, 1995). Given this, analysis using one model may somehow facilitate insight for the other model.

2.2 Model formulation

In order to formulate the vertex *p*-center problem mathematically, consider the following notation:

z, the maximum distance between a demand point and its closest facility

i, index of demand points or service areas, $i = 1, 2, \cdots, n$, entire set denoted I

j, index of potential facility locations, $j = 1, 2, \cdots, m$, entire set denoted J

p, number of facilities to be sited

d_{ij}, distance between demand i and facility j

$$x_{ij} = \begin{cases} 1, \text{ if demand point } i \text{ is assigned to a facility at } j, \\ 0, \text{ otherwise.} \end{cases}$$

$$y_j = \begin{cases} 1, \text{ if facility } j \text{ is sited,} \\ 0, \text{ otherwise.} \end{cases}$$

The vertex *p*-center problem can be formulated as follows (Daskin, 1995):

$$Min \text{ z} \tag{2-1}$$

subject to:

$$\sum_{j=1}^{m} y_j = p \tag{2-2}$$

$$\sum_{j=1}^{m} x_{ij} = 1 \qquad \forall i \in I \tag{2-3}$$

$$x_{ij} \le y_j \qquad \forall i \in I, \forall j \in J \tag{2-4}$$

$$\sum_{j=1}^{m} d_{ij} x_{ij} \le \text{z} \qquad \forall i \in I \tag{2-5}$$

$$x_{ij}, y_j \in \{0,1\} \qquad \forall i \in I, \forall j \in J \tag{2-6}$$

The objective function (2-1) is to minimize the maximum distance between a demand point and its closest facility. The problem is subject to a number of constraints. Constraint (2-2) stipulates that *p* facilities are to be located. Constraint set (2-3) requires that each demand point be assigned to exactly one facility. Constraint set (2-4) restricts demand point assignments only to open facilities. Constraint set (2-5) defines the lower bound on the maximum distance. Constraint set (2-6) establishes the siting decision variables, x_{ij} and y_j, as binary. The non-negativity of d_{ij} and x_{ij} implies a non-negative value of the objective function, that is, $z \geq 0$.

Since the number of potential facility sites is finite, one could solve the vertex *p*-center problem defined in (2-1) to (2-6) by enumerating each possible set of candidate facility locations. This naïve way would require computational time in $O(m^p)$ to solve the problem, where *m* is the total number of candidate facility sites. Clearly, even for moderate values of *m* and *p*, such enumeration is not realistic and more efficient and sophisticated approaches are required for this NP-hard problem[1] (Garey and Johnson, 1979; Daskin, 1995).

As mentioned previously, this research is interested in coverage, so the *p*-center problem is used in the context of service coverage provision. In particular, one can solve and use the *p*-center problem to site facilities when service coverage standards must be adhered to. Alternatively, one can solve an LSCP in order to identify a *p*-center solution under certain conditions.

The LSCP was originally developed by Toregas *et al. (*1971). The LSCP is typically applied in the situation when the demand and facility sites are represented as finite discrete points and coverage of all demand points is required. The problem seeks to identify a minimal set of facilities and their locations so that all demand points are covered. This problem is also recognized as the total covering problem (White and Case,

[1] NP-hard stands for nondeterministic polynomial-time hard (Daskin, 1995). In computational complexity theory, a problem is NP-hard if an algorithm for solving it can be translated into one for solving any NP-problem, that is, it is "at least as hard as the hardest problems in NP" (Daskin, 1995).

1974). Toregas *et al.* (1971) formulated the LSCP to identify the minimal number of emergency service facilities and their locations to ensure travel time response to all areas was not longer than a desired maximal response.

Assuming that the demand in a region to be covered is represented as a collection of spatial entities, the LSCP can be formulated. In order to mathematically state this locational optimization model, consider the following notation, in addition to the indices (*i* and *j*) as used in the vertex *p*-center problem:

$N_i = \left\{ j \mid d_{ij} \le S \right\}$, i.e., the set of potential facilities *j* capable of covering/serving demand *i*, where d_{ij} is the distance from demand *i* to potential site *j* and *S* is the effective coverage distance of a facility;

$$v_j = \begin{cases} 1, \text{ if potential facility site } j \text{ is selected for placement,} \\ 0, \text{ otherwise.} \end{cases}$$

Using the above notation, the LSCP can be structured as an integer-linear formulation (Toregas *et al.* 1971):

$$\min \sum_j v_j \qquad (2\text{-}7)$$

subject to

$$\sum_{j \in N_i} v_j \ge 1 \qquad \forall i \in I \qquad (2\text{-}8)$$

$$v_j = \{0,1\} \qquad \forall j \in J \qquad (2\text{-}9)$$

The objective (2-7) seeks the minimum number of facilities needed to provide complete coverage to all demand. The problem is subject to two sets of constraints. Constraints (2-8) require that each demand point is covered/served by at least one facility, i.e. the distance between each demand point and its closest facility is less than or equal to the desired maximal service distance. Constraints (2-9) specify integer requirements on location decision variables, v_j .

As indicated previously, there is a close relationship between the *p*-center problem and the LSCP. For a study region, suppose the solution to the LSCP, the minimum number of facilities for complete coverage, is p_s. Constraints (2-8) of the LSCP indicate that with the facility configuration corresponding to the LSCP solution, no demand point will be farther than the coverage standard *S* from a facility. In other words, the maximal distance between any demand point and the facility from which service is provided will not be larger than the specified coverage standard *S*. In the vertex *p*-center problem, if the number of facilities to be considered is p_s, then its solution, the minmax distance, must be no longer than *S*. On the other hand, assume the optimal solution to the vertex *p*-center problem is d_p. In the LSCP if the maximal service distance from a facility is specified as d_p, then the solution to the LSCP will not exceed *p*. That is, complete coverage of all demand is possible with *p* facilities for a distance standard of d_p. Therefore, the facility constraint in one of the two models can be treated as a bound on the solution to the other.

The connection to the LSCP has been utilized in many algorithms for solving the vertex *p*-center problem because the LSCP is less computationally intensive. Minieka (1970) proposed an exact solution procedure for the vertex *p*-center problem by solving a series of LSCPs. The algorithm starts with lower and upper bounds on the value of the *p*-center objective function. The average of the lower and upper bounds is used as the coverage radius to solve a LSCP. If the number of facilities needed to cover all demand points at this radius is less than or equal to *p*, the value of upper bound is reset to the coverage radius. Otherwise, the lower bound is reset to the coverage radius plus one. The process is repeated until the lower and upper bounds are equal. Similar to Minieka's approach, those proposed by Daskin (2000) and Elloumi *et al.* (2001) solve successive LSCPs and rely on carrying out an iterative search over coverage distances. Daskin (2000) solves the LSCP by lagrangian relaxation and uses the set covering model as a sub-problem in solving the vertex *p*-center problem. Elloumi *et al.* (2001) find a solution to the *p*-center problem by a greedy heuristic, using the integer programming formulation of the LSCP. Ilhan *et al.* (2002) improve Minieka's algorithm by searching for a tight lower bound on the optimal value in an initial phase. Mladenovic *et al.* (2003) propose variable neighborhood and Tabu search heuristics for the *p*-center problem without the triangle inequality. Chen and Chen (2007) present a variant of an existing relaxation algorithm. At each step of the algorithm, instead of solving a LSCP, they solve a slightly easier problem with the use of the feasibility sub-problem in Ilhan *et al.* (2002).

In general, these existing approaches to solve the vertex p-center problem involve solving a series of LSCP sub-problems. In order to decrease computational effort, it is necessary to reduce the number of sub-problems solved, and most seek to obtain tight lower and upper bounds by investigating the constraint sets of the problem (see Ilhan *et al.* 2002; Chen and Chen, 2007). However, this approach is quite limited and inefficient in large problem instances (e.g., $n = m = 1000$ or more). More discussion on this is provided in the following section.

2.3 Reformulating the vertex *p*-center problem

The problem size of a discrete location model is determined by the number of decision variables and constraints. In the vertex *p*-center problem defined in (2-1) through (2-6), let the number of demand points be n and the number of potential facility locations be m. This gives a total of $nm + m$ variables, including nm demand assignment variables and m facility variables in (2-6). The number of constraints is $nm + 2n + 1$, including one in (2-2), n in (2-3), nm in (2-4), and n in (2-5). For a large *p*-center problem in which $n = m = 1000$, this adds up to 1,001,000 decision variables and 1,002,001 constraints. The large problem size would impose tremendous requirements on memory storage and processing time for any existing optimization algorithm. There are practical limits in using general-purpose software to solve a problem that has so many variables and constraints (Church, 2008). Moreover, the vertex *p*-center problem is NP-hard, implying that for specific problem instances, optimality may be unlikely to attain within reasonable time, similar to the *p*-median problem discussed in Church (2003; 2008). This issue can be resolved if the problem is reformulated using significantly fewer variables and constraints without loss of problem properties, including optimality (Sorensen and Church, 1996).

The *p*-center problem and the *p*-median problem have almost identical constraints. The differences between the two model formulations are twofold. First, their objective functions are different. The objective of the *p*-median problem seeks to minimize the total weighted distance of all demand assignments. Secondly, integer restrictions on variables x_{ij} are not necessary in the *p*-median problem. However, the similarity indicates that methods for downsizing the *p*-median problem can be applied to the *p*-center problem with some modification.

Several approaches have been proposed to trim the problem size of location-allocation models. Rosing *et al. (*1979) propose aggregating constraints (2-4) into one constraints for each facility as follows:

$$\sum_{i=1}^{n} x_{ij} \le n y_j \qquad \forall j \in J \tag{2-10}$$

This reduces the number of constraints significantly. However, constraints (2-10) lack integer friendly properties. In order to preserve such properties, Rosing *et al. (*1979) look to keep constraints (2-4) associated with the *k*-closest facilities to a given demand. This approach does not trim the problem size considerably since the reduction in variables is trivial for most applications.

Rosing and Revelle (1997) use heuristic concentration to downsize the *p*-median formulation. The possible site set is reduced using a heuristic to identify the nodes called a "concentration set", which they argue is superior to those sites not used generally. The use of the concentration set results in a considerably smaller size integer linear programming model. However, this approach does not guarantee that an optimal solution will actually be found since it is a heuristic. Church (2003) uses the COBRA model to reduce the *p*-median model size by identifying and combining redundant assignment variables. But the reduction is limited, which makes it difficult to apply to large problem instances.

Hillsman (1979) proposes a strategy to downsize the formulation of a *p*-median model in a way such that for each demand point only demand-facility assignments in relatively close proximity and associated variables are represented in the model. The rationale for this type of trimming is that in many location models (such as *p*-median and *p*-center), demand is unlikely to be served by facilities located a large distance away, thus demand-facility assignments of greater distance and corresponding variables can be eliminated. Hillsman's method was subsequently used by Densham and Rushton (1992). However, Sorensen and Church (1996) show that this approach does not guarantee finding the optimal unconstrained *p*-median solution.

It is possible to use Hillsman's method for trimming the vertex *p*-center model size while preserving optimality. The key is to find a reasonable cutoff distance. If a small distance is selected as a cutoff, then a demand can only be assigned to a few close facility sites. It is possible that the assignments of the demand points far from the facilities cannot be represented in the model and hence optimality would not be guaranteed. If the cutoff distance chosen is too large, the unlikely assignments of demands to facilities located at a longer distance will also be represented in the model and model size is not reduced sufficiently.

As indicated previously, the cutoff distance must not be less than the optimal solution to the vertex *p*-center problem. Otherwise optimality would not be maintained. Another question to be answered is whether a cutoff distance equal to or larger than the optimal solution to the vertex *p*-center problem ensures optimality. This is assured according to the discussion on the relationship between the vertex *p*-center problem and the LSCP in the previous section. This rational gives rise to the following for trimming the *p*-center formulation while preserving optimality.

The vertex *p*-center problem is reformulated by replacing constraint set (2-1) through (2-6) with (2-11) through (2-16) follows:

$$Min\ \mathrm{z} \tag{2-11}$$

subject to:

$$\sum_{j=1}^{m} y_j = p \tag{2-12}$$

$$\sum_{j \in N_i} x_{ij} = 1 \qquad \forall i \in I \tag{2-13}$$

$$x_{ij} \le y_j \qquad \forall i \in I, \forall j \in N_i^c \tag{2-14}$$

$$\sum_{j \in N_i} d_{ij} x_{ij} \le z \qquad \forall i \in I \tag{2-15}$$

$$x_{ij} \in \{0,1\} \quad \forall i \in I, \forall j \in N_i^c \qquad y_j \in \{0,1\} \quad \forall j \in J \tag{2-16}$$

where N_i^c is the set of candidate facility sites within the cutoff distance S^c from demand *i*, i.e. $N_i^c = \{ j \mid d_{ij} \le S^c \}$.

(2-11) and (2-12) are identical to (2-1) and (2-2), respectively, while the others are different from their counterparts due to the introduction of neighborhood facility set N_i^c. On the one hand, if the cutoff distance is large enough to cover the whole facility set for any demand point, then the new formulation is identical to the original one. On the other hand, if the cutoff distance is small and on average N_i^c is a fraction of *J*, the number of variables and constraints will be substantially reduced in the new formulation. In order to specify N_i^c, we need to obtain a reasonable cutoff distance that does not hinder the optimal solution to the *p*-center problem from being identified. This can be achieved by solving a number of LSCPs as follows:

FINDCUTOFF(S^0, λ)

1) Start with a small effective coverage distance S^0 and solve the LSCP.
2) If the solution to the LSCP, the minimum number of facilities needed for complete coverage, is larger than *p*, increase S^i by λ, i.e. $S^{i+1} = S^i + \lambda$, and solve the LSCP with S^{i+1}.
3) Otherwise, stop.

The initial value S^0 can be determined in a simple way: compute the radius of a circle whose area is equal to 1/p of the total area of the region. Since the cutoff distance does not need to be precise, the step size λ can be relatively large so that the number of iterations is small.

2.4 Application and results

To demonstrate the effectiveness of the solution procedure proposed in Section 3 for solving the vertex p-center problem, an application to siting emergency warning sirens in a central Ohio city is detailed. Figure 2.1 shows the study area, Dubin, Ohio, a northwestern suburb of the city of Columbus, Ohio. This region was utilized in Current and O'Kelly (1992), Murray and O'Kelly (2002) and many others.

We will consider discrete representations of regional demand and potential facility location in this study. The location space is represented through the use of regularly spaced points. Three grid patterns are considered, which are the same as used in Murray and O'Kelly (2002), and are shown in Figure 2.2. The spacing values, R, were chosen from 200m to 450m in 50m increments, giving six representations for each grid representation. This results in 18 regularly spaced representations of the study region. We use these same 18 discrete representations for both service demand and potential facility locations.

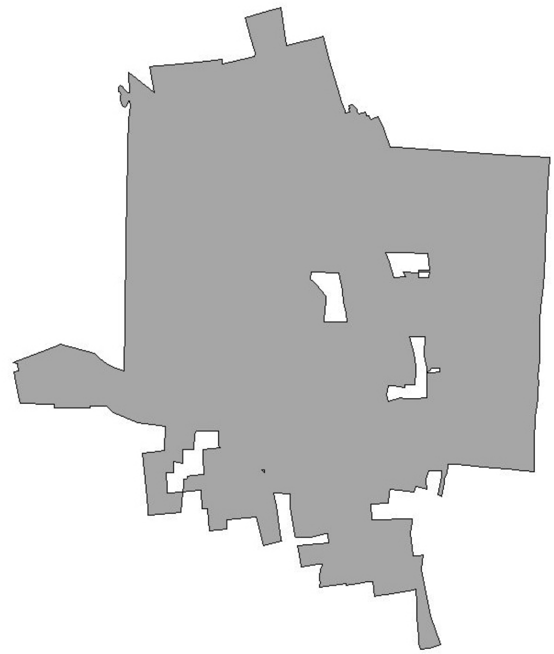


Figure 2.1: The City of Dublin, Ohio

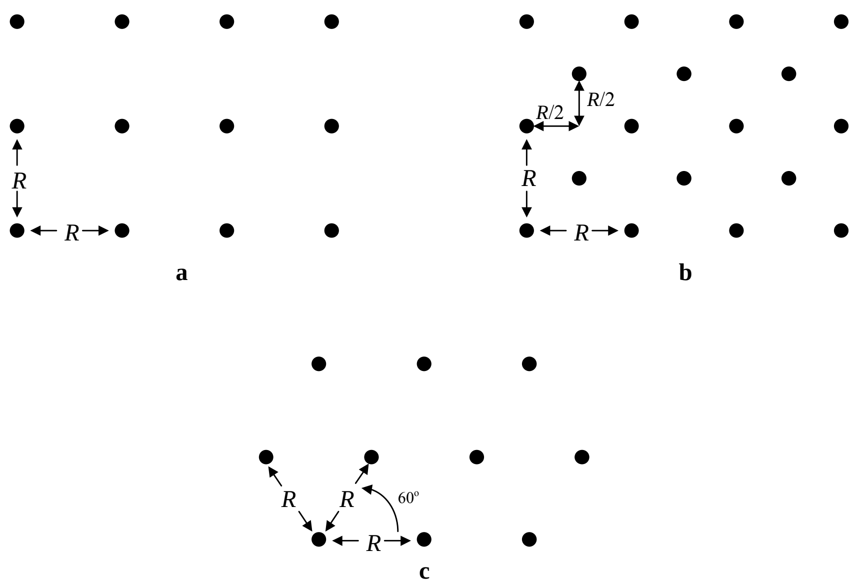


Figure 2.2 (a-c). Alternative grid representations

a: Regular spacing. b: Offest regular spacing. c: geometrically associated spacing

(Source: Murray and O'Kelly, 2002)

The analysis was carried out on a personal computer with a Xeon 3.00GHz processor and 2.0 GB of RAM. ArcView GIS version 3.2 was utilized to manage and manipulate data layers including spatial layers of the service demand, potential facility locations, and coverage or spatial proximity. The optimization problems (the LSCP and the vertex *p*-center problem) were programmed in ArcView with Avenue and written to a text file. The text file was read in and the problem was solved using ILOG CPLEX version 10.1.1, a commercial optimization software package. Results were exported from CPLEX and read into ArcView for display and analysis.

The value of *p* to be considered was chosen as 25. There are several reasons for selecting this value. First, previous research has found that 24 to 26 sirens with effective service radius of 976m would be needed to provide complete service coverage to the Dublin region (Murray *et al.* 2008). Secondly, from an operational point of view, the approach of variable reduction described previously is more effective for a relatively large value of *p*. Moreover, the result based on this value facilitates further study and discussion in following chapters.

A cutoff distance first needs to be determined. This requires solving a small number of LSCPs using the algorithm FINDCUTOFF. The Dublin region is about 45.88 square kilometers in size, which is equivalent to the total area of 25 circles with a radius of $\sqrt{(45.88/25)/\pi} = 0.764$ km, i.e., 764m. Considering the region is relatively compact and unavoidable overlapped area between service coverage will be encountered, we chose 800m as a starting value for the algorithm. The increment was selected as 100m. In all cases, the algorithm stopped after only two or three iterations. This indicates that for some representations, 25 facilities with radius of 900m are sufficient to cover the whole region, while other representations require 25 facilities with a larger service radius for complete coverage. So the cutoff distance could be chosen as 900 m or 1000 m for the corresponding situations. The results were consistent with those of Murray and O'Kelly (2002): for a service radius of 976 m, 19 to 25 warning sirens were needed to completely cover the same region using various regularly spacing representations. For the purpose of simplicity and comparison, a cutoff distance of 1000m was chosen for all regular spacing representations, which assures the existence of an optimal solution to the *p*-center problem for $p = 25$.

The cutoff distance of 1000m was then applied to reformulate the *p*-center problem. Table 2.1 to 2.3 reports the results of variable deduction using regular spacing, offset regular spacing and geometrically associated spacing representations depicted in Figure 2.2. The first column is the discrete representation utilized for demand and potential facility sites. This is followed by the number of variables and constraints in original form defined in (2-1) through (2-6) and that in reduced form reformulated in (2-11) through (2-16). For all spatial representations, the reduction of problem size was substantial. Approximately 94% of the variables and constraints were detected as unnecessary or redundant and thus removed from the formulation. In other words, about 6% of the variables and constraints are sufficient to represent all characteristics of the corresponding *p*-center problem, and thus contain all information needed for solving the problem.

R values	n (m)	Variables	Constraints	Reduced variables		Reduced constraints	
200	1,148	1,319,052	1,320,201	77,782	(5.9%)	78,931	(6.0%)
250	731	535,092	535,824	30,450	(5.7%)	31,182	(5.8%)
300	494	244,530	245,025	15,300	(6.3%)	15,795	(6.4%)
350	365	133,590	133,956	7,858	(5.9%)	8,224	(6.1%)
400	290	84,390	84,681	5,420	(6.4%)	5,711	(6.7%)
450	218	47,742	47,961	2,584	(5.4%)	2,803	(5.8%)

Table 2.1: Variable deduction using regular spacing representation

R values	n (m)	Variables	Constraints	Reduced variables		Reduced constraints	
200	2,291	5,250,972	5,253,264	305,692	(5.8%)	307,984	(5.9%)
250	1,467	2,153,556	2,155,024	123,818	(5.7%)	125,286	(5.8%)
300	1,010	1,021,110	1,022,121	58,324	(5.7%)	59,335	(5.8%)
350	747	558,756	559,504	31,098	(5.6%)	31,846	(5.7%)
400	574	330,050	330,625	18,160	(5.5%)	18,735	(5.7%)
450	448	201,152	201,601	11,282	(5.6%)	11,731	(5.8%)

Table 2.2: Variable deduction using offset regular spacing representation

R values	n (m)	Variables	Constraints	Reduced variables		Reduced constraints	
200	1,322	1,749,006	1,750,329	100,712	(5.8%)	102,035	(5.8%)
250	844	713,180	714,025	43,262	(6.1%)	44,107	(6.2%)
300	585	342,810	343,396	18,648	(5.4%)	19,234	(5.6%)
350	430	185,330	185,761	11,364	(6.1%)	11,795	(6.3%)
400	329	108,570	108,900	5,584	(5.1%)	5,914	(5.4%)
450	264	69,960	70,225	4,452	(6.4%)	4,717	(6.7%)

Table 2.3: Variable deduction using geometrically spacing associated representation

The reformulated *p*-center problems were solved using CPLEX. The application results are presented in Table 2.4 to 2.6 using the three representations for potential siting locations and regional demand. The first column is the discrete representation utilized. The next three columns indicate CPLEX solution characteristics for solving the associated *p*-center problem: *Iterations*, *Nodes*, *Gap*, and *Time* (in seconds). *Gap* represents the distance from the objective function value to the current best MIP bound. The last column shows the mixed integer programming (MIP) solution, the value of the objective function found. CPLEX was forced to stop solving the problem when no significant improvement was found within a large amount of time indicated in the *Time* column.

In four cases, no feasible MIP solution was attained within 20 hours, including R = 200m for all three representations and R = 250m for the offset spacing representation, which is the most spatially expansive among the three representations. Optimal or near-optimal solutions were obtained for only two cases, R = 450m regular spacing and R = 400m geometrically associated spacing representations. With a large positive *Gap* value, the solutions in other cases are feasible and whether they are optimal is unknown. As expected, none of the solutions are larger than 1000, the cutoff distance. The original *p*-center problems with large R values (450m, 400m and 350m) were also solved using CPLEX. CPLEX was allowed to run for 20 hours, much longer than for their counterparts in reduced form. Feasible MIP solutions were obtained for only a few cases when R = 450m and R = 400m using regular spacing and geometrically associated spacing representations. Moreover, the MIP solutions were not better than those for reduced form formulation. This somehow implies that reformulated models well preserve the characteristics of the *p*-center problem.

R values	Iterations	Nodes	Gap	Time	Objective
200	3,221,449	210	Inf	83,726	NA
250	26,713,297	10,581	40.38%	163,400	1,000.0
300	15,717,785	17,595	38.32%	47,968	948.7
350	32,467,547	49,497	38.90%	43,443	989.9
400	60,430,473	182,763	30.87%	44,897	894.4
450	5,415,634	27,860	0.01%	2,392	900.0

Table 2.4: *p*-center solutions for the regular spacing representation

R values	Iterations	Nodes	Gap	Time	Objective
200	NA	NA	NA	NA	NA
250	3,268,888	60	Inf	84,188	NA
300	3,986,607	340	36.51%	77,736	948.7
350	39,520,082	13,885	38.89%	161,462	989.9
400	3,549,197	1,669	32.56%	14,981	894.4
450	5,932,159	7,532	36.61%	13,966	954.6

Table 2.5: *p*-center solutions for the offset regular spacing representation

R values	Iterations	Nodes	Gap	Time	Objective
200	2,577,934	120	Inf	84,013	NA
250	6,945,719	750	40.46%	82,436	1,000.0
300	772,742	262	33.94%	2,074	900.0
350	4,563,954	4,634	35.82%	13,385	926.0
400	3,271,557	4,145	0.00%	3,283	800.0
450	17,578,310	72,467	32.71%	13,621	900.0

Table 2.6: *p*-center solutions for the geometrically associated spacing representation

2.5 Summary

This chapter explored a simple but effective approach for solving large vertex *p*-center problem. By introducing a neighborhood facility set, the *p*-center problem can be reformulated such that redundant variables and constraints are removed but characteristics including the optimality of the problem are preserved. In the application to Dublin region, the problem size of the reformulated model was substantially smaller than in the original form. This enabled the use of general-purpose optimization software (e.g. CPLEX) to solve large vertex *p*-center problems in which the number of vertices was over 1000. This number is larger than that of any recent work on the vertex *p*-center problem attempting to find an optimal solution, as far as we know. On the other hand, a finite set of regularly spacing points is used to approximate the region. Theoretically, the smaller grid size used to discretize the region, the better the solution. However, this would require more intensive computational efforts. More discussion on spatial representation and its effect on model solution will be provided in subsequent chapters. In particular, the results for the vertex *p*-center problem are to be used as a benchmark for its continuous space counterpart. This is to evaluate the effect of discretizing continuous location space.

CHAPTER 3

CONTINUOUS SPACE P-CENTER PROBLEM[1]

3.1 Introduction

The *p*-center problem, a location-allocation model, seeks to locate *p* facilities such that the maximum distance from any demand site to its closest service facility is minimized. This minimax model has been proposed for public facility planning and emergency services management, such as EMS and fire protection (Love *et al.* 1988, Daskin, 1995).

It is well-known that the discrete *p*-center problem is NP-hard (Daskin, 1995), while the continuous or planar counterpart is at least as difficult to solve given that it assumes demand is distributed throughout an area and potential facilities can be located anywhere in the analysis region. It is impossible to enumerate all demand locations and potential facility sites in the continuous case. Thus, it is difficult to extend traditional discrete methods for solving the continuous space *p*-center problem. Attempts have been

[1] This chapter is based upon a paper, "Solving the continuous space *p*-centre problem: planning application issues," published in *IMA Journal of Management Mathematics* (17) and co-authored with Dr. Alan T. Murray and Dr. Ningchuan Xiao.

made to efficiently solve this problem heuristically. However, some limiting conditions are assumed in the use of heuristics.

Suzuki and Okabe (1995) propose a Voronoi diagram heuristic to solve the continuous space *p*-center problem. The heuristic iteratively computes a Voronoi diagram using *p* generator points and moves those *p* points to the centers of resulting Voronoi polygons. This process continues until center locations do not change. Several simplifying assumptions for applying the Voronoi diagram heuristic are established by Suzuki and Okabe (1995): demand is assumed uniformly distributed; the region is assumed to be a rectangle; analysis of a simple Voronoi polygon in solving associated 1-center problems is assumed; and, no restrictions on potential facility locations are assumed. The degree to which these are potentially problematic for practical applications is unknown.

This chapter addresses application oriented issues in solving the continuous space *p*-center problem in support of practical planning concerns. The next section reviews research related to the *p*-center problem and the Voronoi diagram heuristic. Section 3.3 addresses associated practical issues. Section 3.4 explores the complexity of solving the continuous space *p*-center problem, which is followed by the presentation of a spatially restricted version of the *p*-center problem and methods proposed for solving it. Application results are presented in Section 3.5. Finally, a discussion and concluding comments are provided.

3.2 Continuous space *p*-center problem

The continuous space *p*-center problem may be formally stated as follows (Megiddo and Tamir, 1983; Suzuki and Drezner, 1996):

$$\min_{(\hat{x}_j,\, \hat{y}_j) \in A,\ j=1,2,\ldots,p} \{ \max_{(x,\, y) \in A} \{ \min_{j} d_j(x,\, y) \} \} \tag{3-1}$$

where

A = analysis region;

p = number of facilities to be located;

(x, y) = positional reference to a demand location, i.e. $(x, y) \in A$;

$(\hat{x}_j, \hat{y}_j)$ = geographic coordinates of facility j; and

$d_j(x, y)$ = distance from demand location (x, y) to facility j.

This problem implicitly assumes uniformly distributed demand. The objective of the continuous space *p*-center problem is to site *p* "centers" in the plane so as to minimize the maximum distance between any location in *A* and its nearest center. Again, centers could represent any kind of service facility, public or private, such as a fire station, retail outlet, or a warning siren. The problem involves determining the best location for each facility. The optimal solution, $(\hat{x}_1, \hat{y}_1)$, … , $(\hat{x}_p, \hat{y}_p)$, is called a *p*-center and the corresponding longest distance to the closest center is called the *p*-radius (Megiddo and Tamir, 1983). *p* circles with their centers positioned at $(\hat{x}_j, \hat{y}_j)$ and radius equal to the *p*-radius will necessarily enclose (or cover) the entire region *A*, if measured by Euclidean distance. Therefore, the continuous *p*-center problem is equivalent to finding *p* equal circles with the smallest possible radius to cover every point in *A* (Suzuki and Drezner, 1996).

3.2.1 The 1-center problem

As the simplest case of the continuous space *p*-center problem, the 1-center problem ($p = 1$) has received much attention. Solving the 1-center problem involves determining where a circle with minimum radius should be placed in order to cover the entire region. This covering circle is variably referred to as a minimum circle (Hearn and Vijay, 1982), minimum spanning circle (Oommen, 1987), smallest enclosing circle (Xu *et al.* 2003), smallest enclosing disc (Welzl, 1991), or a minimal covering circle (MCC) (Plastria, 2002). With the assumption that *A* is a polygon, locations defining the boundary of *A* are often referred to as vertices of *A*. A circle that covers all the vertices of *A* necessarily covers *A*. Hence, an approach for solving the continuous space 1-center problem can focus on only the vertices of *A*. Efficient algorithms for solving the 1-center problem have been developed, including the work of Elzinga and Hearn (1972), Hearn and Vijay (1982), Oommen (1987), Welzl (1991), and Plastria (2002).

3.2.2 Voronoi diagram

The more general continuous space *p*-center problem remains a challenge to solve. The ability to address this problem has been limited by analytical capabilities associated with geometric computation (Suzuki and Okabe, 1995). However, progress in computational geometry, particularly the development of efficient algorithms for constructing the Voronoi diagram, has enabled researchers to address this problem using heuristic techniques. Recent work includes that of Suzuki and Okabe (1995), Suzuki and Drezner (1996), Okabe and Suzuki (1997), and Plastria (1995, 2002). In particular, Suzuki and Okabe (1995) developed a Voronoi diagram based heuristic to solve the continuous *p*-center problem.

A Voronoi diagram is a spatial tessellation of the plane. Given a finite set of p points in the plane, $P = \{p_1,\ldots, p_p\}$, the two dimensional space can be partitioned such that each location is associated with its closest member of the point set P (as measured by Euclidean distance). The tessellation of the plane consists of a set of polygons, $V = \{V_1,\ldots, V_p\}$, each of which is associated with the corresponding point in P. This tessellation is called the *Voronoi diagram*, the polygons constituting the diagram *Voronoi polygons*, and each member of P is called a *generator* or *generator point*. The boundary shared by two Toronoi polygons is a *Voronoi edge*, and the point where three or more Voronoi edges meet is a *Voronoi point* (Okabe *et al.* 2000). Voronoi edges and Voronoi points have a number of special characteristics, which have been used to develop efficient algorithms in many locational optimization applications. Figure 3.1 shows an example of a Voronoi diagram with generator points bounded by a rectangle. It can be seen that most of Voronoi points are the points where exactly three Voronoi edges meet. This is called a non-degenerate case (Okabe *et al.* 2000).

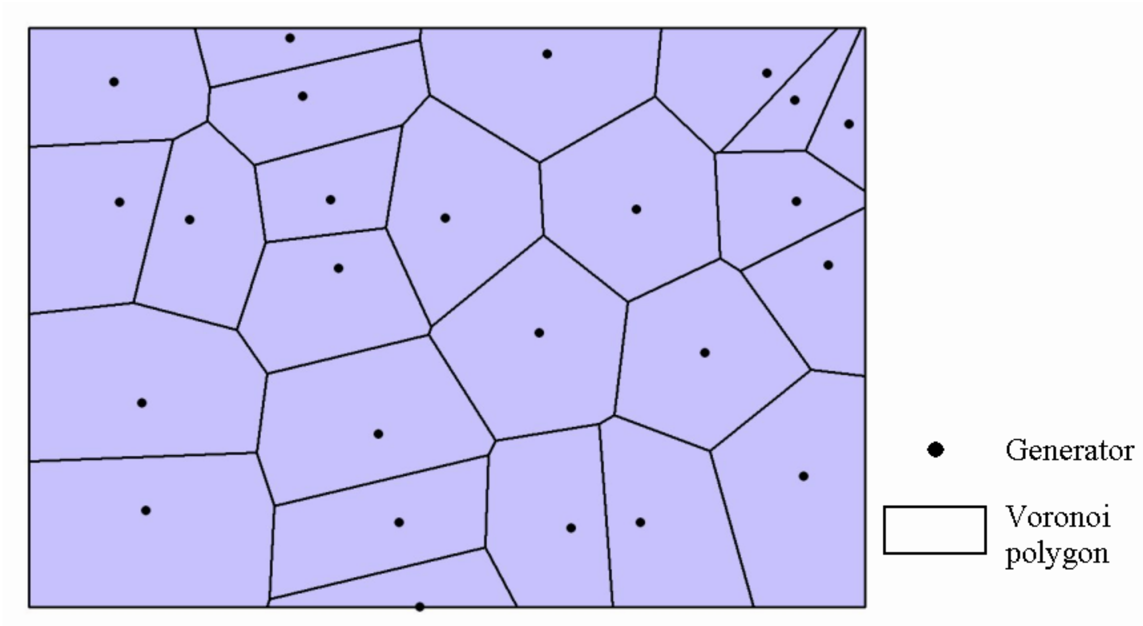


Figure 3.1: Voronoi diagram

A simple, but important, characteristic can be observed from the definition of a Voronoi diagram. If location (x, y) is in Voronoi polygon V_i, then p_i is the nearest generator to this location (Suzuki and Okabe, 1995). For service facilities sited at generator points, the Voronoi polygons (V) represent closest-assignment service areas. Given this property, the Voronoi diagram has been suggested as a tool for solving many continuous space location problems. Okabe and Suzuki (1997) reviewed a class of location optimization problems that can be solved using the Voronoi diagram, including the continuous space p-median problem and p-center problem.

3.2.3 Voronoi diagram based heuristic

The p-center problem is a non-convex optimization problem and it is challenging to find its global optimum. Several heuristic solution methods have been developed to provide near-optimal solutions. These include the Voronoi diagram based heuristic method developed by Suzuki and Okabe (1995).

In the p-center problem we wish to find a configuration of p facilities that give the minimum value of the maximum distance among the distances between users and their nearest facility, i.e., p-radius. Stated differently, we wish to find a configuration of the centers of p disks that cover the region whose radius is the smallest. Suppose a user chooses the closest service facility. Given a configuration of p facilities and corresponding Voronoi diagram, it is easy to see that a user in the ith Vororoi polygon, V_i, uses the ith facility, p_i, because it is the closest facility. Since a Voronoi polygon is convex, among the users in V_i, a user whose distance is the longest from p_i is located at the boundary of V_i. The maximum of the objective function (3-1) is hence achieved at one of the Voronoi points or a point on the region's boundary. Note that the minimum of the longest distance to a point in a polygon is attained at the center of the polygon. Hence, in a near-optimal configuration of the p-center problem, each facility is placed near the center of its Voronoi polygon. This is underlying theory of the Voronoi diagram based heuristic (VDH) to solve the continuous space p-center problem. The heuristic method consists of following steps (Suzuki and Okabe 1995):

1) Generate p centers randomly in A as an initial configuration of location sites.

2) Construct the Voronoi diagram generated by the p centers.

3) Compute the center of each Voronoi polygon (a 1-center problem).

4) If no center has moved more than a pre-specified tolerance distance, or the maximum number of iterations is exceeded, stop. Otherwise, go to Step 2.

The maximum distance between the generator points (facility locations) and the vertices of the associated Voronoi polygons is the maximum distance from a user to its closest center. The maximum distance decreases as the centers disperse in subsequent iterations. Since the p-center problem is a non-convex optimization model, a local minima will be identified by this heuristic upon termination. Suzuki and Okabe (1995) indicate that the quality of a heuristic solution depends on the initial configuration used in Step 1, so they suggest repeating the above process many times with different initial center configurations. Suzuki and Drezner (1996) apply this heuristic to a unit square with different p values, followed by a "finishing-up" algorithm to improve convergence of the maximum distance.

Worth noting is that the VDH requires a Voronoi diagram to be derived at each iteration. Thus, an efficient computational method for constructing the Voronoi diagram is critical. Fortunately, a number of algorithms for constructing a Voronoi diagram have been developed in computational geometry (Okabe *et al.* 2000). Major methods include the plane sweep method, the divide-and-conquer method and the incremental method. The computational time of the divide-and-conquer method is $O(p^2)$ in the worst case and $O(p\log(p))$ on the average where p is the number of generate points.

3.3 Practical issues

Except for uniformly distributed demand, there are three assumptions implicit to the VDH for solving the continuous space *p*-center problem. First, region *A* is a rectangle (or a square). Although Suzuki and Drezner (1996) suggest that there are no restrictions on the shape of region *A*, the VDH is applied to a unit square. Second, a simple Voronoi polygon is assumed for the 1-center problems to be solved. The average number of vertices for Voronoi polygons is suggested as being approximately six in Suzuki and Drezner (1996), but can be significantly more for a non-rectangular region. Finally, there is no restriction on center locations. Facilities can be located anywhere in the plane. The only restriction in the work of Suzuki and Drezner (1996) is that the initial *p* centers are located in *A*. With the convexity assumption of *A*, centers will never migrate outside *A* in the Voronoi diagram heuristic. An important question is what are the implications of these simplifying assumptions for complex planning regions. Each of the three assumptions is now explored in more detail.

It is generally the case that *A* is non-convex, possibly with holes. This means that one or more Voronoi polygons would necessarily be non-convex. Figure 3.2 shows an example of a non-convex region with holes where all four Voronoi polygons are non-convex.

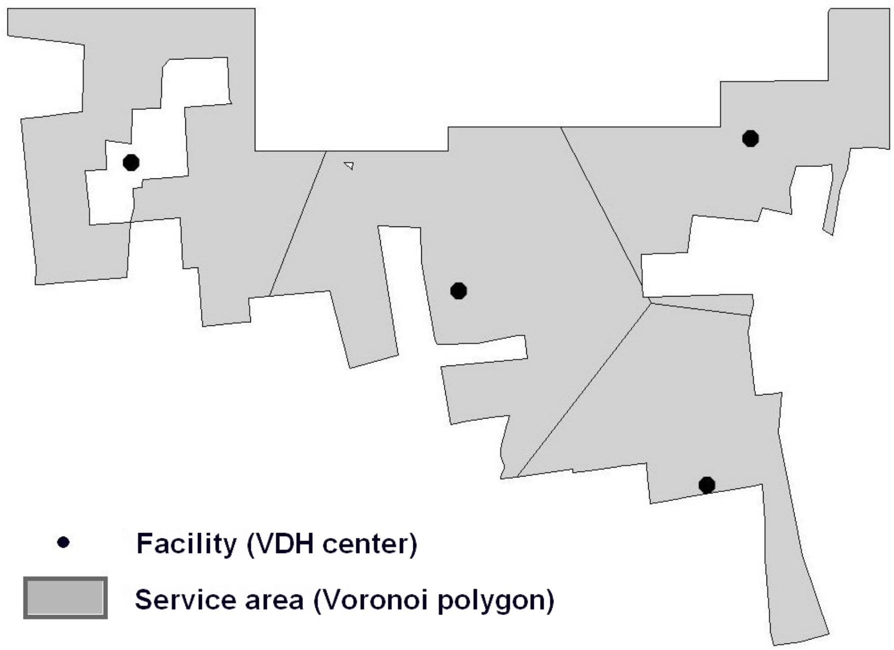


Figure 3.2: A non-convex region

An important issue that arises from the non-convexity of *A*, and associated Voronoi polygons, is that a large number of vertices may be encountered during the iterations of the VDH, in which the computation of 1-center solutions must be conducted thousands, if not more, times. The non-convexity of *A*, therefore, will significantly increase the computational time of the VDH. Thus, solving 1-center problems must be computationally efficient if the VDH is to be successful. For example, the left-most Voronoi polygon in Figure 3.2 contains more than 50 vertices. The MCC (solution to the 1-center problem) is uniquely defined by two or three of the vertices that lie on the circle. A naïve method for finding the MCC is to construct all possible circles and choose the smallest one that covers all the vertices. With time complexity of $O(n^3)$ this would require examining over 10,000 circles in this case for a relatively small number of vertices. This quickly becomes burdensome in an iterative process, and suggests practical limits for solving planning problems. That is, the heuristic might not be able to be applied to a complex region unless an efficient approach is utilized.

The final practical issue to be examined is the unrestricted nature of facility location in the VDH. The reality of facility siting is that feasible or suitable locations for centers may be restricted to A, or sub-regions of A. In this case, the VDH cannot be directly applied since a solution may be outside of A. In Figure 3.2, as an example, the VDH solution demonstrates this issue. The four dots are the centers created by the VDH that does not consider the non-convexity of the area, and the left-most center is not in the region. The configuration of these four centers, therefore, is not feasible when their locations must be in the region.

3.4 Refining the *p*-center formulation

The issue of region non-convexity does not by itself create a problem in (3-1) nor the VDH to solve it. Since the facilities can be located anywhere, this is an unconstrained problem and the maximum distance to the facilities converges as the facilities disperse more and more evenly in the iterative process. However, we will see that non-convexity underlies confounding complexity and feasibility problems.

3.4.1 Exploring complexities

With regard to the issue of complex Voronoi polygons, the implication is that a computationally efficient approach is necessary for solving associated 1-center problems. The method developed by Welzl (1991) is a good choice. Welzl's approach is a simple, but ingenious, randomized incremental method for finding the MCC. The algorithm grows the MCC point by point. A unique circle is initially defined by the first two distinct points forming its diameter. At each step, a new vertex is considered. If it lies outside the current circle, then the circle must be enlarged with the constraint that this new point is on the circle perimeter. The algorithm terminates after a finite number of steps since each successively constructed circle increases in size. It has proven to be computationally efficient with time complexity of $O(n\log(n))$, where n is the number of vertices. Moreover, Welzl's method is simple to implement in a geographic information systems (GIS) environment. As an example, compared to the naïve method to find the MCC for the left-most Voronoi polygon in Figure 3.2, the Welzl approach needs to examine only about 60 circles to optimally solve the MCC. This is a reduction in computational effort of approximately 160 times!

The issue of solution space feasibility complicates the original problem, (3-1), and the Voronoi diagram heuristic. Restricting feasible facility locations in the *p*-center problem makes sense, but actually defines a slightly different optimization problem. Such a constrained problem may be formulated as follows:

$$\min_{(\hat{x}_j,\hat{y}_j)\in S,\ j=1,2,\ldots,p} \{ \max_{(x,y)\in A} \{\min_j d_j(x,y)\}\} \qquad (3\text{-}2)$$

where S represents the feasible siting region. Imposing constraints on $(\hat{x}_j, \hat{y}_j)$ is the only difference from the original formulation. If S is equivalent to A and A is convex, the two problems, (2.4) and (2), are identical. In the cases when S is not equivalent to A or A is non-convex, the problems are structurally different.

Plastria (2002) suggests an approach to determining the constrained MCC (CMCC) for a set of points $\{P_i\}$. If the MCC solution does not lie in S, the optimal site of the CMCC can only be "… either a point of S closest to some point P_i, and then P_i lies on the CMCC, or the point of intersection of the bisector of two points P_i and P_j with the boundary of S, closest to P_i and P_j, and then both P_i and P_j lie on the CMCC" (Plastria, 2002). This argument is problematic for non-convex regions, however. One counter example is shown in Figure 3.3, where P_1, P_2, and P_3 are three points lying on the boundary of the feasible region, S (i.e., the shaded area). b_1, b_2, and b_3 are three bisectors of these three points. According to the method proposed by Plastria (2002), the center of the CMCC that covers all the three points (P_1, P_2, and P_3) must be either one of the vertices at P_1, P_2, or P_3, because they are points of S closest to P_i, or one of N_1, N_2, and N_3 because each of N_1, N_2, and N_3 is the point of intersection of the bisector with the boundary of S closest to the two points that form the bisector. The covering circles would be one of three circles centered at P_1, P_2, or P_3 with zero radii (i.e., P_i itself) or the three circles centered at N_1, N_2, and N_3, as shown in Figure 3.3. None of these circles cover all three vertices, and hence they are not the CMCC. The CMCC should be the circle centering at C_1 and P_1 and P_3 lie on the circle, computed by the algorithm developed in next section. In this case, C_1 is one of the points of intersection of the bisector b_2 with the boundary of *S*. For comparison, Figure 3.3 also shows the MCC with its center at C_0 and all three points lying on the circle.

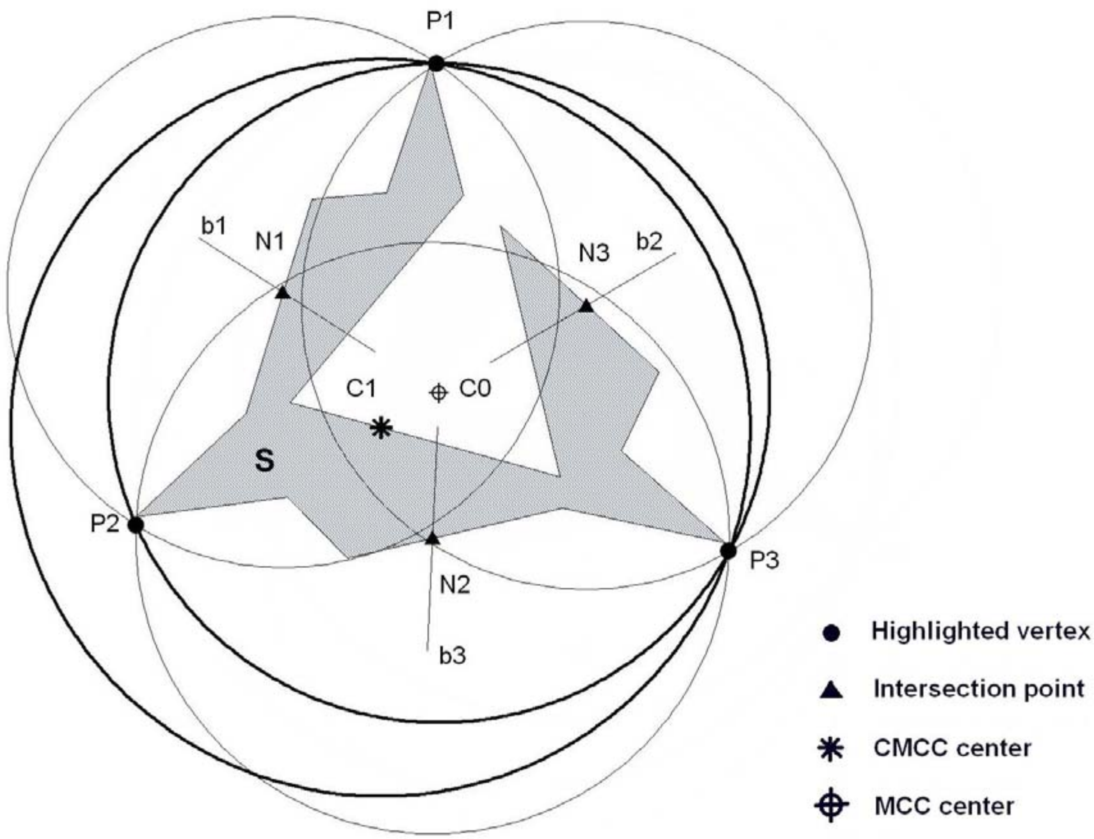


Figure 3.3: Constrained MCC (CMCC)

3.4.2 Extension to the VDH

Extending the VDH to solve the contained *p*-center problem is a logical approach. An issue in (3-2) is that facilities must be in *S*. One way to approach this in the framework of the VDH is to focus on the 1-center sub-problems. This means that a constrained 1-center problem would need to be solved, where solution feasibility is maintained. A circle solving this constrained 1-center problem would then be a constrained minimum covering circle (CMCC), as discussed above. Based on the CMCC, the constrained Voronoi diagram heuristic (CVDH) for solving the constrained *p*-center problem, (2), is summarized as follows:

1) Generate *p* centers randomly in *S* as an initial configuration of location sites.
2) Construct the Voronoi diagram generated by the *p* centers.
3) Compute the CMCC for each Voronoi polygon such that $(\hat{x}_j, \hat{y}_j) \in S$ for all *j*.
4) If no center has moved more than a pre-specified tolerance distance, or the maximum number of iterations is exceeded, stop. Otherwise, go to Step 2.

The important issue in the CVDH is to solve the CMCC in Step 3, which is the only difference from the original VHD. There are properties of the CMCC that will prove useful in developing a solution approach.

Property 1: For a polygon with a set of vertices $\{P_i\}$ and feasible region *S*, the CMCC is one of two cases:

Case (a): The CMCC is equal to the MCC, with its center lying in S, and it's defined by two vertices forming a diameter or three vertices forming an acute triangular.

Case (b): The CMCC is larger than the MCC, with its center lying on the boundary of S.

Proof: Case (a) defines the MCC, where facility sites are unconstrained and satisfy solution space feasibility. Case (b) states that the CMCC cannot be smaller than the MCC. Imposing the siting constraint must result in a larger covering circle being required. Moreover, there are at most two vertices lying on the CMCC in case (b). Otherwise if three or more vertices lie on the circle, this would be the MCC, which it is not. ■

In order to prove case (b), a second property is necessary:

Property 2: When the CMCC lies on the boundary of S, it is defined by one of two possible cases:

Case (i): The CMCC is defined by one vertex (P_i) and its center is one of the closest points of boundary line segments of S to P_i.

Case (ii): The CMCC is defined by two vertices (P_i and P_j) and its center is one of the points of intersection of the bisector defined by P_i and P_j with the boundary of S.

Proof: We first consider case (i) of one vertex (P_i) lying on the CMCC centered at C. The radius of the CMCC is then $d(P_i, C)$. If C is not on the boundary of S, moving C along the direction from C to P_i would result in a smaller $d(P_i, C)$, and hence a smaller covering circle. This is a contradiction with assumption that the covering circle defined by P_i centered at C is smallest.

Case (ii) of only two vertices (P_i and P_j) defining the CMCC (with its center at C) is more complicated. The radius of the CMCC is $d(P_i, C) = d(P_j, C)$. If P_i and P_j forms the diameter of the CMCC, the CMCC is the same as the MCC. Otherwise, C must be on the bisector of P_i and P_j. If C is not on the boundary of S, moving C along the bisector in the direction closer to both P_i and P_j would result in a smaller $d(P_i, C)$, and hence a smaller covering circle. This contradicts the assumption that the covering circle defined by P_i and P_j centered at C is smallest. Therefore, the center of the CMCC in case (b) of property 1 must be on the boundary of S. Suppose the center is on a line segment of the boundary, it must be the closest point of this segment to the one or two vertices defining the CMCC. ■

An example of case (i) of property 2 is illustrated in Figure 3.4 where P_1, P_2, and P_3 are three demand points lying on S. The center of the CMCC, C_1, lies on the line segment L_1. P_1 is the only demand point lying on the CMCC. Each line segment of the boundary of S has a closest point to P_1. C_1 is such a point (the closest point of L_1 to P_1). Figure 3.5 shows an example of case (ii) where the CMCC is centered at C_1 with two demand points, P_1 and P_3, lying on the circle. C_1 is one of the points of intersection of the bisector defined by P_1 and P_3 with S.

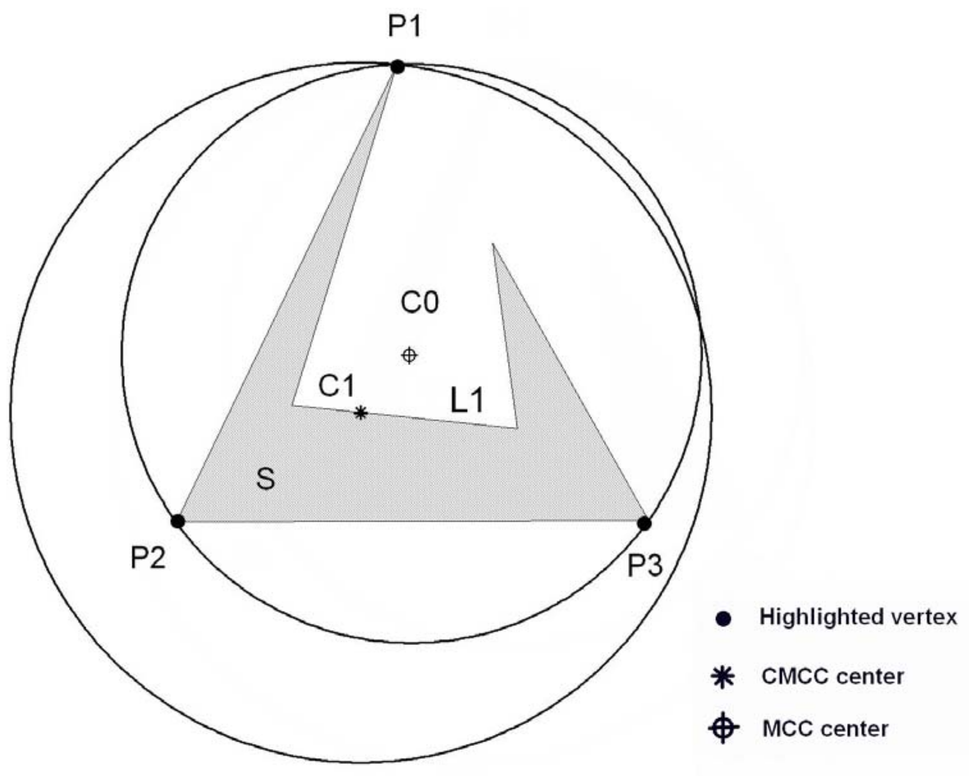


Figure 3.4: The CMCC with one point lying on the CMCC

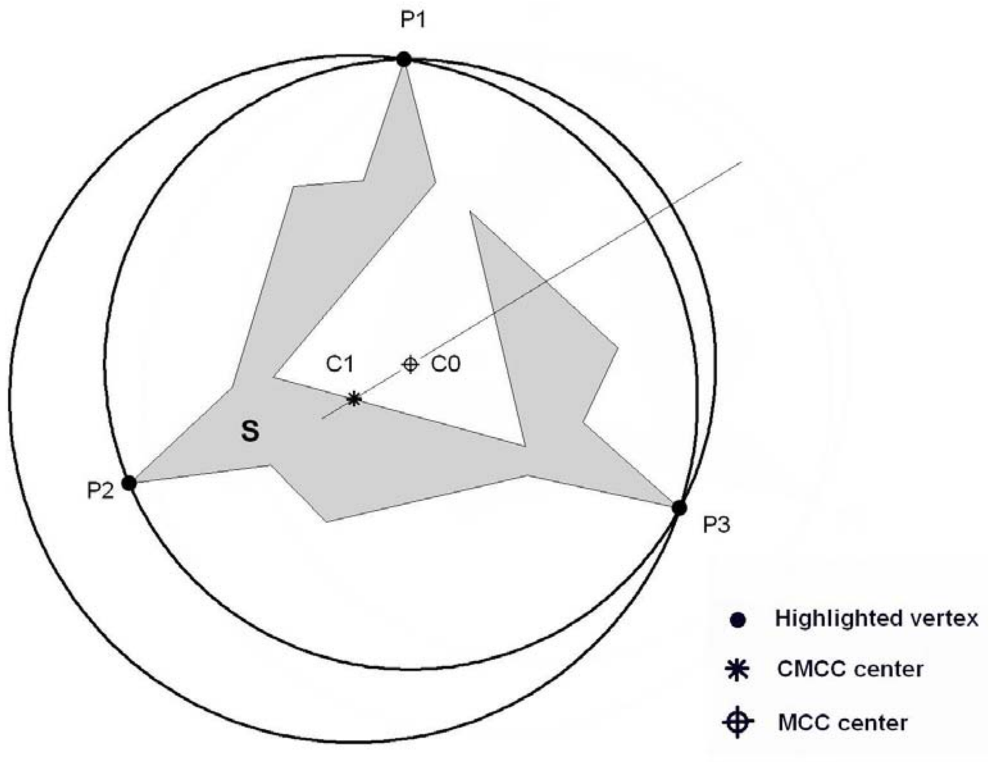


Figure 3.5: The CMCC with two points lying on the CMCC

Given these properties, solving the CMCC can be accomplished as follows:

1) Solve the unconstrained problem, yielding the MCC. If the center of the MCC is in *S*, stop; the solution to the CMCC has been found.
2) Compute the convex hull of the demand region.
3) Evaluate all vertices $\{\hat{P}_i\}$ of the convex hull in turn. For each $\hat{P}_i$, find its closest point on each line segment of the boundary of *S*, namely X_j. If $d(\hat{P}_i, X_j)$ is larger than the MCC radius, then consider the new circle

centered at X_j with a radius $d(\hat{P}_i, X_j)$. If this is a covering circle, save it as a possible candidate CMCC.

4) Evaluate all pairs of vertices of the convex hull. For each pair, $\hat{P}_i$ and $\hat{P}_j$, construct its bisector. Let $\{X_{\text{int}}\}$ be the points of intersection (if they exist) of this bisector with the boundary of *S*. If $d(\hat{P}_i, X_{\text{int}})$ is larger than the MCC, take the circle centered at X_{int} with radius $d(\hat{P}_i, X_{\text{int}}) = d(\hat{P}_j, X_{\text{int}})$. If this circle is a covering circle, save it as a possible candidate CMCC.
5) The CMCC is the smallest circle among the candidate circles found in steps 3 and 4.

This algorithm evaluates all vertices on the convex hull since the covering circle for the hull is the same as that for the original shape. Computational time complexity of constructing the convex hull for a shape with *n* vertices is $O(n\log(n))$. Moreover, the algorithm does not check all possible circles with one or two vertices on the circle to examine if they are covering circles; instead, only those that are larger than the MCC are examined. If a circle is smaller than the MCC, it is not a covering circle.

3.5 Analysis results

Again, our study area is Dublin, Ohio, as shown in Figure 2.1. The study region is a non-convex polygon with holes and an irregular boundary. The VDH and CVDH approaches are first evaluated for solving the associated *p*-center problems. The analysis was carried out on a Pentium IV 3.2 GHz personal computer running Windows XP with 1 GB RAM. ArcView GIS version 3.2 was utilized to manage and manipulate data layers and derive Voronoi diagrams. The VDH and the CVDH were implemented using the Avenue script language provided in ArcView GIS.

The performance and solution quality of the heuristics were first evaluated based on the solutions for the *p* value of 25. For this *p* value, both the VDH and the CVDH were applied 50 times with different randomly generated initial siren locations. A tolerance of $\varepsilon = 0.2$ meters was used in assessing facility convergence. A maximum number of iterations was established at 200.

Initially the VDH was applied for the *p* value of 25. Although the *p* centers are forced to be within the polygon boundary in Step 1, there are no restrictions on the locations of the *p* centers in subsequent steps of the VDH. Solution time for the VDH was 1.2 hours (time for all 50 applications). The average maximum radius for 50 runs was 981.3 meters, with a minimum of 971.8 meters. The results indicate that the maximum radius of covering circles always strictly converges. This implies that the performance of the VDH is rather stable. However, computational experience indicates a major problem with the VDH approach when a non-convex study area is evaluated, that is, facilities are not ensured of being in the region as no constraints enforce this.

The CVDH was also applied to the study region for the *p* value of 25. Facilities are restricted to be in the demand region. Solution time in this case was 2.9 hours to complete all 50 runs. As for the VDH, the maximum radius of covering circles for the CVDH always strictly converges. The average maximum radius for 50 runs was 983.5 meters, with a minimum of 973.6 meters, slightly larger than that from the VDH. The results imply that the CVDH is successful in terms of performance stability, although it is more time consuming.

The performance of the heuristic was further evaluated through the comparison with the results from solving the vertex *p*-center problem by discretizing space discussed in Chapter 2. The comparison is based on the solutions for the *p* value of 25. In terms of computational time, it took the CVDH about 210 seconds to complete one run, i.e. return a local optima, while the solution time for the discrete *p*-center problem was hours in most cases (see Tables 2.4 to 2.6). In order to compare solution quality, we assessed the minmax distance on the continuous space of regional demand for both situations. For the discrete case, the objective function value shown in Tables 2.4 to 2.6 was evaluated in the case of discrete demand and facility locations. The actual minmax distance for continuous demand was obtained in several steps with the use of ArcView GIS. First the facility configuration for the vertex *p*-center problem was used to compute a Voronoi diagram, the maximal distance from each facility to the boundary of the corresponding

Voronoi polygon was computed, and the minimum is the actual minmax distance. Table 3.1 shows the results using regular spacing, offset regular spacing and geometrically associated spacing representations for regional demand. The column *Actual* is the actual minmax distance and the column *Error* is the difference from the objective function value. Note that the errors for all discrete representations are positive, indicating that the assumption of discrete space results in underestimate of the minmax distance in continuous space. In general, the error increased with *R* spacing value, i.e. with the degree of abstraction of the space. The actual minmax distance for the continuous solution from the CVDH (973.6m) was much smaller than that for any case of discretization. The results indicate that the CVDH is an efficient and effective approach in solving the *p*-center problem, in terms of not only computational time but also solution quality.

	Regular		**Geometrically**		**Offset regular**	
R values	Actual	Error	Actual	Error	Actual	Error
200	NA	NA	NA	NA	NA	NA
250	1,173.22	173.22	1,282.37	282.37	NA	NA
300	1,304.63	355.93	1,168.05	268.05	1,111.00	162.30
350	1,378.41	388.51	1,296.75	370.75	1,246.04	256.14
400	1,257.74	363.34	1,226.57	426.57	1,195.79	301.39
450	1,337.07	437.07	1,852.17	952.17	1,371.69	417.09

Table 3.1: Actual minmax distance and absolute error for the vertex p-center problem

3.6 Application

The continuous space *p*-center problem was then utilized for siting emergency warning sirens in the region. The objective is to determine the minimum number of emergency warning sirens and their locations to provide complete coverage of the city, similar to that of Current and O'Kelly (1992) and Murray *et al. (*2005). This is important because of the relatively high cost of purchasing and maintaining sirens. The maximum effective range of the omni-directional siren considered here is 976 meters. As shown in Figure 5, the study region is a non-convex polygon with holes and an irregular boundary.

Murray *et al. (*2005) report the use of the *p*-center problem to ensure complete regional coverage with the fewest facilities by incrementally increasing *p* until the resulting maximum closest distance to a facility is less than or equal to an established coverage standard. Given this, the *p*-center problem is solved for various values of *p* in order to find a configuration of warning sirens that provides complete regional coverage. The VDH and CVDH approaches are evaluated for solving the associated *p*-center problems. The value of *p* was incrementally increased from one. For each *p* value, both the VDH and the CVDH were applied 50 times with different randomly generated initial siren locations. The tolerance value was $\varepsilon = 0.2$ meters and a maximum number of iterations was 200.

Initially the VDH was applied for siting sirens. Complete regional coverage is achieved when 25 sirens are sited (i.e., $p = 25$). Figure 3.6 illustrates a solution for complete coverage. In this case there are two facilities outside city boundary. The CVDH was also applied to the Dublin region for siren siting. Sirens are restricted to be in the demand region. To provide suitable coverage (the maximum radius of covering circles is not larger than 976 meters), the fewest number of sirens was also 25. Figure 3.7 shows a solution to the constrained problem.

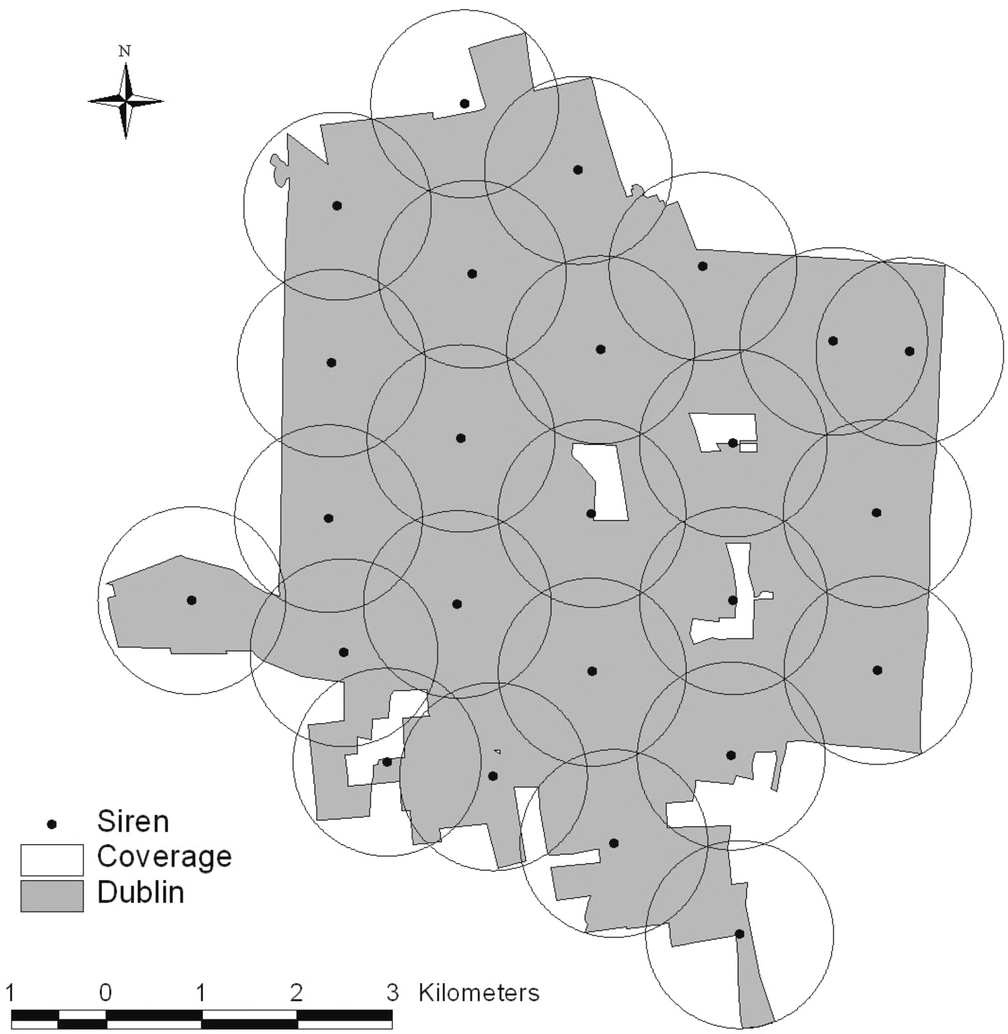


Figure 3.6: Regional coverage of 25 sirens in Dublin (unconstrained)

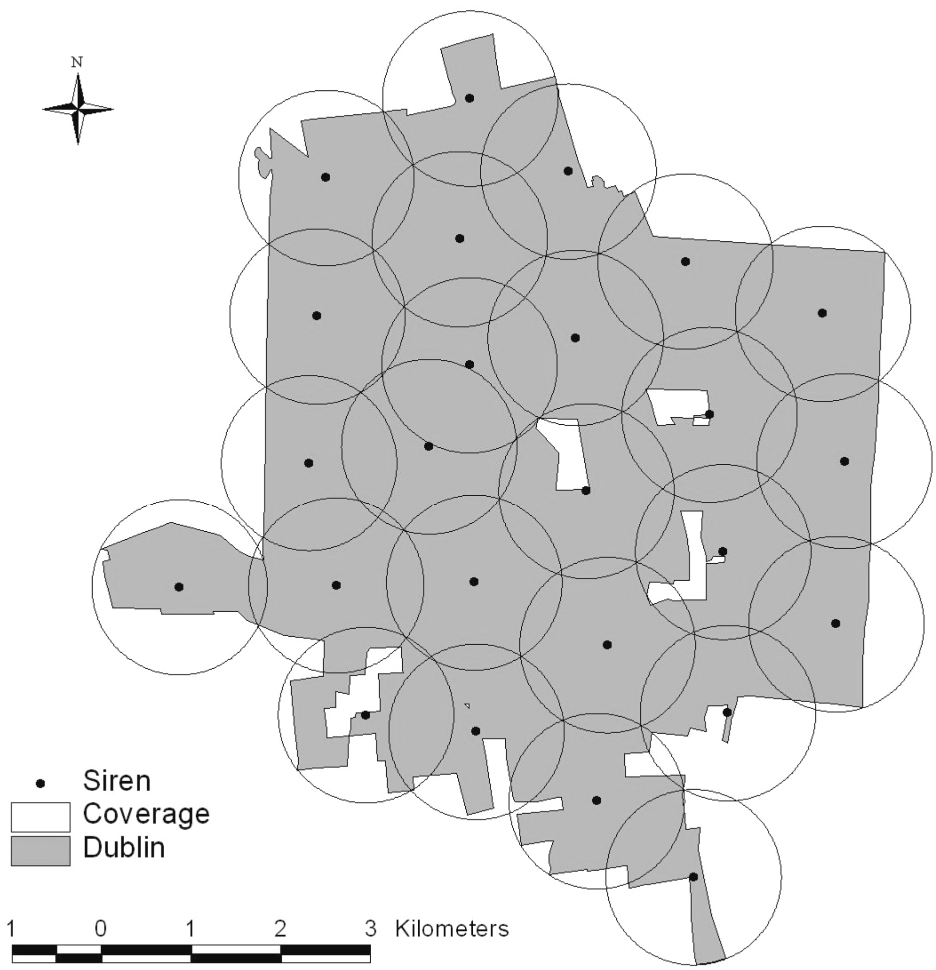


Figure 3.7: Regional coverage of 25 sirens in Dublin (constrained)

3.7 Summary

Comparatively, the CVDH is more computationally intensive because of the need to compute the CMCC. To speed up this step in the CVDH it is possible to develop a heuristic solution to the CMCC. For example, we could move the center of the Voronoi polygon to its closest point in *S* and then define a bigger associated covering circle. Of course, such an approach is approximate for the CMCC, so there are no assurances for either stability or high-quality solutions.

In this chapter, we addressed application oriented issues in solving the continuous space *p*-center problem: non-convex region, complex Voronoi polygons and certain constraints on potential facility locations. Problem complexity in practice is the result of region non-convexity and siting feasibility. The VDH requires a computationally efficient procedure for solving the 1-center problem for the Voronoi polygons when there are non-convex sub-regions. The restrictions placed on facility location lead to a constrained *p*-centerproblem. In this case, we extended the VDH to the CVDH to solve the constrained problem. These heuristics were applied to siren coverage. The CVDH proved to be an effective heuristic in terms of performance stability and solution quality, addressing limitations with the VDH in practical application.

CHAPTER 4

COVERAGE MAXIMIZATION IN CONTINUOUS SPACE[1]

4.1 Introduction

Chapters 2 and 3 discussed the use of *p*-center for complete coverage of a region, identifying the minimal number, and the location, of facilities while ensuring that all demand is covered within the maximal service distance from a facility. However, in many situations, due to the high cost of acquiring and siting service facilities, it is not possible to site a sufficient number of facilities to serve all demand within a desired maximal service distance. In such circumstances, one might instead seek to maximize demand coverage with whatever facilities are available for siting.

Facility location problems generally rely on certain assumptions about where demand is located in a region and where the facilities can be sited. As with other location problems, most applications of the maximal covering models use a collection of discrete points to represent demand and potential facility locations. Such discrete representations

[1] This chapter is based on a paper, "A geocomputational heuristic for coverage maximization in service facility siting", submitted for publication to *Transaction in GIS* and coauthored with Dr. Alan Murray, Dr. Tim Matisziw and Dr. Daoqin Tong.

of space make it possible to formulate these types of location problems as linear integer models and employ standard optimization techniques to obtain model solution. However, point-based assumptions may be too restrictive and result in pronounced measurement and coverage errors (Miller, 1996; Murray *et al.* 2002). Hence, there is a need for more realistic spatial representations of regional demand and potential facility sites. In particular, considering both demand locations and facility sites as continuous is important, i.e. demand is present everywhere and facilities can be sited anywhere in a region. This assumption is not unreasonable in regional planning. For example, some services (e.g. emergency warning, cellular signal) are required to reach all population and demand for some services (e.g. human activity) can be assumed to exist everywhere within a region. Some facilities, like tornado warning sirens, with a relatively small geographic footprint, can be sited practically anywhere in a region. Although such an assumption of continuous space siting is realistic, it is very challenging to solve a continuous space coverage maximization problem.

This chapter focuses on coverage maximization in service facility siting in continuous space assuming demand is continuously distributed. The next section formulates the problem, followed by a literature review relevant to this research. A geocomputational approach is proposed to solve this highly non-linear and non-convex problem. Application results are then presented. Finally, a discussion and concluding comments are provided.

4.2 Background

To address the covering problem of siting facilities in continuous space in order to maximally serve continuously distributed demand, it is necessary to first touch upon the historic research on maximal coverage problems in discrete space, which are at the root of many approaches for continuous space.

4.2.1 Maximal covering problem in discrete space

One of the main problems associated with the LSCP is that the number of facilities that are needed is likely to be excessive since all the demand points need to be covered, regardless of their quantity and density. Complete coverage of demand is sometimes economically infeasible due to limited resources available to a service provider. Another problem is that the LSCP considers all demand points identically. These concerns lead to the Maximal Covering Location Problem (MCLP) model proposed in Church and ReVelle (1974). The MCLP does not require all demand points to be covered. Instead, the model fixes the number of facilities that are to be located and seeks to maximize the coverage of demand. There have been a wide range of applications of the MCLP, including placement of emergency warning sirens (Current and O'Kelly, 1992), emergency response stations (Eaton *et al.* 1986; Adenso-Diaz and Rodriguea, 1997; Kalvenes *et al.* 2004), health centers (Bennett *et al.* 1982), bus stops (Gleason, 1975), bank branches (Sweeney *et al.* 1979), and air pollution monitors (Hougland and Stephens, 1976), to name a few.

Similar to the LSCP, a demand point or area is considered to be covered if it is within a predefined service distance or time from at least one facility. However, since the number of facilities to be located is limited, all demand in the region may not be covered. A budget constraint is incorporated in the MCLP to relax the rigid requirement of complete coverage of all demand in the LSCP. Consider the following notation for stating the problem mathematically:

i, index of demand points or service areas;

j, index of potential facility placement locations;

$N_i = \{ j \mid d_{ij} \le R \}$, i.e., the set of potential facilities *j* capable of covering/serving demand i, where d_{ij} is the distance from demand *i* to potential site *j* and *R* is the effective coverage distance of a facility;

w_i, importance of demand *i*;

p, number of facilities to be sited.

$$x_j = \begin{cases} 1, \text{ if potential facility site } j \text{ is selected for placement,} \\ 0, \text{ otherwise.} \end{cases}$$

$$y_i = \begin{cases} 1, \text{ if demand point } i \text{ is covered,} \\ 0, \text{ otherwise.} \end{cases}$$

Using the above notation, the MCLP can be structured as an integer-linear formulation (Church and ReVelle, 1974):

$$\max \sum_i w_i y_i \tag{4-1}$$

subject to

$$\sum_{j \in N_i} x_j \geq y_i \qquad \forall i, \tag{4-2}$$

$$\sum_{j} x_j = p, \tag{4-3}$$

$$x_j = \{0,1\} \qquad \forall j, \qquad y_i = \{0,1\} \qquad \forall i. \tag{4-4}$$

The objective function of the MCLP, (4-1), is to maximize the total covered demand. This is subject to several sets of constraints. Constraints (4-2) state whether a demand point is covered by sited facilities. Specifically, if a demand point *i* is covered ($y_i = 1$), then there must be at least one facility placed at locations in the cover set N_i. The budget constraint (4-3) requires exactly *p* facilities to be located. Constraints (4-4) impose integer requirements on location and coverage decision variables, which, with the linear objective function, make the MCLP a linear mixed integer optimization program.

Traditionally, due to limited geometric and spatial data handling capabilities, demand locations and potential facility sites have been represented as collections of points in the application of the MCLP (Miller, 1996; Church, 1999). A continuous area, such as a city or a census unit, is usually simplified and aggregated into discrete points. The goal of coverage modeling is based on the evaluation of whether these points are covered by a configuration of facilities. This could result in potential measurement and coverage errors in location problems, as discussed in the work of Daskin *et al.* (1989), Current and Schilling (1990) and Murray and O'Kelly (2002). Murray and O'Kelly (2002) address the abstraction and aggregation of regional demand in a coverage model and find that generally over-estimation of the actual coverage of demand provided by a service facility configuration results. Thus, there is a need for representing potential facility sites and demand locations as a continuous area in coverage modeling such as the MCLP.

4.2.2 Extensions to continuous space

A number of researchers have extended the MCLP to consider siting facilities in continuous space while representing demand as discrete points, including Mehrez (1983), Mehrez and Stulman (1982; 1984), and Church (1984). The work along this track is known as the planar maximal covering (PMC) location problem, or PMCE under the Euclidean distance metric, originally defined in Church (1984). Mehrez and Stulman (1984) use the model for siting fire and radar stations to maximally cover demand distributed across a service region. Since facilities can be located anywhere in space, the relaxation of facility locations makes it possible to increase efficiency in coverage modeling. Given a number of facilities, a greater total coverage can be attained than in the discrete case since there are many more locations available for facility placement. In order to solve the PMCE, Church (1984) exploits the geometric properties of coverage and translates the continuous space problem into a discrete one. The strategy is to identify a finite dominating set of locations as potential facility sites. The point set is the collection of the intersection of all circles with radius equal to maximum service distance centered at each discrete demand location. These points are called the circle intersection point set (CIPS). Church (1984) proves that the CIPS contains the optimal facility sites of the PMCE. The continuous location problem can then be formulated as a mixed integer optimization problem. Although this approach assumes continuous facility location and greater efficiency can be achieved, demand is still represented as discrete points. Thus potential measurement errors and coverage inaccuracies still exist.

Another extension of the MCLP has been approached by Murray and Tong (2007), which enables more flexibility in demand representation and service standard definition. Their approach aims to optimally cover point, line or polygon features representing demand for service in a region (Extended Planar Maximal Covering Location Problem or EPMC), while allowing facilities to be located in continuous space. In particular, Murray and Tong (2007) exploit geometric aspects of the EMPC and introduce a method for identifying a finite set of critical potential facility locations containing an optimal solution to the continuous space problem, thereby reducing an infinite number of potential facility sites to a finite and discrete set of locations. One can then use the MCLP to solve the EPMC with such a discrete set of locations. This method is called Polygon Intersection Point Set (PIPS). In order to find the PIPS, spatial demand objects are first identified and their vertices are extracted. Covering boundaries (areas) for each demand object can then be derived according to the facility service standard. Finally, PIPS, the intersection points of covering boundaries, are identified as potential facility locations. Finding PIPS involves solving for intersecting points for spatial objects, which is supported by standard commercial GIS functionality (Murray and Tong, 2007). One issue raised by this approach is that the number of critical facility locations identified may be rather large, although an additional step could be used to remove dominated locations from the point set, as suggested by Murray and Tong (2007). Thus, computational issues exist if the number of demand objects is large. Another issue is the assumption that facilities can be located anywhere in the plane, while in practice facility locations may be restricted, e.g., within a region boundary. More importantly, a demand object is evaluated in a clear-cut manner, either covered or not covered, by constraints (4-5). This is not an issue when demand is represented as zero-dimensional points. However, when demand is represented as an area (or polygon), there is no attempt in the model to account for whether an area is partially covered. As a result, service coverage may be underestimated, as recognized as an issue for set covering in Murray (2005) and Murray *et al.* (2007). Therefore, there is a need for remedial solutions or an advanced spatial representation to more accurately portray demand for service.

The research on the MCLP, in which both demand and facility locations are represented as continuous, has been limited, due to analytical difficulties in handling geometrical computation in a continuous plane. The continuous *p*-center problem has been introduced for locating facilities in continuous two-dimensional space to completely cover continuously distributed demand (Suzuki and Okabe, 1995; Suzuki and Drezner, 1996; Murray *et al.* 2008). For facilities that have specific service standards, a given number of facilities cannot always ensure complete coverage to the region. Complete coverage can only be attained if the maximum distance from any demand point to its closest facility is less than or equal to the facility service standard, and thus at least a threshold number of facilities must be located to achieve this goal. For a given value of *p* less than the threshold, demand cannot be completely covered. As a result, the *p*-center problem does not address issues of maximizing the total covered demand.

Discussed in Chapter 3, the VDH (CVDH) is an iterative method for solving the continuous space *p*-center problem. However, it is not applicable for coverage maximization with a limited number of facilities since the process focuses on maximal 1-center radius (i.e. distance of demand from a facility), rather than effective service distance of the facilities. To illustrate this point, Figure 4.1 depicts a near-optimal *p*-center solution for *p*=15 using the VDH as well as the effective coverage distance of warning sirens (976 meters) for the Dublin region. For this service standard, at least 25 sirens are actually needed to completely cover the region. The *p*-center solution for *p* = 15 is 2516.4 meters, meaning that the region would be completely covered by these 15 facilities if the effective service radius is equal to or greater than this distance. The service standard for the sirens (976 meters) is clearly less than the *p*-center solution. Examining the shown configuration, it is clear that sirens could be shifted to achieve greater coverage for this number of facilities. Given this, the *p*-center solution clearly does not optimize coverage maximization.

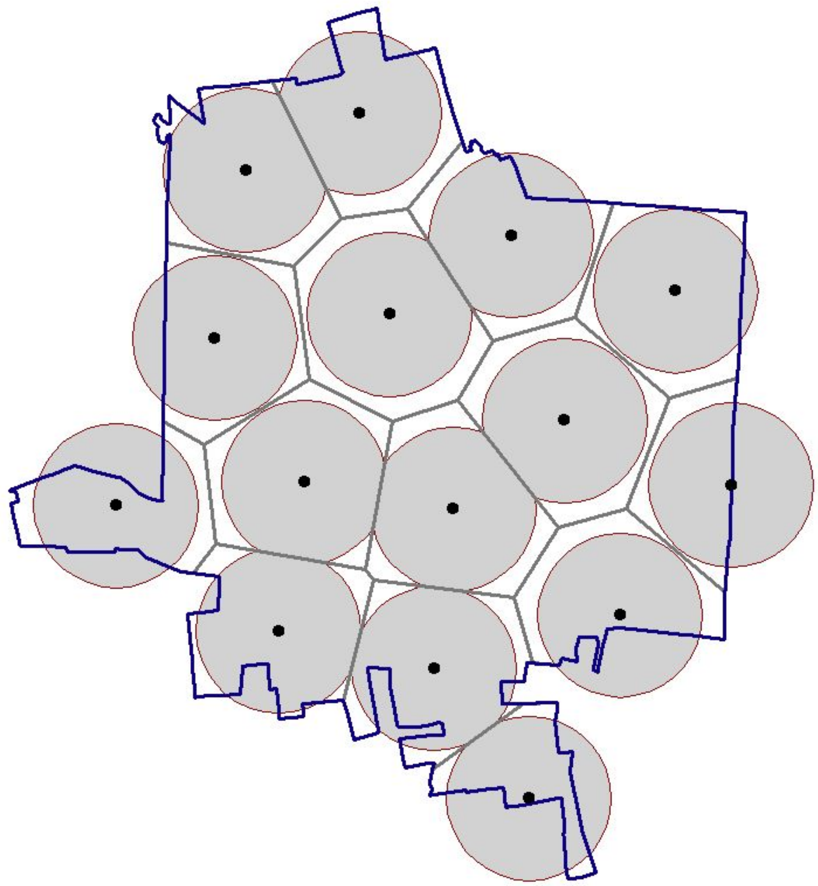


Figure 4.1: VDH *p*-center problem for *p*=15

Some researchers solve location problems in which demand is assumed continuous with the use of double integration (e.g., Drezner and Drezner, 1997). However, such an approach is appropriate for the cases when the region is convex, e.g. square in their application. For a non-convex region, integration would be cumbersome. Other researchers approach continuous space location problems by exploiting geometric properties of a region. For example, Matisziw and Murray (2008) use the medial axes as a geometric representation of a region to solve a continuous single-facility maximal covering problem.

4.2.3 The use of medial axis to represent continuous space

The medial axis is often applied in shape analysis. Given a connected geometric shape *C*, the medial axis is defined in Okabe *et al.* (2000) as the locus of all points with at least two closest points on the boundary of *C*, or alternatively, the locus of the centers of all the interior maximal disks of *C*. Similar definition can be seen in Blum (1967), Pfaltz and Rosenfeld (1967) and Brandt (1994). An example of medial axis is shown in Figure 4.2. The five line segments within the rectangle correspond to the medial axis of the rectangle, which is traced out by the centers of the largest inscribed disks. The medial axis can be expressed mathematically as follows (Okabe *et al.* 2000):

$$M(C) = \Big\{ \boldsymbol{x} \Big| \| \boldsymbol{x} - \boldsymbol{x}_i \| = \| \boldsymbol{x} - \boldsymbol{x}_j \| = \min\{ \| \boldsymbol{x} - \boldsymbol{y} \| \,|\, \boldsymbol{y} \in \boldsymbol{C} \}, \quad \boldsymbol{x}_i \neq \boldsymbol{x}_j, \boldsymbol{x}_i, \boldsymbol{x}_j \in \partial \boldsymbol{C}, \boldsymbol{x} \in \boldsymbol{C} \Big\}. \tag{4-5}$$

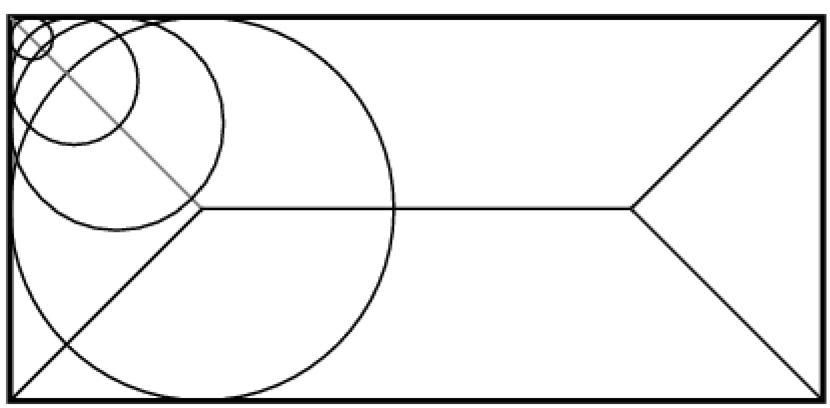


Figure 4.2: The medial axis of a rectangle

Thus, the medial axis represents the shape of objects by finding the topological skeleton, a set of curves which roughly run along the middle of an object. Let $R(\boldsymbol{x})$ be the radius of the maximal disk centered at a point $\boldsymbol{x}$ on the medial axis $M(C)$, and the function $R(\boldsymbol{x})$ is called the radius function of the maximal disk (Okabe *et al.* 2000). As summarized in Okabe *et al.* (2000), a medial axis has the following property: a given shape *C* has a unique medial axis and can be exactly regenerated by taking the union of all maximal disks with radius equal to the radius function $R(\boldsymbol{x})$ on the medial axis $M(C)$, i.e. $C = \cup_{x \in M(C)} D_{\max}(\boldsymbol{x})$. This property is detailed by a number of researchers, such as Lee (1982), Gursoy and Patrikalakis (1991, 1992), and Brandt (1994).

A number of methods have been developed to compute a medial axis. For a polygonal shape, the medial axis can be obtained by the use of the line Voronoi diagram (Kirkpatrick, 1979; Lee, 1982; Brandt and Algazi, 1992). Generally, the boundary of *C* is first decomposed into a set of line segments or a set of points. In so doing, all information necessary to reconstruct the shape is retained. The Voronoi diagram is then generated from the decomposed line segments. Since points on the edges of the Voronoi regions are equally distant from generator lines or points forming the boundaries of *C*, the medial axis of *C* is obtained after removing unnecessary edges of Voronoi regions (See Lee, 1982). Computational methods for deriving the medial axis are different depending on the representation of *C*, vector or raster data. A summary of relevant approaches can be found in Okabe *et al.* (2000).

There have been various applications in geographical analysis in which the medial axis of a polygonal shape is used to represent the region. Examples include the use of the medial axis to characterize the topography of watershed areas from river and elevation data (Vincent and Soille, 1991), and to extract and represent roads and rivers from aerial or satellite images in the automated production of maps (Airault *et al.* 1996; Leymarie *et al.* 1996). Others applications appear in map generalization (Gold *et al.* 1996; Gold and Snoeyink, 2001), and statistical analysis of relative location of points in a bounded region (Sadahiro and Takami, 2001). The use of the medial axis in location optimization problems can be seen in recent work of Matisziw and Murray (2008), who consider a coverage problem in continuous space.

4.2.4 CMCP-1

The one facility case of continuous maximal covering problem (CMCP-1) has been approached by Matisziw and Murray (2008). Unlike traditional methods like the MCLP and PMCE (i.e., discretizing the continuous problem and using a linear-integer programming approach), Matisziw and Murray (2008) look to exploit geometric properties of a region to be covered in order to solve the CMCP-1. With the assumption of uniformly distributed demand and a disk-like service area for the facility, Matisziw and Murray (2008) consider two distinct cases of the CMCP-1: whether or not the service area of a facility can be completely contained within the region. Matisziw and Murray (2008) approach the regional coverage problem relying on a geometric representation of regions as the medial axis. This is based on the property that the medial axis is the locus of a maximum inscribed disk that covers as much or more area than any other disk locally enclosed within the region, which is consistent with the objective of the CMCP-1. Matisziw and Murray (2008) consider three special cases. First, when service standard *S* is greater than or equal to the radius of *1*-center solution, complete coverage can be obtained by solving a *1*-center problem. Second, if *S* is less than or equal to the radius of maximum inscribed disk, at least one optimal facility site exists and the sites along the medial axis with the maximum distance to the boundary coincide with the largest service coverage, i.e. a disk with the radius of *S*. The algorithm for this case obtains the medial axis by generating a line Voronoi diagram, and then identifies points along the medial axis having distances to the boundary larger than or equal to *S*. The third case is in between the first two cases, where points along the medial axis do not necessarily correspond to optimal siting locations. The algorithm for this case is first to approximate the boundary of the region by a set of points, and then apply the CIPS approach of Church (1984) to identify an intersection point associated with maximal demand coverage.

4.3 Formulating continuous maximal covering problem

Matisziw and Murray (2008) address siting a single facility to maximize coverage in the case where both service demand and candidate facility sites can exist anywhere. Siting multiple facilities simultaneous in continuous space to provide maximal coverage of regional demand is no doubt more challenging. This is the problem of interest in this chapter, namely, a general continuous maximal covering problem (CMCP). Following the notation of Matisziw and Murray (2008), the CMCP model can be express as follows:

$$\underset{\mathbf{x}_j,\, j=1,\cdots,p}{Max} \int_{\mathbf{z}\in C} \delta(\mathbf{z})d\mathbf{z} \tag{4-6}$$

where

G, study region;

j, index of potential facility sites;

$\mathbf{x}_j$, decision variable, location of siting facility j.

$\delta(\mathbf{z})$, a continuous density function of demand distribution at location $\mathbf{z}$;

p, the number of facilities to be sited;

C_j, service area of facility j.

$C = \bigcup_{j=1,\cdots,p} C_j \cap G$, which is total coverage area of the p facilities in G, i.e. the intersection of all service areas C_j and region G.

The objective, (4-6), is to maximize the demand in region G that is suitably covered or serviced given p facilities. Demand is considered covered if it is within a desired effective coverage distance of sited facilities. Given a continuous distribution of demand across the region, (4-6) seeks to determine the placement of p facilities so that the total covered demand is maximal. Note that there is no restriction on facility sites in the formulation. This means that candidate facility locations are continuous and exist everywhere, as assumed for demand. Since the cardinality of the sets of demand and candidate facility sites is infinite, it is impossible to evaluate every potential facility location explicitly to determine the demand covered. Thus, standard optimization techniques for discrete location models cannot be applied to the CMCP. Moreover, the computation of total demand in the region involves integrating demand density over *C*, the intersection between all established service areas and the region *G*, which is usually non-convex. Hence the CMCP is a highly non-linear and non-convex model regardless of how simple the demand density is. This makes it challenging to solve.

4.4 Solution approach

Matisziw and Murray (2008) approach the single facility CMCP by exploiting geometric properties of a region to be covered. Such an approach is also essential to the multiple facility siting problem. We propose a geocomputational approach for solving the continuous space maximal covering problem using Voronoi diagrams and medial axes. Thus this is an extension to previous work on Voronoi diagram heuristics as well as the single facility model discussed in Matisziw and Murray (2008).

Equation (4-6) can be reformulated in terms of a Voronoi diagram. Consider the following model:

$$Max \quad \int_{\boldsymbol{x} \in \bigcup_j C_j \cap (\bigcup_j V_j)} \delta(\boldsymbol{x}) d\boldsymbol{x} \tag{4-7}$$

where V_j is the Voronoi region within *G* associated with the *j*th facility.

This is equivalent to:

$$Max \quad \sum_{j=1}^{p} \int_{x \in \bigcup_j C_j \cap V_j} \delta(\boldsymbol{x}) d\boldsymbol{x} \tag{4-8}$$

Models (4-7) and (4-8) are equivalent to (4-6) since the Voronoi regions, V_j, are non-overlapped and their union is equivalent to region *G*. This research focuses on a uniformly distributed demand and thus the problem becomes:

$$Max \quad \sum_{j=1}^{p} (\bigcup_j C_j \cap V_j) \tag{4-9}$$

(4-9) seeks to maximize total area of Voronoi regions covered by the *p* facilities. With a Voronoi diagram, the multiple facility siting problem is decomposed into a set of sub-problems, each associated with the Voronoi region of a facility and treated as a single facility siting problem, similar to that in Matisziw and Murray (2008). Our proposed approach is based on the following rational. The heuristic starts with an initial configuration of *p* facilities in a region *G*. A Voronoi diagram is generated from this set of facility sites and region is partitioned into *p* sub-regions (i.e., *p* Voronoi polygons). Subsequent analysis is based on the exploration of the geometric properties of each Voronoi polygon that is not covered by other existing facilities. In particular, for each of the *p* sub-regions uncovered, the position of the facility that maximizes coverage is identified as a single facility siting problem. The facility corresponding to the greatest increase of regional coverage is relocated and the iteration is complete. If the increase of regional coverage is significant, the heuristic continues, generating a Voronoi diagram from the new facility configuration. Otherwise, the heuristic stops.

Following the notation for the CMCP in (4-6), let S be the facility service standard and γ be a stopping tolerance, and the developed heuristic (MULTICOVER) is detailed as follows:

MULTICOVER (G, G^c, p, S, γ)

1. Locate p facilities in the region G. Compute regional coverage (denoted A) associated with the initial facility configuration, i.e., $A = \bigcup_{j=1}^{p} C_j \cap G$.
2. Use the set of facility sites to generate a Voronoi diagram (denoted V).
3. For each Voronoi polygon V_j, compute the modified polygon, $\hat{V}_j = V_j - \bigcup_{k \in J, k \neq j} C_k \cap V_j$.
4. Compute medial axis $\hat{M}_j$ for the modified Voronoi polygon $\hat{V}_j$.
5. Identify the point on $\hat{M}_j$ corresponding to the largest enclosed disk for $\hat{V}_j$ (this point denoted $\lambda \in \hat{M}_j$) and its associated radius (denoted R_j).
    a. If $S \leq R_j$, compute regional coverage, A_j, associated with moving facility j to λ (denote $C_{j(\lambda)}$ as the coverage of facility j at λ), where $A_j = \left(\bigcup_{k \in J, k \neq j} C_k \right) \cup C_{j(\lambda)} \cap G$.
    b. If $S > R_j$, then compute the 1-center of $\hat{V}_j$ (denoted θ) and compute regional coverage, A_j, associated with moving facility j to θ (denote

$C_{j(\theta)}$ as the coverage of facility *j* at θ), where

$$A_j = \left(\bigcup_{k \in J, k \neq j} C_k \right) \cup C_{j(\theta)} \cap G .$$

6. Obtain the maximum marginal increase of regional coverage ($\max_j A_j - A$). If it is greater than the specified tolerance γ, then relocate facility associated with $\max_j A_j$ to the corresponding λ or θ, set $A = \max_j A_j$, and go to Step 2; otherwise local maxima is found and proceed to Step 7.
7. For each facility *j*, compute the region covered by other existing facilities (i.e., $\hat{V}_j = G - \bigcup_{k \in J, k \neq j} C_k$) as well as the corresponding medial axis $\hat{M}_j$ and the center of the largest enclosed disk λ. Compute regional coverage, A_j, associated with moving facility *j* to λ, where $A_j = \left(\bigcup_{k \in J, k \neq j} C_k \right) \cup C_{j(\lambda)} \cap G$.
8. Obtain ($\max_j A_j - A$). If it is greater than γ, then relocate facility associated with $\max_j A_j$ to the corresponding λ, set $A = \max_j A_j$, and go to Step 2; otherwise stop.

Step 1 of the heuristic initialises the *p* facility sites in region *G*. The initial facility configuration can be generated in many ways, including placing the facilities in a regular spaced form, close to the center of the region, or randomly positioning facility sites, as illustrated in Figure 4.3, respectively. Intuitively, a more dispersed initial configuration would lead to less computational time since the heuristic tend to relocate one facility away from others in order to cover more of uncovered area. Computational experience has show that such difference is not significant. Moreover, the heuristic has appeared to produce the consistent results, regardless of initialisation method used. After the initial p facilities are sited in region G, the effective coverage of this configuration, A, is evaluated.

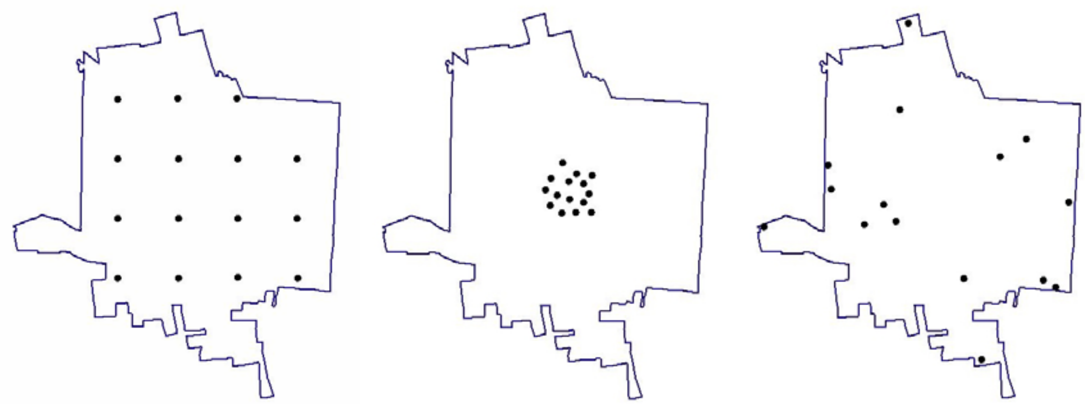


Figure 4.3: Three ways of initializing facility sites for p = 15

(Left: regular spacing; Middle: centered; Right: random)

In Step 2 of the heuristic, the p facility sites are used to compute a Voronoi diagram. The region is then partitioned into p polygons, each associated with a facility site. An example of such a partition is displayed in Figure 4.4, with the Voronoi polygon for facility 2, V_2, highlighted. Following analysis is based on the geometric properties of the sub-regions.

For each Voronoi polygon and associated facility considered, Step 3 computes the portion of the polygon that is not covered by any other existing facility. An example of the modified Voronoi polygon is shown in Figure 4.5. Here $\hat{V}_2 = V_2 - \bigcup_{k \in 2, k \neq 2} C_k \cap V_2$ is the area within V_2 that is not covered by any other sited facility. This modified polygon is treated as a single facility problem to identify whether a move of associated facility increases the coverage.

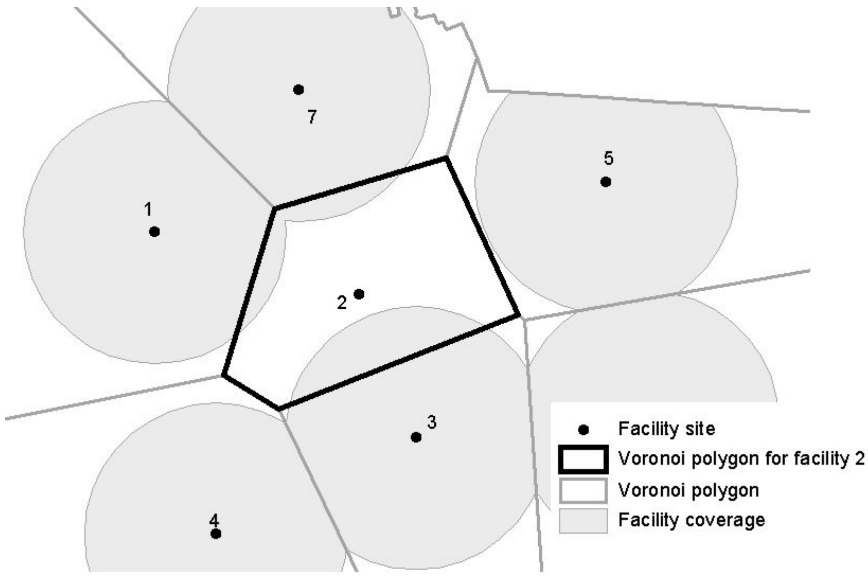


Figure 4.4: A Voronoi polygon

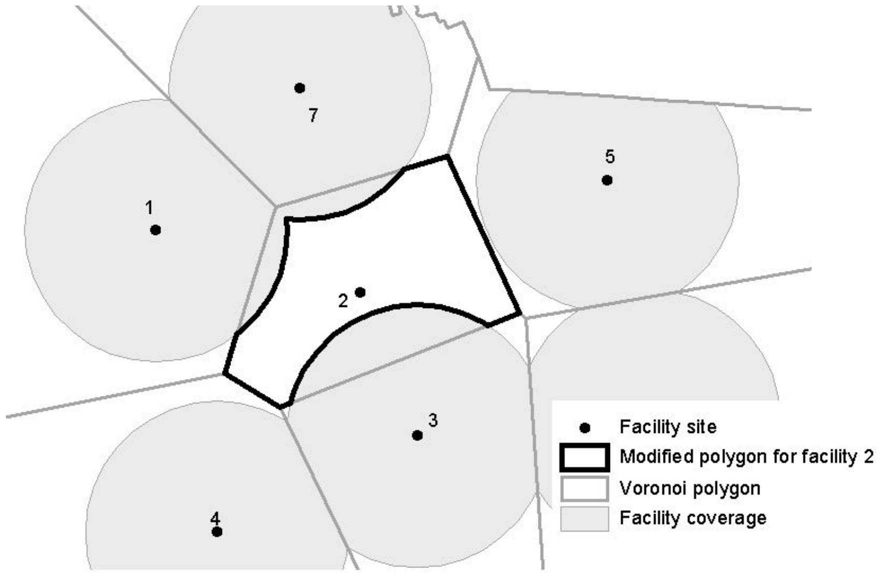


Figure 4.5: A modified Voronoi polygon

The approach to determine the location of a single facility to maximize its coverage is similar to the work of Matisziw and Murray (2008) which explores geometric properties of the region, including medial axis for the largest inscribed circle and 1-center for the minimum covering circle (MCC). Given a modified Voronoi polygon $\hat{V}_j$, in Step 4, the heuristic computes its medial axis $\hat{M}_j$. The computation procedure involves first discretizing the polygon boundary as points with a pre-specified step size (tolerance), calculating the Voronoi diagram based on these points and deleting unnecessary Voronoi edges. Thus the medial axis obtained is neither exact nor fully continuous, but an approximate one. Figure 4.6 shows the medial axis for modified polygon $\hat{V}_2$.

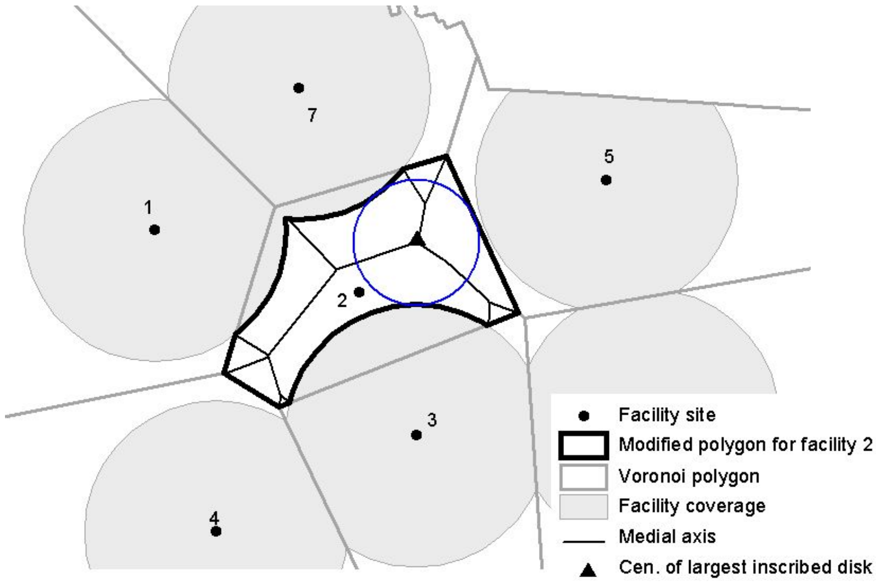


Figure 4.6: Medial axis and the largest inscribed circle for modified polygon

Given the medial axis, the center of the largest inscribed circle for the modified polygon can then be identified by searching the medial axis. This is the task of Step 5. The radius of the largest inscribed circle, R_j, is also obtained. The largest inscribed circle and its center for modified polygon $\hat{V}_2$ are illustrated in Figure 4.6. R_j is important in identifying the location of a facility to maximize its coverage. According to Matisziw and Murray (2008), if R_j is larger than the service standard (*S*) associated with the sited facility, the center of the largest inscribed circle is an optimal solution for maximal coverage. On the other hand, if R_j is less than *S*, a circle with radius of *S* cannot be fully enclosed in the region. However, the optimal facility site for maximal coverage can be identified as the 1-center in the case when *S* is larger of equal to the MCC radius. Based on this theory, the heuristic proceeds with the evaluation of the modified Voronoi polygon $\hat{V}_j$ according to two criteria. First, if $R_j \le S$, then calculate the potential increase in regional coverage corresponding to moving the associated facility to the center of the largest inscribed circle (λ). This is done in Step 5a. Second, if $R_j \le S$, then compute the 1-center of modified polygon $\hat{V}_j$ (θ), and calculate the potential increase in regional coverage corresponding to moving the associated facility to θ. This is the task relegated to Step 5b. Figure 4.7 shows 1-center and MCC for $\hat{V}_2$. Clearly, MCC is larger than the largest inscribed circle shown in Figure 4.6. In sum, Step 5 is to obtain the potential increase in regional coverage for each facility.

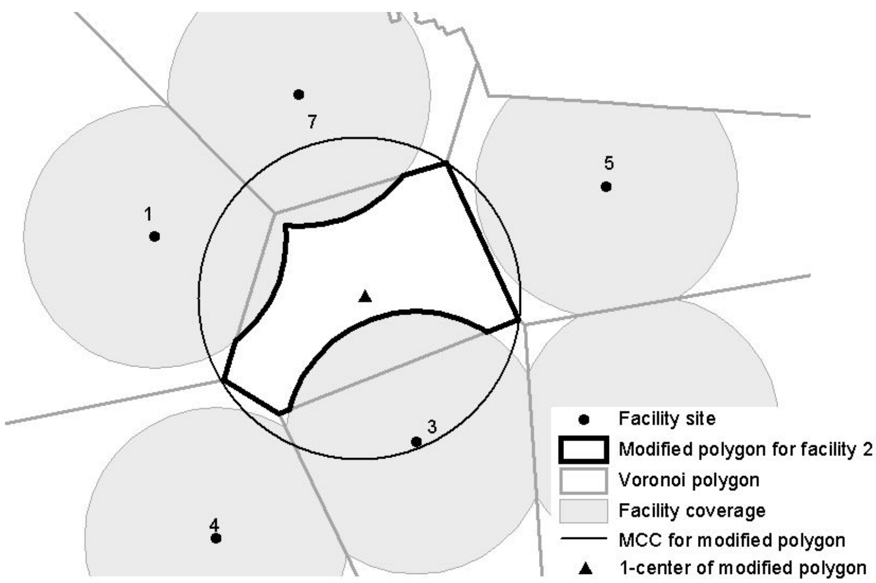


Figure 4.7: 1-center and minimum enclosing circle for modified polygon

After evaluating the p sub-regions sequentially, the heuristic identifies the largest potential increase in regional coverage with respect to the current facility configuration (Step 6). If the maximum coverage increase is larger than pre-specified tolerance γ, the corresponding shift of the facility is accepted and Steps 2-6 are repeated with the new configuration. Otherwise, a local maxima is likely obtained and the heuristic proceed to a perturbation step.

The facility configuration can be perturbed out of the local maxima in a number of ways. Generally these techniques search for "open" space to relocate facilities in the region globally, rather than in local sub-regions. One technique can be shifting some facilities toward the region boundary or away from other facilities. Step 7 details another technique, which is similar to Step 5a. The process first removes each facility and computes the medial axis for the uncovered area of the whole region. This is followed by identifying the center of the largest inscribed disk λ and calculating potential increase in regional coverage associated with relocating the removed facility to λ.

After evaluating the p facilities in turn, the heuristic identifies the largest potential increase in regional coverage with respect to the current facility configuration (Step 8). If the maximal coverage increase is larger than pre-specified tolerance, the corresponding shift of the facility is accepted. The heuristic proceeds with the new configuration and repeat Steps 2-6. Otherwise, the heuristic completes with the maximum coverage for p facilities.

4.5 Application and results

The developed geocomputational heuristic (MULTICOVER) proposed in Section 4.4 is applied to siting emergency warning sirens in an Ohio urban region. Figure 4.8 shows this area, the City of Dublin, Ohio, with isolated areas (holes) in the center of the region removed. The planning concern addressed here is coverage maximization when complete coverage is not affordable due to limited budget resources.

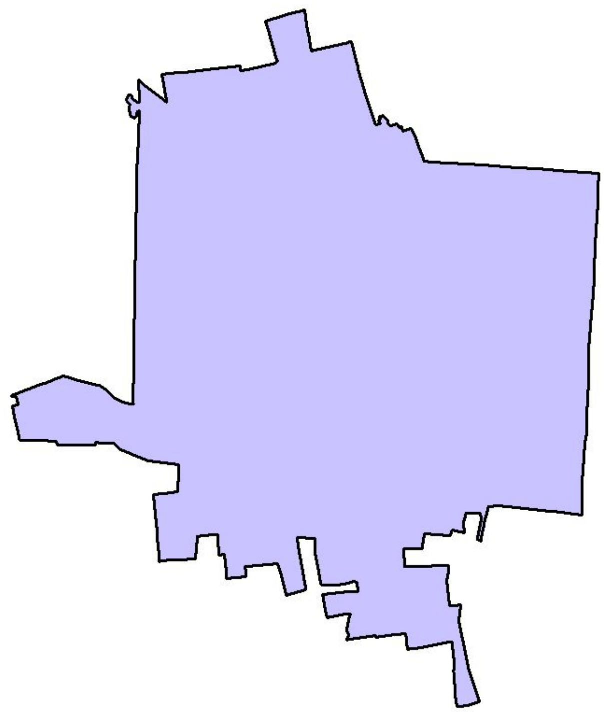


Figure 4.8: Study region

The service standard of an emergency warning siren is 976m. Previous studies show that 24 to 26 sirens are needed to provide complete regional coverage. A range of *p* values (*p*=5 to 20) is considered. Clearly for such a smaller number of facilities, complete regional coverage cannot be attained. The analysis was carried out on a Xeon 3.00GHz GHz personal computer running Windows XP with 2 GB RAM. ArcView GIS version 3.2 was utilized to manage and manipulate data layers (facility sites and coverage), derive Voronoi diagrams, the medial axis and a 1-center solution. The MULTICOVER heuristic was programmed with the use of the Avenue script language provided in ArcView GIS.

For each value of p (5 to 20), the heuristic was applied five times, and each time the heuristic started with a different random siting configuration. The tolerance used to compute a medial axis in the heuristic was 50 meters, which resulted in reasonable solution quality and computational time. Figure 4.9 summarizes the solutions attained and includes the maximum and minimum percentage of regional coverage during the five runs for each value of p. For p less than or equal to 8, the minimum percentage of regional coverage is identical to the maximum. Moreover, for these p values, the regional coverage is equal to p times of the service coverage of a single facility, which is the maximum coverage for p facilities. This indicates that the facility configuration obtained by the heuristic is optimal. A MULTICOVER solution for p=8 is shown in Figure 4.10. Note that the effective coverage disks of the eight sirens are completely enclosed with the region and there is no overlapping area among the disks. In addition, for p larger than 8, the coverage in the worst case was very close to that in the best case (see Figure 4.9). The findings indicate that the heuristic produced consistent results and its performance was stable regardless of the value of p.

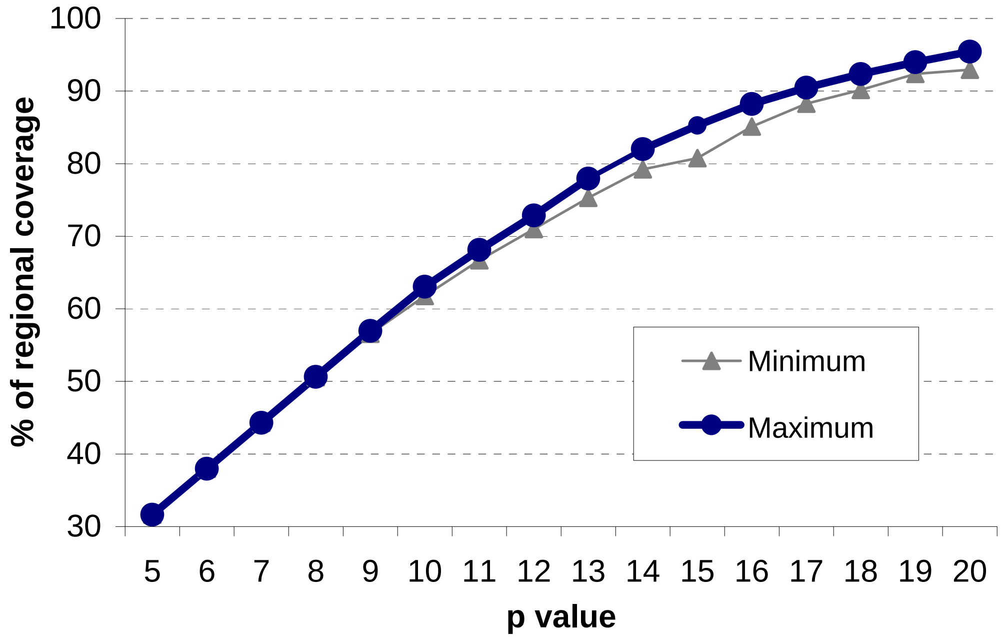


Figure 4.9: Solutions from the MULTICOVER heuristic

Figure 4.10: A MULTICOVER solution for p=8

The effectiveness of the heuristic was further assessed by the comparison with a traditional approach for solving coverage maximization problems. The integer MCLP in (4-1) to (4-5), developed by Church and ReVelle (1974), was used. The model is based on the discretization of the region. Several discrete representations of the region were derived. Regular grid lattices with a range of grid size (500, 400, 300, 200, and 150 meter) were used to generate point representations of the region. Figure 4.10 shows the square tessellation of the region for regular spacing of 400 meters, where the regional demand was represented as square grids and potential facility sites as the corners of grids. The assignment of demand is determined by whether the grid is fully covered by the candidate facility, i.e. within the circle centered at the facility site. Figure 4.11 shows such an assignment method for square grids with spacing of 200 meters and facility service standard of 976 meters. Such an assignment enables programming the MCLP as a mixed integer linear program. The MCLP application instances were solved using ILOG's CPLEX 10.1.1, a commercial optimization software package.

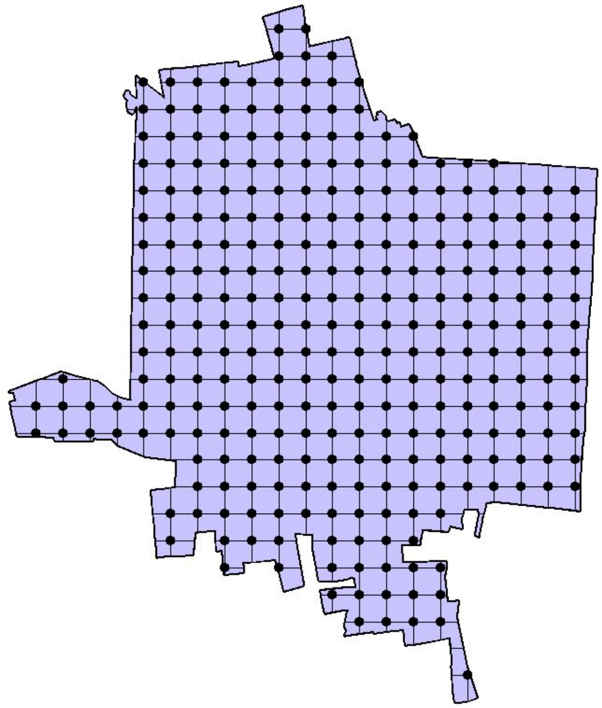


Figure 4.10: Square tessellation of study region and candidate facility sites ($\lambda = 400m$)

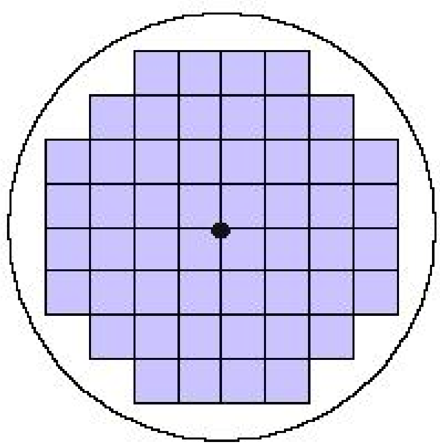


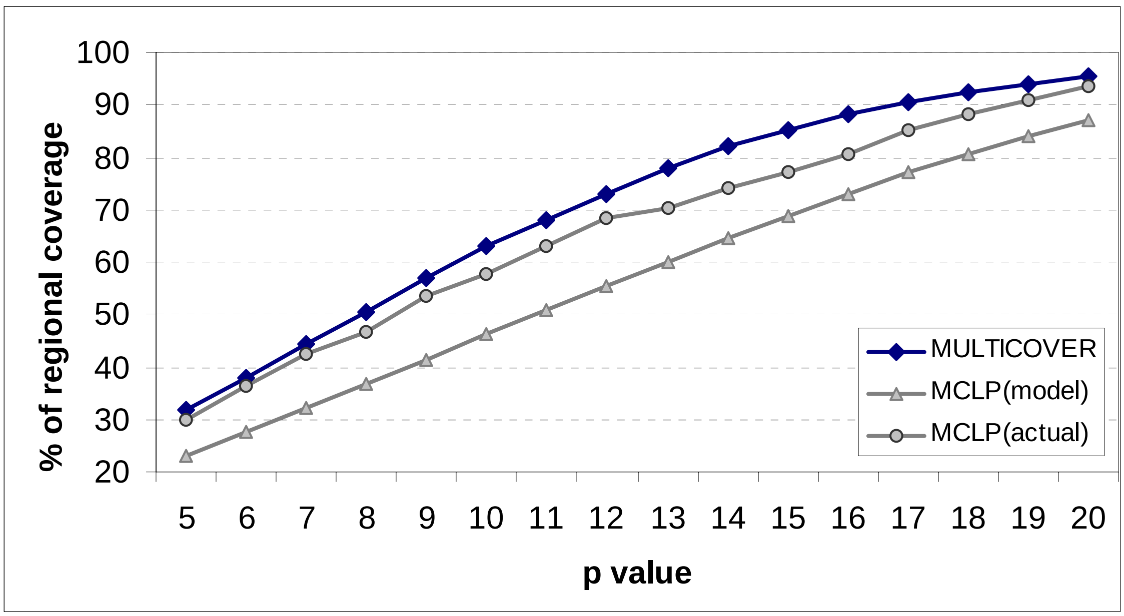
Figure 4.11: Square tessellation packing into a coverage circle (R = 976 m, $\lambda = 200m$)

(Source: Murray *et al.* 2008, page 346)

Clearly, the smaller grid size used to discretize the region, the better the region is represented, and in the MCLP better solution can be identified. The 150 meter representation of the region consisting of 2,357 demand cells and 2,082 facility sites should lead to a better facility configuration than the 500 meter representation of the Dublin region consisting of 236 demand cells and 191 facility sites. Considering there is a gap between the coverage of facilities in the model and actual coverage as shown in Figure 4.11, the coverage for the MCLP was evaluated by computing the actual coverage for facility configuration from the MCLP solution.

Figure 4.12 compares the level of coverage that was attained using siting $p = 5$ to 20 facilities using the MULTICOVER heuristic with those attained using the MCLP and three point representations of the region (150, 200, and 300 meter grid). When siting no more than 8 facilities ($p \le 8$), the MCLP was not able to identify optimal facility configuration. For all values of p using 200 and 300 meter grid representations, the MULTICOVER heuristic was able to identify a configuration that provided more coverage than the MCLP. Using 150 meter grid representation, the heuristic was able to identify solutions better than the MCLP except for three cases when $p = 17$, 18 and 19. The difference in coverage was on average 4.6%, 2.6%, and 0.8% for 300, 200, and 150 meter grid representation respectively, indicating that a decrease of grid size produced better solutions (larger regional coverage). Overall, the solutions of the MULTICOVER heuristic are better than those obtained using the MCLP, indicating good performance of the heuristic in identifying facility configurations for coverage maximization. Figure 4.13

displays a MULTICOVER solution for p=15 using 150 meter grid representation. This facility configuration provides service coverage to about 82.7% of the region.

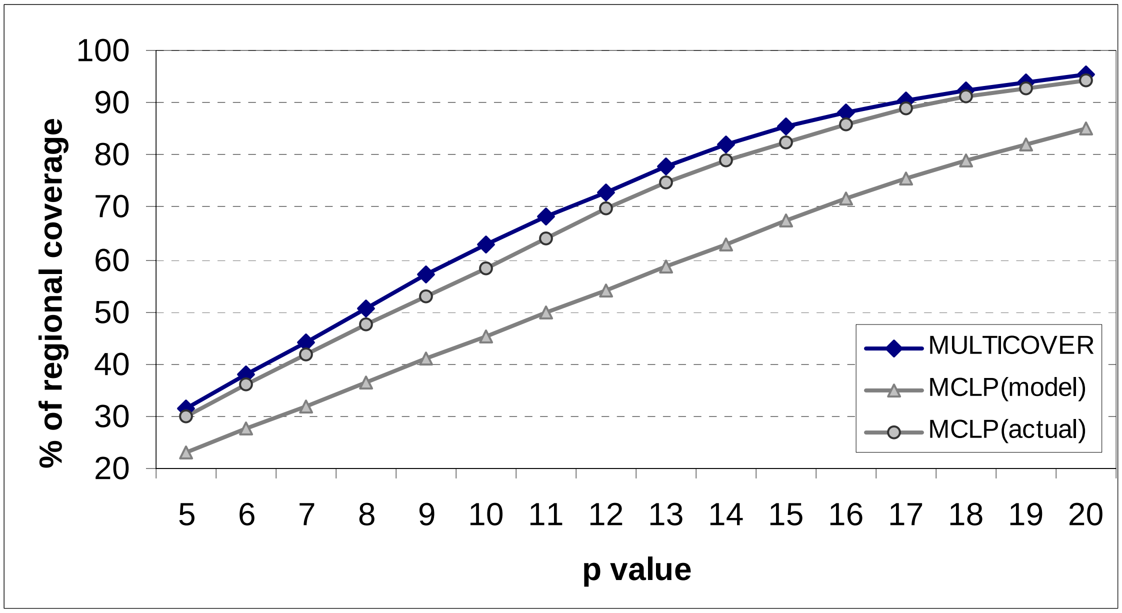


(a) 300 meter grid representation of the region

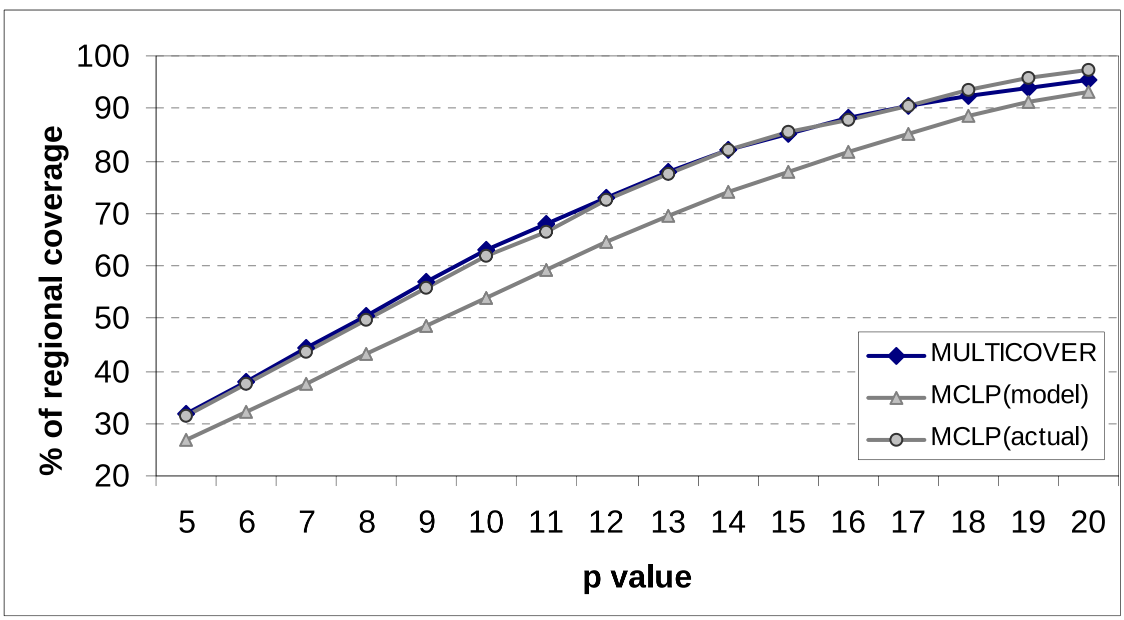
(continued)

Figure 4.12: Solution comparison between MULTICOVER heuristic and MCLP

(Figure 4.12 continued)



(b) 200 meter grid representation of the region



(c) 150 meter grid representation of the region

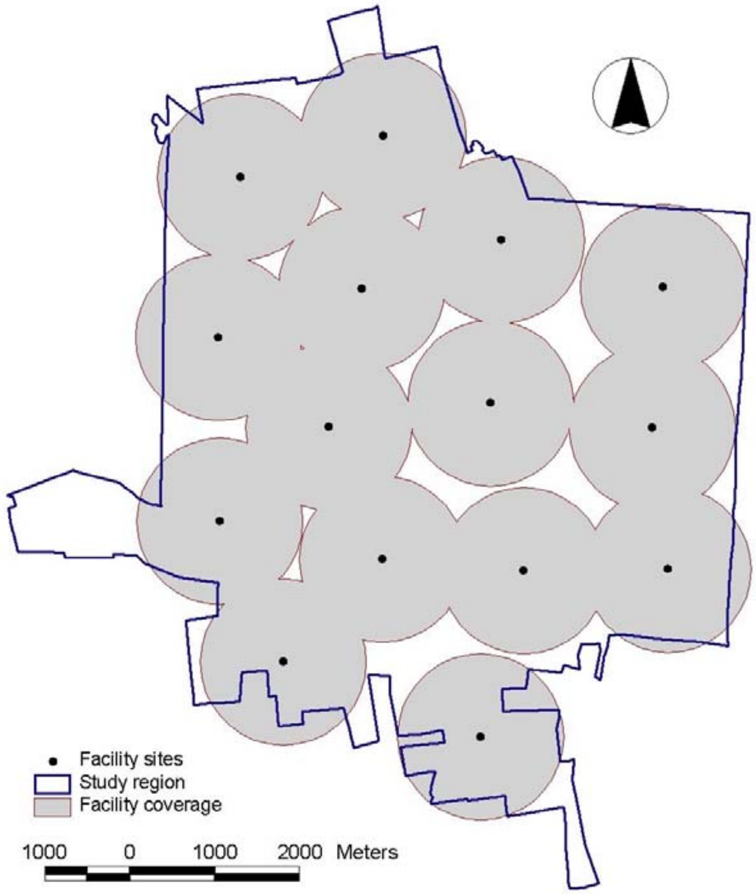


Figure 4.13: A MULTICOVER solution for *p*=15

The computational time was also assessed for the two cases (using the MCLP and MULTICOVER approaches), as summarized in Table 4.1 and illustrated in Figure 4.14. For each value of *p*, the solution time for the MULTICOVER approach was obtained by averaging over the five runs, with consideration of the variation of initial siting configuration and the number of iterations to complete the heuristic. The average solution time for the heuristic was 2,217 seconds, with a minimum of 860 seconds ($p = 5$) and a maximum of 3890 seconds ($p = 19$). In general, solution time increased with the value of p since more sub-problems (single facility problems) needed to solve in an iteration, but the change was not significant. In the case of the MCLP, solution times were considerably small when solving for all p values using 300 meter grid representation, $p \leq 17$ using 200 meter grid representation and $p \leq 12$ using 150 meter grid representation, much less than those in the MULTICOVER heuristic. However, for other larger p values, the solution times of the MCLP increased dramatically, resulting in a higher average solution time overall. Obviously, the problem size of the MCLP increases as the number of demand and potential facility sites increase, so does computational effort to solve it. Computational experience has shown that a slight decrease of grid size (from 200m to 150m) produced slightly better results (larger regional coverage), but the computational time increased dramatically. Therefore, in such cases, the MULTICOVER approach performs well in identifying good facility configuration for coverage maximization within a reasonable time frame.

p value	150m	200m	300m	MULTICOVER
5	47.9	3.8	0.2	860
6	40.4	3.6	0.3	970
7	48.8	3.8	0.3	1360
8	41.1	3.9	0.4	1670
9	48.1	4.6	0.4	1730
10	48.9	4.1	0.4	1789
11	89.8	4.5	0.5	2010
12	39.0	4.4	0.4	1687
13	226.9	4.0	0.5	2312
14	2112.8	4.6	3.2	2487
15	240686.0	4.4	0.5	3033
16	240703.7	36.0	4.2	2856
17	242374.8	67.7	7.4	2790
18	241992.8	1520.3	5.2	3120
19	244387.4	5178.6	0.5	3890
20	414402.3	38976.7	0.5	2911

Table 4.1: Comparison of solution time between MULTICOVER and MCLP

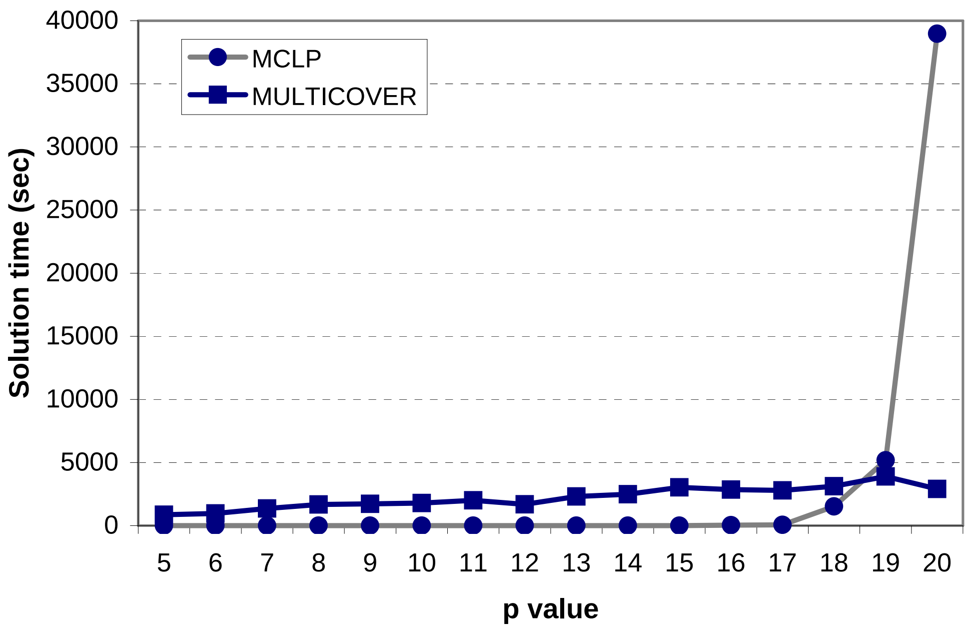


Figure 4.14: Comparison of solution time between MULTICOVER and MCLP (200 meter grid representation)

4.6 Summary

Existing approaches for siting facilities to maximize coverage of regional demand generally involve discretizations of space that can result in measurement errors and hence biased modelling results. In this chapter, we relax assumptions of discrete space representations and seek to maximize coverage of continuous demand through siting facilities in continuous space. The MULTICOVER heuristic was developed to solve such a non-linear and non-convex spatial optimization problem, involving geometric techniques in GIScience such as the Voronoi diagram heuristic, medial axis and 1-center problem. Application results showed that the heuristic performed well in identifying good solutions in a reasonable time frame, particularly with respect to the MCLP, the traditional discrete approach for coverage maximization.

CHAPTER 5

CONCLUSIONS

Facility location models rely on certain assumptions about where demand is located and where facilities can be sited. Traditionally, facility location space and associated demand for service have been represented as a collection of discrete points. This is mainly due to limited geometric capabilities, data availability, and simplification due to modeling. However, representational issues are known to exist with these simplifications. Another approach is to represent location space as continuous, i.e. facilities can be located anywhere and/or service demand is assumed continuously distributed. Although this assumption seems to be more reasonable in many situations, it makes location models more challenging to solve.

The focus of this research was on continuous location problems, including *p*-center problem and coverage maximization problem. In both cases, multiple facilities were considered and a geocomputational heuristic for problem solution was proposed, relying on geometric properties of a region. Relevant discrete location models received considerable attention for the purpose of comparative study.

5.1 Summary

This dissertation addressed the challenges of continuous space location models. It first explored a simple but effective approach for solving large vertex *p*-center problem in Chapter 2. By introducing a neighborhood facility set, the *p*-center problem was reformulated in order to remove redundant variables and constraints while preserving the characteristics of the problem including the optimality. The application results showed that the problem size of the reformulated model was substantially smaller than in original form and large vertex *p*-center problem could be solved with the use of general-purpose optimization software (e.g. CPLEX).

The dissertation then addressed in Chapter 3 application oriented issues in solving the continuous space *p*-center problem: non-convex region, complex Voronoi polygons and certain constraints on potential facility locations. Problem complexity in practice was explored. A Voronoi diagram heuristic (VDH) was extended for the CVDH to solve the constrained *p*-center problem. These heuristics were applied to siren coverage. The CVDH proved to be an effective heuristic in terms of performance stability and solution quality, addressing limitations with the VDH in practical application. The results for the vertex *p*-center problem were used as a benchmark for its continuous space counterpart. The comparison showed that discretizing continuous location space led to significant underestimate of the minmax distance. The application of the VDH and CVDH to siren siting indicated that the developed approaches were successful in terms of both performance stability and solution quality.

Finally this dissertation explored in Chapter 4 approaches for solving the problem of siting service facilities in order to maximize regional coverage in continuous space. Various approaches exist for addressing this particular planning problem given either discrete or continuous representations of potential facility sites and demand to be served. In cases where both candidate facility sites and service demand exist continuously throughout a region, approaches for maximizing regional coverage have only examined the siting of a single facility. This chapter proposed a geocomputational approach for addressing multiple facility siting. Thus, the problem is to site multiple facilities to maximally cover a region, where demand is continuously distributed and facilities may be located anywhere in the region. A Voronoi diagram heuristic was developed to decompose the problem into a number of sub-problems, each of which was solved by exploiting geometric properties of a region. The application results showed that the developed heuristic performed well in identifying facility configurations that maximize regional coverage, while solution time was satisfactory.

5.2 Future research

Existing approaches for solving location problems entail discretizations of space that can spatially bias modelling results. This dissertation relaxes assumptions of discrete location space and focuses on problems of siting facilities in continuous space to provide service to continuous demand. The non-linear and non-convex spatial optimization problems were addressed through geocomputational approaches in this research. While appealing results were obtained, several areas are worthy of further research.

The MULTICOVER algorithm developed in Chapter 4 can be refined in two respects. The algorithm uses a Voronoi diagram to decompose the multi-facility problem into a number of single facility sub-problems. Each sub-problem seeks to relocate the facility to either the center of the largest inscribed disk or as a 1-center (i.e. minimum enclosing disk) for associated Voronoi region. However the optimal solution for the sub-problem is guaranteed only in two extreme situations: either the service standard is smaller than the radius of the largest inscribed disk or it is larger than the radius of the smallest enclosing disk. Clearly, there is a need for further research on identifying an optimal solution for other cases. Another possible refinement is on the perturbation step in the MULTICOVER algorithm. Step 7 involves incrementally removing each facility and computing the medial axis associated with all uncovered areas of the region. This could be a time consuming process if the step size for computing a medial axis is small. The facility configuration could still be trapped in a local maxima in particular situations, e.g. small open space. Other approaches for perturbation are needed to address these cases.

The heuristics developed in this dissertation rely on several assumptions. On the one hand, it is assumed that all facilities have the same service distance and a service area is circular in shape. On the other hand, demand is assumed to be uniformly distributed across space. These assumptions could be simplistic in many applications. Further research is needed to consider other situations.

REFERENCES

Adenso-Diaz, B., and F. Rodriguez, 1997. “A simple search heuristic for the MCLP: application to the location of ambulance bases in a rural region,” *Omega, International Journal of Management Science*, 25(2): 181-187

Airault, S., O. Jamet, and F. Leymarie, 1996. “From manual to automatic stereoplotting: Evaluation of different road network capture processes,” *International Archives of Photogrammetry and Remote Sensing*, 31: 14–18.

Aurenhammer, F., 1991. “Voronoi diagrams: A survey of a fundamental geometric data structure,” *ACM Computing Surveys*, 23: 345-405.

Bennett, C.D., and A. Mirakhor, 1974. “Optimal facility location with respect to several regions,” *Journal of Regional Science*, 14: 131-136.

Bennett, V.L., D.J. Eaton and R.L. Church, 1982. “Selecting sites for rural health workers,” *Social Science and Medicine*, 16, 63-72.

Berman, O. and D. Krass, 2002. “The generalized maximal covering location problem,” *Computers & Operations Research*, 29: 563-581.

Blum, H., 1967. “A transformation for extracting new descriptors of shape,” in W. Walthen-Dunn (ed.), *Models for the Perception of Speech and Visual Form* (pp. 362-380). Cambridge: MIT Press.

Blum, H., 1973. “Biological shape and visual science,” Journal of Theoretical Biology, 38: 205-287.

Brandt, J.W., and V.R. Algazi, 1992. “Continuous skeleton computation by Voronoi diagram,” *CVGIP: Image Understanding*, 55: 329-338.

Chen, D., and R. Chen, 2007. “Revisiting relaxation-based algorithms for the solution of the continuous and discrete *p*-center problems.” Accepted for presentation in EWGLA XVI, Estoril, Portugal, February 2007.

Chou, J.J., 1995. “Voronoi diagrams for planar shapes,” *IEEE Computer Graphics and Applications*, March 1995: 52-59.

Church, R.L., 1984. “The planar maximal covering location problem,” *Journal of Regional Science*, 24: 185-201.

Church, R.L., 1999. “Location modeling and GIS,” in P. Longley, M. Goodchild, D. Maguire, & D. Rhind (Eds.), *Geographical information systems* (second ed.) (pp, 293-303). New York: John Wiley.

Church, R.L., 2003. “COBRA: a new formulation of the classic *p*-median location problem,” *Annals of Operations Research*, 122: 103-120.

Church, R.L., 2008. “BEAMR: An exact and approximate model for the *p*-median problem,” *Computer and Operations Research*, 35: 417-426.

Church, R.L., and C. ReVelle, 1974. “The maximal covering location problem,” *Papers of the Regional Science Association*, 32: 101-118.

Church, R.L. and J.R. Weaver, 1986. “Theoretical links between median and coverage location problems,” *Annals of Operations Research*, *6*: 1-19.

Current, J.R., M.S. Daskin and D. Schilling, 2002. “Discrete Network Location Models,” chapter 3 in Z. Drezner and H. Hamacher (ed.), *Facility Location Theory: Applications and Methods* (pp. 81-118). Berlin: Springer-Verlag.

Current, J.R., and M.E. O’Kelly, 1992. “Locating emergency warning sirens,” *Decision Sciences*, 23: 221-234.

Current, J.R. and D.A. Schilling, 1990. “Analysis of errors due to demand data aggregation in the set covering and maximal covering location problems,” *Geographical Analysis*, 22: 116-126.

Daskin, M.S., 1995. *Network and Discrete Location: Models, Algorithms and Applications*. New York: John Wiley & Sons.

Daskin, M.S., 2000. “A new approach to solving the vertex *p*-center problem to optimality: algorithm and computational results,” *Communications of the Operations Research Society of Japan*, 45: 428-436.

Daskin, M.S., A.E. Haghani, M. Khanal and C. Malandraki, 1989. “Aggregation effects in maximum covering models,” *Annals of Operations Research*, 18: 115-140.

de Berg, M., M. Kreveld, M. Overmars, and O. Schwarzkopf, 2007. *Computational Geometry: Algorithms and Applications*. New York: Springer.

Densham, P., and G. Rushton, 1992. “Strategies for solving large location-allocation problems by heuristic methods,” *Environment and Planning A*, 24: 289-304.

Drezner, Z. (Editor), 1995. *Facility Location: A Survey of Applications and Methods* (pp. 225-262). New York: Springer.

Drezner, T., and Z. Drezner, 1997. “Replacing continuous demand with discrete demand in a competitive location model,” *Naval Research Logistics*, 44(1): 81-95.

Eaton, D., M. Hector, V. Sanchez, R. Latingua and J. Morgan, 1986. “Determining ambulance deployment in Santo Domingo, Dominican Republic,” *Journal of the Operational Research Society*, 37: 113-126

Elloumi, S., M. Labbe, and Y. Pochet, 2001. “New formulation and resolution method for the p-center problem,” http://www.optimization-online.org/DB-HTML/2001/10/394.html.

Elzinga, J., and D.W. Hearn, 1972. “Geometrical solutions for some minimax location problems,” *Transportation Science*, 6: 379-394.

Gleason, J. M., 1975. “A set covering approach to bus stop location,” *Omega*, 3: 605-608.

Gold, C.M., J. Nantel, and W. Yang, 1996. “Outside-in: an alternative approach to forest map digitizing,” *International Journal of Geographical Information Systems*, 10(3): 291-310.

Gold, C.M., and J. Snoeyink, 2001. “A one-step crust and skeleton extraction algorithm,” *Algorithimica*, 30: 144-163.

Gursoy, H.N., and N.M. Patrikalakis, 1991. "Automated interrogation and adaptive subdivision of shape using medial axis transform,” *Advances in Engineering Software*, 56: 287-302.

Gursoy, H.N., and N.M. Patrikalakis, 1992. "An Automatic Coarse and Fine Surface Mesh Generation Scheme Based on Medial Axis Transform: Part I Algorithms; Part II Implementation,” *Engineering With Computers*, 8: 121-137; 179-196.

Hakimi, S.L., 1964. “Optimum location of switching centers and the absolute centers and medians of a graph,” *Operations Research*, 12:450-459.

Hakimi, S.L., 1965. “Optimum distribution of switching centers in a communications network and some related graph-theoretic problems,” *Operations Research* 13: 462-475.

Hearn, D.W., and J. Vijay, 1982. “Efficient algorithms for the (weighted) minimum circle problem,” *Operations Research*, 30: 777-795.

Hershberger, J., 1993. “A faster algorithm for the two-center decision problem,” *Information Processing Letters*, 47: 23–29.

Hillsman, E.L., 1980. “Heuristic solutions to location-allocation problems: a user’s guide to Alloc IV, V, and VI,” Monograph No. 7, Department of Geography, The University of Iowa, Iowa City.

Hochbaum, D.S., and D. Shmoys, 1985. “A best possible heuristic for the k-center problem,” *Mathematics of Operations Research*, 10: 180–184.

Hougland E.S. and N.T. Stephens, 1976. “Air pollutant monitor siting by analytical techniques,” *Journal of the Air Pollution Control Association*, 26: 52-53.

Kalvenes, J., J. Kennington and E. Olinick, 2004. “Hierarchical cellular network design with channel allocation,” *European Journal of Operational Research*, 160: 3-18.

Khuller, S., and Y.J. Sussmann, 2000. “The capacitated k-center problem,” *SIAM Journal of Discrete Math*, 13(3): 403–418.

Kirkpatrick D.G., 1979. “Efficient computation of continuous skeletons,” *Proceedings of the 20th Annual Symposium on Foundations of Computer Science*, 18-27.

Lee, D.T., 1982. “Medial axis transformation of a planar shape,” *IEEE Transactions of Pattern Analysis and Machine Intelligence*, 4: 363-369.

Leymarie, F., N. Boichis, S. Airault, and O. Jamet, 1996. “Towards the automation of road extraction processes,” In *SPIE European Symposium on Satellite and Remote Sensing III*, SPIE-2960: 84–95, Taormina, Italy, September 1996.

Love, R.F., J.G. Morris and G.O. Wesolowsky, 1988. *Facilities Location: Models and Methods*. New York: North-Holland.

Marianov, V., and D. Serra, 2002. “Location models in the public sector”, in Z. Drezner (ed.), *Facility Location: Applications and Theory* (pp. 119-144). New York: Springer.

Matisziw, T.C., and A.T. Murray, 2008. "Siting a facility in continuous space to maximize coverage of a region," *Socio-Economic Planning Science*, forthcoming.

Megiddo, N., and A. Tamir, 1983. "New results the complexity of *p*-center problems," *SIAM Journal on Computing*, 12: 751-758.

Mehrez, A., 1983. "A note on the linear integer formulation of the maximal covering location problem with facility placement on the entire plane," *Journal of Regional Science*, 23: 553-555.

Mehrez, A. and A. Stulman, 1982. "The maximal covering location problem with facility placement on the entire plane," *Journal of Regional Science*, 22: 361-365.

Mehrez, A. and A. Stulman, 1984. "An extended continuous maximal covering location problem with facility placement," *Computers & Operations Research*, 11: 19-23.

Mladenovic, N., M. Labbe, and P. Hansen, 2003. "Solving the *p*-center problem with tabu search and variable neighborhood search," *Networks*, 42: 48-64.

Montanari, U., 1969. "Continuous skeletons from digitized images," *Journal of the Association for Computing Machinery*, 16: 534-549.

Miller H.J., 1996. "GIS and geometric representation in facility location problems," *International Journal of Geographical Information Systems*, 10: 791-816.

Murray A.T., 2003. "Site placement uncertainty in location analysis," *Computer, Environment and Urban Systems*, 27: 205-221.

Murray A.T., 2005. "Geography in coverage modeling: exploiting spatial structure to address complementary partial service of areas," *Annals of the Association of American Geographers*, 95: 761-772.

Murray, A.T., and R.L. Church, 1996. "Applying simulated annealing to location-planning models," *Journal of Heuristics*, 2: 31-53.

Murray, A.T., and M.E. O'Kelly, 2002. "Assessing representation error in point-based coverage modeling," *Journal of Geographical Systems*, 4:171-191.

Murray, A.T., M.E. O'Kelly and R.L. Church, 2008. "Regional service coverage modeling," *Computers and Operations Research*, 35: 339-355.

Murray, A.T., and D. Tong, 2007. "Coverage optimization in continuous space facility siting," *International Journal of Geographical Information Sciences*, 21:757-776.

Okabe, A., and A. Suzuki, 1997. “Locational optimization problems solved through Voronoi diagrams,” *European Journal of Operational Research*, 98: 445-456.

Okabe, A., B. Boots and K. Sugihara, 2000. *Spatial tessellations: concepts and applications of Voronoi diagrams*. Chichester, England; New York: John Wiley & Sons.

Oommen, J, 1987. “An efficient geometric solution to the minimum spanning circle problem,” *Operations Research Society of America*, 35: 80-86.

Pfaltz, J.L., and A. Rosenfeld, 1967. “Computer representation of planar regions by their skeletons,” *Communications of the ACM*, 10: 119-125.

Plastria, F, 1995. “Continuous location problem,” in Z. Drezner (ed.), *Facility Location: A Survey of Applications and Methods* (pp. 225-262). New York: Springer.

Plastria, F, 2002. “Continuous covering location problem,” in Z. Drezner (ed.), *Facility Location: Applications and Theory* (pp. 37-79). New York: Springer.

Preparata, F.P. “The medial axis of a simple polygon,” in G. Goos and J. Hartmanis (ed.), *Mathematical Foundations of Computer Science, Lecture Notes in Computer Science*, V. 53 (pp. 443-450). New York: Springer.

ReVelle, C., 1986. “The maximum capture or ‘sphere of influence’ location problem: Hotelling revisited on a network,” *Journal of Regional Science*, 26(2): 343-385.

ReVelle, C., P. Marks, and J.C. Liebman, 1970. “An analysis of private and public sector location models,” *Management Science*, 16: 692-707.

ReVelle, C. and R. Swain, 1970. “Central facilities location,” *Geographical Analysis*, 2: 30-34.

ReVelle, C., J.C. Williams, and J.J. Boland, 2002. “Counterpart models in facility location science and reserve selection science,” *Environmental Modeling and Assessment*, 7: 71-80.

Rosing, K.E., and C.S. ReVelle, 1997. “Heuristic concentration: two stage solution concentration,” *European Journal of Operational Research*, 97: 75-86.

Rosing, K.E., C.S. ReVelle, and H. Rosing-Vogelaar, 1979. “The *p*-median and its linear programming relaxation: an approach to large problems,” *Journal of the Operational Research Society*, 30: 815-822.

Sadahiro, Y., and K. Takami, 2001. “A statistical method for analyzing the relative location of points in a bounded region,” *Geographical Analysis*, 33(1): 40-57.

Sharir, M., and E. Welzl, 1996. “Rectilinear and polygonal *p*-piercing and *p*-center problems,” *Proceedings of 12th Annual ACM Symposium on Computational Geometry*, Philadelphia, Pennsylvania, United States, p.122–132.

Smith, R.W., 1987. “Computer processing of line images: A survey,” *Pattern Recognition*, 20: 7-15.

Sorensen, P.A., and R.L. Church, 1996. “A comparison of strategies for data storage reduction in location-allocation problems,” *Geographical Systems*, 3: 221-242.

Suzuki, A., and Z. Drezner, 1996. “On the *p*-center location problem in an area,” *Location Science*, 4: 69-82.

Sweeney D.J., R.L. Mairose and R.K. Martin, 1979. “Strategic planning in bank location,” In *AIDS Proceedings*.

Suzuki, A., and A. Okabe, 1995. “Using Voronoi diagrams,” in Z. Drezner (ed.), *Facility Location: A Survey of Applications and Methods* (pp. 103-118). New York: Springer.

Toregas, C., 1971. *Location under maximal travel time constraints*. PhD. Dissertation, Cornell University.

Toregas, C., and C. ReVelle, 1972. “Optimal location under time or distance constraints,” *Papers of the Regional Science Association*, 28: 133-143.

Toregas, C., and C. ReVelle, 1973. “Binary logic solutions to a class of location problems,” *Geographical Analysis*, 5: 145-155.

Toregas, C., R. Swain, C. ReVelle, and L. Bergmann, 1971, "The location of emergency service facilities", *Operations Research*, 19: 1363 - 1373.

Vincent, L., and P. Soille, 1991. “Watersheds in digital spaces: An efficient algorithm based on immersion simulations,” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13(6):583–598.

Weber A., 1909. *Uber den Standort der* (Alfred Weber's theory of the location of industries) (1929). University of Chicago.

Wei, H., A.T. Murray, and N. Xiao, 2006. "Solving the continuous space *p*-center problem: planning application issues," *IMA Journal of Management Mathematics* (special issue), 17: 413-425.

Weiszfeld, E., 1936. "Sur le point pour lequel la somme des distances de n points donnes est minimum," *Tohoku Mathematical Journal*, 43: 355-386.

Wesolowsky, B., 1993. "Locating a noxious facility with respect to several polygonal regions using asymmetric distances," *IIE Transactions*, 25(1): 77-88.

White, J.A., and K.E. Case, 1974. "On covering problems and the central facilities locaiton problem," Geographical Analysis, 6: 281-293.

Xu, S., R. Freund and J. Sun, 2003. "Solution methodologies for the smallest enclosing circle problem," *Computational Optimization and Applications*, 25: 283-292.