DYNAMICS OF BLACK HOLES AND BLACK RINGS IN STRING THEORY

DISSERTATION

Presented in Partial Fulfillment of the Requirements for

the Degree Doctor of Philosophy in the

Graduate School of The Ohio State University

By

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The Ohio State University

2007

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ABSTRACT

This thesis is devoted to a study of the dynamics of two charge and three charge systems in string theory. We analyze properties of two charge systems in various duality frames and carry out perturbative addition of momentum to these systems. This gives us a picture of microscopic origin of entropy of black holes, supporting the Mathur conjecture. We extend the perturbative construction to provide a microstate for the two charge black ring by adding a small amount of momentum as the third charge. We found it to be completely smooth and horizonless in accord with the Mathur conjecture. We further study dynamics of supertubes in both the weak coupling and the strong coupling regimes which suggests a way to distinguish bound states from unbound states in string theory. We apply these results to supertubes in KK monopole background. We construct metrics describing two charge solutions in four dimensions by adding momentum to a system of N coincident KK monopole solution. We find that adding momentum separates the monopoles and resolves the associated Z_N singularities. We also consider branes wrapping cycles in spacetimes generated by two and three charge systems and analyze their supersymmetry properties.

To my parents,

With love, respect and deepest admiration

ACKNOWLEDGMENTS

Doing physics has been a continuous learning process for me and in that I have been constantly rewarded with help and support from numerous individuals. First and foremost, I would like to acknowledge all the help and support I received from my advisor Dr. Samir Mathur. In the past six years, I have received both professional and personal support which have made my research career much easier than it would have been otherwise. I have learnt a lot from him about physics as well as about the process of doing professional research. My collaborators Stefano Giusto and Ashish Saxena have been exceptionally helpful and it was a privelege to work with them. A great deal of work in this thesis was done in collaboration with them and my understanding has been deeply enhanced by their insights. I would also like to thank Borun Dev Chowdhury are several stimulating discussions and help at various times. Discussions with Jeremy Michelson and Steve Avery have improved my understanding of physics in general. I would like to thank my officemate Ben Dundee for very useful discussions and food in times of need.

On a personal front, I would like to express my deepest gratitude to my parents for starting my interest in science and giving me freedom to pursue whichever field I chose. Without their love and sacrifices, I would not have reached this far. My sister Pooja and brother Ankur have given their love, support and constant encouragement since childhood. Through their persistent faith in me, they have helped me more than they know. My friend Ashish has helped me in innumerable ways in all spheres of my life and intellectual discussions with him have deeply enriched my understanding, in physics as well as in other fields. My sister Bogi has been a constant source of love, positive energy and 'dhs' and in general, for having me look at bright side of things. Thanks to Stefano Giusto for his friendship and help in times of sickness and health. Megan Murray has been a good friend to me in past many years and many thanks to her. I would like to thank my friend and roommate Badal Joshi for many discussions and help for all these years. From my undergraduate days, I would like to thank all my teachers and friends. Special thanks to Prof. Ashoke Sen, Prof. Sudhakar Panda, Prof. S.D Joglekar, Prof. Gautam Sengupta and Prof. Pankaj Jain. Anshuman Maharana and Piyush kumar have been great friends and fellow physicists. Many thanks for all the fun and comraderie.

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PUBLICATIONS

Research Publications

S. Giusto, S. D. Mathur and Yogesh Srivastava,“A microstate for 3-charge black ring,” Nucl. Phys. B **763**,60 (2007) [arXiv:hep-th/0601193].

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FIELDS OF STUDY

Major Field: Physics

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CHAPTER 1

OVERVIEW

Our current knowledge of the basic structure of the physical universe is rooted in the frameworks of quantum field theory and general relativity. Empirically, both have been tremendously successful. The standard model of particle physics is an $SU(3) \times SU(2) \times U(1)$ gauge theory. It encapsulates our understanding of the strong and electroweak forces and as of now is fully in accord with experiments. General relativity is a classical field theory for gravity which gives us our most complete understanding of gravitational phenomena and cosmology. One of the deepest problems in theoretical physics is harmonizing general relativity with quantum field theories, or in other words, finding a quantum theory of gravity. General relativity predicts its own destruction because in situations like black holes, smooth initial data can evolve into singular field configurations. A theory of quantum gravity is expected to solve such problems but a perturbative quantization of general relativity yields a non-renormalizable theory. As an effective field theory it can be useful in dealing with low energy phenomena but it fails to cope with strong gravity phenomena like singularities of classical GR which were our main motivation for seeking a theory of quantum gravity. Something more drastic is needed.

The assumption of replacing point particles with higher dimensional objects such as strings or branes leads us to string theory which is currently the most successful theory of quantum gravity. In this chapter, we give a non technical overview of the issues and topics discussed in this thesis. In the next chapter we give a more detailed presentation of some of the background material needed to understand the later chapters.

1.1 Black Holes and String theory

Colloquially, black holes are objects with such a strong gravitational field that even light cannot escape from them. In general relativistic parlance, black holes have an 'event horizon' which is a boundary beyond which no information about the inside region may reach. Black holes are usually formed due to the gravitational collapse of a sufficiently massive star. Surprisingly, when a collapsing star settles into a stationary black hole state, its metric is uniquely determined by a few asymptotic charges like mass and angular momentum. Existence of black holes is quite problematic for the second law of thermodynamics in the rest of the universe because one can decrease the entropy of the outside region by dropping objects in the black hole. One can argue that the entropy is not lost but just hidden inside the black hole. But from the point of view of the rest of the universe, the second law becomes observationally unverifiable since we have no way of knowing how much entropy the black hole has due to matter falling into it.

Bekenstein [1] provided a solution to this dilemma by assigning an entropy to the black hole itself. Based on several 'gedanken' experiments, he determined that the black hole entropy must be proportional to the area of the event horizon. Hawking

[2] fixed the constant of proportionality by considering quantum field theory in the curved classical spacetime of a black hole. The work of Bekenstein and Hawking saved the second law of thermodynamics but it raised profound questions about black holes, thermodynamics and quantum field theory. Hawking discovered that not only do black holes have entropy but they also have a temperature and radiate like any black body at that temperature. Black hole radiation was found to be thermal and it leads to a loss of the mass of the black hole. So unless some new principle arises, black holes can completely evaporate and leave just thermal radiation behind. By its nature, black body radiation does not carry any information (except temperature, which depends only on the total mass) about what went into making the black hole. This, in simplified terms, is known as the *information loss problem*.

A related problem is that unlike any other thermodynamic system we have no microscopic understanding of the Bekenstein-Hawking entropy of a black hole. We do not know how to explain the entropy of black holes as arising due to a coarse graining over some set of microstates. String theoy being our most well developed theory of quantum gravity should shed some light on this profound puzzle. String theory is an interacting relativistic theory of extended objects like strings and solitonic membranes of different dimensionalities. The strength of interaction between objects is governed by a dimensionless coupling constant g. There are also duality symmetries which can convert one set of objects to other or change from a weak coupling regime to a strong coupling regime of the theory. A crucial ingredient of string theory is supersymmetry. In any supersymmetric theory, the mass (M) and charge (Q) satisfy an inequality of the form $M \geq cQ$ for some constant c. States that saturate this bound are called BPS states and have the special property that their mass does not receive any quantum corrections.

In string theory, we try to make black holes from strings and different types of higher dimensional membranes. Strominger and Vafa [3], in their pioneering work made classical sized black holes using three charged objects D1-D5-P. D1 and D5 branes are solitonic objects of one and five dimensions respectively while P refers to momentum carried along their common direction. One problem is that we understand the quantum structure of these objects at weak coupling only while black holes are a strong coupling phenomena. In this situation BPS states come handy. Using them one can make special types of black holes (extremal black holes) and count all possible configurations with given values of charges at weak coupling. Now imagine increasing the coupling. This increases gravity [4] and causes these states to become black holes. Using uniqueness of black holes, we can relate the weak coupling and strong coupling regimes. Since BPS states are protected by supersymmetry, the counting that we did for weak coupling remains valid and logarithm of that gives the entropy even at strong coupling. In one of the famous triumphs of string theory, we get a perfect match between microscopic and macroscopic entropies. Agreement persists if we have near-extremal black holes which are not fully protected by supersymmetry. Calculations have been done for 5 and 4 dimensional black holes including other charges like angular momenta. Small deviations from fully thermal spectrum (grey body factors) have also been shown to match.

1.2 AdS/CFT and the Mathur Conjecture

As the work of Strominger and Vafa showed, string theory was on the right track for a correct theory of quantum gravity. However it did not yield much information about the microstates except their number. Motivated partially, by results on black holes, Maldacena [5] conjectured a very profound duality between string theory on product spaces $AdS_p \times S^q \times M$ and conformal field theories on the conformal boundary of this spacetime. Here AdS_p refers to p dimensional anti-desitter space and S^q denotes q dimensional sphere. We will usually take $M = T^4$ i.e four-torus. For our special case of black holes in 4 or 5 dimensions, we have a duality between $AdS_3 \times S^3$ and some conformal field theory. This $AdS_3 \times S^3$ spacetime occurs as the near horizon limit of black holes and the CFT is the low energy limit of a gauge theory living on the branes making the black hole. Since the CFT is a unitary field theory, there should not be any information loss in black holes. In the CFT one can identify individual states and their counting matches with the black hole entropy. But what if one insists on using a gravitational description? How does one see states on the gravity side if we only have one black hole metric? Based on an abstract argument one can already see that the individual microstates must be horizonless. If microstates had a horizon they will have entropy of their own. We want microstates to explain the entropy of the black hole and not to have entropy of their own. If they do not have horizons they must be non-singular if we do not want naked singularities.

For the two charge systems Mathur and Lunin [6] constructed a family of non-singular horizonless metrics corresponding to states in the CFT which have same asymptotic charges and look like naively constructed two charge metric for black holes outside. But inside they all differ from each other and if we delineate a boundary after which the geometries start differing from each, we find that this boundary has [7],[8] an area which gives the black hole entropy. So "coarse graining" over geometries leads us to the entropy of the black hole. This is a very satisfying picture of the black hole because entropy arises from coarse graining. It must be emphasised that the generic states are expected to be quantum and probably no geometric description would be possible. The idea is to associate a coherent state in the CFT with an asymptotically flat geometry which is smooth and free of horizons, carries the same conserved charges as the black hole and hence constitutes a microstate of the black hole. The two charge system is basically a toy model since it does not have a classically finite horizon. The work of Strominger and Vafa was for three charge systems. It is reasonable to expect the basic picture of black holes to remain unchanged when we go from two to three charges. In chapter 3, we take the first step towards this goal by adding the third charge perturbatively. In that case, we found a smooth perturbation which confirms that the picture holds at least for some special states. After this work was done, the Mathur conjecture was further confirmed when a complete non-perturbative three charge geoemtry was constructed by Giusto, Mathur and Saxena [9] and were found to be smooth and horizonless. Again, the conjecture does not say that all three charge states will have a classical supergravity description.

1.3 Properties of microstate geometries

Given the importance of these smooth geometries, it is worth exploring their properties further. In chapter 5, we study branes wrapping various cycles in these spacetimes and study their supersymmetry properties. In chapter 6, we study the dynamics of supertubes [10],[11] which are basically dualized versions of two charge systems we have been talking about. At weak coupling, supertubes are described by Dirac-Born-Infeld (DBI) action which is proportional to area of the brane. We study fluctuations of these system both at weak coupling (where they are described by a DBI action) and in the strong coupling supergravity limit. From this study we also managed to obtain a conjecture to distinguish between bound states and unbound states. In 5 and 6 dimensions, there has been a lot of recent progress recently and there is now a general scheme [12] to write down solutions to minimal supergravity in these dimensions. Most of these solutions do not correspond to *bound* states. Mathur conjecture deals with *bound* states and hence it is of crucial importance to have a way to distinguish between bound and unbound states, specially for three charge states.

1.4 Systems with KK monopole

Recently, there has been a lot of interest in systems containing the KK monopole. It is a purely gravitational solution in string theory and one of its obvious attractions is that it is a completely regular gravity solution. We will see in the next chapter that the geometry of its non compact part is four dimensional. Hence it is used as an ingredient in constructing black holes in four dimensions. In chapter 7, we use it to construct [13] metrics for 2 charge system in four dimensions. Using a solution generation technique, we add momentum to a system of N coincident KK monopoles. Before adding momentum the system of N coincident KK monopoles has Z_N singularities. But the presence of momentum separates them and since each KK monopole is smooth we get a smooth solution. We do get a singularity when we take KK monopoles as continuously distributed. But this is due to the low number of dimensions. We examine other properties of this system in chapter 7. In chapter 8, we study two charge supertubes moving in a KK monopole background and consider their dynamics. We study this system in both weak coupling (DBI limit) and strong coupling (supergravity) limit. We study the system in different duality frames and with different vibration profiles.

1.5 Black Ring

For four dimensional black holes, classical general relativity is very restrictive and under very general assumptions it is possible to prove certain uniqueness theorems. These uniqueness theorems (see [14] for a review) severely restrict the possible types of black holes, given a set of asymptotic charges and boundary conditions. For example, for four dimensional Einstein-Maxwell theory, a stationary, asymptotically flat black hole is specified by a limited number of parameters. Uniqueness theorems assert that these parameters are precisely those that correspond to conserved charges namely, the mass M and angular momentum J, and possibly the charges Q associated to local gauge symmetries. Hence, the only black hole solution of the four-dimensional Einstein-Maxwell theory is the Kerr-Newman black hole. This result precludes the possibility that a black hole possesses higher multipole moments (for example, a mass quadrupole or a charge dipole) that are not completely fixed by the values of the conserved charges. Physically, during the collapse phase (not described by a stationary metric), a self-gravitating object loses all 'hair' and settles down to a stationary solution described by the Kerr-Newman solution. Anything that can be radiated is radiated away during the collapse process. Deviations from Kerr-Newman solution drive the emission of gravitational radiation. Backreaction from that radiation removes the deviations (the "balding" process). These conclusions are based on perturbative analysis but are confirmed by numerical methods for large deviations as well.

Although the higher dimensional version of the Schwarzschild solution was found long ago [15], it was not until 1986 with the impetus provided by the development of string theory, that the higher-dimensional version of the Kerr solution was constructed by Myers and Perry (MP) [16]. Given that the Kerr black hole solution is unique in four dimensions, it may have seemed natural to expect black hole uniqueness to hold in higher dimensions as well.

Now we know that at least in five dimensions, and very likely in $D \geq 5$ dimensions, this is not the case. A heuristic argument that suggests the possibility of black holes of non-spherical topology is the following. Take a neutral black string in five dimensions, constructed as the direct product of the Schwarzschild solution and a line, so the geometry of the horizon is $\mathbf{R} \times S^2$. Imagine bending this string to form a circle, so the topology is now $S^1 \times S^2$. In principle this circular string tends to contract, decreasing the radius of the S^1, due to its tension and gravitational self-attraction. However, we can make the string rotate along the S^1 and balance these forces against the centrifugal repulsion. Then we end up with a neutral rotating *black ring*: a black hole with an event horizon of topology $S^1 \times S^2$. Ref. [17] obtained an explicit solution of five-dimensional vacuum general relativity describing such an object. This was not only an example of a non spherical horizon topology, but it also turned out to be a counterexample to black hole uniqueness.

Our main interest here is in the supersymmetric black ring discivered in [175]. Since black rings also have an entropy, it is reasonable to ask if the Mathur conjecture picture for black holes also extends to black rings. This would imply that one should be able to find microstates for black rings. For two charges and one dipole moment, black hole and black rings are the same. So one can ask if one can perturbatively add momentum to one of two charge states to get a microstate for a black ring, as was done for black holes in chapter 3. This is what we do in chapter 4 and we do find a smooth perturbations in the black ring regime. This is very encouraging and hopefully more examples can be found soon. The base state around which we perturb has a lot of angular momentum and so one can question its 'genericity' for the case of black holes. But since black rings have angular momentum bounded below, a perturbative state represents a generic situation more closely than in case of black holes.

CHAPTER 2

PRELIMINARIES

In this chapter, we give some background information about issues relating to black holes that we want to discuss in this thesis and how string theory gives us a radically different picture of black holes.

2.1 Black Holes in classical and quantum gravity

In terms of Newtonian gravity, black holes are gravitationally collapsed objects for which escape velocity from the surface becomes greater than the speed of light, so that nothing can escape. In general relativity, the situation is more drastic. A black hole is defined as a region of spacetime that can not communicate with the external universe. The boundary of this region is called the event horizon. Under a fairly reasonable set of assumptions like the validity of Einstein equations and the positivity of energy, it can be shown that a gravitational collapse will produce singularities. Singularities 'clothed' by event horizon lead to black holes as opposed to 'naked' singularities. The conjecture that physically possible singularities in general relativity are always hidden behind event horizon is called Cosmic censorship hypothesis. Even though there is no general proof of it as of now, cosmic censorship is well supported by calculations and numerical work. One might suppose that a realistic black hole which forms due to collapse of all kinds of matter configurations, with arbitrary multipole distributions,magnetic fields etc would be horrendously complicated objected. Surprisingly, during the collapse phase (not described by a stationary metric) , a self-gravitating object looses all 'hair' and settles down to a stationary solution described by a solution described by only three parameters : mass,charge and angular momentum. This solution to vacuum Einstein equations, known as Kerr-Newman solution, is stationary,axisymmetric,asymptotically flat and has a regular event horizon. Anything that can be radiated is radiated away during collapse process. Deviations from Kerr-Newman solution drive the emission of gravitational radiation. Backreaction from that radiation removes the deviation ("balding" process). These conclusions are based on perturbative analysis but are confirmed by numerical methods for large deviations too.

If the black hole has no charge or angular momentum (known as Schwarzschild black hole) then the metric is given by

$$ds^2 = -(1 - \frac{r_H}{r})dt^2 + (1 - \frac{r_H}{r})^{-1}dr^2 + r^2 d\Omega_2^2 \tag{1.1}$$

Here $r_H = 2GM$ and we have set $c = 1$. There is a coordinate singularity at the surface $r = r_H$, called the event horizon, and it can be removed by using a different set of coordinates. Curvature invariants like Ricci scalar are finite at the horizon. Even though there is no real singularity there, the surface $r = r_H$ has the special property of being a one way membrane. Things can go in but cannot come out. One way to see this is that inside the horizon, the radial coordinate becomes time like (coefficient of dr^2 becomes negative) and singularity at $r = 0$ is in our 'future' and we can not avoid hitting it. So once the radius of a collapsing star has dropped below r_H, even light can not escape from its interior, hence the name “black hole”. On the other hand, $r = 0$ is real singularity where curvature invariants diverge. One point to keep in mind is that an object falling into a black hole will not experience any particularly strong force as it crosses the horizon. At the horizon radius $r = r_H$, curvature scales as $1/M^2$ and can be made arbitrarily small by taking sufficiently large mass. However, one peculiarity of event horizon is that for an asymptotic observer watching a falling object, it takes infinite time to reach horizon due to infinite gravitational redshift between horizon and infinity.

In 1971, Hawking proved a theorem [46] which says that the area of a black hole can never decrease with time. If two black holes merge then the area of the new black hole is greater than the sum of the areas of the original black holes. The non-decreasing property of area is reminiscent of entropy in thermodynamics. Another thing to notice about the Schwarzschild metric is that parameter that appears in the metric is total mass. In the general case of Kerr-Newman black hole the metric contains three parameters; mass,charge and angular momentum. We know that black holes are formed by gravitational collapse of some star which contained a huge amount of matter and huge amount of information (like quantum numbers corresponding to various particles, their motion etc). But the end result is an object which only contains information about few parameters like mass,charge etc. So given these parameters, black hole state is unique and hence entropy of a black hole is zero. Black hole formation is a spontaneous process and hence from the outside observer's point of view, second law of thermodynamics is violated in such a collapse. Existence of black holes conflicts with the second law of thermodynamics because one can decrease the entropy of outside universe by throwing things into black hole. One can try to salvage second law by including the entropy of infalling matter in the entropy of outside universe even when it has crossed the horizon. But then the second law becomes observationally unverifiable. One possible way out was suggested by Bekenstein [47]who argued that if we assign an entropy to black hole which is proportional to area then one can define a generalized second law which is valid in all situations. Since area increases when anything falls in black hole, the net entropy of the universe ($S_{matter} + S_{blackhole}$) increases. Using several thought experiments, Bekenstein proposed that entropy of a black hole is proportional to area. Using semi-classical quantum gravity, Hawking [2]fixed the constant of proportionality and we have $S_{BH} = \frac{A}{4G_N}$. Here A is the area of the horizon of black hole. For a black hole of mass equal of mass of earth, this implies that $S_{BH} \sim 10^{66}$ which is an extremely large entropy.

Hawking also found that black holes are not perfect sinks. They radiate like a black body with temperature $T = \frac{\kappa}{2\pi}$. Here κ is the surface gravity of black hole, which for Schwarzschild black hole is $(4M_{BH})^{-1}$. The emitted radiation, called Hawking radiation, is exactly thermal according to semi-classical calculations of Hawking. So black holes emit radiation until they completely evaporate. This poses a problem for quantum mechanical determinism because thermal radiation only carries information about temperature and hence mass from above relation. All knowledge of states which formed black hole is completely lost. So a black hole made from two different sets of objects can evaporate to completely similar end product i. e hawking radiation which depends only on one parameter, namely mass of black hole (or 3 parameters in case of Kerr-Newman black hole). All thermodynamic systems in physics have their entropies explainable in terms of coarse graining of microstates.

If black holes behave similarly then we would expect $N = e^{S_{BH}}$ microstates, according to Boltzmann formula. But as we saw above, black hole uniquenss gives only one state and hence zero entropy. How can one explain such a huge discrepancy?

Hawking's semi-classical calculation is based on treating matter quantum mechanically while keeping gravitational field classical (quantum field theory in curved spaces). Hawking radiation is produced from the quantum fluctuations of the matter vacuum, in the presence of gravitational field of black hole. Near the horizon particle-antiparticle pairs are produced due to vacuum fluctuations. In a black hole background, one member of this pair can fall into the black hole, where it has a net negative energy, while the other member can escape to infinity as real positive energy radiation with thermal spectrum. One might question making such a semi-classical approximation and think that strong quantum gravity effects will invalidate this approximation. But as we say earlier, the gravitational field (curvature) near horizon, where these particle pairs are produced, is not large and can be made arbitrarily small by increasing the mass. So quantum gravity effects as normally understood, should not affect the conclusions of Hawking.

One may suppose that all the information about the $e^{S_{BH}}$ microstates resides in a planck sized region near the singularity (where all infalling matter disppears) and thus the information about states is not visible classically. Infact, this lack of information transfer between this region near singularity and macroscopic horizon (where pair production takes place) manifests itself in thermaliy of hawking radiation and thus in information problem. Our naive intuition about quantum gravity effects being confined in a planck sized region and locality are at the root of this problem. Thus any putative quantum gravity theory, when applied to black holes, should throw light on following issues

(a) It should be possible to resolve singularity at the centre of black hole.

(b) It should be possible to transfer information from interior region of the black hole to the horizon.

(c) It should be possible to construct microstates which give black hole entropy on coarse graining and it should give us some picture of how information is preserved.

String theory is currently the only framework rich enough to let us analyze black holes and information loss problem in any convincing way. So we turn to a description of string theory in the next section.

2.2 String Theory

At a perturbative level(weak coupling) string theory is best thought of as a theory of interacting relativistic strings. We will see later that string theory contains objects of higher dimensionality like D-branes, KK monopoles etc. But rightnow we concentrate on perturbative string theory. In superstring theory, particles are described as vibrational modes of strings. We can have open as well as closed string theories. The energy per unit length of string, string tension, is parametrized as $T_s = \frac{1}{2\pi\alpha'}$ where string length $\sqrt{\alpha'}$ is usually set to be of order 10^{-33} cm. A moving string traces a worldsheet in spacetime and string theory is descibed by a worldsheet action whose bosonic part is proportional to the area of the worldsheet. This $1+1$ dimensional field theory has majorana fermions and worldsheet supersymmetry. Interaction between strings is controlled by a dimensionless coupling constant g_s. Even in weak coupling perturbation theory ($g_s \ll 1$), quantization requires that theory be conformally invariant,supersymmetric and the number of spacetime dimensions be 10. Since the spatial direction of the worldsheet has finite extent, each worldsheet field can be regarded as a collection of infinte number of harmonic osciallators labelled by the quantized momentum along this spatial direction. Different states of the string are obtained by acting on the fock vacuum by these oscillators. This gives an infinte tower of states. For low energy considerations, we are usually only interested in massless fields. Massless sector of open string theory contains gauge fields while massless sector of closed string theories contains graviton $G_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$ and scalar dilaton Φ in their spectra. Bosonic part of action in a general back ground is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left([\sqrt{g} g^{ab} G_{\mu\nu}(X) + \epsilon^{ab} B_{\mu\nu}] \partial_a X^\mu \partial_b X^\nu + \alpha' \sqrt{g} R \Phi(X) \right) \tag{2.2}$$

Here $\alpha' = l_s^2$,with l_s being string length. R is Ricci scalar for worldsheet. Graviton ($G_{\mu\nu}$), B-field($B_{\mu\nu}$) and dilaton (Φ) also act as background fields. Roughly,closed strings describe gravity while open strings describe gauge fields. Action written above is conformally invariant and it turns out that conformal invariance is crucial for the consistency of string theory. In the presence of arbitrary background fields, conformal invariance is broken at quantum level and consistency requires that β function describing this violation is zero. The requirement of vanishing β function gives a set of spacetime equations (order by order in α') which,at the lowest level, are just supergravity equations of motion. Higher order terms provide string theoretic corrections to low energy supergravity results. Depending on field content (bosonic and fermionic) and boundary conditions for this $1+1$ worldsheet theory, we can have five consistent string theories in 10 dimensions. They are known as type IIA,type IIB, type I, $E_8 \times E_8$ heterotic and $SO(32)$ heterotic. In this thesis, we will be interested in type IIA and type IIB theories only. Low energy effective field theory limits of these theories are known as type IIA and type IIB supergravities.

2.2.1 Spectrum and Solitons in type II theories

In type II theories, the worldsheet theory is a free field theory of closed strings containing 8 scalar fields(representing 8 transverse coordinates of string) and 16 majorana-weyl fermions (8 with left handed chirality and 8 with right handed chirality). Scalars always have periodic boundary conditions but for fermions we have choice between periodic (Ramond condition) and antiperiodic (Neveu-Schwarz condition). Since we have two types of fermions (left handed and right handed) , there are four possibilities : NS-NS (bosonic), RR(bosonic),NS-R(fermionic) and R-NS(fermionic). Finally, consistency at 1-loop and desire to get rid of tachyon which occurs in the spectrum, we impose GSO projection which keeps only states with even number of left movers and even number of right movers. If the GSO projections on left and right moving sectors are same then we get type IIB theory. Otherwise we get type IIA theory. Since the two theories differ only in R sector, NS sector bosonic states are same $(G_{\mu\nu}, B_{\mu\nu}, \Phi)$ in both the theories. The RR sector massless states of type IIA string theory consist of a vector and a rank three anti-symmetric tensor. Type IIB RR massless sector consists of a scalar,a rank two antisymmetric tensor, and a rank four antisymmetric tensor (with self dual field strength). Type IIA is non-chiral while type IIB is chiral, both having $N = 2$ susy. Strings do not source these RR fields and from perturbative string theory point of view, they appear a bit mysterious. But analysis of low energy effective action for massless fieds gave us a window to non-perturbative aspects of string theory. Low energy equations for massless fields in these theories are same as respective supergravity equations of motion. From these low energy equations we see that in addition to strings, superstring theory also contains soliton like objects of various internal dimensionalities called Dirichlet Branes (D-branes).

A Dirichlet p-brane(Dp-brane) is a p+1 dimensional hyperplane in 9+1 dimensional space-time where open strings are allowed to end, even in theories where all strings are closed in bulk spacetime[57]. D-brane is like a topological defect: when a closed string touches it, it can open up and turn into open string whose ends are free to move along the D-brane. Open string states with ends lying on D-branes correspond to vibrational modes of D-branes. Excitations of open strings describe gauge theories and hence D-branes naturally realize gauge theories on their world volume. The massless spectrum of open strings living on a Dp-brane is that of a maximally supersymmetric $U(1)$ gauge theory in $p+1$ dimensions. The $9-p$ massless scalar fields present in this supermultiplet are the expected Goldstone modes associated with the transverse oscillations of the Dp-brane, while the photons and fermions may be thought of as providing unique supersymmetric completion. If we consider N parallel D-branes, then there are N^2 different species of open strings because they can begin and end on any D-branes. Open strings with end points on same brane can become massless while open strings stretched between different branes will have masses determined by the separation between two branes. Thus we have $U(1)^N$ gauge symmetry. In the limit of coincident branes, all strings can become massless and $U(1)^N$ symmetry gets promoted to $U(N)$. Separating the branes is then equivalent to Higgsing the gauge theory and giving vacuum expectation values to scalar fields. The overall $U(1) \sim U(N)/SU(N)$ (upto some Z_N identifications which are not important here) corresponds to overall position of branes and may be ignored when considering dynamics on branes. Thus we have $SU(N)$ gauge symmetry. Open string perturbation theory for N Dp-branes is governed by g_sN and is valid if $g_sN < 1$. As we mentioned earlier, there is another description of D branes as solitons of low energy string effective action[52]. For large N, the stack of D branes behaves like a massive object which acts as a source for gravity. These brane solutions occur as solutions to supergravity field equations (which as we mentioned earlier, occur as low energy limit of string equations). These are solitonic, in the sense that their tensions are inversely proportional to string coupling $T_p \sim \frac{1}{g_s l_s^{p+1}}$. D-p brane metric is [52]

$$ds^2 = H^{-1/2}(r)\left(-dt^2 + \sum_{i=1}^{p}(dx^i)^2\right) + H^{1/2}(r)\left(dr^2 + r^2(d\Omega_{8-p})^2\right) \tag{2.3}$$

Dilaton is given by $e^{\Phi} = H^{\frac{3-p}{4}}$. We have not shown RR fields which are not important for arguments below. Here

$$H(r) = 1 + \frac{L^{7-p}}{r^{7-p}} \tag{2.4}$$

L is related to radius of curvature of the spacetime. The gravity description of D branes is valid when $g_sN > 1$. Thus we have a sort of complementarity between gravity and gauge theory descriptions.

2.2.2 Finding supersymmetric solutions

D-branes and other solitonic objects are invariant under half of spacetime supersymmetries. To see this, we briefly go over the concept of killing spinors. A Dirac spinor in D dimensions has $2^{[D/2]}$ complex components where $[D/2]$ denotes the integral part of $D/2$. For on-shell degrees of freedom, the number is halved because spinors satisfy Dirac equation. But spinors which occur in susy transformations and hence in killing spinor equation are off-shell. In dimensions where Majorana condition is possible , we can use this condition to take all components as real. This reduces the real degrees of freedom by half. In ten dimensions, a Dirac spinor has 32 complex components. In type IIB theory, we have two such spinors since $N = 2$. After Majorana condition, we have two spinors with 32 real components each. In even dimensions, we can also have Weyl condition which allows us to consider spinors with definite chirality and further cuts the degrees of freedom by half. If we choose to have two spinors of opposite chirality then we get IIA theory. In type IIB theory, we have spinors of same chirality. In general supergravity equations of motion are pretty complicated second order equations. If we are interested in classical solutions only then situation can be simplified as follows. Classical solutions have fermion fields set to zero (fermions do not produce classical fields because of exclusion principle). Classical solutions are also the solutions of purely bosonic action that one obtains by setting all fermionic fields to zero. Supersymmetry transformations convert a boson into fermion and vice-versa. Schematically, for a susy variation with parameter ϵ

$$\delta_\epsilon B \sim \epsilon F \quad , \quad \delta_\epsilon F \sim \partial\epsilon + B\epsilon \tag{2.5}$$

Here B denotes bosons and F denotes fermions. Since fermions are already zero, the variation of bosons is automatically zero. So we just need to make sure that the variation of fermions is also zero since we do not want to generate fermionic fields by susy transformations on purely bosonic fields. So we want $\delta_\epsilon F = 0$. This gives an equation which can be seen as equation for superisometry in superspace. In analogy with GR, this is called Killing spinor equation. This gives a system of first order equations. Usually this gives an ansatz(very constrained) which can be put in equations of motion to get the full solution. Killing spinor equation also gives us a constraint on vacuum killing spinor and this constraint cuts the no. of components by half.

2.2.3 Dualities

In the perturbative regime, there are 5 consistent string theories. But existence of various dualities connects these theories and it was realized that all five string theories in 10 dimensions are limits of a still imperfectly understood 11 dimesnional theory,called M-theory. One thing that is known about M-theory is that its low energy limit is 11-dimensional supergravity. We will be exclusively concerned with type IIA and type IIB string theories in this report. Since string is an extended object, it can do more things than point particles. If the background contains a spatial circle S^1 of radius R, then there can be momentum modes which have energy proportional to $1/R$ as well as winding modes of strings with energy proportional to R. T-duality is a symmetry which interchanges these two types of modes by taking $R \rightarrow \frac{1}{2\pi R T_s}$ where T_s is string tension and its a peculiar property of string theory that the whole theory is invariant under this transformation. T-duality can also connect a Dp brane to $D(p-1)$ and $D(p+1)$ branes but it takes one from IIA theory to IIB theory and vice-versa. Type IIB theory has another symmetry called S-duality under which string coupling $g_s \rightarrow \frac{1}{g_s}$. S-duality converts strong coupling regime to weak coupling regime and vice-versa. Under S-duality, fundamental string $NS1$ goes to solitonic $D1$ brane. These dualities can be used to transform systems in string theory to equivalent duality related systems which may be more tractable in that regime.

In two charge, which will be our main interest, this trick of transforming to duality related systems has led to significant advances in our understanding.

2.3 Metrics corresponding to various branes

We will mostly interested in gravitational aspects of string theory. So construction of metrics for various objects occuring in string theory is of paramount importance to us. Since all one charge objects in string theory are connected by dualities, we can start with any one of them and using dualities and dimensional reduction we can write down metrics for others. We will be specially interested in KK monopole later on, so we start with a brief description of it and then derive other objects from it via dualities. Five dimensional metric for KK monopole at origin is

$$ds^2 = -dt^2 + H[ds + \chi_j dx^j]^2 + H^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{3.6}$$

$$H^{-1} = 1 + \frac{Q_K}{r} \quad , \quad \vec{\nabla} \times \vec{\chi} = -\vec{\nabla} H^{-1} \tag{3.7}$$

Here x_j with $j = 1, 2, 3$ are transverse coordinates and there is a NUT singularity at $r = 0$ unless $Q_K = \frac{1}{2} N_K R_K$ where N_K corresponds to number of KK monopoles. Near $r = 0$, s circle shrinks to zero. For $N_K = 1$, it does so smoothly while $N_K > 1$, there are Z_{N_K} singularities. We consider $N_K = 1$ case here.

Constant time slices of the above spacetime are Taub-Nut gravitational instanton. Gauge field is clearly that of a monopole. Explicitly, in one chosen patch, it is given by

$$A_\phi = Q_K(1 - \cos\theta) \quad , \quad \vec{B} = \frac{Q_K \vec{r}}{r^3} \tag{3.8}$$

and has a dirac string singularity unless period of s is equal to $2Q_K$. The metric is regular on the half axis $\theta = 0$ but has a singularity at $\theta = \pi$ since the $(1 - \cos\theta)$ term in the metric means that a small loop about this axis does not shrink to zero length at $\theta = \pi$. By a change of coordinates $s' = s + 2Q_K\phi$ the metric becomes regular at $\theta = \pi$ but not a $\theta = 0$. Thus one needs two sets of coordinates to cover the space. Since KK monopoles are 1/2 BPS objects in string theory, they can be put in static equilibrium with each other. So multimonopole metric is just given by same metric as above except

$$H^{-1} = 1 + \sum_{i=1}^{N} \frac{Q_K^{(i)}}{|\vec{x} - \vec{x}_i|} \tag{3.9}$$

If $Q_K^i = Q_K = \frac{1}{2}R_K$ for all i then all the Dirac strings can be made simultaneously unobservable and we have a regular solution with N monopoles sitting at rest at $\vec{x} = \vec{x}_i$. It is a remarkable property of Kaluza-Klein monopoles that they do not interact. From four dimensional point of view i. e in dimensionally reduced form, it is because of exact cancellation between gravitational attraction and repulsion due to scalar field. In four dimensional form, we have

$$ds^2 = -\sqrt{1 - \frac{Q_K}{2r}}dt^2 + \frac{dr^2}{\sqrt{1 - \frac{Q_K}{2r}}} + \sqrt{1 - \frac{Q_K}{2r}}r^2 d\Omega^2 \tag{3.10}$$

$$\psi = \frac{1}{2}ln(1 - \frac{Q_K}{2r}) \qquad \infty \geq r \geq Q_K/2 \tag{3.11}$$

If we consider a massive test particle and its interaction with KK monopole, we see that it interacts both with gravitational field and scalar field. The test particle can remain relatively at rest with respect to KK monopole since newtonian force is exactly cancelled by interaction with scalar field.

In 11 dimensions, the metric for KK monopole is

$$ds^2 = -dt^2 + dy^2 + dz^i dz_i + H^{-1}[ds + A_\phi d\phi]^2 + H[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{3.12}$$

$$H = 1 + \frac{Q}{r} \quad , \quad A_\phi = Q(1 - \cos\theta) \tag{3.13}$$

Here $i = 1, 2, ..5$ and other fields are zero. From this 11 dimensional metric, we can get two different objects in 10 dimensions. If we do not reduce along the compact direction s then we get string theoretic KK monopole. For example, if we first make the direction y compact and then reduce along it then we get the metric

$$ds^2 = -dt^2 + dz^i dz_i + H[ds + A_\phi d\phi]^2 + H^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{3.14}$$

with no dilaton or B-field. If instead we reduce along the compact direction s which is non-trivially fibred with KK monopole then we get $D6$ brane. The metric is

$$ds^2 = -dt^2 + dw^2 + dz^i dz_i + H^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{3.15}$$

$$A_\phi = Q(1 - \cos\theta) \tag{3.16}$$

$$e^C = H \tag{3.17}$$

If we go to string frame then, with $g^s_{ab} = e^{\frac{C}{2}} g^{10}_{ab}$ we get

$$ds^2 = H^{1/2}\left(-dt^2 + dw^2 + dz_i dz^i\right) + H^{-1/2}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{3.18}$$

$$e^{2\Phi} = H^{3/2} \tag{3.19}$$

We take KK monopole metric in 10 dimensions and then do a T-duality along compact direction s. Since there is no dilaton in ten dimensional metric string and Einstein frames are same. Reducing along s we get $e^{C'} = H$. SInce intially the dilaton was zero, new dilaton is $\Phi' = -\frac{1}{2}C'$. Using T-duality rules we get metric in string frame

$$ds_s^2 = -dt^2 + dz_i dz^i + H^{-1}[ds^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{3.20}$$

$$B_{s\mu} = A_\mu \tag{3.21}$$

$$e^{-2\Phi'} = H \tag{3.22}$$

We recognize this as $NS5$ brane. To go to Einstein frame, we multiply by $e^{-\Phi'/2}$. Metric for $NS5$ brane becomes

$$ds_E^2 = H^{1/4}(-dt^2 + dz_i dz^i) + H^{-3/4} dx_j dx^j \tag{3.23}$$

Metric for $D5$ brane is same as above metric (in Einstein frame) with sign of dilaton changed. Two metrics are related by S-duality which in Einstein frame corresponds to changing the sign of dilaton. To write string frame metric for $D5$ brane we multiply by $e^{\Phi'/2}$. Notice the change in sign of dilaton. So we get

$$ds^2 = H^{1/2}(-dt^2 + dz_j^2) + H^{-1/2} dx_i^2 \tag{3.24}$$

2.4 AdS/CFT correspondence

In previous section, we noticed that excitations of open strings describe gauge theories and hence D-branes naturally realize gauge theories on their world volume. The massless spectrum of open strings living on a Dp-brane is that of a maximally supersymmetric $U(1)$ gauge theory in $p+1$ dimensions. The $9-p$ massless scalar fields present in this supermultiplet are the expected Goldstone modes associated with the transverse oscillations of the Dp-brane, while the photons and fermions may be thought of as providing unique supersymmetric completion. If we consider N parallel D-branes, then there are N^2 different species of open strings because they can begin and end on any D-branes. Open strings with end points on same brane can become massless while open strings stretched between different branes will have masses determined by the separation between two branes. Thus we have $U(1)^N$ gauge symmetry. In the limit of coincident branes, all strings can become massless and $U(1)^N$ symmetry gets promoted to $U(N)$. Separating the branes is then equivalent to Higgsing the gauge theory and giving vacuum expectation values to scalar fields. The overall $U(1) \sim U(N)/SU(N)$ (upto some Z_N identifications which are not important here) corresponds to overall position of branes and may be ignored when considering dynamics on branes. Thus we have $SU(N)$ gauge symmetry. Open string perturbation theory for N Dp-branes is governed by $g_s N$ and is valid if $g_s N < 1$. As we mentioned earlier, there is another description of D branes as solitons of low energy string effective action[52]. For large N, the stack of D branes behaves like a massive object which acts as a source for gravity. These brane solutions occur as solutions to supergravity field equations (which as we mentioned earlier, occur as low energy limit of string equations). These are solitonic, in the sense that their tensions are inversely proportional to string coupling $T_p \sim \frac{1}{g_s l_s^{p+1}}$.

D-p brane metric is [52]

$$ds^2 = H^{-1/2}(r) \left(-dt^2 + \sum_{i=1}^{p} (dx^i)^2 \right) + H^{1/2}(r) \left(dr^2 + r^2 (d\Omega_{8-p})^2 \right) \tag{4.25}$$

Dilaton is given by $e^{\Phi} = H^{\frac{3-p}{4}}$. We have not shown RR fields which are not important for arguments below. Here

$$H(r) = 1 + \frac{L^{7-p}}{r^{7-p}} \tag{4.26}$$

L is related to radius of curvature of the spacetime. The gravity description of D branes is valid when $g_sN > 1$. Thus we have a sort of complementarity between gravity and gauge theory descriptions. In some particular spacetimes, it is possible to completely decouple two descriptions and then we have complete duality between gauge theory and gravity descriptions. This is the basis of AdS-CFT correspondence.

To get to such duality, consider N coincident $D3$ branes. Using the metric above for $p = 3$, supergravity description is given by[53]

$$ds^2 = (1 + \frac{L^4}{r^4})^{-1/2}(-dt^2 + \sum_{i=1}^{p}(dx^i)^2) + (1 + \frac{L^4}{r^4})^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right) \tag{4.27}$$

Near horizon limit of this geometry is $r \to 0$ which gives (putting $z = \frac{L^2}{r^2}$)

$$d\tilde{s}^2 = \frac{L^2}{z^2}\left(-dt^2 + dz^2 + \sum_{i=1}^{p}(dx^i)^2\right) + L^2 d\Omega_5^2 \tag{4.28}$$

This geometry is the poincare patch part of $AdS_5 \times S^5$. After Euclidean continuation, we obtain the entire Euclidean AdS space. So roughly, the geometry can be viewed as a semi-infinite throat of radius L which for $r >> L$ opens up into flat space. For gravity desciption to be valid, the curvature scale $L >> l_s$ where l_s is string length. To find a relationship between gravity side and gauge theory side parameters, we equate the AdM tension (mass per unit 3-volume)of the extremal classical 3 brane solution to N times the tension of a single D brane. This gives

$$L^4 \sim g_s N \alpha'^2 \tag{4.29}$$

From this we can see that the range of validity for gravity description ($L >> l_s$) becomes $g_sN >> 1$ as previously mentioned. From the low energy approximation of open string theory on brane, we get $g_{YM}^2 \sim g_s$. Complete action for massless modes for $D3$ branes is

$$S = S_{bulk} + S_{int} + S_{brane} \tag{4.30}$$

Here S_{bulk} is ten dimensional supergravity lagrangian plus higher derivative stringy corrections. S_{brane} is $\mathcal{N} = 4$ super Yang Mills plus higher derivative corrections like $(\alpha'^2 Tr(F^4))$. The last term S_{int} contains terms which come from interaction between brane and bulk modes. For example, if in brane action we put background metric then expanding around some fixed value (like flat space) we get interaction terms. Maldacena[40] considered limit of $\alpha' \to 0$ while keeping all the dimensionless parameters like g_s and N fixed. Gravitational constant $\kappa \sim g_s \alpha'^2 \to 0$. So $S_{int} \to 0$. Also all the higher derivative terms in brane action vanish, leaving $\mathcal{N} = 4$ super Yang Mills. Also supergravity in bulk becomes free. So in this limit we have free gravity in bulk plus 4 dimensional gauge theory.

Now consider the complementary description of branes as solitons in supergravity. Low energy limit $\alpha' \to 0$ here corresponds to $r \to 0$ as the ratio $\frac{r}{\alpha'}$ is kept fixed[40]. The fact that $r \to 0$ corresponds to low energy limit can also be seen from the metric

$$ds^2 = (1 + \frac{L^4}{r^4})^{-1/2}(-dt^2 + \sum_{i=1}^{p}(dx^i)^2) + (1 + \frac{L^4}{r^4})^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right) \qquad (4.31)$$

Energy of an object at a position r, E_r is related to energy relative to infinity by the redshift factor as

$$E_\infty = (1 + \frac{L^4}{r^4})^{-1/4} E_r \qquad (4.32)$$

As $r \to 0$, $E \to 0$. As we saw earlier, geometry becomes $AdS_5 \times S^5$ while asymptotically flat space decouples (free gravity there). Now taking both the complementary descriptions and neglecting free supergravity from both sides, we have a duality between type IIB string theory on $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ super Yang Mills.

In the $AdS_5 \times S^5$ background, worldsheet action becomes

$$S_G = \frac{L^2}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{g} g^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + others\right) \tag{4.33}$$

where $G_{\mu\nu}$ is $AdS_5 \times S^5$ metric with L^2 set to 1. So the effective string coupling is $\frac{L^2}{4\pi\alpha'} = \sqrt{\frac{\lambda}{4\pi}}$. Here $\lambda = g_{YM}^2 N \sim g_s N$. One can consider Maldacena conjecture with varying degrees of confidence. The strong form of conjecture says that for all values of $g_s \sim g_{YM}^2$ and N, full quantum type IIB string theory on $AdS_5 \times S^5$ is fully equivalent to $\mathcal{N} = 4$ super Yang Mills theory. One can perform expansion in $\frac{1}{N}$ on SYM side and g_s expansion on string theory side. In a weaker form of conjecture, one considers 't Hooft limit (fixed λ, $N \to \infty$) on SYM side and takes the leading order contribution which is equivalent to classical type IIB string theory on $AdS_5 \times S^5$. One can also consider large λ limit also, in addition to $N \to \infty$. In this limit, one gets a still weaker form of conjecture. In this limit string theory reduces to classical type IIB supergravity. Let us examine the validity of various approximations in detail. As we saw above, the string expansion parameter(α') becomes ($\sim \lambda^{-1/2}$. We can choose units ($L = 1$ in units of string length) so that $\alpha' = (g_s N)^{-1/2}$. Then gravitational coupling $\sqrt{G_{10}} \sim g_s \alpha'^2 = \frac{1}{N}$. Corrections due to massive string excitations, i. e α' corrections will be of order $O(\frac{1}{\sqrt{g_s N}}$. This is also seen from the fact that the masses of the string states are $M_{string} \sim \sqrt{g_s N}$ and go to infinity as 't Hooft coupling goes to infinity. The masses of Kaluza Klein states is of the order $O(\frac{1}{\sqrt{\alpha'}L}) \sim 1$. String loop effects will be of order $O(g_s^2) \sim \frac{1}{N^2}$ for closed string theory.

2.4.1 The correspondence

The AdS/CFT correspondence states that there is an exact equivalence or duality between between string theory on asymptotically AdS spacetimes (times a compact space) and a quantum field theory that "resides" on the conformal boundary of AdS space. As we saw earlier, Maldacena originally formulated this as duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-yang mills (SYM) in four dimensions. On the string theory side,5 form flux through S^5 is integer N and equal radii L of AdS_5 and S^5 and given by $L^4 \sim N g_s \alpha'^2$. On SYM side, we have gauge coupling $g_{YM}^2 \sim g_s$ and gauge group $SU(N)$. This conjecture is remarkable because its correspondence is between a 10 dimensional theory of gravity and a 4-dimensional theory without gravity at all. The correspondence is called holographic because all the 10 dimensional degrees of freedom can somehow be encoded in a four dimensional theory living at boundary of AdS_5. It is also a weak-strong duality because when string theory is weakly coupled we have a strongly coupled gauge theory and vice-versa.

First thing we need to check is that the global symmetries on both sides match. We first compare bosonic symmetries. On string theory side, we have $SO(2,4)$ isometry group of AdS_5 and $SO(6)$ isometry of S^5. On SYM side we have $SO(2,4)$ conformal symmetry and $SO(6)$ R symmetry. On both sides we have 32 supersymmetries. On SYM side, we have 32 supersymmetries of superconformal algebra while string theory on maximally symmetric background $AdS_5 \times S^5$ has 332 supersymmetries as in flat space. Original $D3$ brane background had only 16 supersymmetries but decoupling of flat space part leaves only near horizon limit of $AdS_5 \times S^5$.

The basic AdS/CFT dictionary is as follows [42] :

1. Gauge invariant operators of the boundary theory are in one-to-one correspondence with bulk fields. For example, the bulk metric corresponds to the stress energy tensor of the boundary theory.

2. The leading boundary behavior of the bulk field is identified with the source of the dual operator.

3. The string partition function (a functional of fields parametrizing the boundary behavior of the bulk fields) is identified with the generating functional of the CFT correlation functions.

On the AdS side, we will decompose all ten dimensional fields onto Kaluza Klein towers on S^5, so effectively all fields are on AdS_5, labelled by their dimension Δ. Away from interaction region we assume that fields are free (except for gravitational interaction). The free field (for simplicity, we consider scalar field)then satisfies

$$(\Box + m_\Delta^2)\phi_\Delta = 0 \tag{4.34}$$

with $m_\Delta^2 = \Delta(\Delta - 4)$(This equation will have two solutions Δ_+ and Δ_-. We will need only Δ_+). Here $\Box$ is laplacian in AdS metric and we will work with poincare patch metric. The two independent solutions have following asymptotic behavior as $x_0 \to 0$:

$$\phi_\Delta(x_0, \vec{x}) \to x_0^\Delta A^\Delta(\vec{x}) + \phi_0^\Delta(\vec{x}) x_0^{4-\Delta} \tag{4.35}$$

The normalizable function $A^\Delta(\vec{x})$determines the vacuum expectation values of operators of associated dimensions and quantum numbers. The non-normalizable solution $\phi_0^\Delta(\vec{x})$ represent the coupling of external sources to string theory and is called associated boundary field. In the example we are considering (compact manifold is S^5),only the bigger root Δ_+ occurs. In some other cases, other root Δ_- is also important. The correspondence between correlators in SYM and dynamics in string theory is given by

$$< exp \left(\int_{boundary} \phi_0^{\Delta} O_{\Delta} \right) >= Z_{string}(\phi_0^{\Delta}) \tag{4.36}$$

Here <> denotes correlation function. Z_{string} is the partition function for full string theory which can be approximated by $exp\left(-I_s(\phi_0^{\Delta})\right)$ where I_s is supergravity action. Beyond supergravity, one would also need to include α' corrections and loop corrections. We do not know full spectrum of type IIB string theory on $AdS_5 \times S^5$ except in supergravity limit. To find the spectrum of type IIB supergravity compactified on $AdS_5 \times S^5$[62], we expand fields in spherical harmonics on S^5, plug them in linearized supergravity equations of motion and then diagonalize them. These fields have masses $m_k^2 L^2 = k(k+4)$. All these fields correspond to BPS operators on field theory side (short representation of superconformal algebra). Thus these states have their masses protected from quantum corrections. These Kaluza Klein modes have masses of order one, as we mentioned previously. String theory also has additional string states with masses of order $\frac{1}{l_s}$. Such states correspond to operators in field theory with dimensions $\Delta \sim (g_s N)^{1/4}$ for large N.

2.4.2 $AdS_3 \times S^3$ case

Uptill now, we have studied AdS/CFT correspondence for the case of duality between string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super yang-mills since it is easier and better understood. But for black holes in 5 or 6 dimensions, we need to consider string theory on $AdS_3 \times S^3 \times M^4$ where for us, $M^4 = T^4$. To derive [83] this, we start with a set of N_1 $D1$ branes along a non-compact direction, and N_5 $D5$ branes wrapping M^4 and sharing the non-compact direction with the $D1$ branes. All the branes are coincident in the transverse non-compact directions. The unbroken Lorentz symmetry of this configuration is $SO(1,1) \times SO(4)$ where $SO(1,1)$ corresponds to boosts along the string and $SO(4)$ is the group of rotations in the four noncompact directions transverse to both the branes. This configuration also preserves $\mathcal{N} = (4,4)$ supersymmetries corresponding to left and right moving spinors. The corresponding supergravity solution for this $D1$-$D5$ system is

$$ds^2_{naive} = \frac{1}{\sqrt{(1+\frac{Q_1}{r^2})(1+\frac{Q_5}{r^2})}}[-dt^2+dy^2]+\sqrt{(1+\frac{Q_1}{r^2})(1+\frac{Q_5}{r^2})}dx_i dx_i+\sqrt{\frac{1+\frac{Q_1}{r^2}}{1+\frac{Q_5}{r^2}}}dz_a dz_a \tag{4.37}$$

where the meaning of designation 'naive' would be clear in the next section. Here Q_1, Q_5 are charges corresponding to N_1, N_5 numbers of $D1$,$D5$ branes respectively. Index i is over noncompact coordinates while $a = 6, 7, 8, 9$ refers to torus coordiantes. Taking the near horizon limit of this, we get

$$ds^2 = \frac{r^2}{Q_1 Q_5}(-dt^2 + dy^2) + \sqrt{Q_1 Q_5}\frac{dr^2}{r^2} + \sqrt{Q_1 Q_5}d\Omega_3^2 + \sqrt{\frac{Q_1}{Q_5}}dz_a dz_a \tag{4.38}$$

This is $AdS_3 \times S^3 \times T^4$ with radius

$$R^2 = R^2_{ads} = R^2_{S^3} = \sqrt{Q_1 Q_5} \tag{4.39}$$

and constant volume for T^4. The dual conformal field theory is the low energy field theory living on the $D1$-$D5$ system which is some $1+1$ dimensional theroy with $\mathcal{N} = (4,4)$ supersymmetry. This CFT has a central charge

$$c = \frac{3R}{2G_N^{(3)}} \tag{4.40}$$

where $G_N^{(3)}$ is three dimensional Newton's constant. We view $D1$ branes as instantons of the low energy SYM theory on $D5$ branes. These instantons live on T^4 and have $SO(1,1)$ isometry along t, y directions. This instanton configuration has moduli (parameters specifying a continuous family of classical instanton configurations, having same energy). Small fluctuations of this configuration are described by fluctuations of instanton moduli. So the low energy dynamics is given by a $1+1$ dimensional sigma model whose target space is the instanton moduli space. Instantons are described in the UV SYM theory as $SU(N_5)$ gauge fields $A_{6789}(\xi^a, z_a)$ with self-dual field strengths satisfying $F = *_4 F$. Here $*_4$ is hodge star on T^4 and ξ^a are moduli parametrizing the family of instantons. The dimension of moduli space for N_1 instantons in $SU(N_5)$ is $4N_1N_5$.

2.5 Supertubes

Supertubes are $1/4$ supersymmetric bound states with $D0$ and $NS1$ brane charges as true charges, along with $D2$-brane as dipole charge. In flat space, Mateos and Townsend [169] first constructed supertubes by using Dirac-Born-Infeld (DBI)[1]effective action for $D2$ brane and turning on worldvolume electric and magnetic fields. Branes corresponding to net charges, $NS1$ and $D0$, are represented as electric and magnetic fluxes on $D2$ brane worldvolume or equivalently, these fields are due to 'dissolved' $NS1$ and $D0$ branes in $D2$ brane worldvolume. $D2$-brane itself carries no net charge but only a dipole charge. Crossed electric and magnetic fields generate Poynting angular momentum which prevents the $D2$ brane from collapsing due to its tension. Another

[1]We will refer various worldvolume actions as DBI even though for strings it would be Polyakov or Nambu-Goto action.

way to describe this is to say that $NS1$ and $D0$ branes expand to $1/4$ supersymmetric $D2$ branes by the addition of angular momentum.

$D2$ supertube has world-volume coordinates $\sigma^0, \sigma^1, \sigma^2$. We choose gauge such that

$$\sigma^0 = t \quad , \quad \sigma^1 = y \quad , \quad X^\mu = X^\mu(\sigma^2) \tag{5.41}$$

Here X^μ are arbitrary functions of σ^2. To stabilize the brane, we introduce gauge field

$$F = E d\sigma^0 \wedge d\sigma^1 + B d\sigma^1 \wedge d\sigma^2 \tag{5.42}$$

Lagrangian is given by

$$\mathcal{L} = -T_2 \sqrt{-det[g+F]} = -T_2 \sqrt{B^2 + R^2(1-E^2)} \tag{5.43}$$

Here g is induced metric and $R^2 = X'^\mu X'_\mu$ and prime denotes differentiation wrt σ^2. We define electric displacement

$$\Pi = \frac{\partial \mathcal{L}}{\partial E} = \frac{T_2 E R^2}{\sqrt{B^2 + R^2(1-E^2)}} \tag{5.44}$$

In terms of this, we write hamiltonian density as

$$\mathcal{H} = E\Pi - \mathcal{L} = \frac{1}{R}\sqrt{(R^2 + \Pi^2)(B^2 + R^2)} \tag{5.45}$$

It can be shown that minimum value for $\mathcal{H}$ is obtained if $T_2 R^2 = \Pi B$ or $E^2 = 1$. These conditions can also be established by supersymmetry analysis. There is no condition on B. By the usual interpretation, fluxes above correspond to $D2$ brane carrying both $D0$ and $F1$(along y direction) charges. We are assuming isometry along y-direction. Charges are given by

$$Q_0 = \frac{T_2}{T_0} \int d\sigma^1 d\sigma^2 B(\sigma^2) \tag{5.46}$$

$$Q_1 = \frac{1}{T_1} \int d\sigma^2 \Pi = \frac{T_2}{T_1} \int d\sigma^2 \frac{E R^2}{\sqrt{B^2 + R^2(1-E^2)}} \tag{5.47}$$

By dualities[2], this $D0$-$NS1$ system can be related to $NS1$-P system, which is given by a $NS1$ string wrapped n_1 times around a circle S^1 and carrying n_p units of momentum along S^1 direction. Initial supertubes were constructed with circular cross-section but it was soon realized [170, 171] that supertubes exist for any arbitrary profile. This fact is a bit obscure in $D0$-$NS1$ language but in the $NS1$-P duality frame, it is just a string carrying a right moving wave with an arbitrary profile. Supertubes can be dualized to $D1$-$D5$ system which corresponds to two charge black holes when corresponding supergravity solution is constructed.

2.5.1 Mathur conjecture

In the DBI description of supertubes, backreaction of branes on spacetime is neglected. Supergravity solution for two-charge supertube was constructed in [185] and in different duality frames in [186, 188, 189]. It turns out that these correspond to simplest of microstate solutions for 2-charge systems in 5-dimensions and is completely smooth and horizon-free in $D1$-$D5$ duality frame. Based on AdS/CFT correspondence for $D1$-$D5$ system and several other evidences, Mathur conjecture [150] is a proposal to associate *bound* states in CFT to smooth,horizon-free geometries (whenever supergravity description is possible). It must be emphasised that generic states are expected to be quantum and probably no geometric description would be possible. The idea is to associate coherent state in the CFT with an asymptotically flat geometry which is smooth, free of horizons, carries the same conserved charges as the black hole, and hence constitutes a microstate of the black hole. For the case of 2-charge systems, all bosonic solutions were constructed by Mathur and Lunin [150, 178]. Geometries with both bosonic and fermionic condensates were considered

[2]We refer to various systems by their true charges without refering to dipole charges explicitly.

in [180] and relationship between gravity and CFT sides has been further explored in [181, 182] recently. Our understanding of the 3-charge systems is less complete but a few examples are known [187, 176, 177, 190, 191]. In four dimensions, smooth solutions for 3 and 4-charges have appeared in the literature [173, 195]. Mathur conjecture emphasizes that microstates of black holes (which are described by smooth, horizon-free geometries when classical supergravity description is possible) correspond to *bound* states only.

Mathur conjecture can be motivated from AdS/CFT considerations. According to AdS/CFT dictionary, for every CFT state there is a corresponding state in string theory which is asymptotically AdS and which encodes the vev of gauge invariant operator in that state. Since pure states in CFT have no entropy, one does not expect corresponding geometry to have horizon. But one should keep in mind the fact, that CFT states are dual to states in full string theory and not in supergravity alone. Hence not all CFT states may have supergravity (geometrical) interpretation. One can distinguish between weak and strong forms of mathur conjecture. According to weak form, black hole microstates are horizon-sized stringy configurations (corresponding to states in CFT and hence with unitary scattering) but they can not be adequately described by supergravity. A stronger version of the conjecture is that among the typical microstates, we have some which are described by supergravity and these are horizonless, smooth geometries. All these configurations (whether or not described by supergravity) look like black holes (naive geometry) from far away and start to differ from each other inside the would-be horizon. In the next section, we study Mathur conjecture concretely for the case of two charge systems.

2.5.2 Generating the 'correct' D1-D5 geometries

Consider IIB string theory compactified on $T^4 \times S^1$. The D1-D5 system can be mapped by a set of S, T dualities to the FP system

$$n_5 \; D5 \text{ branes along } T^4 \times S^1 \;\; \rightarrow \;\; n_5 \text{ units of fundamental string winding along } S^1 \; (F)$$

$$n_1 \; D1 \text{ branes along } S^1 \;\; \rightarrow \;\; n_1 \text{ units of momentum along } S^1 \; (P)$$

The *naive* metric of the FP bound state in string frame is

$$ds^2 = -(1+\frac{Q}{r^2})^{-1}(dudv + \frac{Q'}{r^2}dv^2) + dx_i dx_i + dz_a dz_a \tag{5.48}$$

where $x_i, i = 1 \ldots 4$ are the noncompact directions, $z_a, a = 1 \ldots 4$ are the T^4 coordinates, and we have smeared all functions on T^4. We will also use the definitions

$$u = t + y, \quad v = t - y \tag{5.49}$$

But in fact the bound state of the F and P charges corresponds to a fundamental string 'multiwound' n_5 times around S^1, with all the momentum P being carried on this string as traveling waves. Since the F string has no longitudinal vibrations, these waves necessarily cause the strands of the multiwound string to bend away and separate from each other in the transverse directions. The possible configurations are parametrized by the transverse displacement $\vec{F}(v)$; we let this vibration be only in the noncompact directions x_1, x_2, x_3, x_4. The resulting solution can be constructed using the techniques of [22, 23, 25], and we find for the metric in string frame [189][3]

$$\begin{aligned} ds^2 &= H(-dudv + Kdv^2 + 2A_i dx_i dv) + dx_i dx_i + dz_a dz_a \\ B_{vu} &= -G_{vu} = \frac{1}{2}H, \quad B_{vi} = -G_{vi} = -HA_i, \quad e^{-2\Phi} = H^{-1} \end{aligned} \tag{5.50}$$

[3]We can extend the construction to get additional states by letting the string vibrate along the T^4 directions; these states were constructed in [86].

where

$$H^{-1} = 1 + \frac{Q}{L}\int_0^L \frac{dv}{|\vec{x}-\vec{F}(v)|^2}, \quad K = \frac{Q}{L}\int_0^L \frac{dv(\dot{F}(v))^2}{|\vec{x}-\vec{F}(v)|^2}, \quad A_i = -\frac{Q}{L}\int_0^L \frac{dv\dot{F}_i(v)}{|\vec{x}-\vec{F}(v)|^2} \tag{5.51}$$

($L = 2\pi n_1 R$, the total length of the F string. [4])

Undoing the S,T dualities we find the solutions describing the family of Ramond ground states of the D1-D5 system [150]

$$ds^2 = \sqrt{\frac{H}{1+K}}[-(dt-A_i dx^i)^2 + (dy+B_i dx^i)^2] + \sqrt{\frac{1+K}{H}} dx_i dx_i + \sqrt{H(1+K)} dz_a dz_a \tag{5.52}$$

$$\begin{aligned} e^{2\Phi} &= H(1+K), \quad C^{(2)}_{ti} = \frac{B_i}{1+K}, \quad C^{(2)}_{ty} = -\frac{K}{1+K} \\ C^{(2)}_{iy} &= -\frac{A_i}{1+K}, \quad C^{(2)}_{ij} = C_{ij} + \frac{A_i B_j - A_j B_i}{1+K} \end{aligned} \tag{5.53}$$

where B_i, C_{ij} are given by

$$dB = - *_4 dA, \quad dC = - *_4 dH^{-1} \tag{5.54}$$

and $*_4$ is the duality operation in the 4-d transverse space $x_1 \ldots x_4$ using the flat metric $dx_i dx_i$. The functions H^{-1}, K, A_i are the same as the functions in (2.4)

It may appear the the solution (2.5) will be singular at the points $\vec{x} = \vec{F}(v)$, but it was found in [150] that this singularity reflects all incoming waves in a simple way. The explanation for this fact was pointed out in a nice calculation in [86] where it was shown that the singularity (for generic $\vec{F}(v)$) is a *coordinate* singularity; it is the

[4]Parameters like Q, R are not the same for the FP and D1-D5 systems – they are related by duality transforms. Here we have not used different symbols for the two systems to avoid cumbersome notation and the context should clarify what the parameters mean. For full details on the relations between parameters see [189, 150]).

same coordinate singularity as the one encountered at the origin of a Kaluza-Klein monopole [138].

The family of geometries (2.5) thus have the form pictured in Fig. 3.4. These geometries are to be contrasted with the 'naive' geometry for the D1-D5 system

$$ds^2_{naive} = \frac{1}{\sqrt{(1+\frac{Q_1}{r^2})(1+\frac{Q_5}{r^2})}}[-dt^2+dy^2]+\sqrt{(1+\frac{Q_1}{r^2})(1+\frac{Q_5}{r^2})}dx_idx_i+\sqrt{\frac{1+\frac{Q_1}{r^2}}{1+\frac{Q_5}{r^2}}}dz_adz_a \tag{5.55}$$

The actual geometries (2.5) approximate this naive geometry everywhere except near the 'cap'.

It is important to note that we can perform dynamical experiments with these different geometries that distinguish them from each other. In [150] the travel time Δt_{sugra} was computed for a waveform to travel down and back up the 'throat' for a 1-parameter family of such geometries. Different geometries in the family had different lengths for the 'throat' and thus different Δt_{sugra}. For each geometry we found

$$\Delta t_{sugra} = \Delta t_{CFT} \tag{5.56}$$

where Δt_{CFT} is the time taken for the corresponding excitation to travel once around the 'effective string' in the CFT state dual to the given geometry. Furthermore, the backreaction of the wave on the geometry was computed and shown to be small so that the gravity computation made sense.

In [8] a 'horizon' surface was constructed to separate the region where the geometries agreed with each other from the region where they differed, and it was observed that the entropy of microstates agreed with the Bekenstein entropy that one would associate to this surface[5]

$$S_{micro} \sim \frac{A}{4G} \tag{5.57}$$

Such an agreement was also found for the 1-parameter family of 'rotating D1-D5 systems' where the states in the system were constrained to have an angular momentum J. The horizon surfaces in these cases had the shape of a 'doughnut'.

2.5.3 The geometry for $|0\rangle_R$

The geometry dual to the R sector state $|0\rangle_R$ (which results from the spectral flow of the NS vaccum $|0\rangle_{NS}$) is found by starting with the FP profile

$$f_1(v) = a\cos(\frac{v}{n_5 R}), \quad f_2(v) = a\sin(\frac{v}{n_5 R}), \quad f_3(v) = 0, \quad f_4(v) = 0 \tag{5.58}$$

and constructing the corresponding D1-D5 solution. The geometry for this case had arisen earlier in different studies in [19, 186, 188]. For simplicity we set

$$Q_1 = Q_5 \equiv Q \tag{5.59}$$

which gives the D1-D5 solution

$$\begin{aligned} ds^2 &= -\frac{1}{h}(dt^2 - dy^2) + hf(d\theta^2 + \frac{dr^2}{r^2+a^2}) - \frac{2aQ}{hf}(\cos^2\theta dy d\psi + \sin^2\theta dt d\phi) \\ &+ h[(r^2 + \frac{a^2Q^2\cos^2\theta}{h^2f^2})\cos^2\theta d\psi^2 + (r^2 + a^2 - \frac{a^2Q^2\sin^2\theta}{h^2f^2})\sin^2\theta d\phi^2] + dz_a dz_a \end{aligned} \tag{5.60}$$

where

$$a = \frac{Q}{R}, \quad f = r^2 + a^2\cos^2\theta, \quad h = 1 + \frac{Q}{f} \tag{5.61}$$

[5]In [199] the naive geometry for FP was considered, and it was argued that since the curvature became order string scale below some $r = r_0$, a 'stretched horizon' should be placed at r_0. The area A of this stretched horizon also satisfied $\frac{A}{4G} \sim S_{micro}$. It is unclear, however, how this criterion for a 'horizon' can be used for the duality related D1-D5 system, where the geometry for small r is locally $AdS_3 \times S^3$ and the curvature is *constant* (and small). We, on the other hand have observed that geometries for different microstates *depart* from each other for $r \leq r_0$ and placed the horizon at this location; this gives the same horizon location for all systems related by duality.

The dilaton and RR field are

$$
\begin{aligned}
e^{2\Phi} &= 1, \quad C^{(2)}_{ty} = -\frac{Q}{Q+f}, \quad C^{(2)}_{t\psi} = -\frac{Qa\cos^2\theta}{Q+f} \\
C^{(2)}_{y\phi} &= -\frac{Qa\sin^2\theta}{Q+f}, \quad C^{(2)}_{\phi\psi} = Q\cos^2\theta + \frac{Qa^2\sin^2\theta\cos^2\theta}{Q+f}
\end{aligned}
\tag{5.62}
$$

2.6 Supersymmetric rings and Large radius limit

In this section, we analyze and study supersymmetric black rings. We derive straight ring limit which will be useful in chapter 4. We also study some properties like near horizon limit and relation to two charge Maldacena-Maoz geometry.

2.6.1 Supersymmetric Black ring metric

The five dimensional black ring metric, for 3 charges, Q_1,Q_2,Q_3 and 3 dipole charges, q_1,q_2,q_3 is given by

$$
ds^2 = -f^2(dt+\omega)^2 + f^{-1}\frac{R^2}{(x-y)^2}\left[\frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right]
\tag{6.63}
$$

Here $f^{-1} = (H_1H_2H_3)^{1/3}$ and $\omega = \omega_\phi d\phi + \omega_\psi d\psi$. Functions H_i for $i = 1,2,3$ are not harmonic and are given by

$$
H_1 = 1 + \frac{Q_1 - q_2q_3}{2R^2}(x-y) - \frac{q_2q_3}{4R^2}(x^2-y^2) \tag{6.64}
$$

$$
H_2 = 1 + \frac{Q_2 - q_3q_1}{2R^2}(x-y) - \frac{q_3q_1}{4R^2}(x^2-y^2) \tag{6.65}
$$

$$
H_3 = 1 + \frac{Q_3 - q_1q_2}{2R^2}(x-y) - \frac{q_1q_2}{4R^2}(x^2-y^2) \tag{6.66}
$$

Angular momentum one form components are given by, with $q_iQ_i = q_1Q_1 + q_2Q_2 + q_3Q_3$ and $q = q_1q_2q_3$,

$$
\omega_\phi = -\frac{1}{8R^2}(1-x^2)[q_iQ_i - q(3+x+y)] \tag{6.67}
$$

$$
\omega_\psi = \frac{1}{2}(q_1+q_2+q_3)(1+y) - \frac{1}{8R^2}(y^2-1)[q_iQ_i - q(3+x+y)] \tag{6.68}
$$

The coordinate x ranges in $[-1,1]$ while y has range $[-\infty,-1]$. Angles ϕ,ψ have usual range of $[0,2\pi]$. Asymptotic infinity lies at $x\to y\to -1$. The four dimensional part of the metric, called base metric,

$$ds_4^2 = \frac{R^2}{(x-y)^2}\left[\frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right] \tag{6.69}$$

is actually flat metric, written in unusual coordinates. We want to consider the large radius limit but keeping the charge densities constant. So we define, charge densities $\overline{Q}_i = \frac{Q_i}{2R}$ and coordinates $r=-\frac{R}{y}$, $x=\cos\theta$ and $\eta = R\psi$. Now we take $R\to\infty$ limit. In these coordinates, the metric the flat part of the metric becomes

$$ds_4^2 = \frac{R^2}{(\cos\theta + \frac{R}{r})^2}\left(\frac{R^2dr^2}{r^2(R^2-r^2)} + (\frac{R^2}{r^2}-1)\frac{d\eta^2}{R^2} + \frac{\sin^2\theta d\theta^2}{\sin^2\theta} + \sin^2\theta d\phi^2\right) \tag{6.70}$$

Taking the limit $R\to\infty$ we get

$$ds_4^2 = dr^2 + d\eta^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.71}$$

This is flat metric in cylindrical coordinates. Notice that η has range $-\infty,\infty$ while r ranges from 0 to ∞. In this limit, the functions H_i and ω become

$$H_1 \to (1 + \frac{\overline{Q}_1}{r} + \frac{q_2q_3}{4r^2})\ ,\ \omega_\phi \to 0 \tag{6.72}$$

$$\omega_\psi d\psi \to -\left(\frac{q_1+q_2+q_3}{2r} + \frac{Q_iq_i}{4r^2} + \frac{q}{8r^3}\right)d\eta \tag{6.73}$$

H_2 and H_3 are given by permutations of charges in H_1. We see that this metric has only one angular momentum as compared to full black string metric which has two. Also, since only non-trivial dependence in metric is on r, we can have wave equation factorising in this limit.

Now we have full five dimensional metric, with $\omega_\psi = \omega$ as

$$ds_5^2 = -f^2(r)(dt + \omega_\eta(r)d\eta)^2 + f^{-1}(r)\left(dr^2 + d\eta^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{6.74}$$

2.6.2 Near Horizon Limit

In this section, we try to take the near horizon limit. In the limit of small r, we keep only the leading terms in $1/r$. In this limit, metric functions are

$$H_1 \to (1 + \frac{q_2 q_3}{4r^2}) \tag{6.75}$$

$$\omega \to -\left(\frac{q}{8r^3}\right) \tag{6.76}$$

In this limit, we have

$$f^{-1} = (H_1 H_2 H_3)^{1/3} = \frac{(q_1 q_2 q_3)^{2/3}}{r^2} = \frac{p^2}{4r^2} \tag{6.77}$$

where $p = (q_1 q_2 q_3)^{1/3}$. We can write the metric as

$$ds^2 = f^2 \left[(f^{-3} - \omega^2) d\eta^2 - 2\omega dt d\eta - dt^2 \right] + f^{-1}(dr^2 + r^2 d\Omega_2^2) \tag{6.78}$$

$$ds^2_{nearhorizon} = \frac{16r^4}{p^4} \left[(f^{-3} - \omega^2) d\eta^2 + 2\frac{p^3}{8r^3} dt d\eta - dt^2 \right] + \frac{p^2}{4r^2} dr^2 + \frac{p^2}{4} d\Omega_2^2 \tag{6.79}$$

We saw when writing down the radial equation that the function $f^{-3} - \omega^2$ is less singular than what one would naively expect. With $k_\eta = L = 0$, we can use the same result to write

$$f^{-3} - \omega^2 = \frac{1}{E^2}\left(a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \frac{a_4}{r^4}\right) \tag{6.80}$$

Since there is a prefactor of r^4 in the near horizon limit, we see that only surviving term is a_4 when limit of $r \to 0$ is taken. Let us denote the charge combination $\frac{16a_4}{E^2}$ as m^4 So we have

$$ds^2_{nearhorizon} = \frac{m^4}{p^4} d\eta^2 + 4\frac{r}{p} dt d\eta + \frac{p^2}{4} d\Omega_2^2 + \frac{p^2}{4r^2} dr^2 \tag{6.81}$$

This looks almost like $AdS_3 \times S^2$. To show that this is so, we take the metric

$$ds^2_{nearhorizon} = \frac{m^4}{p^4}d\eta^2 + 4\frac{r}{p}dtd\eta + \frac{p^2}{4r^2}dr^2 \tag{6.82}$$

and make following coordinate transformations

$$r = \rho^2 \ , \ \eta = \frac{x+y}{2\sqrt{p}} \ , \ t = \frac{x-y}{2\sqrt{p}} \tag{6.83}$$

From this we get

$$ds^2_{nearhorizon} = \frac{m^4}{4p^5}(dx+dy)^2 + \frac{\rho^2}{p^2}(dx^2 - dy^2) + \frac{p^2}{\rho^2}d\rho^2 \tag{6.84}$$

This is of the same form as extremal three charge metric. Let us define $T_+ = \frac{m^2}{2\pi p^3\sqrt{p}}$. Then with the coordinate transformations

$$w_+ = \frac{1}{2\pi T_+}e^{2\pi T_+(x+y)} \tag{6.85}$$

$$w_- = (x-y) - \frac{p^4 \pi T_+}{\rho^2} \tag{6.86}$$

$$z = \frac{p^2}{\rho}e^{\pi T_+(x+y)} \tag{6.87}$$

From these we get

$$dw_+ = (2\pi T_+)w_+(dx+dy) = \frac{\rho^2 z^2}{p^4}(dx+dy) \tag{6.88}$$

$$dw_- = (dx-dy) + \frac{2p^4\pi T_+}{\rho^3}d\rho \tag{6.89}$$

$$dz = z\left(\frac{dw_+}{2w_+} - \frac{d\rho}{\rho}\right) \tag{6.90}$$

From the above, we get

$$\frac{dw_+dw_-}{z^2} = \frac{\rho^2(dx+dy)}{p^4}(dx-dy) + \frac{2\pi T_+ d\rho}{\rho}(dx+dy) \tag{6.91}$$

$$\frac{dz^2}{z^2} = \frac{d\rho^2}{\rho^2} + \pi^2 T_+^2 (dx+dy)^2 - \frac{dw_+ d\rho}{\rho w_+} \tag{6.92}$$

Adding the above two, we see that (noting that $\frac{m^4}{p^7} = 4\pi^2 T_+^2$)

$$\frac{m^4}{4p^5}(dx+dy)^2 + \frac{\rho^2}{p^2}(dx^2-dy^2) + \frac{p^2}{\rho^2}d\rho^2 = \frac{p^2}{z^2}\left(dz^2 + dw_+dw_-\right) \tag{6.93}$$

2.6.3 Reduction to four dimensions

Since we have the starting five dimensional metric in Einstein frame, we use KK reduction ansatz to get to Eisntein frame in four dimensions.

$$ds_5^2 = e^{2\alpha\chi}ds_4^2 + e^{2\beta\chi}(d\eta + A_\mu dx^\mu)^2 \tag{6.94}$$

Here $\beta = (2-d)\alpha = (2-4)\alpha$. We are reducing from $d+1$ to d dimensions. The relation between α and β is obtained by requiring that we get Einstein frame metric in d dimension. Absolute value of α can be fixed by requiring the canonical normalization for scalars. Writing the five dimensional metric as

$$ds^2 = f^2\left[(f^{-3}-\omega^2)d\eta^2 - 2\omega dt d\eta - dt^2\right] + f^{-1}(dr^2 + r^2 d\Omega_2^2)$$

$$= (f^{-1} - f^2\omega^2)\left(d\eta - \frac{f^2\omega dt}{f^{-1}-f^2\omega^2}\right)^2 - \frac{f}{f^{-1}-f^2\omega^2}dt^2 + f^{-1}(dr^2 + r^2 d\Omega_2^2) \tag{6.95}$$

Comparing with the reduction ansatz, we get

$$e^{2\beta\chi} = f^{-1} - f^2\omega^2\ , \quad \Rightarrow e^{2\alpha\chi} = (f^{-1} - f^2\omega^2)^{-1/2} \tag{6.96}$$

This gives

$$ds_4^2 = e^{2\alpha\chi}\left[-\frac{f}{f^{-1}-f^2\omega^2}dt^2 + f^{-1}(dr^2+r^2d\Omega_2^2)\right]$$

$$= \; -(f^{-3}-\omega^2)^{-1/2}dt^2 + (f^{-3}-\omega^2)^{1/2}(dr^2+r^2d\Omega_2^2) = -F^{-\frac{1}{2}}dt^2 + F^{\frac{1}{2}}(dr^2+r^2d\Omega_2^2 \tag{6.97}$$

$$A_\mu = A_t = -\frac{f^2\omega}{f^{-1}-f^2\omega^2} = -\frac{\omega}{f^{-3}-\omega^2} = -\frac{\omega}{F} \tag{6.98}$$

Here $F = f^{-3} - \omega^2$.

2.6.4 Two charge Maldacena-Maoz from Black ring

We earlier presented the black ring metric in x and y coordinates. We now consider another coordinate system which would be more suitable study the limit to near region. For this we have $(x, y) \to (r, \theta)$ with

$$\rho^2 = \frac{R^2(1-x)}{x-y}\ ,\ \cos^2\theta = \frac{1+x}{x-y} \tag{6.99}$$

Now $0 \le \rho \le \infty$ and $0 \le \theta \le \frac{\pi}{2}$. In these coordinates, the base metric becomes

$$ds_4^2 = \Sigma(\frac{d\rho^2}{\rho^2+R^2} + d\theta^2) + (\rho^2+R^2)\sin^2\theta d\psi^2 + \rho^2\cos^2\theta d\phi^2 \tag{6.100}$$

where $\Sigma = \rho^2 + R^2\cos^2\theta$. Note that ϕ and ψ here are opposite to conventions in previous papers. Other functions become

$$H_1 = 1 + \frac{Q_1}{\Sigma} - \frac{q_2q_3R^2\cos^2\theta}{\Sigma^2} \tag{6.101}$$

$$\omega_\phi = -\frac{\rho^2\cos^2\theta}{2\Sigma^2}\left[q_iQ_i - q\left(1+\frac{2R^2\cos 2\theta}{\Sigma}\right)\right] \tag{6.102}$$

$$\omega_\psi = -\frac{(\rho^2+R^2)\sin^2\theta}{2\Sigma^2}\left[q_i Q_i - q\left(1+\frac{2R^2\cos 2\theta}{\Sigma}\right)\right] - (q_1+q_2+q_3)\frac{R^2\sin^2\theta}{\Sigma} \quad (6.103)$$

$$A^i = H_i^{-1}(dt+\omega) + \frac{q_i R^2}{\Sigma}(\sin^2\theta d\psi - \cos^2\theta d\phi) \quad (6.104)$$

Other functions H_2 and H_3 are given by obvious permutations of charges. In the limit of two charges and one dipole moment only, say Q_1,Q_2 and q_3 , one angular momentum ω_ϕ becomes zero while rest of the functions become

$$H_i = 1 + \frac{Q_i}{\Sigma}\ ,\ \omega_\psi = -\frac{q_3 R^2\sin^2\theta}{\Sigma} \quad (6.105)$$

$$A^3 = H_3^{-1}(dt+\omega) + \frac{q_3 R^2}{\Sigma}(\sin^2\theta d\psi - \cos^2\theta d\phi) = -\frac{q_3 R^2}{\Sigma}\cos^2\theta d\phi \quad (6.106)$$

Here we have made a gauge transformation to get rid of a constant term in A^3. Lifting this to string frame metric in ten dimensions, we get

$$ds^2 = \frac{1}{\sqrt{H_1 H_2}}\left[-(dt+\omega)^2 + (dz+A^3)^2\right] + \sqrt{H_1 H_2}dx_4^2 + \sqrt{\frac{H_2}{H_1}}dz_4^2 \quad (6.107)$$

If we take maldacena-maoz in and reduce it along y direction, we get the same metric as this after going to Einstein frame. To get the infinite ring metric for the case of two charges and one dipole moment, we go back to the ring coordinates x and y. For the case of two charges, we have following functions.

$$H_1 = 1 + \frac{Q_1}{2R^2}(x-y) \quad (6.108)$$

$$H_2 = 1 + \frac{Q_2}{2R^2}(x-y) \quad (6.109)$$

One form components (related to angular momentum) and gauge field are given by

$$\omega_\phi = 0\ ,\ \omega_\psi = \frac{1}{2}q_3(1+y)\ ,\ A^3_\phi = -\frac{q_3}{2}(1+x) \quad (6.110)$$

To take the infinite ring limit, we again put $y = -\frac{R}{r}$ and $x = \cos\theta$. Again defining the charge densities $\frac{Q_i}{2R} = \overline{Q_i}$, $\eta = R\psi$ and taking the limit $R \to \infty$. We get

$$H_i = 1 + \frac{\overline{Q_i}}{r}\ ,\ \omega = \omega_\eta = -\frac{q_3}{2r} \tag{6.111}$$

So, in the infinte ring limit for two charge (and one dipole moment) ring, we get the following five dimensional metric

$$ds_5^2 = -(H_1H_2)^{-2/3}(dt - \frac{q_3}{r}d\eta)^2 + (H_1H_2)^{1/3}(dr^2 + d\eta^2 + r^2 d\Omega^2) \tag{6.112}$$

Four dimensional Einstein metric, after reduction is

$$ds_4^2 = -F^{-1/2}dt^2 + F^{1/2}(dr^2 + r^2 d\Omega_2^2) \tag{6.113}$$

where $F = 1 + \frac{Q_1+Q_2}{r}$. The corresponding ten dimensional metric is

$$ds^2 = \frac{1}{\sqrt{H_1H_2}}\left[-(dt - \frac{q_3}{2r}d\eta)^2 + (dz - \frac{q_3(1+\cos\theta)d\phi}{2})^2\right]$$

$$+\sqrt{H_1H_2}(dr^2 + d\eta^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) + \sqrt{\frac{H_2}{H_1}}dz_4^2 \tag{6.114}$$

To get the condition for the absence of closed time like curves, we look at the coefficient of $d\eta^2$. Since this was originally a periodic coordinate, we want the coefficient of this to be positive for all values of r. So we want

$$\left(\sqrt{H_1H_2} - \frac{1}{\sqrt{H_1H_2}}\frac{q_3^2}{4r^2}\right) \geq 0 \tag{6.115}$$

This gives $q_3^2 \leq (4\overline{Q}_1\overline{Q}_2)$. For $\gamma = 1$ case in Maldacena-Maoz, we have equality and $q_3^2 = 4\overline{Q}_1\overline{Q}_2$. This case gives regular solution. We see that A_ϕ^3 is not regular at $\theta = 0$ because in a local frame

$$|A^3| = \sqrt{g^{\phi\phi}A_\phi A_\phi} = \frac{f(r)}{\sin\theta}(1+\cos\theta) \tag{6.116}$$

Here $f(r)$ is some function of r. To cure this divergence, we make a gauge transformation such that $A_\phi = \frac{q_3(1-\cos\theta)}{2}$. This gives a coordiante transformation for y such that $z' = z - q_3\phi$. We see that gauge field becomes regular after this gauge transformation. But since z is a compact circle, this implies that one ϕ circle movement must equal some numebr of z circle movements. So $2\pi q_3 = 2\pi n_{KK} R_z$ and hence $q_3 = n_{KK} R_z$. In our case $\gamma = 1$, we have $n_{KK} = 1$. Then small R_z corresponds to taking large R and that is what we have done.

CHAPTER 3

CONSTRUCTING 'HAIR' FOR BLACK HOLES

3.1 Introduction

The Bekenstein-Hawking(BH) entropy of a black hole is

$$S = \frac{A}{4G} \tag{1.1}$$

where A is the area of the horizon. If we are to interpret Black holes as normal thermodynamic systems then it must be possible to find a statistical mechanical interpretation of BH entropy. Statistical mechanics then suggests that the hole should have e^S microstates and BH entropy arises due to coarse graining over these microstates. In their seminal work, Strominger and Vafa used string theory to match the microscopic entropy of $D1-D5-P$ system with macroscopic entropy calculated in supergravity limit. But where are these states? In this chapter we suggest an answer to this question, and support our conjecture by a calculation related to the 3-charge extremal hole.

3.1.1 Black hole 'hair'

String theory computations with extremal and near extremal systems have shown that D-brane states with the same charges and mass as the hole have precisely e^S

states [3, 174]. If we increase the coupling g these states should give black holes [4]. At least for extremal holes supersymmetry tells us that we cannot gain or lose any states when we change g [198, 199]. We are thus forced to address the question: How do the e^S configurations differ from each other in the gravity description?

Early attempts to find 'hair' on black holes were based on looking for small perturbations in the metric and other fields while demanding smoothness at the horizon. One found no such perturbations – the energy in a small deformation of the black hole solution would flow off to infinity or fall into the singularity, and the hole would settle down to its unique metric again. But if we *had* found such hair at the horizon we would be faced with an even more curious difficulty. We would have a set of 'microstates' as pictured in Fig.1(b), each looking like a black hole but differing slightly from other members of the ensemble.

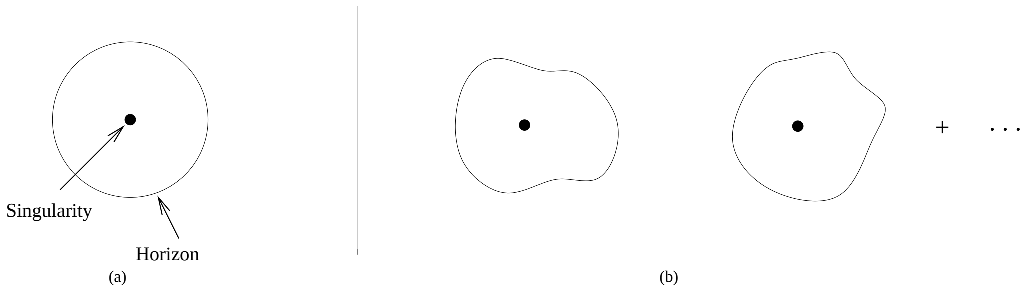

Figure 3.1: (a) The usual picture of a black hole. (b) If the microstates represented small deformations of (a) then each would itself have a horizon.

But if each microstate had a horizon as in the figure, then should'nt we assign an entropy $\approx S$ to it? If we do, then we have e^S configurations, with *each* configuration having an entropy $\approx S$. This makes no sense – we wanted the microstates to *explain* the entropy, not have further entropy themselves. This implies that if we do find the microstates in the gravity description, *then they should turn out to have no horizons themselves.*

We face exactly the same problem if we conjecture that the configurations all look like Fig.1(a) but differ from each other near the singularity; each configuration would again have a horizon, and thus an entropy e^S of its own.

The idea of AdS/CFT duality [5] adds a further twist to the problem. If string states at weak coupling become black holes at larger coupling, then one might think that the strings/branes are somehow sitting at the center $r = 0$ of the black hole. The low energy dynamics of the branes is a CFT. But the standard description of AdS/CFT duality says that the CFT is represented by a geometry that is *smooth* at $r = 0$ (Fig.3.2). In particular there are no sources or singularities near $r = 0$.

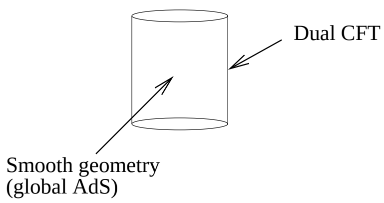

Figure 3.2: The D1-D5 CFT is represented by a *smooth* geometry in the dual representation.

Putting all this together suggests the following requirements for black hole 'hair':

(a) There must be e^S states of the hole.

(b) These individual states should have no horizon and no singularity.

(c) 'Coarse-graining' over these states should give the notion of 'entropy' for the black hole.

This appears to be rather an extreme change in our picture of the black hole, particularly since (b) requires that the geometry of individual states differ significantly from the standard black hole metric everywhere in the interior of the hole, and not just within planck distance of the singularity.

Remarkably though, just such a picture of individual states was found for the 2-charge extremal D1-D5 system in [150][8]. We take n_5 D5 branes wrapped on $T^4 \times S^1$ bound to n_1 D1 branes wrapped on the S^1. CFT considerations tell us that the entropy is $S_{micro} = 2\sqrt{2}\pi\sqrt{n_1 n_5}$, so the extremal ground state is highly degenerate. In the gravity description we should see the same number of configurations, except that in a classical computation this degeneracy would show up as a continuous family of geometries rather than discrete states. The *naive* metric that is usually written down for the D1-D5 state is pictured in Fig.3.3 – it goes to flat space at infinity, and heads to a singularity at $r = 0$. But a detailed analysis shows the following [150, 8]:

(a′) The actual classical geometry of the extremal D1-D5 system is found to be given by a family of states parametrized by a vector function $\vec{F}(v)$; upon quantization this family of geometries should yield the $e^{2\sqrt{2}\pi\sqrt{n_1 n_5}}$ states expected from the entropy.

(b′) Individual members of this family of states have no horizon and no singularity – we picture this in Fig.3.4.

(c′) Suppose we define 'coarse graining' for a family of geometries in the following way. We draw a surface to separate the region where the metrics are all essentially similar from the region where they differ significantly from each other (indicated by the dashed line in Fig.3.5). The area A of this 'horizon' surface satisfies

$$S \approx \frac{A}{4G} \tag{1.2}$$

Note that the properties a′,b′,c′ address directly the requirements a,b,c.

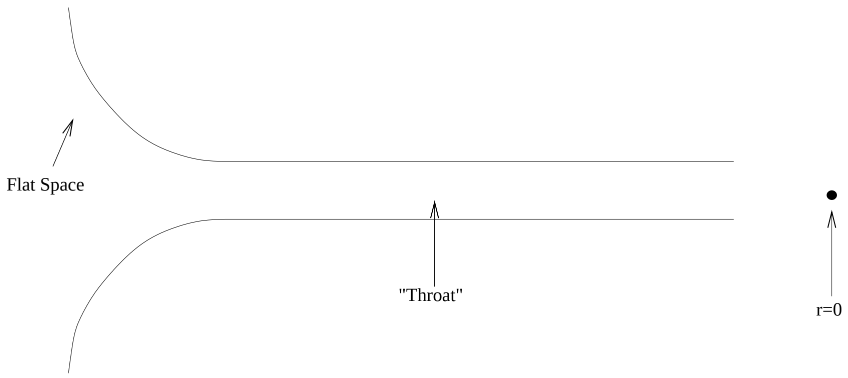

Figure 3.3: The *naive* geometry of the extremal D1-D5 system.

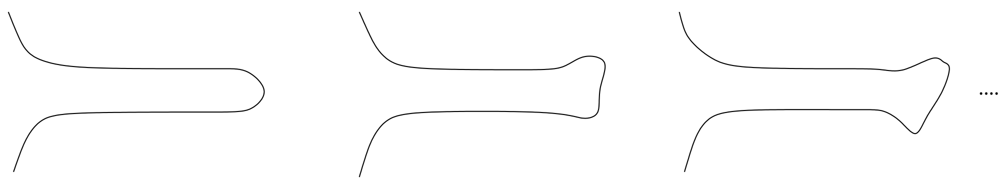

Figure 3.4: *Actual* geometries for different microstates of the extremal D1-D5 system.

3.1.2 The three charge case

The 2-charge D1-D5 extremal system has a 'horizon' whose radius is small compared to other length scales in the geometry, and the entropy of this system is determined from the geometry only upto a factor of order unity (this is the reason for the $\approx$ sign in (1.2)). The 3-charge system which has D1, D5 and P charges (P is momentum along S^1) has a horizon radius that is of the same order as other scales in the geometry, and in the classical limit we get a Reissner-Nordstrom type black hole. The D-brane state entropy S_{micro} exactly equals S_{Bek} [3]. We would therefore like to find individual geometries that describe different states of the 3-charge hole. In line with what was said above, we expect a situation similar to that in Figs.3.3,3.4 – the *naive* D1-D5-P geometry has a horizon at $r = 0$, but *actual* geometries end smoothly (without horizon or singularity) before reaching $r = 0$.

If this description of the 3-charge hole were true then it would imply a simple consequence: There should be smooth perturbations of the 2-charge (D1-D5) system which add a small amount of the third (momentum) charge. Thus we should find small perturbations Ψ around the 2-charge geometries with the following properties

(i) The perturbation has momentum p along the S^1, which implies

$$\Psi \sim e^{i\frac{p}{R}y}, \quad p \in \mathbb{Z} \tag{1.3}$$

where y is the coordinate along S^1 and R is the radius of this S^1.

(ii) The perturbation takes the extremal 2-charge system to an extremal 3-charge so the energy of the perturbation should equal the momentum charge of the perturbation. This implies a t dependence

$$\Psi \sim e^{-i\omega t}, \quad \omega = \frac{p}{R} \tag{1.4}$$

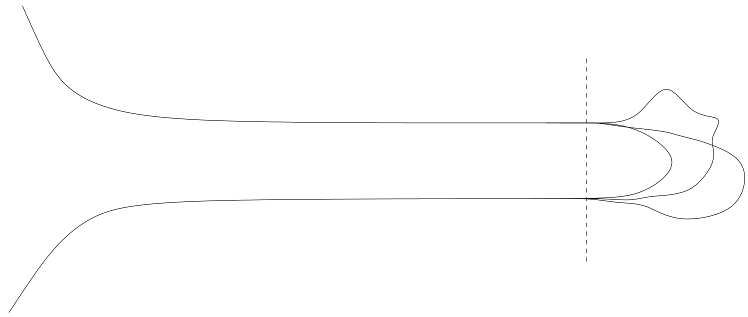

Figure 3.5: Obtaining the 'horizon' by 'coarse-graining'.

(iii) The perturbation must generate no singularity and no horizon, so it must be regular everywhere, and vanishing at $r \to \infty$ so as to be normalizable.

We start with a particular state of the 2-charge extremal system. We have a bound state of D1 and D5 branes, wrapped on a T^4 with volume $(2\pi)^4 V_4$ and an S^1 of radius R, sitting in asymptotically flat $4+1$ transverse spacetime. This system is in the Ramond (R) sector, which has many ground states. We pick the particular one (we call it $|0\rangle_R$) which if spectral flowed to the NS sector yields the NS vacuum $|0\rangle_{NS}$. The geometry for this 2-charge state is pictured in Fig.3.6. The radius of the S^3 in the region III is $(Q_1 Q_5)^{\frac{1}{4}}$. The parameter

$$\epsilon \equiv \frac{(Q_1 Q_5)^{\frac{1}{4}}}{R} \tag{1.5}$$

characterizes, roughly speaking, the ratio $\frac{\text{diameter}}{\text{length}}$ for the 'throat' region III.

In the NS sector we can act with a chiral primary operator on $|0\rangle_{NS}$. Let the resulting state be called $|\psi\rangle_{NS}$. The spectral flow of this state to the R sector gives a state $|\psi\rangle_R$; this will be an R ground state, and will have $L_0 = \bar{L}_0 = \frac{c}{24}$. We will construct the perturbation that will describe the CFT state

$$(J^-_{-1})|\psi\rangle_R \tag{1.6}$$

This state has momentum charge $L_0 - \bar{L}_0 = 1$. We proceed in the following steps:

(A) The regions III and IV are actually a part of global $AdS_3 \times S^3 \times T^4$, and a coordinate change brings the metric here to the standard form [186, 188]. The wavefunction Ψ_{inner} for the state (1.6) in this region can be obtained by rotating a chiral primary perturbation in global $AdS_3 \times S^3$.

(B) We construct the appropriate wavefunction Ψ_{outer} in the regions I, II, III by solving the supergravity equations in this part of the geometry. We choose a solution that decays at infinity.

(C) We find that at leading order ϵ^0 the solutions Ψ_{inner}, Ψ_{outer} agree in the overlap region III.

(D) We extend the computation to order $\epsilon, \epsilon^2, \epsilon^3$ and continue to find agreement in the overlap region; this agreement appears to be highly nontrivial, and we take it as evidence for the existence of the solution satisfying (i), (ii), (iii) above.

After this computation we conclude with some conjectures about the form of 'hair' for generic states of the 3-charge hole, and a discussion of the physics underlying the new picture of the black hole interior that emerges from this structure of microstates.

3.2 The 2-charge system: review

In this section we review the results obtained earlier for the 2-charge D1-D5 system and describe the particular D1-D5 background to which we will add the perturbation carrying momentum charge P.

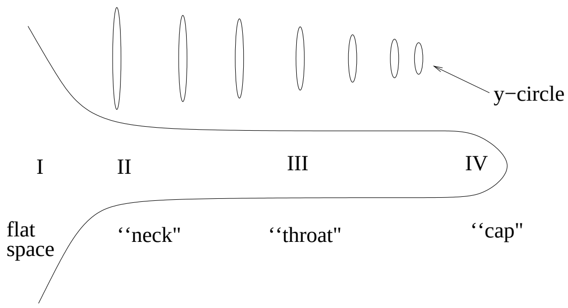

Figure 3.6: Different regimes of the starting 2-charge D1-D5 geometry.

3.2.1 Generating the 'correct' D1-D5 geometries

Consider IIB string theory compactified on $T^4 \times S^1$. The D1-D5 system can be mapped by a set of S, T dualities to the FP system

$$n_5\ D5 \text{ branes along } T^4 \times S^1 \quad \rightarrow \quad n_5 \text{ units of fundamental string winding along } S^1\ (F)$$

$$n_1\ D1 \text{ branes along } S^1 \quad \rightarrow \quad n_1 \text{ units of momentum along } S^1\ (P)$$

The *naive* metric of the FP bound state in string frame is

$$ds^2 = -(1 + \frac{Q}{r^2})^{-1}(dudv + \frac{Q'}{r^2}dv^2) + dx_i dx_i + dz_a dz_a \qquad (2.1)$$

where $x_i, i = 1 \ldots 4$ are the noncompact directions, $z_a, a = 1 \ldots 4$ are the T^4 coordinates, and we have smeared all functions on T^4. We will also use the definitions

$$u = t + y, \quad v = t - y \qquad (2.2)$$

But in fact the bound state of the F and P charges corresponds to a fundamental string 'multiwound' n_5 times around S^1, with all the momentum P being carried on this string as traveling waves. Since the F string has no longitudinal vibrations, these waves necessarily cause the strands of the multiwound string to bend away and separate from each other in the transverse directions. The possible configurations are parametrized by the transverse displacement $\vec{F}(v)$; we let this vibration be only in the noncompact directions x_1, x_2, x_3, x_4. The resulting solution can be constructed using the techniques of [22, 23, 25], and we find for the metric in string frame [189][6,7]

$$\begin{aligned} ds^2 &= H(-dudv + Kdv^2 + 2A_i dx_i dv) + dx_i dx_i + dz_a dz_a \\ B_{vu} &= -G_{vu} = \frac{1}{2}H, \quad B_{vi} = -G_{vi} = -HA_i, \quad e^{-2\Phi} = H^{-1} \end{aligned} \tag{2.3}$$

where

$$H^{-1} = 1 + \frac{Q}{L}\int_0^L \frac{dv}{|\vec{x} - \vec{F}(v)|^2}, \quad K = \frac{Q}{L}\int_0^L \frac{dv(\dot{F}(v))^2}{|\vec{x} - \vec{F}(v)|^2}, \quad A_i = -\frac{Q}{L}\int_0^L \frac{dv\dot{F}_i(v)}{|\vec{x} - \vec{F}(v)|^2} \tag{2.4}$$

($L = 2\pi n_1 R$, the total length of the F string.[8])

Undoing the S,T dualities we find the solutions describing the family of Ramond ground states of the D1-D5 system [150]

$$ds^2 = \sqrt{\frac{H}{1+K}}[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2] + \sqrt{\frac{1+K}{H}} dx_i dx_i + \sqrt{H(1+K)} dz_a dz_a \tag{2.5}$$

$$\begin{aligned} e^{2\Phi} &= H(1+K), \quad C^{(2)}_{ti} = \frac{B_i}{1+K}, \quad C^{(2)}_{ty} = -\frac{K}{1+K} \\ C^{(2)}_{iy} &= -\frac{A_i}{1+K}, \quad C^{(2)}_{ij} = C_{ij} + \frac{A_i B_j - A_j B_i}{1+K} \end{aligned} \tag{2.6}$$

[6]We can extend the construction to get additional states by letting the string vibrate along the T^4 directions; these states were constructed in [86].

[7]The angular momentum bounds of [189] and metrics found in [186, 188, 189] were reproduced in the duality related F-D0 system through 'supertubes' [10]. While supertubes help us to understand some features of the physics we still find that to construct metrics of general bound states of 2-charges and to identify the metrics with their CFT dual states the best way is to start with the FP system.

[8]Parameters like Q, R are not the same for the FP and D1-D5 systems – they are related by duality transforms. Here we have not used different symbols for the two systems to avoid cumbersome notation and the context should clarify what the parameters mean. For full details on the relations between parameters see [189, 150]).

where B_i, C_{ij} are given by

$$dB = - *_4 \, dA, \quad dC = - *_4 \, dH^{-1} \tag{2.7}$$

and $*_4$ is the duality operation in the 4-d transverse space $x_1 \ldots x_4$ using the flat metric $dx_i dx_i$. The functions H^{-1}, K, A_i are the same as the functions in (2.4)

It may appear the the solution (2.5) will be singular at the points $\vec{x} = \vec{F}(v)$, but it was found in [150] that this singularity reflects all incoming waves in a simple way. The explanation for this fact was pointed out in a nice calculation in [86] where it was shown that the singularity (for generic $\vec{F}(v)$) is a *coordinate* singularity; it is the same coordinate singularity as the one encountered at the origin of a Kaluza-Klein monopole [138].

The family of geometries (2.5) thus have the form pictured in Fig.3.4. These geometries are to be contrasted with the 'naive' geometry for the D1-D5 system

$$ds^2_{naive} = \frac{1}{\sqrt{(1+\frac{Q_1}{r^2})(1+\frac{Q_5}{r^2})}}[-dt^2+dy^2]+\sqrt{(1+\frac{Q_1}{r^2})(1+\frac{Q_5}{r^2})}dx_idx_i+\sqrt{\frac{1+\frac{Q_1}{r^2}}{1+\frac{Q_5}{r^2}}}dz_adz_a \tag{2.8}$$

The actual geometries (2.5) approximate this naive geometry everywhere except near the 'cap'.

It is important to note that we can perform dynamical experiments with these different geometries that distinguish them from each other. In [150] the travel time Δt_{sugra} was computed for a waveform to travel down and back up the 'throat' for a 1-parameter family of such geometries. Different geometries in the family had different lengths for the 'throat' and thus different Δt_{sugra}. For each geometry we found

$$\Delta t_{sugra} = \Delta t_{CFT} \tag{2.9}$$

where Δt_{CFT} is the time taken for the corresponding excitation to travel once around the 'effective string' in the CFT state dual to the given geometry. Furthermore, the backreaction of the wave on the geometry was computed and shown to be small so that the gravity computation made sense.

In [8] a 'horizon' surface was constructed to separate the region where the geometries agreed with each other from the region where they differed, and it was observed that the entropy of microstates agreed with the Bekenstein entropy that one would associate to this surface[9]

$$S_{micro} \sim \frac{A}{4G} \tag{2.10}$$

Such an agreement was also found for the 1-parameter family of 'rotating D1-D5 systems' where the states in the system were constrained to have an angular momentum J. The horizon surfaces in these cases had the shape of a 'doughnut'.

3.2.2 The geometry for $|0\rangle_R$

The geometry dual to the R sector state $|0\rangle_R$ (which results from the spectral flow of the NS vaccum $|0\rangle_{NS}$) is found by starting with the FP profile

$$f_1(v) = a\cos(\frac{v}{n_5 R}), \quad f_2(v) = a\sin(\frac{v}{n_5 R}), \quad f_3(v) = 0, \quad f_4(v) = 0 \tag{2.11}$$

and constructing the corresponding D1-D5 solution. The geometry for this case had arisen earlier in different studies in [19, 186, 188]. For simplicity we set

$$Q_1 = Q_5 \equiv Q \tag{2.12}$$

[9]In [199] the naive geometry for FP was considered, and it was argued that since the curvature became order string scale below some $r = r_0$, a 'stretched horizon' should be placed at r_0. The area A of this stretched horizon also satisfied $\frac{A}{4G} \sim S_{micro}$. It is unclear, however, how this criterion for a 'horizon' can be used for the duality related D1-D5 system, where the geometry for small r is locally $AdS_3 \times S^3$ and the curvature is *constant* (and small). We, on the other hand have observed that geometries for different microstates *depart* from each other for $r \leq r_0$ and placed the horizon at this location; this gives the same horizon location for all systems related by duality.

which gives the D1-D5 solution

$$\begin{aligned} ds^2 &= -\frac{1}{h}(dt^2-dy^2)+hf(d\theta^2+\frac{dr^2}{r^2+a^2})-\frac{2aQ}{hf}(\cos^2\theta dyd\psi+\sin^2\theta dtd\phi) \\ &+ h[(r^2+\frac{a^2Q^2\cos^2\theta}{h^2f^2})\cos^2\theta d\psi^2+(r^2+a^2-\frac{a^2Q^2\sin^2\theta}{h^2f^2})\sin^2\theta d\phi^2]+dz_adz_a \end{aligned} \tag{2.13}$$

where

$$a=\frac{Q}{R}, \quad f=r^2+a^2\cos^2\theta, \quad h=1+\frac{Q}{f} \tag{2.14}$$

The dilaton and RR field are

$$\begin{aligned} e^{2\Phi} &= 1, \quad C^{(2)}_{ty}=-\frac{Q}{Q+f}, \quad C^{(2)}_{t\psi}=-\frac{Qa\cos^2\theta}{Q+f} \\ C^{(2)}_{y\phi} &= -\frac{Qa\sin^2\theta}{Q+f}, \quad C^{(2)}_{\phi\psi}=Q\cos^2\theta+\frac{Qa^2\sin^2\theta\cos^2\theta}{Q+f} \end{aligned} \tag{2.15}$$

To construct the 3-charge solution below we will assume that

$$\epsilon\equiv\frac{a}{\sqrt{Q}}=\frac{\sqrt{Q}}{R}<<1 \tag{2.16}$$

which can be achieved by taking the compactification radius R to be large for fixed values of $\alpha', g, n_1, n_5, V_4$. In what follows we will ignore the T^4 and write 6-d metrics only. Since the dilaton Φ and T^4 volume are constant in the above solution the 6-d Einstein metric is the same as the 6-d string metric.

The 'inner' region

For

$$r<<\sqrt{Q} \tag{2.17}$$

the geometry (2.13) becomes

$$\begin{aligned} ds^2 &= -\frac{(r^2+a^2\cos^2\theta)}{Q}(dt^2-dy^2)+Q(d\theta^2+\frac{dr^2}{r^2+a^2}) \\ &\quad -2a(\cos^2\theta dyd\psi+\sin^2\theta dtd\phi)+Q(\cos^2\theta d\psi^2+\sin^2\theta d\phi^2) \end{aligned} \tag{2.18}$$

The change of coordinates

$$\psi_{NS} = \psi - \frac{a}{Q} y, \quad \phi_{NS} = \phi - \frac{a}{Q} t \tag{2.19}$$

brings (2.8) to the form $AdS_3 \times S^3$

$$ds^2 = -\frac{(r^2 + a^2)}{Q} dt^2 + \frac{r^2}{Q} dy^2 + Q \frac{dr^2}{r^2 + a^2} + Q(d\theta^2 + \cos^2 \theta d\psi_{NS}^2 + \sin^2 \theta d\phi_{NS}^2) \tag{2.20}$$

We will call the region (2.7) the *inner region* of the complete geometry (2.13).

The 'outer' region

The region

$$a << r < \infty \tag{2.21}$$

is flat space ($r \to \infty$) going over to the 'Poincare patch' (with $y \to y + 2\pi R$ identification)

$$ds^2 = -\frac{r^2}{Q + r^2}(dt^2 - dy^2) + (Q + r^2)\frac{dr^2}{r^2} + (Q + r^2)[d\theta^2 + \cos^2 \theta d\psi^2 + \sin^2 \theta d\phi^2] \tag{2.22}$$

We will call the region (2.21) the *outer region* of the geometry (2.13). The inner and outer regions have a domain of overlap

$$a << r << \sqrt{Q} \tag{2.23}$$

The spectral flow map

The coordinate transformation (2.9) taking (2.8) to (2.10) gives *spectral flow* [186, 188]. The fermions of the supergravity theory are periodic around the S^1 parametrized by the coordinate y in (2.8), but the transformation (2.9) causes the S^3 to rotate once as we go around this S^1, and the spin of the fermions under the rotation group of this S^3 makes them antiperiodic around y in the metric (2.10). Thus the metric (2.8) gives the dual field theory in the R sector while the metric (2.10) describes the CFT in the NS sector.

3.3 The perturbation carrying momentum

3.3.1 The equations

The fields of IIB supergravity in 10-d give rise to a large number of fields after reduction to 6-d. At the same time we get an enhancement of the symmetry group, as various different fields combine into larger representations of the 6-d theory.[10] In [29] general 4b supergravities in 6-d were studied around $AdS_3 \times S^3$; their perturbation equations however apply to the more general background that we will use. These supergravities have the graviton g_{MN}, self-dual 2-form fields $C^i_{MN}, i = 1 \ldots 5$, anti-self-dual 2-forms $B^r_{MN}, r = 1 \ldots n$ and scalars ϕ^{ir}.

Suppose we have a solution to the field equations with a nontrivial value for the metric and one of the self-dual fields

$$g_{MN} = \bar{g}_{MN}, \quad C^1_{MN} = \bar{C}^1_{MN} \equiv C_{MN} \tag{3.1}$$

The choice $Q_1 = Q_5 = Q$ has made the field $C^{(2)}$ in (2.15) self-dual, and gives us such a background. (This choice simplifies the computations, but we expect that the perturbation we are constructing will exist for general Q_1, Q_5 as well.)

Linear perturbations around the background (3.1) separate into different sets. The anti-self-dual field B^r_{MN} mixes only with the scalar ϕ^{1r}. We set $r = 1$ using the $SO(n)$ symmetry of the theory and write

$$B^1_{MN} \equiv B_{MN}, \quad F_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN}, \quad \phi^{11} \equiv w \tag{3.2}$$

[10]In the actual reduction of IIB from 10-d to 6-d we also get additional fields like $A_\mu \equiv h_{a\mu}$, where $a = 1 \ldots 4$ is a T^4 direction. We do not study these additional fields here.

The field equations are[11] (we write $\bar{H}_{MNP} = \partial_M \bar{C}_{NP} + \partial_N \bar{C}_{PM} + \partial_P \bar{C}_{MN}$)

$$F_{ABC} + \frac{1}{3!}\epsilon_{ABCDEF}F^{DEF} + w\bar{H}_{ABC} = 0 \quad (3.3)$$

$$w_{;A}{}^{;A} - \frac{1}{3}\bar{H}^{ABC}F_{ABC} = 0 \quad (3.4)$$

3.3.2 The (B, w) perturbation at leading order $(O(\epsilon^0))$

In this subsection we construct the desired perturbation to leading order in the inner and outer regions and observe their agreement at this order of approximation.

Inner region: The chiral primary $|\psi\rangle_{NS}$

Consider the equations (3.3),(3.4) in the inner region. In the coordinates (2.10) this region is seen to be just 'global' $AdS_3 \times S^3$. We use $a, b \ldots$ to denote indices on S^3 and $\mu, \nu \ldots$ to denote indices on AdS_3. We find the following solution for these equations in global $AdS_3 \times S^3$

$$w = \frac{e^{-2i\frac{a}{Q}lt}}{Q(r^2 + a^2)^l}\hat{Y}^{(l)}_{NS} \quad (3.5)$$

$$B_{ab} = B\epsilon_{abc}\partial^c\hat{Y}^{(l)}_{NS}, \quad B_{\mu\nu} = \frac{1}{\sqrt{Q}}\epsilon_{\mu\nu\lambda}\partial^\lambda B\ \hat{Y}^{(l)}_{NS} \quad (3.6)$$

where

$$\hat{Y}^{(l)}_{NS} = (Y^{(l,l)}_{(l,l)})_{NS} = \sqrt{\frac{2l+1}{2}\frac{e^{-2il\phi_{NS}}}{\pi}}\sin^{2l}\theta, \quad B = \frac{1}{4l}\frac{e^{-2i\frac{a}{Q}lt}}{(r^2+a^2)^l} \quad (3.7)$$

In (2.14) the tensors ϵ_{abc}, g^{ab} etc are defined using the metric on an S^3 with *unit* radius. This choice simplifies the presentation of spherical harmonics but results in

[11] Our 2-form fields are twice the 2-form fields in [29]. Our normalizations agree with those conventionally used for the 10-D supergravity fields where the action is $-\frac{1}{12}\int F^2$.

the factor (radius of $S^3)^{-1} = \frac{1}{\sqrt{Q}}$ in the definition of $B_{\mu\nu}$. The tensors $\epsilon_{\mu\nu\lambda}, g^{\mu\nu}$ etc. are defined using the t, y, r part of the metric (2.10).[12]

This solution represents a chiral primary of the dual CFT [129]. To see this note the AdS/CFT relations giving charges and dimensions of bulk excitations

$$J_z^{NS} = \frac{i}{2}[\partial_{\psi_{NS}} + \partial_{\phi_{NS}}], \quad \bar{J}_z^{NS} = \frac{i}{2}[-\partial_{\psi_{NS}} + \partial_{\phi_{NS}}] \tag{3.8}$$

$$L_0^{NS} = i\frac{Q}{a}\partial_u, \quad \bar{L}_0^{NS} = i\frac{Q}{a}\partial_v \tag{3.9}$$

The solution (2.13)-(2.15) thus has

$$j^{NS} = l, \quad h^{NS} = l, \quad \bar{j}^{NS} = l, \quad \bar{h}^{NS} = l \tag{3.10}$$

which are the conditions for a chiral primary.

The coordinate transformation (2.9) brings us to the R sector. The scalar in these coordinates is

$$w = \frac{1}{Q(r^2+a^2)^l}\hat{Y}^{(l)}, \qquad \hat{Y}^{(l)} = \sqrt{\frac{2l+1}{2}}\frac{e^{-2il\phi}}{\pi}\sin^{2l}\theta \tag{3.11}$$

so that it has no t or y dependence. The components of B_{AB} similarly do not have any t, y dependence.

The dimensions in the R sector are given by (the partial derivatives this time are with respect to the R sector variables)

$$L_0 = i\frac{Q}{a}\partial_u, \quad \bar{L}_0 = i\frac{Q}{a}\partial_v \tag{3.12}$$

[12]The spherical harmonics are representations of $so(4) \approx su(2) \times su(2)$; the upper labels in $Y^{(l,l)}_{(l,l)}$ give the j values in each $su(2)$, and the lower indices give the j_3 values. Thus $l = 0, \frac{1}{2}, 1, \ldots$. The subscript NS on Y indicates that the arguments are the sphere coordinates in the NS sector, $(\theta, \psi_{NS}, \phi_{NS})$. When we write no such subscript it is to be assumed that the arguments of the spherical harmonic are the R sector coordinates (θ, ψ, ϕ). More details about spherical harmonics are given in Appendix A.

so that we get for our perturbation

$$h = \bar{h} = 0 \tag{3.13}$$

which is expected, since a chiral primary of the NS sector maps under spectral flow to a ground state of the R sector.[13]

Let the CFT state dual to the perturbation (2.13)-(2.15) be called $|\psi\rangle_{NS}$, and let $|\psi\rangle_R$ be its image under spectral flow to the Ramond sector.

Inner region: The state $J_0^-|\psi\rangle_{NS} \;\leftrightarrow\; J_{-1}^-|\psi\rangle_R$

Consider again the inner region in the NS sector coordinates (2.10). We now wish to make the perturbation dual to the NS sector state

$$J_0^-|\psi\rangle_{NS} \tag{3.14}$$

Since the operator J_0^- in the NS sector is represented by just a simple rotation of the S^3, we can immediately write down the bulk wavefunction dual to the above CFT state

$$w = \frac{e^{-2i\frac{a}{Q}lt}}{Q(r^2+a^2)^l} Y_{NS}^{(l)} \tag{3.15}$$

$$B_{ab} = B\epsilon_{abc}\partial^c Y_{NS}^{(l)}, \quad B_{\mu\nu} = \frac{1}{\sqrt{Q}}\epsilon_{\mu\nu\lambda}\partial^\lambda B\; Y_{NS}^{(l)} \tag{3.16}$$

$$Y_{NS}^{(l)} = (Y_{(l-1,l)}^{(l,l)})_{NS} = -\frac{\sqrt{l(2l+1)}}{\pi}\sin^{2l-1}\theta\cos\theta e^{i(-2l+1)\phi_{NS}+i\psi_{NS}}, \quad B = \frac{1}{4l}\frac{e^{-2i\frac{a}{Q}lt}}{(r^2+a^2)^l} \tag{3.17}$$

[13]The full spectral flow relations are $h = h_{NS} - j_{NS} + \frac{c}{24}$, $j = j_{NS} - \frac{c}{12}$. Spectral flow of the background $|0\rangle_{NS}$ gives $h^0 = h^0_{NS} - \frac{c}{24}$, $j^0 = j^0_{NS} - \frac{c}{12}$, so for the perturbation the spectral flow relations are just $h = h_{NS} - j_{NS}, j = j_{NS}$.

This perturbation has

$$j_{NS} = l-1, \quad \bar{j}_{NS} = l, \quad h_{NS} = l, \quad \bar{h}_{NS} = l \tag{3.18}$$

The spectral flow to the R sector coordinates should give

$$h = h_{NS} - j_{NS} = 1, \quad \bar{h} = \bar{h}_{NS} - \bar{j}_{NS} = 0 \tag{3.19}$$

so that we have a state with nonzero $L_0 - \bar{L}_0$, which means that it is a state with momentum. This can be seen explicitly by writing the solution (2.19)-(2.21) in the R sector coordinates. Writing

$$Y^{(l)} = -\frac{\sqrt{l(2l+1)}}{\pi} e^{i(-2l+1)\phi + i\psi} \sin^{2l-1}\theta \cos\theta, \quad u = t + y \tag{3.20}$$

we get

$$w = \frac{1}{Q} \frac{e^{-i\frac{a}{Q}u}}{(r^2+a^2)^l} Y^{(l)} \tag{3.21}$$

$$B_{\theta\psi} = \frac{1}{4l} \frac{e^{-i\frac{a}{Q}u}}{(r^2+a^2)^l} \cot\theta \partial_\phi Y^{(l)} \tag{3.22}$$

$$B_{\theta\phi} = -\frac{1}{4l} \frac{e^{-i\frac{a}{Q}u}}{(r^2+a^2)^l} \tan\theta \partial_\psi Y^{(l)} \tag{3.23}$$

$$B_{\psi\phi} = \frac{1}{4l} \frac{e^{-i\frac{a}{Q}u}}{(r^2+a^2)^l} \sin\theta \cos\theta \partial_\theta Y^{(l)} \tag{3.24}$$

$$B_{t\theta} = -\frac{a}{4l} \frac{e^{-i\frac{a}{Q}u}}{Q(r^2+a^2)^l} \tan\theta \partial_\psi Y^{(l)} \tag{3.25}$$

$$B_{t\psi} = \frac{a}{4l} \frac{e^{-i\frac{a}{Q}u}}{Q(r^2+a^2)^l} \sin\theta \cos\theta \partial_\theta Y^{(l)} \tag{3.26}$$

$$B_{y\theta} = \frac{a}{4l} \frac{e^{-i\frac{a}{Q}u}}{Q(r^2+a^2)^l} \cot\theta \partial_\phi Y^{(l)} \tag{3.27}$$

$$B_{y\phi} = -\frac{a}{4l} \frac{e^{-i\frac{a}{Q}u}}{Q(r^2+a^2)^l} \sin\theta \cos\theta \partial_\theta Y^{(l)} \tag{3.28}$$

$$B_{ty} = -\frac{1}{2Q^2} \frac{r^2 e^{-i\frac{a}{Q}u}}{(a^2+r^2)^l} Y^{(l)} \tag{3.29}$$

$$B_{yr} = \frac{i}{2Q} \frac{r e^{-i\frac{a}{Q}u}}{(r^2+a^2)^{l+1}} Y^{(l)} \tag{3.30}$$

We see that all fields behave as $\sim e^{-i\omega t+i\lambda y}$ with $\omega=|\lambda|$, so we have a BPS perturbation adding a third charge (momentum P$=-1$) to the 2-charge D1-D5 background.

Outer region: Continuing the perturbation $J^-_{-1}|\psi\rangle_R$

We now wish to ask if this solution in the inner region continues out to asymptotic infinity, falling off in a way that makes it a normalizable perturbation. To do this we solve the perturbation equations (3.3),(3.4) in the outer region (2.22). Requiring decay at infinity, we find the solution

$$w=\frac{e^{-i\frac{a}{Q}u}}{(Q+r^2)r^{2l}}Y^{(l)} \tag{3.31}$$

$$B_{ab}=B\epsilon_{abc}\partial^c Y^{(l)},\quad B_{\mu\nu}=\frac{1}{\sqrt{Q+r^2}}\epsilon_{\mu\nu\lambda}\partial^\lambda B\;Y^{(l)},\quad B=\frac{1}{4l}\frac{e^{-i\frac{a}{Q}u}}{r^{2l}} \tag{3.32}$$

where we have chosen the same spherical harmonic $Y^{(l)}$ that appears in (3.20). Again ϵ_{abc}, g^{ab} etc. refer to the metric of a *unit* S^3 (this gives the factor (radius of $S^3)^{-1}=\frac{1}{\sqrt{Q+r^2}}$ in $B_{\mu\nu}$), while $\epsilon_{\mu\nu\lambda}, g^{\mu\nu}$ etc. refer to the t,y,r part of the metric (2.22). Writing explicit components, the above solution becomes

$$\begin{aligned}
w &= \frac{e^{-i\frac{a}{Q}u}}{(Q+r^2)r^{2l}}Y^{(l)} && (3.33)\\
B_{\theta\psi} &= \frac{1}{4l}\frac{e^{-i\frac{a}{Q}u}}{r^{2l}}\cot\theta\,\partial_\phi Y^{(l)} && (3.34)\\
B_{\theta\phi} &= -\frac{1}{4l}\frac{e^{-i\frac{a}{Q}u}}{r^{2l}}\tan\theta\,\partial_\psi Y^{(l)} && (3.35)\\
B_{\psi\phi} &= \frac{1}{4l}\frac{e^{-i\frac{a}{Q}u}}{r^{2l}}\sin\theta\cos\theta\,\partial_\theta Y^{(l)} && (3.36)
\end{aligned}$$

$$\begin{aligned}
B_{ty} &= -\frac{1}{2(Q+r^2)^2}\frac{e^{-i\frac{a}{Q}u}}{r^{2l-2}}Y^{(l)} && (3.37)\\
B_{tr} &= \frac{ia}{r^{2l+1}}\frac{1}{4lQ}e^{-i\frac{a}{Q}u}Y^{(l)} && (3.38)\\
B_{yr} &= \frac{ia}{r^{2l+1}}\frac{1}{4lQ}e^{-i\frac{a}{Q}u}Y^{(l)} && (3.39)
\end{aligned}$$

Matching at leading order

We wish to see if the solutions in the inner and outer regions agree in the domain of overlap $a << r << Q$. In this region we have

$$\frac{a}{\sqrt{Q}} \;<<\; \{\frac{a}{r}, \frac{r}{\sqrt{Q}}\} << 1 \tag{3.40}$$

We can match the solutions around any r in the range $a << r << Q$. To help us organize our perturbation expansion we choose this matching region to be around the geometric mean of a, Q, so that

$$\frac{a}{r} \;\sim\; \frac{r}{\sqrt{Q}} \;\sim\; \epsilon^{\frac{1}{2}} \tag{3.41}$$

In this region the scalar w in the inner region (given by (3.21)) and in the outer region (given by (3.33)) both reduce to the same function

$$w = \frac{e^{-i\frac{a}{Q}u}}{Qr^{2l}}Y^{(l)} + \dots \tag{3.42}$$

so that we get the desired agreement at leading order. We can similarly compare B_{MN}, but note that since B_{MN} is a tensor the components of B_{MN} depend on the coordinate frame. To see the order of a given component B_{MN} we should construct the field strength $F = dB$ from this component and then look at the values of F in an orthonormal frame. For example

$$B_{ty} \rightarrow F_{\hat{t}\hat{y}\hat{r}} \equiv F_{tyr}(g^{tt})^{\frac{1}{2}}(g^{yy})^{\frac{1}{2}})(g^{rr})^{\frac{1}{2}} \sim \frac{1}{Q^{\frac{3}{2}}r^{2l}} \tag{3.43}$$

Note that $\bar{H}_{\hat{t}\hat{y}\hat{r}} \sim \frac{1}{\sqrt{Q}}$, so that the $F_{\hat{t}\hat{y}\hat{r}}$ in the above equation is of the same order as $w\bar{H}_{\hat{t}\hat{y}\hat{r}}$, and thus B_{ty} is a term which we will match at leading order.

We then find that the components surviving at leading order reduce to the following forms for both the inner and outer solutions

$$
\begin{aligned}
B_{\theta\psi} &= \frac{1}{4l}\frac{e^{-i\frac{a}{Q}u}}{r^{2l}}\cot\theta\partial_\phi Y^{(l)} && (3.44)\\
B_{\theta\phi} &= -\frac{1}{4l}\frac{e^{-i\frac{a}{Q}u}}{r^{2l}}\tan\theta\partial_\psi Y^{(l)} && (3.45)\\
B_{\psi\phi} &= \frac{1}{4l}\frac{e^{-i\frac{a}{Q}u}}{r^{2l}}\sin\theta\cos\theta\partial_\theta Y^{(l)} && (3.46)
\end{aligned}
$$

$$
B_{ty} = -\frac{1}{2Q^2}\frac{r^2 e^{-i\frac{a}{Q}u}}{r^{2l}}Y^{(l)} \qquad (3.47)
$$

Other components like $B_{y\phi}$ which do not agree are seen to be higher order terms. We will find agreement for these after we correct the inner and outer region computations by higher order terms.

3.3.3 Nontriviality of the matching

Before proceeding to study the solutions and matching at higher orders in ϵ, we observe that the above match at leading order is itself nontrivial. The dimensional reduction from 10-d to 6-d also gives some massless scalars in 6-d

$$
\Box s = 0 \qquad (3.48)
$$

We show that for such a scalar we *cannot* get any solution that is regular everywhere and decaying at infinity. For the scalar s we can find in the inner region $AdS_3 \times S^3$ a solution analogous to (2.19) [31]

$$
s = \frac{e^{-i(2l+2)\frac{a}{Q}t}}{(r^2+a^2)^{l+1}}Y^{(l)}_{NS} \qquad (3.49)
$$

where we have chosen the same spherical harmonic as in (2.19). Since the scalar generates not a chiral primary but a supersymmetry descendent, we get instead of

(2.22)

$$j_{NS} = l-1, \quad \bar{j}_{NS} = l, \quad h_{NS} = l+1, \quad \bar{h}_{NS} = l+1 \tag{3.50}$$

The solution (3.49) falls off towards the boundary of AdS, but in the complete geometry (2.13) it will not be normalizable at infinity. Using R sector coordinates (which are natural at $r \to \infty$) we find that the t, y dependence is $e^{-i\omega t + i\lambda y} = e^{-i\frac{a}{Q}(3t+y)}$. At large r we then find from the wave equation (3.48) the behavior [32]

$$s \sim \frac{1}{r^{\frac{3}{2}}} e^{-i\frac{a}{Q}(3t+y)} \cos[2\sqrt{2}\frac{a}{Q}r + \text{const}] Y^{(l)} \tag{3.51}$$

The reason for the slowness of the falloff at large r is the following. Since $\omega > |\lambda|$, we find that at large r not all the energy in the perturbation is tied to the S^1 momentum, and the residual energy goes to radial motion; this causes the perturbation to leak away to asymptotic infinity at late times. Normalizability at infinity is thus seen to require

$$\omega = |\lambda| \tag{3.52}$$

If we impose (3.52) on the solution for s, then we see that the solution regular at $r = 0$ is

$$s \sim (r^2 + a^2)^l Y^{(l)} \tag{3.53}$$

For the choice (3.52) there are two solutions in the outer region with radial dependences

$$(i)\ s \sim r^{-(2l+2)}, \quad (ii)\ s \sim r^{2l} \tag{3.54}$$

but the inner region solution matches onto the *growing* solution (ii) of the outer region, and we again get no normalizable solution.[14]

[14]For $\omega = |\lambda|$ the scalar equation (3.48) can be exactly solved in terms of hypergeometric functions, and the non-existence of a normalizable solution can be explicitly seen.

Thus we see that it is quite nontrivial that for the (B, w) system of fields the normalizable solutions of the inner and outer regions match up at leading order. We will now proceed to check the matching to higher orders in ϵ.

3.4 Matching at the next order ($O(\epsilon)$)

We wish to develop a general perturbation scheme that will correct our solution to higher orders in ϵ. It turns out that the inner region solution does not get corrected in a nontrivial way at order ϵ. In this section we first explain the general scheme, then apply it to the outer region to get the $O(\epsilon)$ corrections, and then explain how to match these to the inner region solution so that the entire solution is established to $O(\epsilon)$.

3.4.1 The perturbation scheme

The 'outer region' of our geometry $r >> a$ is described to leading order by the metric (2.22). We must now take into account the corrections that arise because the exact geometry (2.13) departs from this leading order form. In particular we get small 'off-diagonal' components $g_{\mu a}$ in the metric and also small components like $\bar{H}_{\mu\nu a}, \bar{H}_{\mu ab}$ of $\bar{H}_{ABC}$. We develop a systematic way to handle these corrections so that we will get the full solution as a series in ϵ.

We expand the background and perturbations as

$$g_{MN} = g^0_{MN} + g^1_{MN} \tag{4.1}$$

$$H = H_0 + H_1 \tag{4.2}$$

$$F = F_0 + F_1 \tag{4.3}$$

$$* = *_0 + *_1 \tag{4.4}$$

$$w = w_0 + w_1 \tag{4.5}$$

$$\nabla^2 = \nabla^2_0 + \nabla^2_1 \tag{4.6}$$

The metric g^0_{MN} is the metric (2.22) we had written earlier for the outer region. To get g^1_{MN} we take the difference between the full metric (2.13) and the outer region metric (2.22); since we are seeking only the order ϵ corrections at this stage we keep terms of order $\frac{a}{r}$, $\frac{a}{\sqrt{Q}}$ in g^1 and discard higher order corrections. Similarly we obtain $\bar{H}^1$. The operation $*_0$ is defined using the metric g^0, and $*_1$ contains the corrections needed to give the $*$ operation in the full metric (upto the desired order of approximation). ∇^2_0 is the Laplacian on the metric g^0 and ∇^2_1 corrects this (to the desired accuracy) to the Laplacian on the full metric.

To illustrate the general approximation scheme it is convenient to write the perturbation equation (3.3) in form language

$$F + *F + w\bar{H} = 0 \tag{4.7}$$

Inserting the expansions (4.1)-(4.6) in (4.7),(3.4) we get

$$\begin{aligned} F_0 + *_0 F_0 + w_0 \bar{H}_0 &= 0 \\ \nabla_0^2 w_0 - \frac{1}{3}\bar{H}_0^{MNP} F_{0MNP} &= 0 \end{aligned} \tag{4.8}$$

$$\begin{aligned} F_1 + *_0 F_1 + w_1 \bar{H}_0 &= S \\ \nabla_0^2 w_1 - \frac{1}{3}\bar{H}_0^{MNP} F_{1MNP} &= S_w \end{aligned} \tag{4.9}$$

where S_w and S are defined by

$$\begin{aligned} S &= -w_0 \bar{H}_1 - *_1 F_0 \\ S_w &= -\nabla_1^2 w_0 + \frac{1}{3}\bar{H}_1^{MNP} F_{0MNP} \end{aligned} \tag{4.10}$$

Eqs.(4.8) are just the leading order equations that give the leading order solution found for the outer regions in the last section. Eqs.(4.9) give the first order corrections. Note that the LHS of these equations have the same form as the leading order equations, so we need to solve the same equations again but this time with source terms S, S_w. These source terms can be explicitly calculated from the background geometry and the leading order solution.

3.4.2 Expanding in spherical harmonics

Even though the problem does not have exact spherical symmetry, it is convenient to decompose fields into spherical harmonics on S^3. The breaking of spherical symmetry is then manifested by the fact that higher order corrections to the leading order solution contain spherical harmonics that differ from the harmonic chosen at leading order. We write

$$
\begin{aligned}
w &= e^{-i\frac{a}{Q}u}\tilde{w}^{I_1}Y^{I_1} && (4.11)\\
B_{\mu\nu} &= e^{-i\frac{a}{Q}u}b^{I_1}_{\mu\nu}Y^{I_1} && (4.12)\\
B_{\mu a} &= e^{-i\frac{a}{Q}u}b^{I_3}_{\mu}Y^{I_3}_a && (4.13)\\
B_{ab} &= e^{-i\frac{a}{Q}u}b^{I_1}\epsilon_{abc}\partial^c Y^{I_1} && (4.14)
\end{aligned}
$$

The Y^{I_1} are normalized scalar spherical harmonics on the unit 3-sphere. Their orders can be described by writing the rotation group of S^3 as $so(4) = su(2) \times su(2)$. The Y^{I_1} are representations (l,l) of $su(2) \times su(2)$, with $l = 0, \frac{1}{2}, 1, \ldots$. These harmonics satisfy

$$
\nabla^2 Y^{I_1} = -C(I_1)Y^{I_1}, \quad C(I_1) = 4l(l+1) \qquad (4.15)
$$

$$
\nabla_{[a}\nabla_{b]}Y^{I_1} = 0 \qquad (4.16)
$$

The $Y^{I_3}_a$ are normalized vector spherical harmonics. They fall into two classes, one with $su(2) \times su(2)$ representations $(l, l+1)$ and the other with $(l+1, l)$. (Again $l = 0, \frac{1}{2}, 1, \ldots$.) We have

$$
\begin{aligned}
\nabla^a Y^{I_3}_a &= 0 && (4.17)\\
\nabla_a Y^{I_3}_b - \nabla_b Y^{I_3}_a &= \zeta(I_3)\epsilon_{abc}Y^{I_3 c} && (4.18)
\end{aligned}
$$

where

$$
\zeta(I_3) = \begin{cases} -2(l+1), & I_3 = (l+1, l) \\ 2(l+1), & I_3 = (l, l+1) \end{cases} \qquad (4.19)
$$

More details on spherical harmonics are given in Appendix **A**.

3.4.3 Outer region: Solving for the first order corrections

Returning to the field equations (4.9), we compute the sources S, finding

$$\begin{aligned}
S_{tr\theta} &= \frac{Q}{2(Q+r^2)^2}\frac{1}{r^{2l+1}}\partial_\psi Y^{I_1}\tan\theta e^{-i\frac{a}{Q}u} \\
S_{tr\psi} &= -\frac{Q}{(Q+r^2)^2}\frac{1}{2r^{2l+1}}\left[\sin\theta\cos\theta\partial_\theta Y^{I_1}+2\frac{(l+3)r^2+(l+1)Q}{Q+r^2}Y^{I_1}\cos^2\theta\right]e^{-i\frac{a}{Q}u} \\
S_{yr\theta} &= -\frac{Q}{2(Q+r^2)^2}\frac{1}{r^{2l+1}}\partial_\phi Y^{I_1}\cot\theta e^{-i\frac{a}{Q}u} \\
S_{yr\phi} &= \frac{Q}{(Q+r^2)^2}\frac{1}{2r^{2l+1}}\left[\sin\theta\cos\theta\partial_\theta Y^{I_1}-2\frac{(l+3)r^2+(l+1)Q}{Q+r^2}Y^{I_1}\sin^2\theta\right]e^{-i\frac{a}{Q}u}
\end{aligned} \tag{4.20}$$

The source S_w is zero at this order.

We can decompose these sources into scalar and vector spherical harmonics

$$S_{\mu\nu a} = s^{I_3}_{\mu\nu}Y^{I_3}_a + t^{I_1}_{\mu\nu}\partial_a Y^{I_1} \tag{4.21}$$

Substituting this decomposition in (4.9) we get the equations

$$b^{I_1}_{1\mu\nu} - \frac{r}{Q+r^2}\tilde{\epsilon}_{\mu\nu\lambda}\partial^\lambda b^{I_1}_1 = t^{I_1}_{\mu\nu} \tag{4.22}$$

$$\partial_t b^{I_3}_{1y} - \partial_y b^{I_3}_{1t} + \zeta(I_3)\frac{r^3}{(Q+r^2)^2}b^{I_3}_{1r} = 0 \tag{4.23}$$

$$\partial_r b^{I_3}_{1t} - \partial_t b^{I_3}_{1r} + \zeta(I_3)\frac{b^{I_3}_{1y}}{r} = s^{I_3}_{tr} \tag{4.24}$$

$$\partial_y b^{I_3}_{1r} - \partial_r b^{I_3}_{1y} - \zeta(I_3)\frac{b^{I_3}_{1t}}{r} = s^{I_3}_{ry} \tag{4.25}$$

$$\partial_r\left(\frac{r^3}{(Q+r^2)^2}\partial_r b^{I_1}_1\right) + \frac{r}{(Q+r^2)^2}\left[2Q\tilde{w}^{I_1}_1 - C(I_1)b^{I_1}_1\right] = 0 \tag{4.26}$$

$$\frac{1}{r(Q+r^2)}\partial_r\left(r^3\partial_r\tilde{w}^{I_1}_1\right) - \frac{C(I_1)}{(Q+r^2)}\tilde{w}^{I_1}_1 - \frac{8Q}{(Q+r^2)^3}\left[Q\tilde{w}^{I_1}_1 - C(I_1)b^{I_1}_1\right] = 0 \tag{4.27}$$

Eq.(A.1) yields $b^{I_1}_{\mu\nu}$ once we know b^{I_1}; the source components $t^{I_1}_{\mu\nu}$ are listed in Appendix B. Eqs.(A.4),(A.7) allow the trivial solution

$$b_1 = \tilde{w}_1 = 0 \tag{4.28}$$

which we adopt, since other solutions would just amount to shifting the leading order solution taken for b, w. Eq.(A.5) yields $b_r = 0$. Eqns.(A.6), (4.25) are nontrivial and yield the solution $(u = t + y,\ v = t - y)$

$$b_{1ua} = b^{I_3}_{1u} Y^{I_3}_a = \frac{ia}{2}\sqrt{\frac{l}{(2l+1)(l+1)}}\frac{Q}{r^{2l}(Q+r^2)^2} Y_a^{(l+1,l)} + \tag{4.29}$$

$$-\frac{ia}{4r^{2l}}\left(\sqrt{\frac{2l-1}{l(2l+1)}}\frac{Q}{(Q+r^2)^2} - \frac{1}{Q}\sqrt{\frac{4l^2-1}{l^3}}\right) Y_a^{(l-1,l)} \tag{4.30}$$

$$b_{1va} = \frac{ia}{4}\sqrt{\frac{1}{(l+1)}}\frac{Q}{r^{2l}(Q+r^2)^2} Y_a^{(l,l+1)} \tag{4.31}$$

3.4.4 Matching at order ϵ

The inner region solution to order ϵ

Above we have applied the general scheme (4.9) to find the outer region solution to order ϵ. In general we would have to apply a similar scheme to correct the inner region solution as well. But it turns out that the expansion in the inner region goes in powers of ϵ^2. Since at this stage we are only matching terms of order ϵ^0, ϵ^1 we do not need to perform any extra computation for the inner region, and the solution (3.21)-(3.30) is already correct to the desired order. But to effect the comparison with the outer region we perform two manipulations on the inner region solution. First we express the set $B_{ta} = \{B_{t\theta}, B_{t\psi}, B_{t\phi}\}$ and the set B_{ya} in terms of scalar and vector harmonics

$$B_{ta} = \frac{iae^{-i\frac{a}{Q}u}}{2Q(r^2+a^2)^l}\left[\frac{\sqrt{l}Y_a^{(l+1,l)}}{\sqrt{(2l+1)(l+1)}} + \frac{Y_a^{(l,l+1)}}{2\sqrt{l+1}} + \right.$$

$$\left. \frac{l+1}{2l}\sqrt{\frac{2l-1}{l(2l+1)}} Y_a^{(l-1,l)} + \frac{\partial_a Y^{(l)}}{4l^2(l+1)}\right]$$

$$B_{ya} = \frac{iae^{-i\frac{a}{Q}u}}{2Q(r^2+a^2)^l}\left[\frac{\sqrt{l}Y_a^{(l+1,l)}}{\sqrt{(2l+1)(l+1)}} - \frac{Y_a^{(l,l+1)}}{2\sqrt{l+1}} + \right.$$

$$\frac{l+1}{2l}\sqrt{\frac{2l-1}{l(2l+1)}}Y_a^{(l-1,l)} - \frac{(2l-1)\partial_a Y^{(l)}}{4l^2(l+1)}\Bigg] \tag{4.32}$$

Next we perform a gauge transformation on B_{MN}

$$B_{MN} \to B_{MN} + \nabla_M \Lambda_N - \nabla_N \Lambda_M \tag{4.33}$$

Choosing

$$\Lambda_t = \frac{i}{8l^2(l+1)}\frac{a}{Q(r^2+a^2)^l}Y^{(l)}e^{-i\frac{a}{Q}u} \tag{4.34}$$

$$\Lambda_y = -\frac{i(2l-1)}{8l^2(l+1)}\frac{a}{Q(r^2+a^2)^l}Y^{(l)}e^{-i\frac{a}{Q}u} \tag{4.35}$$

we remove the components proportional to $\partial_a Y^{(l)}$ in (4.32), while getting additional terms in other components of B. In particular

$$B_{tr} = \frac{i}{4l(l+1)}\frac{ar}{Q(r^2+a^2)^{l+1}}Y^{(l)}e^{-i\frac{a}{Q}u} \approx \frac{i}{4l(l+1)}\frac{a}{Qr^{2l+1}}Y^{(l)}e^{-i\frac{a}{Q}u} \tag{4.36}$$

$$B_{yr} = \frac{i(2l^2+1)}{4l(l+1)}\frac{ar}{Q(r^2+a^2)^{l+1}}Y^{(l)}e^{-i\frac{a}{Q}u} \approx \frac{i(2l^2+1)}{4l(l+1)}\frac{a}{Qr^{2l+1}}Y^{(l)}e^{-i\frac{a}{Q}u} \tag{4.37}$$

We will see that with this gauge choice we will get a direct agreement of B_{MN} between the outer and inner regions.

The outer region solution to order ϵ

We had solved the field equations to first order in ϵ for the outer region in subsection (3.4.3) above. We list the complete solution thus obtained to order ϵ

$$
\begin{aligned}
w &= \frac{e^{-i\frac{a}{Q}u}}{r^{2l}(Q+r^2)}Y^{(l)} \\
B_{\theta\psi} &= \frac{1}{4l}\frac{e^{-i\frac{a}{Q}u}}{r^{2l}}\cot\theta\partial_\phi Y^{(l)} \\
B_{\theta\phi} &= -\frac{1}{4l}\frac{e^{-i\frac{a}{Q}u}}{r^{2l}}\tan\theta\partial_\psi Y^{(l)} \\
B_{\psi\phi} &= \frac{1}{4l}\frac{e^{-i\frac{a}{Q}u}}{r^{2l}}\sin\theta\cos\theta\partial_\theta Y^{(l)} \\
B_{ty} &= -\frac{1}{2(Q+r^2)^2}\frac{e^{-i\frac{a}{Q}u}}{r^{2l-2}}Y^{(l)} \\
B_{tr} &= -\frac{ia}{r^{2l+1}}\left(\frac{Q}{(Q+r^2)^3}\frac{[(l+2)r^2+lQ]}{4l(l+1)}-\frac{1}{4lQ}\right)Y^{(l)}e^{-i\frac{a}{Q}u} \\
B_{yr} &= \frac{ia}{r^{2l+1}}\left(\frac{(2l-1)Q}{(Q+r^2)^3}\frac{[(l+2)r^2+lQ]}{4l(l+1)}+\frac{1}{4lQ}\right)Y^{(l)}e^{-i\frac{a}{Q}u} \\
B_{ta} &= \frac{iaQe^{-i\frac{a}{Q}u}}{2r^{2l}(Q+r^2)^2}\left[\sqrt{\frac{l}{(2l+1)(l+1)}}Y_a^{(l+1,l)}+\frac{Y_a^{(l,l+1)}}{2\sqrt{l+1}}-\frac{1}{2}\sqrt{\frac{2l-1}{l(2l+1)}}Y_a^{(l-1,l)}\right] \\
&\qquad + \frac{ia}{4Qr^{2l}}\sqrt{\frac{4l^2-1}{l^3}}Y_a^{(l-1,l)} \\
B_{ya} &= \frac{iaQe^{-i\frac{a}{Q}u}}{2r^{2l}(Q+r^2)^2}\left[\sqrt{\frac{l}{(2l+1)(l+1)}}Y_a^{(l+1,l)}-\frac{Y_a^{(l,l+1)}}{2\sqrt{l+1}}-\frac{1}{2}\sqrt{\frac{2l-1}{l(2l+1)}}Y_a^{(l-1,l)}\right] \\
&\qquad + \frac{ia}{4Qr^{2l}}\sqrt{\frac{4l^2-1}{l^3}}Y_a^{(l-1,l)} \qquad (4.38)
\end{aligned}
$$

Comparing the inner and outer solutions at order ϵ

In the region where we match solutions we have to substitute at the present order of approximation

$$
\frac{1}{(r^2+a^2)^l}\approx\frac{1}{r^{2l}}, \qquad \frac{1}{(Q+r^2)}\approx\frac{1}{Q} \qquad (4.39)
$$

We then find agreement between the inner region solution (in the gauge discussed above) and outer region solution (4.38).

3.5 Matching at higher orders

We follow the same scheme to extend the computation to higher orders in ϵ. At each stage the sources S, S_w get contributions from all the terms found at preceeding orders. The computations are straightforward though tedious, and most are done using symbolic manipulation programs.

The solutions obtained for the inner region are listed in Appendix **B**. We have given the solutions in the NS sector coordinates; they must be spectral flowed to the R sector and gauge transformations performed to see directly the agreement with the outer region solutions. As mentioned before the perturbation series in the NS sector of the inner region proceeds in even powers of ϵ, and the odd powers in ϵ result from the spectral flow (2.9).

The solutions obtained for the outer region are listed in Appendix **C**. These are already in R sector coordinates. Note that at each order when we solve the equations with sources we have to choose a homogeneous part to the solution as well, and these parts have been chosen to give regularity everywhere as well as agreement between the inner and outer regions.

We carry out the computation of the solution in each region to order $O(\epsilon^3)$. We find complete agreement between the inner and outer region solutions upto the order investigated. At each stage of the computation there is the possibility of finding that some field is growing at infinity, and it is very nontrivial that this does not happen for any field at any of the orders studied. Thus we expect that the exact solution does exist and is likely to be expressible in closed form.

At all the orders that we have investigated the scalar w can be seen to arise from expansion of the solution

$$w = \frac{e^{-i\frac{a}{Q}u} Y^{(l)}}{(r^2+a^2)^l (Q+f)}, \qquad f = r^2 + a^2\cos^2\theta \tag{5.1}$$

Note that this expression involves just the combinations $(r^2+a^2), f$ which appear in the geometry (2.13). We do not have a similar compact expression for the B field; it is plausible that the compact form would require us to express this 2-form field as part 2-form and part 6-form (the magnetic dual representation). We hope to investigate this issue elsewhere.

3.6 Discussion

We have constructed regular, normalizable supergravity perturbations in the inner and outer regions by a process of successive corrections, and observed that at each order the solutions agree in the domain of overlap. This agreement is very nontrivial, and we take this as evidence for the existence of an exact solution to the problem – i.e. we expect that there exists a regular perturbation on the 2-charge D1-D5 geometry (2.13) which carries one unit of momentum charge and adds one unit of energy (thus yielding an extremal 3-charge solution). We now return to our initial discussion of black hole interiors, and the significance of this solution in that context.

The usual picture of a black hole has a horizon, a singularity at the center, and 'empty space' in between. Abstract arguments given in the introduction suggested a different picture where the interior was nontrivial and exhibited the degrees of freedom contributing to the entropy. The 2-charge extremal system turned out to look like this latter picture – its properties (a′)-(c′) listed in the introduction matched the suggested properties (a)-(c). What about the 3-charge extremal hole? This latter

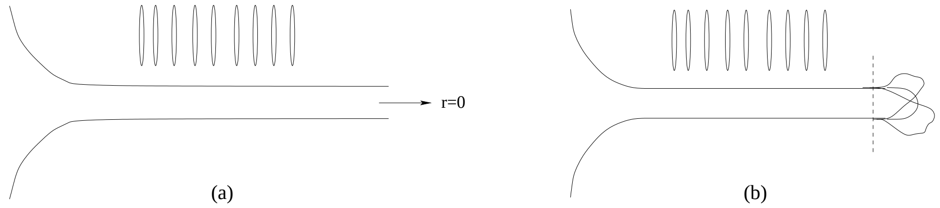

Figure 3.7: (a) Naive geometry for the 3-charge extremal system. (b) Expected structure for the system.

hole has become a benchmark system for understanding black holes, and any lessons deduced here likely extend to all holes in all dimensions.

The metric conventionally written for the D1-D5-P extremal system is

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{(1+\frac{Q_1}{r^2})(1+\frac{Q_5}{r^2})}}[-dudv + \frac{Q_p}{r^2}dv^2] \\ &+ \sqrt{(1+\frac{Q_1}{r^2})(1+\frac{Q_5}{r^2})}[dr^2 + r^2 d\Omega_3^2] + \sqrt{\frac{(1+\frac{Q_1}{r^2})}{(1+\frac{Q_5}{r^2})}}dz_a dz_a \end{aligned} \tag{6.1}$$

This is similar to the 'naive' metric (B.13) of the 2-charge D1-D5 extremal system, except that the y circle stabilizes to a fixed radius as $r \to 0$ instead of shrinking to zero size (we picture the geometry (5.82) in Fig.3.7(a)). The geometry (5.82) has a completion that it continues past the horizon at $r = 0$ to the 'interior' of the black hole, where we have a timelike singularity – the metric is just a 4+1 analogue of the extremal Reissner-Nordstrom black hole.

In a roughly similar manner one might have asked if the 2-charge metric continues past the 'horizon' $r = 0$ to another region, but here we do know the answer – the naive metric (B.13) is incorrect, and the actual geometries 'cap off' before reaching $r = 0$. We are therefore led to ask if a similar situation holds for the 3-charge system, so that the actual geometries 'cap off' before reaching $r = 0$ as in Fig.3.7(b). We would then draw the 'horizon' as a surface which bounds the region where the geometries differ from each other significantly; this surface is indicated by the dashed line in Fig.3.7(b). Note that for the 3-charge system the area of this 'horizon' will give *exactly*

$$\frac{A}{4G} = S_{micro} = 2\pi\sqrt{n_1 n_5 n_p} \tag{6.2}$$

This is because in the naive metric (5.82) the cross sectional area of the throat saturates to a constant A as $r \to 0$, and it is this same value A that will be picked up at the location of the dashed line in Fig.3.7(b). But from [3] we know that this area A satisfies (6.2). (For the 2-charge case we could find A only upto a factor of order unity, since the y circle of the cross section was shrinking with r, and the natural uncertainty in the location of the 'horizon surface' leads to a corresponding uncertainty in A.)

Thus for the 3-charge system the nontrivial issue is not horizon area (which we see will work out anyway) but the nature of the geometry inside the horizon. The computation of this paper has indicated that if we have one unit of P then at least one extremal state

$$|\Psi\rangle = J^-_{-1}|0\rangle_R \tag{6.3}$$

of the 3-charge system is described by a geometry like Fig.3.7(b) and not by Fig.3.7(a). It may be argued though that the 2-charge extremal states and the state (6.3) are not sufficiently like generic black hole states to enable us to conclude that Fig.3.7(b) is the generic geometry of the 3-charge system. Here we give several arguments that counter this possibility:

(a) *Is the 2-charge system like a black hole?* It is sometimes argued that the 2-charge extremal system is not really a black hole since the horizon area vanishes classically. We argue against this view. The microscopic entropy of the 2-charge extremal system (S=$2\sqrt{2}\pi\sqrt{n_1 n_5}$) arises by partitions of $N = n_1 n_5$ in a manner similar to the entropy $2\pi\sqrt{n_1 n_5 n_p}$ of the 3-charge extremal system which arises from partitions of $N = n_1 n_5 n_p$. The 'horizon' that we have constructed for the 2-charge system satisfies $S \approx A/4G$, so this 'horizon' area is $\sim \sqrt{n_1 n_5}$ times $(l_p)^3$, and is thus *not* small at all in planck units.

Why then do we think of this horizon as small? The 2-charge metric has factors like $\sim (1 + \frac{Q_1}{r^2}), (1 + \frac{Q_5}{r^2})$. Assuming $Q_1 \sim Q_5$ and $n_1 \sim n_5 \sim n$ we find that the geometry has a scale, the 'charge radius', which grows with n as $r \sim Q^{\frac{1}{2}} \sim n^{\frac{1}{2}}$. Since the horizon is a 3-dimensional surface, and we have found $S_{micro} \sim n \sim \frac{A}{4G}$, the horizon radius is $r \sim n^{\frac{1}{3}}$. Suppose we take the classical limit $n \to \infty$ and then scale the metric so that the charge radius is order unity. In this limit the horizon radius will *vanish*. For the 3-charge system, both the charge radius and the horizon radius behave as $r \sim n^{\frac{1}{2}}$, so the horizon radius remains nonzero in the analogous classical limit.

But this behavior of classical limits does not imply that the 2-charge system has an ignorable horizon – the horizon does give the correct entropy, and the presence of the other, larger, length scale appears irrelevant to the physics inside this horizon. The region $r \sim Q^{\frac{1}{2}}$ is far removed from the horizon region, and simply governs the changeover from 'throat geometry' to 'flat space'.

(b) *Return time* Δt_{CFT}*:* For the 2-charge system, the naive metric is (B.13). If we throw a test particle down the throat of this naive metric, it does not return after any finite time. In the dual CFT however an excitation absorbed by the 'effective string' can be re-emitted after a time $\Delta t_{CFT} < \infty$. How do we resolve this contradiction?

One might think that nonperturbative effects cause the test particle to turn back from some point in the throat of the naive geometry, but this cannot be the case since the return time Δt_{CFT} is different for different states of the 2-charge system (the length of the components of the effective string are different for different states). The resolution of this puzzle was that the throats were capped; the cap was different for different states [150], and we get (2.1).

The CFT for the 3-charge system is described by the same effective string; we just have additional momentum excitations on the effective string. We would thus again have some finite time Δt_{CFT} after which an excitation should be emitted back from the system, and the requirement (2.1) then suggests that Fig.3.7(b) is the correct picture for the general states of the 3-charge system, rather than Fig.3.7(a).

(c) *Fractionation:* We have argued that the interior of the horizon is not the conventionally assumed 'empty space with central singularity'. How can the classical expectation be false over such large length scales? The key physical effect is 'fractionation'. If we excite a pair of left and right vibrations on a string of length L, the minimum excitation threshold is $\Delta E = \frac{2\pi}{L} + \frac{2\pi}{L} = \frac{4\pi}{L}$. But if we have a bound state of n strings, then we get one long string of length nL, and the threshold drops to $\frac{4\pi}{nL}$ [33]. If we start with 2-charges, n_1 D1 branes and n_5 D5 branes, then the excitations of the third charge, momentum, come in even smaller units, and $\Delta E = \frac{4\pi}{n_1 n_5 L}$ [34]. If we assume more generally that for the bound state of mutually supersymmetric branes the excitations always fractionate in this way, then we find that the excitations of the D1-D5-P hole are such that they extend to a radial distance that is just the horizon scale [76]. For the 2-charge FP where we have explicitly constructed all geometries this fractionation effect can be directly seen – because the momentum waves are fractionally moded on the multiply wound F string, the strands of the F string separate and spread over a significant transverse area, which extends all the way to the 'horizon'.

(d) *Other 3-charge states:* The general perturbations around the 2-charge solution that we have chosen decompose into two classes: The antisymmetric field + scalar perturbations (which we have analyzed) and the metric + self-dual field perturbations. We have checked upto leading order (ϵ^0) that the latter class gives a regular solution as well. Further, the 2-charge solution that we started with may appear special (It has for instance angular momentum $\frac{n_1 n_5}{2}$ in each $su(2)$ factor, while the generic 2-charge state has negligible net angular momentum) but we have also checked that at leading order we get regular perturbations for all starting 2-charge geometries. In principle all these computations could be carried out to higher orders in ϵ, but the technical complexities would be greater due to less symmetry in the starting configuration.

One might think that if we increase the the momentum p then we might get a horizon. For $p = 1$ we have seen that the perturbation is smooth, so there is no hint of an incipient horizon. Suppose for some $p = p_0$ a horizon just about forms; this horizon will be of radius zero at $p = p_0$, and larger at larger p. But what will be the location of the horizon at $p = p_0$? There is no special point in the starting 2-charge geometries; they are just smoothly capped throats. It thus appears more likely that adding momentum will just give more and more complicated configurations, but with no special point which could play the role of a singularity.

(e) *Nonextremal holes:* Having found the above structure for extremal systems, we expect a similar structure for near extremal and also neutral holes, with the difference that the branes in the extremal systems are replaced by a collection of branes and anti-branes. Indeed, for the non-extremal D1-D5-P system it is known that the entropy of holes arbitrarily far from extremality can be reproduced exactly if we assume that the energy is optimally partitioned between branes and anti-branes while reproducing the overall charges and mass [36].

In an interesting recent paper [120] it was argued that the 'black ring' solutions carrying D1-D5-P charges (plus nonextremality) had pathologies like closed timelike curves and thus it was not possible to add momentum by boosting to general rotating D1-D5 states. It was observed however that it might be possible to add momentum in other ways to get a 3-charge state. Our construction *does* take a D1-D5 state with some angular momentum, and adds one unit of momentum. But looking at the form of the perturbation it can be seen that the momentum was not obtained by a boost.

More generally, generating metrics by boosting a 'naive' nonextremal geometry will *not* give the correct states of the system. In such a construction we start with a nonextremal black hole or black ring geometry, where the metric in the interior of the horizon is just the classically expected one (similar in spirit to Fig.1(a) for a black hole). But we have argued that such an interior metric is *not* a correct description for the region inside the horizon; this region we believe is very nontrivial, with details that necessarily depend on the particular state which the system takes (out of the e^S possible states).[15] Instead one should start with one of the 'correct' states for system, and then construct the possible deformations that add momentum.

[15]One should not use the 'correspondence principle' [38] to obtain a qualitative understanding of what might happen inside horizons. It was shown in [76] that at coupling $g < g_c$ the energy added to a string goes to exciting vibrations, while at $g > g_c$ the energy goes to creating *brane-antibrane pairs*. (Here g_c is the coupling at the 'correspondence point' where the string turns to a black hole.) It is these brane-antibrane pairs that have the small energy gaps and large phase space to 'fill up' the interior of the horizon.

We can emphasize this point in another way, using just the 2-charge system. Suppose we start with the *naive* metric for the *nonextremal* F string. This metric will have cylindrical symmetry around the axis of the F string. We can boost and add momentum, still keeping the cylindrical symmetry and getting F and P charges. We can then take the non-extremality to zero. This process will reproduce the *naive* metric (2.1) of the extremal FP system. To get the *correct* metrics for extremal FP starting from non-extremal FP we would have to start with one of the correct interior states for the *nonextremal* FP system.

Clearly what we need next is a construction of the generic 3-charge configuration (i.e. with the P charge not small). It is important that the solutions represent true bound states rather than just multi-center brane solutions that are classically supersymmetric. (Some families of metrics with 3 charges have been constructed before (e.g. [160]) but we are not aware of any set that actually describes the bound states that we wish to study.)[16] It is possible that the generic state is not well approximated by a classical configuration; what we do expect though on the basis of all that was said above is that the region where the different states depart from each other will be of the order the horizon size and not just a planck sized region near the singularity.

[16]We thank D. Mateos and O. Lunin for discussions on this point.

CHAPTER 4

CONSTRUCTING 'HAIR' FOR BLACK RINGS

4.1 Introduction

In chapter 3, we perturbatively added one unit of moemntum to a particular microstate of 2 charge black holes in $4+1$ dimensions. This gave us a perturbative three charge system which was smooth and horizon-free, in accordance with Mathur conjecture for black hole microstates. In 4+1 dimensions we can have not only black holes but also black rings [120]. We would therefore like to construct microstates for the ring. The goal of this chapter is to construct a simple 3-charge extremal state for the ring, where we start with a ring carrying two charges D1,D5 and add a wave carrying one unit of momentum P, the third charge.

There has been a lot of recent progress on black rings. The entropy for the ring can be obtained by computing it for a short straight segment of a ring and multiplying by the total length of the ring [121]. A subset of 3-charge rings can be obtained as supertubes made out of branes [122].

We are interested in the gravity description of microstates. In [123, 9] dual geometries were found for a discrete subset of CFT states. But even though these states have a large angular momentum, they do not look like 'rings', since we cannot find a sphere S^2 that will surround a 'ring' shaped interior. In [124, 125] a method was developed to find large families of 3-charge BPS states, in terms of the choice of locations of poles of certain harmonic functions appearing in the metric [126]. While the CFT states for these geometries are not known, it was argued that these geometries represented bound states because there is a nonzero flux on spheres S^2 linking the poles. Assuming that this argument is correct we can make geometries that are like rings, and that have no horizon and no singularity.

In the present paper we would like to construct a gravity description of microstates in a case where we also know the dual CFT state. Thus in gravity terms our construction will be more modest than the ones obtainable from [124, 125]; the third charge will be only a small perturbation on our 2-charge ring. On the other hand since we will know the microscopic origin of the state, we are assured that we have a bound state and we can also develop some intuition for how CFT operations act in the gravity picture.

In spirit our computation is similar to the computation in [123], where one unit of momentum was added to a D1-D5 extremal state. We will again take the same D1-D5 state, and add a unit of momentum using the same fields, but will be working in a very different limit from the one used in [123]. In [123] we had chosen our moduli so that the D1-D5 geometry had a large AdS type region, which went over to flat space at infinity. This geometry is depicted in Fig.1(a). The wavefunction of the quantum carrying the momentum is peaked in the AdS region, falling off at infinity in a normalizable way. By contrast in the present paper we will take a limit of the moduli so that the D1-D5 state looks like a thin ring, depicted in Fig.1(b). Consider a short segment of this ring, which looks like a straight tube (Fig.1(c)). The bound state wavefunction must now appear as a wavefunction localized in the vicinity of this tube, falling to zero away from this tube, and regular everywhere inside. We find this wavefunction, thus obtaining a simple but explicit example of 'hair' for the black ring. The fact that the wavefunction is regular everywhere suggests that no horizon or singularity should form even for a non-infinitesimal deformation, so the result suppor

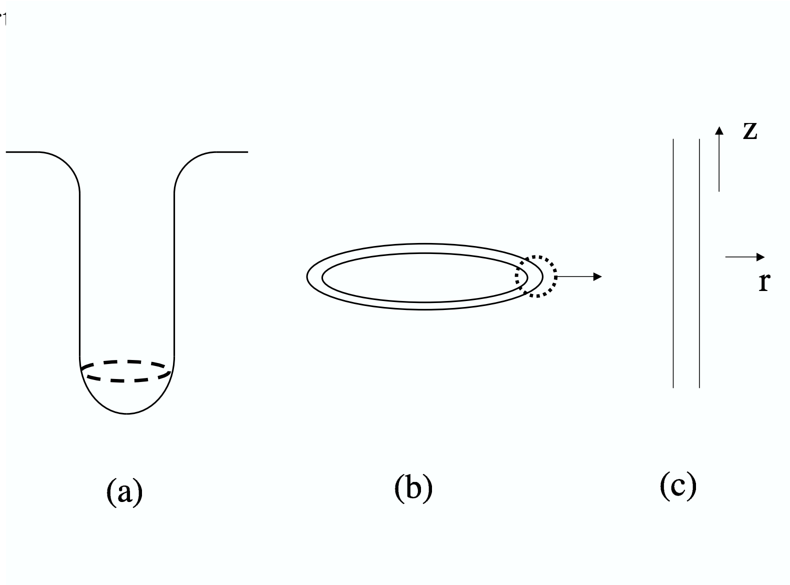

Figure 4.1: (a) The D1-D5 geometry for large values of R_y, the radius of S^1; there is a large AdS region (b) The geometry for small R_y; the metric is close to flat outside a thin ring (c) In the near ring limit we approximate the segment of the ring by a straight line along z.

4.2 The CFT state

It is important for us that the state we construct in the gravity description be known to be a BPS state in the dual CFT. In this section we review the discussion of [123] where this state was described.

4.2.1 The 2-charge geometry

We start with the 2-charge D1-D5 system. We compactify IIB string theory as $M_{9,1} \to M_{4,1} \times T^4 \times S^1$. We wrap n_5 D5 branes on $T^4 \times S^1$, and n_1 D1 branes on S^1. The system has a large class of BPS bound states, out of which we choose a simple one that was first noted in [186, 188]. If we reduce the metric on T^4 we find that the 6-d string metric is the same as the Einstein metric, so we will just call it the 'metric' below. The masses of the D1 and D5 branes are described by parameters $\bar{Q}_1, \bar{Q}_5$ which we will set to be equal

$$\bar{Q}_1 = \bar{Q}_5 = \bar{Q} \tag{2.1}$$

This will simplify our computations, but we expect that the state we construct will exist for all $\bar{Q}_1, \bar{Q}_5$. With the choice (2.1) the dilaton is constant, and the volume of the T^4 is also constant. The metric and gauge field for the solution are given by

$$\begin{aligned} ds^2 &= -H^{-1}(dt^2 - dy^2) + Hf\Big(\frac{d\bar{r}^2}{\bar{r}^2 + a^2} + d\bar{\theta}^2\Big) - 2\frac{a\bar{Q}}{Hf}(\cos^2\bar{\theta} dy d\bar{\psi} + \sin^2\bar{\theta} dt d\bar{\phi}) \\ &\quad + H\Big(\bar{r}^2 + \frac{a^2\bar{Q}^2\cos^2\bar{\theta}}{H^2 f^2}\Big)\cos^2\bar{\theta} d\bar{\psi}^2 + H\Big(\bar{r}^2 + a^2 - \frac{a^2\bar{Q}^2\sin^2\bar{\theta}}{H^2 f^2}\Big)\sin^2\bar{\theta} d\bar{\phi}^2 \end{aligned} \tag{2.2}$$

$$\begin{aligned} C^{(2)} &= -\frac{\bar{Q}}{Hf} dt \wedge dy - \frac{\bar{Q}\cos^2\bar{\theta}}{Hf}(\bar{r}^2 + a^2 + \bar{Q}) d\bar{\psi} \wedge d\bar{\phi} \\ &\quad - \frac{\bar{Q}a\cos^2\bar{\theta}}{Hf} dt \wedge d\bar{\psi} - \frac{\bar{Q}a\sin^2\bar{\theta}}{Hf} dy \wedge d\bar{\phi} \end{aligned} \tag{2.3}$$

where

$$f = \bar{r}^2 + a^2\cos^2\bar{\theta}, \quad H = 1 + \frac{\bar{Q}}{f} \tag{2.4}$$

Here

$$a = \frac{\bar{Q}}{R_y} \tag{2.5}$$

where R_y is the radius of the S^1.

4.2.2 The 'inner' region

Suppose we take

$$\epsilon \equiv \frac{a}{\bar{Q}^{1/2}} = \frac{\bar{Q}^{1/2}}{R_y} << 1 \tag{2.6}$$

(This can be achieved by taking R_y large holding all other moduli and the charges fixed.) We can then look at the 'inner region' of this geometry

$$\bar{r} << \sqrt{\bar{Q}} \tag{2.7}$$

The metric here is

$$\begin{aligned} ds^2 &= -\frac{(\bar{r}^2 + a^2\cos^2\bar{\theta})}{\bar{Q}}(dt^2 - dy^2) + \bar{Q}(d\bar{\theta}^2 + \frac{d\bar{r}^2}{\bar{r}^2 + a^2}) \\ &\quad - 2a(\cos^2\bar{\theta}dyd\bar{\psi} + \sin^2\bar{\theta}dtd\bar{\phi}) + \bar{Q}(\cos^2\bar{\theta}d\bar{\psi}^2 + \sin^2\bar{\theta}d\bar{\phi}^2) \end{aligned} \tag{2.8}$$

The change of coordinates

$$\psi_{NS} = \bar{\psi} - \frac{a}{\bar{Q}}y, \quad \phi_{NS} = \bar{\phi} - \frac{a}{\bar{Q}}t \tag{2.9}$$

brings (2.8) to the form $AdS_3 \times S^3$

$$ds^2 = -\frac{(\bar{r}^2 + a^2)}{\bar{Q}}dt^2 + \frac{\bar{r}^2}{\bar{Q}}dy^2 + \bar{Q}\frac{d\bar{r}^2}{\bar{r}^2 + a^2} + \bar{Q}(d\bar{\theta}^2 + \cos^2\bar{\theta}d\psi_{NS}^2 + \sin^2\bar{\theta}d\phi_{NS}^2) \tag{2.10}$$

This AdS geometry is dual to a 1+1 dimensional CFT. For this CFT we can construct chiral primaries, which are described in the gravity picture by certain BPS configurations. The CFT dual to the geometry (2.10) is in the Neveu-Schwarz (NS) sector. In the original form (2.8) the geometry described the CFT in the Ramond (R) sector, and the coordinate change (2.9) gives the gravity description of the 'spectral flow' between the NS and R sectors [186, 188].

The simplest chiral primaries can be obtained by finding normalizable solutions of the supergravity equations describing linear perturbations around $AdS_3 \times S^3$. The supergravity fields in the 6-d theory separate into different sets (with no coupling at the linear level between sets). One set described an antisymmetric 2-form $B^{(2)}_{MN}$ and a scalar w. We write

$$H^{(3)}_{MNP} = \partial_M C^{(2)}_{NP} + \partial_N C^{(2)}_{PM} + \partial_P C^{(2)}_{MN}, \quad F^{(3)}_{MNP} = \partial_M B^{(2)}_{NP} + \partial_N B^{(2)}_{PM} + \partial_P B^{(2)}_{MN} \quad (2.11)$$

Then the field equations for the perturbation $(B^{(2)}_{MN}, w)$ are

$$\begin{aligned} &F^{(3)} + \star_6 F^{(3)} + w\, H^{(3)} = 0 \\ &\Delta_6 w - \frac{1}{3} H^{(3)\,MNP} F^{(3)}_{MNP} = 0 \end{aligned} \quad (2.12)$$

Here the star operation $\star_6$, the laplacian Δ_6 and index contractions in (2.12) are defined with respect to the metric (4.100).

4.2.3 Constructing the chiral primary

We can solve the equations (2.12) in the 'inner region' geometry (2.10) and obtain normalizable solutions. The solution giving a chiral primary is [129, 123] (the coordinates on S^3 are $a, b, \ldots$ and on the AdS_3 are $\mu, \nu, \ldots$)

$$w_{inner} = \frac{e^{-2i\frac{a}{\bar{Q}}lt}}{\bar{Q}(\bar{r}^2 + a^2)^l} \hat{Y}^{(l)}_{NS} \quad (2.13)$$

$$B^{(2)}_{ab} = B\epsilon_{abc}\, \partial^c \hat{Y}^{(l)}_{NS}, \quad B^{(2)}_{\mu\nu} = [\frac{1}{\sqrt{\bar{Q}}} \epsilon_{\mu\nu\lambda} \partial^\lambda B]\, \hat{Y}^{(l)}_{NS} \quad (2.14)$$

where

$$\hat{Y}^{(l)}_{NS} = (Y^{(l,l)}_{(l,l)})_{NS} = \sqrt{\frac{2l+1}{2}} \frac{e^{-2il\phi_{NS}}}{\pi} \sin^{2l} \bar{\theta}, \quad B = \frac{1}{4l} \frac{e^{-2i\frac{a}{\bar{Q}}lt}}{(\bar{r}^2 + a^2)^l} \quad (2.15)$$

In (2.14) the tensors ϵ_{abc}, g^{ab} etc are defined using the metric on an S^3 with *unit* radius. The spherical harmonics Y are representations of the rotation group $SO(4) \approx SU(2) \times$

$SU(2)$ of the sphere S^3, and $Y^{(l,l')}_{(m,m')}$ has quantum numbers (l,m) in the first $SU(2)$ and (l',m') in the second $SU(2)$. These two $SU(2)$ groups become the R symmetries of the left and right movers respectively in the dual CFT. The perturbation (2.13-2.15) gives a state in the CFT with R charges and dimensions given by

$$j_{NS}=l, \quad h_{NS}=l, \quad \bar{j}_{NS}=l, \quad \bar{h}_{NS}=l \tag{2.16}$$

(Unbarred and barred quantities denote left and right movers respectively.) The quantities $j_{NS}, \bar{j}_{NS}$ are the values of the azimuthal quantum numbers in the two $SU(2)$ groups. The subscript NS denotes that we are in the Neveu-Schwarz sector of the CFT. If we spectral flow this perturbation to the Ramond sector then we will get a perturbation with

$$h=\bar{h}=0 \tag{2.17}$$

which is expected, since a chiral primary of the NS sector maps under spectral flow to a ground state of the R sector.[17]

Let the CFT state dual to the perturbation (2.13)-(2.15) be called $|\psi\rangle_{NS}$, and let $|\psi\rangle_R$ be its image under spectral flow to the Ramond sector.

4.2.4 The state $J_0^-|\psi\rangle_{NS} \;\leftrightarrow\; J_{-1}^-|\psi\rangle_R$

Consider again the inner region in the NS sector coordinates (2.10). We now wish to make the perturbation dual to the NS sector state

$$J_0^-|\psi\rangle_{NS} \tag{2.18}$$

Since the operator J_0^- in the NS sector is represented by just a simple rotation of the S^3, we can immediately write down the bulk wavefunction dual to the above CFT

[17]The full spectral flow relations are $h=h_{NS}-j_{NS}+\frac{c}{24},\ j=j_{NS}-\frac{c}{12}$. Spectral flow of the background $|0\rangle_{NS}$ gives $h^0=h^0_{NS}-\frac{c}{24},\ j^0=j^0_{NS}-\frac{c}{12}$, so for the perturbation the spectral flow relations are just $h=h_{NS}-j_{NS}, j=j_{NS}$.

state

$$w_{inner} = \frac{e^{-2i\frac{a}{\bar{Q}}lt}}{\bar{Q}(\bar{r}^2+a^2)^l} Y_{NS}^{(l)} \tag{2.19}$$

$$B_{ab}^{(2)} = B\epsilon_{abc}\,\partial^c Y_{NS}^{(l)}, \quad B_{\mu\nu}^{(2)} = [\frac{1}{\sqrt{\bar{Q}}}\epsilon_{\mu\nu\lambda}\partial^\lambda B]\; Y_{NS}^{(l)} \tag{2.20}$$

$$Y_{NS}^{(l)} = (Y_{(l-1,l)}^{(l,l)})_{NS} = -\frac{\sqrt{l(2l+1)}}{\pi}\sin^{2l-1}\bar{\theta}\cos\bar{\theta}e^{i(-2l+1)\phi_{NS}+i\psi_{NS}}, \quad B = \frac{1}{4l}\frac{e^{-2i\frac{a}{\bar{Q}}lt}}{(\bar{r}^2+a^2)^l} \tag{2.21}$$

This perturbation has

$$j_{NS} = l-1, \quad \bar{j}_{NS} = l, \quad h_{NS} = l, \quad \bar{h}_{NS} = l \tag{2.22}$$

The spectral flow to the R sector coordinates should give

$$h = h_{NS} - j_{NS} = 1, \quad \bar{h} = \bar{h}_{NS} - \bar{j}_{NS} = 0 \tag{2.23}$$

This spectral flowed state can be written as

$$|\psi\rangle = J_{-1}^{-}|\psi\rangle_R \tag{2.24}$$

This is a state with nonzero $L_0 - \bar{L}_0$, which means that it is a state carrying momentum P along S^1. It is a state in the R sector, which is the sector which we obtain for the CFT if we wrap D1,D5 branes around the compact directions in our original spacetime.

So far we have found the relevant fields only in the 'inner' region (2.8). But in [123] the perturbation equations were also solved in the 'outer' region $a \ll r < \infty$ and it was shown that the inner and outer region solutions agreed with each other to several orders in the small parameter ϵ given in (2.6). This agreement was very nontrivial, and indicated that there was an exact solution to the perturbation problem that was smooth in the inner region and normalizable at spatial infinity. This exact solution would be a state carrying three charges: D1, D5, and one unit of momentum P. Since it is regular everywhere, we learn that it is possible for 3-charge microstate to be nonsingular and horizon free, just like 2-charge microstates.

Even though the solution obtained in [123] was only found by matching inner and outer region solutions to some order in ϵ, it was possible to guess, from the results, a closed form for the scalar w which would conceivably hold for all orders in ϵ:

$$w_{full} = e^{-i\frac{a}{\bar{Q}}(t+y)} e^{-i(2l-1)\bar{\phi}} e^{i\bar{\psi}} \frac{\sin^{2l-1}\bar{\theta}\,\cos\bar{\theta}}{(\bar{r}^2+a^2)^l\,(\bar{Q}+f)} \tag{2.25}$$

We will see that this conjecture for w will help us in obtaining the perturbation for the 3-charge ring.

4.3 The near ring limit

In the above section we took the limit (2.6) which sets $R_y >> \sqrt{\bar{Q}}$; this gives the geometry of Fig.1(a) which has a large AdS type region. Now we will take the opposite limit

$$R_y \ll \sqrt{\bar{Q}} \tag{3.26}$$

In this limit we get a geometry like that in Fig.1(b); we have flat space everywhere except around a thin 'ring'. This ring has radius $a = \bar{Q}/R_y$. Note that we have a large family of bound state D1-D5 geometries; these arise by duality from different vibration profiles of the NS1-P system [8]. In the limit (3.26) all these become thin tubes around the curves generated by the NS1-P profile. The near ring limit of any of these curves looks the same; thus for a local analysis of the perturbation equations we may start with any ring, and we have chosen to start with the 'round' ring because it is the simplest.

In the near ring limit the following coordinates are the natural ones: we take a coordinate z to measure length along the ring, and we introduce spherical polar coordinates r, θ, ϕ in the 3-dimensional space transverse to the ring. (We leave unchanged the coordinate along the compact directions S^1, T^4.) The coordinate change from $(\bar{r}, \bar{\theta}, \bar{\psi}, \bar{\phi})$ to (r, θ, ϕ, z) is described in the Appendix. The result is

$$\bar{r}^2 = \frac{a^2 r(1-\cos\theta)}{a + r\cos\theta}, \quad \sin^2\bar{\theta} = \frac{a-r}{a+r\cos\theta}, \quad \bar{\psi} = \phi, \quad \bar{\phi} = \frac{z}{a} \tag{3.27}$$

The only length scale of the near ring geometry is the parameter characterizing the charge density along the ring

$$Q = \frac{\bar{Q}}{2a} \tag{3.28}$$

The y radius is given in terms of Q by

$$R_y = 2Q \tag{3.29}$$

The near ring region is described by

$$r \ll a \tag{3.30}$$

From (3.27) one finds, in this limit

$$\begin{aligned} &\bar{r}^2 \approx ar(1-\cos\theta), \quad \sin^2\bar{\theta} \approx 1 - \frac{r}{a}(1+\cos\theta), \quad f \approx 2ar \\ &\frac{d\bar{r}^2}{\bar{r}^2 + a^2} + d\bar{\theta}^2 = \frac{1}{2r(a+r\cos\theta)}\left[\frac{a^2}{a^2-r^2}dr^2 + r^2 d\theta^2\right] \approx \frac{dr^2 + r^2 d\theta^2}{2ar} \end{aligned} \tag{3.31}$$

In this limit the metric and RR field become

$$\begin{aligned}
ds^2 &= -H^{-1}\left(dt+\frac{Q}{r}dz\right)^2 + H\,dz^2 + ds^2_{TN} \\
ds^2_{TN} &= H^{-1}\,(dy - Q(1+\cos\theta)d\phi)^2 + H\,(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2) \\
C^{(2)} &= H^{-1}\frac{Q}{r}\,dy\wedge(dt-dz) + H^{-1}Q(1+\cos\theta)\,d\phi\wedge(dt-dz) \qquad (3.32)
\end{aligned}$$

where

$$H = 1 + \frac{Q}{r} \qquad (3.33)$$

4.3.1 The Taub-NUT space

The part of the metric denoted as ds^2_{TN} is Taub-NUT (TN) space: it is smooth due to the relation (3.29). The TN gauge field

$$A = -Q(1+\cos\theta)d\phi \qquad (3.34)$$

satisfies

$$dA = Q\sin\theta\,d\theta\wedge d\phi\,, \quad \star_3 dA = -dH \qquad (3.35)$$

where $\star_3$ is the Hodge dual with respect to the flat $\mathbb{R}^3$ spanned by r, θ, ϕ. A convenient basis of 1-forms on TN is given by $\hat{\sigma}$, dr, $d\theta$, $d\phi$, with

$$\hat{\sigma} = d\hat{y} - \frac{1+\cos\theta}{2}d\phi\,, \quad \hat{y} = \frac{y}{2Q} \qquad (3.36)$$

In terms of these forms the RR field strength can be written as

$$H^{(3)} = \Omega^{(2)}\wedge(dt-dz) \qquad (3.37)$$

with

$$\Omega^{(2)} = -\frac{2Q^2}{H^2r^2}\left[dr\wedge\hat{\sigma} + \frac{Hr^2}{Q}\,d\hat{\sigma}\right] = -2Q\,d(H^{-1}\hat{\sigma}) \qquad (3.38)$$

We choose the orientations of the 6D and TN spaces so that

$$\epsilon_{ty\bar{r}\bar{\theta}\bar{\psi}\bar{\phi}} = \epsilon_{tzyr\theta\phi} = 1\,, \quad \epsilon_{r\theta\phi y} = 1 \tag{3.39}$$

(and thus $\epsilon_{tz} = -1$). Then $H^{(3)}$ is self-dual with respect to the 6-D metric

$$\star_6 H^{(3)} = H^{(3)} \tag{3.40}$$

and $\Omega^{(2)}$ is self-dual with respect to the 4-dimensional TN metric

$$\star\Omega^{(2)} = \Omega^{(2)} \tag{3.41}$$

$\Omega^{(2)}$ is the unique closed and self-dual 2-form on TN.

4.3.2 The scalar w in the near-ring limit

We had noted in section (4.2.4) that the computations of [123] had suggested an exact form for w, given in (2.25).

We would like to take the near ring limit of (2.25). Remember that in the geometry (4.100) the ring is spanned by the coordinate $\bar{\phi}$ and its length is $2\pi a$. From the $\bar{\phi}$ dependence in (2.25) we see that the the wavelength of the perturbation w_{full} in the direction of the ring is

$$\lambda = \frac{2\pi a}{2l-1} \equiv \frac{2\pi}{k} \tag{3.42}$$

We will be interested in the regime in which this wavelength is much shorter than the ring:

$$\lambda \ll a \tag{3.43}$$

This will enable us to take our limit in such a way that we see oscillations of the wavefunction along the z direction even when we take a near-ring limit and see only a short segment of the ring. Eqs.(3.42) and (3.43) imply

$$l \gg 1 \tag{3.44}$$

We can thus approximate $2l - 1 \approx 2l$ in the following. By applying the change of coordinates (3.27) and taking the limits (3.30) and (3.44), we find

$$\cos\bar\theta = \sqrt{\frac{r(1+\cos\theta)}{a + r\cos\theta}} \approx \sqrt{\frac{2}{a}}\,\sqrt{r}\,\cos\frac{\theta}{2} \tag{3.45}$$

and

$$\begin{aligned}\frac{\sin^{2l-1}\bar\theta}{(\bar r^2+a^2)^l} &\approx \Big(\frac{\sin^2\bar\theta}{\bar r^2+a^2}\Big)^l = a^{-2l}\,\Big(\frac{a-r}{a+r}\Big)^l\\ &\approx a^{-2l}\,\Big(1-2\frac{r}{a}\Big)^l \approx a^{-2l}\,\Big(1-\frac{k\,r}{l}\Big)^l \approx a^{-2l}\,e^{-kr}\end{aligned} \tag{3.46}$$

where we have used (3.42) and the identity $(1+\epsilon\alpha)^{1/\epsilon} \approx e^\alpha$. Up to an overall normalization, the near ring limit of w_{full} is then

$$w = e^{-i(pt+kz)}\,e^{-i(\hat y-\phi)}\,\cos\frac{\theta}{2}\,e^{-kr}\,\frac{\sqrt{r}}{Q+r} \tag{3.47}$$

where

$$p = \frac{a}{\bar Q} = \frac{1}{2Q} \tag{3.48}$$

4.4 The perturbation equations

The perturbation we seek carries one unit of momentum along y and is BPS: this fixes the t and y dependence to be of the form $e^{-ip(t+y)}$. We also allow for a generic wave number k along the ring direction z; sometimes we will find it convenient to write this wave number as $k = \kappa/(2Q)$. The perturbation fields then have the form

$$B^{(2)}_{MN} = e^{-ip(t+y)-ikz}\,\tilde B^{(2)}_{MN}(r,\theta,\phi)\,,\quad w = e^{-ip(t+y)-ikz}\,\tilde w(r,\theta,\phi) \tag{4.49}$$

4.4.1 Reducing to equations on TN

In this subsection we reduce the equations (2.12) into a system of equations for a set of p-forms on TN. Indices on TN are denoted by $i, j, \ldots$. Here and in the following d, Δ and $\star$ are the differential, scalar laplacian and Hodge dual on TN. The 2-form $B^{(2)}$ reduces to a 2-form on TN denoted by B, two 1-forms a and b and a scalar Φ:

$$B^{(2)}_{ij} = B_{ij}\,, \quad B^{(2)}_{it} = a_i\,, \quad B^{(2)}_{iz} = b_i\,, \quad B^{(2)}_{tz} = \Phi \tag{4.50}$$

Let $f^{(a)}$ and $f^{(b)}$ be the field strengths of a and b:

$$f^{(a)}_{ij} = \partial_i a_j - \partial_j a_i\,, \quad f^{(b)}_{ij} = \partial_i b_j - \partial_j b_i \tag{4.51}$$

One has the identities

$$F^{(3)}_{ijt} = f^{(a)}_{ij} - ipB_{ij}\,, \quad F^{(3)}_{ijz} = f^{(b)}_{ij} - ikB_{ij}\,, \quad F^{(3)}_{itz} = \partial_i\Phi - ika_i + ipb_i \tag{4.52}$$

We will need the following relations

$$\begin{aligned} &g^{tt} = -H + \frac{Q^2}{Hr^2}\,, \quad g^{tz} = -\frac{Q}{Hr}\,, \quad g^{zz} = \frac{1}{H} \\ &g^{zz} - g^{tz} = 1\,, \quad g^{tt} - g^{tz} = -1 \end{aligned} \tag{4.53}$$

By virtue of these relations we can rewrite the 6D laplacian, acting on w, as

$$\Delta_6 w = \Delta w - (p^2\, g^{tt} + k^2\, g^{zz} + 2pk\, g^{tz})w = \Delta w - (p+k)(pg^{tt} + kg^{zz})w \tag{4.54}$$

We also find

$$H^{(3)\,ijt} = H^{(3)\,ijz} = -\Omega^{(2)\,ij} \tag{4.55}$$

Using (4.52), (4.54) and (4.55), it is easy to see that the equations (2.12) reduce to the following system of equations:

$$\Delta w - (p+k)(pg^{tt} + kg^{zz})w + \Omega^{(2)\,ij}(f^{(a)}_{ij} + f^{(b)}_{ij} - i(p+k)B_{ij}) = 0 \quad (4.56)$$

$$f^{(a)} - ipB - g^{zz} \star (f^{(b)} - ikB) - g^{tz} \star (f^{(a)} - ipB) + w\Omega^{(2)} = 0 \quad (4.57)$$

$$f^{(b)} - ikB + g^{tt} \star (f^{(a)} - ipB) + g^{tz} \star (f^{(b)} - ikB) - w\Omega^{(2)} = 0 \quad (4.58)$$

$$d\Phi - ika + ipb - \star dB = 0 \quad (4.59)$$

If we take the sum of eq. (4.57) and eq. (4.58), use (4.53), and define

$$K = f^{(a)} + f^{(b)} - i(p+k)B \quad (4.60)$$

we find

$$K = \star K \quad (4.61)$$

i.e. K is a self-dual 2-form on TN. Applying d to eq. (4.59) leads to

$$pf^{(b)} - kf^{(a)} = -id \star dB = \frac{d \star dK}{p+k} \quad (4.62)$$

Taking p times eq.(4.58) minus k times eq.(4.57), and using again (4.53), gives

$$pf^{(b)} - kf^{(a)} + \star(pf^{(b)} - kf^{(a)}) + (pg^{tt} + kg^{zz})\,K - (p+k)w\,\Omega^{(2)} = 0 \quad (4.63)$$

and thus, by virtue of (4.62) and (4.61),

$$\Delta K + (p+k)(pg^{tt} + kg^{zz})\,K - (p+k)^2 w\,\Omega^{(2)} = 0 \quad (4.64)$$

where

$$\Delta K = d \star d \star K + \star d \star dK = d \star dK + \star d \star dK \quad (4.65)$$

is the TN laplacian acting on the 2-form K. Eq.(4.56) can also be rewritten in form language via the identity

$$\Omega^{(2)\,ij} K_{ij} = 2 \star (\star\Omega^{(2)} \wedge K) = 2 \star (\Omega^{(2)} \wedge K) \quad (4.66)$$

4.4.2 The equations to be solved

With all this, we have reduced the system (4.56-4.59) to a coupled system of equations for a self-dual 2-form K and scalar w:

$$\Delta w - (p+k)(pg^{tt} + kg^{zz})w + 2 \star (\Omega^{(2)} \wedge K) = 0 \tag{4.67}$$

$$\Delta K + (p+k)(pg^{tt} + kg^{zz})\, K - (p+k)^2 w\, \Omega^{(2)} = 0 \tag{4.68}$$

Moreover, the definition of K (4.60) and eq.(4.59) imply the relations

$$K = f^{(a)} + f^{(b)} - i(p+k)B\,, \qquad \frac{i}{(p+k)} \star dK = d\Phi - ika + ipb \tag{4.69}$$

If K is known, these relations determine B, a, b and Φ, up to gauge transformations.

4.5 Harmonics on Taub-NUT

We would like to solve the above equations by expanding functions on the Taub-NUT space in harmonics on the $(\theta, \phi, \hat{y})$ space. At the core of the Taub-NUT (i.e. at $r \approx 0$) this angular space has the geometry of a 'round' S^3, but for larger r we get a 'squashed sphere'. Forms on the squashed sphere can be expanded in 'monopole spherical harmonics', which have been widely studied; see for example [130]. We will however find it more convenient to develop this expansion in our own notation, in a way that relates it closely to the expansion used for the round sphere in [123].

4.5.1 Symmetries of Taub-NUT

Let us start with the metric on the round sphere

$$ds^2_{S^3} = d\tilde{\theta}^2 + \cos^2\tilde{\theta} d\tilde{\psi}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2 \tag{5.70}$$

The symmetry group is $SO(4) \approx SU(2) \times SU(2)$. We write the elements of $SU(2)$ as $e^{\alpha_a J_a}$, with the antihermitian generators J_a satisfying $[J_a, J_b] = -\epsilon_{abc} J_c$. Writing

$J_\pm = J_1 \pm iJ_2$, we get $[J_3, J_\pm] = \pm iJ_\pm$, $[J_+, J_-] = 2iJ_3$. For the first $SU(2)$ the generators are

$$\begin{aligned}
J_+ &= \frac{1}{2}e^{-i(\tilde{\psi}+\tilde{\phi})}[\partial_{\tilde{\theta}} - i\cot\tilde{\theta}\partial_{\tilde{\phi}} + i\tan\tilde{\theta}\partial_{\tilde{\psi}}] \\
J_- &= \frac{1}{2}e^{i(\tilde{\psi}+\tilde{\phi})}[\partial_{\tilde{\theta}} + i\cot\tilde{\theta}\partial_{\tilde{\phi}} - i\tan\tilde{\theta}\partial_{\tilde{\psi}}] \\
J_3 &= -\frac{1}{2}[\partial_{\tilde{\psi}} + \partial_{\tilde{\phi}}]
\end{aligned} \tag{5.71}$$

and for the second $SU(2)$ they are

$$\begin{aligned}
\bar{J}_+ &= \frac{1}{2}e^{i(\tilde{\psi}-\tilde{\phi})}[\partial_{\tilde{\theta}} - i\cot\tilde{\theta}\partial_{\tilde{\phi}} - i\tan\tilde{\theta}\partial_{\tilde{\psi}}] \\
\bar{J}_- &= \frac{1}{2}e^{-i(\tilde{\psi}-\tilde{\phi})}[\partial_{\tilde{\theta}} + i\cot\tilde{\theta}\partial_{\tilde{\phi}} + i\tan\tilde{\theta}\partial_{\tilde{\psi}}] \\
\bar{J}_3 &= \frac{1}{2}[\partial_{\tilde{\psi}} - \partial_{\tilde{\phi}}]
\end{aligned} \tag{5.72}$$

To relate these generators to Taub-NUT we write the metric (5.70) for the round S^3 in different coordinates. Thus define

$$\theta = 2\tilde{\theta}, \quad \hat{y} \equiv \tilde{\phi}, \quad \phi = \tilde{\phi} - \tilde{\psi} \tag{5.73}$$

This gives

$$ds^2_{S^3} = \frac{1}{4}d\theta^2 + \frac{1}{4}\sin^2\theta d\phi^2 + \left[d\hat{y} - \frac{1}{2}(1+\cos\theta)d\phi\right]^2 \tag{5.74}$$

The generators (2.27),(5.72) become

$$\begin{aligned}
J_+ &= \frac{1}{2}e^{-i(2\hat{y}-\phi)}[2\partial_\theta - i(\cot\frac{\theta}{2} + \tan\frac{\theta}{2})\partial_\phi - i\cot\frac{\theta}{2}\partial_{\hat{y}}] \\
J_- &= \frac{1}{2}e^{i(2\hat{y}-\phi)}[2\partial_\theta + i(\cot\frac{\theta}{2} + \tan\frac{\theta}{2})\partial_\phi + i\cot\frac{\theta}{2}\partial_{\hat{y}}] \\
J_3 &= -\frac{1}{2}\partial_{\hat{y}}
\end{aligned} \tag{5.75}$$

$$\begin{aligned}
\bar{J}_+ &= \frac{1}{2}e^{-i\phi}[2\partial_\theta - i(\cot\frac{\theta}{2} - \tan\frac{\theta}{2})\partial_\phi - i\cot\frac{\theta}{2}\partial_{\hat{y}}] \\
\bar{J}_- &= \frac{1}{2}e^{i\phi}[2\partial_\theta + i(\cot\frac{\theta}{2} - \tan\frac{\theta}{2})\partial_\phi + i\cot\frac{\theta}{2}\partial_{\hat{y}}] \\
\bar{J}_3 &= -\frac{1}{2}[\partial_{\hat{y}} + 2\partial_\phi]
\end{aligned} \tag{5.76}$$

In the Taub-NUT metric if we fix r then we get a 3-dimensional surface with metric of the form

$$ds^2 = A(d\theta^2 + \sin^2\theta d\phi^2) + 4B\Big[d\hat{y} - \frac{1}{2}(1+\cos\theta)d\phi\Big]^2 \tag{5.77}$$

At the center of Taub-NUT we get $A = B$, and the metric becomes that of a round S^3. For larger r we have $A \neq B$, and this gives the squashed sphere. We can now check that the vector fields (5.76) are Killing vectors of (5.77), for all A, B. But out of the vector fields (5.75) only J_3 is a Killing vector if $A \neq B$. Thus the $SU(2) \times SU(2)$ symmetry of the round sphere is broken to $U(1) \times SU(2)$.

4.5.2 Harmonics on the squashed sphere

On the round sphere we can expand any form in spherical harmonics, which are characterized by quantum numbers $(j, m), (j', m')$ in the two $SU(2)$ factors. On the squashed sphere, we can use the *same* functions, in the following sense. We take the map from the squashed S^3 to the round S^3 which sends each point $(\theta, \phi, \hat{y})$ on the former to the point with the same coordinates on the latter. The harmonics on the round S^3 then give harmonics on the squashed S^3 via the pullback under this map. These pulled back harmonics can be used to expand any form on the squashed sphere, though the harmonics are not orthogonal to each other as they were on the round sphere.

The quantum numbers m and (j', m') correspond to symmetries of the squashed sphere, and so are 'good' quantum numbers in the sense that we can restrict all terms in an equation to have the same values of these numbers. On the other hand a form of order p on the round S^3 was characterized by *four* quantum numbers, $(j, m), (j', m')$. In the latter case the quantum numbers uniquely specify the form. A form on the squashed sphere will therefore be a sum

$$\omega_{(m,j',m')} = \sum_j C_j\, \omega_{(j,m),(j',m')} \tag{5.78}$$

It turns out however that if ω is a p-form then for its harmonics on the round sphere we must have $|j - j'| \leq p$. This tells us that the sum in (5.78) will be a finite one, and this makes the expansion in harmonics useful for the squashed sphere.

As an application of this approach consider the scalar w in (3.47): its angular dependence is captured by the function

$$\omega_0 = e^{-i(\hat{y}-\phi)}\, \cos\frac{\theta}{2} = e^{-i\tilde{\psi}}\, \cos\tilde{\theta} \tag{5.79}$$

All scalars on S^3 have quantum numbers $(j,m),(j,m')$; i.e. $j = j'$. From the $\tilde{\psi}$ dependence of (5.79) we find that $m = \frac{1}{2}, m' = -\frac{1}{2}$. The lowest j this can come from is $j = \frac{1}{2}$, so we look at the scalar spherical harmonic on the round S^3 given by the quantum numbers $(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})$. Such harmonics were given explicitly in [123], and we find that indeed the function (5.79) is proportional to the required scalar harmonic.

Now consider 1-forms. On the round S^3, there are two kinds of 1-forms. The first kind are obtained by just applying d to the scalar harmonics, so these have quantum numbers $(j,m),(j,m')$. The second kind have $j - j' = \pm 1$, so they come in two varieties: with quantum numbers $(j+1,m),(j,m')$, and $(j,m),(j+1,m')$. Let us examine these 1-forms for our problem.

Since m, j', m' are good quantum numbers for the problem these must be the same for the 1-forms as for the scalar w. Thus the first kind of 1-form must be

$$d\omega_0 = -e^{-i(\hat{y}-\phi)}\Big[\frac{1}{2}\sin\frac{\theta}{2}d\theta + i\cos\frac{\theta}{2}(d\hat{y} - d\phi)\Big] \tag{5.80}$$

For the second kind of 1-form we find only one set of quantum numbers that are consistent with the given m, j', m': the set $(\frac{3}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})$. The corresponding harmonic was constructed in [123]

$$\tilde{\omega}_1 = e^{-i\tilde{\psi}}\,[\sin\tilde{\theta}\,d\tilde{\theta} - i\cos\tilde{\theta}(3\cos^2\tilde{\theta} - 1)d\tilde{\psi} - 3i\cos\tilde{\theta}\sin^2\tilde{\theta}d\tilde{\phi}] \tag{5.81}$$

In the coordinates $(\theta, \hat{y}, \phi)$ this is

$$\tilde{\omega}_1 = e^{-i(\hat{y}-\phi)}\Big[\frac{1}{2}\sin\frac{\theta}{2}d\theta + i\cos\frac{\theta}{2}(3\cos^2\frac{\theta}{2} - 1)d\phi - 2i\cos\frac{\theta}{2}d\hat{y}\Big] \tag{5.82}$$

4.5.3 Decomposing along base and fiber

At this stage we may think of expanding the angular components of our 1-forms using (5.80) and (5.82), and the dr component using the scalar harmonic ω_0. But actually we can do better, by exploiting the spherical symmetry of the background in the r, θ, ϕ space. The Taub-NUT has such a spherical symmetry, though any choice of coordinates prevents this symmetry from being manifest.

Consider the squashed sphere at any r. This space can be regarded as a S^1 fiber (parameterized by $\hat{y}$) over a S^2 base (parametrized by θ, ϕ). We can geometrically identify the fiber direction over any point, and thus also the 2-plane orthogonal to the fiber. We can thus decompose any 1-form into two parts: $\omega_1 = \alpha + \beta$. The part α will have no component along the base; thus $\langle v, \alpha\rangle = 0$ for all v perpendicular to $\partial_{\hat{y}}$. The part β will have no component along the fiber; thus $\langle \partial_{\hat{y}}, \beta\rangle = 0$. We find that α must be proportional to

$$\hat{\sigma} = d\hat{y} - \frac{1+\cos\theta}{2}d\phi \tag{5.83}$$

while β is just characterized by having no term proportional to $d\hat{y}$.

Let us now apply this decomposition to our 1-form. The part α can be written as $\alpha = f\hat{\sigma}$ where f is a function on the squashed sphere. This function must carry the quantum numbers $(m', j, m) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. Since f is a scalar it must have $j = j'$ and so it must actually be proportional to the scalar harmonic that is the pullback of $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})$, which is just ω_0. So we find that $\alpha = f_0(r)\omega_0\hat{\sigma}$. To find β we take the linear combination of (5.80), (5.82) which has no component $d\hat{y}$. A conveniently normalized choice for this combination is

$$\omega_1 = -\frac{4}{3}\Big[d\omega_0 - \frac{1}{2}\tilde{\omega}_1\Big] = e^{-i(\hat{y}-\phi)} \sin\frac{\theta}{2}[d\theta - i\sin\theta d\phi] \tag{5.84}$$

To summarize, our 1-form must have the form $f_0(r)\omega_0\hat{\sigma} + f_1(r)\omega_1$.

4.5.4 Some relations on forms

A decomposition of the type $\omega_1 = \alpha + \beta$ which we did for 1-forms can be done for any p-form ω on the squashed 3-sphere. The $\hat{y}$ dependence of our forms is $e^{-i\hat{y}}$. Using this fact we find

$$d\omega = -i\hat{\sigma} \wedge \omega + D\omega \tag{5.85}$$

where

$$D\omega \equiv d_2\omega - i\frac{1+\cos\theta}{2}d\phi \wedge \omega \tag{5.86}$$

is the covariant derivative of ω and d_2 denotes the differential with respect to θ, ϕ. The square of D is proportional to the monopole field strength

$$D^2\omega = \Big(i\frac{\sin\theta}{2}\, d\theta \wedge d\phi\Big) \wedge \omega \tag{5.87}$$

If we denote by $\star_2$ the Hodge dual with respect to the S^2 metric,[18] the monopole harmonics ω_0 and ω_1 satisfy

$$\begin{aligned}
\star_2\omega_1 &= -i\omega_1 \\
D\omega_0 &= -\frac{1}{2}\,\omega_1 \\
D^2\omega_0 &= i\,\frac{\sin\theta}{2}\,\omega_0\,d\theta\wedge d\phi \\
D\omega_1 &= -2D^2\omega_0 = -i\sin\theta\,\omega_0\,d\theta\wedge d\phi
\end{aligned} \tag{5.88}$$

4.5.5 The 2-form K

We can use the structure above to write a general ansatz for the 2-form K. Any self-dual 2-form on TN can be written as

$$dr\wedge\tilde{\omega}+\star(dr\wedge\tilde{\omega}) \tag{5.89}$$

where $\tilde{\omega}$ is a 1-form on TN. The form dr has all angular quantum numbers zero, so the quantum numbers of the perturbation must be carried by $\tilde{\omega}$. But we have seen in section (4.5.3) that any such 1-form, with quantum numbers $(m', j, m) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, is of the form

$$\tilde{\omega} = f_0(r)\,\omega_0\hat{\sigma} + f_1(r)\,\omega_1 \tag{5.90}$$

The 2-form K is self-dual, as a form on TN, and depends on t and z as in (4.49): it can thus be written as

$$K = e^{-i(t+\kappa z)/(2Q)}\,(K_0+K_1) \tag{5.91}$$

where the K_0, K_1 parts correspond to the first and second parts on the RHS of (5.90)

$$\begin{aligned}
K_0 &= f_0(r)\,\omega_0[dr\wedge\hat{\sigma}+\star(dr\wedge\hat{\sigma})] = f_0(r)\,\omega_0\left[dr\wedge\hat{\sigma}+\frac{Hr^2}{Q}\,d\hat{\sigma}\right] \\
K_1 &= f_1(r)\,[dr\wedge\omega_1+\star(dr\wedge\omega_1)] = f_1(r)\left[dr\wedge\omega_1 - i\frac{2Q}{H}\,\hat{\sigma}\wedge\omega_1\right]
\end{aligned} \tag{5.92}$$

[18]Note that $(\star_2)^2 = -1$ on 1-forms. We have $\epsilon_{\theta\phi} = 1$.

Note that we have used only the scalar ω_0 and the 1-form ω_1 in our expansion, and avoided a separate coefficient function for the 1-form $d\omega_0$

4.6 The radial equations and their solution

The ansatz (6.136) reduces the unknowns to two functions of r: $f_0(r)$ and $f_1(r)$. In this section we will derive the system of differential equations these functions have to satisfy, and then see how they are solved.

4.6.1 Obtaining the radial equations

Let us start from eq. (4.67). We note that

$$\Omega^{(2)} \wedge K_1 = 0 \tag{6.93}$$

and that, by comparing (6.136) with (3.38),

$$K_0 = -f_0\,\omega_0\,\frac{(rH)^2}{2Q^2}\,\Omega^{(2)} \tag{6.94}$$

Thus

$$\star(\Omega^{(2)} \wedge K) = -e^{-i(t+\kappa z)/(2Q)}\,f_0\,\omega_0\,\frac{(rH)^2}{2Q^2}\,\star(\Omega^{(2)} \wedge \Omega^{(2)}) \tag{6.95}$$

An easy computation gives

$$\Omega^{(2)} \wedge \Omega^{(2)} = \frac{4Q^3}{H^3 r^2}\,\sin\theta\,dr \wedge d\theta \wedge d\phi \wedge \hat{\sigma} \tag{6.96}$$

and

$$\star(\Omega^{(2)} \wedge \Omega^{(2)}) = \frac{2Q^2}{(Hr)^4} \tag{6.97}$$

Using this in (6.95) we find

$$\star(\Omega^{(2)} \wedge K) = -e^{-i(t+\kappa z)/(2Q)}\,f_0\,\omega_0\,\frac{1}{(Hr)^2} \tag{6.98}$$

Using the expression for w in (3.47), it is also straightforward to compute (for example with the help of Mathematica)

$$\Delta w - (p+k)(pg^{tt} + kg^{zz})w = -e^{-i(t+\kappa z)/(2Q)}\, e^{-\kappa r/(2Q)}\, \frac{\sqrt{r}}{(Hr)^4}\, [(3+\kappa)Q + (1+\kappa)r]\, \omega_0 \tag{6.99}$$

Using (6.98) and (6.99), we see that equation (4.67) becomes

$$e^{-\kappa r/(2Q)}\, \frac{\sqrt{r}}{(Hr)^2}\, [(3+\kappa)Q + (1+\kappa)r] + 2f_0 = 0 \tag{6.100}$$

Let us now turn to eq. (4.68). We first need to compute

$$\begin{aligned} \Delta K_a &= \star d \star dK_a + d \star dK_a \\ &= -\Big(\nabla^k \nabla_k K_{a\,ij} + [\nabla^k, \nabla_i] K_{a\,jk} - [\nabla^k, \nabla_j] K_{a\,ik}\Big)\, dx^i \wedge dx^j \end{aligned} \tag{6.101}$$

for $a = 0, 1$. The covariant derivatives and index contractions in the second line of (6.101) are done with the TN metric. A lengthy but straightforward computation, that makes use of identities (5.88), leads to

$$\begin{aligned} \Delta K_0 &= -\omega_0\, [dr \wedge \hat{\sigma} + \star(dr \wedge \hat{\sigma})] \Big[\frac{f_0''}{H} + 2\frac{f_0'}{Hr} - \frac{r^4 + 4Qr^3 + 16Q^2r^2 - 8Q^3 r + 3Q^4}{4Q^2 r(Q+r)^3}\, f_0\Big] \\ &\quad - \frac{i}{2Q}\, [dr \wedge \omega_1 + \star(dr \wedge \omega_1)]\, \frac{f_0}{Hr} \\ \Delta K_1 &= -[dr \wedge \omega_1 + \star(dr \wedge \omega_1)] \Big[\frac{f_1''}{H} + 2\frac{f_1' Q}{(Hr)^2} - \frac{r^4 + 4Qr^3 + 8Q^2r^2 + 16Q^3 r + 3Q^4}{4Q^2 r(Q+r)^3}\, f_1\Big] \\ &\quad + i4Q\, [dr \wedge \hat{\sigma} + \star(dr \wedge \hat{\sigma})]\, \frac{f_1}{(Hr)^3} \end{aligned} \tag{6.102}$$

The full wave operator is

$$\Delta_6 K_a \equiv \Delta K_a + (p+k)(pg^{tt} + kg^{zz})\, K_a\,,\;\; a = 0, 1 \tag{6.103}$$

and we find

$$\begin{aligned}
\Delta_6 K_0 &= -\omega_0\,[dr\wedge\hat\sigma+\star(dr\wedge\hat\sigma)]\Big[\frac{f_0''}{H}+2\frac{f_0'}{Hr}-\Big(\frac{r^4+4Qr^3+16Q^2r^2-8Q^3r+3Q^4}{r(Hr)^3}\\
&\quad -(1+\kappa)\,\frac{2Q+(1-\kappa)r}{Hr}\Big)\,\frac{f_0}{(2Q)^2}\Big]-\frac{i}{2Q}\,[dr\wedge\omega_1+\star(dr\wedge\omega_1)]\,\frac{f_0}{Hr}\\
\Delta_6 K_1 &= -[dr\wedge\omega_1+\star(dr\wedge\omega_1)]\Big[\frac{f_1''}{H}+2\frac{f_1'Q}{(Hr)^2}-\Big(\frac{r^4+4Qr^3+8Q^2r^2+16Q^3r+3Q^4}{r(Hr)^3}\\
&\quad -(1+\kappa)\,\frac{2Q+(1-\kappa)r}{Hr}\Big)\,\frac{f_1}{(2Q)^2}\Big]+i4Q\,[dr\wedge\hat\sigma+\star(dr\wedge\hat\sigma)]\,\frac{f_1}{(Hr)^3}
\end{aligned} \tag{6.104}$$

In (4.68) the first two terms constitute Δ_6; thus the last term will act as a 'source term' for this Laplacian. The source term is

$$-(p+k)^2\,w\Omega^{(2)}=\frac{(1+\kappa)^2}{2}\,e^{-i(t+\kappa z)/(2Q)}\,e^{-\kappa r/(2Q)}\frac{\sqrt{r}}{(Hr)^3}\,\omega_0\,[dr\wedge\hat\sigma+\star(dr\wedge\hat\sigma)] \tag{6.105}$$

Collecting the terms proportional to $\omega_0\,[dr\wedge\hat\sigma+\star(dr\wedge\hat\sigma)]$ and to $[dr\wedge\omega_1+\star(dr\wedge\omega_1)]$ in eq. (4.68), we find the following system of equations for f_0 and f_1:

$$\begin{aligned}
&\frac{f_0''}{H}+2\frac{f_0'}{Hr}-\Big(\frac{r^4+4Qr^3+16Q^2r^2-8Q^3r+3Q^4}{r(Hr)^3}-(1+\kappa)\frac{2Q+(1-\kappa)r}{Hr}\Big)\,\frac{f_0}{(2Q)^2}\\
&\qquad -i\,4Q\,\frac{f_1}{(Hr)^3}-\frac{(1+\kappa)^2}{2}\,e^{-\kappa r/(2Q)}\,\frac{\sqrt{r}}{(Hr)^3}=0
\end{aligned} \tag{6.106}$$

$$\begin{aligned}
&\frac{f_1''}{H}+2\frac{f_1'Q}{(Hr)^2}-\Big(\frac{r^4+4Qr^3+8Q^2r^2+16Q^3r+3Q^4}{r(Hr)^3}-(1+\kappa)\frac{2Q+(1-\kappa)r}{Hr}\Big)\,\frac{f_1}{(2Q)^2}\\
&\qquad +\frac{i}{2Q}\,\frac{f_0}{(Hr)}=0
\end{aligned} \tag{6.107}$$

4.6.2 Solving the radial equations

Eq. (6.100) can be readily solved for f_0, giving:

$$f_0=-e^{-kr}\,\frac{(3+\kappa)Q+(1+\kappa)r}{2}\,\frac{\sqrt{r}}{(Hr)^2} \tag{6.108}$$

By substituting f_0 into (6.106) we can derive f_1. With the help of Mathematica we compute:

$$\frac{f_0''}{H} + 2\frac{f_0'}{Hr} - \Big(\frac{r^4 + 4Qr^3 + 16Q^2r^2 - 8Q^3r + 3Q^4}{r(Hr)^3} - (1+\kappa)\frac{2Q + (1-\kappa)r}{Hr}\Big)\frac{f_0}{(2Q)^2}$$
$$= e^{-kr}\,\frac{2 + (1+\kappa)^2}{2}\,\frac{\sqrt{r}}{(Hr)^3} \qquad (6.109)$$

and thus, from (6.106), we obtain

$$f_1 = -\frac{i}{4Q}\,e^{-kr}\,\sqrt{r} \qquad (6.110)$$

Eq. (6.107) is a consistency condition for our previously determined values of f_0 and f_1. By Mathematica we compute

$$\frac{f_1''}{H} + 2\frac{f_1'Q}{(Hr)^2} - \Big(\frac{r^4 + 4Qr^3 + 8Q^2r^2 + 16Q^3r + 3Q^4}{r(Hr)^3} - (1+\kappa)\frac{2Q + (1-\kappa)r}{Hr}\Big)\frac{f_1}{(2Q)^2}$$
$$= \frac{i}{4Q}\,e^{-kr}\,[(3+\kappa)Q + (1+\kappa)r]\,\frac{\sqrt{r}}{(Hr)^3} \qquad (6.111)$$

Substituting this in (6.107) and using (6.108), we see that (6.107) is satisfied.

To summarize, we have found the solution

$$w = e^{-i(pt+kz)}\,e^{-kr}\,\frac{\sqrt{r}}{Q+r}\,\omega_0 \qquad (6.112)$$
$$K = e^{-i(pt+kz)}\,(K_0 + K_1)$$
$$K_0 = -e^{-kr}\,\frac{\sqrt{r}}{2(Q+r)^2}[(3Q+r) + \kappa(Q+r)]\omega_0\Big[dr \wedge \hat{\sigma} + \frac{Hr^2}{2Q}\,\sin\theta d\theta \wedge d\phi\Big]$$
$$K_1 = -ie^{-kr}\,\frac{\sqrt{r}}{4Q}\Big[dr \wedge \omega_1 - i\frac{2Q}{H}\,\hat{\sigma} \wedge \omega_1\Big] \qquad (6.113)$$

Knowing K we can derive the values of a, b, B and Φ. Before we do this we discuss the gauge invariance of our problem.

Gauge invariance

One can easily check, using the identities (4.52), that the following two gauge transformations leave all the components of $F^{(3)}$ invariant:

$$B \to B + d\lambda^{(1)}\,, \quad a \to a + ip\,\lambda^{(1)}\,, \quad b \to b + ik\,\lambda^{(1)} \tag{6.114}$$

$$\Phi \to \Phi + \lambda^{(0)}\,, \quad a \to a + d\lambda_a^{(0)}\,, \quad b \to b + d\lambda_b^{(0)} \quad \text{with} \quad \lambda^{(0)} - ik\,\lambda_a^{(0)} + ip\,\lambda_b^{(0)} = 0 \tag{6.115}$$

Here $\lambda^{(1)}$ is a 1-form and $\lambda^{(0)}$, $\lambda_a^{(0)}$ and $\lambda_b^{(0)}$ are 0-forms on TN. The 2-form K is gauge invariant.

Deriving the gauge fields

By making use of the transformation (6.115), we can set $\Phi = 0$. Then the second equation in (4.69) implies

$$b - \kappa a = (2Q)^2 \frac{\star dK}{1+\kappa} \tag{6.116}$$

One can compute

$$\frac{\star dK}{1+\kappa} = e^{-i(pt+kz)}\, e^{-kr}\, \sqrt{r}\left[-i\frac{1}{8Q^2}\,\omega_1 + i\frac{1}{8Q^2\,r}\,\omega_0\,dr + \frac{1}{4(Q+r)^2}\left(1-\kappa\frac{r}{Q}\right)\omega_0\,\hat\sigma\right] \tag{6.117}$$

Then a solution of (6.116) for a and b is

$$\begin{aligned} a &= e^{-i(pt+kz)}\, e^{-kr}\, \sqrt{r}\,\frac{Qr}{(Q+r)^2}\,\omega_0\,\hat\sigma \\ b &= e^{-i(pt+kz)}\, e^{-kr}\, \sqrt{r}\left[\frac{i}{2r}\omega_0\,dr - \frac{i}{2}\,\omega_1 + \frac{Q^2}{(Q+r)^2}\,\omega_0\,\hat\sigma\right] \end{aligned} \tag{6.118}$$

By picking this solution we have fixed the gauge freedom implied by the transformation (6.114).

Substituting these values of a and b into the first equation in (4.69) we derive B

$$B = e^{-i(pt+kz)}\, e^{-kr}\, \sqrt{r}\left[\frac{1}{2}dr\wedge\omega_1 - \frac{ir}{2}\,\omega_0\,\sin\theta d\theta\wedge d\phi\right] \tag{6.119}$$

4.7 Regularity of the solution

We show that the solution given in (6.112,4.27,6.119) is both regular and normalizable.

4.7.1 Normalizability

For $k > 0$ normalizability is guaranteed by the exponential fall off e^{-kr}. Note however that waves with $k \leq 0$ give rise to non-normalizable perturbations. This is obvious for $k < 0$. For $k = 0$ let us look, for example, at the scalar w: its large r behavior is

$$w \approx e^{-ipt} \frac{1}{\sqrt{r}} \omega_0 \tag{7.120}$$

and thus

$$|w|^2 \sim 1/r \tag{7.121}$$

Since the volume element of the space transverse to the ring grows as $\sim r^2 dr$ for large r, the norm of w is quadratically divergent at $r \to \infty$. This shows that for $k = 0$ the perturbation leaks out to the center of the ring ($r \sim a$) and does not stay confined to the vicinity of the tube.[19] For $k > 0$ the wavefunction becomes confined closer to the ring, and in the limit (3.30) we find a normalizable solution in the near ring limit.

The fact that positive and negative k behave differently is to be expected; the 2-charge background does not have the symmetry $z \leftrightarrow -z$. In the NS1-P picture the geometry is created by a string carrying a wave, and the strands of the string carry momentum along the ring, thus breaking the $z \leftrightarrow -z$ symmetry. In [11] it was found that there are 'left-moving' non-BPS perturbations that move in one direction along

[19] If we construct the exact wavefunction for the ring (instead of just constructing it for the near ring limit) then we expect to have a solution normalizable at spatial infinity, since the state exists in the dual CFT.

the ring, while 'right-moving' perturbations create time independent distortions of the 2-charge geometry. For our present problem note that in w_{full} (eq. (2.25)) we have $l \geq \frac{1}{2}$, so we must have $k \geq 0$, and negative values of k do not appear.[20] We need to take large $|k|$ to be able to use the 'straight segment' limit of the ring so the case $k = 0$ is not relevant for our discussion, and we naturally find ourselves at large positive k.

4.7.2 Regularity at $\theta = 0, \pi$

Let us now consider the regularity of the solution. The fields w, a, b and B are manifestly regular away from the points where our system of coordinates degenerates. This degeneration happens at $\theta = 0$ or π and at $r = 0$. Around $\theta = 0, \pi$ it is convenient to change to S^3 coordinates (5.73). From the expression of ω_0 in (5.79) it is apparent that ω_0 is regular: indeed for $\tilde{\theta} = \pi/2$, where the $\tilde{\psi}$ coordinate degenerates, the coefficient of $e^{-i\tilde{\psi}}$ vanishes. The second identity in (5.88) expresses ω_1 as the covariant derivative of ω_0, and thus ω_1 is regular too. The 1-form along the fiber $\hat{\sigma}$ can be expressed in S^3 coordinates as

$$\hat{\sigma} = \sin^2 \tilde{\theta}\, d\tilde{\phi} + \cos^2 \tilde{\theta}\, d\tilde{\psi} \tag{7.122}$$

which is also regular. Since the angular dependence of w, a, b and B is entirely expressed in terms of ω_0, ω_1 and $\hat{\sigma}$, this proves that our solution is regular at $\theta = 0, \pi$.

[20] Spherical harmonics for the scalar have $l = 0, \frac{1}{2}, 1, \ldots$, but for $l = 0$ we get zero if we apply J_0^-, and so we cannot construct the required perturbation of section (4.2.4).

4.7.3 Regularity at $r = 0$

At $r \to 0$ the TN space becomes flat $\mathbb{R}^4$: the change of coordinates that brings the TN metric into explicitly flat form is (5.73) for the angular variables and

$$r = \frac{\rho^2}{4Q} \tag{7.123}$$

for the radial coordinate. In these coordinates the $r \to 0$ limit of w is

$$w \sim \rho\, e^{-i\tilde{\psi}} \cos\tilde{\theta} = x_1 - i x_2 \tag{7.124}$$

where x_i, with $i = 1, \ldots, 4$ are Cartesian coordinates[21] in $\mathbb{R}^4$. This shows that w is regular at $r \to 0$. Similarly, the gauge fields a and B behave like

$$\begin{aligned} a &\sim \rho^3\, e^{-i\tilde{\psi}} \cos\tilde{\theta}\, (\sin^2\tilde{\theta}\, d\tilde{\phi} + \cos^2\tilde{\theta}\, d\tilde{\psi}) = (x_1 - i x_2)\, [x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3] \\ B &\sim -i\rho^2\, e^{-i\tilde{\psi}} \sin\tilde{\theta}\, [i d\rho \wedge d\tilde{\theta} + \cos\tilde{\theta}(\sin\tilde{\theta} d\rho + \rho\cos\tilde{\theta} d\tilde{\theta}) \wedge (d\tilde{\phi} - d\tilde{\psi})] \\ &= -i[(x_1 - i x_2)(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) - i \sum_i x_i dx_i \wedge (dx_1 - i dx_2)] \end{aligned} \tag{7.126}$$

and are hence regular. Regularity of b is not manifest in the form in which it appears in (4.27). This form was derived after making the arbitrary gauge choice $\Phi = 0$. By using the transformation (6.115) we can change Φ and b and write them in an explicitly smooth form; since a was already shown to be smooth, we can take $\lambda_a^{(0)} = 0$ in (6.115) and leave it unchanged. If we choose

$$\lambda^{(0)} = -e^{-i(pt+kz)}\, e^{-kr}\, \frac{\sqrt{r}}{2Q}\, \omega_0\,, \quad \lambda_b^{(0)} = \frac{i}{p}\, \lambda^{(0)} \tag{7.127}$$

[21]Explicitly,

$$x_1 + i x_2 = \rho\, e^{i\tilde{\psi}} \cos\tilde{\theta}\,, \quad x_3 + i x_4 = \rho\, e^{i\tilde{\phi}} \sin\tilde{\theta} \tag{7.125}$$

in (6.115), then the fields Φ and b are changed into

$$\begin{aligned}\Phi &= -e^{-i(pt+kz)}\, e^{-kr}\, \frac{\sqrt{r}}{2Q}\, \omega_0 \\ b &= e^{-i(pt+kz)}\, e^{-kr}\, \sqrt{r}\, \Big[ik\, \omega_0\, dr + \Big(\frac{Q^2}{(Q+r)^2} - 1\Big)\, \omega_0\, \hat{\sigma}\Big]\end{aligned} \tag{7.128}$$

At $r \to 0$ both Φ and b are now explicitly regular:

$$\begin{aligned}\Phi &\sim \rho\, e^{-i\tilde{\psi}}\, \cos\tilde{\theta} = x_1 - ix_2 \\ b &\sim \rho\, e^{-i\tilde{\psi}}\, \cos\tilde{\theta}\Big[\frac{ik}{2Q}\rho\, d\rho - \frac{\rho^2}{2Q^2}\, (\sin^2\tilde{\theta}\, d\tilde{\phi} + \cos^2\tilde{\theta}\, d\tilde{\psi})\Big] \\ &= (x_1 - ix_2)\, \Big[\frac{ik}{2Q} \sum_i x_i dx_i - \frac{x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3}{2Q^2}\Big]\end{aligned} \tag{7.129}$$

4.8 Summary of the solution

We summarize here the full solution for w and $B^{(2)}_{MN}$, in the gauge of section 4.7.3 where all fields are regular:

$$
\begin{aligned}
w &= e^{-\frac{1}{2Q}(t+y)}\, e^{i(\phi-kz)}\, \cos\frac{\theta}{2}\, e^{-kr}\, \frac{r^{1/2}}{Q+r} \\
B^{(2)}_{tz} &= -e^{-\frac{1}{2Q}(t+y)}\, e^{i(\phi-kz)}\, \cos\frac{\theta}{2}\, e^{-kr}\, \frac{r^{1/2}}{2Q} \\
B^{(2)}_{yt} &= e^{-\frac{1}{2Q}(t+y)}\, e^{i(\phi-kz)}\, \cos\frac{\theta}{2}\, e^{-kr}\, \frac{r^{3/2}}{2(Q+r)^2} \\
B^{(2)}_{\phi t} &= -e^{-\frac{1}{2Q}(t+y)}\, e^{i(\phi-kz)}\, \cos^3\frac{\theta}{2}\, e^{-kr}\, \frac{Q\, r^{3/2}}{(Q+r)^2} \\
B^{(2)}_{yz} &= e^{-\frac{1}{2Q}(t+y)}\, e^{i(\phi-kz)}\, \cos\frac{\theta}{2}\, e^{-kr}\, \frac{r^{1/2}}{2Q}\left[\frac{Q^2}{(Q+r)^2}-1\right] \\
B^{(2)}_{\phi z} &= -e^{-\frac{1}{2Q}(t+y)}\, e^{i(\phi-kz)}\, \cos^3\frac{\theta}{2}\, e^{-kr}\, r^{1/2}\left[\frac{Q^2}{(Q+r)^2}-1\right] \\
B^{(2)}_{rz} &= ik\, e^{-\frac{1}{2Q}(t+y)}\, e^{i(\phi-kz)}\, \cos\frac{\theta}{2}\, e^{-kr}\, r^{1/2} \\
B^{(2)}_{r\theta} &= e^{-\frac{1}{2Q}(t+y)}\, e^{i(\phi-kz)}\, \sin\frac{\theta}{2}\, e^{-kr}\, \frac{r^{1/2}}{2} \\
B^{(2)}_{r\phi} &= -i\, e^{-\frac{1}{2Q}(t+y)}\, e^{i(\phi-kz)}\, \sin\frac{\theta}{2}\, \sin\theta\, e^{-kr}\, \frac{r^{1/2}}{2} \\
B^{(2)}_{\theta\phi} &= -i\, e^{-\frac{1}{2Q}(t+y)}\, e^{i(\phi-kz)}\, \cos\frac{\theta}{2}\, \sin\theta\, e^{-kr}\, \frac{r^{3/2}}{2}
\end{aligned}
\tag{8.130}
$$

or in form language

$$
\begin{aligned}
B^{(2)} &= e^{-\frac{1}{2Q}(t+y)}\, e^{i(\phi-kz)}\, e^{-kr}\, r^{1/2}\Big\{-\frac{1}{2Q}\,\cos\frac{\theta}{2}\, dt\wedge dz \\
&\quad +\frac{r}{2(Q+r)^2}\,\cos\frac{\theta}{2}\,[dy-Q(1+\cos\theta)d\phi]\wedge\Big[dt-\frac{2Q+r}{Q}\,dz\Big] \\
&\quad +ik\,\cos\frac{\theta}{2}\, dr\wedge dz+\frac{1}{2}\sin\frac{\theta}{2}\, dr\wedge[d\theta-i\sin\theta d\phi]-\frac{i}{2}\, r\,\cos\frac{\theta}{2}\sin\theta\, d\theta\wedge d\phi\Big\}
\end{aligned}
$$

4.9 Relation to D0-D6 bound states

We have added one unit of P to a D1-D5 bound state. For the moduli we have used this is a 'threshold bound' state; i.e., the mass of the bound state is the mass of the D1-D5 plus the mass of the P.

In [131] it was noted that a D0 brane is repelled by a D6 brane,[22] but if a suitable flux F was turned on in the D6 worldvolume then the D0 and D6 will form a bound state. In this section we find a relation between our threshold bound state and the condition in [131] which separates the domain of bound states from unbound states in the D0-D6 system.

4.9.1 The near ring limit

Strictly speaking, we have a threshold bound state between the entire ring shaped D1-D5 and the entire P wavefunction. The threshold nature of this state follows from the general supersymmetry relation between D1,D5,P charges. But we have seen that for large k the P wavefunction is confined to the vicinity of the ring, so we expect to get threshold binding between a short straight segment of the ring (like that in Fig.1(c)) and the wavefunction carrying P that we found in this short segment approximation. We will take the further step of identifying the two ends of our ring segment; this will enable us to perform a T-duality in the z direction.

Let us review the charges carried by this segment of the ring:

(a) We have the KK monopole charge that can be measured by a S^2 surrounding the tube; the nontrivially fibered circle of this KK is the y circle, and the directions

[22] A similar system was also studied in [132].

T^4, z are 'homogeneous directions', so they behave like directions along the 'KK-brane'.

(b) We have the 'true charge' D5 along $T^4 \times S^1$.

(c) We have the 'true charge' D1 along the S^1.

(d) We have 'dipole momentum' along the ring direction z; we call this P_z.

(e) The 'test quantum' that we seek to bind to this background is a unit of P_y (momentum along S^1).

The dipole charges are created automatically by the binding of the true charges, and so there are relations between the values of the true and dipole charges. To find these relations it is convenient to dualize the D5-D1 charges to NS1-P. The near ring limit of NS1-P was discussed in some detail in [11]; we reproduce some relevant details here.

The NS1 string carries the momentum P through transverse oscillations, described by a profile $\vec{F}(t-y)$. In Fig.2(a) we open up this multiwound NS1 to exhibit this vibration profile. Since the NS1 is wrapped many times on the S^1 we find that a short segment of this oscillating string looks like Fig.2(b). We can see that the winding along the direction y is due to the 'winding charge' of the NS1, while the slant along the z direction is due to the 'derivative of the transverse displacement' $\frac{d\vec{F}}{dy}$ which we take to point along the z direction.

For our present discussion we have compactified the z direction, so that we can assign well defined 'charges' to all elements. We find the following charges in the NS1-P frame:

(a) We have one unit of $NS1_z$, winding charge of the NS1 in the z direction. This is the dual of the KK dipole charge in the D1-D5 frame. Thus we write $n_1^{dipole} = 1$.

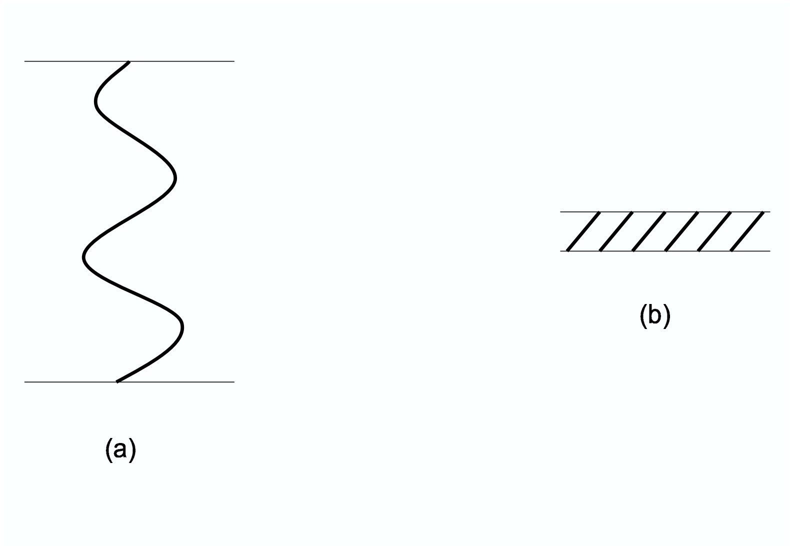

Figure 4.2: (a) The NS1 carrying a transverse oscillation profile in the covering space of S^1. (b) The strands of the NS1 as they appear in the actual space.

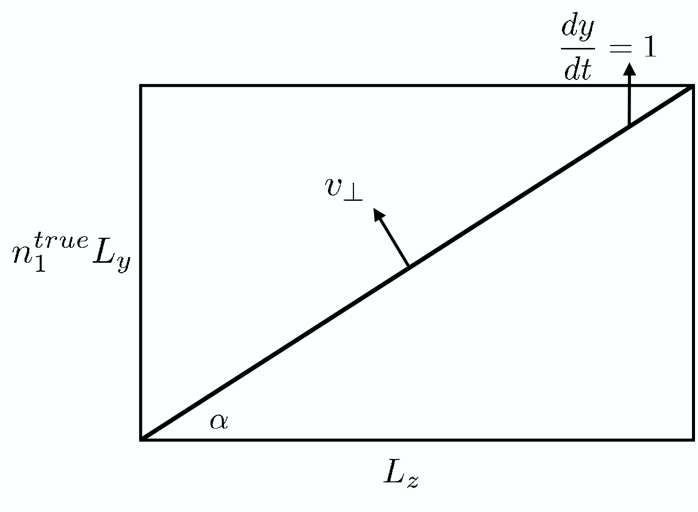

Figure 4.3: The winding and momentum charges of a segment of the NS1; we have used a multiple cover of the S^1 so that the NS1 looks like a diagonal line.

(b) The 'true' D5 charge becomes winding along S^1. We write this as n_1^{true} units of $NS1_y$.

(c) The 'true' D1 charge becomes momentum along S^1. We write this as n_p^{true} units of P_y.

(d) We have momentum P_z along the dipole direction (this has been unchanged in the duality from D1-D5). The number of units of this momentum we call n_p^{dipole}.

(e) The quantum that we wish to bind to the background changes from P_y to an NS5 along $S^1 \times T^4$.

Note that we must choose the compactification lengths of the y, z directions judiciously so that we get integer values for all charges. This can be done by choosing L_z, L_y so that n_1^{true}, n_p^{true} are integers. We will see below (eq. (9.133)) that this will set n_p^{dipole} to be integral. Note that $n_1^{dipole} = 1$ so it is already integral.

4.9.2 Relations between true and dipole charges

Note on notation: In this section we will encounter three different duality related systems: D1-D5, NS1-P, and a system where these true charges become D4 branes. We will not need to compute in the D1-D5 frame. For the NS1-P frame we use unprimed symbols for all quantities (for example lengths are L_y, L_z etc.). These should not be confused with unprimed symbols used in earlier sections of this paper; the computations here will not use results from those sections. For the frame using D4 branes we use primes on all symbols (e.g. L'_y, L'_z).

Let us ignore for now the charge (e) in the above list and look at the other charges which together give the background geometry of the ring. These charges are depicted in Fig.3. We have a NS1 moving in a direction perpendicular to itself with some velocity $v_\perp$; this gives all the four charges (a)–(d) above. We have denoted the lengths of the y, z directions by L_y, L_z.

We will now derive the relations between the true charges and the dipole charges. The first relation comes from the fact that the momentum carried by the NS1 is in a direction perpendicular to the NS1. Indeed if the momentum had a component along along the NS1 then there would be oscillations along the NS1 and a corresponding entropy. The entire NS1-P bound state does has an entropy, which is manifested in different possible shapes for the entire ring. But we are now zooming in on a short segment of the ring, and so by definition should have no entropy visible in oscillations of this segment.

The NS1 winds in a direction given by the vector

$$\vec{W} = L_z n_1^{dipole} \hat{z} + L_y n_1^{true} \hat{y} = L_z \hat{z} + L_y n_1^{true} \hat{y} \tag{9.131}$$

The momentum vector is

$$\vec{P} = \frac{2\pi n_p^{dipole}}{L_z} \hat{z} + \frac{2\pi n_p^{true}}{L_y} \hat{y} \tag{9.132}$$

Requiring $\vec{W} \cdot \vec{P} = 0$ gives

$$n_1^{dipole} n_p^{dipole} + n_1^{true} n_p^{true} = 0, \qquad \Rightarrow \qquad n_p^{dipole} = -n_1^{true} n_p^{true} \tag{9.133}$$

The second condition comes from the fact that the waveform $\vec{F}(t-y)$ moves along the y direction at the speed of light $v = 1$. This implies that the velocity of the NS1 in the direction normal to itself is

$$v_\perp = \cos\alpha = \frac{L_z}{\sqrt{(L_z)^2 + (n_1^{true} L_y)^2}} \tag{9.134}$$

The mass of the NS1 is

$$M = T\sqrt{(L_z)^2 + (n_1^{true} L_y)^2} \tag{9.135}$$

where $T = 1/2\pi\alpha'$ is the tension of the NS1. The momentum of the NS1 is in the direction normal to itself, and has magnitude

$$|\vec{P}| = \sqrt{\Big(\frac{2\pi n_p^{dipole}}{L_z}\Big)^2 + \Big(\frac{2\pi n_p^{true}}{L_y}\Big)^2} \tag{9.136}$$

Setting $|\vec{P}| = \frac{Mv_\perp}{\sqrt{1-v_\perp^2}}$ we get

$$\sqrt{\Big(\frac{2\pi n_p^{dipole}}{L_z}\Big)^2 + \Big(\frac{2\pi n_p^{true}}{L_y}\Big)^2} = T\sqrt{(L_z)^2 + (n_1^{true}L_y)^2}\frac{L_z}{n_1^{true}L_y} \tag{9.137}$$

Using (9.133) gives

$$\sqrt{\Big(\frac{2\pi n_p^{dipole}}{L_z}\Big)^2 + \Big(\frac{2\pi n_p^{true}}{L_y}\Big)^2} = \frac{(2\pi)n_p^{true}}{L_yL_z}\sqrt{(L_z)^2 + (n_1^{true}L_y)^2} \tag{9.138}$$

We thus find that that (9.137) is equivalent to

$$[Tn_1^{true}L_y]\Big[\frac{2\pi n_p^{true}}{L_y}\Big] = [TL_z]^2 \tag{9.139}$$

which tells us that

$$[\text{Mass of true NS1 charge}]\times[\text{Mass of true P charge}] = [\text{Mass of NS1 dipole charge}]^2 \tag{9.140}$$

In this form the condition is valid in all duality frames, with only the names of the charges changing under the dualities.

To summarize we have two relations between the true and dipole charges. The relation (9.133) comes from requiring that there be no entropy in the ring segment after we have zoomed into a sufficiently small region of the ring. The other condition (9.140) is related to the supersymmetry of the charges distributed along the ring. The supersymmetry is assured by the fact that the entire waveform moves in one direction with the speed of light. Different parts of the NS1 have different slopes and different velocities $v_\perp$, but for a profile of the form $\vec{F}(t-y)$ the slope and velocity are always correlated in such a way that the different parts are mutually BPS.

4.9.3 Dualizing to D6-D0

We now wish to perform dualities that will map the dipole charge of the ring (KK in the case of D1-D5, NS1 in the case of NS1-P) to a D6 brane charge. The quantum carrying one unit of P_y will be converted to a D0. Since we have found the relations between true and dipole charges in the NS1-P frame let us start with NS1-P and perform the required dualities:

$$\begin{pmatrix} NS1_z \\ NS1_y \\ P_y \\ P_z \\ NS5_{y1234} \end{pmatrix} \xrightarrow{S} \begin{pmatrix} D1_z \\ D1_y \\ P_y \\ P_z \\ D5_{y1234} \end{pmatrix} \xrightarrow{T_{yz12}} \begin{pmatrix} \overline{D3}_{y12} \\ D3_{z12} \\ NS1_y \\ NS1_z \\ \overline{D3}_{z34} \end{pmatrix} \xrightarrow{S} \begin{pmatrix} \overline{D3}_{y12} \\ D3_{z12} \\ D1_y \\ D1_z \\ \overline{D3}_{z34} \end{pmatrix} \xrightarrow{T_{z34}} \begin{pmatrix} D6_{zy1234} \\ D4_{1234} \\ D4_{yz34} \\ D2_{34} \\ D0 \end{pmatrix} \quad (9.141)$$

The true charges n_1^{true}, n_p^{true} have become D4 branes which can be described by fluxes in the D6:

$$\begin{aligned} n_1^{true} &= n_4^{(1234)} = \frac{1}{2\pi}\int_{zy} F = \frac{L'_z L'_y}{2\pi} F_{zy} \\ n_p^{true} &= n_4^{(yz34)} = -\frac{1}{2\pi}\int_{12} F = -\frac{L'_1 L'_2}{2\pi} F_{12} \end{aligned} \quad (9.142)$$

where L'_i are the lengths of cycles after the dualities. The minus sign in the expression for n_p^{true} arises from the orientation of the D6: the positive orientation is $(zy1234)$ while the n_4 is oriented as $(yz34)$. The presence of the above components of F also induces a D2 charge

$$n_2^{(34)} = \frac{1}{2}\frac{1}{(2\pi)^2}\int_{zy12} F \wedge F = \frac{L'_z L'_y L'_1 L'_2}{(2\pi)^2} F_{zy} F_{12} = -n_4^{(1234)} n_4^{(zy34)} \quad (9.143)$$

Since under the dualities $n_p^{dipole} = n_2^{(34)}$, we observe that (4.104) is equivalent to (9.133). In other words the relation (9.133) translates in the D6 duality frame to the statement that the D2 charge comes entirely from the fluxes needed to induce the required D4 charges; there is no 'additional' D2 charge.

4.9.4 The condition of [131]

Consider a D6 brane along the directions $(zy1234)$. Suppose that there is a background NS-NS 2-form turned on; by a suitable change of coordinates we can bring this to a form where the nonzero components are $b_1 = B_{zy}, b_2 = B_{12}, b_3 = B_{34}$. Write

$$e^{2\pi i v_a} = \frac{1 + i b_a}{1 - i b_a} \quad a = 1, 2, 3 \tag{9.144}$$

The threshold value of B, beyond which a D0 will bind to the D6, is given by [131]

$$v_1 + v_2 + v_3 = \frac{1}{2} \tag{9.145}$$

In terms of the b_a this condition becomes

$$b_1 b_2 + b_1 b_3 + b_2 b_3 = 1 \tag{9.146}$$

We can replace the B field with a field strength on the D6:

$$b_a \rightarrow 2\pi\alpha' F_a \tag{9.147}$$

We take $\alpha' = 1$ in the following. In our case we have $b_3 = 0$ and $b_1 = 2\pi F_{zy}$, $b_2 = 2\pi F_{12}$. One has the freedom to change the orientation in each of the 2-planes (z, y), $(1, 2)$, $(3, 4)$: this flips the sign of v_a and b_a, and thus the sign of each of the terms in (9.146) is actually arbitrary. Taking this into account the threshold condition of [131] for our case is

$$(2\pi)^2 |F_{zy} F_{12}| = 1 \tag{9.148}$$

4.9.5 Checking the threshold condition

Let us now see if the condition (9.148) is satisfied by our ring segment. From (9.142) we find that

$$(2\pi)^2 F_{zy}F_{12} = -\frac{(2\pi)^4}{L'_z L'_y L'_1 L'_2} n_1^{true} n_p^{true} \tag{9.149}$$

Under the dualities (9.141) the moduli change as follows (primed quantities refer to D6 frame, unprimed to the NS1-P frame, and $L_i = 2\pi R_i$)

$$\begin{aligned} g' &= g\sqrt{\frac{R_z}{R_y R_1 R_2}\frac{1}{R_3 R_4}}, \quad R'_3 = \frac{g}{R_4\sqrt{R_z R_y R_1 R_2}}, \quad R'_4 = \frac{g}{R_3\sqrt{R_z R_y R_1 R_2}} \\ R'_y &= \sqrt{\frac{R_z R_1 R_2}{R_y}}, \quad R'_z = \sqrt{\frac{R_z}{R_y R_1 R_2}}, \quad R'_1 = \sqrt{\frac{R_z R_y R_2}{R_1}}, \quad R'_2 = \sqrt{\frac{R_z R_y R_1}{R_2}} \end{aligned} \tag{9.150}$$

Thus

$$(2\pi)^2 F_{zy}F_{12} = -\frac{(2\pi)^4 n_1^{true} n_p^{true}}{L'_z L'_y L'_1 L'_2} = -\frac{(2\pi)^2 n_1^{true} n_p^{true}}{L_z^2} \tag{9.151}$$

If we now use the relation (9.139) we find

$$(2\pi)^2 F_{zy}F_{12} = -1 \tag{9.152}$$

We thus see that the charges carried by our ring satisfy the condition (9.148) noted in [131].

4.9.6 Depth of the tachyon potential

Let us see what we have learned. The 2-charge system has true charges and dipole charges, and these satisfy the relations (9.133),(9.140). The system can be mapped to a D6 brane carrying fluxes, and the fluxes have a value which puts the system at the boundary of the domain where a D0 brane will bind to the D6.

In the D1-D5 picture the analogue of the D0 is the P charge. In section (4.9.1) we listed the charges carried by the 2-charge D1-D5 system and the charge P carried by the wavefunction we are trying to construct. But there is one more charge carried by the wavefunction, which comes from the momentum of this wavefunction along the z direction. In the wavefunction this momentum arises from the factor e^{-ikz}. So we would label this charge as an additional amount of P_z, carried by the quantum that we are trying to bind to the 2-charge D1-D5 ring segment.

In the D6 duality frame this additional P_z becomes a $D2_{34}$. Thus the quantum that we are trying to bind to the D6 is not just a D0, but a 'D0 plus some $D2_{34}$'. We now draw some conclusions about the D0-D6 bound state from our construction of the wavefunction (8.130).

The case $k = 0$

Since k is a free parameter, we can try to set $k = 0$. This would correspond to letting the test quantum be just the D0 (not bound to any $D2_{34}$), and asking if at the threshold value of fluxes (9.148) we get a good bound state with the D6. But from the discussion of section (4.7.1) we see that the wavefunction is *not* normalizable for the case $k = 0$, so there is no bound state in this case. We therefore conclude that for a D6 wrapped on a torus T^6 carrying fluxes at the threshold value (9.148) we do not get a bound state with the D0. As argued in [131] we would of course get a bound state for larger values of F and no bound state for smaller F, but our explicit construction of the wavefunction (in the dual D1-D5 case) tells us the situation at the threshold value of F.

The case $k > 0$

In this case the test quantum to be bound has some $D2_{34}$ branes along with the D0. The mass of a 'D0 plus some $D2_{34}$' is obviously more than the mass of just the

D0. But after we bind the 'D0 plus some $D2_{34}$' to the D6 carrying fluxes, the final mass of the bound state is independent of the mass of the $D2_{34}$ branes coming with the D0, since in the D1-D5 frame the energy of the wavefunction is given by $e^{-i\frac{t}{2Q}}$ for all values of k.

This observation tells us the binding energy of the $D2_{34}$ branes in the situation where we have a D6 carrying fluxes equal to their 'threshold' value (9.148). The binding energy ΔE must be equal to the mass M_2 of the $D2_{34}$ in order that these branes do not show up in the final result for the mass of the composite:

$$\Delta E = M_2 \tag{9.153}$$

Note that the D0 is repelled by a D6, is neutral with respect to the D4's in the D6, and is attracted by the D2 charge in the D6. At the 'threshold' value of fluxes it becomes neutral with respect to the 'D6-D4-D4-D2' bound state created by the D6 with fluxes. By contrast the $D2_{34}$ is neutral with respect to the D6, is attracted to the D4's in the D6, and is neutral with respect to the D2 in the D6. Thus we expect a binding energy ΔE for the $D2_{34}$, and our construction of the wavefunction tells us that this energy is (9.153). In CFT terms we get a tachyon in the open string spectrum between the $D2_{34}$ and the D6 with fluxes. For the threshold value of these fluxes the depth of the tachyon potential must equal the mass of the $D2_{34}$.

4.10 Discussion

We have constructed a simple case of '3-charge hair' for the BPS black ring, by starting with a D1-D5 ring and adding a perturbation carrying one unit of P. A normalizable perturbation carrying this P was expected to exist because there was a corresponding state in the dual CFT. After constructing the perturbation we observe that it is smooth everywhere, so the result supports a 'fuzzball' picture for black ring microstates.

A similar perturbation was constructed (up to several orders in a small parameter) for the 3-charge black *hole* in [123], and solutions dual to specific CFT states carrying nonperturbative amounts of P were found in [9]. But these solutions carried a large amount of angular momentum. Thus it may be said that they did not give generic microstates for the 3-charge hole. By contrast, the black ring is *supposed* to carry a sizable amount of angular momentum, which gives it the 'ring shape'. Thus even though we have only one unit of P in our present construction, the hair we have constructed might be considered a good indicator of the nature of generic states of the ring.

In [133] 'ring-like' 2-charge states were considered, and it was observed that the area of a 'horizon' drawn around such states has an area satisfying a Bekenstein type relation $A/4G \sim \sqrt{n_1 n_5 - J} \sim S$ where S is the entropy of these states and J is the angular momentum of the ring. (Such 2-charge systems have been further studied recently [134, 135, 136, 137].) In the present chapter we have taken the simplest of the 2-charge ring states and added one unit of P. We have made the wavefunction only in the near ring limit, where the segment of the ring looked like a straight line. But we will get a similar near ring limit from any sufficiently smooth microstate out of the collection used in [133], so our wavefunction adding P should describe the nature of P excitations for any of these 2-charge microstates.

A large class of 3-charge BPS solutions for the black hole and black ring were found in [124, 125]. While the explicit examples studied there had axial symmetry (and thus a nontrivial amount of rotation) one may be able to construct nonrotating solutions by extending such techniques. Thus this approach may lead to generic nonperturbative hair for the black hole as well as for the black ring. It would therefore be very interesting to identify microstates in this approach. In the perturbative construction of the present paper we have excited the NS-NS 2-form gauge field, which was not excited in the solutions of [124, 125]. It would be interesting to find an extension of the solutions of [124, 125] which give nonperturbative hair involving this gauge field.

CHAPTER 5

BRANES WRAPPING BLACK HOLES

5.1 Introduction

In this chapter, we start exploring the microstate geometries by considering probe branes in them. Only for certain situations will the brane be in a supersymmetric configuration with the background. The dynamics of stable extended branes in backgrounds containing fluxes have played an important role in exploring non-perturbative aspects of string theory. A particularly important class of such objects are dielectric branes which are extended objects formed by a collection of lower dimensional extended objects moving in a transverse dimension via Myers' effect [102]. Dielectric branes wrap *contractible* cycles in the space-time and therefore do not carry any net charge appropriate to its dimensionality, but has nonvanishing higher multipole moments. In a class of backgrounds (e.g. AdS space-times or their plane wave limits and certain D-brane backgrounds) the energy due to the tension of the dielectric brane is completely cancelled by the effect of the background flux, so that its dispersion relation is that of a massless particle, which is why they are called giant gravitons [103]-[106] . The energetics of such branes are usually determined by a classical analysis : however these branes are BPS states which renders the classical results exact.

Recently a different class of extended brane configurations have been found in the near-horizon geometry of four dimensional extremal black holes [107]-[109] constructed e.g. from intersecting $D4$ branes and some additional $D0$ charge. The near-horizon geometry is $AdS_2 \times S^2 \times K$ where K is a suitable six dimensional internal space (e.g. Calabi-Yau). These are branes of various dimensionalities wrapping *non-contractible* cycles of the compact directions. The branes which are wrapped on cycles in K have a net charge in the full geometry and are similar to giant gravitons - the tension of the brane is cancelled and one is left with the dynamics of gravitons. More interestingly, there are BPS $D2$ branes wrapped on the S^2 with a worldvolume flux providing a $D0$ brane charge, and posessing momentum along K. These branes do not have net $D2$ charges in the full geometry - they only contribute a net $D0$ charge. The ground state is static in global time, located at a radial coordinate determined by the $D0$ charge. These configurations preserve half of the enhanced supersymmetries of the near-horizon geometry, but *do not preserve any supersymmetry of the full geometry.*

In [109] it has been argued that such brane configurations provide a natural understanding of the entropy of the black hole background. The presence of a magnetic type flux in the compact direction means that such a static brane carries a nonzero momentum and is in fact in the lowest Landau level. This means that the ground state is degenerate. It turns out that this degeneracy is *independent of the* $D0$ *charge* q_0 *of the background.* The idea then is to "construct" a black hole by starting from the set of $D4$ branes and then add $D0$ charges. However the $D0$ charge appear as these $D2$ branes which wrap the S^2, and each such $D2$ has a ground state degeneracy. The problem then reduces to a partitioning problem of distributing a given $D0$ charge

N among $D2$ branes - the various possible ways of doing this give rise to the entropy of the final black hole. This argument has been extended to “small” black holes in [110].

In this chapter, we show that such brane configurations are quite generic not only in near-horizon geometries of black holes, but in the *full* asymptotically flat geometries of certain black hole microstates. While these are supersymmetric states in near-horizon regions of black holes and near-cap regions of microstates, they break all the supersymmetries of the asymptotically flat backgrounds. We find that in all cases they have a universal dispersion relation characteristic of threshold bound states : the total energy is just the sum of the energies due to various brane charges. In near-horizon regions this simple dispersion relation follows from supersymmetry and conformal algebras. However, we have not been able to find a good reason why the same dispersion relation holds in the full microstate geometries.

One key feature of the examples which we provide is that the background does not have to posess the same kind of charge as the brane itself. This feature could be relevant for the proposal of [109], though we have reservations about this proposal as it stands.

In section (5.2) we consider generic $AdS_m \times S^n \times \mathcal{M}$ space-times with a brane wrapped around S^n and moving along a AdS direction with momentum P and derive the universal dispersion relation

$$E = P + M_n \tag{1.1}$$

where M_n denotes the mass of the brane.

Specific examples of solutions of M theory and Type IIA string theory which lead to $AdS_m \times S^n \times \mathcal{M}$ spacetimes are described in section (5.3). Our main example involves five dimensional black strings in M theory compactified on T^6 (in section (5.3.1)) and their dimensional reduction to four dimensional black holes in IIA theory (in section (5.3.4)). In section (5.3.2) it is argued that the dispersion relation (1.1) follows from the underlying conformal algebra. This is explicitly shown for AdS_3, but the considerations should generalize to other AdS_m. These branes are static in global time. In Poincare time, they correspond to branes coming out of the horizon upto a maximum distance and eventually returning back to the horizon. However, we find that for AdS_3 (section (5.3.3)) and for AdS_2 (section (5.3.4), the relation (1.1) is valid *both in global and Poincare coordinates*. Furthermore in AdS_3 the *Poincare momentum is equal to the global momentum* P. We argue that the equality of global and Poincare energies and momenta signifies that the brane is in a highest weight state of the conformal algebra.

The second class of backgrounds where we find such brane configurations with identical dispersion relations are geometries which represent microstate of 2-charge and 3-charge systems. In the examples of section (5.3) the existence of these brane configurations appears to be special to near-horizon limits. This is because they are states of lowest value of the global AdS energy and not of the Poincare energy and it is the latter which coincides with the energy defined in the full asymptotically flat geometry. In contrast, the microstate geometries are asymptotically flat and go over to a *global patch* of AdS in the interior. The time in the asymptotic region continues to the *global time* of the interior AdS. Consequently, the notion of energy is unambigious.

In sections (5.4) and (5.5) we find that the lowest energy states of such branes are indeed static configurations with dispersion relations given by (1.1). Section (5.4) deals with a T-dualized version of the 2-charge microstate geometry with D3 branes wrapping the S^3. We show, in section (5.4.3) that the energy has an interesting implication for the conformal field theory dual. In section (5.4.4) we determine the spectrum of vibrations of the brane and find a remarkably simple equispaced excitation spectrum with spacing determined only by the AdS scale - reminiscent of the spectrum of giant gravitons found in [105]. Section (5.5) deals with analogous treatments of a special 3-charge microstate geometry.

In section (5.6) we calculate the field produced by such a probe brane in the 2-charge microstate geometry and show that this leads to a *constant* field strength in the asymptotic region, pretty much like a domain wall.

In section (5.7) we examine the supersymmetry properties of these brane configurations. Section (5.7.1) deals with the case of D2 branes in the background of 4d black holes, which is the background of section (5.3.4). We show that in the near horizon limit this D2 brane preserves half of the supersymmetries. We calculate the topological charge on the brane and show that the supersymmetry algebra leads to our simple dispersion relation. It is then explicitly shown that the brane does not preserve any supersymmetry of the full black hole geometry. In section (5.7.2) we investigate the question in the 2 charge microstate geometry and show that while the near-cap limit (which is again $AdS_3 \times S^3$) the brane preserves supersymmetry, it breaks all the supersymmetries of the full background.

In an appendix we examine the validity of the near-horizon approximation our brane trajectories for the case of 4D black holes and show that the approximation is indeed valid when the energy due to D0 charge of the D2 brane is smaller than the D2 brane mass.

5.2 Spherical branes $AdS \times S \times M$ space-times

The simplest space-times in which these brane configurations occur are of the form $AdS_m \times S^n \times \mathcal{M}$, where $\mathcal{M}$ is some internal manifold.

Let us first consider branes in M-theory backgrounds. The metric is given by

$$ds^2 = R^2[-\cosh^2\chi\ d\tau^2 + d\chi^2 + \sinh^2\chi\ d\Omega_{m-2}^2] + \tilde{R}^2 d\Omega_n^2 + g_{ij}dy^i dy^j \qquad (2.2)$$

where $R, \tilde{R}$ are length scales, g_{ij} is the metric on $\mathcal{M}$ and $d\Omega_p^2$ denotes the line element on a unit S^p. We will choose coordinates (θ_k, φ) on S^{m-2} leading to a metric

$$d\Omega_{m-2}^2 = d\theta_1^2 + \sin^2\theta_1\ d\theta_2^2 + \sin^2\theta_2 \sin^2\theta_1\ d\theta_3^2 + \cdots + \sin^2\theta_{m-3}\cdots\sin^2\theta_1\ d\varphi^2 \quad (2.3)$$

The background could have m-form and n form gauge field stengths which will not be relevant for our purposes.

In addition, the background contains $(n+1)$-form gauge potentials ($n = 2$ or $n = 5$) of the form

$$A^{(n+1)} = A_i(y^i)\ d\omega_n \wedge dy^i \qquad (2.4)$$

where $d\omega_n$ denotes the volume form on the sphere. We will see explicit examples of these geometries later.

Consider the motion of a n-brane which is wrapped on the S^n, rotating in the S^{m-2} contained in the AdS_m and in general moving along both χ and y^i. The bosonic part of the brane action is of the form

$$S = -\mu_n \int d^{n+1}\xi\ \sqrt{\det G} + \mu_n \int P[A^{(n+1)}] \qquad (2.5)$$

where G denotes the induced metric, the symbol P stands for pullback to the world-volume and μ_n is the tension of the n-brane.

Let us fix a static gauge where the worldvolume time is chosen to be the target space time and the worldvolume angles are chosen to the angles on S^n. The remaining worldvolume fields are $\chi, y^i, \theta_k, \varphi$. When these fields are independent of the angles on the worldvolume, the dynamics is that of a point particle. The Hamiltonian can be easily seen to be

$$H = \cosh\chi\sqrt{M_n^2 + \frac{P_\chi^2}{R^2} + \frac{\Lambda^2}{R^2\sinh^2\chi} + g^{ij}(P_i - M_nA_i)(P_j - M_nA_j)} \tag{2.6}$$

where M_n is the mass of the brane

$$M_n = \mu_n \tilde{R}^n \Omega_n \tag{2.7}$$

Ω_n being the volume of unit S^n. Λ denotes the conserved angular momentum on S^{m-2}

$$\Lambda^2 = p_{\theta_1}^2 + \frac{p_{\theta_2}^2}{\sin^2\theta_1} + \frac{p_{\theta_3}^2}{\sin^2\theta_1\sin^2\theta_2} + \cdots \frac{p_\varphi^2}{\sin^2\theta_1\cdots\sin^2\theta_{m-3}} \tag{2.8}$$

Consider the lowest energy state for some given $|\Lambda|$. In the internal space this means that $P_i = M_nA_i$. (This can be considered to be the description of the lowest Landau level in the classical limit). In AdS this has a fixed value of the global coordinate $\chi = \chi_0$ determined by minimizing the hamiltonian :

$$\sinh^2\chi_0 = \frac{|\Lambda|}{R\ M_n} \tag{2.9}$$

The motion on the S^{m-1} contained in AdS_{m+1} is along an orbit with

$$p_{\theta_k} = 0 \quad \theta_k = \frac{\pi}{2} \qquad k = 1\cdots(m-3) \tag{2.10}$$

The ground state energy is

$$E_{global} = \frac{|\Lambda|}{R} + M_n \tag{2.11}$$

Finally it is easy to check that in this state

$$\dot{\varphi} = 1 \tag{2.12}$$

While the above formulae have been given for M-branes, they apply equally well for $D5$ branes in $AdS_5 \times S^5$ backgrounds of Type IIB string theory. This in fact provides the simplest example of such configurations. We will give a general explanation below for the simple form (2.11) of the energy E.

5.3 Extremal Black Strings in M theory and Black Holes in String Theory

In this section we will provide some concrete examples where branes in $AdS \times S \times \mathcal{M}$ appear.

5.3.1 5D Black Strings and 4D Black Holes

A specific example of interest is the geometry of an extremal black string in M-theory copmpactified on T^6 whose coordinates are denoted by $y^1 \cdots y^6$. The background is produced by three sets of M5 branes which are wrapped on the directions $y\,y^3\,y^4\,y^5\,y^6$, $y\,y^1\,y^2\,y^5\,y^6$ and $y\,y^1\,y^2\,y^3\,y^4$ and carrying momentum q_0 along y. The numbers n_i and charges p_i of the M5 branes are related as

$$p_i = \frac{2\pi^2\,n_i}{M_{11}^3\,T^{(i)}}\ , \ i = 1, 2, 3 \tag{3.13}$$

where $T^{(1)}$, $T^{(2)}$, $T^{(3)}$ are the volumes of the 2-tori $(1,2)$, $(3,4)$ and $(5,6)$.

The metric and gauge fields produced by this system of branes is

$$\begin{aligned} ds^2 &= h^{-1/3}\left[-dt^2 + dy^2 + \frac{q_0}{r}\,(dt - dy)^2\right] + h^{2/3}\left[dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2)\right] \\ &+ h^{-1/3}\sum_{i=1,2,3} H_i\,ds^2_{T^{(i)}} \end{aligned} \tag{3.14}$$

$$A^{(3)} = \sin\theta \; d\theta d\phi \, [p_3 \; \frac{y^5 dy^6 - y^6 dy^5}{2} + p_2 \; \frac{y^3 dy^4 - y^4 dy^3}{2} + p_1 \; \frac{y^1 dy^2 - y^2 dy^1}{2}] \quad (3.15)$$

where we have defined

$$h = H_1 \, H_2 \, H_3 \,, \quad H_i = 1 + \frac{p_i}{r} \,, \; i = 1,2,3 \,, \quad H_0 = 1 + \frac{q_0}{r} \quad (3.16)$$

$ds^2_{T^{(i)}}$ is the flat metric on the 2-torus of volume $T^{(i)}$.

Near-horizon limit with $q_0 = 0$

When $q_0 = 0$ the near-horizon limit is given by $AdS_3 \times S^2 \times T^6$. This may be seen by re-defining coordinates

$$y = \lambda x \qquad t = \lambda T \qquad r = 4\lambda \; u^2 \quad (3.17)$$

where we have defined

$$\lambda \equiv (p_1 \; p_2 \; p_3)^{1/3} \quad (3.18)$$

Then for $r \ll p_i$ and $q_0 = 0$ the metric (3.14) becomes

$$ds^2 = (2\lambda)^2 \Big[\frac{du^2}{u^2} + u^2 \, (-dT^2 + dx^2) + \frac{1}{4} \, (d\theta^2 + \sin^2\theta \, d\phi^2) \Big] + \frac{1}{\lambda} \sum_{i=1,2,3} p_i \, ds^2_{T^{(i)}} \quad (3.19)$$

which is $AdS_3 \times S^2 \times T^6$ in Poincare coordinates.

One can further continue the metric to global AdS_3 using the transformations

$$T = \frac{\cosh\chi \, \sin\tau}{\cosh\chi \, \cos\tau - \sinh\chi \, \sin\varphi} \,, \quad x = \frac{\sinh\chi \, \cos\varphi}{\cosh\chi \, \cos\tau - \sinh\chi \, \sin\varphi}$$
$$u = \cosh\chi \, \cos\tau - \sinh\chi \, \sin\varphi \quad (3.20)$$

The resulting metric is

$$ds^2 = (2\lambda)^2 \Big[d\chi^2 - \sinh^2\chi \, d\tau^2 + \cosh\chi^2 \, d\varphi^2 + \frac{1}{4} \, (d\theta^2 + \sin^2\theta \, d\phi^2) \Big] + \frac{1}{\lambda} \sum_{i=1,2,3} p_i \, ds^2_{T^{(i)}} \quad (3.21)$$

We can now consider a M2 brane wrapped around the S^2 and apply the general results in equations (2.5)-(2.11). For some given momentum P_φ in the φ direction, the lowest value of the global energy is given by

$$E_{gs} = P_\varphi + 8\pi\mu_2\lambda^3 \tag{3.22}$$

which corresponds to a brane which is static in global time.

Near-horizon limit with $q_0 \neq 0$

The near-horizon geometry for $q_0 \neq 0$ is again $AdS_3 \times S^2 \times T^6$. For $r \ll q_0, p_i$ we have, from (3.14)

$$ds^2 = \lambda^2[\rho'(-dT'^2 + dx'^2) + (dT' - dx')^2 + \frac{du'^2}{u'^2}] \tag{3.23}$$

$$+ \lambda^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{\lambda}\sum_{i=1,2,3} p_i\, ds^2_{T^{(i)}} \tag{3.24}$$

where we have defined

$$y = (\frac{\lambda^3}{q_0})^{1/2}\ x' \qquad t = (\frac{\lambda^3}{q_0})^{1/2}\ T' \qquad r = q_0 u' \tag{3.25}$$

With a further change of coordinates ([111])

$$\bar{T} - \bar{x} = e^{T'-x'} \qquad \bar{T} + \bar{x} = T' + x' + \frac{2}{u'} \qquad \bar{u} = \frac{\sqrt{u'}}{2}\ e^{-(T'-x')/2} \tag{3.26}$$

the metric reduces to the standard form of the Poincare metric on $AdS_3 \times S^2 \times T^6$

$$ds^2 = (2\lambda)^2\Big[\frac{d\bar{u}^2}{\bar{u}^2} + \bar{u}^2\,(-d\bar{T}^2 + d\bar{x}^2) + \frac{1}{4}(d\theta^2 + \sin^2\theta\, d\phi^2)\Big] + \frac{1}{\lambda}\sum_{i=1,2,3} p_i\, ds^2_{T^{(i)}} \tag{3.27}$$

which is identical to the metric (3.19). As before, one can pass to the global AdS_3 using the formulae above.

Thus we see that whether the background has momentum in the y direction or not the near-horizon geometry has the local form $AdS_3 \times S^2 \times \mathcal{M}$, so the dynamics of the M2 brane will be similar in the two cases.

5.3.2 An explanation of the dispersion relation

For branes moving in flat space, we expect that the total energy E arises from the 'rest energy' M and the momentum P by a relation of the type $E = \sqrt{M^2 + P^2}$. But for the branes studied here we get a linear relation of the type $E = P + M$. The momentum P causes a shift in radial position of the brane, where the redshift factor is different, and in the end we end up with this simple energy law.

As we will see in a later section the brane configuration considered above is a BPS state which preserves half of the supersymmetries of the background. The dispersion relation then follows from the supersymmetry algebra.

It turns out that there is a simple derivation of this linear relation for branes in AdS spacetime based on the bosonic part of the conformal algebra. We will present this for the case of $AdS_3 \times S^n$. We suspect that similar considerations would hold for arbitrary AdS_m.

A n-brane wrapped on S^n becomes a point massive particle in AdS_3. Its lagrangian

$$L = -m[-\frac{\partial X^\mu}{\partial \tilde{\tau}} \frac{\partial X_\mu}{\partial \tilde{\tau}}]^{\frac{1}{2}} \tag{3.28}$$

where m is the mass of the brane and $\tilde{\tau}$ denotes the worldline parameter. The lagrangian is invariant under the $SL(2,R) \times SL(2,R)$ isometries of the background. Denoting the global AdS_3 coordinates by τ, χ, φ and defining $z = \tau + \varphi,\ \bar{z} = \tau - \varphi$

the generators are

$$\begin{aligned} L_0 &= i\ \partial_z \\ L_{-1} &= i\ e^{-iz}\ [\frac{\cosh 2\chi}{\sinh 2\chi}\partial_z - \frac{1}{\sinh 2\chi}\partial_{\bar z} + \frac{i}{2}\partial_\chi] \\ L_1 &= i\ e^{iz}\ [\frac{\cosh 2\chi}{\sinh 2\chi}\partial_z - \frac{1}{\sinh 2\chi}\partial_{\bar z} - \frac{i}{2}\partial_\chi] \end{aligned} \quad (3.29)$$

and

$$\begin{aligned} \bar L_0 &= i\ \partial_{\bar z} \\ \bar L_{-1} &= i\ e^{-i\bar z}\ [\frac{\cosh 2\chi}{\sinh 2\chi}\partial_{\bar z} - \frac{1}{\sinh 2\chi}\partial_z + \frac{i}{2}\partial_\chi] \\ \bar L_1 &= i\ e^{i\bar z}\ [\frac{\cosh 2\chi}{\sinh 2\chi}\partial_{\bar z} - \frac{1}{\sinh 2\chi}\partial_z - \frac{i}{2}\partial_\chi] \end{aligned} \quad (3.30)$$

We have the algebra

$$[L_0, L_{-1}] = L_{-1}, \quad [L_0, L_1] = -L_1, \quad [L_1, L_{-1}] = 2L_0 \quad (3.31)$$

$$[\bar L_0, \bar L_{-1}] = \bar L_{-1}, \quad [\bar L_0, \bar L_1] = -\bar L_1, \quad [\bar L_1, \bar L_{-1}] = 2\bar L_0 \quad (3.32)$$

The conserved quantities corresponding to these isometries are given by the replacement

$$-i\partial_\mu \ \rightarrow \ P_\mu \quad (3.33)$$

in (3.29),(3.30). The global coordinate energy E_{global} and momentum P_φ of the brane are related to the conserved charges under translations of t, φ

$$E_{global} = -P_\tau \quad P = P_\varphi \quad (3.34)$$

Denote the parameter on the worldline of the particle by $\tilde\tau$. The kind of solution we have been considering is of the form

$$\chi = \chi_0, \quad t = \tilde\tau, \quad \varphi = \tilde\tau \quad (3.35)$$

This is a geodesic in AdS_3. The isometries of AdS_3 will move this to other geodesics. The key property of our solution is that

$$\bar{z} = \tau - \varphi = \text{constant} \tag{3.36}$$

By a choice of the zero of τ we can choose this trajectory to be along $\bar{z} = 0$. On this trajectory the isometry $\bar{L}_1 - \bar{L}_{-1} = -\partial_\chi$ leads to a shift of the radial coordinate χ. Therefore applying this isometry transformation we will get a new solution to the equations of motion of the form

$$\chi = \chi_0 + \epsilon \quad \tau = \tilde{\tau}, \quad \varphi = \tilde{\tau} \tag{3.37}$$

the meomenta conjugate to $z, \bar{z}$ are

$$P_z = \frac{1}{2}(P_\tau + P_\varphi) = \frac{1}{2}(P - E_{global}), \quad P_{\bar{z}} = \frac{1}{2}(P_\tau - P_\varphi) = -\frac{1}{2}(E_{global} + P) \tag{3.38}$$

while the isometry $Q \equiv \bar{L}_1 - \bar{L}_{-1}$ is given by

$$Q = -e^{-i\bar{z}}\,[\frac{\cosh 2\chi}{\sinh 2\chi}P_{\bar{z}} - \frac{1}{\sinh 2\chi}P_z + \frac{i}{2}P_\chi] + e^{i\bar{z}}\,[\frac{\cosh 2\chi}{\sinh 2\chi}P_{\bar{z}} - \frac{1}{\sinh 2\chi}P_z - \frac{i}{2}P_\chi] \tag{3.39}$$

where we have used (3.33).

We now observe that

$$\{P_z, Q\} = 0 \tag{3.40}$$

so $P - E_{global}$ does not change under the shift. We thus see that for our family of solutions given by (3.37) we will have

$$E_{global} = P + \text{constant} \tag{3.41}$$

To fix the constant we can go to the geodesic at $\chi = 0$ which has $P = 0$. Then we just get the energy of the brane wrapped on the S^n, sitting at the center of AdS_3, Calling this energy M_n, we get

$$E_{global} = P + M_n \tag{3.42}$$

giving the simple additive relation between the mass and momentum contributions to the energy.

5.3.3 Poincare coordinate energies and momenta

The brane discussed above is static in global coordinates and would therefore correspond to a moving brane in Poincare time. In this subsection we discuss some properties of dynamical quantities in Poincare coordinates for branes in $AdS_3 \times S^n$. The coordinate transformations are given in equations (3.20).

A trajectory $\chi = \chi_0,\ \varphi = \tau$ becomes the following trajectory in Poincare coordinates

$$x = \tanh\chi_0\ (1 + T\tanh\chi_0) \tag{3.43}$$

$$u = \frac{\cosh\chi_0}{\sqrt{T^2(1+\tanh^2\chi_0) + 2T\ \tanh\chi_0 + 1}} \tag{3.44}$$

Thus the brane pops out of the horizon $u = 0$ at $T = -\infty$, goes out to a maximum distance u_{max} and returns back to the horizon at $T = -\infty$. At the same time the coordinate x increases monotonically with T. The total elapsed proper time is *finite*. The value of u_{max} can be calculated from the above trajectory and one gets

$$u_{max} = \sqrt{\cosh\ 2\chi_0} \tag{3.45}$$

The Poincare energy is given by

$$E_{Poincare} = \frac{M_n|g_{TT}|}{\sqrt{|g_{TT}| - g_{xx}\dot{x}^2 - g_{uu}\dot{u}^2}} = \frac{M_n u^2}{\sqrt{u^2(1-\dot{x}^2) - \frac{1}{u^2}\dot{u}^2}} \tag{3.46}$$

Using the value of χ_0 in (2.9) one finds that

$$E_{Poincare} = M_n \, \cosh^2 \chi_0 = \frac{|L|}{R} + M_n = E_{global} \tag{3.47}$$

In an analogous way one can verify that the momentum in global coordinates, P_φ, equals the momentum in Poincarè coordinates, P_x:

$$P_x = M_n \frac{u^2 \, \dot{x}}{\sqrt{u^2(1-\dot{x}^2) - \frac{1}{u^2}\dot{u}^2}} = M_n \, \cosh^2 \chi_0 \, \tanh^2 \chi_0 = P_\varphi \tag{3.48}$$

where we have used the fact that, for the above trajectory $\dot{x} = \tanh^2 \chi_0$.

The trajectory $\chi = \chi_0, \varphi = \tau$ clearly does not have the smallest possible value of $E_{Poincare}$. The lowest value of $E_{Poincare}$ is in fact zero and corresponds to the brane being pushed to the horizon $u = 0$.

The equality of global and Poincare energies can be understood from the symmetries of AdS. The generators of the $SL(2,R) \times SL(2,R)$ isometries of the background have been given in global coordinates in equation (3.29) and (3.30). The generators in Poincare coordinates are given in terms of $w = T + x$ and $\bar{w} = T - x$ by

$$\begin{aligned} H_{-1} &= i\,\partial_w \\ H_0 &= i\left[w\,\partial_w - \frac{u}{2}\,\partial_u\right] \\ H_1 &= i\left[w^2\,\partial_w - w\,u\,\partial_u - \frac{1}{u^2}\,\partial_{\bar{w}}\right] \end{aligned} \tag{3.49}$$

and analogous ones with $H_i \to \bar{H}_i$ and $w \to \bar{w}$.

The relation between the two sets of generators is

$$H_0 = \frac{L_1 + L_{-1}}{2}\,, \quad H_{\pm 1} = L_0 \mp i\,\frac{L_1 - L_{-1}}{2} \tag{3.50}$$

Since the global energy E_{global} and the global momentum P_φ are equal to the Poincare energy $E_{Poincare}$ and the Poincare momentum P_x we must have

$$E_{global} = L_0 + \bar{L}_0 = E_{Poincare} = H_{-1} + \bar{H}_{-1}\,, \quad P_\varphi = -L_0 + \bar{L}_0 = P_x = -H_{-1} + \bar{H}_{-1} \tag{3.51}$$

which implies

$$L_1 - L_{-1} = \bar{L}_1 - \bar{L}_{-1} = 0 \tag{3.52}$$

The relations (3.52) may be readily verified for the trajectory under question by calculating the corresponding Noether charges. Computation of these charges require some care : since the transformations involve time, we cannot compute the charges starting from the static gauge lagrangian. Rather we should compute this *before* we choose the worldvolume time equal to the target space time. However we can choose the worldvolume angles equal to the target space angles as before. This partially gauge fixed lagrangian is given by

$$L = -\frac{M_n}{2}\sqrt{\dot{z}^2 + \dot{\bar{z}}^2 + 2\cosh 2\chi\, \dot{z}\,\dot{\bar{z}} - (2\dot{\chi})^2} \tag{3.53}$$

where the dot denotes derivative with respect to the worldvolume time $\tilde{\tau}$. The Noether charges corresponding to the $SL(2,R) \times SL(2,R)$ generators (3.29) and (3.30) are obtained by the substitutions

$$-i\partial_z = P_z\,, \quad -i\partial_{\bar{z}} = P_{\bar{z}}\,, \quad -i\partial_\chi = P_\chi \tag{3.54}$$

The momenta P_z, $P_{\bar{z}}$ and P_χ for a given configuration are given by

$$\begin{aligned}
P_z &= -\frac{M_n}{2}\frac{\dot{z} + \cosh 2\chi\,\dot{\bar{z}}}{\sqrt{\dot{z}^2 + \dot{\bar{z}}^2 + 2\cosh 2\chi\,\dot{z}\,\dot{\bar{z}} - (2\dot{\chi})^2}}\\
P_{\bar{z}} &= -\frac{M_n}{2}\frac{\dot{\bar{z}} + \cosh 2\chi\,\dot{z}}{\sqrt{\dot{z}^2 + \dot{\bar{z}}^2 + 2\cosh 2\chi\,\dot{z}\,\dot{\bar{z}} - (2\dot{\chi})^2}}\\
P_\chi &= M_n\frac{2\,\dot{\chi}}{\sqrt{\dot{z}^2 + \dot{\bar{z}}^2 + 2\cosh 2\chi\,\dot{z}\,\dot{\bar{z}} - (2\dot{\chi})^2}}
\end{aligned} \tag{3.55}$$

For our configuration with $\chi = \chi_0$, $z = 2\,\tilde{\tau}$, $\bar{z} = 0$ we find

$$P_z = -\frac{M_n}{2} \quad P_{\bar{z}} = -\frac{M_n}{2}\cosh 2\chi_0\,, \quad P_\chi = 0 \tag{3.56}$$

and thus the Noether charges evaluate to[23]

$$\begin{aligned}
L_0 &= -P_z = \frac{M_n}{2} \\
L_{-1} &= -e^{-iz}\,[\frac{\cosh 2\chi}{\sinh 2\chi}P_z - \frac{1}{\sinh 2\chi}P_{\bar z} + \frac{i}{2}P_\chi] = 0 \\
L_1 &= -e^{iz}\,[\frac{\cosh 2\chi}{\sinh 2\chi}P_z - \frac{1}{\sinh 2\chi}P_{\bar z} - \frac{i}{2}P_\chi] = 0 \\
\bar L_0 &= -P_{\bar z} = \frac{M_n}{2}\cosh 2\chi_0 \\
\bar L_{-1} &= -e^{-i\bar z}\,[\frac{\cosh 2\chi}{\sinh 2\chi}P_{\bar z} - \frac{1}{\sinh 2\chi}P_z + \frac{i}{2}P_\chi] = \frac{M_n}{2}\sinh 2\chi_0 \\
\bar L_1 &= -e^{i\bar z}\,[\frac{\cosh 2\chi}{\sinh 2\chi}P_{\bar z} - \frac{1}{\sinh 2\chi}P_z - \frac{i}{2}P_\chi] = \frac{M_n}{2}\sinh 2\chi_0
\end{aligned} \tag{3.57}$$

From the expressions above we verify that $L_1 - L_{-1} = 0$ and $\bar L_1 - \bar L_{-1} = 0$, which explains the equality of E, P between the global and Poincare systems. We also note that the charges satisfy the constraints

$$L_0^2 - L_1\,L_{-1} = \bar L_0^2 - \bar L_1\,\bar L_{-1} = \frac{M_n^2}{4} \tag{3.58}$$

Further, note that $L_1 = L_{-1} = 0$, so the configuration is a highest weight state of one of the $SL(2,R)$ aglebras. This gives $L_0 = \frac{M_n}{2}$, which yields $E = P + M_n$, the linear relation observed for the energy of the brane.

5.3.4 Reduction to IIA Black Holes

The geometry (3.14)-(3.16) can be reduced to IIA theory by a Kaluza Klein reduction along the y direction. Using the standard relation

$$ds_{11}^2 = e^{\frac{-2\Phi}{3}}ds_{10}^2 + e^{\frac{4\Phi}{3}}[dy - A_\mu dx^\mu]^2 \tag{3.59}$$

where ds_{10}^2 is the string metric, Φ is the dilaton and A_μ is the RR 1-form gauge field, it is straightforward to see that we get a 4-charge extremal black hole in four

[23]Note that, for $\chi_0 \neq 0$, our configuartion is not symmetric under exchange of z and $\bar z$: this is obviously because we have chosen $\dot\varphi = 1$. Another solution can be obtained with the choice $\dot\varphi = -1$.

dimensions

$$\begin{aligned} ds^2 &= -(H_0 h)^{-1/2} dt^2 + (H_0 h)^{1/2} [dr^2 + r^2 d\Omega_2^2] + (\frac{H_0}{h})^{1/2} \sum_i H_i ds_{T_i}^2 \end{aligned} \tag{3.60}$$

$$A^{(1)} = (1 - \frac{1}{H_0})\; dt \tag{3.61}$$

$$A^{(3)} = \sin\theta\; d\theta d\phi\, [p_3\, \frac{y^5 dy^6 - y^6 dy^5}{2} + p_2\, \frac{y^3 dy^4 - y^4 dy^3}{2} + p_1\, \frac{y^1 dy^2 - y^2 dy^1}{2}] \tag{3.62}$$

$$e^{\Phi} = \frac{H_0^3}{h} \tag{3.63}$$

The near-horizon limit of this IIA metric depends on whether or not q_0 is non-vanishing. For $q_0 = 0$ this has a null singularity at $r = 0$. Note that this limiting metric is *not* the dimensional reduction of the metric (3.19).

For $q_0 \neq 0$ the geometry is $AdS_2 \times S^2 \times T^6$. This may be seen by looking at the above formulae for $r \ll q_0, p_i$. The resulting metric, 1-form potential and dilaton are given by

$$ds^2 = -\frac{r^2}{R_{IIA}^2}dt^2 + \frac{R_{IIA}^2}{r^2}dr^2 + R_{IIA}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.64}$$

$$+ \sqrt{\frac{q_0 p_1}{p_2 p_3}}((dy^1)^2 + (dy^2)^2) + \sqrt{\frac{q_0 p_2}{p_3 p_1}}((dy^3)^2 + (dy^4)^2) \tag{3.65}$$

$$+ \sqrt{\frac{q_0 p_3}{p_1 p_2}}((dy^5)^2 + (dy^6)^2) \tag{3.66}$$

$$A^{(1)} = [1 - \frac{r}{q_0}]\ dt \tag{3.67}$$

$$A^{(3)} = \sin\theta\ d\theta d\phi\,[p_3\ \frac{y^5 dy^6 - y^6 dy^5}{2} + p_2\ \frac{y^3 dy^4 - y^4 dy^3}{2} + p_1\ \frac{y^1 dy^2 - y^2 dy^1}{2}] \tag{3.68}$$

$$e^{\Phi} = \frac{q_0}{R_{IIA}} \tag{3.69}$$

where

$$R_{IIA} = (q_0 p_1 p_2 p_3 p_4)^{1/4} \tag{3.70}$$

If we replace the internal torus with a Calabi-Yau manifold, this is the background which is used in [107]- [109].

Equation (3.69) is the metric in Poincare coordinates. The coordinate transformations

$$\frac{R_{IIA}}{r} = \frac{1}{\cosh\chi\cos\tau + \sinh\chi} \tag{3.71}$$

$$t = \frac{R_{IIA}\cosh\chi\sin\tau}{\cosh\chi\cos\tau + \sinh\chi} \tag{3.72}$$

can be used to continue this metric to global coordinates

$$ds^2 = R_{IIA}^2(-\cosh^2\chi\ d\tau^2 + d\chi^2) + R_{IIA}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.73}$$

$$+\sqrt{\frac{q_0 p_1}{p_2 p_3}}((dy^1)^2 + (dy^2)^2) + \sqrt{\frac{q_0 p_2}{p_3 p_1}}((dy^3)^2 + (dy^4)^2) \tag{3.74}$$

$$+ \sqrt{\frac{q_0 p_3}{p_1 p_2}}((dy^5)^2 + (dy^6)^2) \tag{3.75}$$

and one can choose a gauge in which the 1-form potential becomes

$$A^{(1)} = -\frac{R_{IIA}}{q_0}[1 - \sinh\chi]d\tau \tag{3.76}$$

In the IIA language the M2 brane becomes a D2 brane and the momentum along the y direction becomes a D0 charge because of the presence of a worldvolume gauge field

$$F = \frac{f}{2\pi\alpha'}\sin\theta d\theta \wedge d\phi \tag{3.77}$$

The contribution to the D0 brane charge to the mass of this brane in string metric is

$$M_0 = 4\pi\mu_2 f \tag{3.78}$$

where μ_2 is the D2 brane tension. The global hamiltonian may be written down using standard methods

$$H = \cosh\chi[(M_2^2 + M_0)^2\ e^{-2\Phi} + P_\chi^2 + \frac{(P_i - 4\pi\mu_2 A_i)^2}{g_{ii}}]^{\frac{1}{2}} + M_0\ e^{-\Phi}[1 - \sinh\chi] \tag{3.79}$$

where in writing down the last term we have used the explicit form of the dilaton in (3.69). (Here $A^{(3)} \equiv A_i\ \cos\theta\ d\theta d\phi dy^i$.) We have also denoted the mass of the D2 brane by M_2

$$M_2 = 4\pi R_{IIA}^2\mu_2 \tag{3.80}$$

A static solution is obtained at a value of $\chi = \chi_0$ given by

$$\tanh \chi_0 = \frac{M_0}{\sqrt{M_2^2 + M_0^2}} \tag{3.81}$$

and the value of the energy is

$$E = (M_0 + M_2)e^{-\Phi} \tag{3.82}$$

which is what we expect from the dimensional reduction of the M theory result.

Note that the magnitude of the energy depends on the gauge choice for $A^{(1)}$. We have intentionally chosen a gauge which leads to an energy which is identical to the M-theory result. A gauge transformation on $A^{(1)}$ translates to a *coordinate* transformation in the M theory which redefines the coorinate y and therefore changes the Killing vector along which dimensional reduction is performed to obtain the IIA theory. For example instead of the choice in (3.69) we could have chosen

$$A^{(1)\prime} = -\frac{r}{q_0}\, dt \tag{3.83}$$

which is related to the original potential by a gauge transformation. From (3.59) it is easy to see that this corresponds to a coordinate transformation on y, $y \to y + t$. Thus this gauge potential would arise from a KK reduction of the 11 dimensional metric along $y + t$ rather than y. In this situation we do not of course expect the energy as calculated in IIA to agree with the energy as calculated in M theory.

The expression for the hamiltonian, (3.79) is not a sum of positive terms and it is not evident that the static solution has the lowest energy. However it is not hard to see that this is indeed the ground state, using the trick of [106]. It is convenient to use coordinates $\rho = \sinh \chi$ so that the metric of the AdS part becomes

$$ds^2 = -(1 + \rho^2)\, d\tau^2 + \frac{d\rho^2}{1 + \rho^2} \tag{3.84}$$

The expression for the energy is

$$E = \frac{\sqrt{M_2^2 + M_0^2}|g_{\tau\tau}|e^{-\Phi}}{\sqrt{|g_{\tau\tau}| - g_{\rho\rho}(\partial_\tau \rho)^2}} + M_0 e^{-\Phi}(1-\rho) \tag{3.85}$$

This equation may be now re-written as

$$(\partial_\tau \rho)^2 + 2U(\rho) = 0 \tag{3.86}$$

where

$$2U(\rho) = \frac{(M_0^2 + M_2^2)(1+\rho^2)^3}{((Ee^{\Phi} - M_0) + M_0\rho)^2} - (1+\rho^2)^2 \tag{3.87}$$

The relativistic dynamics of the $D2$ brane is thus identical to the *non-relativistic* dynamics of a particle of unit mass moving in a potential $U(\rho)$. The energy of this analog non-reltivistic problem is zero.

A solution to this non-relativistic problem will exist only if $U(\rho) = 0$ for some real ρ. From (3.87) we see that this happens when

$$M_2^2\rho^2 - 2M_0(Ee^{\Phi} - M_0)\ \rho - (Ee^{\Phi} - M_0)^2 + (M_2^2 + M_0^2) = 0 \tag{3.88}$$

This has a real solution only if

$$E \geq (M_2 + M_0)e^{-\Phi} \tag{3.89}$$

which establishes the lower bound on the energy. When the energy saturates this bound the solution is static.

Poincare energies

The Poincare energies and momenta for this $D2$ brane are again equal to the global energies and momenta. The transformations are given in (3.72). The trajectory is then given by

$$\sinh \chi_0 = \frac{u}{2R_{IIA}}[1 - (\frac{R_{IIA}^2}{u^2} - \frac{t^2}{R_{IIA}^2})] \tag{3.90}$$

This is again a trajectory which comes out of the horizon and returns to it in finite proper time. The maximum value of u now turns out to be

$$u_{max} = R_{IIA}\ e^{\chi_0} \tag{3.91}$$

The value of the Poincare energy for this trajectory is

$$E_{Poincare} = \frac{M g_{tt} e^{-\Phi}}{\sqrt{g_{tt} - g_{rr}(\partial_\tau r)^2}} + M_0 e^{-\Phi}[1 - \frac{r}{R_{IIA}}] = (M_0 + M_2)\ e^{-\Phi} \tag{3.92}$$

which is again *exactly* equal to the global energy E_{global}.

Just as in the subsection (5.3.3), the equality of Poincare and global energies has a group theoretic significance. In terms of light cone coordinates $t_\pm = t \pm \frac{R_{IIA}^2}{r}$ the generators of the $SL(2,R)$ conformal isometries of AdS_2 are

$$h = L_{-1} = \frac{\partial}{\partial t_+} + \frac{\partial}{\partial t_-}, \quad d = L_0 = t_+\frac{\partial}{\partial t_+} + t_-\frac{\partial}{\partial t_-}, \quad k = L_1 = t_+^2\frac{\partial}{\partial t_+} + t_-^2\frac{\partial}{\partial t_-} \tag{3.93}$$

and the transformation to global coordinates is given by

$$t_\pm = \tan\ [\frac{1}{2}(\tau \pm \frac{1}{\cosh\chi})] \tag{3.94}$$

The global hamiltonian H is then

$$H_{global} = \frac{\partial}{\partial\tau} = h + k \tag{3.95}$$

Since the configurations we discussed have $H_{global} = h$ these must have $k = 0$. k is the generator of conformal boosts and the standard $SL(2,R)$ algebra obeyed by $L_\pm, L_0$ then implies that this state is a highest weight state.

The computations of these conserved charges follow the procedure of subsection (5.3.3). The partially gauge fixed action (for lowest Landau level orbits on the T^6)

$$S = -\frac{4\pi\mu_2 R_{IIA}^2}{q_0}\int \frac{d\tau}{v(\tau)}[\sqrt{R_{IIA}^4 + f^2}\ \sqrt{(\partial_\tau t)^2 - (\partial_\tau v)^2} - f(\partial_\tau t)] \tag{3.96}$$

The conserved charges in the static gauge are [24].

$$h = \frac{4\pi\mu_2 R_{IIA}^2}{q_0 v}[\frac{A}{\sqrt{1-\dot{v}^2}} - f] \tag{3.97}$$

$$d = \frac{4\pi\mu_2 R_{IIA}^2}{q_0 v}[-\frac{At}{\sqrt{1-\dot{v}^2}} + \frac{Av\dot{v}}{\sqrt{1-\dot{v}^2}} + ft] \tag{3.98}$$

$$k = \frac{4\pi\mu_2 R_{IIA}^2}{q_0 v}[\frac{A}{\sqrt{1-\dot{v}^2}}(tv\dot{v} - \frac{1}{2}(t^2+v^2)) - \frac{f}{2}(v^2-t^2)] \tag{3.99}$$

Substituting the trajectory (3.90) we find that k evaluates to zero.

Validity of the near-horizon approximation

The branes we discussed so far were shown to be stable and static in global time in the near horizon geometry of the 4d extremal black hole. From the point of view of black hole physics these would be of interest only if they exist in the full asymptotically flat geometry. In the full geometry, the near-horizon region is a Poincare patch of AdS and we have seen that in Poincare coordinates the brane comes out of the horizon and goes back into it. This is what one would expect in the full geometry as well. However we have to check whether the approximation of restriction to the near-horizon limit is self-consistent. In the Appendix, this is done for four dimensional black hole geometry of section (5.3.4). We find that the brane remains in the near-horizon region so long as $M_0 \ll M_2$, but goes out of this region otherwise

5.3.5 Examples in Type IIB String Theory

Another example is provided by extremal black strings in Type IIB string theory compactified on T^4 formed by two sets of D3 branes intersecting along a line together

[24]Note that the lagrangian is not invariant under special conformal transformations, though the action is - this results in an additional contribution to the Noether charge

with some momentum along the intersection, and its dimensional reduction to five dimensional black holes. The physics is identical to black strings in M theory and their reduction to four dimensional black holes considered above. The calculations are identical and will not be repeated here.

5.4 $D3$ branes in 2-charge microstate geometries

We have examined branes in $AdS_m \times S^n$ spaces, and computed their energy. But we can make a symmetry transformation in the AdS, and change what we call E. If on the other hand we had an asymptotically flat spacetime then we might get a physically unique definition of energy. Note also that the goal of [109] was to study black hole states. Black holes have asymptotically flat geometries, and we measure the energy of different excitations using the time at infinity. So it would be helpful if we could study branes in spacetimes which have the *global* $AdS_m \times S^n$ structure in some region (we will wrap the test branes on the S^n) but which go over to asymptotically flat spacetime at large r.

Interestingly, such geometries are given by microstates of the 2-charge system. In [186, 188] it was found that metrics carrying D1 and D5 charges and a certain amount of rotation had the above mentioned property: they were asymptotically flat at large r but were $AdS_3 \times S^3 \times T^4$ in the small r region. The point to note is that the geometries were not just *locally* $AdS_3 \times S^3$ in the small r region; rather the small r region had the shape of a 'cap' which looked like the region $r < r_0$ of *global* $AdS_3 \times S^3$.

In detail, we take type IIB string theory, compactified on $T^4 \times S^1$. We wrap D1 branes on the S^1 and we wrap D5 branes on $S^1 \times T^4$. Let the S^1 be parametrized by y, with $0 < y < 2\pi R$, and the T^4 be parametrized by coordinates y^1, y^2, y^3, y^4

with a overall volume V. For our present purposes we will do two T-dualities, in the directions y^1, y^2, so that the system is composed of two sets of D3 branes. These branes extend along y^1, y^2, y and along y^3, y^4, y respectively. This does not change the nature of the geometry that we have described above.

We can now consider a D3 brane wrapped over the S^3, and let it move in the direction y. This situation with the D3 brane is very similar to the case of the M2 brane that we had studied above, and we expect to get similar results on the energy . But now we can extend our analysis to a spacetime which is asymptotically flat, so we can identify the charges which correspond to the energy E (conjugate to time t at infinity) and the momentum P (conjugate to the variable y).

We can extend the analysis to a class of geometries that carry *three charges*: the two D3 brane charges as above as well as momentum P along S^1. The geometries for specific microstates of this system were constructed in [114], and these again have an AdS type region at small r and go over to flat space at infinity.

In each of the above cases we find, somewhat surprisingly, that we again get a relation of the form $E = P + Constant$. This might suggest that there is again an underlying symmetry that rotates orbits of the wrapped brane, but we have not been able to identify such a symmetry.

5.4.1 The 2-charge microstate geometry

The string frame metric is given by

$$\begin{aligned}
ds^2 &= -h^{-1}(dt^2-dy^2)+hf\left(\frac{dr^2}{r^2+a^2\gamma^2}+d\theta^2\right)\\
&+ h\Big(r^2+\frac{Q_1Q_2a^2\gamma^2\cos^2\theta}{h^2f^2}\Big)\cos^2\theta\,d\psi^2+h\Big(r^2+a^2\gamma^2-\frac{Q_1Q_2a^2\gamma^2\sin^2\theta}{h^2f^2}\Big)\sin^2\theta\,d\phi^2\\
&- 2\frac{a\gamma\sqrt{Q_1Q_2}}{hf}\cos^2\theta\,d\psi\,dy-2\frac{a\gamma\sqrt{Q_1Q_2}}{hf}\sin^2\theta\,d\phi\,dt\\
&+ \sqrt{\frac{Q_2}{Q_1}}(dy_1^2+dy_2^2)+\sqrt{\frac{Q_1}{Q_2}}(dy_3^2+dy_4^2)\\
h &= \sqrt{\Big(1+\frac{Q_1}{f}\Big)\Big(1+\frac{Q_2}{f}\Big)},\quad f=r^2+a^2\gamma^2\cos^2\theta
\end{aligned}\tag{4.100}$$

while the dilaton field vanishes. There is a 4-form potential given by

$$\begin{aligned}
A^{(4)} &= \Big[-\frac{Q_1}{f+Q_1}dt\wedge dy-\frac{Q_2(r^2+a^2\gamma^2+Q_1)}{f+Q_1}\cos^2\theta\,d\psi\wedge d\phi\\
&- \frac{a\gamma\sqrt{Q_1Q_2}}{f+Q_1}\cos^2\theta\,dt\wedge d\psi-\frac{a\gamma\sqrt{Q_1Q_2}}{f+Q_1}\sin^2\theta\,dy\wedge d\phi\Big]\wedge dy^1\wedge dy^2
\end{aligned}\tag{4.101}$$

However the experience of the previous sections show that the only role of this is to put a probe $D3$ brane in a Lowest Landau level orbit on the T^4. We will therefore ignore this in the following discussion.

This geomtery reduces to the asymptotically flat space-time $M^{1,5}\times T^4$ in the large r limit. In the limit $r^2, a^2 \ll \sqrt{Q_1Q_2}$ the metric becomes

$$\begin{aligned}
ds^2 &= \sqrt{Q_1Q_2}\Big(\frac{dr^2}{r^2+a^2\gamma^2}+\frac{r^2}{Q_1Q_2}dy^2-\frac{r^2+a^2\gamma^2}{Q_1Q_2}dt^2\Big)\\
&+ \sqrt{Q_1Q_2}(d\theta^2+\cos^2\theta\,d\psi'^2+\sin^2\theta\,d\phi'^2)+\sqrt{\frac{Q_2}{Q_1}}(dy_1^2+dy_2^2)+\sqrt{\frac{Q_1}{Q_2}}(dy_3^2+dy_4^2)
\end{aligned}\tag{4.102}$$

where ψ' and ϕ' are "NS sector coordinates"

$$\psi'=\psi-\frac{a\gamma}{\sqrt{Q_1Q_2}}y\,,\quad \phi'=\phi-\frac{a\gamma}{\sqrt{Q_1Q_2}}t \tag{4.103}$$

For $\gamma = 1$, this is precisely global $Ad_3 \times S^3 \times T^4$ as may be seen by making the coordinate transformations to

$$\tau = \frac{a\gamma t}{\sqrt{Q_1 Q_2}} \qquad \varphi = \frac{a\, y\gamma}{\sqrt{Q_1 Q_2}} \qquad r = a\gamma\ \sinh\chi \tag{4.104}$$

For $\gamma = 1/k$, with k integer greater than 1, the "near horizon" geometry is an orbifod space of the type $(Ad_3 \times S^3)/Z_k \times T^4$.

The geometry therefore smoothly interpolates between *global AdS* (or an orbifold of it) and flat space. The key fact about this geometry is that in the small r region t is the *global* time in AdS_3, while in the large r region the same t is the usual Minkowski time in the asymptoically flat space-time. This is in contrast to the geometry of three charge black holes in five dimensions where the Minkowski time of the asymptotic region becomes the *Poincare* time of the near-horizon region. Therefore we can address the question of wrapped $D3$ branes in the full geometry.

5.4.2 $D3$ branes in 2-charge microstate geometry

In the geometry described above, consider a D3 brane wrapping the angular S^3 and carrying momentum P along the circle y. This brane couples to the background $F^{(5)}$ flux, which extends in the S^3 directions as well as two of the directions of T^4, and hence behaves like a charged particle moving in a magnetic field on T^4. This system represents thus a five dimensional analogue of the S^2 wrapped D2 brane in a 4d black hole, studied in section 2.

Choosing t, θ, ψ and ϕ as worldvolume coordinates, the square root of the determinant of the metric induced on the D3 brane can be written in the form

$$\sqrt{-\mathrm{det}P(g)} = \sin\theta\,\cos\theta\,\sqrt{(r^2 + a^2\gamma^2)\,F_1 - r^2\,F_2\,\dot{y}^2} \tag{4.105}$$

where we have defined

$$F_1 = r^2\,(f + Q_1 + Q_2) + Q_1 Q_2\,, \quad F_2 = (r^2 + a^2\,\gamma^2)\,(f + Q_1 + Q_2) + Q_1 Q_2 \tag{4.106}$$

It is then straightforward to compute the D3 brane Lagrangian

$$L = -\mu_3\,(2\pi)^2 \int d\theta\,\sin\theta\,\cos\theta\,\sqrt{(r^2 + a^2\,\gamma^2)\,F_1 - r^2\,F_2\,\dot{y}^2} \tag{4.107}$$

the momentum conjugate to y

$$P = \mu_3\,(2\pi)^2 \int d\theta\,\sin\theta\,\cos\theta\,\frac{r^2\,F_2\,\dot{y}}{\sqrt{(r^2 + a^2\,\gamma^2)\,F_1 - r^2\,F_2\,\dot{y}^2}} \tag{4.108}$$

and the energy of the D3 brane

$$E = \mu_3\,(2\pi)^2 \int d\theta\,\sin\theta\,\cos\theta\,\frac{(r^2 + a^2\,\gamma^2)\,F_1}{\sqrt{(r^2 + a^2\,\gamma^2)\,F_1 - r^2\,F_2\,\dot{y}^2}} \tag{4.109}$$

Though the θ integrals could be explicitly computed, we find it more convenient to perform integrations only after having minimized the energy.

The location at which the D3 brane stabilizes can be found by either minimizing E with respect to r^2 keeping P fixed or minimizing L with respect to r^2 keeping $\dot{y}$ fixed. The second way is the most convenient and yelds the following, surprinsingly simple, result:

$$\frac{\partial L}{\partial r^2} = 0 \;\Rightarrow\; \partial_{r^2}[(r^2 + a^2\,\gamma^2)F_1] - \partial_{r^2}[r^2 F_2]\,\dot{y}^2 = 0 \tag{4.110}$$

$$\Rightarrow \dot{y}^2 = \frac{r^2(f + Q_1 + Q_2) + Q_1Q_2 + (r^2 + a^2\,\gamma^2)(f + r^2 + Q_1 + Q_2)}{(r^2 + a^2\,\gamma^2)(f + Q_1 + Q_2) + Q_1Q_2 + r^2(f + r^2 + a^2\,\gamma^2 + Q_1 + Q_2)} = 1$$

The location at which the D3 brane sits is then found by putting $\dot{y} = 1$ in the expression (4.108) for P and solving with respect to r. Note that for $\dot{y} = 1$ the square root which appears in the expression for P and E simplifies

$$\sqrt{(r^2 + a^2\,\gamma^2)\,F_1 - r^2\,F_2} = a\,\gamma\,\sqrt{Q_1 Q_2} \tag{4.111}$$

The expressions for the momentum and energy of the D3 at its stable point are then

$$
\begin{aligned}
P &= \frac{\mu_3\,(2\pi)^2}{a\,\gamma\,\sqrt{Q_1 Q_2}} \int d\theta\, \sin\theta\, \cos\theta\, r^2 F_2 \\
&= \frac{\mu_3\,(2\pi)^2}{2\,a\,\gamma\,\sqrt{Q_1 Q_2}}\, r^2 \left[Q_1 Q_2 + (r^2 + a^2\,\gamma^2)\left(r^2 + \frac{a^2\,\gamma^2}{2} + Q_1 + Q_2\right)\right] \\
E &= \frac{\mu_3\,(2\pi)^2}{a\,\gamma\,\sqrt{Q_1 Q_2}} \int d\theta\, \sin\theta\, \cos\theta\, (r^2 + a^2\,\gamma^2) F_1 \\
&= \frac{\mu_3\,(2\pi)^2}{2\,a\,\gamma\,\sqrt{Q_1 Q_2}}\, (r^2 + a^2\,\gamma^2) \left[Q_1 Q_2 + r^2\left(r^2 + \frac{a^2\,\gamma^2}{2} + Q_1 + Q_2\right)\right] \quad (4.112)
\end{aligned}
$$

From the expressions above we see that the dispersion relation of the D3 brane is

$$
E = P + 2\pi^2\,\mu_3\,\sqrt{Q_1 Q_2}\,a\,\gamma \tag{4.113}
$$

Remarkably, this is *identical* to the formula we would have obtained if we performed the analysis in the AdS limit. This may be easily seen from the general formulae of section (5.2) and noting that the standard AdS coordinates are related to the coordinates r, t, y by the equations in (4.104) and that the AdS scale is given by $(Q_1 Q_2)^{1/4}$.

We would like to emphasize that the definition of energy is completely unambigious in this geometry because of the presence of an asymptotically flat region. Furthermore from general grounds we know that if we simply added pure momentum to the 2-charge microstate geometry the additional ADM energy is simply equal to the momentum. This is what happens if we take the formal limit $\mu_3 = 0$ in (4.113) which shows we have taken the zero of the energy correctly.

Even though the dispersion relation is the same as in the AdS limit, the location of the brane obtained by solving the first equation of (4.112) has a modified dependence on the momentum P. We would like to determine the range of parameters for which this location lies in the AdS region. We have not obtained the general solution of the

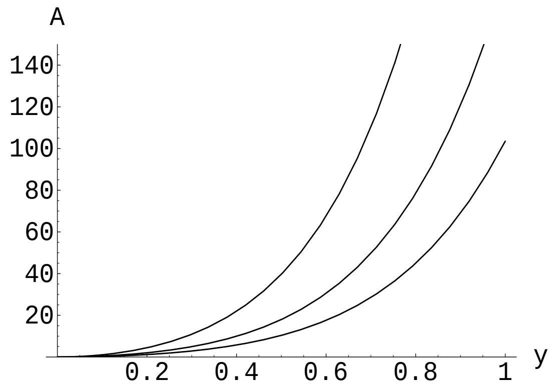

Figure 5.1: The ratio A plotted as a function of y. The curves have $b = 0.1, 0.15, 0.2$ starting from top to bottom

equation. However to get an idea we examine the solution for $Q_1 = Q_2 = \lambda^2$. The quantity λ will become the scale of the AdS in the appropriate region. In this case it is useful to express this equation in terms of the following quantities

$$A \equiv \frac{P}{2\pi^2 a\gamma\lambda^2\mu_3} \qquad y \equiv \frac{r}{\lambda} \qquad b = \frac{a\gamma}{\lambda} \tag{4.114}$$

Note that A is the ratio of the contributions to the from the momentum and the $D3$ brane (as in (4.113). The AdS region of the solution corresponds to $y, b \ll 1$.

The first equation of (4.112) then becomes

$$A = (\frac{y}{b})^2 \; [1 + (y^2 + b^2)(y^2 + \frac{1}{2}b^2 + 2)] \tag{4.115}$$

Figure (5.1) shows a plot of A versus y for various values of b. The brane location moves further away from the center of AdS as we increase the ratio A, and for a given value of A, the brane location $r = r_0$ is larger for larger values of a. This shows that for small values of b there is a large range of values of A for which the brane sits in the AdS region of small y.

5.4.3 CFT Duals

In order to gain some insight on the dual CFT significance of the D3 brane configuarion discussed here, let us rewrite the expression above in terms of microscopic quantities. If R is the radius of the y circle, $V = L_1 L_2 L_3 L_4$ is the volume of T^4, g the string coupling and n_1 and n_2 are the numbers of D3 branes wrapped on y^1, y^2, y and y^3, y^4, y, one has

$$a = \frac{\sqrt{Q_1 Q_2}}{R}, \quad \mu_3 = \frac{1}{(2\pi)^3 \alpha'^2 g}, \quad Q_1 = \frac{(2\pi)^2 g \alpha'^2}{L_3 L_4} n_1, \quad Q_2 = \frac{(2\pi)^2 g \alpha'^2}{L_1 L_2} n_2 \tag{4.116}$$

and thus

$$E = P + 2\pi^2 \mu_3 \frac{Q_1 Q_2}{k R} = P + \frac{4\pi^3 \alpha'^3 g}{V} \frac{1}{R} \frac{n_1 n_2}{k} \tag{4.117}$$

While the significance of this result is not clear to us, it s interesting that the powers of the charges are integral, so we get a quantity $\frac{n_1 n_2}{k}$ that counts the number of 'component strings' in the CFT microstate (see [115] for a discussion of the microstate in terms of component strings). Further the energy comes in units of $\frac{1}{R}$ which is the natural qantum of energy in the CFT which lives on a circle of radius R.

5.4.4 Vibration modes

Let us look at the D3 brane cosidered above, and restrict attention to the small r region where the geometry is $AdS_3 \times S^3$. We have found the energy E of the brane in a specific configuration (which minimised E for a given P), but we can now ask for the properties of small vibratons of the brane around this configuration. We will only consider oscillations in the AdS_3 directions, so that, in a static gauge, we can write

$$\chi = \chi_0 + \epsilon\, \delta\chi(\tau, \theta_i), \quad y^5 = \dot{y}\, \tau + \epsilon\, \delta y^5(\tau, \theta_i), \quad y^i = y^i_0, i = 1 \ldots, 4 \tag{4.118}$$

where we have denoted coordinates on S^3 by θ_i, $i = 1, \ldots, 3$ and the metric on a S^3 of unit radius by g_3 .

We will compute the action of the D3 brane up to quadratic order in ϵ. Having suppressed oscillations in the T^4 directions, only the DBI term contributes. The term of first order in ϵ is

$$S_1 = \epsilon\, \mu_3\, \lambda^4 \int d\tau\, d^3\theta_i\, \sqrt{g_3}\, \frac{\sinh \chi_0}{\sqrt{\cosh^2 \chi_0 - \dot{y}^2 \sinh^2 \chi_0}} \left[(\dot{y}^2 - 1) \cosh \chi\, \delta\chi + \dot{y}\, \sinh \chi_0\, \partial_\tau \delta y^5\right] \quad (4.119)$$

The term proportional to $\delta\chi$ vanishes for $\dot{y}^2 = 1$, while the coefficient of $\partial_\tau \delta y^5$ is a constant and thus this term does not contribute to the equations of motion. Restricting to $\dot{y}^2 = 1$, and performing the change of coordinates $\rho = \sinh \chi$, the term of second order in ϵ is

$$\begin{aligned} S_2 \;=\; & -\epsilon^2\, \mu_3\, \lambda^4 \int d\tau\, d^3\theta_i\, \sqrt{g_3} \Big[\frac{g_3^{ij}}{2} \frac{\partial_i\delta\rho\, \partial_j\delta\rho}{\rho_0^2 + 1} - \frac{1}{2} \frac{(\partial_t\delta\rho)^2}{\rho_0^2 + 1} \\ & + \frac{g_3^{ij}}{2} \rho_0^2(\rho_0^2 + 1)\, \partial_i\delta y^5\, \partial_j\delta y^5 - \frac{1}{2}\, \rho_0^2(\rho_0^2 + 1)\, (\partial_t\delta y^5)^2 - 2\, \rho_0\, \delta\rho\, \partial_t\delta y^5 \Big] \end{aligned} \quad (4.120)$$

If one expands the perturbations $\delta\rho$ and δy^5 as

$$\delta\rho(\tau, \theta_i) = \delta\tilde{\rho}\, e^{-i\omega\tau}\, Y_l(\theta_i)\,, \quad \delta y^5(\tau, \theta_i) = \delta\tilde{y}^5\, e^{-i\omega\tau}\, Y_l(\theta_i) \quad (4.121)$$

where Y_l are spherical harmonics on S^3

$$\frac{1}{\sqrt{g_3}}\, \partial_i \left(g_3^{ij} \partial_j Y_l(\theta_i)\right) = -Q_l\, Y_l(\theta_i)\,, \quad Q_l = l(l + 2) \quad (4.122)$$

the equations of motion derived from the action (4.120) become

$$\begin{pmatrix} (\rho_0^2 + 1)^{-1}\, (-Q_l + \omega^2) & -2\, i\, \omega\, \rho_0 \\ 2\, i\, \omega\, \rho_0 & \rho_0^2\, (\rho_0^2 + 1)\, (-Q_l + \omega^2) \end{pmatrix} = \begin{pmatrix} \delta\tilde{\rho} \\ \delta\tilde{y}^5 \end{pmatrix} \quad (4.123)$$

The vibration frequencies are then

$$\omega^2 = Q_l + 2 \pm 2\sqrt{Q_l + 1} = l(l + 2) + 2 \pm 2(l + 1) \quad (4.124)$$

or equivalently

$$\omega = l + 2 \quad \text{and} \quad \omega = l \tag{4.125}$$

Note that ω denotes the conjugate of the dimensionless coordinate τ. This is related to the physical energies by a suitable factor of the AdS scale. We therefore see that *the frequencies are universal.* They depend only on the AdS scale of the background and not on the value of the momentum P of the brane. This is similar to what happens for giant gravitons [105].

5.5 D3 branes in 3-charge microstates

By applying a spectral flow to the two charge microstate of the previous subsections one obtains a geometry dual to a three charge microstate. This is described, in the string frame, by the following metric and dilaton [114]

$$\begin{aligned} ds^2 &= -\frac{1}{h}(dt^2 - dy^2) + \frac{Q_p}{hf}(dt-dy)^2 + hf\left(\frac{dr^2}{r^2 + (\gamma_1+\gamma_2)^2\eta} + d\theta^2\right) \\ &+ h\Big(r^2 + \gamma_1(\gamma_1+\gamma_2)\eta - \frac{Q_1Q_2(\gamma_1^2-\gamma_2^2)\eta\cos^2\theta}{h^2f^2}\Big)\cos^2\theta d\psi^2 \\ &+ h\Big(r^2 + \gamma_2(\gamma_1+\gamma_2)\eta + \frac{Q_1Q_2(\gamma_1^2-\gamma_2^2)\eta\sin^2\theta}{h^2f^2}\Big)\sin^2\theta d\phi^2 \\ &+ \frac{Q_p(\gamma_1+\gamma_2)^2\eta^2}{hf}\big(\cos^2\theta d\psi + \sin^2\theta d\phi\big)^2 \\ &- \frac{2\sqrt{Q_1Q_2}}{hf}\big(\gamma_1\cos^2\theta d\psi + \gamma_2\sin^2\theta d\phi\big)(dt-dy) \\ &- \frac{2\sqrt{Q_1Q_2}(\gamma_1+\gamma_2)\eta}{hf}\big(\cos^2\theta d\psi + \sin^2\theta d\phi\big)dy + \sqrt{\frac{H_1}{H_2}}(dy_1^2+dy_2^2) + \sqrt{\frac{H_2}{H_1}}(dy_3^2+dy_4^2) \end{aligned} \tag{5.126}$$

$$e^{2\Phi} = \frac{H_1}{H_2} \tag{5.127}$$

where

$$\eta = \frac{Q_1 Q_2}{Q_1 Q_2 + Q_1 Q_p + Q_2 Q_p}$$
$$f = r^2 + (\gamma_1 + \gamma_2)\,\eta\,\big(\gamma_1 \sin^2\theta + \gamma_2 \cos^2\theta\big)$$
$$H_1 = 1 + \frac{Q_1}{f}\,, \quad H_2 = 1 + \frac{Q_2}{f}\,, \quad h = \sqrt{H_1 H_2} \tag{5.128}$$

For the solution obtained by spectral flow from the 2-charge microstate geometry, the parameters γ_1 and γ_2 take the values

$$\gamma_1 = -a\,n\,, \quad \gamma_2 = a\left(n + \frac{1}{k}\right), \quad a = \frac{\sqrt{Q_1 Q_2}}{R} \tag{5.129}$$

where R is the y radius and n and k are integers. Geometries corresponding to other values of γ_1 and γ_2 can be obtained by S and T dualities.

In this geometry, consider a D3 brane wrapping the angular S^3 and rotating along y. The determinant of the induced metric in static gauge can be cast the the form

$$\sqrt{-\det P(g)} = -\sin\theta\cos\theta\,\sqrt{c_0 + \dot{y}\,c_1 + \dot{y}^2\,c_2} \tag{5.130}$$

where c_0, c_1 and c_2 are functions of r and θ that can be computed using Mathematica. As we did not manage to bring these functions to reasonably simple form, we do not give their explicit expressions here. We can however proceed with the help of Mathematica and verify that the r-derivative of the Lagrangian

$$L = -\mu_3\,(2\pi)^2 \int d\theta\,\sin\theta\,\cos\theta\,\sqrt{c_0 + \dot{y}\,c_1 + \dot{y}^2\,c_2} \tag{5.131}$$

at fixed $\dot{y}$ vanishes for $\dot{y} = 1$ (note that in this case the invariance under $y \to -y$ is broken by the momentum carried by the background metric (5.126) and $\dot{y} = -1$ is not a local minimum). For this value of $\dot{y}$ the determinant of the induced metric simplifies to

$$\sqrt{-\det P(g)} = -\sin\theta\cos\theta\,(\gamma_1 + \gamma_2)\,\eta\,\sqrt{Q_1 Q_2} \tag{5.132}$$

Following the same steps as in the previous subsection, one can compute the energy and momentum conjugate to y at the stable point $\dot{y} = 1$:

$$\begin{aligned} E &= \frac{\mu_3\,(2\pi)^2}{(\gamma_1+\gamma_2)\,\eta\,\sqrt{Q_1Q_2}} \int d\theta\,\sin\theta\,\cos\theta\,\frac{2c_0+c_1}{2} \\ P &= -\frac{\mu_3\,(2\pi)^2}{(\gamma_1+\gamma_2)\,\eta\,\sqrt{Q_1Q_2}} \int d\theta\,\sin\theta\,\cos\theta\,\frac{2c_2+c_1}{2} \end{aligned} \tag{5.133}$$

Neither E or P have particularly good looking expressions, but their difference is simply given by

$$E = P + 2\pi^2\,\mu_3\,\sqrt{Q_1Q_2}\,(\gamma_1+\gamma_2)\,\eta = P + 2\pi^2\,\mu_3\,\frac{\sqrt{Q_1Q_2}\,a}{k}\,\eta \tag{5.134}$$

where in the last equality we have used the values (5.129) for γ_1 and γ_2.

We thus conclude that the dispersion relation of the D3 brane in the three charge geometry differs from that in the two charge geometry only by a factor of η.

5.6 The field produced by the wrapped brane

In this section we look at the gauge field produced by the D3 brane that we wrap on the S^3, in the asymptotically flat 2-charge microstate geometry. If we think of the brane as a small perturbation of strength ϵ on the background, then the field strength produced by the brane is also of order ϵ, and the energy carried by this field is $O(\epsilon^2)$. But we find that the field strength goes to a constant at large r, so that its overall energy would diverge. The brane wrapped on the S^3 appears to behave like a domain wall in the spacetime, making the field nonzero on the outside everywhere.

The action for the 4-form RR field $A^{(4)}$ sourced by the D3 brane is

$$S = \frac{1}{2}\int F^{(5)} \wedge \star F^{(5)} + \mu_2 \int dr\,dy\,dt\,d\theta\,d\phi\,d\psi\,dV\,\delta(r-r_0)\,[A^{(4)}_{t\theta\phi\psi} + A^{(4)}_{y\theta\phi\psi}] \tag{6.135}$$

($dV = dy^1 \wedge \ldots \wedge dy^4$ is the volume form on T^4). We have assumed the brane to be smeared along y and the torus directions y^i and, in writing the source term, we have taken into account that the brane moves with velocity $\dot{y} = 1$ along y.

We will make the following ansatz for $A^{(4)}$

$$\begin{aligned} A^{(4)} &= A^{(4)}_{t\theta\phi\psi}\, dt \wedge d\theta \wedge d\phi \wedge d\psi + A^{(4)}_{t\theta\phi y}\, dt \wedge d\theta \wedge d\phi \wedge dy \\ &+ A^{(4)}_{y\theta\phi\psi}\, dy \wedge d\theta \wedge d\phi \wedge d\psi + A^{(4)}_{y\theta t\psi}\, dy \wedge d\theta \wedge dt \wedge d\psi \end{aligned} \tag{6.136}$$

(At the same order in μ_2, the gauge field also has components $A^{(4)}_{\mu\nu y^1 y^2}$, where $\mu, \nu = t, y, \psi, \phi$ and y^1, y^2 are directions in T^4: these components arise from the fact that the background metric is perturbed by the D3 brane together with the fact that the unperturbed background has non-zero values of $A^{(4)}_{\mu\nu y^1 y^2}$. Since the equations of motion do not mix the components $A^{(4)}_{\mu\nu y^1 y^2}$ with the ones contained in the ansatz (6.136), we can consistently ignore these extra components in the following).

Ona has

$$F^{(5)} = dr \wedge \partial_r A^{(4)} \tag{6.137}$$

The star operation in a geometry with $t\phi$ and $y\psi$ mixings is given by[25]

$$\begin{aligned} \star\,(dr \wedge dt \wedge d\theta \wedge d\phi \wedge d\psi) &= \sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt} g^{\phi\phi} - g^{t\phi} g^{t\phi})\, (g^{\psi\psi}\, dy - g^{\psi y}\, d\psi) \wedge dV \\ \star\,(dr \wedge dt \wedge d\theta \wedge d\phi \wedge dy) &= \sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt} g^{\phi\phi} - g^{t\phi} g^{t\phi})\, (g^{\psi y}\, dy - g^{yy}\, d\psi) \wedge dV \\ \star\,(dr \wedge dy \wedge d\theta \wedge d\phi \wedge d\psi) &= -\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy} g^{\psi\psi} - g^{y\psi} g^{y\psi})\, (g^{\phi\phi}\, dt - g^{\phi t}\, d\phi) \wedge dV \\ \star\,(dr \wedge dy \wedge d\theta \wedge dt \wedge d\psi) &= -\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy} g^{\psi\psi} - g^{y\psi} g^{y\psi})\, (g^{\phi t}\, dt - g^{tt}\, d\phi) \end{aligned} \tag{6.138}$$

The equations of motion are

$$d \star F^{(5)} + \mu_2\, \delta(r - r_0)\, dr \wedge (dy - dt) \wedge dV = 0 \tag{6.139}$$

[25]We are using the orientation $\epsilon_{tyr\theta\phi\psi} = 1$.

which yeld

$$\partial_r[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})\, (g^{\psi\psi}\partial_r A^{(4)}_{t\theta\phi\psi} + g^{\psi y}\partial_r A^{(4)}_{t\theta\phi y})] + \mu_2\, \delta(r - r_0) = 0$$
$$\partial_r[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})\, (g^{yy}\partial_r A^{(4)}_{t\theta\phi y} + g^{\psi y}\partial_r A^{(4)}_{t\theta\phi\psi})] = 0$$
$$\partial_r[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})\, (g^{\phi\phi}\partial_r A^{(4)}_{y\theta\phi\psi} + g^{\phi t}\partial_r A^{(4)}_{y\theta t\psi})] + \mu_2\, \delta(r - r_0) = 0$$
$$\partial_r[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})\, (g^{tt}\partial_r A^{(4)}_{y\theta t\psi} + g^{\phi t}\partial_r A^{(4)}_{y\theta\phi\psi})] = 0 \qquad (6.140)$$

$$\partial_\theta[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})\, (g^{\psi\psi}\partial_r A^{(4)}_{t\theta\phi\psi} + g^{\psi y}\partial_r A^{(4)}_{t\theta\phi y})] = 0$$
$$\partial_\theta[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})\, (g^{yy}\partial_r A^{(4)}_{t\theta\phi y} + g^{\psi y}\partial_r A^{(4)}_{t\theta\phi\psi})] = 0$$
$$\partial_\theta[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})\, (g^{\phi\phi}\partial_r A^{(4)}_{y\theta\phi\psi} + g^{\phi t}\partial_r A^{(4)}_{y\theta t\psi})] = 0$$
$$\partial_\theta[\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})\, (g^{tt}\partial_r A^{(4)}_{y\theta t\psi} + g^{\phi t}\partial_r A^{(4)}_{y\theta\phi\psi})] = 0 \qquad (6.141)$$

Their solution is

$$\begin{aligned}
F^{(5)}_{rt\theta\phi\psi} &= \frac{a_\pm\, g_{\psi\psi} + b_\pm\, g_{\psi y}}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})}\\
F^{(5)}_{rt\theta\phi y} &= \frac{a_\pm\, g_{\psi y} + b_\pm\, g_{yy}}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt}g^{\phi\phi} - g^{t\phi}g^{t\phi})}\\
F^{(5)}_{ry\theta\phi\psi} &= \frac{c_\pm\, g_{\phi\phi} + d_\pm\, g_{\phi t}}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})}\\
F^{(5)}_{ry\theta t\psi} &= \frac{c_\pm\, g_{\phi t} + d_\pm\, g_{tt}}{\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy}g^{\psi\psi} - g^{y\psi}g^{y\psi})}
\end{aligned} \qquad (6.142)$$

where $a_\pm$, $b_\pm$, $c_\pm$ and $d_\pm$ are r and θ independent constants: the subscript $+$ applies to the region $r > r_0$ while the subscript $-$ applies to $r < r_0$. Because of the delta function source we have $a_+ - a_- = -\mu_2$, $b_+ - b_- = 0$, $c_+ - c_- = -\mu_2$, $d_+ - d_- = 0$.

In order to fix the values of these constants let us impose regularity of the field strength. It will be convenient to work in "NS-sector coordinates"

$$\phi' = \phi - \frac{a}{\sqrt{Q_1 Q_2}}\, t\,, \quad \psi' = \psi - \frac{a}{\sqrt{Q_1 Q_2}}\, y \qquad (6.143)$$

Consider first regularity at $\theta = 0, \pi/2$. One has

$$\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{tt} g^{\phi'\phi'} - g^{t\phi'} g^{t\phi'}) = -\frac{r}{hf}\, \frac{\cos\theta}{\sin\theta}$$
$$\sqrt{-g}\, g^{rr}\, g^{\theta\theta}\, (g^{yy} g^{\psi'\psi'} - g^{y\psi'} g^{y\psi'}) = \frac{r^2 + a^2}{r\, hf}\, \frac{\sin\theta}{\cos\theta} \tag{6.144}$$

Moreover $g_{\phi'\phi'} \sim \sin^2\theta$, $g_{t\phi'} \sim \sin^2\theta$, $g_{\psi'\psi'} \sim \cos^2\theta$, $g_{y\psi'} \sim \cos^2\theta$ while g_{tt} and g_{yy} go to some finite non-zero values as $\theta \to 0, \pi/2$. We thus see that the term proportional to $b_\pm$ in $F^{(5)}_{rt\theta\phi y}$ is singular for $\theta = \pi/2$ and the term proportional to $d_\pm$ in $F^{(5)}_{ry\theta t\psi}$ is singular at $\theta = 0$. Therefore we have to take $b_\pm = d_\pm = 0$.

Consider now the behaviour around $f = 0$ (i.e. $r = 0$ and $\theta = \pi/2$), where the metric goes to

$$\begin{aligned}\frac{ds^2}{\sqrt{Q_1 Q_2}} \;\approx\;& \frac{dr^2}{r^2 + a^2} + \frac{r^2}{Q_1 Q_2}\, dy^2 - \frac{r^2 + a^2}{Q_1 Q_2} \Big(1 - 2\frac{a^2}{\sqrt{Q_1 Q_2}}\Big) + d\theta^2 + \cos^2\theta\, d\psi'^2 \\ +\;& \sin^2\theta\, d\phi'^2 \Big(1 + 2\frac{a^2}{\sqrt{Q_1 Q_2}}\Big) + 4\, \frac{a\, r^2}{\sqrt{Q_1 Q_2}}\, \cos^2\theta\, dy\, d\psi' + 4\, \frac{a\,(r^2 + a^2)}{\sqrt{Q_1 Q_2}}\, \sin^2\theta\, dt\, d\phi'\end{aligned} \tag{6.145}$$

Then we have

$$F^{(5)}_{rt\theta\phi'\psi'} \approx -a_-\, Q_1 Q_2\, \frac{\sin\theta\cos\theta}{r}$$
$$F^{(5)}_{rt\theta\phi' y} \approx -2a_-\, a\, r\, \sin\theta\, \cos\theta$$
$$F^{(5)}_{ry\theta\phi'\psi'} \approx c_-\, Q_1 Q_2\, \Big(1 + 2\frac{a^2}{\sqrt{Q_1 Q_2}}\Big)\, \frac{\sin\theta\cos\theta\, r}{r^2 + a^2}$$
$$F^{(5)}_{ry\theta t\psi'} \approx 2c_-\, a\, r\, \sin\theta\, \cos\theta \tag{6.146}$$

Regularity at $f = 0$ then requires $a_- = 0$ (and thus $c_+ = -\mu_2$), while c_- is left arbitrary.

Let us now consider the behaviour of the field strength at asymptotic infinity:

$$\begin{aligned}
F^{(5)} &= a_+ \, dr \wedge dt \wedge d\theta \wedge d\phi' \wedge \frac{g_{\psi'\psi'} \, d\psi' + g_{\psi' y} \, dy}{\sqrt{-g} \, g^{rr} \, g^{\theta\theta} \, (g^{tt} g^{\phi\phi} - g^{t\phi} g^{t\phi})} \\
&+ c_+ \, dr \wedge dy \wedge d\theta \wedge \frac{g_{\phi'\phi'} \, d\phi' + g_{\phi' t} \, dt}{\sqrt{-g} \, g^{rr} \, g^{\theta\theta} \, (g^{yy} g^{\psi\psi} - g^{y\psi} g^{y\psi})} \wedge d\psi' \\
&\approx -a_+ \, r^3 \sin\theta \, \cos\theta \, dr \wedge dt \wedge d\theta \wedge d\phi \wedge d\psi \\
&+ c_+ \, r^3 \sin\theta \, \cos\theta \, dr \wedge dy \wedge d\theta \wedge d\phi \wedge d\psi
\end{aligned} \tag{6.147}$$

The formula above shows that, asymptotically, the field strength is constant in local orthonomal coordinates.

We have the freedom to choose the constant c_- to have any value that we want; this freedom corresponds to adding a smooth magnetic field everywhere to the background. A simple choice of c_- would be the one that makes $c_+ = 0$, so that this magnetic field vanishes at infinity. Then we get

$$\begin{aligned}
F^{(5)} &= -\mu_2 \, dr \wedge dt \wedge d\theta \wedge d\phi' \wedge \frac{g_{\psi'\psi'} \, d\psi' + g_{\psi' y} \, dy}{\sqrt{-g} \, g^{rr} \, g^{\theta\theta} \, (g^{tt} g^{\phi'\phi'} - g^{t\phi'} g^{t\phi'})} \\
&= \mu_2 \, \frac{\sin\theta \, \cos\theta}{r} \, dr \wedge dt \wedge d\theta \wedge d\phi' \wedge \Big[h^2 \, f \Big(r^2 + \frac{Q_1 Q_2 \, a^2 \, \cos^2\theta}{h^2 f^2} \Big) d\psi' \\
&\qquad + \frac{a}{\sqrt{Q_1 Q_2}} \, r^2 \, (f + Q_1 + Q_2) \, dy \Big] \, , \ \text{for} \, r > r_0 \\
F^{(5)} &= -\mu_2 \, dr \wedge dy \wedge d\theta \wedge d\psi' \wedge \frac{g_{\phi'\phi'} \, d\phi' + g_{\phi' t} \, dt}{\sqrt{-g} \, g^{rr} \, g^{\theta\theta} \, (g^{tt} g^{\psi'\psi'} - g^{y\psi'} g^{y\psi'})} \\
&= -\mu_2 \, \frac{r \, \sin\theta \, \cos\theta}{r^2 + a^2} \, dr \wedge dy \wedge d\theta \wedge d\psi' \wedge \Big[h^2 \, f \Big(r^2 + a^2 - \frac{Q_1 Q_2 \, a^2 \, \sin^2\theta}{h^2 f^2} \Big) d\phi' \\
&\qquad + \frac{a}{\sqrt{Q_1 Q_2}} \, (r^2 + a^2) \, (f + Q_1 + Q_2) \, dt \Big] \, , \ \text{for} \, r < r_0
\end{aligned} \tag{6.148}$$

For any choice of c_- we find the the stress tensor of the field goes to a constant rather than vanish at infinity. We can thus generate a uniform cosmological type contribution in the spacetime dimensionally reduced on the y direction. The only way to cancel this contribution would be to have a $\overline{D3}$ brane in the 'throat' of the microstate geometry, or in the throat of a different microstate geometry located at some other spacetime point. We have to be aware that the energy computed from the DBI action in the above sections does not iclude this (possibly divergent) field contribution.

5.7 Supersymmetry properties of the branes

The simple expressions for the energies as a sum of the contribution from individual charges signifies a threshold bound state. As is usual in such situations, this usually follows from supersymmetry and BPS bounds. In this section we will examine the supersymmetry properties of these brane configurations.

5.7.1 Supersymmetry of the $D2$ brane

In this section we will examine the supersymmetry properties for the case of $D2$ branes in IIA theory. The considerations can be easily generalized to the M-branes.

Killing spinors of the near-horizon background

We work in global coordinates. The metric, dilaton, RR 1-form, and RR 3-forms of the near-horizon background were given in (3.63). We use $m, n... = \tau, \chi, \theta, \phi, 1, ..., 6$ as the ten-dimensional curved space indices, $a, b... = \hat{\tau}, \hat{\chi}, \hat{\theta}, \hat{\phi}, \hat{1}, ..., \hat{6}$ (or sometimes, equivalently, $a, b... = \underline{0}, ..., \underline{9}$) as the tangent space indices. The Clifford algebra is

$$\left\{\Gamma^a, \Gamma^b\right\} = 2\eta^{ab} \tag{7.149}$$

with η^{ab} having signature $(-, +, ..., +)$, and the gamma matrices Γ^a's are 32 by 32 real matrices ($\Gamma^{\hat{\tau}}$ being antisymmetric, and $\Gamma^{\hat{\chi}}, ..., \Gamma^{\hat{6}}$ being symmetric). $\Gamma^{\underline{10}} \equiv \Gamma^{\underline{0}...\underline{9}}$ and $(\Gamma^{\underline{10}})^2 = 1$. We use 32-component real spinors y, and define $\bar{y} \equiv y^T \Gamma^{\underline{0}}$.

The local supersymmetry variation of the dilatino, parameterized by a 32-component real spinor ϵ, is

$$\delta\lambda = \frac{1}{8}e^{\Phi}\left(\frac{3}{2!}F^{(2)}_{ab}\Gamma^{ab}\Gamma^{\underline{\varphi}} + \frac{1}{4!}F^{(4)}_{abcd}\Gamma^{abcd}\right)\epsilon \tag{7.150}$$

and the gravitino variation is

$$\delta\psi_m = \left[\partial_m + \frac{1}{4}\omega_{mab}\Gamma^{ab} + \frac{1}{8}e^{\Phi}\left(\frac{1}{2!}F^{(2)}_{ab}\Gamma^{ab}\Gamma_m\Gamma^{\underline{\varphi}} + \frac{1}{4!}F^{(4)}_{abcd}\Gamma^{abcd}\Gamma_m\right)\right]\epsilon \tag{7.151}$$

where $\Gamma^{\underline{\varphi}} \equiv -\Gamma^{\underline{10}} = -\Gamma^{\hat{\tau}\hat{\chi}\hat{\theta}\hat{\phi}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}}$. Plugging in the expressions of the RR field strength, we get

$$\delta\lambda = \frac{1}{R}N\epsilon, \quad \delta\psi_m = \left[\partial_m + \frac{1}{4}\omega_{mab}\Gamma^{ab} + \frac{1}{R}M\Gamma_m\right]\epsilon \tag{7.152}$$

where the matrices N and M are given by

$$\begin{aligned} N &= \frac{1}{8}\left[3\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}} + \Gamma^{\hat{\theta}\hat{\phi}}\left(\Gamma^{\hat{1}\hat{2}} + \Gamma^{\hat{3}\hat{4}} + \Gamma^{\hat{5}\hat{6}}\right)\right] \\ M &= \frac{1}{8}\left[-\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}} + \Gamma^{\hat{\theta}\hat{\phi}}\left(\Gamma^{\hat{1}\hat{2}} + \Gamma^{\hat{3}\hat{4}} + \Gamma^{\hat{5}\hat{6}}\right)\right] \end{aligned} \tag{7.153}$$

(note the only nonvanishing $\frac{1}{4}\omega_{mab}\Gamma^{ab}$'s are $\frac{1}{4}\omega_{\tau ab}\Gamma^{ab} = -\frac{\sinh\chi}{2}\Gamma^{\hat{\tau}\hat{\chi}}$ and $\frac{1}{4}\omega_{\phi ab}\Gamma^{ab} = \frac{\cos\theta}{2}\Gamma^{\hat{\phi}\hat{\theta}}$; also note that $\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}} = -\left(\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}\right)\left(\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}\right)\left(\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}\right)$.) Next we solve $\delta\lambda = 0$ and $\delta\psi_m = 0$ to find the Killing spinors.

Let's divide the 32-dimensional vector space of ϵ into eight subspaces of simultaneous eigenvectors of $\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}$, and $\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}$, labeled as $(\pm\pm\pm)$ (with the $\pm$'s denotes the ± 1 eigenvalues of these three matrices, respectively). Each of these subspaces is four-dimensional by itself. It is easy to see that, $\delta\lambda = 0$ if and only if

$$\epsilon = \epsilon_+ + \epsilon_- \tag{7.154}$$

with $\epsilon_+ \in (+++)$ and $\epsilon_- \in (---)$.

Plugging the above expression for ϵ into $\delta\psi_m$ and integrating, we then get the explicit expression of the eight Killing spinors of $AdS_2 \times S^2 \times T^6$, four of them being

$$\epsilon_1 = \left[e^{-\frac{1}{2}\chi\Gamma^{\hat{\chi}}} e^{\frac{1}{2}\tau\Gamma^{\hat{\tau}}} \sin\frac{\theta}{2} e^{\frac{1}{2}\phi\Gamma^{\hat{\phi}\hat{\theta}}} + e^{\frac{1}{2}\chi\Gamma^{\hat{\chi}}} e^{-\frac{1}{2}\tau\Gamma^{\hat{\tau}}} \left(-\cos\frac{\theta}{2} \right) \Gamma^{\hat{\theta}} e^{\frac{1}{2}\phi\Gamma^{\hat{\phi}\hat{\theta}}} \right] \Phi_0 \tag{7.155}$$

with Φ_0 being an arbitrary constant 32-component real spinor in the four-dimensional $(+ + +)$ subspace, i.e. $\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}\Phi_0 = \Phi_0$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}\Phi_0 = \Phi_0$, and $\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}\Phi_0 = \Phi_0$; and the other four being

$$\epsilon_2 = \left[e^{-\frac{1}{2}\chi\Gamma^{\hat{\chi}}} e^{\frac{1}{2}\tau\Gamma^{\hat{\tau}}} \cos\frac{\theta}{2} e^{-\frac{1}{2}\phi\Gamma^{\hat{\phi}\hat{\theta}}} + e^{\frac{1}{2}\chi\Gamma^{\hat{\chi}}} e^{-\frac{1}{2}\tau\Gamma^{\hat{\tau}}} \sin\frac{\theta}{2} \Gamma^{\hat{\theta}} e^{-\frac{1}{2}\phi\Gamma^{\hat{\phi}\hat{\theta}}} \right] \Phi_0' \tag{7.156}$$

with Φ_0' being another arbitrary constant 32-component real spinor in the four-dimensional $(+ + +)$ subspace. A general Killing spinor is given by $\epsilon = \epsilon_1 + \epsilon_2$.

Supersymmetric D2 configuration

Next we show that the D2 trajectory considered in Subsection 5.3.4 preserves half of the background supersymmetries. Recall that the trajectory is

$$\tau = \chi^0,\ \theta = \chi^1,\ \phi = \chi^2,\ \chi = \chi_0,\ y_1 = 0, ..., y_6 = 0 \tag{7.157}$$

for which the κ projection matrix as given in [116] evaluates to

$$\Gamma = \frac{-1}{\cosh\chi_0} \left(1 + \sinh\chi_0 \Gamma^{\hat{\theta}\hat{\phi}} \Gamma^{\underline{10}} \right) \Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}} \tag{7.158}$$

The supersymmetries preserved by the D2 brane are the Killing spinors ϵ that satisfy

$$(1 - \Gamma)\epsilon = 0 \tag{7.159}$$

After some manipulation, one finds that there are four supersymmetries preserved, with two of the corresponding Killing spinors given by eqn. (7.155) constrained by $(1 + \Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}})\Phi_0 = 0$, and the other two given by eqn. (7.156) constrained by $(1 + \Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}})\Phi_0' = 0$. Note that these projection conditions turn out to be independent of the D0 charge (i.e., independent of the value of χ_0).

Topological charge of the brane

In [117] p-forms constructed from background Killing spinors are integrated over probe branes' spatial worldvolumes to give topological charges in M-theory. [118] generalize this to IIA theory, whose approach we shall now apply to the above D2 brane. We shall find a central charge $C_{D2} = M_2 e^{-\Phi} + M_0 e^{-\Phi}$ in the superalgebra, which equals the D2's global energy and shows that the D2 indeed saturates a BPS bound.

After being sandwiched between ϵ^T and ϵ (where ϵ is a Killing spinor, and is treated as a commuting rather than anti-commuting variable), the superalgebra with the probe brane can be written as

$$(Q\epsilon)^2 = \int d^2\chi K_\mu p^\mu \pm \int \omega_{D2} \tag{7.160}$$

where the integrals are over the spatial worldvolume of the brane, K is a one-form defined as a bilinear of ϵ

$$K = \bar{\epsilon}\Gamma_a\epsilon\ e^a \tag{7.161}$$

(e^a being the vielbein one-form) and ω_{D2} is a closed two-form also constructed from bilinears of ϵ. The choice of ω_{D2} is background-specific[26], and we shall take the one used in [118] to consider supertubes

$$\omega_{D2} = \mu_2 \left(e^{-\Phi}\Omega + K \cdot A^{(3)} + \tilde{K} \wedge A^{(1)} - 2\pi\alpha' F \right) \tag{7.162}$$

with the $\cdot$ denoting the inner product of q-forms with p-forms $(p < q)$ $(\alpha_p \cdot \beta_q)_{a_1 \ldots a_{q-p}} = (1/p!)\,\alpha^{b_1 \ldots b_p}\beta_{b_1 \ldots b_p a_1 \ldots a_{q-p}}$, and

$$\Omega = \frac{1}{2}\bar{\epsilon}\Gamma_{ab}\epsilon\ e^{ab}, \quad \tilde{K} = \bar{\epsilon}\Gamma_a\Gamma^{\underline{10}}\epsilon\ e^a \tag{7.163}$$

[26]For a string probe, there is a general expression for the closed one-form ω_{string}, see [118] for details.

Note that our choice for the contribution of the worldvolume field strength to ω_{D2} differs from that of [118] by a minus sign. Due to ϵ's being a Killing spinor, K turns out to be a Killing vector, and $K, \Omega, \tilde{K}$ satisfy the following differential relations (which are obtained by plugging our background into equations (3.18) and (3.19) of [118])

$$d\tilde{K} = 0, \quad d\left(e^{-\Phi}\Omega\right) = \tilde{K} \wedge F^{(2)} + K \cdot F^{(4)} \tag{7.164}$$

Using these relations one finds

$$d\omega_{D2} = K \cdot F^{(4)} + d\left(K \cdot A^{(3)}\right) = \mathcal{L}_K A^{(3)} \tag{7.165}$$

Hence ω_{D2} will be closed if $A^{(3)}$ is invariant under the Lie derivative $\mathcal{L}_K$, and now we turn our attention to K.

One readily sees that $K_{\hat{1}} = 0$, since ϵ only has components in $(+++)$ and $(---)$ while $\Gamma^{\tau\hat{1}}$ takes $(+++)$ to $(-++)$ and $(---)$ to $(+--)$, and orthogonality of the subspaces then gives $\epsilon^T \Gamma^{\hat{\tau}\hat{1}} \epsilon = 0$. Similiarly, $K_{\hat{2}}, ..., K_{\hat{6}}$ all vanish.

After some algebra, one finds

$$K_{\hat{\chi}} = \epsilon^T \Gamma^{\hat{\tau}\hat{\chi}} \epsilon = \cos\tau \left(\Phi_0^T \Gamma^{\hat{\tau}\hat{\chi}} \Phi_0 + \Phi'^T_0 \Gamma^{\hat{\tau}\hat{\chi}} \Phi'_0\right) + \sin\tau \left(\Phi_0^T \Gamma^{\hat{\chi}} \Phi_0 + \Phi'^T_0 \Gamma^{\hat{\chi}} \Phi'_0\right) \tag{7.166}$$

$$K_{\hat{\theta}} = \epsilon^T \Gamma^{\hat{\tau}\hat{\theta}} \epsilon = 2\Phi_0^T \Gamma^{\hat{\tau}} \exp\left(-\phi \Gamma^{\hat{\phi}\hat{\theta}}\right) \Phi'_0 \tag{7.167}$$

$$K_{\hat{\phi}} = \epsilon^T \Gamma^{\hat{\tau}\hat{\phi}} \epsilon = 2\cos\theta \Phi'^T_0 \Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}} \exp\left(\phi \Gamma^{\hat{\phi}\hat{\theta}}\right) \Phi_0 + \sin\theta \left(\Phi_0^T \Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}} \Phi_0 - \Phi'^T_0 \Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}} \Phi'_0\right) \tag{7.168}$$

Now let's pick out a unique Killing spinor by further imposing the projection and normalization conditions

$$\begin{aligned}
&\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}} \Phi_0 = -\Phi_0, \quad \Gamma^{\hat{\tau}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}} \Phi_0 = \Phi_0 \\
&\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}} \Phi'_0 = -\Phi'_0, \quad \Gamma^{\hat{\tau}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}} \Phi'_0 = -\Phi'_0 \\
&\Phi_0^T \Phi_0 = \frac{\Delta}{2}, \quad \Phi'^T_0 \Phi'_0 = \frac{\Delta}{2}
\end{aligned} \tag{7.169}$$

where Δ is some positive normalization number whose value shall be determined soon. Note that this Killing spinor is preserved by the D2 (see subsection 5.7.1). For this Killing spinor, one immediately finds

$$K_{\hat{\chi}} = 0,\ K_{\hat{\theta}} = 0,\ K_{\hat{\phi}} = 0, and,\ K_{\hat{\tau}} = \epsilon^T \epsilon = \Delta \cosh \chi \tag{7.170}$$

i.e. $K = \frac{-\Delta}{R}\frac{\partial}{\partial \tau}$, which is the Killing vector generating global time translation. For this K $\mathcal{L}_K A^{(3)}$ vanishes, and we then find ω_{D2} is indeed closed. (Actually, the story here is quite trivial: since $A^{(3)}, F^{(4)}$ don't have any τ-component, $K \cdot A^{(3)}, K \cdot F^{(4)}$ both vanish.)

Having established the closedness of ω_{D2}, we now integrate it over the spatial worldvolume of the D2. Since $A^{(1)} \sim d\tau$ and $K \cdot A^{(3)}$ vanishes, only the $e^{-\Phi}\Omega$ term and the worldvolume flux term contributes to the integral

$$\begin{aligned} \int_{S^2} \omega_{D2} &= \mu_2 \int_{S^2} \frac{R}{q_0}\left(\epsilon^T \Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}} \epsilon\right) R^2 \sin\theta d\theta \wedge d\phi - \mu_2 2\pi\alpha' \int_{S^2} F \\ &= -4\pi\mu_2 \Delta \frac{R^3}{q_0} - M_0 = -\Delta M_2 e^{-\Phi} - M_0 \end{aligned} \tag{7.171}$$

where in the second line we've used the fact that $\epsilon^T \Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}} \epsilon$ evaluates to $-\Delta$ for the particular Killing spinor we've chosen. Since $\int d^2\chi K_\mu p^\mu = K^\tau P_\tau = -\frac{\Delta}{R} P_\tau = -\Delta E$ (recall that the physical energy is $E = P_\tau / R$), and the particular ϵ is perserved by the D2, the supersymmetry algegra (7.160) becomes

$$-\Delta E = \mp \left(-\Delta M_2 e^{-\Phi} - M_0\right) \tag{7.172}$$

which upon taking the lower sign on the r.h.s. gives

$$E = C_{D2} = M_2 e^{-\Phi} + \frac{M_0}{\Delta} \tag{7.173}$$

From this we see that we should take the normalization number Δ to be $e^{\Phi} = \frac{q_0}{R}$, which results in $C_{D2} = M_2 e^{-\Phi} + M_0 e^{-\Phi}$. This is the same as the global energy we computed earlier for this D2 trajectory and shows this D2 saturates the BPS bound.

Supersymmetry of D2 in the full black hole geometry

In this subsection, we show that the D2 considered above does not preserve any of the supersymmetries of the full black hole geometry (except in the $\chi_0 \to \infty$ limit where it is effectively a bunch of D0 branes), and is thus not really a stable configuration in the full geometry. First let's work out the Killing spinors of the full geometry.

Recall that the metric of the full geometry is given by

$$\begin{aligned} ds^2 = \quad & \frac{-1}{\sqrt{H_0 H_1 H_2 H_3}} dt^2 + \sqrt{H_0 H_1 H_2 H_3}\left(dr^2 + r^2 d\Omega_2^2\right) + \sqrt{\frac{H_0 H_1}{H_2 H_3}}\left(dy_1^2 + dy_2^2\right) \\ & + \sqrt{\frac{H_0 H_2}{H_1 H_3}}\left(dy_3^2 + dy_4^2\right) + \sqrt{\frac{H_0 H_3}{H_1 H_2}}\left(dy_5^2 + dy_6^2\right) \end{aligned} \tag{7.174}$$

where $H_0 = 1 + \frac{q_0}{r}$, $H_i = 1 + \frac{p_i}{r}, i = 1, 2, 3$. The nonvanishing components of the RR four-form and two-form field strengths are given by

$$\begin{aligned} F^{(4)}_{\theta\phi 12} &= -\frac{dH_1}{dr} r^2 \sin\theta, \;\; F^{(4)}_{\theta\phi 34} = -\frac{dH_2}{dr} r^2 \sin\theta, \quad F^{(4)}_{\theta\phi 56} = -\frac{dH_3}{dr} r^2 \sin\theta \\ F^{(2)}_{rt} &= -\frac{1}{(H_0)^2}\frac{dH_0}{dr} \end{aligned} \tag{7.175}$$

And the dilaton is

$$e^{\Phi} = \left(\frac{H_1 H_2 H_3}{(H_0)^3}\right)^{-1/4} \tag{7.176}$$

Note that the dilaton is no longer constant once we go beyond the near-horizon region. Now the local supersymmetry variation of the dilatino is given by

$$\delta\lambda = \left[\frac{1}{2}\Gamma^m \partial_m \Phi + \frac{1}{8} e^{\Phi}\left(\frac{3}{2!} F^{(2)}_{ab}\Gamma^{ab}\Gamma^{\underline{\varphi}} + \frac{1}{4!} F^{(4)}_{abcd}\Gamma^{abcd}\right)\right]\epsilon \tag{7.177}$$

which after plugging in the expression of the RR fields becomes

$$\delta\lambda = \frac{1}{8}\left(H_0H_1H_2H_3\right)^{-1/4}\left\{-\left(\sum_{i=1}^{3}\frac{1}{H_i}\frac{dH_i}{dr}-\frac{3}{H_0}\frac{dH_0}{dr}\right)\Gamma^{\hat{r}}+\left[\frac{-3}{H_0}\frac{dH_0}{dr}\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}} + \left(-\frac{1}{H_1}\frac{dH_1}{dr}\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}-\frac{1}{H_2}\frac{dH_2}{dr}\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}-\frac{1}{H_3}\frac{dH_3}{dr}\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}\right)\right]\right\} \tag{7.178}$$

Now we divide the 32-component spinor ϵ into sixteen subspaces labeled by $(s_1s_2s_3w)$ with $s_1, s_2, s_3 = \pm 1$ being eigenvalues of $\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}$, and $w = \pm 1$ being eigenvalue of $\Gamma^{\hat{r}}$. It is then easy to see that, $\delta\lambda = 0$ if and only if

$$\epsilon = \epsilon_{+++-} + \epsilon_{---+} \tag{7.179}$$

where the subscripts denote the subspace the spinors belong to. This gives us the four Killing spinors of the full black hole geometry, and we shall denote them as ϵ_{full}. The concrete coordinate-dependence of ϵ_{full} can be worked out by requiring the vanishing of the gravitino variation, however we don't need this detailed knowledge for the analysis below.

Now let's look at the kappa-projection matrix Γ given in eqn. (7.158) in the near-horizon region. Note that

$$\Gamma^{\hat{\theta}\hat{\phi}}\Gamma^{\underline{10}}\Gamma^{\hat{\tau}\hat{\theta}\hat{\phi}} = \Gamma^{\hat{\chi}}\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}} \tag{7.180}$$

and that $\Gamma^{\hat{\chi}}$ is the same as $\Gamma^{\hat{r}}$ because both are tangent-indiced gamma matrices. We see that Γ commutes with $\Gamma^{\hat{\theta}\hat{\phi}\hat{1}\hat{2}}$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{3}\hat{4}}$, $\Gamma^{\hat{\theta}\hat{\phi}\hat{5}\hat{6}}$. Hence requiring the supersymmetry of the full geometry to be preserved by D2, i.e.,

$$\Gamma\epsilon_{full} = \epsilon_{full} \tag{7.181}$$

is equivalent to requiring

$$\Gamma\epsilon_{+++-} = \epsilon_{+++-},\ \textit{and}\ \ \Gamma\epsilon_{---+} = \epsilon_{---+} \tag{7.182}$$

which is immediately seen to be impossible to satisfy for any finite value of χ_0, because

$$\Gamma\epsilon_{+++-} = \frac{-1}{\cosh\chi_0}\Gamma^{\hat{r}\hat{\theta}\hat{\phi}}\epsilon_{+++-} + \tanh\chi_0\ \epsilon_{+++-}$$
$$\Gamma\epsilon_{---+} = \frac{-1}{\cosh\chi_0}\Gamma^{\hat{r}\hat{\theta}\hat{\phi}}\epsilon_{---+} + \tanh\chi_0\ \epsilon_{---+} \quad (7.183)$$

(where the identity (7.180) has been used) and we see that the first terms on the right hand sides have the wrong eigenvalue under $\Gamma^{\hat{r}}$ (because $\Gamma^{\hat{r}\hat{\theta}\hat{\phi}}$ anticommutes with $\Gamma^{\hat{r}}$). This proves our claim that, for finite χ_0 the D2 brane doesn't preserve any of the four usual supersymmetries of the full geometry (the four supersymmetries preserved by the D2 as shown in subsection 5.7.1 have to be formed out of linear combinations of the usual supersymmetries of the full geometry and the conformal supersymmetries that are present only in the near-horizon region). What about the case $\chi_0 \to \infty$? In this case, the first terms on the right hand sides of eqn. (7.183) vanish, and the second terms become ϵ_{+++-} and ϵ_{---+} respectively, giving exactly what is needed for $\Gamma\epsilon_{full} = \epsilon_{full}$. This comes as no surprise since in the infinite χ_0 limit the D2 has an infinite D0 charge and is effectively just a bunch of D0 branes, which is known to preserve all the four usual supersymmetries of the full black hole geometry.

5.7.2 Supersymmetry of D3 branes in Microstate geometry

In this subsection we examine supersymmetry properties of D3 brane in the 2 charge microstate geometry discussed in section (5.4). Analogously to the D2 case considered above, we shall find that the D3 brane preserves half of the supersymmetries of the near-hoziron geometry, but doesn't preserve any of the supersymmetries of the full asymptotically flat geometry.

As in the above IIA case, we use $m, n... = t, y, r, \theta, \phi, \psi, y^1, y^2, y^3, y^4$ to denote curved space indices, and $a, b... = \hat{0}, \hat{1}, ..., \hat{9}$ to denote tangent space indices. $\hat{\Gamma}_a$ are ten dimensional Gamma matrices, which we will decompose into direct products of 6-d Gamma matrices denoted as $\tilde{\Gamma}_a$ and 4-d Gamma matrices denoted as Γ_a. The analysis in the near-horizon region $AdS_3 \times S^3 \times T^4$ is similar to the D2 case, hence instead of giving all the details here we will simply quote the near-horizon results when needed without proof.

Let us consider the D3 brane at its stable point $\dot{y} = -1$. (In Section 5.4 the choice of $\dot{y} = +1$ was made. This difference in choices does not affect the conclusion of the analysis below, because they just correspond to conjugate Killing spinors preserved by the D3 brane). As shown in Section 5.4, for $\dot{y} = -1$ the determinant of the metric induced on the brane simplifies to $\sqrt{-g} = a\sqrt{Q_1 Q_2}\sin\theta\cos\theta$. Then the kappa symmetry condition (after getting rid of antisymmetrization and combinatorial factors) becomes

$$\gamma_t\gamma_\theta\gamma_\psi\gamma_\phi\xi = -ia\sqrt{Q_1 Q_2}\sin\theta\cos\theta\xi \tag{7.184}$$

where γ_i are the pull backs on the brane worldvolume of the space time Gamma matrices. Using the vielbeins for the six dimensional 2-charge microstate metric

$$e^{\hat{0}} = \frac{1}{\sqrt{h}}\Big(dt + \frac{a\sqrt{Q_1 Q_2}}{f}\sin^2\theta d\phi\Big) \ , \ e^{\hat{1}} = \frac{1}{\sqrt{h}}\Big(dy - \frac{a\sqrt{Q_1 Q_2}}{f}\cos^2\theta d\psi\Big) \tag{7.185}$$

$$e^{\hat{2}} = \sqrt{\frac{hf}{r^2+a^2}}dr \ , \ e^{\hat{3}} = \sqrt{hf}d\theta \ , \ e^{\hat{4}} = \sqrt{h}r\cos\theta d\psi \ , \ e^{\hat{5}} = \sqrt{h(r^2+a^2)}\sin\theta d\phi \tag{7.186}$$

the induced gamma matices are found to be

$$\gamma_\theta = e^{\hat{3}}_\theta\tilde{\Gamma}_{\hat{3}} \ , \ \gamma_\phi = e^{\hat{5}}_\phi\tilde{\Gamma}_{\hat{5}} + e^{\hat{0}}_\phi\tilde{\Gamma}_{\hat{0}} \ , \ \gamma_t = e^{\hat{0}}_t\tilde{\Gamma}_{\hat{0}} + \dot{y}e^{\hat{1}}_y\tilde{\Gamma}_{\hat{1}} \ , \ \gamma_\psi = e^{\hat{4}}_\psi\tilde{\Gamma}_{\hat{4}} + e^{\hat{1}}_\psi\tilde{\Gamma}_{\hat{1}} \tag{7.187}$$

Setting $\dot{y} = -1$ in the expression for γ_t and using using $e^{\hat{0}}_t = e^{\hat{1}}_y$ we can then rewrite the kappa symmetry matrix in terms of constant Gamma matrices and vielbeins as

$$\gamma_{t\theta\psi\phi}\,\xi = e^{\hat{0}}_t e^{\hat{3}}_\theta\Big[(e^{\hat{1}}_\psi e^{\hat{5}}_\phi\tilde{\Gamma}_{\hat{3}}\tilde{\Gamma}_{\hat{5}} - e^{\hat{4}}_\psi e^{\hat{0}}_\phi\tilde{\Gamma}_{\hat{3}}\tilde{\Gamma}_{\hat{4}})(1-\tilde{\Gamma}_{\hat{0}\hat{1}}) + (e^{\hat{4}}_\psi e^{\hat{5}}_\phi\tilde{\Gamma}_{\hat{0}}\tilde{\Gamma}_{\hat{3}}\tilde{\Gamma}_{\hat{4}}\tilde{\Gamma}_{\hat{5}} - e^{\hat{1}}_\psi e^{\hat{0}}_\phi\tilde{\Gamma}_{\hat{0}}\tilde{\Gamma}_{\hat{3}})(1+\tilde{\Gamma}_{\hat{0}\hat{1}})\Big]\,\xi \tag{7.188}$$

Now let's look at a Killing spinor for the asymptotically flat metric generated by the background $D3$ branes. We know that it will be of the form $\xi = g(x)\xi_0$ (see, e.g., [119]). Here $g(x)$ is some spacetime dependent part which will cancel from both sides of kappa symmetry matrix as it doesn't depend on gamma matrices (unlike the near horizon case). The constant part ξ_0 satisfies projection conditions corresponding to two orthogonal sets of $D3$ branes. Our $D3$ branes are along directions $y67$ and $y89$, hence

$$\xi_0 + i\hat{\Gamma}_{\hat{0}\hat{1}\hat{6}\hat{7}}\xi_0 = 0 \ , \ \xi_0 + i\hat{\Gamma}_{\hat{0}\hat{1}\hat{8}\hat{9}}\xi_0 = 0 \tag{7.189}$$

We decompose the constant spinor ξ_0 as $\xi^{(0)}_{M6} \otimes \xi^{(0)}_{T4}$. This gives three constraints

$$\xi^{(0)}_{M6} + \tilde{\Gamma}_{\hat{0}\hat{1}}\xi^{(0)}_{M6} = 0 \ , \ \xi^{(0)}_{T4} + i\Gamma_{\hat{6}\hat{7}}\xi^{(0)}_{T4} = 0 \ , \ \xi^{(0)}_{T4} + i\Gamma_{\hat{8}\hat{9}}\xi^{(0)}_{T4} = 0 \tag{7.190}$$

The second and third constraints can be seen to be satisfied as in the near horizon case by using an explicit representation of gamma matrices. For now we concentrate on the $M6$ part. Using the first constraint in (7.190), we see that the term containg $(I + \tilde{\Gamma}_{\hat{0}\hat{1}})\xi$ in the kappa symmetry matrix gives zero. Plugging in the values of vielbeins, we get, from the remaining term,

$$\frac{1}{\sqrt{f}}(\sqrt{r^2+a^2}\cos\theta\tilde{\Gamma}_{\hat{3}\hat{5}} + r\sin\theta\tilde{\Gamma}_{\hat{3}\hat{4}})(1 - \tilde{\Gamma}_{\hat{0}\hat{1}})\xi^{(0)}_{M6} = -\xi^{(0)}_{M6} \tag{7.191}$$

It is apparent that the kappa symmetry condition cannot be satisfied for $r \neq 0$. For $r = 0$ we get a projection condition on $\xi^{(0)}_{M6}$ that can be easily seen to be inconsistent with the first of the constraints in (7.190). We conclude that the D3 brane is not supersymmetric in the full asymptotically flat geometry for any value of r.

Let's ask why the supersymmetry of the D3 brane is broken in full 2-charge microstate geometry. We have seen that the Killing spinors of the full six dimensional background geometry of $D3 - D3$ system satisfy the projection condition

$(I + \tilde{\Gamma}_{\hat{0}\hat{1}})\xi^{(0)}_{M6} = 0$. In the near horizon region, the geometry neatly separates into AdS and sphere parts, hence we can write gamma matrices for AdS part and they act on the AdS part of Killing spinor. So we have

$$\tilde{\Gamma}_{\hat{0}\hat{1}}\xi^{(0)}_{ads} = -\xi^{(0)}_{ads} \tag{7.192}$$

In the near horizon region we have two types of supersymmetries. In addition to ordinary supersymmetries, there are also the superconformal supersymmetries. Only ordinary supersymmetries continue to the far, i.e., asymptotically flat region. Now we want to see if the projection condition (7.192) is compatible with the kappa symmetry condition for $D3$ brane wrapping the sphere in the near horizon region. The condition one finds in the near-horizon region is, with $\xi^{(0)} = \epsilon_0$,

$$\tilde{\Gamma}_{\hat{0}}\epsilon_0 = -\epsilon_0 \tag{7.193}$$

The condition to be continuable to the far region is that it be in an eigenvector of $\tilde{\Gamma}_{\hat{0}\hat{1}} = \tilde{\Gamma}_{\hat{2}}$ i.e

$$\tilde{\Gamma}_{\hat{2}}\epsilon_0 = -\epsilon_0 \tag{7.194}$$

In three dimensions, $\tilde{\Gamma}_{\hat{0},\hat{1},\hat{2}}$ are just Pauli matrices which don't commute. Hence it is not possible for them to have simultaneous eigenvectors. As a result, the two conditions (7.193) and (7.194) are not compatible and hence Killing spinors in the far region that are preserved by this $D3$ brane do not exist.

5.8 Discussion

In [109] the 4-charge black hole was considered. The charges were D4-D4-D4-D0. It was argued that the D0 branes swell up into D2 branes which wrap the horizon, and which occupies a Landau level on the torus. The different ways to partition the D0 branes into such groups gave the entropy of the hole.

We must however ask if the energy of the D0 branes remains the same when we try to make them form a D2 brane; since we are looking at the states of an extremal hole we do not have any 'extra' energy to make the D2 brane. It is not clear to us how this would work in general, since in the limit where we have a very small D0 charge the mass of the D2 brane would seem to be just the area of the horizon times the tension, and this is much more than the mass of the D0 branes attached to it. In fact in the work of [109] the global energy which follows from the supersymmetry algebra turned out to be equal to the mass of the D2 brane *with no contribution to the D0 charge*! This is what would follow in our treatment if we chose a gauge for the 1-form potential of the background to be $A^{(1)} = \frac{R_{IIA}}{q_0} \sinh\chi \ d\tau$ rather than (3.76). As we have noted, there is always an ambiguity in calculating energies from brane actions.

The situation would be clearer if we had an asymptotically flat space-time. With this in mind we have looked at 2-charge microstates which have a similar structure to the system of [109], but where the AdS space inside goes over to asymptotically flat space at large r. We find that the mass of the D3+P system (which is analogous to the D2+D0 system) is given by the *sum* of two contributions: the energy carried by P and an energy coming from the tension of the D3. It is interesting that the energy is given by such a simple relation, because this configuration is not supersymmetric in the full asymptotically flat geometry. This suggests that there is some hidden symmetry in this 2-charge background, but we do not have any clear understanding of this as yet. But this also raises a puzzle about the relation of this computation with that of [109], since the mass of the D3+P system is more than the mass of the P charge alone.

We also computed the gauge field produced by the D3 brane wrapped on the S^3 in the full asymptotically flat geometry, and found that the field strength went to a nonzero constant at infinity. This suggests a divergent total energy for the field produced by the brane, or alternatively, that the D3 branes and anti-branes wrapped in this way are ‘confined’ and cannot be separated to large distances without generating a uniform energy density in the intervening spacetime. Note that the energy $E = P + M$ computed using the DBI action ignores this field energy. The field energy is quadratic in the test brane charge, and would be ignored in a linear analysis if it were finite.

CHAPTER 6

DYNAMICS OF SUPERTUBES

6.1 Introduction

In chapter 2, we saw that a supersymmetric brane in Type II string theory is a 1/2 BPS object. The bound state of N identical branes (wrapped on a torus) can be mapped by duality to a massless quantum with momentum $P = N/R$ on a circle of radius R. Thus the bound state has degeneracy 256, regardless of N.

The situation is very different for 1/4 BPS states. Such states can be made in many duality related ways: NS1-P, NS1-D0, D0-D4, D1-D5 etc. If the two charges are n_1, n_2 then the degeneracy of the bound state is $Exp[2\pi\sqrt{2}\sqrt{n_1 n_2}]$ [198, 199]. In the classical limit $n_1, n_2 \to \infty$ this degeneracy manifests itself as a continuous family of solutions. Examples are the 2-charge D1-D5 solutions found in [150] and the supertubes constructed in [10, 67, 68]. These 2-charge states are important because they give the simplest example of a black hole type entropy [199].

In this chapter we address the question: What is the low energy dynamics of such 1/4 BPS states? We will perform some calculations to arrive at a conjecture for the answer. The behavior of the system can depend on whether the coupling is small or large, and whether we have bound states or unbound states. For this reason we first give an overview of *possible* dynamical behaviors, and then summarize our computations and conclusions.

6.1.1 Possibilities for low energy dynamics

In the following it will be assumed that all compactifications are toroidal, and all branes are wrapped on these compact directions in a way that preserves 1/4 supersymmetry.

(a) *'Drift' on moduli space:* A D0 brane can be placed at rest near a D4 brane; there is no force between the branes. Thus we have a moduli space of possibilities for the relative separation. If we give the branes a small relative velocity v then we get a $\sim v^2$ force, and the resulting motion can be described by 'motion on moduli space' [69]. More generally, we can make 3-charge black holes that are 1/8 BPS and their slow motion will be described by motion on a moduli space [70, 71]. For later use we make the required limits explicit: The velocity v is $O(\epsilon)$, the time over which we follow the motion is $O(1/\epsilon)$, and the distance in moduli space over which the configuration 'drifts' is $O(1)$

$$v \sim \epsilon, \quad \Delta t \sim \frac{1}{\epsilon}, \quad \Delta x \sim 1, \qquad (\epsilon \to 0) \tag{1.1}$$

Note that the different branes or black holes involved here are not bound to each other.

(b) *Oscillations:* Consider branes carrying just one charge, and let these be NS1 for concreteness. Then the force between branes is $\sim v^4$. So for the relative motion between such branes we again get 'drift' on moduli space except that the moduli space is flat [72]. But we can also focus on just one brane and study its low energy excitations. These will be vibration modes along the brane, with the amplitude for each harmonic behaving like a harmonic oscillator. Calling the amplitude for a given harmonic $A_n \equiv x$ we note that x will have the time evolution $x = \bar{x}\cos(\omega t + \phi)$. Setting $\bar{x} = \epsilon$ for a small deformation, the analogue of (1.1) is

$$v \sim \epsilon, \quad \Delta t \sim 1, \quad \Delta x \sim \epsilon, \qquad (\epsilon \to 0) \tag{1.2}$$

where we have assumed that we are not looking at a zero mode $\omega = 0$. For the zero mode we will have the behavior

$$x = x_0 + vt \tag{1.3}$$

and we get 'drift' over configuration space with characteristics given by (1.1).

(c) *'Quasi-oscillations':* Consider a charged particle free to move in the $x - y$ plane in a uniform magnetic field $F_{xy} = B$. The particle can be placed at rest at any position on the plane, and it has the same energy at all these points. Thus far its behavior looks like that of a system with a zero mode. But if we give the particle a small velocity then it describes a small circle near its original position, instead of 'drifting' along the plane. The motion is described by

$$v \sim \epsilon, \quad \Delta t \sim 1, \quad \Delta x \sim \epsilon \qquad (\epsilon \to 0) \tag{1.4}$$

Thus even though we may have a continuous family of energetically degenerate configurations, this does not mean that the dynamics will be a 'drift' along this space.

(d) *Gravitational radiation:* We are going to give our system a small energy above extremality. But the system is coupled to Type II supergravity, and there are massless quanta in this theory. Thus any energy we place on our branes can leave the branes and become radiation flowing off to infinity. There will of course always be some radiation from any motion in the system, but the relevant issue here is the time scale over which energy is lost to radiation. If the time scale relevant to the dynamics is Δt then as $\epsilon \to 0$ we have to ask what fraction of the energy is lost to radiation in time Δt. If the fraction is $O(1)$, then the system is strongly coupled to the radiation field and cannot be studied by itself while ignoring the radiation. If on the other hand the fraction of energy lost to radiation goes to zero as $\epsilon \to 0$ then the radiation field decouples and radiation can be ignored in the dynamics.

(e) *Excitations trapped near the brane:* In the D1-D5 system we can take a limit of parameters such that the geometry has a deep 'throat' region. In [189, 150] it was found that excitations of the supergravity field can be trapped for long times in this throat; equivalently, we can make standing waves that leak energy only slowly to the radiation modes outside the throat [133]. These are oscillation modes of the supergravity fields and thus could have been listed under (b) above. We list them separately to emphasize that the fields excited need not be the ones making the original brane state; thus the excitation is not in general a collective mode of the initial fields.

6.1.2 Results and conjectures

Consider first the D1-D5 bound state geometries found in [150]. These geometries are flat space at infinity, they have a locally $AdS_3 \times S^3 \times T^4$ 'throat', and this throat ends smoothly in a 'cap'. The geometry of the cap changes from configuration to configuration, and is parametrized by a function $\vec{F}(v)$. All the geometries have the same mass and charges, are 1/4 BPS, and yield (upon quantization) different bound states of the D1-D5 system.

What happens if we take one of these geometries and add a small energy? The bound state of D1-D5 branes has a nontrivial transverse size, so one may say that the brane charges have separated away from each other in forming the bound state. If the charges indeed behave like separate charges then we would expect 'drift on moduli space' dynamics, (type (a) in our list). Or does the bound state fragment into a few unbound states, which then drift away from each other? This is in principle possible, since the D1-D5 system is threshold bound. Do we stay within the class of bound geometries of [150] but have 'drift on moduli space' (1.1) between different bound state configurations (i.e. drift on the space $\vec{F}(v)$)? Or do we have one of the other possibilities (b)-(e)?

Now consider the opposite limit of coupling: Take a supertube in flat space. The supertube carries NS1-D0 charges, and develops a D2 'dipole' charge. This D2 brane can take on a family of possible profiles in space, giving a continuous family of 1/4 BPS configurations. What happens if we take a supertube in any given configuration and add a small amount of energy? Is there a 'drift' among the family of allowed configurations, or some other kind of behavior?

In [75] the 'round supertube' was considered, and the low energy behavior yielded excitations with time dependence $\sim e^{-i\omega t}$. Can we conclude that there is no 'drift' among supertube configurations? Any drift can occur only between states that have the same values of conserved quantities. The round supertube has the maximal possible angular momentum J for its charges, and is the only configuration with this J. So 'drifting' is not an allowed behavior if we give a small excitation to this particular supertube, and we must look at the generic supertube to know if periodic behavior is the norm.

We now list our computations and results:

(i) First we consider the 2-charge systems in flat space (i.e., we set $g = 0$). It turns out that the simplest system to analyze is NS1-P, which is given by a NS1 string wrapped n_1 times around a circle S^1, carrying n_p units of momentum along the S^1. The added excitation creates further vibrations on the NS1. But this is just a state of the free string, and can be exactly solved (the classical solution is all we need for our purpose). Taking the limit $n_1, n_p \to \infty$ we extract the dynamical behavior of the supertube formed by NS1-P charges. In this way we get not only the small perturbations but arbitrary excitations of the supertube.

We then dualize from NS1-P to D0-NS1 which gives us the traditional supertube. This supertube can be described by a DBI action of a D2 brane carrying fluxes. We verify that the solution found through the NS1-P system solves the dynamical equations for the D2, both at the linear perturbation level and at the nonlinear level.

Even before doing the calculation it is easy to see that there is no 'drift' over configurations in the NS1-P dynamics. The BPS string carries a right moving wave, and the excitation just adds a left moving perturbation. Since right and left movers can be separated, the perturbation travels around the string and the string returns to its initial configuration after a time $\Delta t \sim 1$. But this behavior of the 'supertube' is not an oscillation of type (b); rather it turns out to be a 'quasi-oscillation' of type (c). This can be seen from the fact that even though we move the initial tube configuration towards another configuration of the same energy, the resulting motion is periodic rather than a 'drift' which would result from a zero mode (1.3).

(ii) Our goal is to move towards larger values of the coupling g. At $g = 0$ the gravitational effect of the supertube does not manifest itself at any distance from the supertube. Now imagine increasing g, till the gravitational field is significant over distances $\sim Q$ from the supertube. Let the radius of the curve describing the supertube profile be $\sim a$. We focus on the domain

$$Q \ll a \tag{1.5}$$

Then we can look at a small segment of the supertube which looks like a straight line. But this segment is described by a geometry, and we look for small perturbations of the geometry. We solve the linearized supergravity equations around this 'straight line supertube' and note that the resulting periods of the solutions agree with (i) above.

We note however that far from the supertube the gravity solution will be a perturbation on free space with some frequency satisfying $\omega^2 > 0$. The only such solutions are traveling waves. For small Q/a we find that the amplitude of the solution when it reaches the approximately flat part of spacetime is small. Thus we expect that the radiation into modes of type (d) will be suppressed by a power of Q/a.

(iii) Now imagine increasing g to the point where

$$Q \sim a \tag{1.6}$$

In this situation we see no reason why the part of the wavefunction leaking into the radiation zone should be suppressed. Thus we expect that the excitation will not be confined to the vicinity of the branes, but will be a gravitational wave that will flow off to infinity over a time of order the crossing time across the diameter of the supertube.

(iv) We increase the coupling further so that

$$Q \gg a \tag{1.7}$$

Now the geometry has a deep 'throat' and as mentioned above we find excitations which stay trapped in this throat for long times, with only a slow leakage to radiation at infinity. What is the relation between these excitations and those found in (ii)? We argue that these two kinds of excitations are different, and represent the excitations in two different phases of the 2-charge system. These two phases were identified by looking at microscopic degrees of freedom as a function of g in [76, 77], and what we see here appears to be a gravity manifestation of the transition.

(v) All the above computations were for *bound states* of the 2-charge system. But we have seen above that if have *unbound* states – two different 2-charge black holes for example – then we get 'drift' modes of type (a). It looks reasonable to assume that in any coupling domain if we have two or more different bound states then the relative motion of these components will be a 'drift'. For example at $g = 0$ we can have two supertubes that will move at constant velocity past each other.

It is intriguing to conjecture that this represents a basic difference between bound and unbound states: Bound states have no 'drift' modes and unbound states do have one or more such modes. The importance of this conjecture is that 3-charge systems are very similar to 2-charge ones, so we would extend the conjecture to the 3-charge case as well. While all bound states can be explicitly constructed for the 2-charge case, we only know a few 3-charge bound states [9]. There is a way to construct *all* 3-charge supersymmetric solutions [12] but the construction does not tell us which of these solutions are bound states. Since these bound states are the microstates of the 3-charge extremal black hole, it is very important to be able to select the bound states out of all the possible supersymmetric solutions. The above conjecture says that those states are bound which do not have any 'drift' type modes of excitations, and the others are unbound. If this conjecture is true, then we have in principle a way to identify all 3-charge black hole states.

Note: We will use the term 'supertube' or just 'tube' for 2-charge bound states in all duality frames, and at all values of the coupling. The supertube made from D0,NS1 charges carries a D2 dipole charge, and we will call this the D0-NS1 supertube or the D2 supertube. When we use charges NS1,P we will call the object the NS1-P supertube.

6.2 The NS1-P bound state in flat space

We will find that the most useful representation of the 2-charge system will be NS1-P. We compactify a circle S^1 with radius R_y; let $X^1 \equiv y$ be the coordinate along this S^1. The elementary string (NS1) is wrapped on this S^1 with winding number n_1, and n_p units of momentum run along the S^1. We are interested in the *bound* state of these charges. This corresponds to the NS1 being a single 'multiwound' string with wrapping number n_1, and the momentum is carried on this NS1 by its transverse oscillations.[27]

Consider first the BPS states of this system. Then all the excitations carry momentum in one direction; we set this to be the positive y direction and call these excitations 'right moving'. In Fig.6.1(a) we open up the multiwound string to its

[27] The momentum can also be carried by the fermionic superpartners of these oscillation degrees of freedom, but we will not focus on the fermions in what follows. For a discussion of fermion modes in the 2-charge system see for example [78] .

covering space where we can see the transverse oscillation profile. As explained in [79] these oscillations cause the n_1 strands to separate from each other and the bound state acquires a transverse 'size'. For the generic state of this 2-charge system the radius of the state is $\sim \sqrt{\alpha'}$ and the surface area of this region gives the entropy of the state by a Bekenstein type relation [8]

$$\frac{A}{4G} \sim S_{micro} = 2\pi\sqrt{2}\sqrt{n_1 n_5} \tag{2.1}$$

To understand the generic state better it is useful to look at configurations that have a much larger transverse size, and later take the limit where we approach the generic state. The relevant limit is explained in [80]. In this limit the wavelength of the vibration on the multiwound string is much larger than the radius of the S^1, so locally the strands of the NS1 look like Fig.6.1(b). In the classical limit $n_1 n_p \to \infty$ these strands will form a continuous 'strip', which will be described by (i) the profile of the strip in the space transverse to the S^1 and (ii) the 'slope' of the strands at any point along the profile.[28]

An S-duality gives NS1-P $\to$ D1-P, and a further T-duality along y gives D1-P $\to$ D0-NS1. But note that locally the string is slanted, and the T-duality also generates a local $D2$ charge. Thus we get a 'supertube'[29] where the D0-NS1 have formed a D2 [10]. There is of course no *net* D2 charge; rather the D2 is a 'dipole' charge. Note also that the slope of the NS1 in the starting NS1-P configuration implies that the momentum is partly along the direction of the 'strip'. Since we do no dualities in the strip direction we will end up with momentum being carried along the D2 supertube.

[28]Note that the separation between successive strands is determined by the slope, since the radius of the S^1 is fixed at R_y.

[29]In [81] the same dualities were performed in the reverse order.

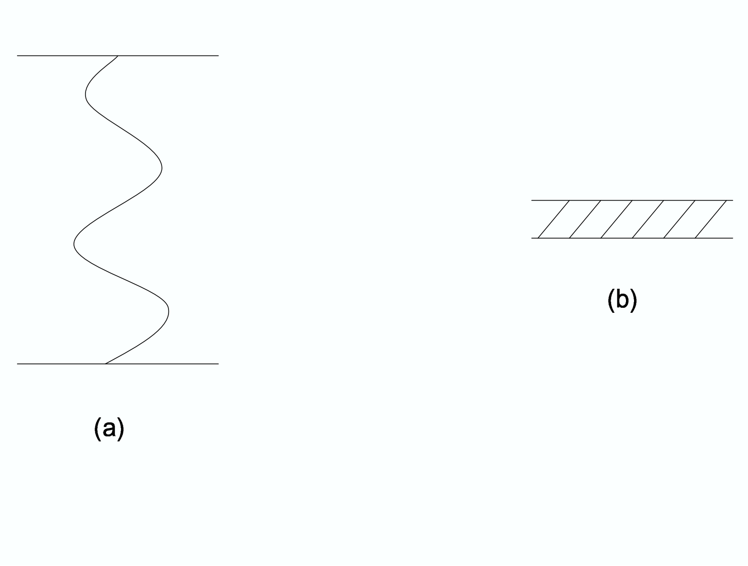

Figure 6.1: (a) The NS1 carrying a transverse oscillation profile in the covering space of S^1. (b) The strands of the NS1 as they appear in the actual space.

In this BPS configuration the D2 supertube is stationary. If we add some extra energy to the tube (while keeping its true (i.e. non-dipole) charges fixed) then we will get the dynamics of the supertube. But we can study the dynamics in the NS1-P picture and dualize to the D2 supertube at the end if we wish. In the NS1-P picture we just have to study a free, classical string. Here the left and right movers decouple and the problem can be solved exactly. Let us review this solution and extract the dynamics of the 'supertube' in the limit of large charges.

6.2.1 The classical string solution

The string dynamics in flat space is described by the Nambu-Goto action

$$S_{NG} = -T\sqrt{-\det[\frac{\partial X^\mu}{\partial \chi^a}\frac{\partial X_\mu}{\partial \chi^b}]} \tag{2.2}$$

where

$$T = \frac{1}{2\pi\alpha'} \tag{2.3}$$

We can get an equivalent dynamics by introducing an auxiliary metric on the world sheet (this gives the Polyakov action)

$$S_P = -\frac{T}{2}\int d^2\chi\sqrt{-g}\frac{\partial X^\mu}{\partial\chi^a}\frac{\partial X_\mu}{\partial\chi^b}g^{ab} \tag{2.4}$$

The variation of g_{ab} gives

$$\frac{\partial X^\mu}{\partial\chi^a}\frac{\partial X_\mu}{\partial\chi^b} - \frac{1}{2}g_{ab}\frac{\partial X^\mu}{\partial\chi^c}\frac{\partial X_\mu}{\partial\chi^d}g^{cd} = 0 \tag{2.5}$$

so g_{ab} must be proportional to the induced metric. Substituting this g_{ab} in (2.4) we get back (2.2), thus showing that the two actions are classically equivalent.

The X^μ equations give

$$\partial_a[\sqrt{-g}\frac{\partial X_\mu}{\partial\chi^b}g^{ab}] = 0 \tag{2.6}$$

Note that the solution for the X^μ does not depend on the conformal factor of g_{ab}.

We choose coordinates $\chi^0 \equiv \hat{\tau}$, $\chi^1 \equiv \hat{\sigma}$ on the world sheet so that $g_{ab} = e^{2\rho}\eta_{ab}$ for some ρ. Writing

$$\chi^+ = \chi^0 + \chi^1, \quad \chi^- = \chi^0 - \chi^1 \tag{2.7}$$

we have

$$g_{++} = 0, \quad g_{--} = 0 \tag{2.8}$$

Since the induced metric must be proportional to g_{ab} we get

$$\frac{\partial X^\mu}{\partial\chi^+}\frac{\partial X_\mu}{\partial\chi^+} = 0, \quad \frac{\partial X^\mu}{\partial\chi^-}\frac{\partial X_\mu}{\partial\chi^-} = 0 \tag{2.9}$$

Thus in these coordinates we get a solution if the X_μ are harmonic functions

$$X_{\mu\ a}^{;a} = 0 \tag{2.10}$$

and they satisfy (2.9). The equations (2.10) imply that the coordinates X^μ can be expanded as

$$X^\mu = X_+^\mu(\chi^+) + X_-^\mu(\chi^-) \tag{2.11}$$

We can use the residual diffeomorphism symmetry to set the harmonic function X^0 to[30]

$$X^0 = \hat{a} + \hat{b}\hat{\tau} = \hat{a} + \hat{b}\frac{1}{2}(\chi^+ + \chi^-) \tag{2.12}$$

Let the coordinate along the S^1 be called y. We can solve the constraints to express the terms involving y in terms of the other variables. We find

$$\partial_+ y_+ = \pm\sqrt{\frac{\hat{b}^2}{4} - \partial_+ X^i_+ \partial_+ X^i_+}\,, \quad \partial_- y_- = \pm\sqrt{\frac{\hat{b}^2}{4} - \partial_- X^i_- \partial_- X^i_-} \tag{2.13}$$

where $X^i, i = 1 \ldots 8$ are the spatial directions transverse to the S^1. The parameter $\hat{b}$ should be chosen in such a way that the coordinate y winds n_w times around a circle of length R_y when $\hat{\sigma} \to \hat{\sigma} + 2\pi$. There is no winding around any other direction. We also use a reference frame in which the string has no momentum in any direction transverse to the S^1. We let $0 \le \hat{\sigma} < 2\pi$. Then the target space coordinates can be expanded as

$$\begin{aligned} y &= \frac{\alpha' n_p}{R_y}\hat{\tau} + n_w\, R_y\, \hat{\sigma} + \sum_{n\neq 0}(c_n\, e^{i\,n\,\chi^-} + d_n e^{i\,n\,\chi^+}) \\ X^i &= \sum_{n\neq 0}(c^i_n\, e^{i\,n\,\chi^-} + d^i_n e^{i\,n\,\chi^+}) \end{aligned} \tag{2.14}$$

Define

$$S_+ = \sqrt{\frac{\hat{b}^2}{4} - \partial_+ X^i_+ \partial_+ X^i_+}\,, \quad S_- = \sqrt{\frac{\hat{b}^2}{4} - \partial_- X^i_- \partial_- X^i_-} \tag{2.15}$$

From the energy and winding required of the configuration we find that the choice of signs in (2.13) should be

$$\partial_+ y_+ = S_+, \qquad \partial_- y_- = -S_- \tag{2.16}$$

[30]Note that it is more conventional to set a light cone coordinate X^+ to be linear in $\hat{\tau}$. Using a light cone coordinate allows the constraints (2.9) to be solved without square roots, but for us this is not important since we will not need to quantize the string.

After an interval

$$\Delta\hat{\tau} = \pi \tag{2.17}$$

all the X^i return to their original values. This can be seen by noting that $\hat{\sigma}$ is only a parameter that labels world sheet points, so the actual configuration of the system does not depend on the origin we choose for $\hat{\sigma}$. Thus consider the change

$$(\hat{\tau} = \hat{\tau}_1,\ \hat{\sigma}) \quad \to \quad (\hat{\tau} = \hat{\tau}_1 + \pi,\ \hat{\sigma} + \pi) \tag{2.18}$$

From (2.14) we see that the X^i are periodic with period $\hat{\tau} = \pi$. The coordinate y does not return to its original value, but in the classical limit that we have taken to get the 'supertube' we have smeared over this direction and so the *value* of y is not involved in describing the configuration of the supertube. But the *slope* of the NS1 at a point in the supertube *is* relevant, and is given by

$$s = |\frac{\partial X^i}{\partial\hat{\sigma}}| \Big/ (\frac{\partial y}{\partial\hat{\sigma}}) \tag{2.19}$$

But

$$\frac{\partial y}{\partial\hat{\sigma}} = n_w R_y + \sum_{n\neq 0} [(-in)c_n e^{in(\hat{\tau}-\hat{\sigma})} + (in)d_n e^{in(\hat{\tau}+\hat{\sigma})}] \tag{2.20}$$

We see that $\partial y/\partial\hat{\sigma}$ is periodic under (2.18) and thus so is (2.19).

From (2.12) we see that when $\hat{\tau}$ changes by the above period then

$$\Delta X^0 = \hat{b}\Delta\hat{\tau} = \hat{b}\pi \tag{2.21}$$

and the supertube configuration returns to itself. But

$$\hat{b} = \alpha' P^0 \equiv \alpha' E \tag{2.22}$$

where E is the energy of the configuration. We therefore find that the motion of the supertube is periodic in the target space time coordinate with period

$$\Delta t = \Delta X^0 = \alpha' \pi E \tag{2.23}$$

For the NS1-P system the dipole charge is NS1 – this arises from the fact that the NS1 slants as shown in Fig.6.1(b) and so there is a local NS1 charge along the direction of the supertube. The tension of the NS1 is $T = 1/(2\pi\alpha')$. This is thus the mass of the dipole charge per unit length

$$m_d = \frac{1}{2\pi\alpha'} \tag{2.24}$$

We then see that (4.101) can be recast as

$$\Delta t = \frac{1}{2}\frac{E}{m_d} \tag{2.25}$$

This form for the period will be of use to us later, because we will find that it holds in other duality frames as well.

6.2.2 The linearized perturbation

We can solve the NS1-P system exactly and we have thus obtained the exact dynamics of the supertube in flat space. For some purposes it will be useful to look at the small perturbations to the stationary tube configurations. We now study these small perturbations, starting in a slightly different way from the above analysis.

Consider first the string in a BPS configuration: The wave on the string is purely right moving. We know that in this case the waveform travels with the speed of light in the positive y direction. Let us check that this is a solution of our string equations. This time we know the solution in the *static* gauge on the worldsheet:

$$t = b\tilde{\tau}, \quad y = b\tilde{\sigma} \tag{2.26}$$

Writing $\xi^{\pm} = \tilde{\tau} \pm \tilde{\sigma}$ and noting that a right moving wave is a function of ξ^- we expect the following to be a solution

$$t = b\,\frac{\xi^+ + \xi^-}{2}, \quad y = b\,\frac{\xi^+ - \xi^-}{2}, \quad X^i = X^i(\xi^-) \tag{2.27}$$

In these worldsheet coordinates the induced metric is

$$ds^2 = -b^2\, d\xi^+\, d\xi^- + (X^{i\prime} X^{i\prime})\, (d\xi^-)^2 \tag{2.28}$$

so it does not satisfy (2.8). (Here prime denotes differentiation with respect to ξ^-). However we can change to new coordinates on the worldsheet

$$(\tilde{\xi}^+, \xi^-) = (\xi^+ - f(\xi^-), \xi^-) \tag{2.29}$$

with

$$f'(\xi^-) = \frac{(X^{i\prime} X^{i\prime})(\xi^-)}{b^2} \tag{2.30}$$

This brings the metric to the conformally flat form

$$ds^2 = -b^2\, d\xi^-\, d\tilde{\xi}^+ \tag{2.31}$$

Moreover, rewriting (2.27) in terms of $(\tilde{\xi}^+, \xi^-)$

$$t = b\,\frac{\tilde{\xi}^+ + \xi^- + f(\xi^-)}{2}\,, \quad y = b\,\frac{\tilde{\xi}^+ - \xi^- + f(\xi^-)}{2}\,, \quad X^i = X^i(\xi^-) \tag{2.32}$$

one sees that the configuration is of the form

$$X^\mu = x_+^\mu(\tilde{\xi}^+) + x_-^\mu(\xi^-) \tag{2.33}$$

so that the X^μ are harmonic in the coordinates $(\tilde{\xi}^+, \xi^-)$. Thus the coordinates $(\tilde{\xi}^+, \xi^-)$ are conformal coordinates for the problem and we have verified that (2.27) is a solution of the equations of motion.

We now proceed to adding a small right moving perturbation, which was our goal. Consider the perturbed configuration

$$\begin{aligned} &t = b\,\tilde{\tau} = b\,\frac{\tilde{\xi}^+ + \xi^- + f(\xi^-)}{2}\,, \quad y = b\,\tilde{\sigma} = b\,\frac{\tilde{\xi}^+ - \xi^- + f(\xi^-)}{2} \\ &X^i = X^i(\xi^-) + x^i(\tilde{\xi}^+) \end{aligned} \tag{2.34}$$

where x^i is assumed small. Then the induced metric on the worldsheet is

$$ds^2 = -(b^2 - 2X^{i\prime}x^{i\prime})\, d\xi^-\, d\tilde{\xi}^+ + O(x')^2\,(d\tilde{\xi}^+)^2 \qquad (2.35)$$

so that it is conformally flat to first order in the perturbation. The target space coordinates X^μ in (2.34) are clearly of the form (2.33) so they are harmonic, and we have found a solution of the string equations of motion.

6.2.3 Summary

We can get a general solution of the NS1-P system by taking arbitrary harmonic functions X^i in (2.14) and determining X^0, y by (2.12), (2.13). Taking the classical limit where the strands of the string forms a continuum gives the arbitrary motion of the supertube, and the period of this motion is given by (2.25). The conformal gauge coordinate $\hat{\sigma}$ that is used on the string is not very intuitive, since it is determined by the state of the string. We next looked at the linearized perturbation to a BPS state, and this time we started with an intuitively simple coordinate on the string – the static gauge coordinate $\tilde{\sigma}$ proportional to the spacetime coordinate y. We found the explicit map (2.16) to the conformal gauge coordinates. The solution to the linearized problem was then given by an arbitrary choice of the x^i in (2.34).

We will now see that these solutions reproduce the behavior of the D2 brane supertube at the exact and linearized levels respectively. The NS1-P system is the easiest way to solve the problem, since it exhibits the separation of the dynamics into a left and a right mover; this separation is not obvious in the other duality frames.

6.3 Perturbations of the D0-NS1 supertube

In this section we will consider the more conventional definition of the supertube: The true charges are D0-NS1 and the dipole charge is a D2. The dynamics is given by the DBI action of the D2 with worldvolume fields corresponding to the true charges. In [75] perturbations were considered around the 'round supertube' which has as its profile a circle in the (X_1, X_2) plane. This supertube has the maximum possible angular momentum J for its charges. So if we add a small perturbation to it we know that we will not get a 'drift' through a set of supertube configurations – J is conserved and there are no other configurations with this value of J. So even though periodic excitations were found for this supertube we cannot conclude from this that small perturbations to the generic supertube will also be periodic. Thus we wish to extend the computation of [75] to the generic supertube. We will write the equations of motion for the generic case, but instead of solving them directly we will note that we have already solved the problem in NS1-P language and we will just dualize the solution there and check that it solves the equations for the D0-NS1 supertube.[31]

We work in flat space with a compact S^1 of length $\tilde{L}_y = 2\pi\tilde{R}_y$, parametrized by the coordinate y. We have already obtained the general motion of the supertube in the NS1-P description, and below will verify that this solves the general D2 supertube equations as well. But first we check the behavior of small perturbations, and for this purpose we model our presentation as close to that of [75] as possible. Thus we let the supertube lie along a closed curve γ in the (X_1, X_2) plane, but γ need not be a circle as in [75]. The worldvolume of the D2 will be $\gamma \times S^1$.

[31]The fact that for given charges there is a range of possible configurations around a generic supertube was also noted in [82].

Let R and σ be the radial and angular coordinates in the (X_1, X_2) plane. We will denote by Z_a all the coordinates other than t, X_1, X_2, y. We will also sometimes use the notation $\mathbf{X}^I = \{X_1, X_2, Z_a\}$. We fix a gauge in which the world volume coordinates on the D2 brane are t, σ, y. Thus the angular coordinate in the supertube plane serves as the parameter along the supertube curve γ. On the D2 world volume we have a gauge field, for which we adopt the gauge

$$A_t = 0 \tag{3.1}$$

Thus the gauge field has the form

$$A = A_\sigma \, d\sigma + A_y \, dy \tag{3.2}$$

The D2-brane Lagrangian density is given by usual Born-Infeld term:

$$\mathcal{L} = -T_2 \sqrt{-\det(g + F)} \tag{3.3}$$

where T_2 is the D2 brane tension, g is the metric induced on the D2 world volume and F is the field strength of A. There are no background fields, so there is no Chern-Simons term in the action.

We want to consider fluctuations around a static configuration described by the curve

$$R = \bar{R}(\sigma) \, , \quad Z_a = 0 \tag{3.4}$$

and field strength

$$F = \bar{E} \, dt \wedge dy + \bar{B}(\sigma) \, dy \wedge d\sigma \tag{3.5}$$

It is known [10] that this configuration[32] satisfies the equations of motion and is supersymmetric for arbitrary $\bar{R}(\sigma), \bar{B}(\sigma)$ if $\bar{E}^2 = 1$ and $\text{sign}\, \bar{B}(\sigma) = \pm 1$. Without

[32]Since the configuration is independent of t, y, the Bianchi identity requires that $\bar{E}$ be a constant. There is no restriction on $\bar{B}$ and it is an arbitrary function of σ.

any loss of generality, we will take $\bar{E} = 1$ and $\text{sign}\,\bar{B}(\sigma) = 1$ in what follows. The electric field $\bar{E}$ induces a NS1 integer charge given by

$$n_1 = \frac{1}{T}\int d\sigma\,\Pi_y = \frac{1}{T}\int d\sigma\,\frac{\partial\mathcal{L}}{\partial(\partial_t A_y)} \tag{3.6}$$

where T is the NS1 tension. The magnetic field $\bar{B}$ induces a D0 integer charge equal to

$$n_0 = \frac{T_2}{T_0}\int dy d\sigma\,\bar{B}(\sigma) \tag{3.7}$$

where T_0 is the D0 brane mass.

We want to study fluctuations around the configuration described above. So we expand the Lagrangian up to quadratic order. We will assume that the fluctuations do not depend on y. We parametrize the D2-brane world volume as

$$R = \bar{R}(\sigma) + r(\sigma, t)\,, \quad Z_a = z_a(\sigma, t) \tag{3.8}$$

and the field strength as

$$\begin{aligned} F &= E\,dt\wedge dy + B\,dy\wedge d\sigma + \partial_t a_\sigma\,dt\wedge d\sigma \\ &= (\bar{E} + \partial_t a_y)\,dt\wedge dy + (\bar{B}(\sigma) - \partial_\sigma a_y)\,dy\wedge d\sigma + \partial_t a_\sigma\,dt\wedge d\sigma \end{aligned} \tag{3.9}$$

where lower case quantities denote the fluctuations. The metric induced on the D2 brane world volume is

$$ds^2 = -dt^2 + (\partial_\sigma\bar{R}\,d\sigma + \partial_\sigma r\,d\sigma + \partial_t r\,dt)^2 + (\bar{R}(\sigma) + r)^2\,d\sigma^2 + dy^2 + (\partial_\sigma z_a\,d\sigma + \partial_t z_a\,dt)^2 \tag{3.10}$$

The Lagrangian density $\mathcal{L}$ for the system is given by

$$
\begin{aligned}
-\frac{\mathcal{L}}{T_2} &= \sqrt{-\det(g+F)} \\
&= \{ -|\partial_t \mathbf{X}|^2 |\partial_\sigma \mathbf{X}|^2 + (\partial_t \mathbf{X} \cdot \partial_\sigma \mathbf{X})^2 + (1-E^2)|\partial_\sigma \mathbf{X}|^2 + B^2(1-|\partial_t \mathbf{X}|^2) \\
&\quad -2EB(\partial_t \mathbf{X} \cdot \partial_\sigma \mathbf{X}) - (\partial_t a_\sigma)^2 \}^{1/2} \\
&= \{-R^2[(\partial_t R)^2 + |\partial_t z_a|^2] - (\partial_t R)^2 |\partial_\sigma z_a|^2 - (\partial_\sigma R)^2 |\partial_t z_a|^2 + 2\partial_t R \partial_\sigma R \, \partial_t z_a \partial_\sigma z_a \\
&\quad +(1-E^2)[R^2 + (\partial_\sigma R)^2 + |\partial_\sigma z_a|^2] + B^2[1 - (\partial_t R)^2 - |\partial_t z_a|^2] \\
&\quad -2EB[\partial_t R \partial_\sigma R + \partial_t z_a \, \partial_\sigma z_a] + (\partial_t z_a \, \partial_\sigma z_a)^2 - |\partial_\sigma z_a|^2 |\partial_t z_a|^2 \\
&\quad -(\partial_t a_\sigma)^2\}^{1/2}
\end{aligned}
\tag{3.11}
$$

We wish to find the equations of motion up to linear order in the perturbation. To do this we expand $\mathcal{L}$ up to second order in r, a_y, a_σ:

$$
\frac{\mathcal{L}}{T_2} = L^{(0)} + L^{(1)} + L^{(2)} \tag{3.12}
$$

We find

$$
L^{(0)} = -\bar{B} \tag{3.13}
$$

$$
L^{(1)} = \left[\partial_\sigma a_y + \frac{\bar{R}^2 + (\partial_\sigma \bar{R})^2}{\bar{B}} \, \partial_t a_y + \partial_\sigma \bar{R} \, \partial_t r\right] \tag{3.14}
$$

We see that at first order in the perturbation the Lagrangian reduces to a total derivative in σ and t; this verifies the fact that our starting configuration satisfies the equations of motion. The term quadratic in the perturbation is

$$
\begin{aligned}
L^{(2)} &= -\frac{1}{2}\Big[-\frac{\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2}{\bar{B}}\,(\partial_t r)^2 - 2\partial_t r\,\partial_\sigma r \\
&- 2\,\frac{\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2}{\bar{B}^2}\,\partial_\sigma\bar{R}\,\partial_t r\,\partial_t a_y - 4\frac{\partial_\sigma\bar{R}}{\bar{B}}\,\partial_\sigma r\,\partial_t a_y - 4\frac{\bar{R}}{\bar{B}}\,r\,\partial_t a_y \\
&- \frac{(\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2)\,(\bar{R}^2+(\partial_\sigma\bar{R})^2)}{\bar{B}^3}\,(\partial_t a_y)^2 - 2\frac{\bar{R}^2+(\partial_\sigma\bar{R})^2}{\bar{B}^2}\,\partial_t a_y\,\partial_\sigma a_y \\
&- \frac{\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2}{\bar{B}}\,(\partial_t z_a)^2 - 2\partial_t z_a\,\partial_\sigma z_a - \frac{|\partial_t a_\sigma|^2}{\bar{B}}\Big] \qquad (3.15)
\end{aligned}
$$

From this Lagrangian we find the following equations of motion for the linearized perturbation:

$$
\begin{aligned}
&\frac{\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2}{\bar{B}}\,\partial_t^2 r + 2\partial_t\partial_\sigma r + \left(\frac{\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2}{\bar{B}}\,\partial_t^2 a_y + 2\partial_t\partial_\sigma a_y\right)\frac{\partial_\sigma\bar{R}}{\bar{B}} \\
&\qquad\qquad -2\frac{\bar{R}}{\bar{B}}\,\partial_t a_y + 2\partial_\sigma\left(\frac{\partial_\sigma\bar{R}}{\bar{B}}\right)\partial_t a_y = 0 \\
&\left(\frac{\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2}{\bar{B}}\,\partial_t^2 a_y + 2\partial_t\partial_\sigma a_y\right)\frac{\bar{R}^2+(\partial_\sigma\bar{R})^2}{\bar{B}^2} \\
&\qquad + \left(\frac{\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2}{\bar{B}}\,\partial_t^2 r + 2\partial_t\partial_\sigma r\right)\frac{\partial_\sigma\bar{R}}{\bar{B}} + 2\frac{\bar{R}}{\bar{B}}\,\partial_t r + \partial_\sigma\left(\frac{\bar{R}^2+(\partial_\sigma\bar{R})^2}{\bar{B}^2}\right)\partial_t a_y = 0 \\
&\frac{\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2}{\bar{B}}\,\partial_t^2 z_a + 2\partial_t\partial_\sigma z_a = 0 \\
&\partial_t^2 a_\sigma = 0 \qquad (3.16)
\end{aligned}
$$

We have an additional equation coming from the variation of A_t; this is the Gauss law which says

$$\partial_\sigma E_\sigma \equiv \partial_\sigma\partial_t a_\sigma = 0 \qquad (3.17)$$

The last equation in (2.9) and (3.17) together say that we can add an electric field along the σ direction but this field will be constant in both σ and t. We will henceforth set this additional E to zero, and thus $a_\sigma = 0$ for the rest of the calculation.

Note that only time derivatives of fields occur in the equations; there are no terms where the fields appear without such time derivatives. Thus any time independent perturbation is a solution to the equations. This tells us that we can make arbitrary time independent deformations of the supertube, reproducing the known fact that the supertube has a family of time independent solutions.

The D0 and NS1 integer charges of the perturbed configuration are

$$\begin{aligned} n_0 &= \frac{T_2}{T_0}\int dy d\sigma\,(\bar{B}(\sigma)-\partial_\sigma a_y) = \frac{T_2}{T_0}\int dy d\sigma\,\bar{B}(\sigma) \\ n_1 &= \frac{T_2}{T}\int d\sigma\,\Big[\frac{\bar{R}^2+(\partial_\sigma\bar{R})^2}{\bar{B}} + 2\frac{\bar{R}}{\bar{B}}\,r \\ &\quad + \frac{(\bar{R}^2+(\partial_\sigma\bar{R})^2)(\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2)}{\bar{B}^3}\,\partial_t a_y + \frac{\bar{R}^2+(\partial_\sigma\bar{R})^2}{\bar{B}^2}\,\partial_\sigma a_y \\ &\quad + \frac{\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2}{\bar{B}^2}\,\partial_\sigma\bar{R}\,\partial_t r + 2\frac{\partial_\sigma\bar{R}}{\bar{B}}\,\partial_\sigma r\Big] \end{aligned} \tag{3.18}$$

We see that the D0 charge is unchanged by the perturbation. This charge in fact is a topological invariant of the gauge field configuration. For the NS1 charge we can check conservation by explicitly computing the time derivative and verifying that it vanishes.

The angular momentum in the (X_1, X_2) plane is

$$J = \int d\sigma\, dy\,(\Pi_2 X_1 - \Pi_1 X_2) \tag{3.19}$$

where

$$\Pi_i = \frac{\partial\mathcal{L}}{\partial(\partial_t X_i)}\ ,\ i = 1,2 \tag{3.20}$$

From the Lagrangian (4.82) we find

$$\Pi_i = -T_2^2\left(\frac{\partial_t X_i\,[(\partial_\sigma\mathbf{X})^2+B^2] - \partial_\sigma X_i\,[(\partial_t\mathbf{X}\partial_\sigma\mathbf{X}) - EB]}{\mathcal{L}}\right) \tag{3.21}$$

Expanding J up to first order in the perturbation we get

$$J = T_2\,(2\pi\tilde{R}_y)\int d\sigma\,\Big[\bar{R}^2 + 2\bar{R}\,r + \frac{\bar{R}^2\,(\bar{R}^2+(\partial_\sigma\bar{R})^2+\bar{B}^2)}{\bar{B}^2}\,\partial_t a_y\Big] \tag{3.22}$$

6.3.1 Using the NS1-P solution: Linear perturbation

The equations (2.9) for the perturbations to the D2 brane look complicated, but we will obtain the solution by dualizing the NS1-P solution found above. Recall that we have split the spatial coordinates transverse to the S^1 (i.e. the $\mathbf{X}$) as $\mathbf{X} = \{X_1, X_2, Z_a\}$. To arrive at the D2 supertube in the (X_1, X_2) plane we assume that the right moving wave on the NS1 has its transverse oscillations only in the (X_1, X_2) plane. This solution is then perturbed by a small left-moving wave. Recall that we had defined static gauge coordinates $\tilde{\tau}, \tilde{\sigma}$ (2.13) on the world sheet and then obtained the conformal coordinates $\tilde{\xi}^+, \xi^-$. We will find it convenient to use as independent variables $\tilde{\tau}$ and ξ^-. This is because from (2.13) we see that $\tilde{\tau}$ directly gives the target space time t, and ξ^- is the variable in terms of which we have the basic right moving wave $\mathbf{X}(\xi^-)$ that gives the unperturbed solution. Thus we have

$$\tilde{\xi}^+ = \xi^+ - f(\xi^-) = 2\tilde{\tau} - \xi^- - f(\xi^-)\,, \qquad f' = \frac{(X_1')^2 + (X_2')^2}{b^2} \tag{3.23}$$

For the NS1-P solution the target space coordinates are given by

$$\begin{aligned} t &= b\,\tilde{\tau} \\ y &= b\,\tilde{\sigma} = b\,(\tilde{\tau} - \xi^-) \\ X_i(\xi^-, \tilde{\tau}) &= X_i(\xi^-) + x_i(\tilde{\xi}^+)\ ,\ i = 1, 2 \\ Z_a(\xi^-, \tilde{\tau}) &= z_a(\tilde{\xi}^+) \end{aligned} \tag{3.24}$$

where x_i, z_a are small perturbations.

We perform an S-duality to go from NS1-P to D1-P, and then a T-duality along S^1 to get the D0-NS1 supertube. The S^1 coordinate y goes, under these changes, to the component A_y of the gauge field on the D2. In the normalization of the gauge field A used in the action (3.3) we just get

$$y \;\to\; A_y \tag{3.25}$$

so from (3.24) we have

$$A_y = t - b\,\xi^- \tag{3.26}$$

In this solution derived by duality from NS1-P the natural coordinates on the D2 are $(\tilde{\tau}, \xi^-, y)$.[33] But when we wrote the DBI action for the D2 the natural coordinates were (t, σ, y), where σ was the angle in the (X_1, X_2) plane

$$\tan\sigma = \frac{X_2(\xi^-, \tilde{\tau})}{X_1(\xi^-, \tilde{\tau})} \tag{3.27}$$

The coordinates t and $\tilde{\tau}$ are related by a constant, so there is no difficulty in replacing the t by $\tilde{\tau}$ in converting the NS1-P solution to a D0-NS1 supertube solution. But the change $\xi^- \to \sigma$ is more complicated, and will necessitate the algebra steps below. Inverting (3.27) gives

$$\xi^- = \xi^-(\sigma, \tilde{\tau}) \tag{3.28}$$

so we see that the change $\xi^- \to \sigma$ depends on time as well, if the supertube is oscillating. The variables describing the supertube configuration will be

$$\begin{aligned} R(\sigma, \tilde{\tau}) &= \sqrt{X_1^2(\xi^-(\sigma, \tilde{\tau}), \tilde{\tau}) + X_2^2(\xi^-(\sigma, \tilde{\tau}), \tilde{\tau})} \\ A_y(\sigma, \tilde{\tau}) &= t - b\,\xi^-(\sigma, \tilde{\tau}) \\ Z_a(\sigma, \tilde{\tau}) &= Z_a(\xi^-(\sigma, \tilde{\tau}), \tilde{\tau}) \end{aligned} \tag{3.29}$$

which should satisfy the equations for the D0-NS1 supertube.

[33]In these coordinates we can see that the electric field is $E = \partial_t A_y = 1$, as expected for the stationary supertube configurations.

First consider the unperturbed configuration. Here the transformation (3.28) does not depend on $\tilde{\tau}$. For the variables of the unperturbed configuration we write

$$\bar{X}_i = X_i(\bar{\xi}^-(\sigma)) , \quad \bar{X}'_i = X'_i(\bar{\xi}^-(\sigma)) \ , \ i = 1, 2 \tag{3.30}$$

where the prime denotes a derivative with respect to the argument $\bar{\xi}^-$. The function $\bar{\xi}^-(\sigma)$ will be the solution of the equation

$$\tan\sigma = \frac{\bar{X}_2(\bar{\xi}^-)}{\bar{X}_1(\bar{\xi}^-)} \tag{3.31}$$

and the stationary configuration will be given by

$$\begin{aligned} \bar{R}(\sigma) &= \sqrt{\bar{X}_1^2 + \bar{X}_2^2} \\ \bar{B}(\sigma) &= -\partial_\sigma A_y = b\,\partial_\sigma \bar{\xi}^-(\sigma) \end{aligned} \tag{3.32}$$

From the above definitions we can derive the identities

$$\begin{aligned} \frac{\bar{B}}{\bar{R}^2} &= \frac{b}{\bar{X}_1 \bar{X}'_2 - \bar{X}_2 \bar{X}'_1} \\ \partial_\sigma \bar{R} &= \frac{\bar{B}}{b\,\bar{R}} (\bar{X}_1 \bar{X}'_1 + \bar{X}_2 \bar{X}'_2) \end{aligned} \tag{3.33}$$

Using these identities one can prove a relation that will be important in the following

$$\bar{R}^2 + (\partial_\sigma \bar{R})^2 = \bar{B}^2 \, \bar{f}' \tag{3.34}$$

where $\bar{f}' = f'(\bar{\xi}^-(\sigma))$. Now consider the small perturbation on the supertube. We will keep all quantities to linear order in the x_i, z_a. Inverting the relation (3.27) gives us

$$\xi^- = \bar{\xi}^- + \hat{\xi}^- , \quad \hat{\xi}^-(\sigma, \tilde{\tau}) = -\frac{\bar{X}_1 \, \tilde{x}_2 - \bar{X}_2 \, \tilde{x}_1}{\bar{X}_1 \bar{X}'_2 - \bar{X}_2 \bar{X}'_1} \tag{3.35}$$

where

$$\tilde{x}_i = x_i(2\tilde{\tau} - \bar{\xi}^- - f(\bar{\xi}^-)) \ , \ i = 1, 2 \tag{3.36}$$

Using (3.35), the first identity in (3.33), and performing an expansion to first order in x_i, we find the perturbation around the static configuration

$$
\begin{aligned}
r(\sigma, \tilde{\tau}) &= R(\sigma, \tilde{\tau}) - \bar{R}(\sigma) = \frac{\bar{B}}{b\,\bar{R}}\,(\tilde{x}_1 \bar{X}_2' - \tilde{x}_2 \bar{X}_1') \\
&= \tilde{x}_1\,\cos\sigma + \tilde{x}_2\,\sin\sigma + \frac{\partial_\sigma \bar{R}}{\bar{R}}\,(\tilde{x}_1\,\sin\sigma - \tilde{x}_2\,\cos\sigma) \\
a_y(\sigma, \tilde{\tau}) &= -b\,\hat{\xi}^- = -\frac{\bar{B}}{\bar{R}^2}\,(\tilde{x}_1 \bar{X}_2 - \tilde{x}_2 \bar{X}_1) \\
&= -\frac{\bar{B}}{\bar{R}}\,(\tilde{x}_1\,\sin\sigma - \tilde{x}_2\,\cos\sigma) \\
z_a(\sigma, \tilde{\tau}) &= z_a(2\tilde{\tau} - \bar{\xi}^- - f(\bar{\xi}^-)) \;\equiv\; \tilde{z}_a
\end{aligned}
\tag{3.37}
$$

We would like to check that the functions r, a_y and z_a defined above satisfy the equations of motion (2.9). For this purpose, some useful identities are

$$
\partial_\sigma \tilde{x}_i' = -\frac{\bar{B}}{b}(1 + \bar{f}')\,\tilde{x}_i''\,, \quad \partial_\sigma \tilde{z}_a' = -\frac{\bar{B}}{b}(1 + \bar{f}')\,\tilde{z}_a'' \tag{3.38}
$$

We can simplify some expressions appearing in the equations of motion (2.9)

$$
\begin{aligned}
\frac{\bar{R}^2 + (\partial_\sigma \bar{R})^2 + \bar{B}^2}{\bar{B}}\,\partial_t^2 r + 2\partial_t \partial_\sigma r &= \frac{4}{b}\left[-\tilde{x}_1'\,\sin\sigma + \tilde{x}_2'\,\cos\sigma + \frac{\partial_\sigma \bar{R}}{\bar{R}}\,(\tilde{x}_1'\,\cos\sigma + \tilde{x}_2'\,\sin\sigma)\right] \\
&\quad + \frac{4}{b}\partial_\sigma\left(\frac{\partial_\sigma \bar{R}}{\bar{R}}\right)(\tilde{x}_1'\,\sin\sigma - \tilde{x}_2'\,\cos\sigma) \\
\frac{\bar{R}^2 + (\partial_\sigma \bar{R})^2 + \bar{B}^2}{\bar{B}}\,\partial_t^2 a_y + 2\partial_t \partial_\sigma a_y &= -4\frac{\bar{B}}{b\,\bar{R}}\,(\tilde{x}_1'\,\cos\sigma + \tilde{x}_2'\,\sin\sigma) \\
&\quad - \frac{4}{b}\partial_\sigma\left(\frac{\bar{B}}{\bar{R}}\right)(\tilde{x}_1'\,\sin\sigma - \tilde{x}_2'\,\cos\sigma) \\
\frac{\bar{R}^2 + (\partial_\sigma \bar{R})^2 + \bar{B}^2}{\bar{B}}\,\partial_t^2 z_a + 2\partial_t \partial_\sigma z_a &= 0
\end{aligned}
\tag{3.39}
$$

The last identity proves that the equations for z_a are satisfied. For the equations involving r and a_y some more work is needed. The l.h.s. of the first equation in (2.9) is equal to

$$\frac{4}{b}(\tilde{x}_1' \sin\sigma - \tilde{x}_2' \cos\sigma)\Big[-1 + \partial_\sigma\Big(\frac{\partial_\sigma \bar{R}}{\bar{R}}\Big) - \partial_\sigma\Big(\frac{\bar{B}}{\bar{R}}\Big)\frac{\partial_\sigma \bar{R}}{\bar{B}} + \frac{\bar{B}}{\bar{R}}\Big(\frac{\bar{R}}{\bar{B}} - \partial_\sigma\Big(\frac{\partial_\sigma \bar{R}}{\bar{B}}\Big)\Big)\Big]$$
$$+\frac{4}{b}(\tilde{x}_1' \cos\sigma + \tilde{x}_2' \sin\sigma)\left[\frac{\partial_\sigma \bar{R}}{\bar{R}} - \frac{\bar{B}}{\bar{R}}\frac{\partial_\sigma \bar{R}}{\bar{R}}\right] \tag{3.40}$$

which, after some algebra, is seen to vanish. The l.h.s. of the second equation in (2.9) is

$$\frac{4}{b}(\tilde{x}_1' \sin\sigma - \tilde{x}_2' \cos\sigma)\Big[-\frac{\bar{R}^2 + (\partial_\sigma \bar{R})^2}{\bar{B}^2}\partial_\sigma\Big(\frac{\bar{B}}{\bar{R}}\Big) - \frac{\partial_\sigma \bar{R}}{\bar{B}}\Big(1 - \partial_\sigma\Big(\frac{\partial_\sigma \bar{R}}{\bar{R}}\Big)\Big)$$
$$+\frac{\bar{R}}{\bar{B}}\frac{\partial_\sigma \bar{R}}{\bar{R}} - \frac{1}{2}\frac{\bar{B}}{\bar{R}}\partial_\sigma\Big(\frac{\bar{R}^2 + (\partial_\sigma \bar{R})^2}{\bar{B}^2}\Big)\Big]$$
$$+\frac{4}{b}(\tilde{x}_1' \cos\sigma + \tilde{x}_2' \sin\sigma)\left[-\frac{\bar{B}}{\bar{R}}\frac{\bar{R}^2 + (\partial_\sigma \bar{R})^2}{\bar{B}^2} + \frac{(\partial_\sigma \bar{R})^2}{\bar{B}\bar{R}} + \frac{\bar{R}}{\bar{B}}\right] \tag{3.41}$$

which also vanishes.

We thus find that the expressions (3.37), with x_i, z_a arbitrary functions of their arguments, satisfy the equations (2.9).

6.3.2 Period of oscillation

We would like to determine the period of the oscillations of the solution (3.37). The world sheet coordinate $\tilde{\sigma}$ has a period 2π. The time dependence of the solution (3.37) is contained in functions $x_i(2\tilde{\tau} - \bar{\xi}^- - f(\bar{\xi}^-))$ and $z_a(2\tilde{\tau} - \bar{\xi}^- - f(\bar{\xi}^-))$. The quantity $(\mathbf{X}')^2(\bar{\xi}^-)$ which appears in the definition of $f(\bar{\xi}^-)$ will be the sum of a constant term, $\tilde{R}^2$, plus terms periodic in $\bar{\xi}^-$:

$$(\mathbf{X}')^2(\bar{\xi}^-) = \tilde{R}^2 + \sum_{n\neq 0}(a_n e^{in\bar{\xi}^-} + c.c.) \tag{3.42}$$

which implies that f has the form

$$f(\bar{\xi}^-) = \frac{\tilde{R}^2}{b^2}\bar{\xi}^- + \sum_{n\neq 0}(b_n e^{in\bar{\xi}^-} + c.c.) \tag{3.43}$$

The functions x_i, z_a are functions of the coordinate $\bar{\xi}^-$ along the supertube. This supertube is a closed loop, so all functions on it are periodic under the shift $(\tilde{\tau}, \bar{\xi}^-) \to (\tilde{\tau}, \bar{\xi}^- + 2\pi)$. This implies

$$x_i\Big(2\tilde{\tau} - \bar{\xi}^- - f(\bar{\xi}^-)\Big) = x_i\Big(2\tilde{\tau} - \bar{\xi}^- - f(\bar{\xi}^-) - 2\pi\Big(1 + \frac{\tilde{R}^2}{b^2}\Big)\Big) \tag{3.44}$$

where we have used (3.43) to get the change in $f(\bar{\xi}^-)$.

We have a similar relation for $z_a(2\tilde{\tau} - \bar{\xi}^- - f(\bar{\xi}^-))$. Thus the period of the oscillations is given by

$$\Delta t = b\Delta\tilde{\tau} = \pi\,\frac{b^2 + \tilde{R}^2}{b} \tag{3.45}$$

This form of the period is similar to that found in [75]; it reduces to the period found there when the radius $\bar{R}$ is a constant.

To arrive at our more general form (2.25) we write

$$\bar{\xi}^- + f(\bar{\xi}^-) = \int_0^{\bar{\xi}^-} d\chi(1 + f'(\chi)) \tag{3.46}$$

So the change in $\bar{\xi}^- + f(\bar{\xi}^-)$ when $\bar{\xi}^-$ increases by 2π can be written as $\int_0^{2\pi} d\chi(1 + f'(\chi))$. We then find that the argument of x_i, z_a are unchanged when $(\tilde{\tau}, \bar{\xi}^-) \to (\tilde{\tau} + \Delta\tilde{\tau}, \bar{\xi}^- + 2\pi)$ with

$$2\Delta\tilde{\tau} - \int_0^{2\pi} \big(1 + f'(\bar{\xi}^-)\big)\, d\bar{\xi}^- = 0 \tag{3.47}$$

Using the identity (3.34) we write the above as

$$\Delta\tilde{\tau} = \frac{1}{2}\int_0^{2\pi} \left(1 + \frac{\bar{R}^2 + (\partial_\sigma \bar{R})^2}{\bar{B}^2}\right) d\bar{\xi}^- \tag{3.48}$$

Now using the fact that $\bar{B} = b\,\partial_\sigma\bar{\xi}^-$, $\Delta t = b\Delta\tilde{\tau}$ and changing variables from $\bar{\xi}^-$ to σ we get

$$\Delta t = \frac{1}{2}\int d\sigma \left(\bar{B} + \frac{\bar{R}^2 + (\partial_\sigma \bar{R})^2}{\bar{B}}\right) \tag{3.49}$$

Now we express (3.49) in terms of the NS1 and D0 charges (we can use the unperturbed values of these quantities), using

$$n_1 = \frac{T_2}{T} \int d\sigma \, \frac{\bar{R}^2 + (\partial_\sigma \bar{R})^2}{\bar{B}} \quad , \quad n_0 = \frac{T_2}{T_0} \int dy \, d\sigma \, \bar{B} \tag{3.50}$$

We get

$$\Delta t = \frac{1}{2} \left(\frac{n_0 T_0 + n_1 T \tilde{L}_y}{T_2 \tilde{L}_y} \right) \tag{3.51}$$

where $\tilde{L}_y = 2\pi \tilde{R}_y$ is the length of y circle in the D0-NS1 duality frame.

Note that $n_0 T_0 + n_1 T \tilde{L}_y$ is the mass of the BPS state and since we have added only an infinitesimal perturbation it is to leading order the energy E of the configuration. Further $T_2 \tilde{L}_y$ is the mass of the D2 dipole charge per unit length of the supertube curve γ. Thus we see that the period again has the form (2.25)

$$\Delta t = \frac{1}{2} \frac{E}{m_d} \tag{3.52}$$

6.3.3 Using the NS1-P solution: Exact dynamics

Now consider the exact NS1-P solution (i.e. not perturbative around a BPS configuration). We again perform the required dualities to transform this solution into a solution of the D2 supertube. We will use as world-volume coordinates for the D2 brane $(\hat{\tau}, \hat{\sigma}, y)$. Then the D2 solution is given by

$$\begin{aligned} X^i &= X^i_+(\chi^+) + X^i_-(\chi^-) \, , \quad A_y = y_+(\chi^+) + y_-(\chi^-) \\ E &= \partial_{\hat{\tau}} A_y = \partial_+ y_+ + \partial_- y_- = S_+ - S_- \\ B &= -\partial_{\hat{\sigma}} A_y = -\partial_+ y_+ + \partial_- y_- = -(S_+ + S_-) \end{aligned} \tag{3.53}$$

In this subsection X^i denotes all coordinates other than t and y. We wish to prove that (3.53) satisfies the dynamical equations of the D2 brane. The DBI lagrangian density is given by

$$
\begin{aligned}
\frac{\mathcal{L}}{T_2} &= -\sqrt{-\det(g+F)} \\
&= -[-(\partial_{\hat{\tau}}X)^2(\partial_{\hat{\sigma}}X)^2 + (\partial_{\hat{\tau}}X\partial_{\hat{\sigma}}X)^2 + (b^2-E^2)(\partial_{\hat{\sigma}}X)^2 + B^2(b^2-(\partial_{\hat{\tau}}X)^2) \\
&\quad -2EB(\partial_{\hat{\tau}}X\partial_{\hat{\sigma}}X)]^{1/2}
\end{aligned} \tag{3.54}
$$

The equations of motion are

$$
\begin{aligned}
&\partial_{\hat{\tau}}\Big[\frac{\partial_{\hat{\tau}}X^i[(\partial_{\hat{\sigma}}X)^2+B^2] - \partial_{\hat{\sigma}}X^i[(\partial_{\hat{\tau}}X\partial_{\hat{\sigma}}X) - EB]}{\mathcal{L}}\Big] \\
&\qquad + \partial_{\hat{\sigma}}\Big[\frac{\partial_{\hat{\sigma}}X^i[(\partial_{\hat{\tau}}X)^2+E^2-b^2] - \partial_{\hat{\tau}}X^i[(\partial_{\hat{\tau}}X\partial_{\hat{\sigma}}X) - EB]}{\mathcal{L}}\Big] = 0
\end{aligned} \tag{3.55}
$$

$$
\partial_{\hat{\tau}}\Big[\frac{E(\partial_{\hat{\sigma}}X)^2 + B(\partial_{\hat{\tau}}X\partial_{\hat{\sigma}}X)}{\mathcal{L}}\Big] - \partial_{\hat{\sigma}}\Big[\frac{B[(\partial_{\hat{\tau}}X)^2-b^2] + E(\partial_{\hat{\tau}}X\partial_{\hat{\sigma}}X)}{\mathcal{L}}\Big] = 0
$$

To verify that these equations are satisfied by the configuration (3.53) we need following identities:

$$
\begin{aligned}
-\frac{\mathcal{L}}{T_2} &= \Big[\frac{b^4}{2} + 4(\partial_+X_+)^2(\partial_-X_-)^2 + 4(\partial_+X_+\partial_-X_-)^2 \\
&\quad + b^2[2S_+S_- - 2(\partial_+X_+\partial_-X_-) - (\partial_+X_+)^2 - (\partial_-X_-)^2] - 8S_+S_-(\partial_+X_+\partial_-X_-)\Big]^{1/2} \\
&= \frac{b^2}{2} + 2S_+S_- - 2(\partial_+X_+\partial_-X_-)
\end{aligned} \tag{3.56}
$$

$$
\begin{aligned}
(\partial_{\hat{\sigma}} X)^2 + B^2 &= -[(\partial_{\hat{\tau}} X)^2 + E^2 - b^2] = \frac{b^2}{2} + 2S_+S_- - 2(\partial_+ X_+ \partial_- X_-) \\
(\partial_{\hat{\tau}} X \partial_{\hat{\sigma}} X) - EB &= 0 \\
E(\partial_{\hat{\sigma}} X)^2 + B(\partial_{\hat{\tau}} X \partial_{\hat{\sigma}} X) &= 2S_+[(\partial_- X_-)^2 - (\partial_+ X_+ \partial_- X_-)] \\
&- 2S_-[(\partial_+ X_+)^2 - (\partial_+ X_+ \partial_- X_-)] \\
&= (S_+ - S_-)\left[\frac{b^2}{2} + 2S_+S_- - 2(\partial_+ X_+ \partial_- X_-)\right] \\
B[(\partial_{\hat{\tau}} X)^2 - b^2] + E(\partial_{\hat{\tau}} X \partial_{\hat{\sigma}} X) &= 2S_+\left[\frac{b^2}{2} - (\partial_- X_-)^2 - (\partial_+ X_+ \partial_- X_-)\right] \\
&- 2S_-\left[-\frac{b^2}{2} + (\partial_+ X_+)^2 + (\partial_+ X_+ \partial_- X_-)\right] \\
&= (S_+ + S_-)\left[\frac{b^2}{2} + 2S_+S_- - 2(\partial_+ X_+ \partial_- X_-)\right] \qquad (3.57)
\end{aligned}
$$

Then the equations (3.55),(3.56) become

$$
\begin{aligned}
&\partial_{\hat{\tau}}^2 X^i - \partial_{\hat{\sigma}}^2 X^i = 4\partial_+ \partial_- X^i = 0 \\
&\partial_{\hat{\tau}}(S_+ - S_-) - \partial_{\hat{\sigma}}(S_+ + S_-) = 2\partial_- S_+ - 2\partial_+ S_- = 0 \qquad (3.58)
\end{aligned}
$$

which are seen to be satisfied due the harmonic nature of the fields X^i, y.

6.3.4 'Quasi-oscillations'

In the introduction we termed the periodic behavior of the supertube a 'quasi-oscillation'. In a regular 'oscillation' there is an equilibrium point; if we displace the system from this point then there is a force tending to restore the system to the equilibrium point. But in the supertube we can displace a stationary configuration to a nearby stationary configuration, and the system does not try to return to the first configuration. The only time we have such a behavior for a usual *oscillatory* system is when we have a 'zero mode' (1.3). Such zero modes allow a 'drift' in which we give the system a small initial velocity and then we have an evolution like (1.1). But the supertube does not have this behavior either; there is no 'drift'.[34]

Now consider a different system, a particle with charge e and mass m placed in a uniform magnetic field $F_{xy} = B$. With the gauge potential $A_y = x$ we have the lagrangian

$$L = \frac{m}{2}[(\dot{x})^2 + (\dot{y})^2] + e\vec{A}\cdot\vec{v} = \frac{m}{2}[(\dot{x})^2 + (\dot{y})^2] + e\,x\dot{y} \tag{3.59}$$

The equations of motion are

$$\ddot{x} = \frac{e}{m}\dot{y}, \qquad \ddot{y} = -\frac{e}{m}\dot{x} \tag{3.60}$$

Since each term in the equation has at least one time derivative, any constant position $x = x_0$, $y = y_0$ is a solution. But if we perturb the particle slightly then the particle does not drift over this space of configurations in the manner (1.1); instead it describes a circle with characteristics (1.4). So while this motion is periodic the physics is not that of usual oscillations, and we call it a 'quasi-oscillation'.

Now we wish to show that the motion of the supertube is also a 'quasi-oscillation'. We will take a simple configuration of the D2 brane to illustrate the point. Let the D2 brane extend along the $z - y$ plane and oscillate in one transverse direction x. We will restrict to motions which are invariant in y and thus described by a field $x = x(t, z)$. We will also turn on a (y-independent) world volume gauge field, for which we choose the $A_t = 0$ gauge:

$$A = A_z(t,z)\,dz + A_y(t,z)\,dy \tag{3.61}$$

$$F = \dot{A}_z\,dt\wedge dz + \dot{A}_y\,dt\wedge dy + A'_y\,dz\wedge dy \equiv \dot{A}_z\,dt\wedge dz + E\,dt\wedge dy - B\,dz\wedge dy$$

[34]By contrast, 'giant gravitons' have usual vibration modes [84]. The giant graviton in $AdS_3\times S^3$ has a zero mode corresponding to changing the radius of the giant graviton, and we find a 'drift' over the values of this radius. In [133] giant gravitons were studied for AdS_3 and it was argued that they give *unbound* states where one brane is separated from the rest [133, 85].

Using t, z and y as world volume coordinates, the DBI lagrangian density is

$$\frac{\mathcal{L}}{T_2} = -\sqrt{-\det(g+F)} = -[1 - \dot{x}^2 + x'^2 + B^2(1-\dot{x}^2) - E^2(1+x'^2) - 2EB\,\dot{x}x' - \dot{A}_z^2]^{1/2} \tag{3.62}$$

In order to have a qualitative understanding of the dynamics induced by this lagrangian, let us expand it around a classical stationary solution with $x = 0$, $\bar{E} = 1$, $B = \bar{B}$ and $A_z = 0$. We denote by $a_y(t, z)$ the fluctuation of the gauge field A_y, so that

$$E = 1 + \dot{a}_y\,, \quad B = \bar{B} - a'_y \tag{3.63}$$

As the gauge field A_z decouples from all other fields we will set it to zero. Keeping terms up to second order in x and a_y, we find the quadratic lagrangian density to be

$$L^{(2)} = -\bar{B} + \frac{\dot{a}_y}{\bar{B}} + a'_y + \frac{1+\bar{B}^2}{2\bar{B}}\,\dot{x}^2 + \dot{x}\,x' + \frac{1}{\bar{B}^2}\left(\frac{1+\bar{B}^2}{2\bar{B}}\,\dot{a}_y^2 + \dot{a}_y\,a'_y\right) \tag{3.64}$$

The terms of first order in a_y are total derivatives (with respect to t and z) and do not contribute to the action. The fields x and a_y are decoupled, at this order, and both have a lagrangian of the form

$$L^{(2)}_\phi = \frac{1+\bar{B}^2}{2\bar{B}}\,\dot{\phi}^2 + \dot{\phi}\,\phi' \tag{3.65}$$

(with $\phi = x$ or a_y). As we can see the lagrangian (3.65) has no potential terms (terms independent of $\dot{\phi}$) and we find that any time independent configuration solves the equations of motion. There is however a magnetic-type interaction ($\dot{\phi}\,\phi'$), which is responsible for the fact that all time dependent solutions are oscillatory. Indeed, the equations of motion for ϕ are

$$\frac{1+\bar{B}^2}{2\bar{B}}\,\ddot{\phi} + \dot{\phi}' = 0 \tag{3.66}$$

whose solution is

$$\phi = e^{ik\,z - i\omega\,t}\,, \qquad \omega = 2\,\frac{\bar{B}}{1+\bar{B}^2}\,k \tag{3.67}$$

One can make the analogy between the interaction $\dot{\phi}\,\phi'$ and the toy problem of a particle in a magnetic field more precise by discretizing the z direction on a lattice of spacing a. Then we have

$$\begin{aligned}\int dz\, L^{(2)}_\phi &\approx a\sum_n \Big(\frac{m}{2}\,\dot{\phi}_n^2 + \dot{\phi}_n\,\frac{\phi_{n+1}-\phi_n}{a}\Big)\\ &\approx a\sum_n \Big(\frac{m}{2}\,\dot{\phi}_n^2 + \frac{\dot{\phi}_n\,\phi_{n+1}}{a}\Big)\end{aligned} \tag{3.68}$$

where in the second line we have discarded a total time-derivative and $m = (1+\bar{B}^2)/\bar{B}$. The term $\dot{\phi}_n\,\phi_{n+1}$ is just like the term $x\dot{y}$ in (3.59) induced by a constant magnetic field where the variables ϕ_n, ϕ_{n+1} play the role of x, y.

6.3.5 Summary

We have obtained the full dynamics of the D2 supertube, by mapping the problem to a free string which can be exactly solved. In the D2 language it is not obvious that the problem separates into a 'right mover' and a 'left mover', but (3.53) exhibits such a break up. This breakup needs a world sheet coordinate $\hat{\sigma}$ that is a conformal coordinate on the string world sheet, and is thus not an obvious coordinate in the D2 language. The D2 has a natural parametrization in terms of the angular coordinate on the spacetime plane (X_1, X_2), and the difficulties we encountered in mapping the NS1-P solution to the D2 supertube all arose from the change of parametrization.

6.4 The thin tube limit of the gravity solution

So far we have ignored gravity in our discussion of the supertube, so we were at vanishing coupling $g = 0$. If we slightly increase g then the gravitational field of

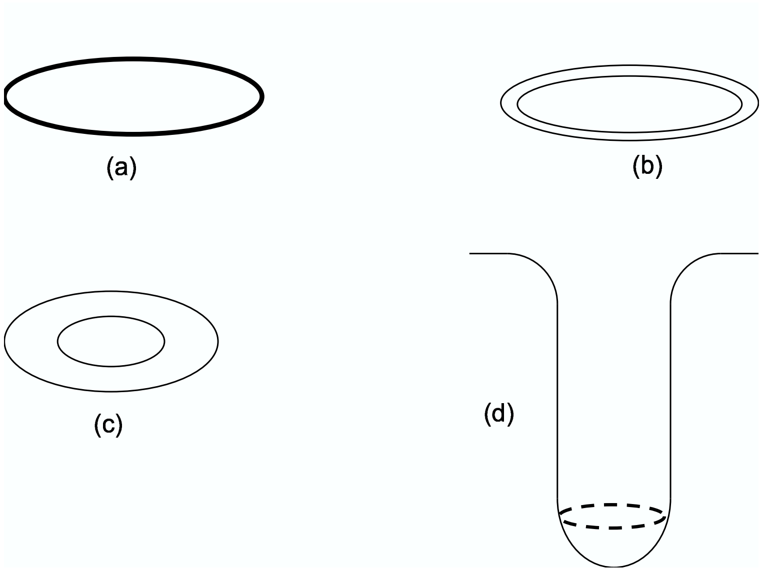

Figure 6.2: (a) The supertube at $g \to 0$, described by a worldsheet action. (b) The 'thin tube' at weak coupling. (c) The 'thick tube' reached at larger coupling. (d) At still larger coupling we get a 'deep throat' geometry; the strands of the NS1 generating the geometry run along the dotted curve.

the supertube will extend to some distance off the tube, but for small enough g this distance will be much less than the radius of the supertube. We will call this the 'thin tube limit', and we picture it in Fig.6.2(b).

We expect that in this thin tube the dynamics should not be too different from that found at $g \to 0$, and we will find that such is the case; we will find periodic excitations with frequency agreeing with that found from the free string computation and the D2 brane DBI action. But by doing the problem in a gravity description we move from the worldsheet theory to a spacetime one, which will help us to understand what happens when we increase the coupling still further.

Let us recall the 2-charge BPS geometries made in the NS1-P duality frame [150]. Start with type IIB string theory and take the compactification $M_{9,1} \to M_{4,1} \times S^1 \times T^4$. As before the coordinate along S^1 is y and the coordinates $z_a, a = 1\ldots4$ are the coordinates on T^4. The S^1 has length $L_y = 2\pi R_y$ and the T^4 has volume $(2\pi)^4 V$. The four noncompact spatial directions are called $\bar{x}_i, i = 1\ldots4$. We also write $u = t + y$, $v = t - y$.

The NS1 is wrapped n_1 times around the S^1, and carries n_p units of momentum along the S^1. This momentum is carried by transverse traveling waves; we assume that the polarization of the wave is in the four noncompact directions and is described by a function $\vec{F}(v)$. Then the string frame metric, B-field and dilaton are

$$\begin{aligned} ds^2_{string} &= H^{-1}[-dudv + K\,dv^2 + 2A_i\,dv\,d\bar{x}_i] + d\bar{x}_i d\bar{x}_i + dz_a dz_a \\ B &= \frac{H^{-1}-1}{2}\,du \wedge dv + H^{-1}\,A_i\,dv \wedge d\bar{x}_i \\ e^{2\Phi} &= H^{-1} \end{aligned} \tag{4.1}$$

with

$$\begin{aligned} H &= 1 + \frac{\bar{Q}_1}{L_T}\int_0^{L_T} \frac{dv}{\sum_i(\bar{x}_i - F_i(v))^2} \\ K &= \frac{\bar{Q}_1}{L_T}\int_0^{L_T} dv\,\frac{\sum_i(\dot{F}_i(v))^2}{\sum_i(\bar{x}_i - F_i(v))^2} \\ A_i &= -\frac{\bar{Q}_1}{L_T}\int_0^{L_T} dv\,\frac{\dot{F}_i(v)}{\sum_i(\bar{x}_i - F_i(v))^2} \end{aligned} \tag{4.2}$$

Here $L_T = 2\pi\,n_1\,R_y$ is the total length of the multiply wound string.

The points on the NS1 spread out over a region in the noncompact directions with size of order $\sim |\vec{F}(v)|$. On the other hand the gravitational field of the NS1-P system is characterized by the length scales $(\bar{Q}_1)^{1/2}$, $(\bar{Q}_p)^{1/2}$ where

$$\bar{Q}_p = \frac{\bar{Q}_1}{L_T}\int_0^{L_T} dv\,\sum_i(\dot{F}_i(v))^2 \tag{4.3}$$

In terms of microscopic quantities we have

$$\bar{Q}_1 = \frac{g^2\alpha'^3}{V} n_1 \,, \quad \bar{Q}_p = \frac{g^2\alpha'^4}{VR_y^2} n_p \tag{4.4}$$

Thus when we keep other parameters fixed and take g very small then the gravitational field of the supertube gets confined to a small neighborhood of the supertube and we get a 'thin tube' like that pictured in Fig.6.2(b). If we increase g large then we pass to a 'thick tube' like Fig.6.2(c) and then to the 'deep throat' geometry of Fig.6.2(d). We can thus say that Fig.6.2(a) is 'weak coupling' and Fig.6.2(d) is 'strong coupling' but note that for 'strong coupling' g itself does not need to be large since the charges n_1, n_p are large in (4.4). Thus to be more correct we should say that Fig.2(d) is obtained for large 'effective' coupling.

In this section we will consider the 'weak coupling' case so that we have a 'thin tube'. Then to study the nontrivial part of the metric we have to go close to a point on the tube, so the tube looks essentially like an infinite straight line. Let z be a coordinate along this line (not to be confused with z_a, which are coordinates on T^4) and r the radial coordinate for the three-space perpendicular to the ring. The NS1-P profile was described by a function $\vec{F}(v)$; let $v = v_0$ correspond to the point $z = 0$ along the ring and choose the orientation of the z line such that z increases when y increases. Then we have

$$z \approx -|\dot{\vec{F}}(v_0)|(v - v_0) \,, \quad \sum_i (\bar{x}_i - F_i(v))^2 \approx z^2 + r^2 \tag{4.5}$$

Since we are looking at distances r from the ring which are much smaller than the size of the ring we have

$$r \ll |\vec{F}(v_0)| \tag{4.6}$$

We can thus make the following approximations

$$
\begin{aligned}
H &\approx 1 + \frac{\bar{Q}_1}{L_T\,|\dot{\vec{F}}(v_0)|}\int_{-\infty}^{\infty}\frac{dz}{z^2+r^2} = 1 + \frac{\bar{Q}_1\,\pi}{L_T\,|\dot{\vec{F}}(v_0)|}\,\frac{1}{r} \\
K &\approx \frac{\bar{Q}_1\,|\dot{\vec{F}}(v_0)|\,\pi}{L_T}\,\frac{1}{r}\,, \quad A_z \approx \frac{\bar{Q}_1\,\pi}{L_T}\,\frac{1}{r}
\end{aligned}
\tag{4.7}
$$

Define the charge densities

$$
Q_1 \equiv \frac{\bar{Q}_1\,\pi}{L_T\,|\dot{\vec{F}}(v_0)|}\,, \quad Q_p \equiv \frac{\bar{Q}_1\,|\dot{\vec{F}}(v_0)|\,\pi}{L_T} \tag{4.8}
$$

Then we get the geometry (in the string frame)

$$
\begin{aligned}
ds^2_{string} &= H^{-1}\left[-2dt\,dv + \tilde{K}\,dv^2 + 2A\,dv\,dz\right] + dz^2 + dx_i dx_i + dz_a dz_a \\
B &= (H^{-1}-1)\,dt\wedge dv + H^{-1}\,A\,dv\wedge dz \\
e^{2\Phi} &= H^{-1}
\end{aligned}
\tag{4.9}
$$

$$
H = 1 + \frac{Q_1}{r}\,, \quad \tilde{K} = 1 + K = 1 + \frac{Q_p}{r}\,, \quad A = \frac{\sqrt{Q_1\,Q_p}}{r} \tag{4.10}
$$

Here we use x_i, $i = 1, 2, 3$, to denote the three spatial noncompact directions transverse to the tube.

We are looking for a perturbation of (4.9) corresponding to a deformation of the string profile. The profile could be deformed either in the non-compact x_i directions or in the T^4 directions. We consider deformations in one of the directions of the T^4; this maintains symmetry around the tube in the noncompact directions and is thus easier to work with. We thus consider deforming the string profile in one of the T^4 directions, denoted $\bar{a}$. We will also smear the perturbed metric on T^4, so that our fields will be independent on z_a.

The BPS geometry (4.9) carries a wave of a definite chirality: let us call it right moving. If the deformation we add also corresponds to a right moving wave, the resulting geometry can be generated by Garfinkle-Vachaspati transform [13, 86]. This will alter the metric and B-field as follows:

$$ds^2_{string} \to ds^2_{string} + 2\,\mathcal{A}^{(1)}\,dz_{\bar{a}}\,, \quad B \to B + \mathcal{A}^{(2)} \wedge dz_{\bar{a}} \tag{4.11}$$

where

$$\mathcal{A}^{(1)} = \mathcal{A}^{(2)} = H^{-1}\,a_v\,dv \tag{4.12}$$

and a_v is a harmonic function on $\mathbb{R}^3 \times S^1_z$, whose form will be given in section (6.4.2). If we also add a left moving deformation, thus breaking the BPS nature of the system, we do not have a way to generate the solution. Note, however, that the unperturbed system has a symmetry under

$$z_{\bar{a}} \to -z_{\bar{a}} \tag{4.13}$$

and the perturbation will be odd under such transformation. We thus expect that only the components of the metric and B-field which are odd under (4.13) will be modified at first order in the perturbation. We can thus still write the perturbation in the form (4.11), with $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ some gauge fields on $\mathbb{R}^{(3,1)} \times S^1_z \times S^1_y$, not necessarily given by (4.12).

To find the equations of motion for $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ we look at the theory dimensionally reduced on T^4, using the results of [111]. At first order in the perturbation the dimensionally reduced metric g_6 is simply given by the six-dimensional part of the unperturbed metric (4.9). The part of the action involving the gauge fields is

$$S_{\mathcal{A}} = \int \sqrt{-g_6}\, e^{-2\Phi} \left[-\frac{1}{4}\,(F^{(1)})^2 - \frac{1}{4}\,(F^{(2)})^2 - \frac{1}{12}\,\tilde{H}^2 \right] \tag{4.14}$$

where all the index contractions are done with g_6. $F^{(1)}$ and $F^{(2)}$ are the usual field strengths of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ while the field strength $\tilde{H}$ of the dimensionally reduced B-field $\tilde{B}$ includes the following Chern-Simons couplings:

$$\tilde{H}_{\mu\nu\lambda} = \partial_\mu \tilde{B}_{\nu\lambda} - \frac{1}{2}(\mathcal{A}^{(1)}_\mu F^{(2)}_{\nu\lambda} + \mathcal{A}^{(2)}_\mu F^{(1)}_{\nu\lambda}) + \text{cyc. perm.}$$
$$\tilde{B}_{\mu\nu} = B_{\mu\nu} + \frac{1}{2}(\mathcal{A}^{(1)}_\mu \mathcal{A}^{(2)}_\nu - \mathcal{A}^{(2)}_\mu \mathcal{A}^{(1)}_\nu) \tag{4.15}$$

Using $\tilde{B}$, $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ as independent fields, we find that the linearized equations of motion for the gauge fields are

$$\nabla^\mu(e^{-2\Phi} F^{(1)}_{\mu\lambda}) + \frac{1}{2}e^{-2\Phi} H^{\mu\nu}{}_\lambda F^{(2)}_{\mu\nu} = 0\,, \quad \nabla^\mu(e^{-2\Phi} F^{(2)}_{\mu\lambda}) + \frac{1}{2}e^{-2\Phi} H^{\mu\nu}{}_\lambda F^{(1)}_{\mu\nu} = 0 \tag{4.16}$$

These can be rewritten as decoupled equations as

$$\mathcal{A}^{\pm} = \mathcal{A}^{(1)} \pm \mathcal{A}^{(2)} \tag{4.17}$$

$$\nabla^\mu(e^{-2\Phi} F^{\pm}_{\mu\lambda}) \pm \frac{1}{2}e^{-2\Phi} H^{\mu\nu}{}_\lambda F^{\pm}_{\mu\nu} = 0 \tag{4.18}$$

Our task is to find the solutions of these equations representing non-BPS oscillations of the two charge system (4.9).

6.4.1 Solution in the 'infinite wavelength limit'

The geometry (4.1) has a singularity at the curve $\vec{x} = \vec{F}(v)$, which is the location of the strands of the oscillating NS1. Since we wish to add perturbations to this geometry, we must understand what boundary conditions to impose at this curve. The wavelength of the oscillations will be of order the length of the tube. Since the tube is 'thin' and we look close to the tube, locally the tube will look like a straight line even after the perturbing wave is added. The wave can 'tilt' the tube, and give it a velocity. So in this subsection we write the metric for a straight tube which has been rotated and boosted by infinitesimal parameters α, β. In the next subsection we will require that close to the axis of the tube (where the singularity lies) all fields match onto such a rotated and boosted straight tube solution.

We will consider oscillations of the supertube in one of the T^4 directions. Since we smear on the T^4 directions, the solution will remain independent of the torus coordinates z_a but we will get components in the metric and B field which reflect the 'tilt' of the supertube. We are using the NS1-P description. The unperturbed configuration looks, locally, like a NS1 that is a slanted line in the $y-z$ plane, where z is the coordinate along the tube. The perturbation tilts the tube towards a T^4 direction $z_{\bar{a}}$. We will find it convenient to start with the NS1 along y, first add the tilt and boost corresponding to the perturbation, and then add the non-infinitesimal tilt in the $y-z$ plane (and the corresponding boost).

We start from the one charge system

$$\begin{aligned} ds^2_{string} &= H^{-1}\,[-(d\tilde{t}'')^2 + (d\tilde{y}'')^2] + (d\tilde{z}'')^2 + dx_i dx_i + d\tilde{z}''_{\bar{a}} d\tilde{z}''_{\bar{a}} + \sum_{a\neq\bar{a}} dz_a dz_a \\ B &= -(H^{-1}-1)\, d\tilde{t}'' \wedge d\tilde{y}'' \\ e^{2\Phi} &= H^{-1} \end{aligned} \tag{4.19}$$

and perform the following operations: An infinitesimal boost in the direction $\tilde{z}''_{\bar{a}}$, with parameter β

$$\tilde{t}'' = \tilde{t}' - \tilde{z}'_{\bar{a}}\,\beta\,, \quad \tilde{z}''_{\bar{a}} = \tilde{z}'_{\bar{a}} - \tilde{t}'\,\beta\,, \quad \tilde{y}'' = \tilde{y}'\,, \quad \tilde{z}'' = \tilde{z}' \tag{4.20}$$

and an infinitesimal rotation in the $(\tilde{y}', \tilde{z}'_{\bar{a}})$ plane, with parameter α:

$$\tilde{y}' = \tilde{y} + \tilde{z}_{\bar{a}}\,\alpha\,, \quad \tilde{z}'_{\bar{a}} = \tilde{z}_{\bar{a}} - \tilde{y}\,\alpha\,, \quad \tilde{t}' = \tilde{t}\,, \quad \tilde{z}' = \tilde{z} \tag{4.21}$$

These operations give

$$
\begin{aligned}
ds^2_{string} &= H^{-1}\left[-d\tilde{t}^2 + d\tilde{y}^2 - 2\alpha\,(H-1)\,d\tilde{y}\,d\tilde{z}_{\bar{a}} - 2\beta\,(H-1)\,d\tilde{t}\,d\tilde{z}_{\bar{a}}\right] \\
&+ d\tilde{z}^2 + dx_i dx_i + d\tilde{z}_{\bar{a}} d\tilde{z}_{\bar{a}} + \sum_{a\neq\bar{a}} dz_a dz_a \\
B &= -(H^{-1}-1)\,d\tilde{t}\wedge d\tilde{y} + \beta\,H^{-1}\,(H-1)\,d\tilde{y}\wedge d\tilde{z}_{\bar{a}} + \alpha\,H^{-1}\,(H-1)\,d\tilde{t}\wedge d\tilde{z}_{\bar{a}} \\
e^{2\Phi} &= H^{-1} \qquad (4.22)
\end{aligned}
$$

We can read off from (4.22) the gauge fields $\mathcal{A}^{\pm}$:

$$
\mathcal{A}^{+} = (\alpha-\beta)\,H^{-1}\,(H-1)\,d\tilde{v}\,,\quad \mathcal{A}^{-} = -(\alpha+\beta)\,H^{-1}\,(H-1)\,d\tilde{u} \quad (4.23)
$$

($\tilde{u} = \tilde{t} + \tilde{y}$ and $\tilde{v} = \tilde{t} - \tilde{y}$). The part of the perturbation proportional to $\alpha - \beta$ represents a right moving wave, in which case only the $\mathcal{A}^{+}$ gauge field is excited. The reverse happens for the left moving perturbation, proportional to $\alpha + \beta$.

We would now like to add a finite amount of momentum Q_p to the system (4.22). This momentum is carried by a right moving wave moving with the speed of light in the positive y direction, with polarization in the direction z. The result will give us a geometry representing a small perturbation of the system (4.9). We can reach the desired configuration from (4.22) by performing a boost in the direction $\tilde{z}$ with parameter $\bar{\beta}$

$$
\tilde{t} = t'\,\cosh\bar{\beta} - z'\,\sinh\bar{\beta}\,,\quad \tilde{z} = z'\,\cosh\bar{\beta} - t'\,\sinh\bar{\beta}\,,\quad \tilde{y} = y'\,,\quad \tilde{z}_{\bar{a}} = z'_{\bar{a}} \quad (4.24)
$$

followed by a rotation in the (y', z') plane, with parameter $\bar{\alpha}$:

$$
y' = y\,\cos\bar{\alpha} + z\,\sin\bar{\alpha} \quad z' = z\,\cos\bar{\alpha} - y\,\sin\bar{\alpha}\,,\quad t' = t\,,\quad z'_{\bar{a}} = z_{\bar{a}} \quad (4.25)
$$

The parameters $\bar{\alpha}, \bar{\beta}$ are related. This is because the segment of string under consideration is supposed to be a short piece of the string in a state like that in

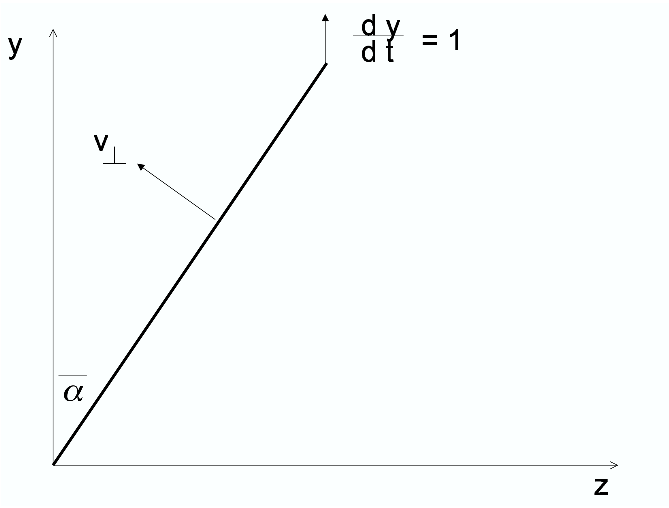

Figure 6.3: A short segment of the NS1 moving at the speed of light in the y direction. This yields a velocity v for the segment in the direction perpendicular to itself.

Fig.6.1(a), where the traveling wave is moving in the positive y direction with the speed of light. We depict this segment in Fig.6.3. We can ask how fast the string segment must be moving in a direction *perpendicular* to itself to yield $dy/dt = 1$, and we find

$$v_\perp \equiv -\tanh \bar{\beta} = \sin \bar{\alpha} \tag{4.26}$$

This implies

$$\sinh \bar{\beta} = -\tan \bar{\alpha}\,, \quad \cosh \bar{\beta} = \frac{1}{\cos \bar{\alpha}} \tag{4.27}$$

The final configuration is given by

$$\begin{aligned}
ds^2_{string} &= H^{-1}\Big[-2dt\,dv + [1+\sinh^2\bar\beta\,(H-1)]\,dv^2 - 2\sinh\bar\beta\,(H-1)\,dv\,dz \\
&+ 2(H-1)\Big(\frac{\alpha\,\cos^2\bar\alpha - \beta\,\sin^2\bar\alpha}{\cos\bar\alpha}\,dv\,dz_{\bar a} - (\alpha+\beta)\,\cos\bar\alpha\,dt\,dz_{\bar a} \\
&- (\alpha+\beta)\,\sin\bar\alpha\,dz\,dz_{\bar a}\Big)\Big] + dz^2 + dx_i dx_i + dz_{\bar a}dz_{\bar a} + \sum_{a\neq\bar a} dz_a dz_a \\
B &= (H^{-1}-1)\,dt\,dv - H^{-1}\,(H-1)\,\sinh\bar\beta\,dv\wedge dz \\
&+ H^{-1}\,(H-1)\Big(\frac{\alpha\,\sin^2\bar\alpha - \beta\,\cos^2\bar\alpha}{\cos\bar\alpha}\,dv\wedge dz_{\bar a} + (\alpha+\beta)\,\cos\bar\alpha\,dt\wedge dz_{\bar a} \\
&+ (\alpha+\beta)\,\sin\bar\alpha\,dz\wedge dz_{\bar a}\Big) \\
e^{2\Phi} &= H^{-1}
\end{aligned} \tag{4.28}$$

We note that, for $\alpha = \beta = 0$, we obtain the system (4.9) with[35]

$$Q_p = Q_1\,\sinh^2\bar\beta \tag{4.29}$$

The perturbation is proportional to α and β and is encoded in the gauge fields

$$\begin{aligned}
&\mathcal{A}^+_v = (\tilde\alpha - \tilde\beta)\,H^{-1}\,\frac{Q_1}{r}\,, \quad \mathcal{A}^+_t = 0\,, \quad \mathcal{A}^+_z = 0 \\
&\mathcal{A}^-_v = (\tilde\alpha + \tilde\beta)\,H^{-1}\,\frac{Q_1 - Q_p}{r}\,, \quad \mathcal{A}^-_t = -2(\tilde\alpha + \tilde\beta)\,H^{-1}\,\frac{Q_1}{r} \\
&\mathcal{A}^-_z = -2(\tilde\alpha + \tilde\beta)\,H^{-1}\,\frac{\sqrt{Q_1 Q_p}}{r}
\end{aligned} \tag{4.30}$$

where we have redefined

$$\tilde\alpha - \tilde\beta = \frac{\alpha - \beta}{\cos\bar\alpha}\,, \quad \tilde\alpha + \tilde\beta = (\alpha+\beta)\,\cos\bar\alpha \tag{4.31}$$

and we have used (4.29) and (4.27). We see that, as before, $\mathcal{A}^+$ comes from right moving perturbations, proportional to $\tilde\alpha - \tilde\beta$, and $\mathcal{A}^-$ comes from left moving perturbations, proportional to $\tilde\alpha + \tilde\beta$.

[35]With our conventions $\bar\alpha > 0$ and $\bar\beta < 0$. Thus $\sqrt{Q_p} = -\sqrt{Q_1}\,\sinh\bar\beta$.

6.4.2 Solution for $\mathcal{A}^+$

Let us look for a solution of (4.18), in the $\mathcal{A}^+$ sector, which matches the configuration (4.30) when $k \to 0$. We learned from (4.30) that $\mathcal{A}^+$ receives contributions only from the BPS (right moving) part of the wave and that, at least in the long wavelength limit, only the component $\mathcal{A}^+_v$ is non-vanishing. One can thus look for a solution of the form

$$\mathcal{A}^+_v = H^{-1}\, a^+_v\,, \quad \mathcal{A}^+_t = 0\,, \quad \mathcal{A}^+_z = 0\,, \quad \mathcal{A}^+_i = 0 \tag{4.32}$$

Equation (4.18) implies the following conditions for a^+_v (here $\triangle = \partial_i \partial_i$ is the ordinary Laplacian in the 3-dimensional space of the x_i)

$$\lambda = t: \quad \partial_t^2 a^+_v = 0 \tag{4.33}$$

$$\lambda = v: \quad \triangle\, a^+_v + \partial_z^2 a^+_v - 2A\, \partial_t \partial_z a^+_v = 0 \tag{4.34}$$

$$\lambda = z: \quad \partial_t \partial_z a^+_v = 0 \tag{4.35}$$

$$\lambda = i: \quad \partial_t \partial_i a^+_v = 0 \tag{4.36}$$

It is thus clear that a t-independent a^+_v satisfying

$$\triangle\, a^+_v + \partial_z^2 a^+_v = 0 \tag{4.37}$$

solves the linearized equations of motion. The general solution of (4.37), with momentum

$$k = \frac{n}{R_z} \tag{4.38}$$

along z, is

$$a^+_v = e^{ikz}\, \frac{c_+\, e^{k\,r} + c_-\, e^{-k\,r}}{r} + \text{c.c.} \tag{4.39}$$

Without loss of generality let us set n to be positive. To have a converging field at large r one should take $c_+ = 0$. Matching with (4.30) fixes c_-:

$$c_- = (\tilde{\alpha} - \tilde{\beta})\, Q_1 \tag{4.40}$$

so that

$$\mathcal{A}^+_v = (\tilde{\alpha} - \tilde{\beta})\, H^{-1}\, \frac{Q_1}{r}\, e^{ik\, z - k\, r} + \text{c.c.} \tag{4.41}$$

The above result is consistent with the form of $\mathcal{A}^+$ derived by Garfinkle-Vachaspati transform: Consider a string carrying a right moving wave described by the profile $F_i(v)$ in the non-compact directions $\bar{x}_i$ and $f_{\bar{a}}(v)$ in the T^4 direction $z_{\bar{a}}$. After smearing over z_a, Garfinkle-Vachaspati transform predicts a gauge field

$$\mathcal{A}^+_v = \mathcal{A}^{(1)}_v + \mathcal{A}^{(2)}_v$$

$$\mathcal{A}^{(1)}_v = \mathcal{A}^{(2)}_v = -H^{-1}\, \frac{\bar{Q}_1}{L_T} \int_0^{L_T} dv\, \frac{\dot{f}_{\bar{a}}(v)}{\sum_i (\bar{x}_i - F_i(v))^2} \tag{4.42}$$

Eq. (4.42) is analogous to the relation (4.2) for A_i, applied to the case in which the profile extends in the T^4 directions. Let us take the near ring limit of (4.42) for a profile $f_{\bar{a}}$ of the form

$$f_{\bar{a}}(v) = \xi_{\bar{a}}\, e^{-i\bar{k}\, v} + c.c. \tag{4.43}$$

Around some point v_0 on the ring we write

$$z = -|\dot{\vec{F}}(v_0)|\, (v - v_0)\,, \quad z_0 = -|\dot{\vec{F}}(v_0)|\, v_0 \tag{4.44}$$

so that we can write

$$f_{\bar{a}}(v) = \xi_{\bar{a}}\, e^{i\bar{k}\, (z+z_0)/|\dot{\vec{F}}(v_0)|} + c.c. \equiv \xi_{\bar{a}}\, e^{ik\, (z+z_0)} + c.c. \tag{4.45}$$

and

$$\dot{f}_{\bar{a}}(v) = -i\xi_{\bar{a}}\,|\dot{\vec{F}}(v_0)|\,k\,e^{ik\,(z+z_0)} + c.c. \tag{4.46}$$

In the near ring limit one can approximate

$$\begin{aligned} \mathcal{A}^+_v &\approx 2H^{-1}\,\frac{i\bar{Q}_1\,\xi_{\bar{a}}\,k}{L_T}\,e^{ik\,z_0}\int_{-\infty}^{+\infty} dz\,\frac{e^{ik\,z}}{r^2+z^2} + \text{c.c.} \\ &= 2H^{-1}\,\frac{i\bar{Q}_1\,\pi\,\xi_{\bar{a}}\,k}{L_T}\,\frac{e^{ik\,z_0-k\,r}}{r} + \text{c.c.} \end{aligned} \tag{4.47}$$

Using (4.8) to relate $\bar{Q}_1$ and Q_1 we see that (4.47) coincides with (4.41), with

$$(\tilde{\alpha}-\tilde{\beta}) = 2i\,\xi_{\bar{a}}\,|\dot{\vec{F}}(v_0)|\,k \tag{4.48}$$

The time-independent solution (4.41) represents the response of the system to a BPS right moving wave. Since the $\mathcal{A}^+$ part of the gauge field should only be sensitive to BPS deformations, we expect that equation (4.18) for $\mathcal{A}^+$ should not admit time-dependent solutions consistent with the boundary condition (4.30). In an appendix we prove this fact for the more general $\mathcal{A}^+$ ansatz.

6.4.3 Solution for $\mathcal{A}^-$

We now look at the $\mathcal{A}^-$ sector, where we expect to find the time-dependent configurations corresponding to left moving non-BPS perturbations.

Consider an ansatz of the form

$$\mathcal{A}^-_v = H^{-1}\,a^-_v\,,\quad \mathcal{A}^-_t = H^{-1}\,a^-_t\,,\quad \mathcal{A}^-_z = H^{-1}\,a^-_z\,,\quad \mathcal{A}^-_i = 0 \tag{4.49}$$

By spherical symmetry $\mathcal{A}^-_i$ only has a radial component $\mathcal{A}^-_r$ and we chose our gauge to set $\mathcal{A}^-_r = 0$. (Such a gauge can have difficulties at $r = 0$ but we can consider it as an ansatz and see later that we obtain a good solution.) The equations for a^-_v,

a_t^- and a_z^-, obtained by using the ansatz (4.49) in (4.18) and using the background (4.9), are (we list the equations in the order $\lambda = t, v, z, i$)

$$\triangle\, a_t^- + H\,\partial_t^2 a_v^- + \partial_z(\partial_z a_t^- - \partial_t a_z^-) + A\,\partial_t(\partial_z a_t^- - \partial_t a_z^-) = 0 \qquad (4.50)$$

$$\begin{aligned}&\triangle\, a_v^- + \partial_z^2 a_v^- - [(H\tilde{K} - A^2)\,\partial_t^2 a_v^- - 2A\,\partial_t\partial_z a_v^-]\\ &+H^{-2}\,\partial_i H\partial_i H\,(2a_v^- + \tilde{K}\,a_t^-) - H^{-1}\,\partial_i H\,\partial_i(2a_v^- + \tilde{K}\,a_t^-) + \partial_i a_t^-\,\partial_i\tilde{K} = 0\end{aligned} \qquad (4.51)$$

$$\begin{aligned}&\triangle\, a_z^- + H\,\partial_t\partial_z a_v^- + (H\tilde{K} - A^2)\,\partial_t(\partial_z a_t^- - \partial_t a_z^-) - A\,\partial_z(\partial_z a_t^- - \partial_t a_z^-)\\ &+2H^{-2}\,\partial_i H\partial_i H(a_z^- + A\,a_t^-) - 2H^{-1}\,\partial_i H\,\partial_i\,(a_z^- + A\,a_t^-) + 2\partial_i a_t^-\,\partial_i A = 0\end{aligned} \qquad (4.52)$$

$$\begin{aligned}&H\,\partial_t\partial_i a_v^- - \partial_z\partial_i a_z^- + [(H\tilde{K} - A^2)\,\partial_t\partial_i a_t^- - A\,\partial_z\partial_i a_t^- - A\,\partial_t\partial_i a_z^-]\\ &+H^{-1}\,\partial_i H\,[\partial_z a_z^- + A\,(\partial_t a_z^- + \partial_z a_t^-) - (H\tilde{K} - A^2)\,\partial_t a_t^-]\\ &-\partial_i A\,(\partial_z a_t^- - \partial_t a_z^-) - 2\partial_i H\,\partial_t a_v^- = 0\end{aligned} \qquad (4.53)$$

Inspired by the limiting solution (4.30), we make the following ansatz for a_v^-, a_t^- and a_z^-:

$$\begin{aligned}a_v^- &= (Q_1 - Q_p)\,e^{ik\,z - i\omega\,t}\,f(r)\\ a_t^- &= -2Q_1\,e^{ik\,z - i\omega\,t}\,f(r)\,, \quad a_z^- = -2\sqrt{Q_1 Q_p}\,e^{ik\,z - i\omega\,t}\,f(r)\end{aligned} \qquad (4.54)$$

Substituting this ansatz in eq. (4.50) we find an equation for $f(r)$:

$$-2Q_1\,\triangle\,f - \omega^2\,(Q_1 - Q_p)\,f + 2k\,(kQ_1 + \omega\,\sqrt{Q_1 Q_p})\,f - \frac{\omega\,Q_1\,f}{r}[2\sqrt{Q_1 Q_p}\,k + (Q_1 + Q_p)\,\omega] = 0 \qquad (4.55)$$

This equation can be simplified by taking

$$f = \frac{\tilde{f}}{r} \tag{4.56}$$

after which we get

$$-2Q_1\,\tilde{f}'' - \omega^2\,(Q_1 - Q_p)\,\tilde{f} + 2k\,(kQ_1 + \omega\,\sqrt{Q_1 Q_p})\,\tilde{f} - \frac{\omega\,Q_1\,\tilde{f}}{r}[2\sqrt{Q_1 Q_p}\,k + (Q_1 + Q_p)\,\omega] = 0 \tag{4.57}$$

According to the boundary condition (4.30), we want $\tilde{f}$ to go to a constant when $r \to 0$; this is only possible if the $1/r$ term in (4.57) vanishes and this determines the frequency of oscillation to be

$$\omega = -k\,\frac{2\sqrt{Q_1 Q_p}}{Q_1 + Q_p} \tag{4.58}$$

Using this value of ω back in (4.57) we find that $\tilde{f}$ satisfies

$$\tilde{f}'' - \tilde{k}^2 \tilde{f} = 0 \tag{4.59}$$

with

$$\tilde{k}^2 = k^2 - \omega^2 = k^2 \left(\frac{Q_1 - Q_p}{Q_1 + Q_p}\right)^2 \tag{4.60}$$

and thus

$$\tilde{f} = c_+\,e^{+|\tilde{k}|\,r} + c_-\,e^{-|\tilde{k}|\,r} \tag{4.61}$$

In order to have a converging solution for large r one needs $c_+ = 0$ and to match with (4.30) one needs $c_- = \tilde{\alpha} + \tilde{\beta}$. To summarize we find

$$\mathcal{A}_v^- = (\tilde{\alpha} + \tilde{\beta})\,H^{-1}\,(Q_1 - Q_p)\,e^{ik\,z - i\omega\,t}\,\frac{e^{-|\tilde{k}|\,r}}{r} \tag{4.62}$$

$$\mathcal{A}_t^- = -2(\tilde{\alpha} + \tilde{\beta})\,H^{-1}\,Q_1\,e^{ik\,z - i\omega\,t}\,\frac{e^{-|\tilde{k}|\,r}}{r}\,, \quad \mathcal{A}_z^- = -2(\tilde{\alpha} + \tilde{\beta})\,H^{-1}\sqrt{Q_1 Q_p}\,e^{ik\,z - i\omega\,t}\,\frac{e^{-|\tilde{k}|\,r}}{r}$$

It is a lengthy but straightforward exercise to verify that (4.62) solves the remaining equations (4.51-4.53).

6.4.4 Period of the oscillations

The speed of the left-moving wave on the supertube is

$$v = \frac{\omega}{|k|} = 2\frac{\sqrt{Q_1 Q_p}}{Q_1 + Q_p} \tag{4.63}$$

The direction z used above is the coordinate along the supertube. So even though z looked like an infinite direction in the 'near tube' limit, this direction is actually a closed curve with a length L_z. The time for the wave to travel around this closed curve is

$$\Delta t = \int_0^{L_z} \frac{dz}{v} = \int_0^{L_z} dz \frac{Q_1 + Q_p}{2\sqrt{Q_1 Q_p}} = \frac{1}{2}\int_0^{L_z} dz[\sqrt{\frac{Q_1}{Q_p}} + \sqrt{\frac{Q_p}{Q_1}}] \tag{4.64}$$

We have

$$Q_1 = \frac{\bar{Q}_1\,\pi}{L_T}\,\frac{1}{\eta}, \quad Q_p = \frac{\bar{Q}_1\,\pi}{L_T}\,\eta \tag{4.65}$$

with

$$\eta^{-1} = \frac{1}{|\dot{\vec{F}}|} = \frac{dy}{dz} \tag{4.66}$$

This gives

$$\begin{aligned} \Delta t &= \frac{1}{2}\int_0^{L_z} dz(\eta^{-1} + \eta) \\ &= \frac{1}{2}\int_0^{L_z} dz[\frac{dy}{dz} + \frac{dy}{dz}(\frac{dz}{dy})^2] \\ &= \frac{1}{2T}(Tn_1 L_y) + \frac{1}{2T}(T\int |\dot{F}|^2 dy) \\ &= \frac{1}{2T}(M_{NS1} + M_P) \end{aligned} \tag{4.67}$$

where M_{NS1} is the mass contributed by the NS1 charge and M_P is the mass of the momentum charge. We see that this period Δt agrees with the period (2.25) found from the NS1-P system at $g = 0$.

We offer an intuitive explanation for the time period (2.25). We have

$$\frac{Q_1}{Q_p} = \eta^{-2} = (\frac{dy}{dz})^2 \tag{4.68}$$

Thus we can write (4.63) as

$$v = 2\frac{\frac{dy}{dz}}{1 + (\frac{dy}{dz})^2} \tag{4.69}$$

Consider a segment of the NS1 before the perturbation is added. In section (6.4.1) we had seen, (with the help of Fig.6.3) that because this segment represents a wave traveling in the y direction with $dy/dt = 1$, the velocity of this segment *perpendicular* to itself was

$$v_\perp = \sin\bar{\alpha} = \frac{1}{\sqrt{1 + (\frac{dy}{dz})^2}} \tag{4.70}$$

So we have a segment of a NS1, moving at a certain velocity transverse to itself. Go to the rest frame of this segment. Then any small perturbation on the segment will move to the right or to the left with speed unity. Consider the perturbation going left.

Now return to the original reference frame, and look at this perturbation on the segment. The distances along the segment are not affected by the change of frame (since the boost is perpendicular to the segment) but there is a time dilation by a factor $\gamma = 1/\sqrt{1 - v_\perp^2}$. This means that the perturbation will be seen to be moving along the strand at a speed

$$v_L = \gamma^{-1} = \frac{\frac{dy}{dz}}{\sqrt{1 + (\frac{dy}{dz})^2}} \tag{4.71}$$

We are interested in the motion of the perturbation in the z direction, so we look at the z component of this velocity

$$v_{L,z} = v_L \sin\bar{\alpha} = \frac{\frac{dy}{dz}}{1 + (\frac{dy}{dz})^2} \tag{4.72}$$

What we actually observe as the wave on the supertube is a deformation moving along the tube, so we wish to measure the progress of the waveform as a function of the coordinate z. A given point on our NS1 segment moves in the direction of the velocity v, so it moves towards smaller z values at a speed

$$v_z = v_\perp \cos\bar{\alpha} = \frac{\frac{dy}{dz}}{1 + (\frac{dy}{dz})^2} \tag{4.73}$$

Thus if we measure the speed of the left moving perturbation with respect to a a frame where z is fixed then we find the velocity

$$v_L^{pert} = v_{L,z} + v_z = 2\frac{\frac{dy}{dz}}{1 + (\frac{dy}{dz})^2} \tag{4.74}$$

which agrees with (4.69).

Similarly if we look at the right moving perturbation then we find

$$v_R^{pert} = -v_{L,z} + v_z = 0 \tag{4.75}$$

This agrees with the fact that if we add a further right moving wave to the NS1 then we just get another BPS tube configuration, which is stationary and so does not change with time.

6.5 Coupling to radiation modes

The perturbations of the 'thin' tube in the 'infinite line limit' is seen to fall off exponentially with the distance from the tube axis. Note however that if we take the longest wavelengths on the supertube, then the term $e^{-|\tilde{k}|r}$ is not really significant. For such modes $|\tilde{k}| \sim 1/a$ where a is the radius of the tube. So $e^{-|\tilde{k}|r} \sim 1$ for $r \ll a$, and for $r \gtrsim a$ we cannot use the infinite line limit of the thin tube anyway. If however we look at higher wavenumbers on the tube then $|\tilde{k}| \sim n/a$ and then the factor $e^{-|\tilde{k}|r}$

is indeed significant in describing the fall off of the perturbation away from the tube axis.

We now wish to look at the behavior of the perturbation far from the entire supertube, i.e. for distances $\bar{r} \gg a$. Here we use the symbol $\bar{r}$ for the radial coordinate in the 4 dimensional noncompact space, to distinguish it from the radial distance r from the tube axis that we used in the last section when looking at the 'infinite line limit'. For $\bar{r} \gg a$ we get flat space. Suppose we were studying a scalar field $\Box\Psi = 0$ in the supertube geometry. We can write

$$\Psi = e^{-i\omega t}\,\mathcal{R}(\bar{r})\,Y^{(l)}(\theta, \phi, \psi) \tag{5.76}$$

If $\omega^2 < 0$ then we get solutions $\sim e^{\pm|\omega|t}$; these are not allowed because they will not conserve energy. For $\omega^2 > 0$, we get the behavior (see Appendix (F.1))

$$\mathcal{R} = \frac{r_+\, e^{i\omega\,\bar{r}} + r_-\, e^{-i\omega\,\bar{r}}}{\bar{r}^{3/2}}(1 + O(\bar{r}^{-1})) \tag{5.77}$$

This solution describes traveling waves that carry flux to and from spatial infinity. Thus if we start with an excitation localized near the supertube then the part of its wavefunction that extends to large $\bar{r}$ will lead to the energy of excitation flowing off to infinity as radiation.

Let us see how significant this effect is for the 'thin tube'. Let us set Q_1, Q_p to be of the same order. From (4.62) we see that the magnitude of the perturbation behaves as

$$\mathcal{A} \sim H^{-1}\frac{Q_1}{r} \sim \frac{Q_1}{r + Q_1} \tag{5.78}$$

Thus if the perturbation is order unity at the ring axis then at distances $r \gtrsim a$ we will have

$$\mathcal{A} \lesssim \frac{Q_1}{a} \tag{5.79}$$

But the thin tube limit is precisely the one where the ratio Q_1/a is small, so the part of the wavefunction reaching large r is small. Thus the rate of leakage of energy to the radiation field is small, and the excitations on the 'thin tube' will be long lived. This is of course consistent with the fact that in the limit $g \to 0$ we can describe the system by just the free string action or the D2 brane DBI action, and here there is *no* leakage of energy off the supertube to infinity.

As we keep increasing g we go from the 'thin' tube of Fig.6.2(b) to the 'thick tube' of Fig.6.2(c). Now $Q_1/a \sim 1$ and the strength of the perturbation reaching the radiation zone is *not* small. We thus expect that the energy of excitation will flow off to infinity in a time of order the oscillation time of the mode. Thus we expect that the oscillations of the supertube become 'broad resonances' and cease to be well defined oscillations as we go from Fig.6.2(a) to Fig.6.2(c).

In the above discussion we referred to the excitation as a scalar field, but this is just a toy model; what we have is a 1-form field in 5+1 spacetime. In Appendix (F.1) we solve the field equations for this 1-form field at infinity, and find again a fall off at infinity that gives a non-zero flux of energy. We also find the next correction in $1/\bar{r}$, and show how a series expansion in $1/\bar{r}$ may be obtained in general. These corrections do not change the fact that the leading order term carries flux out to infinity. It is important that the first correction to flat space is a potential $\sim 1/\bar{r}^2$ and not $\sim 1/\bar{r}$; this avoids the appearance of a logarithmic correction at infinity.

It is to be noted that such series solutions in $1/\bar{r}$ are asymptotic expansions rather than series with a nonzero radius of convergence [88], so these arguments are not a rigorous proof for the absence of infinitely long lived oscillations. The wave equations for a given ω are similar in structure to the Schrodinger equation (in 4+1 dimensions) with a potential V falling to zero at infinity

$$-\mathcal{R}'' + V(\bar{r}, \omega)\mathcal{R} = \omega^2 \mathcal{R} \tag{5.80}$$

Note that because $\omega^2 > 0$ our wavefunction would be like a positive energy eigenstate of the Schrodinger equation; i.e. we need an energy eigenvalue embedded in the continuum spectrum. For the Schrodinger equation there are several results that exclude such eigenvalues on general grounds [89]. The required results come from two kinds of theorems. First we need to know that there is no 'potential well with infinitely high walls' near the origin; if there was such a well then we can have a positive energy eigenstate which has no 'tail' outside the well. Next, given that there *is* a tail outside the well we need to know that the potential falls off to zero fast enough and does not 'oscillate' too much; such oscillations of the potential can cause the wavefunction to be back-scattered towards the origin repeatedly and die off too fast to carry a nonzero flux at infinity. We cannot directly apply these results to our problem because our equations are not exactly the Schrodinger equation, but the potential like terms in our equations do not appear to be of the kind that will prevent flux leakage to infinity.

To summarize, we conjecture that as we increase g to go from Fig.6.2 (a) to Fig.6.2(c) the periodic oscillations present at $g \to 0$ merge into the continuum spectrum of bulk supergravity. Thus for $g > 0$ the energy of excitation placed on the supertube eventually leaks off to infinity, with the rate of leakage increasing as we go from the 'thin tube' to the 'thick tube'.

6.6 Long lived excitations at large coupling

Let us increase the coupling still further. Then the supertube geometry becomes like that pictured in Fig.6.2(d) [186, 188, 150]. The metric is flat space at infinity, then we have a 'neck' region, this leads to a deep 'throat' which ends in a 'cap' near $r = 0$. Supergravity quanta can be trapped in the 'throat' bouncing between the cap and the neck for long times before escaping to infinity. We first consider the gravity description, then a microscopic computation, and finally suggest a relation between the two.

6.6.1 The geometry at large effective coupling

Consider an NS1 wrapped n_1 times on the S^1 with radius R_y, and give it the transverse vibration profile

$$X_1 = a\cos\frac{(t-y)}{n_1 R_y}, \quad X_2 = a\sin\frac{(t-y)}{n_1 R_y} \tag{6.1}$$

Thus the string describes a 'uniform helix with one turn' in the covering space of the S^1. At weak coupling $g \to 0$ we get a ring with radius a in flat space, while at strong coupling we get a geometry like Fig.6.2(d) with the circle (4.77) sinking deep into the throat (the dotted line in the figure).

In [150] the computations were done in the D1-D5 duality frame, so let us start with that frame and dualize back to NS1-P later. We will denote quantities in the D1-D5 frame by primes. The time for a supergravity quantum to make one trip down the throat and back up is

$$\Delta t_{osc} = \pi R'_y \tag{6.2}$$

where R'_y is the radius of S^1 in the D1-D5 frame. When the quantum reaches the neck there is a probability P that it would escape to infinity, and a probability $1-P$ that it would reflect back down the throat for another cycle. For low energy quanta in the l-th spherical harmonic this probability P is given by [92, 189]

$$P_l = 4\pi^2 (\frac{\bar{Q}'_1 \bar{Q}'_5 \omega'^4}{16})^{l+1} [\frac{1}{(l+1)!l!}]^2 \tag{6.3}$$

where ω' is the energy of the quantum. We see that the escape probability is highest for the s-wave, so we set $l = 0$. Then the expected time after which the trapped quantum will escape is

$$\Delta t_{escape} = P_0^{-1} \Delta t_{osc} \tag{6.4}$$

The low energy quanta in the throat have $\omega' \sim 2\pi/\Delta t_{osc}$ [150, 133] so for our estimate we set

$$\omega' = \frac{2}{R'_y} \tag{6.5}$$

We then find

$$\alpha \equiv \frac{\Delta t_{escape}}{\Delta t_{osc}} = \frac{1}{(2\pi)^2} \frac{R'^4_y}{\bar{Q}'_1 \bar{Q}'_5} = \frac{1}{(2\pi)^2} [\frac{(\bar{Q}'_1 \bar{Q}'_5)^{\frac{1}{4}}}{a'}]^4 \tag{6.6}$$

where $a' = (\bar{Q}'_1 \bar{Q}'_5)^{1/2}/R'_y$ is the radius obtained from a after the dualities to the D1-D5 frame [150]. In this frame the cap+throat region has the geometry of global $AdS_3 \times S^3 \times T^4$. The curvature radius of the AdS_3 and S^3 is $(\bar{Q}'_1 \bar{Q}'_5)^{1/4}$. The ratio

$$\beta \equiv \frac{(\bar{Q}'_1 \bar{Q}'_5)^{\frac{1}{4}}}{a'} \tag{6.7}$$

gives the number of AdS radii that we can go outwards from $r = 0$ before reaching the 'neck' region.[36] Thus β is a measure of the depth of the throat compared to its diameter.

[36]This can be seen from the metric for the profile (4.77) [186, 188, 80].

While all lengths in the noncompact directions are scaled under the dualities, the ratio of such lengths is unchanged. Thus in the NS1-P duality frame

$$\beta = \frac{(\bar{Q}_1 \bar{Q}_p)^{\frac{1}{4}}}{a} \tag{6.8}$$

We note that

$$\alpha \sim \beta^4 \tag{6.9}$$

Thus when the throat becomes deep the quanta trapped in the throat become long lived excitations of the system.

For completeness let us also start from the other limit, where the coupling is weak and we have a thin long tube as in Fig.6.2(b). The radius of the ring described by (4.77) is a. The gravitational effect of NS1,P charges extends to distances Q_1, Q_p from the ring. In the definitions (4.8) we put in the profile (4.77), and find

$$Q_1 = \frac{\bar{Q}_1}{2a}, \qquad Q_p = \frac{\bar{Q}_p}{2a} \tag{6.10}$$

If we take for the 'thickness' of the ring the length scale $\sqrt{Q_1 Q_p}$ then from (6.10) we find

$$\frac{(Q_1 Q_p)^{1/2}}{a} = \frac{(\bar{Q}_1 \bar{Q}_p)^{1/2}}{2a^2} = \frac{\beta^2}{2} \tag{6.11}$$

so we see again that the ring 'thickness' becomes comparable to the ring radius when $\beta \sim 1$. For $\beta \ll 1$ we have a 'thin ring' and for $\beta \gg 1$ we have a 'deep throat'.

Instead of using $\sqrt{Q_1 Q_p}$ as a measure of the ring thickness we can say that the ring is thin when

$$a \gtrsim Q_1, \quad a \gtrsim Q_p \tag{6.12}$$

This can be encoded in the requirement

$$a \gtrsim \frac{Q_1 Q_p}{Q_1 + Q_p} \tag{6.13}$$

From the first equality in (4.64) we get an expression for Δt, which we equate to the expression found in (2.25), to get

$$\frac{1}{2}L_z\frac{Q_1+Q_p}{\sqrt{Q_1Q_p}}=\Delta t=\pi\alpha' M_T \tag{6.14}$$

where $L_z = 2\pi a$ is the length of the ring and M_T is its total mass. Using this and (6.10) we can rewrite (6.13) as

$$\alpha' M_T \gtrsim (Q_1Q_p)^{1/2}=\frac{(\bar{Q}_1\bar{Q}_p)^{1/2}}{2a} \tag{6.15}$$

Expressing the macroscopic parameters in terms of the microscopic charges and moduli[37]

$$\bar{Q}_1=\frac{g^2\alpha'^3n_1}{V},\quad \bar{Q}_p=\frac{g^2\alpha'^4n_p}{VR_y^2},\quad a=\sqrt{n_1n_p\alpha'} \tag{6.16}$$

we find that the ring is 'thin' when

$$\alpha' M_T \gtrsim \frac{g^2\alpha'^3}{VR_y} \tag{6.17}$$

This version of the criterion for ring thickness will be of use below.

6.6.2 The phase transition in the microscopic picture

We now turn to the microscopic description of the system. Consider first the BPS bound state in the D1-D5 duality frame. Suppose we add a little bit of energy to take the system slightly above extremality. From the work on near-extremal states [174, 92, 150, 77] we know that the energy will go to exciting vibrations that run up and down the components of the effective string

$$D1-D5\ +\ \Delta E\ \ \rightarrow\ \ D1-D5\ +\ P\bar{P} \tag{6.18}$$

[37]The expression for a is obtained by using the profile (4.77) in (4.3) and use the expressions (4.4).

where we call the excitations $P\bar{P}$ since they carry momentum charge in the positive and negative S^1 directions. For the geometry made by starting with the NS1-P profile (4.77) we have no 'fractionation'; i.e. the effective string formed in the D1-D5 bound state has $n_1 n_5$ 'singly wound' circles [150]. Thus the minimum energy needed to excite the system is the energy of one left and one right mover on the effective string

$$\Delta E^{D1D5}_{P\bar{P}} = \frac{1}{R'_y} + \frac{1}{R'_y} = \frac{2}{R'_y} \tag{6.19}$$

The charges D1-D5-P can be permuted into each other, so we can map D1-D5 to P-D1, and then the dual of (6.18) is

$$P - D1 \; + \; \Delta E \quad \rightarrow \quad P - D1 \; + \; D5\overline{D5} \tag{6.20}$$

A further S duality brings the system to the NS1-P system that we are studying, and then we get

$$P - NS1 \; + \; \Delta E \quad \rightarrow \quad P - NS1 \; + \; NS5\overline{NS5} \tag{6.21}$$

This may look strange, since it says that if we excite an oscillating string the energy of excitations goes to creating pairs of NS5 branes; we are more used to the fact that energy added to a string just creates more oscillations of the string. Dualizing (6.19) gives for the excitation (6.21) the minimum energy threshold

$$\Delta E^{NS1P}_{NS5\overline{NS5}} = 2m_{NS5} = 2\frac{VR_y}{g^2\alpha'^3} \tag{6.22}$$

Thus at small g these excitations are indeed heavy and should not occur. For comparison, we find the minimum energy required to excite *oscillations* on the NS1-P system. For small g we use the spectrum of the free string which gives

$$M^2 = (\frac{R_y n_1}{\alpha'} + \frac{n_p}{R_y})^2 + \frac{4}{\alpha'}N_L = (\frac{R_y n_1}{\alpha'} - \frac{n_p}{R_y})^2 + \frac{4}{\alpha'}N_R \tag{6.23}$$

The lowest excitation is given by $\delta N_L = \delta N_R = 1$. This gives

$$\Delta E^{NS1P}_{oscillations} = \Delta M = \frac{2}{\alpha'}\frac{1}{M_T} \tag{6.24}$$

where M_T is the total mass of the BPS NS1-P state.

We now observe that oscillations on the NS1-P system are lighter than NS5 excitations only when

$$\frac{1}{\alpha' M_T} \lesssim \frac{V R_y}{g^2 \alpha'^3} \tag{6.25}$$

Thus for very small g the lightest excitation on the NS1-P system is an oscillation of the string. But above a certain g the $NS5\overline{NS5}$ pairs are lighter and so will be the preferred excitation when we add energy to the system.

6.6.3 Comparing the gravity and microscopic pictures

We now observe that the conditions (6.17) and (6.25) are the *same*. Thus we see that when the ring is thin then in the corresponding microscopic picture we have '2-charge excitations'; i.e. the third charge NS5 is not excited and the string giving the NS1-P state just gets additional excitations which may be interpreted as pairs of NS1 and P charges. But when we increase the coupling beyond the point where the ring becomes 'thick' and the geometry is better described as a throat, then the dual CFT has '3-charge excitations' which are pairs of NS5 branes. When g is small and the ring is thin then the oscillations of the supertube are long lived because they couple only weakly to the radiation modes of the gravity field. When the tube becomes very *thick* then the oscillation modes are again long lived – we get $\beta \gg 1$ and by (6.9) this implies a very slow leakage of energy to infinity.

Thus we see that the modes at small and large coupling should not be seen as the 'same' modes; rather the '2-charge modes' at weak coupling disappear at larger

g because of coupling to the radiation field, and at still larger g the '3-charge modes' appear. For these latter modes one might say that the gravitational field of the system has 'trapped' the excitations of the metric from the region $\bar{r} \lesssim (\bar{Q}_1)^{1/2}, (\bar{Q}_p)^{1/2}$, so that these modes have in some sense been extracted from the radiation field.

6.7 A conjecture on identifying bound states for the 3-charge extremal system

Consider a D0 brane placed near a D4 brane. The force between the branes vanishes. But now give the D0 a small velocity in the space transverse to the D4. The force between the branes goes as $\sim v^2$, and the motion of the D0 can be described as a geodesic on the moduli space of its static configurations [69]. This moduli space would be flat if we took a D0-D0 system (which is 1/2 BPS) but for the D0-D4 case (which is 1/4 BPS) the metric is a nonflat hyperkahler metric.

We can look at more complicated systems, for example 3-charge black holes in 4+1 spacetime. Now the system is 1/8 BPS. The positions of the black holes give coordinates on moduli space, and the metric on moduli space was computed in [70]. If we set to zero one of the three charges then we get a 1/4 BPS system.

It is easy to distinguish 'motion on moduli space' from the kinds of oscillatory behavior that we have encountered in the dynamics of supertubes. As mentioned in the introduction, when we have motion on moduli space we take the limit of the velocity going to zero, and over a long time Δt the system configuration changes by order unity. Using Δx as a general symbol for the change in the configuration[38] we

[38]For example x could be the separation of two black hole centers.

have for 'drift on moduli space'

$$v \sim \epsilon, \quad \Delta t \sim \frac{1}{\epsilon}, \quad \Delta x \sim 1, \qquad (\epsilon \to 0) \tag{7.1}$$

On the other hand for the periodic behavior that we have found for both the weak coupling and strong coupling dynamics of bound states, we have

$$v \sim \epsilon, \quad \Delta t \sim 1, \quad \Delta x \sim \epsilon \qquad (\epsilon \to 0) \tag{7.2}$$

Note that for the motion (1.1) the energy lost to radiation during the motion vanishes as $\epsilon \to 0$, so the dynamics (1.1) is unlike any of the cases that we have discussed for the bound state.

While the moduli space metric in [70] was found for spherically symmetric black holes ('naive geometries' in the language of [150]) we expect that a similar 'drifting' motion would occur even if we took two 'actual' geometries of the 2-charge system and gave them a small relative velocity with respect to each other.[39] Thus such unbound systems would have a dynamical mode *not* present for the bound states.

For the bound state 3-charge geometries that have been constructed [9] the structure is very similar to the structure of 2-charge geometries. It is therefore reasonable to conjecture that 3-charge geometries will have a similar behavior: Unbound systems will have 'drift' modes like (1.1) while bound systems will have no such modes. If true, this conjecture could be very useful for the following reason. It is known how to write down the class of *all* 3-charge supersymmetric geometries [12, 94, 95]. But we do not know which of these are bound states. On the other hand the microstates of the 3-charge black hole [96] are bound states of three charges. If we can select

[39]The motion of the centers of the two states could be accompanied by a slow change in the internal configurations of the states.

the bound state geometries from the unbound ones by some criterion then we would have a path to understanding all the microstates of the 3-charge black hole. This is important because the 3-charge hole has a classical horizon and our results on this hole should extend to all holes.

To summarize, it seems a reasonable conjecture that out of the class of all supersymmetric 3-charge geometries the bound states are those that have no 'drift' modes (1.1). It would be interesting to look for 'drift' modes for the 3-charge geometries constructed recently in [97, 124]; here the CFT dual is not known so we do not know a priori if the configuration is a bound state. The same applies for geometries made by adding KK-monopole charge to BPS systems carrying a smaller number of charges [99]. It would also be useful to extend these considerations to the suggested construction of 3-charge supertubes and their geometries [100, 101].

In the introduction we have also asked the question: Can the bound state break up into two or more unbound states under a small perturbation? Since the bound state is only threshold bound, such a breakup is allowed on energetic grounds. But for the 2-charge system we see that bound states are not 'close' to unbound states. The bound states are described by a simple closed curve traced out by the locations $\vec{x} = \vec{F}(v)$ for $0 \leq v < L_T$. A superposition of *two* such bound states has *two* such simple closed curves. The curve can break up into two curves if it self-intersects, but in a generic state the curve is not self-intersecting. If we add a little energy to a 2-charge bound state then we have seen that the configuration does not 'drift' through the space of bound states, so the curve will not drift to a curve with a self-intersection and then split. Thus we expect that generic bound states are stable to small perturbations; energy added to them causes small oscillations for a while and the energy is eventually lost to infinity as radiation.

6.8 Discussion

Let us summarize our arguments and conjectures. If we have two BPS objects with 1/4 susy then they feel no force at rest, but their low energy dynamics is a slow relative motion described by geodesics on a moduli space. If we look at just one 1/4 BPS bound state then it has a large degeneracy, which in the classical limit manifests itself as a continuous family of time-independent solutions. If we add a small energy to the BPS bound state, then what is the evolution of the system?

Based on the behavior of unbound objects one might think that there will again be a 'drift' over the family of configurations, described by some metric on the moduli space of configurations. But we have argued that this is not what we should expect. We first looked at the 1/4 BPS configurations at zero coupling, where we get 'supertubes' described by a DBI action. We saw that the best way to get the dynamics of such 1/4 BPS objects is to use the NS1-P picture, which is a 'multiwound string carrying a traveling wave'. For this zero coupling limit we found that instead of a 'drift' over configurations' we get oscillatory behavior. These oscillations are not described by a collection of simple harmonic oscillators. Rather they are like the motion of a charged particle in a magnetic field where each term in the equation of motion has at least one time derivative, and there is a continuous family of equilibrium configurations. We found a simple expression (2.25) for the period of oscillations with arbitrary amplitude, which reduced to the period found in [75] for the case of small oscillations of the round supertube.

If we increase the coupling a little then we get a gravity description of the supertube, but with gravitational field of the tube extending only to distances small compared to the circumference of the tube. Thus we get a 'thin' long tube. Zooming in to a point of the tube we see an essentially straight segment, and we studied the perturbations to this geometry. We found excitations that agree in frequency with those found from the zero coupling analysis.

We noted that the part of the excitation that leaks out to spatial infinity will have the form of a traveling wave. As we increase the coupling the amplitude of the wave reaching this region becomes larger. Thus there will be an energy flux leaking out to infinity, and the excitation will not remain concentrated near the supertube. But as we increase the coupling still further we find that the geometry develops a deep 'throat' and we get a new kind of long lived excitation: Supergravity modes can be trapped in this throat for long times, only slowly leaking their energy to infinity.

We argued that the different kinds of excitations found at weak and strong coupling reflect the phase transition that had been noted earlier from the study of black holes [76, 77]. At weak coupling the excitations on such a system creates pairs of the charges already present in the BPS state. But at larger coupling the excitation energy goes to creating pairs of a *third* kind of charge. The value of g where this transition occurs was found to have the same dependence on V, R, n_i as the value of g where the supertube stops being 'thin'; i.e. where the gravitational effect of the tube starts extending to distances comparable to the radius of the tube.

We have noted that bound states do not exhibit a 'drift' over a moduli space of configurations, while unbound states do. If 3-charge systems behave qualitatively in the same way as 2-charge ones then this fact can be used to distinguish bound states from unbound ones for the class of 1/8 BPS states; such bound states would give microstates of the 3-charge extremal hole.

CHAPTER 7

BOUND STATES OF KK MONOPOLE AND MOMENTUM

7.1 Introduction

In previous chapters, we have expressed two charge systems in several duality frames like $D1-D5$ and $NS1-P$. In this chapter, we consider two charge system of KK monopole carrying momentum wave. This would correspond to a simple system of 2-charges in 4-dimensions and would be the first example of such a metric. As discussed in the introduction, Kaluza-Klein(KK) monopole solution has attracted considerable attention since it was first proposed by Gross and Perry in [138]. It is a purely gravitational solution in string theory and one of its obvious attractions is that it is a completely regular solution in string theory. Recently, there has been much interest in studying solutions containing KK monopole [173, 195, 190]. Also, as recent work shows, it can be used to connect black rings in five dimensions to black holes in four dimensions [184, 175]. Studies of black rings in Taub-NUT space [173, 190] led to supersymmetric solutions carrying angular momentum in four dimensional asymptotically flat space [175]. KK monopoles also occur in 4-dimensional string theoretic black holes.

Note that our solution cannot be obtained by setting one of the charges in the known three charge solultion to zero and U-dualizing. For example, setting D5 charge to zero in D1-D5-KK solution of [190], we get D1-KK which can be U-dualized to KK-P. When we try to put one charge to zero in the geometry of [190], one finds that it reduces to the 'naive' 2-charge geometry and on dualization, it gives the naive KK-P geometry. Here, by 'naive' we mean geometries obtained by applying the harmonic-superposition rule. The black-ring structure of the geometry is destroyed when one of the charges (other than the KK monopole) is set to zero. This raises the question whether this geometry has all three charges bound and whether this 3-charge system is 'symmetric' between the charges. One of the motivation for the present work is to understand, in a simplified setting, if the solution constructed in [190] is a true bound state or not. Our construction of KK-P is manifestly bound and if it can be related by dualities to the solution of Bena and Kraus (with one charge set to zero) then, at least in this simplified setting, we can be confident that this is a bound state.

Note that in this system we add momentum along one of the isometry directions, different from KK monopole fibre direction. Hence this system is still supersymmetric and is not dual to $D0-D6$ system as studied in [157] which was non-supersymmetric and would correspond to momentum along fibre direction.

Since all two charge systems are related by string dualities, one may ask the reason for constructing KK-P *ab initio* when it can obtained by dualities from F1-P. We will also construct it by dualities from F1-P solution constructed in [150] in section 2. The reason we also obtain it using Garfinkle-Vachaspati transformation is that it gives us unsmeared solutions which carry t and y dependence, y being the direction of wave. We show complete smoothness of this N-monopole solution carrying momentum.

When we try to get a solution independent of t, y by smearing then we will see that singularities develop which are similar to singularities in solution obtained via dualities. Since number of KK monopoles is always discrete (even classically), we know that singularities are an artifact of smearing and discrete solution is always smooth (even classically). One particular feature of these solutions is that orbifold singularities of multiple KK monopoles are also resolved and they are completely smooth.

7.1.1 Outline of the chapter

The plan for present chapter is as follows.

- In $\Sigma 1$, we add momentum to KK monopole by the method of Garfinkle-Vachaspati (GV) transform.
- In $\Sigma 2$, we concentrate on the smoothness of N monopoles solution. Specifically, we consider the case of two monopole solution with momentum. We demonstrate how KK monopoles get separated by the addition of momentum and discuss the regularity of solution.
- In $\Sigma 3$, we get the same solution as above by performing dualities on general two charge solutions constructed in [150].
- In $\Sigma 4$, we perform T-duality to convert this to KK-F1 solution.
- In $\Sigma 5$, we consider the KK-D1-D5 metric obtained by Bena and Kraus in the near-horizon limit and try to see if it is duality symmetric. It turns out that it is not. This is not surprising as Buscher duality rules used are valid only at the

supergravity level and as mentioned earlier and discussed in [190], more refined duality rules will be required.

- We give our T-duality conventions and a discussion of Garfinkle-Vachaspati (GV) transform in two appendices.

7.2 Adding momentum to KK monopoles by GV transformation

In this section, we take the metric of a single KK monopole and add momentum to it along one of isometry directions (not the fibre direction) using the procedure of Garfinkle and Vachaspati. Using the linearity of various harmonic functions appearing in metric, we can superpose harmonic functions to get multi-monopole metric with momentum.

7.2.1 KK monopole metric

Ten dimensional metric for KK monopole at origin is

$$ds^2 = -dt^2 + dy^2 + \sum_{i=6}^{9} dz^i dz_i + H[ds + \chi_j dx^j]^2 + H^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \quad (2.1)$$

$$H^{-1} = 1 + \frac{Q_K}{r} \quad , \quad \vec{\nabla} \times \vec{\chi} = -\vec{\nabla} H^{-1} \quad (2.2)$$

Here y is compact with radius R_5 while x_j with $j = 1, 2, 3$ are transverse coordinates while z_i with $i = 6, 7, 8, 9$ are coordinates for torus T^4. Here $Q_K = \frac{1}{2} N_K R_K$ where N_K corresponds to number of KK monopoles. Near $r = 0$, s circle shrinks to zero. For $N_K = 1$, it does so smoothly while $N_K > 1$, there are Z_{N_K} singularities. First we consider just $N_K = 1$ case. Introducing the null coordinates $u = t + y$ and $v = t - y$, above metric reads

$$ds^2 = -dudv + \sum_{i=6}^{9} dz^i dz_i + H[ds + \chi_j dx^j]^2 + H^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \quad (2.3)$$

We want to add momentum to this using Garfinkle-Vachaspati (GV) transform method [158, 159].

7.2.2 Applying the GV transform

Given a space-time with metric $g_{\mu\nu}$ satisfying the Einstein equations and a null, killing and hypersurface orthogonal vector field k_μ i.e. satisfying the following properties

$$k^\mu k_\mu = 0, \; k_{\mu;\nu} + k_{\nu;\mu} = 0, \; k_{\mu;\nu} = \frac{1}{2}(k_\mu A_{,\nu} - k_\nu A_{,\mu}) \quad (2.4)$$

for some scalar function A is some scalar function, one can construct a new exact solution of the equations of motion by defining

$$g'_{\mu\nu} = g_{\mu\nu} + e^A \Phi k_\mu k_\nu \quad (2.5)$$

The new metric $g'_{\mu\nu}$ describes a gravitational wave on the background of the original metric provided the matter fields if any. satisfy some conditions [160] and the function Φ satisfies

$$\nabla^2\Phi = 0 \; , \; k^\mu \partial_\mu \Phi = 0 \quad (2.6)$$

Some more details about Garfinkle-Vachaspati transform are given in appendix. Note that all this is in Einstein frame but it can be rephrased in string frame very easily. In our case, there are no matter fields and the dilaton is zero so there is no difference between the string and Einstein frames. We take $(\frac{\partial}{\partial u})^\mu$ as our null, killing vector. Since $g_{uu} = 0$, it is obviously null and since the metric coefficients do not depend on

u, it is also killing. One can also check that this vector field is hypersurface orthogonal for a constant A which may be absorbed in Φ. Applying the transform we get

$$ds^2 = -(dudv + T(v,\vec{x})dv^2) + \sum_{i=6}^{9} dz^i dz_i + H[ds + \chi_j dx^j]^2 + H^{-1}[\sum_{j=1}^{3} dx_j^2)] \quad (2.7)$$

where $T(v,\vec{x})$ satisfies the three dimensional Laplace equation. General solution for T is

$$T(v,\vec{x}) = \sum_{l\geq 0} \sum_{m=-l}^{l} [a_l(v) r^l + b_l(v) r^{-l+1}] Y_{lm} \quad (2.8)$$

Here Y_{lm} are the usual spherical harmonics in three dimensions. Constant terms can be removed by a change of coordinates. To see this, we consider $T(v,\vec{x}) = g(v)Y_{0m}$. We can go to a new set of coordinates $du' = du - g(v)Y_{0m}dv$ and other coordinates remaining same. If we want [40] a regular (at origin) and asymptotically flat solution (after dimensional reduction along the fibre) then the only surviving term is $T(v,\vec{x}) = \vec{f}(v)\cdot\vec{x}$. This is apparently not asymptotically flat but can be made so by the following coordinate transformations

$$v = v' \quad (2.9)$$

$$\vec{x} = \vec{x'} - \vec{F} \quad (2.10)$$

$$u = u' - 2\dot{F}_i x'_i + 2\dot{F}_i F_i - \int^{v'} \dot{F}^2(v)dv \quad (2.11)$$

Here $\vec{f}(v) = -2\ddot{\vec{F}}$ and dot refers to derivative with respect to v. Making this change of coordinates, the terms in metric change as follows

$$dudv = du'dv' - 2\dot{F}_i dx'_i dv' + \dot{F}^2(v')dv'^2 \quad (2.12)$$

$$dx_j dx_j = dx'_j dx'_j + \dot{F}^2(v')dv'^2 - 2\dot{F}_i dx'_i dv' \quad (2.13)$$

[40]We are excluding vibrations along the fibre direction. One could include such excitations but making the corresponding solution asymptotically flat turns out to be difficult. Perhaps a formalism different than GV transform might be better suited for that purpose

So the final metric is

$$ds^2 = -du'dv' + 2\dot{F}_i(1-H^{-1})dx'_i dv' - \dot{F}^2(1-H^{-1})dv'^2 + \sum_{i=6}^{9} dz^i dz_i$$

$$+H[ds + \chi'_j dx'_j - \chi'_j \dot{F}_j dv]^2 + H^{-1}[\sum_{j=1}^{3} dx_j'^2)] \tag{2.14}$$

Removing the primes,we write the above metric in the form of chiral-null model as

$$ds^2 = -dudv + dz_i dz_i + H^{-1}dx_j^2 + H(ds + V_j dx^j + Bdv)^2 + 2A_j dx_j dv + Kdv^2 \tag{2.15}$$

Here we have introduced the notation

$$H^{-1} = 1 + \frac{Q_K}{|\vec{x} - \vec{F}(v)|^2} \qquad B = -\vec{\chi} \cdot \vec{F}(v) \tag{2.16}$$

$$K(x,v) = \frac{Q_K|\dot{\vec{F}}(v)|^2}{|\vec{x} - \vec{F}(v)|^2} \quad , \quad A_i = -\frac{Q_K \dot{F}_i(v)}{|\vec{x} - \vec{F}(v)|^2} \tag{2.17}$$

$$\chi_1 = -\frac{Q_K(x_2 - F_2(v))}{(x_1 - F_1(v))^2 + (x_2 - F_2(v))^2}\left(\frac{(x_3 - F_3(v))}{|\vec{x} - \vec{F}(v)|}\right) \tag{2.18}$$

$$\chi_2 = \frac{Q_K(x_1 - F_1(v))}{(x_1 - F_1(v))^2 + (x_2 - F_2(v))^2}\left(\frac{(x_3 - F_3(v))}{|\vec{x} - \vec{F}(v)|}\right) \tag{2.19}$$

We have written harmonic functions above for the case of single KK monopole. But because of linearity, we can superpose the harmonic functions to get the metric for the multi-monopole solution, with each monopole carrying it's wave profile $\vec{F}^{(p)}(v)$ and having a charge $Q^{(p)} = \frac{Q_K}{N_K}$. Functions appearing in the metric then become

$$H^{-1} = 1 + \sum_p \frac{Q_K^{(p)}}{|\vec{x} - \vec{F}^{(p)}|} \tag{2.20}$$

$$K(x,v) = \sum_p \frac{Q_K^{(p)}|\dot{\vec{F}}^{(p)}|^2}{|\vec{x} - \vec{F}^{(p)}|} \quad , \quad A_i = -\sum_p \frac{Q_K^{(p)} \dot{F}_i^{(p)}}{|\vec{x} - \vec{F}^{(p)}|} \tag{2.21}$$

Besides the above functions, there are $\vec{\chi}$ and $\chi_i \dot{F}_i$ with

$$\chi_1 = -\sum_p \frac{Q_K^{(p)}(x_2 - F_2^{(p)})}{(x_1 - F_1^{(p)})^2 + (x_2 - F_2^{(p)})^2} \left(\frac{(x_3 - F_3^{(p)})}{|\vec{x} - \vec{F}^{(p)}|} \right) \tag{2.22}$$

$$\chi_2 = \sum_p \frac{Q_K^{(p)}(x_1 - F_1^{(p)})}{(x_1 - F_1^{(p)})^2 + (x_2 - F_2^{(p)})^2} \left(\frac{(x_3 - F_3^{(p)})}{|\vec{x} - \vec{F}^{(p)}|} \right) \tag{2.23}$$

7.3 Smoothness of solutions

To show smoothness, we concentrate on simple case of two monopoles. So in this section, we consider the simple case of two monopoles carrying waves. Normally (i.e without momentum), one would expect the system of two monopoles two have orbifold type Z_2 singularities. But since momentum is expected to separate the monopoles, this solution would be smooth, without any singularities. As we saw earlier, the metric for a single monopole carrying a wave is

$$\begin{aligned} ds^2 &= -dudv + 2\dot{F}_i(1 - H^{-1})dx_i dv - \dot{F}^2(1 - H^{-1})dv^2 + \sum_{i=1}^{4} dz^i dz_i \\ &\quad + H[ds + \chi_j dx_j - \chi_j \dot{F}_j dv]^2 + H^{-1}[\sum_{j=1}^{3} dx_j'^2)] \end{aligned} \tag{3.1}$$

After the change of coordinates, we have

$$H^{-1} = 1 + \frac{Q_K}{|\vec{x} - \vec{F}(v)|} \tag{3.2}$$

For a single monopole, $Q_K = \frac{R_K}{2}$. For two monopoles, we take profile function with $F(v)$ with range from 0 to $4\pi R_5$ where R_5 is the radius of y circle. From 0 to $2\pi R_5$ it gives profile function $F_1(v)$ for the first monopole while from $2\pi R_5$ to $4\pi R_5$ it gives profile function $F_2(v)$ for second monopole. For two monopoles, harmonic functions need to be superposed. So we have

$$H^{-1} = 1 + \frac{Q_K}{|\vec{x} - \vec{F}^{(1)}(v)|} + \frac{Q_K}{|\vec{x} - \vec{F}^{(2)}(v)|} \tag{3.3}$$

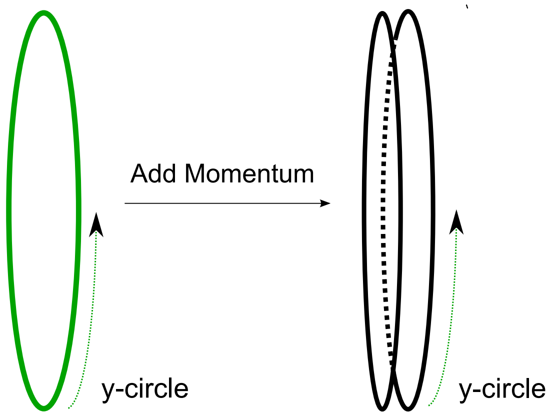

Figure 7.1: Monopole-strings i.e KK monopoles reduced on T^4 (a) 2 coincident monopole-strings(b) 2 single monopole-string separated in transverse directions.

Since $\vec{\nabla} \times \vec{\chi} = -\vec{\nabla} H^{-1}$ is a linear equation, the function $\vec{\chi}$ also gets superposed and

$$\vec{\chi} = \vec{\chi}^{(1)} + \vec{\chi}^{(2)} \tag{3.4}$$

Profile functions F_1 and F_2 are given in terms of a single profile function in the covering space $F(v)$ which goes from 0 to $4\pi R$ such that

$$F^{(1)}(v) = F(v) \text{ for } v = [0, 2\pi R] \tag{3.5}$$

$$F^{(2)}(v) = F(v - 2\pi R) \text{ for } v = [2\pi R, 4\pi R] \tag{3.6}$$

$$F^{(1)}(v = 2\pi R) = F^{(2)}(v = 0) \tag{3.7}$$

Notice that since one monopole goes right after the other Q_K is same for both parts of the harmonic function and equal to Q_K for single KK monopole. To check the regularity of the two monopole solution, we make the following observations. Apparent singularities are at the locations $x = F^{(1)}(v)$ and $x = F^{(2)}(v)$. We can go near any one of them and it is like a single KK monopole (containing terms which do not contain dv or dv^2) and hence smooth. Notice that it is important that poles are not at the same location to avoid conical defects. Locally, we can make a coordinate transformation

$$u' = u + f(x_i, v) \quad \text{so that } du' = du + \partial_i f dx^i + \partial_v f dv \tag{3.8}$$

$$-dudv + 2A_i dx^i dv + K dv^2 = dv(-du + 2A_i dx^i + K dv) = -du' dv \tag{3.9}$$

by suitably choosing $f(x_i, v)$. Since such a coordinate transformation can always by locally done, we will only see single KK monopole which is smooth. Basically, momentum separates a monopole with N-unit of charge into N monopoles of unit charge, each of which is smooth. If we go near any one, we see only that monopole. For the same reason of monopole separation due to momentum, N-monopoles with momentum are also smooth.

7.4 Continuous distribution of monopoles

In this section, we get t, y independent solution by smearing over v which corresponds to metric for multiple KK monopoles distributed continuously. But since we have a three dimensional base space, smearing over v gives elliptic function. To see this we use three dimensional spherical polar coordinates

$$x_1 = \tilde{r} \sin\tilde{\theta} \cos\phi \ , \ x_2 = \tilde{r} \sin\tilde{\theta} \sin\phi \ , \ x_3 = \tilde{r} \cos\tilde{\theta} \tag{4.1}$$

and following profile function

$$F_1 = F \cos(\omega v + \alpha) \ , \ F_2 = F \sin(\omega v + \alpha) \tag{4.2}$$

This is the profile function used for simplest metric for D1-D5 system. It is possible that choosing different profile function may lead to regular behavior. But the point is that for D1-D5 system, all metrics (for generic profile functions) were regular while that will not be the case here. The smeared harmonic function would be

$$H^{-1} = 1 + \frac{Q_K}{2\pi}\int_0^{2\pi}\frac{d\alpha}{|\vec{x}-\vec{F}|} = 1 + \int_0^{2\pi}\frac{d\alpha}{\sqrt{\tilde{r}^2 + F^2 - 2F\tilde{r}\sin\tilde{\theta}\cos(\omega v + \alpha - \phi)}} \quad (4.3)$$

Using the periodicity of the integral, this reduces to

$$H^{-1} = 1 + \frac{Q_K}{2\pi}\int_0^{2\pi}\frac{d\beta}{\sqrt{\tilde{r}^2 + F^2 - 2F\tilde{r}\sin\tilde{\theta}\cos\beta}} \quad (4.4)$$

To do the integral, we switch from $\tilde{r}, \tilde{\theta}$ to coordinates r, θ which are defined by

$$\tilde{r}^2 = r^2 + F^2\sin^2\theta \quad , \quad \tilde{r}\cos\tilde{\theta} = r\cos\theta \quad , \quad \tilde{r}^2\sin^2\tilde{\theta} = (r^2 + F^2)\sin^2\theta \quad (4.5)$$

Using these, we write

$$H^{-1} = 1 + \frac{Q_K}{2\pi}\frac{1}{\sqrt{r^2+F^2}}\int_0^{2\pi}\frac{d\beta}{\sqrt{1 + \frac{F^2 sin^2\theta}{r^2+F^2} - \frac{2F\sin\theta\cos\beta}{\sqrt{r^2+F^2}}}} \quad (4.6)$$

Writing $p = \frac{F\sin\theta}{\sqrt{r^2+F^2}}$, we get

$$H^{-1} = 1 + \frac{Q_K}{\pi}\frac{K(p)}{\sqrt{r^2+F^2}} \quad (4.7)$$

where $K(p)$ is elliptic integral of the first kind and

$$2K(p) = \int_0^{2\pi}\frac{d\beta}{\sqrt{1+p^2-2p\cos\beta}} \quad (4.8)$$

$K(p)$ diverges when $p = 1$ i.e

$$r^2 + F^2 = F^2\sin^2\theta \ \text{ or } \ r^2 + F^2\cos^2\theta = 0 \quad (4.9)$$

which is the same place where there is an apparent singularity in the geometry of [186, 188]. It is known that elliptic integral $K(p)$ diverges logarithmically as $p \to 1$. One can, of course, add a suitable harmonic counterterm to cancel the singularity but then the solution will not be asymptotically flat. Integral for function K appearing in the metric is very similar and it gives

$$K = \frac{Q_K F^2 \omega^2 K(p)}{\pi \sqrt{r^2 + F^2}} \tag{4.10}$$

Similarly, components of A_i are given by

$$A_\phi = \frac{2Q_K}{\pi} \sqrt{r^2 + F^2} \left(\frac{K(p) - E(p)}{F} \right) \tag{4.11}$$

with other components zero. This is in untilde coordinates. Expressions for functions $\chi_1,\ \chi_2$ can be obtained by solving the equation $\vec{\nabla} \times \vec{\chi} = -\vec{\nabla} H^{-1}$. In the next section, we would connect the above functions to functions obtained by dualizing D1-D5 system.

7.4.1 Singularities

Due to the presence of elliptic functions and their attendant singularities the solution above, in the smeared case, is not smooth. In section 2, we saw that solution with two KK-monopoles with momentum added is smooth. The calculation goes through for N-monopole case. This is similar to case of fundamental string and momentum system [150] where adding momentum leads to separation of previously coincident strings. One can ask, what causes singularities to develop in the case when a continuum of KK monopoles carry momentum. The reason is that harmonic functions like H^{-1} corresponding to three-dimensional transverse space go like $1/r$ and one further integration (for smearing) effectively converts them into harmonic functions in a two-dimensional transverse space [41] which are known to diverge logarithmically. Elliptic integral $K(p)$, for example, also diverges logarithmically as p goes to 1.

[41] One may guess that if we had allowed vibrations along fibre direction, harmonic functions would be different and smoothness would be maintained even after smearing

Physically also, we can see that smeared case which corresponds to a continuous distribution of KK monopoles is expected to have troubles. Normally, we can consider continuous distribution of sources like branes, fundamental string etc in constructing metrics in supergravity approximation. Discreteness emerges when we use quantization conditions from our knowledge of string theory sources and BPS condition. KK monopole solution is different because here discreteness is inbuilt as smoothness of single KK monopole forces definite periodicity for compact direction and gives $Q_K = \frac{1}{2}N_k R_K$. So considering a continuous distribution of KK monopoles can give singularities even in cases where where situation is smooth for large but discrete distribution of KK monopoles. Even though continuous solution is not smooth, it is still less singular than 'naive' KK-P solution. 'Naive' solution is

$$ds^2 = -dt^2 + dy^2 + \frac{k}{r}(dt+dy)^2 + ds_{T^4} + H[ds + \chi_j dx^j]^2 + H^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{4.12}$$

In smeared solution, we have logarithmic singularity due to elliptic integrals occurring in solution. We note that those singularities are milder than what we get in 'naive' solution.

7.5 Connecting F-P to KK-P via dualities

In this section, we connect KK-P metric found above to 2-charge metrics constructed in [8] by doing various dualities. This will also help in interpreting various quantities appearing in the metric. We start with F1-P metric, written in the form of chiral null model. Introducing the null coordinates $u = t + y$ and $v = t - y$, the metric reads

$$ds^2 = H\left(-dudv + Kdv^2 + 2A_i dx_i dv\right) + dx_i dx_i + dz_j dz_j \tag{5.1}$$

$$B_{uv} = -\frac{(H-1)}{2},\ B_{vi} = HA_i,\ e^{-2\Phi} = H^{-1} = 1 + \frac{Q}{|\vec{x}-\vec{F}|^2} \tag{5.2}$$

$$K(x,v) = \frac{Q|\dot{\vec{F}}|^2}{|\vec{x}-\vec{F}|^2} \quad , \quad A_i = -\frac{Q\dot{F}_i}{|\vec{x}-\vec{F}|^2} \tag{5.3}$$

Here y is compact with radius R_5 while x_i with $j = 1, 2, 3, 4$ are transverse coordinates while z_j with $i = 6, 7, 8, 9$ are coordinates for torus T^4. Summation over repeated indices is implied. We have written harmonic functions above for the case of single string. But because of the linearity of chiral null model, we can superpose the harmonic functions to the metric for the multi-wound string, with each strand carrying its wave profile. Functions appearing in the metric then become

$$H^{-1} = 1 + \sum_p \frac{Q^{(p)}}{|\vec{x}-\vec{F}^{(p)}|^2} \tag{5.4}$$

$$K(x,v) = \sum_p \frac{Q^{(p)}|\dot{\vec{F}}^{(p)}|^2}{|\vec{x}-\vec{F}^{(p)}|^2} \quad , \quad A_i = -\sum_p \frac{Q^{(p)}\dot{F}_i^{(p)}}{|\vec{x}-\vec{F}^{(p)}|^2} \tag{5.5}$$

Here we have smeared along torus directions so that nothing depends on these coordinates and these are isometry directions along which T-duality can be performed. To go from fundamental string carrying momentum (FP) system to KK-P system we perform following chain of dualities

$$F(y)P(y) \xrightarrow{S} D1(y)P(y) \xrightarrow{T6789} D5(y6789)P(y) \xrightarrow{S} NS5(y6789)P(y) \xrightarrow{T4} KK(4y6789)P(y)$$

In the above we start with type IIB theory and metric above is in string frame. In the final step we will need to smear along x_4 direction so that harmonic functions becomes 3 dimensional harmonic functions for KK monopole. Direction $x_4 = s$ which is now compact becomes non-trivially fibred with other non-compact directions to give KK monopole metric. To perform S-duality we first need to go to Einstein frame and there the effect of S-duality is to reverse the sign of dilaton and B-field going to RR field. For NS fields the net effect in string frame is that dilaton changes sign and metric gets multiplied by $e^{-\Phi} = H^{-1/2}$. Also B-field becomes RR field. Now we apply four T-dualities along z_i directions for $i = 6, 7, 8, 9$. Since there is no B-field here, only change is in the metric along torus directions. In the absence of B-field, RR field only picks up extra indices. So we have following fields for D5-P system.

$$ds^2 = H^{1/2}\left(-dudv + Kdv^2 + 2A_i dx_i dv\right) + H^{-1/2} dx_i dx_i + dz_j dz_j \tag{5.6}$$

$$e^{-2\Phi} = H^{-1} \quad , \quad C_{uv6789} = -\frac{(H-1)}{2} \quad , \quad C_{vi6789} = HA_i \tag{5.7}$$

We need to dualize this 6 form field to 2 form field using this metric. First we write down field strengths corresponding to above RR fields.

$$G_{uv6789i} = \partial_u C_{v6789i} + (-1)^6 \partial_v C_{6789iu} + ... + (-1)^6 \partial_i C_{uv6789} = -\frac{1}{2}\partial_i H \tag{5.8}$$

$$G_{vi6789j} = \partial_v C_{i6789j} + (-1)^6 \partial_i C_{6789jv} + ... + (-1)^6 \partial_j C_{vi6789} = \partial_j (HA_i) - \partial_i (HA_j) \tag{5.9}$$

Here we have used the fact that direction u and torus directions are isometries. To dualize this we use

$$G^{\mu_1 \ldots \mu_{p+1}} = \frac{\epsilon^{\mu_1 \ldots \mu_{p+1} \nu_1 \ldots \nu_{9-p}}}{(9-p)!\sqrt{-g}} G_{\nu_1 \ldots \nu_{9-p}} \tag{5.10}$$

and we normalize ϵ by $\epsilon^{tyjkli6789} = 1$. For our metric we have $\sqrt{-g} = \sqrt{H}$. Also, in terms of lightcone coordinates, our epsilon tensor is normalized as $\epsilon^{uvjkli6789} = -2$.

Using these, we get dual 3-form field strengths.

$$G^{jkl} = \frac{\epsilon^{jkluv6789i}}{7!\sqrt{H}} G_{uv6789i} = \frac{\epsilon^{jkli}\partial_i H}{\sqrt{H}} \tag{5.11}$$

$$G^{ukl} = \frac{\epsilon^{uklvi6789j}}{7!\sqrt{H}} G_{vi6789j} = \frac{-2\epsilon^{klij}[\partial_j(HA_i) - \partial_i(HA_j)]}{\sqrt{H}} \tag{5.12}$$

we have also reduced ten dimensional epsilon symbol to epsilon tensor in four flat Euclidean dimensions. Now we can use metric to lower the indices. We get

$$G_{mnp} = g_{mj}g_{nk}g_{lp}G^{jkl} = \frac{\epsilon_{mnpi}\partial^i H}{H^2} = -\epsilon_{mnpi}\partial^i H^{-1} \tag{5.13}$$

$$G_{vmn} = g_{uv}g_{mk}g_{nl}G^{ukl} + g_{vj}g_{mk}g_{nl}G^{jkl} = \epsilon_{mnij}\partial^i A^j \tag{5.14}$$

where we have flat Euclidean metric in four dimensional space. After this we perform an S-duality to get to NS5-P system.

$$ds^2 = \left(-dudv + Kdv^2 + 2A_i dx_i dv\right) + H^{-1}dx_i dx_i + dz_j dz_j \tag{5.15}$$

$$e^{2\Phi} = H^{-1} \tag{5.16}$$

Under S-duality, RR field go to NS-NS B-field. To go to KK monopole we apply a T-duality along a direction perpendicular to NS5. Let us choose that to be $x^4 = s$. Rightnow, x^4 is not an isometry direction since H, A_i and K depend on x^4. To remedy this we smear along x^4 direction. Smearing converts four dimensional harmonic functions into three dimensional harmonic functions. Now we do T-duality using the Buscher T-duality rules given in the appendix. We can write the T-dual metric in general as

$$ds_T^2 = ds'^2 - \frac{G_{\mu s}G_{\nu s}dx^\mu dx^\nu}{G_{ss}} + \frac{(ds + B_{\mu s})^2}{G_{ss}} \tag{5.17}$$

where ds'^2 is the original metric minus the G_{ss} part. Then metric in generic form for $KK-P$ (since we do not completely know the values of B-field yet) is

$$ds^2 = -dudv + dz_i dz_i + H^{-1}dx_i^2 + H(ds + B_{\mu s}dx^\mu)^2 + 2A_j dx_j dv + (K - HA_s^2)dv^2 \quad (5.18)$$

After duality we also have $B'_{\mu s} = HG_{\mu s}$. Since KK-P is a purely gravitational solution we do not want any B-field. So we must dualize in a direction in which $\dot{F}_j$ is zero. So $G_{\mu s} = 0$ and $A_s = 0$. Since we have smeared along $x_4 = s$ direction, any derivatives along s give zero. For field strengths of $NS5-P$, we had

$$G_{ijk} = -\epsilon_{ijkl}\partial^l H^{-1} \ , \ G_{vij} = \epsilon_{ijkl}\partial^k A^l \quad (5.19)$$

Consider $l = s$ case first. Since we have smeared along s, we get $G_{ijk} = 0$ and hence $B^{(2)} = B_{ij}$ becomes pure gauge after smearing and can be set to zero. So to have non-zero field strength, we must have one s index. Suppose $i = s$. Then using isometry along s, we have

$$G_{sjk} = \partial_s B_{jk} + \partial_k B_{sj} + \partial_j B_{ks} = \partial_k \chi_j - \partial_j \chi_k \quad (5.20)$$

where $\chi_j = B_{sj}$ is a three dimensional vector field. We see that

$$\vec{\nabla} \times \vec{\chi} = -\vec{\nabla} H^{-1} \quad (5.21)$$

where $\vec{\nabla}$ is three-dimensional gradient. From other components of field strength, we get

$$G_{vij} = \partial_v B_{ij} + \partial_j B_{vi} + \partial_i B_{jv} = \epsilon_{ijkl}\partial^k A^l \quad (5.22)$$

Since $A^s = 0$ and derivative with respect to s gives zero, we have, for 3-dimensional indices i, j

$$G_{vij} = \partial_v B_{ij} + \partial_j B_{vi} + \partial_i B_{jv} = (dC)_{ij} = 0 \quad (5.23)$$

where first term is zero because as we saw earlier, B_{ij} with both indices on 3-dim. space are zero (upto gauge transformations). Here $B_{iv} = C_i$ is a three dimensional vector field which is again gauge equivalent to zero as can be seen above. So one of the indices i, j must be s to get no-zero right hand side. Then we get

$$G_{vis} = \partial_v B_{is} + \partial_s B_{vi} + \partial_i B_{sv} = \partial_i B - \partial_v \chi_i = \epsilon_{iskl}\partial^k A^l = -\epsilon_{ikl}\partial^k A^l \tag{5.24}$$

where $B_{sv} = B$ is a three dimensional scalar which satisfies above equation. So KK-P metric is

$$ds^2 = -dudv + dz_i dz_i + H^{-1}dx_j^2 + H(ds + \chi_j dx^j + Bdv)^2 + 2A_j dx_j dv + Kdv^2 \tag{5.25}$$

Here $i = 1, 2, 3$ and all harmonic functions are in three dimensions. z_i for $i = 6, ..9$ are torus coordinates. Using Garfinkle-Vachaspati transform, we got $B = -\vec{\chi} \cdot \dot{\vec{F}}$. Let us check that it satisfies the equation for B written above. By using the rules of three dimensional vector calculus we have

$$\nabla(\vec{\chi}\cdot\dot{\vec{F}}) = (\dot{\vec{F}}\cdot\nabla)\vec{\chi}+(\vec{\chi}\cdot\nabla)\dot{\vec{F}}+\dot{\vec{F}}\times(\nabla\times\vec{\chi})+\vec{\chi}\times(\nabla\times\dot{\vec{F}}) = (\dot{\vec{F}}\cdot\nabla)\vec{\chi}+\dot{\vec{F}}\times(\nabla\times\vec{\chi}) \tag{5.26}$$

where we have used that $\vec{F}$ only depends on v. Comparing this with equation for B, we see that it is same when we realize that

$$\vec{\nabla} \times \vec{\chi} = -\vec{\nabla} H^{-1} \quad , \quad A_j = (1 - H^{-1})\dot{F}_j$$

Only non-trivial step is to show that

$$(\dot{\vec{F}} \cdot \nabla)\vec{\chi} = -\partial_v \vec{\chi}$$

To see this we write

$$\partial_v = \dot{F}_i \partial_{F_i} = -\dot{F}_i \nabla_i$$

where we have used the fact that derivatives with respect to observation point x_i and source F_i can be interchanged at the cost of a minus sign. One can reduce the expression for B to a quadrature but it is not easy to carry out the integration explicitly. For completeness, we give the formal expression below.

$$B = \frac{Q_K}{2} r \cos\theta \int_0^1 g dg \int_0^{2\pi} d\alpha \frac{1}{(r^2 + F^2 g^2 - 2Fgr \sin\theta \cos\alpha)^{3/2}} \tag{5.27}$$

7.6 Properties of solution

In this section, we discuss some properties of the new solutions.

1. Smoothness: As we mentioned earlier, N-monopole solutions are smooth. Z_n singularities associated with coincident KK monopoles are lifted by adding momentum carrying gravitational wave. System behaves as N_K single monopoles and is smooth. But as we saw earlier, smeared solution has singularities. From the analysis of two monopole case, the reason is apparent. When we consider continuum of KK monopoles, any two monopoles come arbitrarily close to each other and separation due to momentum is not enough to prevent singularities. Even though logarithmic singularity encountered is quite mild, it is not removable by coordinate identification, as was possible in single KK monopole case.

2. KK electric charge: In the metric

$$ds^2 = -dudv + dz_i dz_i + H^{-1} dx_j^2 + H(ds + \chi_j dx^j + Bdv)^2 + 2A_j dx_j dv + K dv^2 \tag{6.1}$$

   we have term B which corresponds to momentum along fibre direction s even though we started with no profile-function component along the fibre. On dimensional reduction, this gives a KK electric field, in addition, to magnetic field due to KK monopole. Adding momentum to KK monopoles has caused this electric field. To see its origin from other duality related systems, note that $B = B_{sv}$ where B_{sv} is the component of B-field in the fibre direction. Physically, the angular momentum present in 5-dimensional metric (NS5-P) becomes momentum along fibre direction and manifests itself as electric field in reduced theory.

7.7 Comparison to recent works

7.7.1 Work of Bena-Kraus

Recently, Bena and Kraus [190] constructed a metric for D1-D5-KK system which, according to them, corresponds to a microstate. These solutions are smooth and are related to similar studies of black rings with Taub-NUT space as the base space [190, 175]. In these metrics, KK monopole charge is separated from D1 and D5 charges and is treated differently from the other two charges. In this section, we want to check whether Bena-Kraus metric for D1-D5-KK is symmetric under permutation of charges by duality. For simplicity, we consider near horizon limit of Bena-Kraus(BK) metric and perform dualities to permute the charges. BK metric and gauge field (after correcting the typos), in the near horizon limit, is

$$ds^2 = \frac{1}{\sqrt{Z_1 Z_5}}\left[-(dt+k)^2 + (dy-k-s)^2\right] + \sqrt{Z_1 Z_5}ds^2_{KK} + \sqrt{\frac{Z_1}{Z_5}}ds^2_{T^4} \quad (7.1)$$

$$k = \frac{l^2}{4\Sigma}\frac{\Sigma - r - \tilde{R}}{Q_K}\left(d\psi - \frac{Q_K}{R_K}d\phi\right) \quad s = -\frac{l^2}{2\Sigma Q_K}\left((\Sigma - r)d\psi + \frac{Q_K}{R_K}\tilde{R}d\phi\right) \quad (7.2)$$

$$ds^2_{KK} = Z_K(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2) + \frac{1}{Z_K}(R_K d\psi + Q_K\cos\theta d\phi)^2 \quad (7.3)$$

Here we have used the notation

$$\Sigma = \sqrt{r^2 + \tilde{R}^2 + 2\tilde{R}r\cos\theta} \quad , \quad \tilde{R} = \frac{R_K^2}{4Q_K} \tag{7.4}$$

$$Z_K = \frac{Q_K}{r} \quad , \quad Z_{1,5} = \frac{Q_{1,5}}{\Sigma} \quad , \quad l^2 = 4Q_K\sqrt{Q_1Q_5} \tag{7.5}$$

RR two form field is given by

$$C^{(2)} = \frac{1}{Q_1}\Bigg((r-\tilde{R})dt\wedge dy - \frac{l^2}{4}(\tilde{R}+\alpha)\left[dt\wedge(\frac{d\psi}{Q_K}+\frac{d\phi}{R_K}) - dy\wedge(\frac{d\psi}{Q_K}-\frac{d\phi}{R_K})\right]$$

$$-\frac{(\tilde{R}+\alpha)l^4}{4Q_KR_K}d\psi\wedge d\phi\Bigg) \tag{7.6}$$

Here we have used the notation $\alpha = \Sigma - r$. Bena-Kraus notation Z_K is our H^{-1}. Their $R_K\psi$ is our s coordinate. Correspondingly, periodicity of ψ is 2π while period of s was $2\pi R_K$. Dilaton is given by $e^{2\Phi} = \frac{Q_1}{Q_5}$. Fields given above are for $D1_yD5_{y6789}KK_{\psi y6789}$ system.

7.7.2 Dualities

We can perform an S-duality to go to F1-NS5-KK system and then perform a T-duality along the fibre direction ψ permute the charges of KK monopole and NS5 brane. Since dualities map near horizon region of one metric to near horizon region of other metric, we expect an interchange of KK and 5-brane charges. After S-duality, we get the following metric for F1-NS5-KK system.

$$ds^2 = \frac{1}{\sqrt{Z_1Z_5}}\left[-(dt+k)^2 + (dy-k-s)^2\right] + \sqrt{Z_1Z_5}ds^2_{KK} + \sqrt{\frac{Z_1}{Z_5}}ds^2_{T^4} \tag{7.7}$$

NS-NS two form field is given by

$$B^{(2)} = \frac{1}{Q_1}\Bigg((r-\tilde{R})dt\wedge dy - \frac{l^2}{4}(\tilde{R}+\alpha)\left[dt\wedge(\frac{d\psi}{Q_K}+\frac{d\phi}{R_K}) - dy\wedge(\frac{d\psi}{Q_K}-\frac{d\phi}{R_K})\right]$$

$$-\frac{(\tilde{R}+\alpha)l^4}{4Q_K R_K}d\psi\wedge d\phi\Bigg) \tag{7.8}$$

Dilaton is given by $e^{2\Phi} = \frac{Q_5}{Q_1}$. Details of final T-duality are given in the appendix. Metric, after T-duality, is given by

$$ds^2 = \frac{Q_5 Q_K}{r\Sigma}(dr^2+r^2 d\theta^2)+\frac{1}{4Q_5\tilde{R}}d\psi^2+\frac{2Q_5Q_K}{\tilde{R}}(\alpha+\tilde{R})d\phi^2+\frac{2Q_K}{\tilde{R}R_K}(\tilde{R}+\alpha)d\phi d\psi-\frac{r}{Q_1}dt^2+\frac{\Sigma}{Q_1}$$

$$-\frac{1}{Q_1}(\tilde{R}+\alpha)dtdy+\frac{4(\tilde{R}+\alpha)Q_K}{R_K}\sqrt{\frac{Q_5}{Q_1}}d\phi(dy-dt)+\frac{(\tilde{R}+\alpha)}{2\tilde{R}\sqrt{Q_5Q_1}}d\psi(dy-dt) \tag{7.9}$$

Dilaton is given by $e^{2\Phi'} = \frac{1}{4Q_1\tilde{R}}$. Non-zero components of B-field are given by

$$B'_{t\psi} = \frac{\tilde{R}-\alpha}{4\tilde{R}\sqrt{Q_1Q_5}} \quad , \quad B'_{y\psi} = \frac{\tilde{R}+\alpha}{4\tilde{R}\sqrt{Q_1Q_5}} \tag{7.10}$$

$$B'_{\phi\psi} = \frac{R_K\alpha}{4\tilde{R}^2} \quad , \quad B'_{ty} = \frac{\Sigma+r-\tilde{R}}{2Q_1} \tag{7.11}$$

Since metric after T-duality is not of same form, we see that this metric naively does not look like a bound state. For a bound state, one would expect just a permutation of charges under duality like done above. But since we performed a T-duality along fibre direction to permute NS5 and KK6 using Buscher rules which as shown in [163] are insufficient to give correct answer. So the situation remains open. The question which we want to discuss is whether the solution constructed in [190] has KK monopole bound to other two charges or it just acts as a background. Since KK monopole is much heavier than other two components, D1 and D5 branes, it may look as if acting as background for other two and difference might not be apparent at supergravity level. But still one would think that in the S-dual system F1-NS5-KK where at least NS5 and KK both have masses going like $1/g^2$ (actual masses will depend on compactification radii) , it should be possible to permute these two charges. Analysis similar to [163] could be done for 3-charge system to completely fix this issue. We intend to look further in this matter in a future publication.

7.8 T-duality to KK-F1

We now convert KK-P system to KK-F1 by T-dualizing along the w direction where $v = t - w$ and $u = t + w$. We are in IIA supergravity since we can connect this to D1-D5 by S-duality followed by T-duality along a perpendicular direction. Writing the metric as

$$ds^2 = -(dt^2 - dw^2) + 2C_i dx_i (dt - dw) + K(dt - dw)^2 + H[ds + A_j dx_j - A_j \dot{F}_j dt + A_j \dot{F}_j dw]^2$$

$$+ H^{-1} dx_j dx_j + dz_l dz_l \tag{8.12}$$

where we have written $C_i = (1 - H^{-1})\dot{F}_i$ and $K = -(1 - H^{-1})\dot{F}^2$. From now on we will leave the trivial torus coordinates z_l in what follows. It is easier to write the T-dual metric using

$$ds_T^2 = ds'^2 - \frac{G_{\mu w} G_{\nu w} dx^\mu dx^\nu}{G_{ww}} + \frac{(dw + B_{\mu w} dx^\mu)^2}{G_{ww}} \tag{8.13}$$

Since there is no B-field in KK-P, we get

$$ds^2 = -dt^2 + 2C_i dx_i dt + K dt^2 + H[ds + A_j dx_j - A_j \dot{F}_j dt]^2 + H^{-1} dx_j dx_j + dz_l dz_l$$

$$+ \frac{dw^2 - [(K + H(A_l \dot{F}_l)^2) dt + H(A_l \dot{F}_l) ds]^2 - (H A_i A_l \dot{F}_l - C_i)(H A_j A_l \dot{F}_l - C_j) dx^i dx^j}{[1 + K + H(A_l \dot{F}_l)^2]} \tag{8.14}$$

7.9 Conclusion

We summarize our results and look at possible directions for future work.

7.9.1 Results

We have found gravity solutions describing multiple KK monopoles carrying momentum by applying the solution-generating transform of Garfinkle and Vachaspati and also by using various string dualities on the known two charge solutions. The second method only yields smeared solution which are logarithmically singular. One important feature of these solutions is that adding momentum to multiple KK monopoles leads to the separation of previously coincident KK monopoles. Hence orbifold type singularities of coincident KK monopoles are resolved and the solution is smooth. One can also superpose a continuum of KK monopoles carrying momentum and replace the summation by an integral. Doing this, one gets stationary solutions (no t dependence) with isometry along y(compact coordinate along which the wave is travelling). The continuous case however, turns out to be singular. Singularity occurs at the same location where it occurs in the solution obtained by applying dualities. In the case with y-isometry, we also dualized it to a KK-F1 system.

Our reasons for studying these geometries were, in part, motivated by recent work of Bena and Kraus [173, 190] in which they constructed a smooth solution carrying D1, D5 and KK charges. This solution is supposed to represent one of the microstates of this system. But in these solutions, KK monopole is separated from D1 and D5 branes and acts more like a background in which the D1-D5 bound states live. One effect of this is that the system does not appear to be duality symmetric i.e one can not permute the charges by performing various string dualities. We performed a specific duality sequence (in the near horizon limit, for simplicity) to permute the charges and found that the solution is not symmetric. But since we used only Buscher T-duality rules, our result does not conclusively show the unboundedness of the geometry and it is possible that duality rules going beyond supergravity will restore the symmetry. Our KK-F1 geometry is also different from the two charge geometries one would get from Bena-Kraus geometries by setting one charge to zero.

Study of two charge systems with KK monopole is far from over. We have constructed time-dependent KK-P geometries which are perfectly smooth and it would be interesting to study them further. On the microscopic side, one can perform DBI analysis on KK-brane [164] . First thing that needs to be checked is that brane-side gives same value for conserved quantities like angular momentum as the gravity side. It is expected that on the microscopic side, system would be dual to usual supertubes. One can also do perturbation analysis on brane-side and gravity side as done in [165]. Perturbation calculation for other two charge systems were also done in [172] and yielded results in agreement with microscopic expectations. Microscopic side of KK-branes is not very well understood, as far as we know. So calculations on gravity side should give us information about microscopic side and vice-versa. We will carry out some of these computations in the next chapter. It would also be interesting to explore further the connection between these solutions and black rings in KK monopole backgrounds as found in [175] as that might suggest ways to add the third charge to these two-charge systems.

CHAPTER 8

PERTURBATIONS OF SUPERTUBES IN KK MONOPOLE BACKGROUND

8.1 Introduction

In this chapter, we continue our study of systems with KK monopole. Our aim is to further explore the question of boundedness of the metrics constructed by Bena and Kraus. In Chapter 4, [172] we studied adding perturbations to BPS supertubes (both in $D0$-$NS1$ and $NS1$-P duality frames) and found classical solutions at both linear and non-linear levels (in the $NS1$-P language these are just vibrations of fundamental string and it's trivial to write down the full solution). Based on several evidences, we formulated a conjecture which allows us to distinguish bound states from unbound states. The conjecture says that bound states are characterized by the absence of 'drift' modes where by 'drift' modes, we mean slow motion on moduli space of configurations. So when we have motion on moduli space we take the limit of the velocity going to zero, and over a long time Δt the system configuration changes by order unity. Using Δx as a general symbol for the change in the configuration we have for 'drift' on moduli space'

$$v \sim \epsilon, \quad \Delta t \sim \frac{1}{\epsilon}, \quad \Delta x \sim 1, \qquad (\epsilon \to 0) \tag{1.1}$$

On the other hand for the periodic behavior of bound states, we have

$$v \sim \epsilon, \quad \Delta t \sim 1, \quad \Delta x \sim \epsilon \qquad (\epsilon \to 0) \tag{1.2}$$

If the geometries of Bena and Kraus are really one of the microstates for 3 charge black holes in four dimensions then they must be bound states of the corresponding charges.Mathur conjecture emphasizes that microstates of black holes (which are described by smooth, horizon-free geometries when classical supergravity description is possible) correspond to *bound* states only.

Unlike the case of supertubes[42]where system was obviously bound, in the systems with KK monopole it is not a priori obvious that we are considering *bound* states. Since we have a conjectured test which can distinguish, at least in principle, bound states from unbound ones, we would like to apply it to some of the systems with KK monopole. This involves solving perturbation equations in KK monopole background. Due to non-trivial background, non-linear perturbation equations, even at DBI level, are quite difficult to solve and hence we would restrict ourselves to linearized perturbations. Thus our considerations are geared to analyze some of these systems, especially one corresponding to geometry given in [173] and to study it's boundedness properties using conjecture of [172]. We found that there are no 'drift' modes at the DBI level and system shows 'quasi-oscillations' as discussed in [172]. On the gravity side, we construct near ring limit of the geometry and we were able to show that near ring limit is identical to near ring limit of 2-charge systems considered [172] except for the periodicity of the ring circle. Then we consider torus perturbations as in [172] and find that results agree with DBI analysis.

[42]In $D0$-$NS1$ language, both charges were induced in a single higher brane $D2$ or in the $NS1$-P language these are just vibrations of fundamental string.

8.1.1 Outline of this chapter

Plan for the present chapter is as follows.

- In Σ2, we set up equations which describe motion of classical string in a general curved background and then apply them for the KK monopole background. We linearize about BPS solution of a fundamental string carrying a right moving wave and write down perturbation equations.

- In Σ3, we write the string profile corresponding to metric of Bena and Kraus (BK) [173]. In our system of 3-charges in four dimensions different geometries correspond to different profile functions [197] of a one dimensional string, as was also the case for 2-charge systems. Here we will work in $NS1-P$ duality frame. Then we solve perturbation equations for this profile.

- In Σ4, we discuss the construction of $D0$-$NS1$ supertube in KK monopole background. Then we study perturbations to this.

- In Σ5, we study a profile different from BK profile in both $NS1$-P language and $D0$-$NS1$ language.

- In Σ6, we study the supergravity side of the system. We take near ring limit of BK geometry and show how the perturbation analysis can be reduced to one done in a previous paper [172] and consequently time-period of torus vibrations also matches with the one considered in [172].

- In Σ7, we conclude with a discussion of our results and directions for future investigations.

8.2 Oscillating string in KK monopole background

In this section, we consider Polyakov string in KK monopole background. We first set up equations of motion and constraint equations in a general background. Our action is

$$S = -\frac{T_1}{2}\int d\sigma d\tau \sqrt{g} g_{\alpha\beta}(\sigma,\tau) G_{AB}(X)\partial^\alpha X^A \partial^\beta X^B \tag{2.3}$$

Here σ, τ are worldsheet coordinates and α, β are worldsheet indices. Index A for spacetime coordinates X^A goes from $0, ..9$. For worldsheet metric $g_{\alpha\beta}$ we have $g = -det(g_{\alpha\beta})$. Varying the action with respect to coordinates X^A, we get

$$\frac{\delta S}{\delta X^A} = 0 = \partial_\alpha[\sqrt{g} G_{AB}\partial^\alpha X^B] - \frac{1}{2}(\partial_A G_{CD})\partial_\alpha X^C \partial^\alpha X^D \tag{2.4}$$

In the conformal gauge on worldsheet, we have $g_{\alpha\beta} = e^{2f}\eta_{\alpha\beta}$. So we get

$$\partial_\alpha[(\partial^\alpha X^B) G_{AB}] - \frac{1}{2}(\partial_A G_{CD})(\partial_\alpha X^C \partial^\alpha X^D) = 0 \tag{2.5}$$

Contracting with G^{AP}, we get

$$\partial_\alpha \partial^\alpha X^P + G^{AP}[(\partial_C G_{AB})\partial^\alpha X^C \partial_\alpha X^B - \frac{1}{2}(\partial_A G_{CD}\partial^\alpha X^C \partial_\alpha X^D] = 0 \tag{2.6}$$

In [194] (see also the references given there), general string equations of motion in curved background were given in a slightly different form. To match with those, we write the combination of derivatives as christoffel symbols. Writing

$$(\partial_C G_{AB})\partial^\alpha X^C \partial_\alpha X^B = \frac{1}{2}[\partial_C G_{AB} + \partial_B G_{AC}]\partial^\alpha X^C \partial_\alpha X^B$$

and recognising the combination of derivatives as christoffel symbols, we get

$$\partial_\alpha \partial^\alpha X^P + \Gamma^P_{CB}\partial^\alpha X^C \partial_\alpha X^B = 0 \tag{2.7}$$

Constraint equations are given by

$$\frac{\delta S}{\delta g^{\alpha\beta}} = T_{\alpha\beta} = G_{AB}[\partial_\alpha X^A \partial_\beta X^B - \frac{1}{2} g_{\alpha\beta} \partial_\gamma X^A \partial^\gamma X^B] = 0 \tag{2.8}$$

If we choose lightcone variables $\xi^\pm = \tilde{\tau} \pm \tilde{\sigma}$ then $g_{++} = g_{--} = 0$ and we get

$$\partial_+ \partial_- X^A + \Gamma^A_{BC} \partial_+ X^B \partial_- X^C = 0 \tag{2.9}$$

$$G_{AB} \partial_\pm X^A \partial_\pm X^B = 0 \tag{2.10}$$

$\partial_\pm$ denotes derivative with respect to $\xi^\pm$ in previous equation.

Ten dimensional metric for KK monopole at origin is

$$ds^2 = -dt^2 + dy^2 + \sum_{i=6}^{9} dz^i dz_i + V[ds + \chi_j dx^j]^2 + V^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2.11}$$

$$V^{-1} = 1 + \frac{Q}{r} \quad , \quad \vec{\nabla} \times \vec{\chi} = -\vec{\nabla} V^{-1} \tag{2.12}$$

Here y is compact with radius R_5 while x_j with $j = 1, 2, 3$ are transverse coordinates while z_i with $i = 6, 7, 8, 9$ are coordinates for torus T^4. Here $Q = \frac{1}{2} N_K R_K$ where N_K corresponds to number of KK monopoles. Near $r = 0$, s circle shrinks to zero. For $N_K = 1$, it does so smoothly while $N_K > 1$, there are Z_{N_K} singularities. Here we just consider $N_K = 1$ case. General problem of classical string propagation in KK monopole background is quite difficult to solve and so we will restrict ourselves to considering linearized perturbations about a given string configuration satisfying equations of motion. Our base configuration (about which we want to perturb) is fundamental string wrapped along y-circle and carrying a right moving wave or in other words, supertube in $NS1$-P duality frame. We know that in this case the waveform travels with the speed of light in the y direction. Let us check that this is a solution of our string equations. This time we know the solution in the *static* gauge on the worldsheet:

$$t = b\tilde{\tau}, \quad y = b\tilde{\sigma} \tag{2.13}$$

Writing $\tilde{\xi}^{\pm} = \tilde{\tau} \pm \tilde{\sigma}$ and noting that a right moving wave is a function of $\tilde{\xi}^-$ we expect the following to be a solution

$$t = b\frac{\tilde{\xi}^+ + \tilde{\xi}^-}{2} \quad , \quad y = b\frac{\tilde{\xi}^+ - \tilde{\xi}^-}{2} \quad , \quad X^\mu = x^\mu(\tilde{\xi}^-) \tag{2.14}$$

We see that this satisfies equation of motion. But it doesn't satisfy constraint equations i.e the induced metric on worldsheet is not conformal to flat metric.

$$ds^2 = -b^2 d\tilde{\xi}^+ d\tilde{\xi}^- + G_{\mu\nu}(x'^\mu x'^\nu)(d\tilde{\xi}^-)^2 \tag{2.15}$$

where primes denote differentiation wrt. $\tilde{\xi}^-$. However, as done in [172], we change coordinates to

$$(\xi^+, \xi^-) = (\tilde{\xi}^+ - f(\tilde{\xi}^-), \tilde{\xi}^-) \tag{2.16}$$

with

$$f'(\xi^-) = \frac{G_{\mu\nu} x'^\mu x'^\nu}{b^2} \tag{2.17}$$

Here prime now denotes derivative with respect to ξ^- and index μ denotes directions along taub-nut part of KK monopole. In terms of these new coordinates, we have a conformally flat metric on the worldsheet.

$$ds^2 = -b^2\, d\xi^-\, d\xi^+ \tag{2.18}$$

So configuration

$$t = b\frac{\xi^+ + \xi^- + f(\xi^-)}{2} \quad , \quad y = b\frac{\xi^+ - \xi^- + f(\xi^-)}{2} \quad , \quad x^\mu = x^\mu(\xi^-) \tag{2.19}$$

satisfies the equations of motion. For t, y coordinates, there is separation between left and right movers and hence equation of motion is trivially satisfied. We will take vibrations along torus coordinates to be zero in the base configuration. For coordinates along taub-nut, christoffel symbols are non-zero but since $\partial_+ x^\mu(\xi^-) = 0$, equations of motion are satisfied. This is to be expected since KK monopole is an exact background for string theory. Now we consider linearized perturbations about this configuration

$$X^\mu = x^\mu(\xi^-) + \epsilon Y^\mu(\xi^+, \xi^-) \quad , \quad z_j = \epsilon Z_j(\xi^+, \xi^-) \tag{2.20}$$

where ϵ is a small parameter. We will neglect terms of higher order in ϵ in what follows. In a curved spacetime, left and right movers are mixed and hence Y^μ depends on both $\xi^\pm$. Expanding eqn. 2.9 to first order in ϵ perturbation equation will be

$$\partial_+ \partial_- Y^\mu + \Gamma^\mu_{\nu\rho} \partial_+ Y^\nu \partial_- x^\rho = 0 \tag{2.21}$$

Here christoffel symbols are calculated using zeroth order background metric evaluated for the base configuration and hence it depends only on ξ^-. Hence we have following first order equation after first integration.

$$\partial_- Y^\mu + \Gamma^\mu_{\nu\rho} Y^\nu \partial_- x^\rho = h^\mu(\xi^-) \tag{2.22}$$

For directions z_j, christoffel symbols are zero and hence perturbations are of the form

$$Z_j = Z_-(\xi^-) + Z_+(\xi^+) \tag{2.23}$$

Before trying to solve the equation 2.22 , let us try to see the form of constraint equations 2.10 for these solutions. First Constraint equation becomes

$$0 = G_{AB} \partial_+ X^A \partial_+ X^B = -\frac{b^2}{4} + \frac{b^2}{4} + \epsilon^2 G_{\mu\nu} \partial_+ Y^\mu \partial_+ Y^\nu + \epsilon^2 G_{ji} \partial_+ Z^j \partial_+ Z^i \tag{2.24}$$

and second non-trivial one becomes

$$G_{AB}\partial_- X^A \partial_- X^B = -\frac{b^2}{4}(1+f'(\xi^-))^2 + \frac{b^2}{4}(-1+f'(\xi^-))^2 + \epsilon^2 G_{ji}\partial_+ Z^j \partial_+ Z^i$$
$$+G_{\mu\nu}\partial_- X^\mu \partial_- X^\nu = 0 \quad (2.25)$$

We see that upto first order in ϵ, first constraint is satisfied. Now we manipulate order ϵ terms in second equation a bit to get the constraint on h^μ implied by the second equation. We first expand all the terms into base quantities and perturbations.

$$G_{\mu\nu}\partial_- X^\mu \partial_- X^\nu = (\overline{G}_{\mu\nu} + \epsilon h_{\mu\nu})(x'^\mu + \epsilon Y'^\mu)(x'^\nu + \epsilon Y'^\nu) \quad (2.26)$$

Here $\overline{G}_{\mu\nu}$ is the four dimensional taub-nut part of base metric evaluated for base configuration x^μ and $h_{\mu\nu}$ is the linearized perturbation in metric. Prime denotes derivative with respect to ξ^-. Putting this in constraint equation and considering terms upto order ϵ only, we get

$$-b^2 f'(\xi^-) + \overline{G}_{\mu\nu}x'^\mu x'^\nu + \epsilon[h_{\mu\nu}x'^\mu x'^\nu + 2\overline{G}_{\mu\nu}x'^\mu Y'^\nu] = 0 \quad (2.27)$$

At zeroth order in ϵ, terms vanish by the definition of $f(\xi^-)$. To further massage first order terms, we put Y'^μ from the equation of motion in the constraint equation.

$$[h_{\mu\nu}x'^\mu x'^\nu + 2\overline{G}_{\mu\nu}x'^\mu(-\Gamma^\nu_{\rho\sigma}Y^\rho x'^\sigma + h^\nu(\xi^-)] = 0 \quad (2.28)$$

Here we have set

$$-b^2 f'(\xi^-) + \overline{G}_{\mu\nu}x'^\mu x'^\nu = 0 \quad (2.29)$$

giving $f'(\xi^-)$. Putting the definition of christoffel symbols , we get

$$2\overline{G}_{\mu\nu}x'^\mu(\Gamma^\nu_{\rho\sigma}Y^\rho x'^\sigma) = x'^\mu x'^\sigma Y^\rho \overline{G}_{\mu\nu}\overline{G}^{\nu\alpha}(\partial_\rho \overline{G}_{\alpha\sigma} + \partial_\sigma \overline{G}_{\alpha\rho} - \partial_\alpha \overline{G}_{\rho\sigma}) = x'^\mu x'^\sigma Y^\rho \partial_\rho \overline{G}_{\mu\sigma} \quad (2.30)$$

Putting this in eqn. 2.28 we get

$$x'^{\mu}x'^{\sigma}(h_{\mu\sigma} - Y^{\rho}\partial_{\rho}\overline{G}_{\mu\sigma}) + 2\overline{G}_{\mu\nu}h^{\nu}x'^{\mu} = 0 \tag{2.31}$$

This gives constraint on h^{ν} as first term automatically vanishes. Thus

$$\overline{G}_{\mu\nu}h^{\nu}x'^{\mu} = 0 \tag{2.32}$$

is the final form of constraint equation which we will use later.

8.3 Perturbations for BK string profile

Now we have set up our equations of motion and constraint equations. So we can use these to find linearized perturbations for given base configurations. Our interest is in systems which correspond to supertubes in KK monopole background. Geometries constructed in [173, 190, 175] correspond to such situations. In [197], it is shown that geometry corresponding to Bena-Kraus (BK) metric is generated by considering a particular string profile in KK monopole background and other string profiles give different geometries, generalizing those of [150] (where general geometries correspond to string profile in four dimensional flat space) to the case of 3-charges in four dimensions. Since we finally want to consider supergravity perturbations in Bena-Kraus geometry, it would be necesary to consider same string profile (which generates BK geometry in supergravity limit) as our base configuration about which we add perturbations. So in this section, we determine the profile corresponding to BK geometry in coordinates appropriate for KK monopole background. Since taub-nut space becomes flat space for small distances, we can find the profile corresponding to BK metric by considering the profile near the center of KK monopole. For this we need conversion between flat space coordinates and taub-nut coordinates. Since we have $\cos\theta$ as the gauge field for taub-nut instead of usual $1-\cos\theta$, we give the calculation for our case. Taub-nut metric is

$$ds^2 = \frac{1}{V}(dz + \cos\theta d\phi)^2 + V(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2) \qquad (3.33)$$

Here $V = 1 + \frac{Q}{r}$ and $0 \leq \theta \leq \pi$. The periodicities of angular coordinates are $\delta\phi = 2\pi$ and $\delta z = 2\pi R_K$ where $Q = \frac{1}{2}N_K R_K$. Here R_K is the asymptotic radius of z-circle and N_K is the number of monopoles. For $r << Q$, we have

$$ds^2 \approx Q(\frac{dr^2}{r} + r d\theta^2 + r\sin^2\theta d\phi^2) + \frac{r}{Q}(dz + A\cos\theta d\phi)^2 \qquad (3.34)$$

Now we make change of variables by defining

$$\rho = 2\sqrt{Qr} \quad , \quad \tilde{\theta} = \frac{\theta}{2} \qquad (3.35)$$

Now metric becomes

$$ds^2 = d\rho^2 + \rho^2 d\tilde{\theta}^2 + \frac{\rho^2}{4}\sin^2 2\tilde{\theta} d\phi^2 + \frac{\rho^2}{4Q^2}(dz + Q\cos 2\tilde{\theta} d\phi)^2 \qquad (3.36)$$

$$= d\rho^2 + \rho^2 d\tilde{\theta}^2 + \frac{\rho^2}{4}d\phi^2 + \frac{\rho^2}{4}dz^2 + \frac{\rho^2}{2Q}\cos 2\tilde{\theta} d\phi dz \qquad (3.37)$$

Inserting $1 = sin^2\tilde{\theta} + \cos^2\tilde{\theta}$, we get

$$ds^2 = d\rho^2 + \rho^2[d\tilde{\theta}^2 + \frac{\cos^2\tilde{\theta}}{4}(\frac{1}{Q}dz + \phi)^2 + \frac{\sin^2\tilde{\theta}}{4}(\frac{1}{Q}dz - \phi)^2] \qquad (3.38)$$

We define following combinations

$$2\tilde{\psi} = \frac{1}{Q}dz + \phi \quad , \quad 2\tilde{\phi} = \frac{1}{Q}dz - \phi \qquad (3.39)$$

In terms of these quantities, we have flat metric

$$ds^2 = d\rho^2 + \rho^2[d\tilde{\theta}^2 + \cos^2\tilde{\theta} d\tilde{\psi}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2] \qquad (3.40)$$

In this flat space metric, cartesian coordinates are defined by

$$x_1 = \rho \sin\tilde{\theta}\cos\tilde{\phi},\ x_2 = \rho\sin\tilde{\theta}\sin\tilde{\phi}\ ,\ x_3 = \rho\cos\tilde{\theta}\cos\tilde{\psi}\ ,\ x_4 = \rho\cos\tilde{\theta}\sin\tilde{\psi} \qquad (3.41)$$

BK metric for 3-charges in 4 dimensions is analogous to supertube metric [185, 192, 188, 189] for 2-charges in 5 dimensions. Near the centre of KK monopole, we know that KK monopole metric reduces to flat space as we saw above. There string profile of BK metric must be same as string profile of supertube metric. As show in [189, 178], we have circular profile function

$$F_1 = a\cos\omega v \quad , \quad F_2 = a\sin\omega v \qquad (3.42)$$

for geometry corresponding to simplest supertube. Here $v = t - y$.We see that we have following coordinates for profile function in polar coordinates

$$\rho = a \quad , \quad \tilde{\theta} = \frac{\pi}{2} \quad , \quad \tilde{\phi} = \omega v \qquad (3.43)$$

The value of $\tilde{\psi}$ is indeterminate. To simplify things we take $\tilde{\psi} = \omega v$. In terms of taub-nut coordinates these values translate to

$$r = \frac{a^2}{4Q} \quad , \quad \theta = \pi \quad , \quad \phi = 0 \quad , \quad z = 2Q\omega v \qquad (3.44)$$

Here $\omega = \frac{1}{nR_y}$ for the state we are considering, n being the number of times string winds around y-circle of radius R_y.

8.3.1 Perturbations of BK profile

In section 2, we determined the equation of motion 2.22 and constraint equation 2.32 for a general base configuration given by profile $x^\mu(\xi^-)$ in taub-nut directions. In this subsection, we apply these for the case of BK profile in $NS1$-P duality frame.

From section 2, equation of motion for perturbations Y^A are

$$\partial_- Y^A + \overline{\Gamma}^A_{BC} Y^B \partial_- x^C = h^A(\xi^-) \tag{3.45}$$

In the flat directions z_j, solution is like in flat space i.e

$$Z_j = Z_-(\xi^-) + Z_+(\xi^+) \tag{3.46}$$

$$t = b\frac{\xi^+ + \xi^- + f(\xi^-)}{2} \quad , \quad y = b\frac{\xi^+ - \xi^- + f(\xi^-)}{2} \tag{3.47}$$

where

$$-b^2 f'(\xi^-) + \overline{G}_{\mu\nu} x'^\mu x'^\nu = 0 \tag{3.48}$$

gives $f'(\xi^-)$. In the taub-nut directions, we will only consider perturbations along r and z. Since there are coordinate singularities at $\theta = \pi$, we can work at $\pi - \delta$ and then take [43] limit $\delta \to 0$. In what follows, we will set $z = R_K \psi$ to simplify some calculations. We will need following components of connection in what follows

$$p = \overline{\Gamma}^\psi_{r\psi} = \frac{Q}{2r(Q+r)} \quad , \quad -q = \overline{\Gamma}^r_{\psi\psi} = -\frac{QrR_K^2}{2(Q+r)^3} \tag{3.49}$$

We consider the case where base configuration has non-constant radius. We consider taub-nut directions as perturbations in other directions (whose connection components vanish) are same as above. Base configuration in this case is

$$r = R(\xi^-) \quad , \quad \theta = \pi \quad , \quad \phi = 0 \quad , \quad \psi = \frac{2\omega Q}{R_K}\xi^- = \alpha\xi^- \tag{3.50}$$

We consider perturbations only along r, ψ directions. Then

$$\partial_- \overline{X}^\psi = \alpha \quad , \quad \partial_- \overline{X}^r = R' = \frac{dR(\xi^-)}{d\xi^-} \tag{3.51}$$

[43] Since these coordinates have singularities at $\theta = \pi$, we should change to other coordinate patch to cover the point $\theta = \pi$ (in that patch $\theta = 0$ will have problem). In other coordinate system, similar conclusions follow and hence we will not worry about these spurious singularities any further

Putting these in

$$\partial_- Y^A + \overline{\Gamma}^A_{BC} Y^B \partial_- x^C = h^A(\xi^-) \tag{3.52}$$

we get following two equations for the perturbations

$$\partial_- Y^r + \overline{\Gamma}^r_{rr} Y^r R' + \overline{\Gamma}^r_{\psi\psi} \alpha Y^\psi = h^r(\xi^-) \tag{3.53}$$

$$\partial_- Y^\psi + \overline{\Gamma}^\psi_{r\psi} \alpha Y^r + \overline{\Gamma}^r_{\psi r} R' Y^\psi = h^\psi(\xi^-) \tag{3.54}$$

Apart from equation of motion, we also have constraint equation 2.32 which give following relation between h^r and h^ψ.

$$\overline{G}_{rr} h^r R' + \overline{G}_{\psi\psi} h^\psi \alpha = 0 \tag{3.55}$$

Putting the values of appropriate connection components, we get

$$\partial_- Y^r - \frac{QR'}{2R(Q+r)} Y^r - \frac{QR\alpha}{2(Q+R)^3} Y^\psi = h^r(\xi^-) \tag{3.56}$$

$$\partial_- Y^\psi + \frac{Q}{2R(Q+R)} (\alpha Y^r + R' Y^\psi) = h^\psi(\xi^-) \tag{3.57}$$

Multiplying the first equation by $\sqrt{V}$, dividing the second by $\sqrt{V}$ and using expression 8.3 for V, we can write the two equations as

$$\partial_-(\sqrt{V} Y^r) - \frac{Q\alpha}{2(Q+R)^2} \left(\frac{Y^\psi}{\sqrt{V}} \right) = (\sqrt{V} h^r)(\xi^-) \tag{3.58}$$

$$\partial_- \left(\frac{Y^\psi}{\sqrt{V}} \right) + \frac{Q\alpha}{2(Q+R)^2} (\sqrt{V} Y^r) = \left(\frac{h^\psi}{\sqrt{V}} \right) (\xi^-) \tag{3.59}$$

Defining new dependent and independent variables

$$\tilde{Y}^r = \sqrt{V} Y^r \quad , \quad \tilde{Y}^\psi = \frac{Y^\psi}{\sqrt{V}} \tag{3.60}$$

$$\tilde{\xi} = \frac{Q\alpha}{2} \int \frac{d\xi^-}{(Q+R(\xi^-))^2} \tag{3.61}$$

we get

$$\tilde{\partial}_-(\tilde{Y}^r) - \tilde{Y}^\psi = G^r(\tilde{\xi}) \tag{3.62}$$

$$\tilde{\partial}_-(\tilde{Y}^\psi) + \tilde{Y}^r = G^\psi(\tilde{\xi}) \tag{3.63}$$

where $\tilde{\partial}_-$ denotes derivative with respect to $\tilde{\xi}$ and new arbitrary functions G^r, G^ψ are now expressed as functions of $\tilde{\xi}$. Since we will be needing it later also, let us solve equations of motion in a general form. We can express the above coupled first order inhomogeneous equations as matrix equation

$$\tilde{\partial}\vec{Y} = \mathbf{A}\vec{Y} + \vec{G} \tag{3.64}$$

where $\mathbf{A}$ is a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.65}$$

Solution to matrix equations like 3.64 is found by diagonalizing the matrix $\mathbf{A}$. If λ_j are eigenvalues and $\vec{S}_j$ are eigenvectors, with $j = 1, 2$ then solution is given by

$$\vec{Y} = \sum_j c_j \vec{S}_j e^{\lambda_j \tilde{\xi}} + \sum_j e^{\lambda_j \tilde{\xi}} \vec{S}_j \int e^{-\lambda_j \tilde{\xi}} \tilde{G}_j(\tilde{\xi}) d\tilde{\xi} \tag{3.66}$$

Here $\vec{\tilde{G}} = S^{-1}\vec{G}$ and $S = [S_1\ S_2]$ is the matrix of eigenvectors as column vectors. For our case, we get

$$\begin{pmatrix} \tilde{Y}^r \\ \tilde{Y}^\psi \end{pmatrix} = c_1(\xi^+) \begin{pmatrix} i \\ 1 \end{pmatrix} e^{i\tilde{\xi}} + c_2(\xi^+) \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-i\tilde{\xi}}$$
$$+ e^{i\tilde{\xi}} \begin{pmatrix} i \\ 1 \end{pmatrix} \int e^{-i\tilde{\xi}} \tilde{G}^r(\tilde{\xi}) d\tilde{\xi} + e^{-i\tilde{\xi}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \int e^{i\tilde{\xi}} \tilde{G}^\psi(\tilde{\xi}) d\tilde{\xi} \tag{3.67}$$

Solution can then be combined to schematically write down

$$\tilde{Y}^r + i\tilde{Y}^\psi = B(\xi^+) e^{-i\tilde{\xi}} + G_1(\tilde{\xi}) \tag{3.68}$$

$$\tilde{Y}^r - i\tilde{Y}^\psi = A(\xi^+) e^{i\tilde{\xi}} + G_2(\tilde{\xi}) \tag{3.69}$$

8.4 D0-F1 supertube in KKM background

In previous section, we found perturbed solution corresponding to oscillating string in KK monopole background. We know that this system is dual to usual $D0-NS1$ supertube. In this section, we study supertube in $D0-NS1$ duality frame. Since we are in a non-trivial background (KK monopole), it is not clear how to do dualities required to go from $NS1$-P to $D0$-$NS1$ frame as done in [172]. So we perform calculation of linearized perturbation separately in this duality frame. Static case of $D0$-$NS1$ supertube was considered in [196]. Here we will review their construction for the case of round supertube. In the next subsection we will add perturbations to it.

$D2$ supertube has world-volume coordinates $\sigma^0, \sigma^1, \sigma^2 = \sigma$. We embed supertube in such a way that

$$\sigma^0 = t \quad , \quad \sigma^1 = y \quad , \quad X^\mu = X^\mu(\sigma^2) \tag{4.70}$$

Here X^μ are arbitrary functions of σ. To stabilize the brane against contraction due to brane-tension, we introduce gauge field

$$F = Ed\sigma^0 \wedge d\sigma^1 + B(\sigma^2)d\sigma^1 \wedge d\sigma^2 = Edt \wedge dy + B(\sigma)dy \wedge d\sigma \tag{4.71}$$

For $D2$-brane of tension T_2, Lagrangian is given by

$$\mathcal{L} = -T_2\sqrt{-det[g+F]} = -T_2\sqrt{B^2 + R^2(1-E^2)} \tag{4.72}$$

Here g is induced metric and $R^2 = G_{\mu\nu}X'^\mu X'^\nu$ and prime denotes differentiation wrt σ. Background metric $G_{\mu\nu}$ for KK monopole is given by 2.11. We define electric displacement as

$$\Pi = \frac{\partial \mathcal{L}}{\partial E} = \frac{T_2 E R^2}{\sqrt{B^2 + R^2(1-E^2)}} \tag{4.73}$$

In terms of this, we write hamiltonian density as

$$\mathcal{H} = E\Pi - \mathcal{L} = \frac{1}{R}\sqrt{(R^2 + \Pi^2)(B^2 + R^2)} \tag{4.74}$$

It is easy to see that minimum value for $\mathcal{H}$ is obtained if $T_2 R^2 = \Pi B$ or $E^2 = 1$. These conditions agree with what one gets from supersymmetry analysis. As in flat space, $B(\sigma)$ is an arbitrary function of σ. By the usual interpretation, fluxes above correspond to $D2$ brane carrying both $D0$ and $F1$(along y direction) charges. We are assuming isometry along y-direction. Charges are given by

$$Q_0 = \frac{T_2}{T_0}\int dy d\sigma B(\sigma) \tag{4.75}$$

$$Q_1 = \frac{1}{T_1}\int d\sigma \Pi(\sigma) = \frac{T_2}{T_1}\int d\sigma \frac{ER^2}{\sqrt{B^2 + R^2(1 - E^2)}} \tag{4.76}$$

Round supertube in KK is given by

$$\sigma^0 = t \quad , \quad \sigma^1 = y \quad , \quad R_0 = \frac{a^2}{4Q} \quad , \quad \theta = \pi \quad , \quad \phi = 0 \quad , \quad z = 2Q\omega\sigma \tag{4.77}$$

We have chosen parameters in such a way as to facilitate comparison with $NS1$-P duality frame. In terms of flat space (or near the center of KK monopole), this corresponds to a circular profile in say, (X_1, X_2) plane. We are not perturbing along torus directions. For this configuration

$$R^2 = X'^{\mu} X'_{\mu} = G_{zz}(2Q\omega)^2 \tag{4.78}$$

So the supersymmetry condition gives a relationship between all three charges and the compactification radius. Now consider perturbation of this configuration.

8.4.1 Perturbations in D0-F1 picture

Let R and σ be the radial and angular coordinates in the (X_1, X_2) plane. We choose the gauge $A_t = 0$ for the worldvolume gauge field. Thus the gauge field has the form

$$A = A_\sigma d\sigma + A_y dy \tag{4.79}$$

$$F = Edt \wedge dy + Bdy \wedge d\sigma + \partial_t a_\sigma dt \wedge d\sigma \tag{4.80}$$

We want to study fluctuations around the configuration given by 4.77,4.79 with $\overline{E} = 1$ and $b = \overline{B}(\sigma)$. Lagrangian is given by

$$\mathcal{L} = -T_2\sqrt{-det[g + F]} \tag{4.81}$$

Putting values from 4.77,4.79, we get

$$\mathcal{L} = -T_2\sqrt{(1 - E^2)X'^2 - \dot{X}^2 X'^2 + (\dot{X} \cdot X')^2 - \dot{a}_\sigma^2 + B^2(1 - \dot{X}^2) - 2EB\dot{X} \cdot X'} \tag{4.82}$$

We perturbed as

$$R = R_0 + \epsilon r(\sigma, t) \quad , \quad E = 1 + \epsilon \dot{a}_y \quad , \quad B = b - \epsilon a'_y \quad , Z = \alpha\sigma + \epsilon z \tag{4.83}$$

where lower case quantities denote fluctuations. Field strength becomes

$$F = (1 + \epsilon \dot{a}_y)dt \wedge dy + (b - \epsilon a'_y)dy \wedge d\sigma + \partial_t a_\sigma dt \wedge d\sigma \tag{4.84}$$

We have put perturbations along θ, ϕ directions to be zero. Putting these in Lagrangian and expanding upto second order

$$\frac{\mathcal{L}}{T_2} = L^{(0)} + \epsilon L^{(1)} + \epsilon^2 L^{(2)} \tag{4.85}$$

we find

$$L^{(0)} = -b \tag{4.86}$$

$$L^{(1)} = \frac{1}{b}\left(\frac{\alpha^2 \dot{a}_y}{V_0} + \frac{\alpha b \dot{z}}{V_0} + b\dot{a}_y\right) \tag{4.87}$$

We see that first order perturbation is a total derivative. This follows from the fact that our unperturbed configuration satisfies equations of motion. Term second order in ϵ is given by

$$L^{(2)} = \frac{\alpha^2 + V_0 b^2}{2b}\dot{r}^2 + \frac{\alpha^2(\alpha^2 + V_0 b^2)}{2b^3 V_0^2}\dot{a}_y^2 + \frac{(\alpha^2 + V_0 b^2)}{2bV_0^2}\dot{z}^2 + V_0 \dot{r} r' + \frac{1}{V_0}\dot{z}z' + \frac{\alpha^2}{b^2 V_0}\dot{a}_y a_y'$$

$$+\frac{Q\alpha^2}{bV_0^2 R^2} r\dot{a}_y + \frac{\alpha(\alpha^2 + V_0 b^2)}{V_0^2 b^2}\dot{a}_y \dot{z} + \frac{\alpha Q}{V_0^2 R^2} r\dot{z} + \frac{2\alpha}{bV_0}\dot{a}_y z' \tag{4.88}$$

From this , we get following equations of motion

$$\left(\frac{\alpha^2 + V_0 b^2}{b}\right)\ddot{r} + 2V_0 \partial_\sigma \dot{r} - \frac{\alpha Q}{V_0^2 R^2}(\dot{z} + \frac{\alpha}{b}\dot{a}_y) = 0 \tag{4.89}$$

$$\left(\frac{\alpha(\alpha^2 + V_0 b^2)}{V_0^2 b^2}\right)[\ddot{z} + \frac{\alpha}{b}\ddot{a}_y] + \frac{2\alpha}{V_0 b}\partial_\sigma(\dot{z} + \frac{\alpha}{b}a_y) + \frac{\alpha^2 Q}{bV_0^2 R^2}\dot{r} = 0 \tag{4.90}$$

$$\left(\frac{(\alpha^2 + V_0 b^2)}{V_0^2 b}\right)[\ddot{z} + \frac{\alpha}{b}\ddot{a}_y] + \frac{2}{V_0}\partial_\sigma(\dot{z} + \frac{\alpha}{b}a_y) + \frac{\alpha Q}{V_0^2 R^2}\dot{r} = 0 \tag{4.91}$$

We see that second and third equations are same. If we define $x = z + \frac{\alpha}{b}a_y$ then we have following equations

$$\left(\frac{\alpha^2 + V_0 b^2}{b}\right)\ddot{r} + 2V_0 \partial_\sigma \dot{r} - \frac{\alpha Q}{V_0^2 R^2}\dot{x} = 0 \tag{4.92}$$

$$\left(\frac{\alpha^2 + V_0 b^2}{b}\right)\ddot{x} + 2V_0 \partial_\sigma \dot{x} + \frac{\alpha Q}{R^2}\dot{r} = 0 \tag{4.93}$$

We notice that as in the case of supertube in flat space, we only have time derivatives of field in the equations of motion. Thus any time independent perturbation is a solution, confirming that supertube in KK monopole background also has a family of time independent solutions. Solution to above equations is given by

$$r = c_1(\xi^+)\cos a\sigma + c_2(\xi^+)\sin a\sigma \tag{4.94}$$

$$x = k\left[c_1(\xi^+)\sin a\sigma - c_2(\xi^+)\cos a\sigma\right] \tag{4.95}$$

where

$$\xi^{+} = \frac{2t}{b} - \sigma - \frac{\alpha^2}{V_0 b^2}\sigma \quad , \quad k = -V_0 \quad , \quad a = \frac{\alpha Q}{V_0^2 R^2} \tag{4.96}$$

We see that here perturbations along z direction add with the gauge field and only a combination occurs in equations of motion. This occurs because as in $NS1$-P duality frame, we have only two independent degrees of freedom. It's important to note that frequencies of oscillation agree in both the duality frames, as one expects. Motion is periodic and as in flat space case, we did not find 'drift' modes. So according to conjecture of [172], this would correspond to bound state

8.4.2 Period of oscillation

We had earlier defined static gauge coordinates $\tilde{\tau}$ and $\tilde{\sigma}$ in equation 2.13 and then obtained conformal coordinates ξ^{+}, ξ^{-} from them using equations 2.16,2.17 respectively. For finding the period of oscillation, it would be convenient to take $\tilde{\tau}$ and ξ^{-} as our basic variables. Relationship between target space time t and $\tilde{\tau}$ is just by a simple multiplicative factor b while ξ^{-} gives the parametrization of unperturbed base configuration. So in terms of these, we have

$$\xi^{+} = \tilde{\xi}^{+} - f(\xi^{-}) = 2\tilde{\tau} - \xi^{-} - f(\xi^{-}) \tag{4.97}$$

The time dependence of the solution (4.95) is contained in functions like $A(2\tilde{\tau} - \xi^{-} - f(\xi^{-}))$ and similarly for torus directions $Z_j(2\tilde{\tau} - \xi^{-} - f(\xi^{-}))$. We write

$$\xi^{-} + f(\xi^{-}) = \int_0^{\xi^{-}} d\chi (1 + f'(\chi)) \tag{4.98}$$

So the change in $\xi^{-} + f(\xi^{-})$ when ξ^{-} increases by 2π can be written as $\int_0^{2\pi} d\chi (1 + f'(\chi))$. We then find that the argument of A, Z_j are unchanged when $(\tilde{\tau}, \xi^{-}) \rightarrow$

$(\tilde{\tau} + \Delta\tilde{\tau}, \xi^- + 2\pi)$ with

$$2\Delta\tilde{\tau} - \int_0^{2\pi} \left(1 + \bar{f}'(\bar{\xi}^-)\right) d\bar{\xi}^- = 0 \tag{4.99}$$

Using expression for $f'(\xi^-)$ as given in 2.29, we get

$$\Delta\tilde{\tau} = \frac{1}{2}\int_0^{2\pi} \left(1 + \frac{\overline{G}_{\mu\nu} x'^{\mu} x'^{\nu}}{\bar{b}^2}\right) d\bar{\xi}^- \tag{4.100}$$

For torus directions, situation is similar to flat space case. In case of only torus vibrations, we get back flat space result

$$\Delta t = \frac{1}{2}\left(\frac{M_{D0} + M_{NS1}}{M_{D2}}\right) \tag{4.101}$$

8.5 Profile in 3-d part of KK

Uptill now, we have considered BK profile only. In this section, we consider a different profile which seems natural for KK monopole background. We consider unperturbed profile with $z = \overline{X}_3 = 0$ and perturbations only along X_1, X_2 directions only, with X_1, X_2 being arbitrary functions. First consider the perturbations in $NS1$-P language. Again we use equations of motion 2.22 and constraint equation 2.32 as derived in section 2 previously[44].

$$\partial_- Y^i + \Gamma^i_{jk} Y^{\nu} \partial_- x^k = h^i(\xi^-) \tag{5.102}$$

Relevant christoffel connections in this case are

$$\Gamma^i_{jk} = \frac{1}{2V}\left[(\partial_j V)\delta^i_k + (\partial_k V)\delta^i_j - (\partial_l V)\delta^{il}\delta_{jk}\right] \tag{5.103}$$

In other directions, christoffel symbols are zero and hence perturbation equations are trivial as shown in section 2. Here we concentrate on fluctuations along three

[44] We will denote directions along three dimensional part of taub-nut (i.e excluding fibre direction) by latin letters

dimensional part of taub-nut which is conformal to flat space. We put relevant connection coefficients in perturbation equation for taub-nut directions and get

$$\partial_- Y^i + \frac{1}{2V}\left[(\partial_k V)\partial_- \overline{X}^k Y^i + (\partial_j V) Y^j \partial_- \overline{X}^i - (\partial_l V)\delta^{il}\delta_{jk}\partial_- \overline{X}^k Y^j\right] = h^i(\xi^-) \quad (5.104)$$

Writing $S = lnV$ and

$$w^i = e^{\frac{1}{2}\int(\partial_- S)d\xi^-} Y^i = \sqrt{V} Y^i \quad , \quad \partial_k V \partial_- \overline{X}^k = \partial_- V \quad (5.105)$$

we get

$$\partial_- w^i + \frac{1}{2}[(\partial_j S) w^j \partial_- \overline{X}^i - \partial_l S \delta^{il}\delta_{jk}\partial_- \overline{X}^k w^j] = H^i(\xi^-) \quad (5.106)$$

Here $H^i = \sqrt{V} h^i$. In terms of vector notation, this can be written as

$$\frac{d\vec{w}}{d\xi^-} + \frac{1}{2}\vec{w} \times \vec{B} = H^i(\xi^-) \quad (5.107)$$

where $\vec{B} = \partial_- \vec{X} \times \nabla S$. We first consider a circular profile

$$\overline{X}_1 = R\cos\xi^- \quad , \quad \overline{X}_2 = R\sin\xi^- \quad (5.108)$$

Then we get following equations

$$\frac{dw^{(1)}}{d\xi^-} + \frac{Q}{2VR} w^{(2)} = H^1(\xi^-) \quad (5.109)$$

$$\frac{dw^{(2)}}{d\xi^-} - \frac{Q}{2VR} w^{(1)} = H^1(\xi^-) \quad (5.110)$$

Solution to these equations is

$$w^{(1)} + iw^{(2)} = C_1(\xi^+)e^{-i\alpha\xi^-} + G^{(1)}(\xi^-) \quad (5.111)$$

$$w^{(1)} - iw^{(2)} = C_2(\xi^+)e^{i\alpha\xi^-} + G^{(2)}(\xi^-) \quad (5.112)$$

Here $\alpha = \frac{Q}{2VR}$ and G_1, G_2 are arbitrary functions. Now we consider the case when in base configuration has non-constant radius.

$$\overline{X}_1 = R(\xi^-)\cos\xi^- \quad , \quad \overline{X}_2 = R(\xi^-)\sin\xi^- \quad (5.113)$$

Now putting this in

$$\partial_- w^i + \frac{1}{2}[(\partial_j S) w^j \partial_- \overline{X}^i - \partial_l S)\delta^{il}\delta_{jk}\partial_- \overline{X}^k w^j] = H^i(\xi^-) \tag{5.114}$$

we get same equations

$$\frac{dw^{(1)}}{d\xi^-} + \frac{Q}{2VR} w^{(2)} = H^1(\xi^-) \tag{5.115}$$

$$\frac{dw^{(2)}}{d\xi^-} - \frac{Q}{2VR} w^{(1)} = H^1(\xi^-) \tag{5.116}$$

Only change is that $R = R(\xi^-)$. Terms containing derivatives of R cancel. We can change the independent variable to $\tilde{\xi} = \tilde{\xi}(\xi^-)$ such that

$$\frac{d}{d\xi^-} = \frac{d\tilde{\xi}}{d\xi^-}\frac{d}{d\tilde{\xi}} \quad , \quad \frac{2VR}{Q}\frac{d\tilde{\xi}}{d\xi^-} = 1 \tag{5.117}$$

Then the equation becomes like constant coefficient case. New variable $\tilde{\xi}$ is given by

$$\tilde{\xi} = \frac{Q}{2}\int \frac{d\xi^-}{Q + R(\xi^-)} \tag{5.118}$$

Solution is

$$w^{(1)} + iw^{(2)} = C_1(\xi^+)e^{-i\tilde{\xi}} + G^{(1)}(\tilde{\xi}) \tag{5.119}$$

$$w^{(1)} - iw^{(2)} = C_2(\xi^+)e^{i\tilde{\xi}} + G^{(2)}(\tilde{\xi}) \tag{5.120}$$

Again G_1, G_2 are arbitrary functions.

8.5.1 D0-F1-KK picture

Now we consider same profile in $D0$-$NS1$ duality frame. In polar coordinates, we have

$$R = \overline{R} + \epsilon r \quad , \quad E = 1 + \epsilon \dot{a}_y \quad , \quad B = \overline{B} - \epsilon a'_y \tag{5.121}$$

We have put $X_3 = 0$ or $\theta = \frac{\pi}{2}$. Putting these in Lagrangian 4.82 and as in previous section, expanding upto second order in ϵ, we get

$$L^{(2)} = \frac{-T_2}{2\overline{B}} \Bigg(\dot{r}^2 (V_0^2 \overline{R}^2 + V_0 \overline{B}^2) + \dot{a}_y^2 (V_0 \overline{R}^2 + \frac{V_0^2 \overline{R}^4}{\overline{B}^2}) + \frac{2\dot{a}_y a'_y V_0 \overline{R}^2}{\overline{B}} + (4\overline{R} r V_0 + 2Qr)\dot{a}_y + 2\overline{B} V_0 \dot{r} r' \Bigg) \tag{5.122}$$

From this we get following equations of motion

$$\frac{V_0^2 \overline{R}^2 + V_0 \overline{B}^2}{\overline{B}} \ddot{r} + 2V_0 \partial_t \partial_{\sigma_2} r - \frac{\dot{a}_y}{\overline{B}} (2V_0 \overline{R} - Q) = 0 \tag{5.123}$$

$$\frac{V_0^2 \overline{R}^4 + V_0 \overline{R}^2 \overline{B}^2}{\overline{B}^3} \ddot{a}_y + \frac{2V_0 \overline{R}^2}{\overline{B}^2} \partial_t \partial_{\sigma_2} a_y + \frac{\dot{r}}{\overline{B}} (2V_0 \overline{R} - Q) = 0 \tag{5.124}$$

Simplifying, we get

$$\frac{V_0 \overline{R}^2 + \overline{B}^2}{\overline{B}} \ddot{r} + 2\partial_t \partial_{\sigma_2} r - \frac{2\overline{R} \dot{a}_y}{\overline{B}} (1 - \frac{Q}{2V_0 \overline{R}}) = 0 \tag{5.125}$$

$$\frac{V_0 \overline{R}^2 + \overline{B}^2}{\overline{B}} \ddot{a}_y + 2\partial_t \partial_{\sigma_2} a_y + \frac{2\overline{B} \dot{r}}{\overline{R}} (1 - \frac{Q}{2V_0 \overline{R}}) = 0 \tag{5.126}$$

As in flat space case, we see that only time derivatives of the perturbations r and a_y occur. Hence any static deformation is a solution. Solution to above equations can be written as

$$r = c_1(\xi^+) \cos(1 - \alpha)\sigma + c_2(\xi^+) \sin(1 - \alpha)\sigma \tag{5.127}$$

$$a_y = -\frac{\overline{B}}{\overline{R}} \big(c_1(\xi^+) \cos(1 - \alpha)\sigma + c_2(\xi^+) \sin(1 - \alpha)\sigma \big) \tag{5.128}$$

Here $\alpha = \frac{Q}{2V_0 \overline{R}}$.

8.6 Near ring limit of Bena-Kraus metric

Till now we have considered, DBI description of supertubes in KK monopole background. In this description, string coupling g_s is zero and backreaction of supertube on geometry is not considered. Now we increase g_s so that we have a gravitational description (supergravity). A metric for 3-charge system $D1$-$D5$-KK was given by Bena and Kraus in [173] and generalized to include more dipole charges in [190, 175]. Since this description, corresponds to supertube in KK monopole background, we consider perturbations in this system. The type IIB string frame solution is [173]

$$ds_{10}^2 = \frac{1}{\sqrt{Z_1 Z_5}}\left[-(dt+k)^2 + (dy-k-s)^2\right] + \sqrt{Z_1 Z_5}\, ds_K^2 + \sqrt{\frac{Z_1}{Z_5}}\, ds_{T^4}^2 \quad (6.129)$$

$$e^{\Phi} = \sqrt{\frac{Z_1}{Z_5}} \quad , \quad F^{(3)} = d[Z_1^{-1}(dt+k)\wedge(dy-s-k)] - *_4 dZ_5 \quad (6.130)$$

where $*_4$ is taken with respect to the metric ds_K^2 and

$$ds_K^2 = Z_K(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2) + \frac{1}{Z_K}(R_K d\psi + Q\cos\theta d\phi)^2 \quad (6.131)$$

$$Z_K = 1 + \frac{Q}{r} \quad , \quad Z_{1,5} = 1 + \frac{Q_{1,5}}{\Sigma} \quad , \quad \Sigma = \sqrt{r^2 + R^2 + 2Rr\cos\theta} \quad (6.132)$$

From singularity analysis, bena-kraus derived following periodicity condition also

$$y \cong y + 2\pi R_y \quad , \quad R_y = \frac{2\sqrt{Q_1 Q_5 \tilde{Z}_K}}{n} \quad , \quad \tilde{Z}_K = 1 + \frac{Q}{R} \quad (6.133)$$

One forms s and k have following components

$$s_\psi = -\frac{\sqrt{Q_1 Q_5 \tilde{Z}_K} R_K}{Z_K r \Sigma}\left[\Sigma - r + \frac{r\Sigma}{Q\tilde{Z}_K}\right] \quad (6.134)$$

$$s_\phi = -\frac{\sqrt{Q_1 Q_5 \tilde{Z}_K}}{\Sigma}\left[R - \frac{(\Sigma - \frac{\Sigma}{Z_K} - r)}{Z_K}\cos\theta\right] \quad (6.135)$$

$$k_\psi = \frac{\sqrt{Q_1 Q_5 \tilde{Z}_K} R_K Q}{2R\tilde{Z}_K Z_K r\Sigma}\left[\Sigma - r - R - \frac{2rR}{Q}\right] \quad (6.136)$$

$$k_\phi = -\frac{\sqrt{Q_1 Q_5 \tilde{Z}_K} Q}{2R\tilde{Z}_K \Sigma}\left[\Sigma - r - R + \frac{\Sigma - r + R}{\tilde{Z}_K}\cos\theta\right] \quad (6.137)$$

$$(6.138)$$

Chrages are quantized according to

$$Q = \frac{1}{2} N_K R_K \quad , \quad Q_1 = \frac{(2\pi)^4 g \alpha'^3 N_1}{2 R_K V_4} \quad , \quad Q_5 = \frac{g\alpha' N_5}{2R_K} \tag{6.139}$$

These coordinates are centred at KK monopole. It would be much more arduous task to consider perturbations in full geometry above. So as in [172], we take near ring or thin tube limit of the above geometry. To take the near ring limit, it's better to use coordinates centred on ring. We define new coordinates by

$$\rho = \Sigma \quad , \quad \phi = \phi \quad , \quad \psi = \psi \quad , \quad \cos\theta_1 = \frac{R + r\cos\theta}{\Sigma} \tag{6.140}$$

Another way to write is

$$\rho \sin\theta_1 = r \sin\theta \quad , \quad r^2 = \rho^2 + R^2 - 2R\rho\cos\theta_1 \tag{6.141}$$

It's easy to see that for this change of coordinates

$$dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta = d\rho^2 + \rho^2 d\theta_1^2 + \rho^2 \sin^2\theta_1 d\phi^2 \tag{6.142}$$

We want to take the limit $R \to \infty$ keeping ρ, θ_1 fixed. It's easy to see that under this limit

$$r \sim R \quad , \quad \cos\theta \to -1 \quad , \quad Z_K \to 1 \quad , \quad \tilde{Z}_K \to 1 \tag{6.143}$$

Now $Z_{1,5} = 1 + \frac{Q_{1,5}}{\rho}$ and ds_K^2 becomes

$$(R_K d\psi - Q d\phi)^2 + d\rho^2 + \rho^2 d\theta_1^2 + \rho^2 \sin^2\theta_1 d\phi^2 \tag{6.144}$$

Defining $z = R_K\psi - Q\phi$, we see that this is a metric for $R^3 \times S^1$. One forms becomes

$$k_\psi = -\frac{\sqrt{Q_1Q_5}R_K}{\rho} \tag{6.145}$$

$$k_\phi = \frac{\sqrt{Q_1Q_5}Q}{\rho} \tag{6.146}$$

$$s_\psi = \frac{\sqrt{Q_1Q_5}R_K}{\rho} - \frac{\sqrt{Q_1Q_5}R_K}{Q} \tag{6.147}$$

$$s_\phi = -\frac{\sqrt{Q_1Q_5}Q}{\rho} - \sqrt{Q_1Q_5}\cos\theta_1 \tag{6.148}$$

Now combining them, we get

$$k = k_\psi d\psi + k_\phi d\phi = -\frac{\sqrt{Q_1Q_5}}{\rho}(R_K d\psi - Q d\phi) \tag{6.149}$$

$$k + s = (k_\psi + s_\psi)d\psi + (k_\phi + s_\phi)d\phi = -\frac{\sqrt{Q_1Q_5}R_K}{Q}d\psi - \sqrt{Q_1Q_5}\cos\theta_1 d\phi \tag{6.150}$$

Defining $z = R_K\psi - Q\phi$, we have

$$k = -\frac{\sqrt{Q_1Q_5}}{\rho}dz \quad , \quad -(k+s) = \frac{\sqrt{Q_1Q_5}}{Q}dz + \sqrt{Q_1Q_5}(1-\cos\theta)d\phi \tag{6.151}$$

In terms of these quantities, the metric becomes

$$ds_{10}^2 = \frac{1}{\sqrt{Z_1Z_5}}\left[-(dt - \frac{\sqrt{Q_1Q_5}}{\rho}dz)^2 + (dy + \frac{\sqrt{Q_1Q_5}}{Q}dz + \sqrt{Q_1Q_5}(1-\cos\theta_1)d\phi)^2\right]$$

$$+\sqrt{Z_1Z_5}ds_K^2 + \sqrt{\frac{Z_1}{Z_5}}ds_{T^4}^2 \tag{6.152}$$

Define $\tilde{y} = y + \frac{\sqrt{Q_1Q_5}}{Q}z$ and then we see that it is same as near ring limit of Maldacena-Maoz

$$ds_{10}^2 = \frac{1}{\sqrt{Z_1Z_5}}\left[-(dt - \frac{\sqrt{Q_1Q_5}}{\rho}dz)^2 + (d\tilde{y} + \sqrt{Q_1Q_5}(1-\cos\theta_1)d\phi)^2\right]$$

$$+\sqrt{Z_1Z_5}ds_K^2 + \sqrt{Z_1}Z_5 ds_{T^4}^2 \tag{6.153}$$

KK monopole structure fixes periodicity of $\tilde{y}$.

$$R_y = \frac{2\sqrt{Q_1 Q_5}}{n} \tag{6.154}$$

Now we write down the RR field. We have

$$F^{(3)} = d[Z_1^{-1}(dt + k) \wedge (dy - s - k)] - *_4 dZ_5 \tag{6.155}$$

Since

$$dZ_5 = -\frac{Q_5}{\rho^2} d\rho \tag{6.156}$$

we have

$$*_4 dZ_5 = -\frac{Q_5}{\rho^2}\sqrt{g_4}g^{\rho\rho}\epsilon_{\rho\theta_1\phi z} d\theta_1 \wedge d\phi \wedge dz = -Q_5 \sin\theta_1 d\theta_1 \wedge d\phi \wedge dz \tag{6.157}$$

where we have put $\epsilon_{\rho\theta_1\phi z} = 1$ as in KK monopole space. Since four dimensional base space is $R^3 \times S^1$ we have used flat metric. Writing

$$\sigma = dy + \sqrt{Q_1 Q_5}(1 - \cos\theta_1) \quad , \quad d\sigma = \sqrt{Q_1 Q_5}\sin\theta_1 d\theta_1 \wedge d\phi \tag{6.158}$$

we can write

$$*_4 dZ_5 = -d(Q_5 \sigma \wedge dz) \tag{6.159}$$

So $F^{(3)} = dC^{(2)}$ where

$$C^{(2)} = \frac{1}{Z_1}\left[(dt - \frac{\sqrt{Q_1 Q_5}}{\rho}) \wedge \sigma\right] + Q_5 \sigma \wedge dz \tag{6.160}$$

We see that only effect of KK monopole on supertube geometry is that of compactifying R^4 to $R^3 \times S^1$ with radius of S^1 determined in terms of KK monopole charges. One can easily dualize this to 'thin tube' limit of $NS1$-P system. So the results of [172] involving near ring limit can be taken over for this system. In [172], we only considered fluctuations along torus directions when considering near ring or 'thin tube' limit of $NS1$-P geometry. In our case here, period of oscillation on gravity side will be same as period calculated in [172]. In terms of masses, gravity calculation gave

$$\Delta t = \frac{1}{2T_{NS1}}(M_{NS1} + M_P) \tag{6.161}$$

This is same as equation 4.101 from DBI analysis after one does the dualities. For fluctuations in torus directions in DBI limit, there is no difference between flat background and KK monopole background except that R_0^2 for supertube would be calculated using KK monopole metric. So period of oscillations match. Fluctuations along taub-nut directions are difficult to solve and we postpone that work to a future publication.

8.7 Results and discussion

We studied supertubes in various profiles moving in a KK monopole background. At the DBI level, profile which corresponds to Bena-Kraus metric in gravity limit was analyzed in both $NS1$-P and $D0$-$NS1$ duality frames. We considered perturbations of supertube with this profile and found that motion of supertube in KK monopole background is not a drifting motion but more like quasi-oscillations as considered in [172]. This can be taken as evidence for the bound state nature of system corresponding to BK profile. But since conjecture of [172] was based on flat background geometry (at DBI level), one should be cautious in considering this as definitive for the bound state nature of the system.

Near ring or thin tube limit of $D1$-$D5$-KK turned out to be identical to near ring limit of $D1$-$D5$ supertube alone, only change occuring in the periodicity of the ring circle. In our present case of $D1$-$D5$-KK, the periodicity of the ring is determined by the monopole charge while with just $D1$-$D5$, it could be arbitrary. Calculations of period of oscillation at DBI level, for torus directions, match with the gravity analysis. Both are very similar to $D1$-$D5$ supertube case dealt in [172]. Only difference comes from the fact that in $D1$-$D5$-KK case, radius of supertube is calculated using KK monopole metric rather than flat metric.

Substantial difference from flat space case occurs when one considers form of fluctuation and not just the periodicity. In KK monopole background, even at linear level, there is no separation of dependences on ξ^+ and ξ^- and thus left-movers and right-movers are invariably mixed. This effect is due to curvature of background. We also analyzed, at DBI level, perturbations which have profile functions different from BK profile even though in these cases no gravity description is known and so can not be compared with DBI analysis.

It would be interesting to analyze KK-P system constructed in [193] and discussed in previous chapter using the linearized perturbation formalism as developed in the present chapter. Since we know that KK-P is a bound state, it can give us insight about how the conjecture of [172] works in presence of KK monopoles. One can also work with other three charge systems which do not contain KK monopole like $D1$-$D5$-P system constructed in [176, 177] and are known to be bound states.

APPENDIX A

T-DUALITY FORMULAE

In this thesis, we perform T dualities following the notation of [168]. Let us summarize the relevant formulae. We call the T-duality direction s. For NS–NS fields, one has

$$G'_{ss} = \frac{1}{G_{ss}}, \qquad e^{2\Phi'} = \frac{e^{2\Phi}}{G_{ss}}, \qquad G'_{\mu s} = \frac{B_{\mu s}}{G_{ss}}, \qquad B'_{\mu s} = \frac{G_{\mu s}}{G_{ss}}$$
$$G'_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu s}G_{\nu s} - B_{\mu s}B_{\nu s}}{G_{ss}}, \qquad B'_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu s}G_{\nu s} - G_{\mu s}B_{\nu s}}{G_{ss}}, \quad \text{(A.1)}$$

while for the RR potentials we have:

$$C'^{(n)}_{\mu\ldots\nu\alpha s} = C^{(n-1)}_{\mu\ldots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|s}G_{|\alpha]s}}{G_{ss}}, \quad \text{(A.2)}$$

$$C'^{(n)}_{\mu\ldots\nu\alpha\beta} = C^{(n+1)}_{\mu\ldots\nu\alpha\beta s} + nC^{(n-1)}_{[\mu\ldots\nu\alpha}G_{\beta]s} + n(n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|s}B_{|\alpha|s}G_{|\beta]s}}{G_{ss}}. \quad \text{(A.3)}$$

APPENDIX B

GARFINKLE-VACHASPATI TRANSFORM

Wave-generating transform found by Garfinkle and Vachaspati belongs to the class of generalized Kerr-Schild transformations. If one has a vector field k_μ which has following properties

$$k^\mu k_\mu = 0,\ k_{\mu;\nu} + k_{\nu;\mu} = 0,\ k_{\mu;\nu} = \frac{1}{2}(k_\mu A_{,\nu} - k_\nu A_{,\mu}) \qquad \text{(B.1)}$$

where A is some scalar function and covariant derivatives are with respect to some base metric $g_{\mu\nu}$. Then one has a new metric

$$g'_{\mu\nu} = g_{\mu\nu} + e^A \Phi k_\mu k_\nu \qquad \text{(B.2)}$$

which describes a gravitational wave travelling on the original metric provided matter fields satisfy some conditions and the function Φ satisfies

$$\nabla^2 \Phi = 0\ ,\ k^\mu \partial_\mu \Phi = 0 \qquad \text{(B.3)}$$

Nullity of the vector field allows us to 'linearize' the Einstein equations. By employing additional conditions (killing, hypersurface-orthogonality) on the vector field, Garfinkle and Vachaspati found that Einstein equations reduce to simple harmonicity of a scalar function and some conditions on the matter field. Authors of [160] discuss conditions on matter fields in the context of low energy effective action in string theory.

Consider the action

$$S = \int d^D x \sqrt{-g} \left(R - \frac{1}{2} \sum_a h_a(\phi)(\nabla \phi_a)^2 - \frac{1}{2} \sum_p f_p(\phi) F^2_{(p+1)} \right) \tag{B.4}$$

Here we have included a set of scalar fields ϕ_a with arbitrary (non-derivative) couplings $h_a(\phi)$ and $f_p(\phi)$. Degree of p-forms appearing depends on whether we are in type IIA or type IIB theory. Since we want the vector field k to yield an invariance of the full solution, we impose the following conditions on the matter fields

$$L_k \phi_a = k^\mu \partial_\mu \phi_a = 0 \tag{B.5}$$

$$L_k F_{(p+1)} = (di_k + i_k d) F_{(p+1)} = di_k F_{(p+1)} = 0 \tag{B.6}$$

where L_k denotes Lie-derivative with respect to vector field k and i_k denotes interior product. In the second equation, we have used the identity $L_k = di_k + i_k d$ and also the Bianchi identity $dF_{p+1)} = 0$ for forms. We also require a transversality condition

$$i_k F_{(p+1)} = k \wedge \theta_{(p-1)} \tag{B.7}$$

where $p-1$ form $\theta_{(p-1)}$ necessarily satisfies $i_k \theta_{(p-1)}$ since $i_k^2 F_{(p+1)} = 0$. This transversality condition ensures that the operation of raising and lowering the indices does not change the $p+1$ form field strength. With these conditions, the matter field equations of motion remain unchanged. Hence if the set (g, ϕ_a, A_p) is a solution to supergravity equations then so is (g', ϕ_a, A_p). Note are that all this is in Einstein frame but it can be rephrased in string frame very easily. Only change is that if Einstein and string metrics are related by

$$g^S_{ab} = e^C g^E_{ab} \tag{B.8}$$

then Laplacian condition above becomes

$$\partial_\mu (e^{\frac{(2-D)C}{2}} \sqrt{g^S} g_S^{\mu\nu} \partial_\nu \Phi) = 0 \tag{B.9}$$

Our null, killing vector is $(\frac{\partial}{\partial u})^a$. Since nothing depends on u, v and it is a light-like direction, it is obvious that this is null and killing. Since there is no mixing between u, v and other terms, this is also hypersurface orthogonal with $e^A = H^{-1}$. To see this we explicitly check hypersurface-orthogonality condition. We have $k^u = 1$ and $k_v = g_{uv}k^u = g_{uv}$ as the only non-zero component. We use this in the hypersurface orthogonality condition

$$\partial_\nu k_\mu - \Gamma^\lambda_{\nu\mu} k_\lambda = \frac{1}{2}(k_\nu \partial_\mu A - k_\mu \partial_\nu A) \tag{B.10}$$

Now we consider various cases. We use the fact that nothing depends on u or v. We have following connection components which we will need.

$$\Gamma^v_{u\nu} = 0 \ , \ \Gamma^v_{v\nu} = \frac{1}{2}\partial_\nu \ln g_{uv} \ , \ \Gamma^v_{iv} = \frac{1}{2}\partial_i \ln g_{uv} \tag{B.11}$$

For $\mu = u$, we see that hypersurface orthogonality condition is trivially satisfied as all the terms vanish on both sides. For $\mu = i$, we only have non-zero terms for $\nu = v$ and in that case

$$-\frac{1}{2}\partial_i \ln g_{uv} k_v = \frac{1}{2} k_v \partial_\mu A \tag{B.12}$$

This gives $e^A = g^{uv} = (g_{uv})^{-1}$. From the other case $\mu = v$, we get the same value for A and hence equations are consistent. With this value, we get

$$e^A k_\mu k_\nu dx^\mu dx^\nu = \frac{1}{g_{uv}} g_{uv} dv g_{uv} dv = g_{uv} dv^2 \tag{B.13}$$

So new metric is

$$ds^2 = -(dudv + Tdv^2) + H^{-1}dx_i dx_i + H(ds + V_j dx^j)^2 \tag{B.14}$$

We need to solve Laplace equation in the Taub-NUT geometry. Since derivatives with respect to u or v and along torus directions are zero, we have only Laplace equation

$$\frac{1}{\sqrt{g_{TN}}}\partial_i(\sqrt{g_{TN}} g^{ij} \partial_j T) = 0 \tag{B.15}$$

For Taub-NUT metric, we have

$$\sqrt{g_{TN}} = H^{-1} r^2 \sin\theta \ , \ g^{rr} = H \ , \ g^{\theta\theta} = \frac{H}{r^2} \ , \qquad \text{(B.16)}$$

$$g^{\phi\phi} = \frac{H}{r^2 \sin^2\theta} \ , \ g^{s\phi} = -\frac{HQ_K \cos\theta}{r^2 \sin^2\theta} = g^{\phi s} \ , \ g^{ss} = H^{-1} + \frac{HQ_K^2 \cot^2\theta}{r^2} \qquad \text{(B.17)}$$

Using these, we write down Laplace equation for T as

$$\partial_r(Hr^2 \sin\theta \frac{1}{H} \partial_r T) + \partial_\theta(Hr^2 \sin\theta \frac{1}{Hr^2} \partial_\theta T) + \partial_s(Hr^2 \sin\theta \frac{(H^{-2}r^2 + Q_K^2 \cot^2\theta}{Hr^2} \partial_s T)$$

$$+ \partial_\phi^2(H^{-1} r^2 \sin\theta \frac{1}{H^{-1} r^2 \sin^2\theta} T) - 2\partial_\phi \partial_s(H^{-1} r^2 \sin\theta \frac{HQ_K \cos\theta}{r^2 \sin^2\theta} T) = 0 \qquad \text{(B.18)}$$

Dividing by $\sqrt{g_{TN}}$, we get

$$\frac{1}{r^2}\partial_r(r^2 \partial_r T) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta \partial_\theta T) + \frac{1}{\sin^2\theta}(Q_K^2 \partial_s^2 T + \partial_\phi^2 T - 2Q_K \cos\theta \partial_\phi \partial_s T)\right)$$

$$+ \frac{1}{r^2}(H^{-2} r^2 - Q_K^2)\partial_s^2 T = 0 \qquad \text{(B.19)}$$

If we assume that $\partial_s T = 0$ then we simply get three dimensional Laplace equation whose solution is given in the main part of the paper.

We will also need a theorem proved in [160] which says that the scalar curvature invariants of metrics $g_{\mu\nu}$ and $g'_{\mu\nu}$ in Garfinkle-Vachaspati transform are exactly identical.

APPENDIX C

SPHERICAL HARMONICS ON S^3

In this Appendix we list the explicit forms of the various spherical harmonics encountered in the solutions presented in chapter 3. The metric on the unit 3-sphere is

$$ds^2 = d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2 \tag{A.1}$$

The harmonics will be orthonormal

$$\begin{aligned} \int d\Omega\ (Y^{I_1})^* Y^{I_1'} &= \delta^{I_1,I_1'} \\ \int d\Omega\ (Y_a^{I_3})^* Y^{I_3' a} &= \delta^{I_3,I_3'} \end{aligned} \tag{A.2}$$

In order to simplify notation we have used the following abbreviations

$$\hat{Y}^{(l)} \equiv Y^{(l,l)}_{(l,l)} \tag{A.3}$$

$$Y^{(l)} \equiv Y^{(l,l)}_{(l-1,l)} \tag{A.4}$$

$$Y^{(l+1)} \equiv Y^{(l+1,l+1)}_{(l-1,l)} \tag{A.5}$$

$$Y^{(l+1,l)}_a \equiv Y^{(l+1,l)}_{a(l-1,l)} \tag{A.6}$$

$$Y^{(l,l+1)}_a \equiv Y^{(l,l+1)}_{a(l-1,l)} \tag{A.7}$$

$$Y^{(l-1,l)}_a \equiv Y^{(l-1,l)}_{a(l-1,l)} \tag{A.8}$$

$$Y^{(l+2,l+1)}_a \equiv Y^{(l+2,l+1)}_{a(l-1,l)} \tag{A.9}$$

$$Y^{(l+1,l+2)}_a \equiv Y^{(l+1,l+2)}_{a(l-1,l)} \tag{A.10}$$

C.1 Scalar Harmonics

The scalar harmonics we use are (in explicit form)

$$\hat{Y}^{(l)} = \sqrt{\frac{2l+1}{2}\frac{e^{-2il\phi}}{\pi}}\sin^{2l}\theta \tag{A.11}$$

$$Y^{(l)} = -\frac{\sqrt{l(2l+1)}}{\pi}e^{-i(2l-1)\phi+i\psi}\sin^{2l-1}\theta\cos\theta \tag{A.12}$$

$$Y^{(l+1)} = \frac{\sqrt{(2l+1)(2l+3)}}{2\pi}e^{-i(2l-1)\phi+i\psi}((l-1)+(l+1)\cos 2\theta)\sin^{2l-1}\theta\cos\theta \tag{A.13}$$

C.2 Vector Harmonics

The vector harmonics are given by

$$Y_\theta^{(l+1,l)} = -\frac{e^{-i(2l-1)\phi+i\psi}}{4\pi}\frac{\sin^{2l-2}\theta}{\sqrt{l+1}}\left((2l^2-l+1)+(l-1)(2l+1)\cos 2\theta\right) \quad (A.14)$$

$$Y_\psi^{(l+1,l)} = i\frac{e^{-i(2l-1)\phi+i\psi}}{4\pi}\frac{\sin^{2l-1}\theta\cos\theta}{\sqrt{l+1}}\left((2l^2+3l-1)+(l+1)(2l+1)\cos 2\theta\right) \quad (A.15)$$

$$Y_\phi^{(l+1,l)} = -i\frac{e^{-i(2l-1)\phi+i\psi}}{4\pi}\frac{\sin^{2l-1}\theta\cos\theta}{\sqrt{l+1}}\left((2l^2-5l-1)+(2l^2+3l+1)\cos 2\theta\right) \quad (A.16)$$

$$Y_\theta^{(l,l+1)} = -\frac{e^{-i(2l-1)\phi+i\psi}}{4\pi}\sqrt{\frac{4l(2l+1)}{l+1}}\sin^{2l-2}\theta\left((l-1)+l\cos 2\theta\right) \quad (A.17)$$

$$Y_\psi^{(l,l+1)} = i\frac{e^{-i(2l-1)\phi+i\psi}}{4\pi}\sqrt{\frac{4l(2l+1)}{l+1}}\sin^{2l-1}\theta\cos\theta\left(l+(l+1)\cos 2\theta\right) \quad (A.18)$$

$$Y_\phi^{(l,l+1)} = i\frac{e^{-i(2l-1)\phi+i\psi}}{4\pi}\sqrt{\frac{4l(2l+1)}{l+1}}\sin^{2l-1}\theta\cos\theta\left((l+2)+(l+1)\cos 2\theta\right) \quad (A.19)$$

$$Y_\theta^{(l-1,l)} = \frac{e^{-i(2l-1)\phi+i\psi}}{2\pi}\sqrt{2l-1}\sin^{2l-2}\theta \quad (A.20)$$

$$Y_\psi^{(l-1,l)} = -i\frac{e^{-i(2l-1)\phi+i\psi}}{2\pi}\sqrt{2l-1}\sin^{2l-1}\theta\cos\theta \quad (A.21)$$

$$Y_\phi^{(l-1,l)} = -i\frac{e^{-i(2l-1)\phi+i\psi}}{2\pi}\sqrt{2l-1}\sin^{2l-1}\theta\cos\theta \quad (A.22)$$

$$\begin{aligned} Y_\theta^{(l+2,l+1)} &= -\frac{e^{-i(2l-1)\phi+i\psi}}{8\pi}\sqrt{\frac{3}{l+2}}\sin^{2l-2}\theta\Big[(l-1)(2l^2+l+1) \\ &\quad +\frac{2(4l^3-l+3)\cos 2\theta}{3}+\frac{(l-1)(l+1)(2l+3)\cos 4\theta}{3}\Big] \quad (A.23) \end{aligned}$$

$$\begin{aligned} Y_\psi^{(l+2,l+1)} &= -i\frac{e^{-i(2l-1)\phi+i\psi}}{4\pi}\sqrt{\frac{3}{l+2}}\sin^{2l-1}\theta\cos\theta\Big[\frac{l(2l^2+5l-1)}{2} \\ &\quad +\frac{1}{3}(l+1)(4l^2+8l-3)\cos 2\theta+\frac{(l+1)(l+2)(2l+3)}{6}\cos 4\theta\Big] \quad (A.24) \end{aligned}$$

$$\begin{aligned} Y_\phi^{(l+2,l+1)} &= i\frac{e^{-i(2l-1)\phi+i\psi}}{4\pi}\sqrt{\frac{3}{l+2}}\sin^{2l-1}\theta\cos\theta\Big[\frac{(2l^3-3l^2+3l+4)}{2} \\ &\quad +\frac{1}{3}(4l^3-13l-9)\cos 2\theta+\frac{(l+1)(l+2)(2l+3)}{6}\cos 4\theta\Big] \quad (A.25) \end{aligned}$$

APPENDIX D

SOLUTION – INNER REGION

In this appendix, we give the solution for the inner region for calculations in chapter 3.The supergravity equations are expressed in terms of the fields B_{MN} and w. It is convenient to divide the B_{MN} into three classes – $B_{ab}, B_{\mu a}$ and $B_{\mu\nu}$ where B_{ab} is an antisymmetric tensor on S^3, $B_{\mu a}$ is a vector on S^3 and $B_{\mu\nu}$ is a scalar on S^3. At a given order ϵ^n, the corrections to B_{ab} and $B_{\mu\nu}$ at that order can be expressed in terms of a single scalar field b and the antisymmetric tensor $t_{\mu\nu}$:

$$B_{ab} = \epsilon_{abc} e^{-2il\frac{a}{Q}t} \partial^c b \tag{B.1}$$

$$B_{\mu\nu} = \frac{r}{Q} \tilde{\epsilon}_{\mu\nu\lambda} \partial^\lambda \left(e^{-2il\frac{a}{Q}t} b \right) + e^{-2il\frac{a}{Q}t} t_{\mu\nu} \tag{B.2}$$

Here ϵ_{abc} is the usual Levi-Civita tensor on the unit S^3 (with $\epsilon_{\theta\psi\phi} = \sqrt{g}$), while $\tilde{\epsilon}_{\mu\nu\lambda}$ is the Levi-Civita tensor *density* on the t, y, r part of the metric (2.10); thus $\tilde{\epsilon}_{tyr} = 1$. Below we will give the values of b and $t_{\mu\nu}$ at each order in the perturbation. The 1-forms B_{ta}, B_{ya} and B_{ra} will be given explicitly. To avoid cumbersome notation we do not put labels on the fields indicating the order of perturbation; rather we list the order of all fields in the subsection heading.

In this Appendix the solutions are in the NS sector coordinates. In order to compare with the outside we need to spectral flow these to the R sector using the coordinate transformation

$$\psi_{NS} = \psi - \frac{a}{Q}y, \quad \phi_{NS} = \phi - \frac{a}{Q}t \tag{B.3}$$

The perturbation expansion in the NS sector coordinates has only even powers of ϵ. The spectral flow (B.3) to R sector coordinates generates odd powers in ϵ. Thus the $O(\epsilon^0)$ NS sector computation gives $O(\epsilon^0), O(\epsilon^1)$ in the R coordinates.

The solution to a given order ϵ^n is given by the sum of the corrections at all orders $\leq n$.

D.1 Leading Order ($O(\epsilon^0) \to O(\epsilon^0), O(\epsilon^1)$)

$$b = \frac{1}{4l}\frac{1}{(r^2+a^2)^l}Y^{(l)}_{NS} \tag{B.4}$$

$$w = \frac{1}{Q(r^2+a^2)^l}Y^{(l)}_{NS}e^{-2il\frac{a}{Q}t} \tag{B.5}$$

$$B_{ta} = B_{ya} = B_{ra} = 0 \tag{B.6}$$

$$t_{\mu\nu} = 0 \tag{B.7}$$

D.2 Second Order ($O(\epsilon^2) \to O(\epsilon^2), O(\epsilon^3)$)

$$b = \frac{a^2}{Q(r^2+a^2)^l}\left[\frac{(3l-1)-2l(l+1)\cos^2\theta}{4l(l+1)^2}\right]Y_{NS}^{(l)} \tag{B.8}$$

$$w = -\frac{1}{Q(r^2+a^2)^l}\frac{r^2+a^2\cos^2\theta}{Q}Y_{NS}^{(l)}e^{-2il\frac{a}{Q}t} \tag{B.9}$$

$$\begin{aligned} B_{ta} = & -\frac{ia}{Q^2(r^2+a^2)^l}\left[\left(\sqrt{\frac{l}{(l+1)^5(2l+1)}}\right)\left[(2l+1)a^2+(l+1)^2r^2\right](Y_a^{(l+1,l)})_{NS}\right. \\ & +\left(\frac{1}{2(l+1)^2}\sqrt{\frac{1}{(l+1)}}\right)\left[(3l+1)a^2+(l+1)^2r^2\right](Y_a^{(l,l+1)})_{NS} \\ & \left.-\left(\frac{1}{4l}\sqrt{\frac{2l-1}{l(2l+1)}}\right)\left[a^2+2lr^2\right](Y_a^{(l-1,l)})_{NS}\right]e^{-2il\frac{a}{Q}t} \end{aligned} \tag{B.10}$$

$$\begin{aligned} B_{ya} = & -\frac{ia}{Q^2(r^2+a^2)^l}\left[\left(\sqrt{\frac{l}{(l+1)(2l+1)}}\right)r^2(Y_a^{(l+1,l)})_{NS}-\frac{1}{2\sqrt{l+1}}r^2(Y_a^{(l,l+1)})_{NS}\right. \\ & \left.-\left(\frac{1}{4l}\sqrt{\frac{2l-1}{l(2l+1)}}\right)\left[a^2+2lr^2\right](Y_a^{(l-1,l)})_{NS}\right]e^{-2il\frac{a}{Q}t} \end{aligned} \tag{B.11}$$

$$\begin{aligned} B_{ra} = & \frac{a^2}{Q(r^2+a^2)^{l+1}}\left[\left(\sqrt{\frac{l^5}{(l+1)^5(2l+1)}}\right)r(Y_a^{(l+1,l)})_{NS}+\left(\frac{l(l-1)}{2(l+1)^{\frac{5}{2}}}\right)r(Y_a^{(l,l+1)})_{NS}\right. \\ & \left.-\left(\frac{1}{4l}\sqrt{\frac{2l-1}{l(2l+1)}}\right)\frac{1}{r}\left[a^2+2lr^2\right](Y_a^{(l-1,l)})_{NS}\right]e^{-2il\frac{a}{Q}t} \end{aligned} \tag{B.12}$$

$$t_{ty} = \frac{r^2}{Q^3(r^2+a^2)^l}\left[\left(\frac{(2l+1)a^2+(l+1)^2r^2}{(l+1)^2}\right)Y_{NS}^{(l)}+a^2\frac{l}{(l+1)^2}\sqrt{\frac{l}{(2l+3)}}Y_{NS}^{(l+1)}\right] \tag{B.13}$$

$$\begin{aligned} t_{yr} = & i\frac{ar}{Q^2(r^2+a^2)^{l+1}}\left(\frac{(l^2+2l-1)a^2-(l^2-1)(2l-1)r^2}{2l(l+1)^2}\right)Y_{NS}^{(l)} \\ & -i\frac{a^3r}{Q^2(r^2+a^2)^{l+1}}\frac{l}{(l+1)^2}\sqrt{\frac{l}{(2l+3)}}Y_{NS}^{(l+1)} \end{aligned} \tag{B.14}$$

$$t_{tr} = i\frac{ar}{Q^2(r^2+a^2)^l}\frac{l-1}{2l(l+1)}Y_{NS}^{(l)} \tag{B.15}$$

APPENDIX E

SOLUTION – OUTER REGION

As was done for the inner region in previous appendix, we divide the field B_{MN} into three classes – $B_{ab}, B_{\mu a}$ and $B_{\mu\nu}$. At a given order ϵ^n, the corrections to B_{ab} and $B_{\mu\nu}$ at that order can be expressed in terms of a single scalar field b and the antisymmetric tensor $t_{\mu\nu}$:

$$\begin{aligned} B_{ab} &= e^{-i\frac{a}{Q}u}\epsilon_{abc}\partial^c b \\ B_{\mu\nu} &= \left(\frac{r}{Q+r^2}\tilde{\epsilon}_{\mu\nu\lambda}\partial^\lambda b + t_{\mu\nu}\right)e^{-i\frac{a}{Q}u} \end{aligned} \tag{C.1}$$

Again ϵ_{abc} is the Levi-Civita tensor on the unit S^3 while $\tilde{\epsilon}_{\mu\nu\lambda}$ is the Levi-Civita tensor *density* on the t, y, r part of the metric (2.22); thus $\tilde{\epsilon}_{tyr} = 1$. We give $b, t_{\mu\nu}$ at each order. We also write

$$\begin{aligned} B_{\mu a} &= e^{-i\frac{a}{Q}u}b_{\mu a} \\ w &= e^{-i\frac{a}{Q}u}\tilde{w} \end{aligned} \tag{C.2}$$

We will give $b_{\mu a}, \tilde{w}$ at each order.

The solution to a given order ϵ^n is given by the sum of the corrections at all orders $\leq n$.

E.1 Leading Order ($O(\epsilon^0)$)

$$
\begin{aligned}
b &= \frac{1}{4l}\frac{1}{r^{2l}}Y^{(l)} && \text{(C.3)}\\
\tilde{w} &= \frac{1}{r^{2l}(Q+r^2)}Y^{(l)} && \text{(C.4)}\\
b_{ta} &= b_{ya} = b_{ra} = 0 && \text{(C.5)}\\
t_{\mu\nu} &= 0 && \text{(C.6)}
\end{aligned}
$$

E.2 First Order ($O(\epsilon^1)$)

$$
\begin{aligned}
b &= \tilde{w} = 0 && \text{(C.7)}\\
b_{ua} &= \frac{ia}{2}\sqrt{\frac{l}{(2l+1)(l+1)}}\frac{Q}{r^{2l}(Q+r^2)^2}Y_a^{(l+1,l)} \\
&\quad -\frac{ia}{4}\frac{1}{r^{2l}}\sqrt{\frac{2l-1}{l(2l+1)}}\frac{Q}{(Q+r^2)^2}Y_a^{(l-1,l)} + \frac{ia}{4Qr^{2l}}\sqrt{\frac{4l^2-1}{l^3}}Y_a^{(l-1,l)} && \text{(C.8)}\\
b_{va} &= i\frac{a}{4}\sqrt{\frac{1}{(l+1)}}\frac{Q}{r^{2l}(Q+r^2)^2}Y_a^{(l,l+1)} && \text{(C.9)}
\end{aligned}
$$

$$
\begin{aligned}
t_{ty} &= 0 && \text{(C.10)}\\
t_{rt} &= ia\left(\frac{Q}{r^{2l+1}(Q+r^2)^3}\frac{[(l+2)r^2+lQ]}{4l(l+1)} - \frac{1}{4lQr^{2l+1}}\right)Y^{(l)} && \text{(C.11)}\\
t_{yr} &= ia\left(\frac{(2l-1)Q}{r^{2l+1}(Q+r^2)^3}\frac{[(l+2)r^2+lQ]}{4l(l+1)} + \frac{1}{4lQr^{2l+1}}\right)Y^{(l)} && \text{(C.12)}\\
& && \text{(C.13)}
\end{aligned}
$$

E.3 Second Order ($O(\epsilon^2)$)

$$b = \frac{a^2}{r^{2l}}\left(-\frac{1}{4r^2}+\frac{2Q+r^2}{(Q+r^2)^2}\left(\frac{(3l-1)-2l(l+1)\cos^2\theta}{8l(l+1)^2}\right)\right)Y^{(l)} \tag{C.14}$$

$$\tilde{w} = \frac{a^2}{r^{2l}(Q+r^2)}\left(-\frac{l}{r^2}-\frac{\cos^2\theta}{(Q+r^2)}\right)Y^{(l)} \tag{C.15}$$

$$\begin{aligned} b_{ra} &\equiv b_r^{I_3}Y_a^{I_3} = \frac{a^2}{2r^{2l+1}(Q+r^2)^3}\left(2l^2Q^2+3l(l+1)Qr^2+l(l+1)r^4\right) \\ &\times\left[\frac{\sqrt{l}Y_a^{(l+1,l)}}{\sqrt{(2l+1)(l+1)^5}}+\frac{l-1}{2l(l+1)^{\frac{5}{2}}}Y_a^{(l,l+1)}-\frac{1}{2l^2}\sqrt{\frac{2l-1}{l(2l+1)}}Y_a^{(l-1,l)}\right] \end{aligned} \tag{C.16}$$

$$b_{ua} = b_{va} = 0 \tag{C.17}$$

$$\begin{aligned} t_{ty} &= -\frac{a^2}{4l(l+1)^2r^{2l}(Q+r^2)^5}\left[l(l+1)(2l+3)Q^3\right. \\ &\left.+\,l(6l^2+9l+7)Q^2r^2+(6l^3+4l^2+l+3)Qr^4+(2l^3-l+1)r^6\right]Y^{(l)} \\ &+\frac{a^2Q}{2r^{2l}(l+1)^2(Q+r^2)^5}\sqrt{\frac{l}{2l+3}}\left[(l+1)Q^2+(3l+1)Qr^2+2(l+1)r^4\right]Y^{(l)} \end{aligned} \tag{C.18}$$

$$t_{yr} = 0, \quad t_{rt} = 0 \tag{C.19}$$

E.4 Third Order ($O(\epsilon^3)$)

$$b = \tilde{w} = b_{ra} = 0 \tag{C.20}$$

$$\begin{aligned} b_{ua} &= \left(\frac{ia^3Q}{2(Q+r^2)^3r^{2l}(l+1)(2l+3)}\sqrt{\frac{3l(2l+1)}{l+2}}\right)Y_a^{(l+2,l+1)} \\ &+ \left(\frac{ia^3Q}{2(Q+r^2)^2r^{2l}(l+1)^{\frac{3}{2}}}\left[\frac{(l-1)(2Q+r^2)}{4Q^2(l+1)} - \frac{2l}{(2l+3)(Q+r^2)}\right]\right)Y_a^{(l,l+1)} \\ &- \frac{ia^3}{4Q(Q+r^2)^3r^{2l+2}}\sqrt{\frac{l}{(l+1)^5(2l+1)}}\left[(4l^2+2l+4)Q^2r^2 + (6l^2-3l)Qr^4 + \right. \\ &\left.(2l^2-l)r^6 + 2l(Q+r^2)\left((l+1)^2Q^2 - 2lQr^2 - lr^4\right)\right]Y_a^{(l+1,l)} \\ &+ \frac{ia^3}{Qr^{2l+2}}\sqrt{\frac{2l-1}{l(2l+1)}}\left(-\frac{1}{4(Q+r^2)^2}\left((l+1)(Q^2+4Qr^2+2r^4)\right) + \frac{r^2}{4lQ}\right. \\ &\left.+ \frac{r^2}{8l(l+1)(Q+r^2)^3}\left(2(2l^2+3l-1)Q^2 + (3Qr^2+r^4)(2l^2+l-1)\right)\right)Y_a^{(l-1,l)} \end{aligned} \tag{C.21}$$

$$\begin{aligned} b_{va} &= -\left(\frac{iQa^3}{2(Q+r^2)^3r^{2l}}\sqrt{\frac{l(2l+1)}{l+1)}}\frac{1}{(2l+3)(l+1)}\right)Y_a^{(l+1,l)} \\ &- \left(\frac{iQa^3}{8\sqrt{(l+1)^3}(Q+r^2)^3r^{2l+2}}\left[2l(l+1)Q + \left((l+1) + (2l^2+l+3)\right)r^2\right]\right)Y_a^{(l,l+1)} \\ &+ \left(\frac{iQa^3}{2(Q+r^2)^3r^{2l}}\frac{1}{(2l+3)}\sqrt{\frac{4l}{(l+1)(l+2)}}\right)Y_a^{(l+1,l+2)} \end{aligned} \tag{C.22}$$

$$\begin{aligned} t_{yr} &= -\left(\frac{ia^3Q(2l-1)\left(lQ+(l+3)r^2\right)}{r^{2l+1}(Q+r^2)^4l(l+1)^2} + \frac{ia^3}{4r^{2l+3}Q(Q+r^2)^3l}\times\right. \\ &\left.\left[(l+1)(Q+r^2)^3 + l(2l-1)Q^3 + \frac{l(2l-1)(l+3)Q^2r^2}{l+1}\right]\right)Y^{(l)} \\ &+ \frac{ia^3Q\left(lQ+(l+3)r^2\right)}{2r^{2l+1}(Q+r^2)^4}\left(\frac{(2l-1)}{(l+1)^2(l+2)}\sqrt{\frac{l}{(2l+3)}}\right)Y^{(l+1)} \end{aligned} \tag{C.23}$$

$$
\begin{aligned}
t_{rt} &= \left(-\frac{ia^3Q\,(lQ+(l+3)r^2)}{r^{2l+1}(Q+r^2)^4 l(l+1)^2} + \frac{ia^3}{4r^{2l+3}Q(Q+r^2)^3}\times \right. \\
&\left. \left[1 - \frac{((l^2-1)Q^3 + (l^2-3)Q^2r^2 - 3(l+1)Qr^4 - (l+1)r^6)}{l(l+1)(Q+r^2)^3}\right]\right) Y^{(l)} \\
&+ \left(\frac{ia^3Q\,(lQ+(l+3)r^2)}{2r^{2l+1}(Q+r^2)^4}\frac{1}{(l+1)^2(l+2)}\sqrt{\frac{l}{(2l+3)}}\right) Y^{(l+1)} \qquad \text{(C.24)} \\
t_{ty} &= 0 \qquad \text{(C.25)}
\end{aligned}
$$

APPENDIX F

ANALYSIS OF THE $\mathcal{A}^+$ EQUATIONS

In this appendix we look at the field equations (4.18) for the field $\mathcal{A}^+$. We have found the expected time-independent solutions in section (6.4.2). We will now consider a general ansatz for the solution and argue that there are no time *dependent* solutions for this field, if we demand consistency with the long wavelength limit (4.30).

Let us write

$$\mathcal{A}_v^+ = H^{-1}\,a_v^+\,, \quad \mathcal{A}_t^+ = a_t^+\,, \quad \mathcal{A}_z^+ = a_z^+\,, \quad \mathcal{A}_i^+ = a_i^+ \tag{A.1}$$

Since we have spherical symmetry in the space spanned by the coordinates i all fields will be functions only of the radial coordinate r in this space; further, the $\mathcal{A}_i^+$ can have only the component $\mathcal{A}_r^+$. Putting this ansatz into (4.18) we obtain the coupled system of equations (we list the equations in the order $\lambda = t, v, z, i$)

$$\triangle\, a_t^+ + \partial_t^2 a_v^+ - \partial_t\partial_i a_i^+ + \partial_z(\partial_z a_t^+ - \partial_t a_z^+) + A\,\partial_t(\partial_z a_t^+ - \partial_t a_z^+) = 0 \tag{A.2}$$

$$\begin{aligned}&\triangle\, a_v^+ + \partial_z^2 a_v^+ - [(H\tilde{K} - A^2)\,\partial_t^2 a_v^+ - 2A\,\partial_t\partial_z a_v^+]\\ &+[\partial_i(H\tilde{K} - A^2)\partial_i a_t^+ - 2\partial_i A\,\partial_i a_z^+] - [\partial_i(H\tilde{K} - A^2)\partial_t a_i^+ - 2\partial_i A\,\partial_z a_i^+] = 0\end{aligned} \tag{A.3}$$

$$\triangle\, a_z^+ + \partial_t \partial_z a_v^+ - \partial_z \partial_i a_i^+ + (H\tilde{K} - A^2)\, \partial_t (\partial_z a_t^+ - \partial_t a_z^+)$$
$$-A\, \partial_z (\partial_z a_t^+ - \partial_t a_z^+) = 0 \qquad \text{(A.4)}$$

$$\partial_t \partial_i a_v^+ - \partial_z \partial_i a_z^+ + \partial_z^2 a_i^+ - \partial_j (\partial_i a_j^+ - \partial_j a_i^+) + \partial_i A\, (\partial_z a_t^+ - \partial_t a_z^+)$$
$$+[(H\tilde{K} - A^2)\, \partial_t \partial_i a_t^+ - A\, \partial_z \partial_i a_t^+ - A\, \partial_t \partial_i a_z^+]$$
$$-[(H\tilde{K} - A^2) \partial_t^2 a_i^+ - 2A \partial_t \partial_z a_i^+] = 0 \qquad \text{(A.5)}$$

We look for a gauge field $\mathcal{A}^+$ having the same z and t dependence as $\mathcal{A}^-$. We write

$$a_v^+ = e^{ik\, z - i\omega\, t}\, f_v(r)\,, \quad a_t^+ = e^{ik\, z - i\omega\, t}\, f_t(r)$$
$$a_z^+ = e^{ik\, z - i\omega\, t}\, f_z(r)\,, \quad a_r^+ = ie^{ik\, z - i\omega\, t}\, \partial_r \Lambda \qquad \text{(A.6)}$$

where the notation we have used for a_r^+ will be helpful in what follows.

Consider first equations (A.2) and (A.4):

$$\triangle\, f_t - \omega^2\, f_v - \omega\, \triangle \Lambda - (k\, f_t + \omega\, f_z)\, (k - \omega\, A) = 0$$
$$\triangle\, f_z + \omega k\, f_v + k\, \triangle\, \Lambda + (k\, f_t + \omega\, f_z)\, [k\, A + \omega\, (H\tilde{K} - A^2)] = 0 \qquad \text{(A.7)}$$

Taking k times the first equation plus ω times the second we get

$$\triangle\, (k\, f_t + \omega\, f_z) - [k^2 - 2\omega k\, A - \omega^2 (H\tilde{K} - A^2)]\, (k\, f_t + \omega\, f_z) = 0 \qquad \text{(A.8)}$$

We note that the coefficient $k^2 - 2\omega k\, A - \omega^2 (H\tilde{K} - A^2)$ has the form $c + d/r$. If $d \neq 0$ then by an argument similar to that leading to (4.58) we find that there is no solution with the required behavior at $r = 0$. Setting $d = 0$ tells us that we need to have the same relation between ω and k that we have seen before (4.58)

$$\omega = -k\,\frac{2\sqrt{Q_1 Q_p}}{Q_1 + Q_p} \tag{A.9}$$

Using (A.9) one finds

$$k^2 - 2\omega k\,A - \omega^2(H\tilde{K} - A^2) = k^2\left(\frac{Q_1 - Q_p}{Q_1 + Q_p}\right)^2 \;\equiv\; \tilde{k}^2 \tag{A.10}$$

and thus

$$\triangle\,(k\,f_t + \omega\,f_z) - \tilde{k}^2\,(k\,f_t + \omega\,f_z) = 0 \tag{A.11}$$

The solution of the above equation that converges at infinity is

$$k\,f_t + \omega\,f_z = \tilde{c}\,\frac{e^{-|\tilde{k}|r}}{r} \tag{A.12}$$

Let us now look at equation (A.3). Using (A.9) and (A.12) we find

$$\begin{aligned}
&[(H\tilde{K} - A^2)\,\partial_t^2 a_v^+ - 2A\,\partial_t\partial_z a_v^+] = -\omega^2\,f_v \\
&[\partial_i(H\tilde{K} - A^2)\partial_i a_t^+ - 2\partial_i A\,\partial_i a_z^+] = -\frac{Q_1 + Q_p}{r^2}\,\frac{\tilde{c}}{k}\,\partial_r\Big(\frac{e^{-|\tilde{k}|r}}{r}\Big) \\
&[\partial_i(H\tilde{K} - A^2)\partial_t a_i^+ - 2\partial_i A\,\partial_z a_i^+] = 0
\end{aligned} \tag{A.13}$$

Then equation (A.3) becomes

$$\triangle f_v - \tilde{k}^2\,f_v - \frac{Q_1 + Q_p}{r^2}\,\frac{\tilde{c}}{k}\,\partial_r\Big(\frac{e^{-|\tilde{k}|r}}{r}\Big) = 0 \tag{A.14}$$

We find that unless

$$\tilde{c} = 0 \tag{A.15}$$

f_v will be too singular to agree with (4.30) at small r. The vanishing of $\tilde{c}$ also makes (A.12) agree with (4.30) at small r.

From (A.14) and using the short distance limit implied by (4.30) we get

$$f_v = (\tilde{\alpha} - \tilde{\beta})\, Q_1\, \frac{e^{-|\tilde{k}| r}}{r} \tag{A.16}$$

Consider now the last equation (A.5) (for $i = r$, the only non-trivial component). The fact that $\tilde{c} = 0$ implies

$$\partial_z a_t^+ - \partial_t a_z^+ = 0 \tag{A.17}$$

Moreover one has

$$\begin{aligned} &[(H\tilde{K} - A^2)\, \partial_t \partial_r a_t^+ - A\, \partial_z \partial_r a_t^+ - A\, \partial_t \partial_r a_z^+] = -i\omega\, \partial_r f_t \\ &[(H\tilde{K} - A^2) \partial_t^2 a_r^+ - 2A \partial_t \partial_z a_r^+] = -i\omega^2 \partial_r \Lambda \end{aligned} \tag{A.18}$$

Equation (A.5) then gives

$$\partial_r f_v + \left(1 - \frac{k^2}{\omega^2}\right) \partial_r (f_t - \omega\, \Lambda) = 0 \tag{A.19}$$

Equation (A.2) also simplifies to

$$\triangle\, (f_t - \omega\, \Lambda) - \omega^2\, f_v = 0 \tag{A.20}$$

We see that (A.19) implies (A.20). But at this stage we would like to compare the solution we have found with the limits required by (4.30). First consider the case $Q_1 = Q_p$. Then $\omega = k$ and (A.19) is not compatible with (A.16). Now consider $Q_1 \neq Q_p$. Eq. (A.19) implies

$$f_t - \omega\, \Lambda = \frac{\omega^2}{k^2 - \omega^2}\, f_v + \text{const.} \tag{A.21}$$

This is again incompatible with the limit (4.30), according to which $f_t - \omega\, \Lambda$ should *vanish* for small r. We conclude that there are no time-dependent solutions for $\mathcal{A}^+$ consistent with the limit (4.30).

F.1 Asymptotic behavior of the perturbation

In this part we will study the behavior of perturbations on the 2-charge geometries near spatial infinity. We wish to see how fields fall off with $\bar{r}$, and in particular to check that they carry energy flux off to infinity. We will first look at a scalar field to get an idea of the behaviors involved, and then address the 1-form gauge field that is actually excited in our problem.

Consider first the case in which the perturbation is represented by a minimally coupled scalar Ψ. We take the metric (4.1) and look at its large $\bar{r}$ limit. We denote by $\bar{r}$, θ, ϕ, ψ the spherical coordinates in the 4 noncompact dimensions $\bar{x}_i$ $(i = 1, \ldots, 4)$. The 5D Einstein metric is

$$ds^2 = -h^{-4/3}\, dt^2 + h^{2/3}\, (d\bar{r}^2 + \bar{r}^2\, d\theta^2 + \bar{r}^2\, \sin^2\theta\, d\phi^2 + \bar{r}^2\, \cos^2\theta\, d\psi^2) \tag{B.1}$$

with

$$h = \sqrt{\Big(1 + \frac{\bar{Q}_1}{\bar{r}^2}\Big)\Big(1 + \frac{\bar{Q}_p}{\bar{r}^2}\Big)} \tag{B.2}$$

We set momentum along the S^1 to zero and expand in angular harmonics

$$\Psi = e^{-i\omega\, t}\, \mathcal{R}(\bar{r})\, Y^{(l)}(\theta, \phi, \psi) \tag{B.3}$$

where $Y^{(l)}$ the l-th scalar spherical harmonic. The wave equation for Ψ

$$\Box\Psi = 0 \tag{B.4}$$

implies

$$\frac{1}{\bar{r}^3}\partial_{\bar{r}}(\bar{r}^3\partial_{\bar{r}}\mathcal{R}) + \omega^2\, \Big(1 + \frac{\bar{Q}_1 + \bar{Q}_p}{\bar{r}^2} + \frac{\bar{Q}_1\bar{Q}_p}{\bar{r}^4}\Big)\mathcal{R} - \frac{l(l+2)}{\bar{r}^2}\, \mathcal{R} = 0 \tag{B.5}$$

We can understand the behavior of $\mathcal{R}$ at large $\bar{r}$ as follows. If we define

$$\mathcal{R} = \frac{\tilde{\mathcal{R}}}{\bar{r}^{3/2}} \tag{B.6}$$

the equation for $\tilde{\mathcal{R}}$ is

$$\partial_{\bar r}^2 \tilde{\mathcal{R}} + \omega^2 \tilde{\mathcal{R}} + \frac{(\bar Q_1 + \bar Q_p)\omega^2 - l(l+2) - 3/4}{\bar r^2}\tilde{\mathcal{R}} + \frac{\bar Q_1 \bar Q_p}{\bar r^4}\omega^2 \tilde{\mathcal{R}} = 0 \tag{B.7}$$

At leading order in $1/\bar r$ the terms proportional to $1/\bar r^2$ and $1/\bar r^4$ can be neglected and we have the solution

$$\tilde{\mathcal{R}} = r_+ e^{i\omega \bar r} + r_- e^{-i\omega \bar r} \quad \Rightarrow \quad \mathcal{R} = \frac{r_+ e^{i\omega \bar r} + r_- e^{-i\omega \bar r}}{\bar r^{3/2}} \tag{B.8}$$

which corresponds to traveling waves carrying a nonzero flux. When the terms of higher order in $1/\bar r$ are included, (B.7) can be recursively solved as a formal power series in $1/\bar r$:

$$\mathcal{R} \;\approx\; r_+ \frac{e^{i\omega \bar r}}{\bar r^{3/2}} \left(1 + \sum_{n=1}^{\infty} \frac{r_+^{(n)}}{\bar r^n}\right) + r_- \frac{e^{-i\omega \bar r}}{\bar r^{3/2}} \left(1 + \sum_{n=1}^{\infty} \frac{r_-^{(n)}}{\bar r^n}\right) \tag{B.9}$$

The coefficients $r_\pm^{(n)}$ in this expansion are determined by the recursion relation

$$r_\pm^{(n)} = \mp \frac{i}{2\omega}\left[r_\pm^{(n-1)}\left(n - 1 + \frac{(\bar Q_1 + \bar Q_p)\omega^2 - l(l+2) - 3/4}{n}\right) + r_\pm^{(n-3)}\frac{\bar Q_1 \bar Q_p\, \omega^2}{n}\right] \tag{B.10}$$

The $r_\pm^{(n)}$ are finite for any value of ω and any n. However, since at large n one has

$$\frac{r_\pm^{(n)}}{r_\pm^{(n-1)}} \approx \mp \frac{i(n-1)}{2\omega} \tag{B.11}$$

the series (B.9) has zero radius of convergence. Equations like (B.5) lead instead to asymptotic series in $1/r$ [88], and we expect that the above expansion is to be interpreted as an asymptotic series, which accurately describes the behavior of $\mathcal{R}$ at sufficiently large $\bar r$. From (B.9) we can still conclude that the perturbation $\mathcal{R}$ radiates a finite amount of flux at infinity. Note that, in order to avoid logarithms in the expansion (B.9), it is crucial that the next to leading corrections to the equation (B.7) are of order $1/\bar r^2$.

We find a similar situation for the case in which the perturbation is represented by a vector on the six dimensional space $\mathbb{R}^{(5,1)} \times S^1$. As we showed in section (6.4) the perturbation on the 2-charge system is a vector field with wave equation

$$\nabla_\mu (e^{-2\Phi}\, F^{\pm\,\mu\lambda}) \pm \frac{1}{2} e^{-2\Phi}\, H^{\mu\nu\lambda}\, F^{\pm}_{\mu\nu} = 0 \tag{B.12}$$

The gauge fields $\mathcal{A}^+_\mu$ and $\mathcal{A}^-_\mu$ represent respectively BPS and non-BPS perturbations. Since we are interested in time-dependent, non-BPS, perturbations, we will only look at the equation for $\mathcal{A}^-_\mu$ in this section. For the metric, dilaton and B-field appearing in (B.12) we will take the large $\bar{r}$ limits

$$\begin{aligned} ds^2 &= \bar{H}^{-1}\,[-dt^2 + dy^2 + \bar{K}\,(dt - dy)^2] + d\bar{r}^2 + \bar{r}^2\, d\theta^2 + \bar{r}^2\, \sin^2\theta\, d\phi^2 + \bar{r}^2\, \cos^2\theta\, d\psi^2 \\ B &= -(\bar{H}^{-1} - 1)\, dt \wedge dy\,, \quad e^{2\Phi} = \bar{H}^{-1} \end{aligned} \tag{B.13}$$

with

$$\bar{H} = 1 + \frac{\bar{Q}_1}{\bar{r}^2}\,, \quad \bar{K} = \frac{\bar{Q}_p}{\bar{r}^2} \tag{B.14}$$

The spherical symmetry of the background (B.13) allows us to expand the vector field components into spherical harmonics: Denoting by $Y^{(l)}$ and $Y^{(l)}_\alpha$ the scalar and vector spherical harmonics on S^3, we can write

$$\mathcal{A}^-_I = e^{-i\omega\, t}\, \mathcal{R}_I(\bar{r})\, Y^{(l)}(\theta,\phi,\psi)\,, \quad \mathcal{A}^-_\alpha = e^{-i\omega\, t}\, [\mathcal{R}_s(\bar{r})\, \partial_\alpha Y^{(l)}(\theta,\phi,\psi) + \mathcal{R}_v(\bar{r})\, Y^{(l)}_\alpha(\theta,\phi,\psi)] \tag{B.15}$$

with $I = t, y, \bar{r}$ and $\alpha = \theta, \phi, \psi$. We will need the following spherical harmonic identities

$$\Box' Y^{(l)} = -l(l+2)\, Y^{(l)}\,, \quad \Box' Y^{(l)}_\alpha = (2 - (l+1)^2)\, Y^{(l)}_\alpha \equiv -c(l)\, Y^{(l)}_\alpha\,, \quad \nabla'^\alpha Y^{(l)}_\alpha = 0 \tag{B.16}$$

where "primed" quantities refer to the metric on an S^3 of unit radius. (We use a notation in which $l = 0, 1, \ldots$ for the scalar harmonics and $l = 1, 2, \ldots$ for the vector harmonics). The components with $\lambda = \alpha$ in (B.12) give

$$\frac{1}{\bar r}\,\partial_I\Big[\bar r\, g^{IJ}\,\partial_J(e^{-i\omega\, t}\,\mathcal{R}_v)\Big] - \frac{c(l)+2}{\bar r^2}\,e^{-i\omega\, t}\,\mathcal{R}_v = 0$$

$$\frac{1}{\bar r}\,\partial_I\Big[\bar r\, g^{IJ}\,\Big(e^{-i\omega\, t}\,\mathcal{R}_J - \partial_J(e^{-i\omega\, t}\,\mathcal{R}_s)\Big)\Big] = 0 \tag{B.17}$$

and the components with $\lambda = I$ give

$$\frac{1}{\bar r^3}\partial_K\Big[\bar r^3\, g^{KL}\, g^{IJ}\Big(\partial_L(e^{-i\omega\, t}\,\mathcal{R}_J) - \partial_J(e^{-i\omega\, t}\,\mathcal{R}_L)\Big)\Big] \tag{B.18}$$

$$-\frac{g^{IJ}\, l(l+2)}{\bar r^2}\Big(e^{-i\omega\, t}\,\mathcal{R}_J - \partial_J(e^{-i\omega\, t}\,\mathcal{R}_s)\Big) - \epsilon^{IJK}\frac{\bar Q_1}{\bar r^3}\Big(\partial_J(e^{-i\omega\, t}\,\mathcal{R}_K) - \partial_K(e^{-i\omega\, t}\,\mathcal{R}_J)\Big) = 0$$

with $\epsilon^{\bar r t y} = 1$. As expected from group theory considerations, the component $\mathcal{R}_v$ decouples from all others, while $\mathcal{R}_s$ and $\mathcal{R}_I$ satisfy a coupled system of differential equations. We want to show that, in spite of these mixings, $\mathcal{R}_v$, $\mathcal{R}_s$ and $\mathcal{R}_I$ admit an $1/\bar r$ expansion analogous to (B.9). Putting in the explicit value of g_{IJ} in (B.17) and (B.18) and using the gauge

$$\mathcal{A}^-_t = 0 \tag{B.19}$$

we obtain the following system of equations

$$\frac{1}{\bar r}\partial_{\bar r}(\bar r\partial_{\bar r}\mathcal{R}_v) + \omega^2\Big(1 + \frac{\bar Q_1 + \bar Q_p}{\bar r^2} + \frac{\bar Q_1 \bar Q_p}{\bar r^4}\Big)\,\mathcal{R}_v - \frac{c(l)+2}{\bar r^2}\,\mathcal{R}_v = 0 \tag{B.20}$$

$$\frac{1}{\bar r}\partial_{\bar r}(\bar r\partial_{\bar r}\mathcal{R}_s) + \omega^2\Big(1 + \frac{\bar Q_1 + \bar Q_p}{\bar r^2} + \frac{\bar Q_1 \bar Q_p}{\bar r^4}\Big)\mathcal{R}_s - \frac{1}{\bar r}\partial_{\bar r}(\bar r\mathcal{R}_{\bar r}) - i\omega\frac{\bar Q_p}{\bar r^2}\Big(1 + \frac{\bar Q_1}{\bar r^2}\Big)\mathcal{R}_y = 0 \tag{B.21}$$

$$\frac{l(l+2)}{\bar r^2}\,(\partial_{\bar r}\mathcal{R}_s - \mathcal{R}_{\bar r}) + \omega^2\Big(1 + \frac{\bar Q_1 + \bar Q_p}{\bar r^2} + \frac{\bar Q_1 \bar Q_p}{\bar r^4}\Big)\,\mathcal{R}_{\bar r}$$

$$-i\omega\,\frac{\bar Q_p}{\bar r^2}\Big(1 + \frac{\bar Q_1}{\bar r^2}\Big)\,\partial_{\bar r}\mathcal{R}_y + 2i\omega\,\frac{\bar Q_1}{\bar r^3}\,\mathcal{R}_y = 0 \tag{B.22}$$

$$\frac{1}{\bar r^3}\partial_{\bar r}(\bar r^3\mathcal{R}_{\bar r}) - \frac{l(l+2)}{\bar r^2}\,\mathcal{R}_s + i\omega\,\frac{\bar Q_p}{\bar r^2}\Big(1 + \frac{\bar Q_1}{\bar r^2}\Big)\,\mathcal{R}_y$$

$$-\frac{2}{\bar r^3}\,\Big(1 + \frac{\bar Q_1}{\bar r^2}\Big)^{-1}(\bar Q_1 + \bar Q_p)\,\Big(\mathcal{R}_{\bar r} - \frac{i}{\omega}\,\partial_{\bar r}\mathcal{R}_y\Big) = 0 \tag{B.23}$$

$$\frac{1}{\bar{r}^3}\partial_{\bar{r}}(\bar{r}^3\partial_{\bar{r}}\mathcal{R}_y) - \frac{l(l+2)}{\bar{r}^2}\mathcal{R}_y + \omega^2\Big(1 + \frac{\bar{Q}_1 + \bar{Q}_p}{\bar{r}^2} + \frac{\bar{Q}_1\bar{Q}_p}{\bar{r}^4}\Big)\mathcal{R}_y$$
$$-\frac{2}{\bar{r}^3}\Big(1+\frac{\bar{Q}_1}{\bar{r}^2}\Big)^{-1}(\bar{Q}_1 - \bar{Q}_p)\Big(i\omega\,\mathcal{R}_{\bar{r}} + \partial_{\bar{r}}\mathcal{R}_y\Big) = 0 \qquad \text{(B.24)}$$

(Eq. (B.22) is the $I = \bar{r}$ component of (B.18); eqs. (B.23) and (B.24) are linear combinations of the $I = t, y$ components of (B.18).)

Eq. (B.20) for $\mathcal{R}_v$ is analogous to eq. (B.7): it thus admits an analogous asymptotic expansion, of the form

$$\mathcal{R}_v \approx r_{v,+}\frac{e^{i\omega\,\bar{r}}}{\bar{r}^{1/2}}\Big(1 + \sum_{n=1}^{\infty}\frac{r_{v,+}^{(n)}}{\bar{r}^n}\Big) + r_{v,-}\frac{e^{-i\omega\,\bar{r}}}{\bar{r}^{1/2}}\Big(1 + \sum_{n=1}^{\infty}\frac{r_{v,-}^{(n)}}{\bar{r}^n}\Big) \qquad \text{(B.25)}$$

When expressed in local orthonormal coordinates, the contribution of $\mathcal{R}_v$ to $\mathcal{A}^-$ is of the type

$$\mathcal{A}^-_{\hat{\alpha}} \sim \frac{e^{\pm i\omega\,\bar{r}}}{\bar{r}^{3/2}}(1 + O(\bar{r}^{-1})) \qquad \text{(B.26)}$$

and thus it again gives rise to a wave carrying finite flux at infinity.

The remaining eqs. (B.21-B.24) are four relations for the three unknowns $\mathcal{R}_s$, $\mathcal{R}_{\bar{r}}$ and $\mathcal{R}_y$: this is so because we have used gauge invariance to eliminate one unknown, $\mathcal{R}_t$. It then must be that only three of the four eqs. (B.21-B.24) are linearly independent, and indeed one can check that eq. (B.23), for example, follows from (B.21) and (B.22). We are thus left to solve the coupled system of equations (B.21), (B.22) and (B.24). We can do this by using the following strategy: solve eq. (B.22) for $\mathcal{R}_{\bar{r}}$ and substitute into (B.21) and (B.24), which can then be solved for $\mathcal{R}_s$ and $\mathcal{R}_y$, iteratively in $1/\bar{r}$. To make the behavior at large $\bar{r}$ more transparent we also write $\mathcal{R}_s$ and $\mathcal{R}_y$ as

$$\mathcal{R}_s = \frac{\tilde{\mathcal{R}}_s}{\bar{r}^{1/2}}\,, \quad \mathcal{R}_y = \frac{\tilde{\mathcal{R}}_y}{\bar{r}^{3/2}} \qquad \text{(B.27)}$$

We find

$$\mathcal{R}_{\bar{r}} = -\frac{l(l+2)}{\omega^2}\frac{\partial_{\bar{r}}\tilde{\mathcal{R}}_s}{\bar{r}^{5/2}} + \frac{1}{\bar{r}^{7/2}}\Big(\frac{l(l+2)}{2\omega^2}\tilde{\mathcal{R}}_s + i\frac{\bar{Q}_p}{\omega}\,\partial_{\bar{r}}\tilde{\mathcal{R}}_y\Big) + O(\bar{r}^{-9/2}) \qquad \text{(B.28)}$$

and

$$
\begin{aligned}
&\partial_{\bar{r}}^2\tilde{\mathcal{R}}_s + \omega^2\,\tilde{\mathcal{R}}_s + \frac{\omega^2(\bar{Q}_1+\bar{Q}_p)+1/4}{\bar{r}^2}\,\tilde{\mathcal{R}}_s + \frac{l(l+2)}{\omega^2\,\bar{r}^2}\,\partial_{\bar{r}}^2\tilde{\mathcal{R}}_s + O(\bar{r}^{-3}) = 0\\
&\partial_{\bar{r}}^2\tilde{\mathcal{R}}_y + \omega^2\,\tilde{\mathcal{R}}_y + \frac{\omega^2(\bar{Q}_1+\bar{Q}_p)-l(l+2)-3/4}{\bar{r}^2}\,\tilde{\mathcal{R}}_y + O(\bar{r}^{-3}) = 0 \qquad \text{(B.29)}
\end{aligned}
$$

In (B.28) and (B.29) we have organized the powers of $1/\bar{r}$ by assuming that $\tilde{\mathcal{R}}_s$, $\tilde{\mathcal{R}}_y$ and their $\bar{r}$-derivatives are of order $\bar{r}^0$: By looking at (B.29), we see that this assumption is actually implied by the equations themselves. Note also that the next to leading corrections in (B.29) are of oder $1/\bar{r}^2$. We can thus conclude that $\mathcal{R}_s$ and $\mathcal{R}_y$ have the form

$$
\begin{aligned}
\mathcal{R}_s &\approx \frac{e^{i\omega\,\bar{r}}}{\bar{r}^{1/2}}\sum_{n=0}^{\infty}\frac{r_{s,+}^{(n)}}{\bar{r}^n} + \frac{e^{-i\omega\,\bar{r}}}{\bar{r}^{1/2}}\sum_{n=0}^{\infty}\frac{r_{s,-}^{(n)}}{\bar{r}^n}\\
\mathcal{R}_y &\approx \frac{e^{i\omega\,\bar{r}}}{\bar{r}^{3/2}}\sum_{n=0}^{\infty}\frac{r_{y,+}^{(n)}}{\bar{r}^n} + \frac{e^{-i\omega\,\bar{r}}}{\bar{r}^{3/2}}\sum_{n=0}^{\infty}\frac{r_{y,-}^{(n)}}{\bar{r}^n} \qquad \text{(B.30)}
\end{aligned}
$$

Analogously to the case of the scalar perturbation, the coefficients $r_{s,\pm}^{(n)}$ and $r_{y,\pm}^{(n)}$ for $n > 1$ are recursively determined from $r_{s,\pm}^{(0)}$ and $r_{y,\pm}^{(0)}$, and are finite for any value of ω and any finite n. Substituting in (B.28) we have the solution for $\mathcal{R}_{\bar{r}}$

$$
\mathcal{R}_{\bar{r}} \approx \frac{e^{i\omega\,\bar{r}}}{\bar{r}^{5/2}}\sum_{n=0}^{\infty}\frac{r_{r,+}^{(n)}}{\bar{r}^n} + \frac{e^{-i\omega\,\bar{r}}}{\bar{r}^{5/2}}\sum_{n=0}^{\infty}\frac{r_{r,-}^{(n)}}{\bar{r}^n} \qquad \text{(B.31)}
$$

where $r_{\bar{r},\pm}^{(n)}$ are determined in terms of $r_{s,\pm}^{(n)}$ and $r_{y,\pm}^{(n)}$ (the leading coefficients $r_{\bar{r},\pm}^{(0)}$ vanish for $l = 0$).

The components $\mathcal{R}_s$ and $\mathcal{R}_y$ give rise to nonvanishing energy flux at infinity while $\mathcal{R}_{\bar{r}}$ does not contribute to the flux at leading order. Note that because $\mathcal{R}_{\bar{r}}$ is zero if both $\mathcal{R}_s$ and $\mathcal{R}_y$ vanish, it is not possible to have a solution in which only $\mathcal{R}_{\bar{r}}$ is excited, and thus all solutions carry some flux at infinity.

APPENDIX G

COORDINATES FOR THE RING

In this appendix we explain the geometric meaning of the coordinates (3.27) useful in describing the ring, and also obtain the near ring limit used in our analysis. The coordinates we define are constructed on the lines of the coordinates used in [120], and are related to them by a simple transformation.

The D1-D5 geometry (4.100) can be generated by starting with an NS1-P system where the NS1 describes one turn of a uniform helix. Let this helix lie in the $x_1 - x_2$ plane of the noncompact 4-dimensional space x_1, x_2, x_3, x_4. We introduce polar coordinates in this space

$$x_1 = \tilde{r}\sin\tilde{\theta}\cos\tilde{\phi}, \qquad x_2 = \tilde{r}\sin\tilde{\theta}\sin\tilde{\phi}$$
$$x_3 = \tilde{r}\cos\tilde{\theta}\cos\tilde{\psi}, \qquad x_4 = \tilde{r}\cos\tilde{\theta}\sin\tilde{\psi} \tag{A.1}$$

Then the coordinates $\bar{r}, \bar{\theta}$ appearing in (4.100) are related to $\tilde{r}, \tilde{\theta}$ by [150]

$$\tilde{r} = \sqrt{\bar{r}^2 + a^2\sin^2\bar{\theta}}, \quad \cos\tilde{\theta} = \frac{\bar{r}\cos\bar{\theta}}{\sqrt{\bar{r}^2 + a^2\sin^2\bar{\theta}}}, \quad \tilde{\phi} = \bar{\phi}, \quad \tilde{\psi} = \bar{\psi} \tag{A.2}$$

In these coordinates the ring is easy to see; the center of the 'tube' runs along the circle at $\tilde{r} = a, \tilde{\theta} = \pi/2$. We will start by defining our ring coordinates with the help of these variables, and later convert to the coordinates $(\bar{r}, \bar{\theta}, \bar{\phi}, \bar{\psi})$.

G.1 New Coordinates

In this section, we want to define coordinates near the ring such that the direction along the ring becomes a linear coordinate

$$z = a\bar{\phi} \tag{A.3}$$

We now wish to choose coordinates in the 3-dimensional space perpendicular to the ring. Choose a point $P = (a\cos\tilde{\phi}, a\sin\tilde{\phi}, 0, 0)$ on the ring. Close to the ring we would like these to be spherical polar coordinates r, θ, ϕ centered at P, with the direction $\theta = 0$ pointing towards the center of the ring. Close to the ring the coordinate r should measure distance from the ring, but when $r \sim a$ we will see the diametrically opposite point $P' = (-a\cos\tilde{\phi}, -a\sin\tilde{\phi}, 0, 0)$ on the ring, and should use a radial coordinate that vanishes at P'. Consider all points that have azimuthal coordinate $\bar{\phi} = \tilde{\phi}$ and for these points define

$$\frac{1}{r} = \frac{1}{2a}(\frac{r_P}{r_{P'}} + \frac{r_{P'}}{r_P}) \tag{A.4}$$

where $r_P, r_{P'}$ measure distances from the points P, P' respectively

$$r_P = \sqrt{\tilde{r}^2 + a^2 - 2a\tilde{r}\sin\tilde{\theta}}, \quad r_{P'} = \sqrt{\tilde{r}^2 + a^2 + 2a\tilde{r}\sin\tilde{\theta}} \tag{A.5}$$

If we approach the point P we have $r_P \to 0$, and

$$\frac{1}{r} \approx \frac{1}{2a}\frac{r_{P'}}{r_P} \approx \frac{1}{r_P} \tag{A.6}$$

So we see that $r \approx r_P$ near P, and similarly $r \approx r_{P'}$ near P'.

Note that

$$r_P^2 r_{P'}^2 = (a^2 + \tilde{r}^2)^2 - 4\tilde{r}^2 a^2 \sin^2\tilde{\theta} = (a^2 - \tilde{r}^2)^2 + 4\tilde{r}^2 a^2 \cos^2\tilde{\theta} \tag{A.7}$$

Thus

$$|a^2 - \tilde{r}^2| \leq r_P r_{P'} \tag{A.8}$$

with equality only for points on the ring diameter passing through P, P'. Thus we can define

$$\cos\theta = \frac{(a^2 - \tilde{r}^2)}{r_P r_{P'}} \tag{A.9}$$

Near P we have

$$r_P \approx r, \quad r_{P'} \approx 2a, \quad a^2 - \tilde{r}^2 = (a+\tilde{r})(a-\tilde{r}) \approx 2a(a-\tilde{r}) \tag{A.10}$$

Close to P we have

$$\tilde{r} = \sqrt{(x_1^2 + x_2^2) + (x_3^2 + x_4^2)} \approx \sqrt{x_1^2 + x_2^2} \tag{A.11}$$

where we have kept terms up to linear order in the displacement from P. Thus $a - \tilde{r}$ measures the distance d from the P along the diameter through P (with d positive for points inside the ring). We then see that

$$\cos\theta \approx \frac{(2a)d}{(2a)r} = \frac{d}{r} \tag{A.12}$$

and thus θ is the desired polar coordinate near P. Finally note that the $x_3 - x_4$ plane is perpendicular to the ring and also to the diameter through P, so we define the azimuthal angle

$$\phi = \tan^{-1}\frac{x_4}{x_3} = \tilde{\psi} \tag{A.13}$$

Using (A.1) we write the ring coordinates in terms of the coordinates $(\bar{r}, \bar{\theta}, \bar{\phi}, \bar{\psi})$

$$r = a\Big(\frac{\bar{r}^2 + a^2\cos^2\bar{\theta}}{\bar{r}^2 + a^2 + a^2\sin^2\bar{\theta}}\Big), \quad \cos\theta = \frac{a^2\cos^2\bar{\theta} - \bar{r}^2}{a^2\cos^2\bar{\theta} + \bar{r}^2}, \quad z = a\bar{\phi}, \quad \phi = \bar{\psi} \tag{A.14}$$

The inverse of these relations gives (3.27)

$$\bar{r}^2 = \frac{a^2 r(1-\cos\theta)}{a + r\cos\theta}\,, \quad \sin^2\bar{\theta} = \frac{a-r}{a+r\cos\theta}\,, \quad \bar{\psi} = \phi\,, \quad \bar{\phi} = \frac{z}{a} \tag{A.15}$$

APPENDIX H

TRAJECTORIES IN FULL BLACK HOLE BACKGROUND

Consider the motion of a brane in the full four dimensional black hole geometry which has an energy (as measured in terms of the time in the asymptotically flat region) which is given by $E = (M_2 + M_0)\frac{R}{q_0}$, i.e. the same energy which we found in the near-horizon approximation. We will verify that this brane comes out of the horizon and goes back and examine the parameter space for which the brane remains in the near-hroizon region. In this analysis we will set the motion along the T^6 to zero from the beginning, so that we will deal with the four dimensional part of the geometry.

The black hole solution is described in terms of harmonic functions

$$H_0(r) = 1 + \frac{q_0}{r} \qquad H_i(r) = 1 + \frac{p_i}{r} \quad (i = 1, \cdots 3) \tag{A.1}$$

The (four dimensional part) string metric, dilaton and the 1-form RR fields are given by

$$ds^2 = -\frac{dt^2}{[H(r)]^2} + [H(r)]^2 \; [dr^2 + r^2 d\Omega_2^2] \tag{A.2}$$

$$A_t = 1 - \frac{1}{H_0(r)} \tag{A.3}$$

$$e^{\Phi} = \frac{H_0(r)}{H(r)} \tag{A.4}$$

where we have defined

$$H(r) = (H_0 H_1 H_2 H_3)^{\frac{1}{4}} \tag{A.5}$$

The lagrangian for a $D2$ brane which is wrapped on the S^2 at some value of r then becomes

$$S = -\mu(r)\sqrt{[H(r)]^{-2} - [H(r)]^2(\dot{r})^2} + \frac{M_0}{H_0(r)} \tag{A.6}$$

where we have defined

$$\mu(r) = 4\pi\mu_2 \frac{H(r)}{H_0(r)}\sqrt{(H(r))^4 r^4 + f^2} \tag{A.7}$$

and the other quantities have been defined above.

The expression for the energy is

$$E = \frac{\mu(r)[H(r)]^{-2}}{\sqrt{[H(r)]^{-2} - [H(r)]^2(\dot{r})^2}} - \frac{M_0}{H_0(r)} \tag{A.8}$$

H.1 Behavior of the Potential

Following the strategy of section (2.2) we will cast the problem as that of a non-relativistic particle in some potential with the non-relativistic energy equal to zero.

The equation of motion may be written using (A.8) as

$$\frac{1}{2}(\dot{r})^2 + W(r) = 0 \tag{A.9}$$

where

$$W(r) = -\frac{1}{2H^2(r)}\,[\frac{1}{H^2(r)} - \frac{\mu^2(r)}{H^4(r)(E + \frac{M_0}{H_0(r)})^2}] \tag{A.10}$$

The potential $W(r)$ behaves as $-r^4$ for small r and $+r^4$ for large r and has a single minimum. For any E the brane therefore starts from the horizon, goes upto a maximum distance $r = r_0$ given by the point $W(r_0) = 0$ and turns back to the horizon.

The near-horizon region has $r \ll q_0, p_i$ and we want to examine whether r_0 lies in this region. The general problem is difficult to analyze. However we get some indication by looking at the simpler case where

$$q_0 = p_1 = p_2 = p_3 \equiv q \tag{A.11}$$

so that $H_0(r) = H_1(r) = H_2(r) = H_3(r) = H(r)$. In this case

$$\mu^2(r) = M_2^2(1 + \frac{r}{q})^4 + M_0^2 \tag{A.12}$$

where M_2 is the $D2$ mass of the previous subsections.

In terms of the dimensionless distance

$$y \equiv \frac{r}{q} \tag{A.13}$$

the potential $W(r)$ becomes

$$W(y) = \frac{y^4}{2(1+y)^3}\frac{y^2(1+y)^3 - \epsilon^2(1+y) - 2\alpha\epsilon y}{(\epsilon(1+y) + \alpha y)^2} \tag{A.14}$$

where we have defined

$$\epsilon \equiv \frac{E}{M_2} \qquad \alpha = \frac{M_0}{M_2} \tag{A.15}$$

We want to examine only the special trajectory with $E = M_2$. The function $W(y)$ for $E = M_2$ is shown in Figure (H.1) for various values of the ratio $\alpha = M_0/M_2$

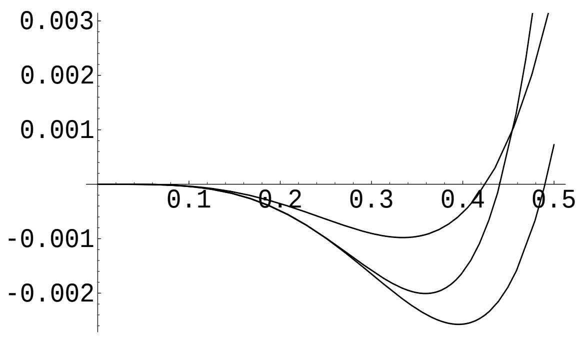

Figure H.1: The potential $W(y)$ as a function of y for $E = M_2$. The curves have $\frac{M_0}{M_2} = 0, 1, 6$ starting from the top

The trajectory will proceed to the zero of $W(y)$ at $y = y_0(\alpha) \neq 0$. The function $W(y)$ is plotted against y for various values of α in Figure (H.1). It is clear that the value of y_0 increases as α increases and becomes *greater than unity* for sufficiently large α. Thus the $D2$ brane goes beyond the near-horizon region for large enough α and strictly speaking the near-horizon approxiomation can be trusted only when $M_0 \ll M_2$.

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