SLIDING MODE CONTROLLERS AND OBSERVERS

A Thesis

Presented in Partial Fulfillment of the Requirements for

the Degree Master of Science in the

Graduate School of The Ohio State University

By

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1996

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ABSTRACT

In this thesis, sliding mode controllers and observers are studied in terms of their robustness properties and implementation concerns. First, a twofold discrete time sliding mode control action is proposed defining a boundary layer which divides the original state space fictitiously into two regions. Outside the boundary layer, the velocity vector of the state trajectories is directed towards the boundary layer with a proper control action so as to achieve the outside stability. Inside the boundary layer, a modified equivalent control law is applied for disturbance rejection. The resulting controller eliminates the disturbance from the generalized control variable completely if its dynamics are known. If the disturbance dynamics are not available, a disturbance model is fitted using deviations from the sliding manifold and this model is incorporated into the control law. Second, a continuous time sliding mode observer design procedure is proposed for the reconstruction of the original state vector in finite time. Equivalent control concept is exploited for this purpose. A discrete time counterpart to the continuous time case is also provided using discrete time sliding mode control theory. The resulting observer shows a deadbeat response and is also robust to disturbances with known dynamics. The robustness properties of the discrete time sliding mode controllers and finite time converging characteristics of the sliding mode observers are verified on a truck-semitrailer system during typical highway maneuvers in simulations.

Dedicated to my parents ...

ACKNOWLEDGMENTS

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CHAPTER 1

INTRODUCTION

The robustness of control systems to parametric uncertainties and external disturbances is one of the most considerable research interests of the today's control engineers. The theory of variable structure systems (VSS) has been studied in great depth for this purpose beginning from the publication of the first paper [1] related to this topic in 1977. The major reasons for variable structure control being of exceptional significance in recent control studies can be summarized as follows:

- The element which is used to induce sliding mode switches at high frequency in sliding mode. Its input is approximately zero while the output takes finite values and therefore resembles a high gain feedback controller which is considered as the conventional method of disturbance rejection.

- In sliding mode, state trajectories are confined to a reduced order manifold. Therefore, the effective system order reduces and the original control problem can be decoupled into smaller order, independent subproblems.

- The original system dynamics are replaced by user-defined sliding mode dynamics while in sliding mode. Therefore the system can be enforced to behave in a desired manner whatever the complexity of the system is.

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- The usage of variable structure theory is not limited to just controller design purposes. It's general principles are successfully applicable for other control problems such as linearization, finite time converging state observers, order reduction and so on and so forth.

1.1   Overview

The variable structure theory and its applications have formed one of the most attractive research areas of the last two decades. The main characteristic of a variable structure controller is to switch the system structure so as to confine state trajectories to a prespecified manifold and to create a new type of motion, so called sliding mode, on this manifold. Any variable structure controller design procedure consists of two independent steps in common.

First, a sliding manifold is constructed so that the system satisfies desired design objectives (stability, optimality, order reduction, linearization, etc.) while in sliding mode. There are several sliding manifold design methods for linear systems in the literature. Utkin and Young proposed a manifold design procedure based on pole placement and optimal control [3]. In this method, the original system is transformed into a special canonical form and the design is carried out in the new coordinates. This idea has been exploited from different perspectives and formed a basis for many related work in the future [4], [5], [6]. In 1993, Young and Özgüner introduced a new manifold design procedure in the frequency shaping technique and the optimal control framework [7]. In this setting, the proposed manifold appears as a linear operator on states rather than a static linear transformation of them. The resulting manifold acts as a filter in the closed loop and prevents the excitation of the high frequency

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vibrational modes of the system. Later on, W. Su, S. Drakunov and Özgüner have approached to the manifold design problem from a Lyapunov point of view [9]. In this approach, Lyapunov second method is used in the original state space realization of the system. The system is not necessarily partitioned into a canonical form and the proposed method is also applicable to nonlinear systems and linear systems with time delays successfully.

Once a proper manifold which achieves the desired design objectives has been chosen, a control law is found to generate a sliding mode on that manifold. The variable structure controller takes different actions at the different locations of the state space to direct the state trajectories towards the manifold and to keep them continuously on it. Sliding mode takes place a finite time later and trajectories slide to the origin along the manifold. The overall system motion consists of two consecutive phases. In the first phase, trajectories are brought to the manifold from an initial location which is usually referred to as the reaching phase. The second phase describes the motion in sliding mode in which trajectories are confined to the manifold and slide to the origin through it. However, the system motion is indefinite just on the manifold, and some further analysis are required to analyze it [17]. To keep the trajectories on the manifold, the controller switching frequency should be theoretically infinite.

The problem of state estimation using variable structure control theory has also been considered for several years and uses the same design theory and reasonings with the variable structure controllers. However, different from the controller design, the input variable which induces sliding mode on a proper manifold is not the real input

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of the system anymore. The superiority of a sliding mode observer over a classical Luenberger observer results from the finite time converging characteristic of the former. A standard Luenberger observer reconstructs the original state vector asymptotically. However, in sliding mode observers, a nonlinear auxiliary term is used to guarantee the finite time convergence by deliberately introducing of sliding mode. Most of the efforts for this subject have been concentrated on continuous time systems and the first sliding mode observer has appeared as the sliding mode realization of a reduced order asymptotic observer [2]. In this formulation, the original system is first transformed into a canonical form in which the output variables form a part of the state vector. Sliding mode is induced in the known output variables in finite time and the remaining observer states converge to the associated plant states asymptotically. Later, the equivalent control methodology has been exploited from different perspectives and used in the observer design of a linear system transformed into the "block observable form" using sequential application of the state transformations [12], [13], [14].

For any sliding mode scheme, the closed loop system is insensitive to matched parametric uncertainties and external disturbances because of the motion on a pre-specified surface in sliding mode. However, unmatched portions of disturbances and neglected actuator and sensor dynamics prevent the occurrence of the ideal sliding mode and state trajectories cannot be kept continuously on the manifold. Instead, they oscillate randomly around it. These oscillations are referred to as chattering in the literature. Chattering is highly undesirable in practise, because these rapid oscillations can excite the unmodeled high frequency dynamics of the system and

deteriorate the control accuracy. They may also decrease the life time of system elements, wear moving mechanical parts and cause heat losses in power circuits [2]. When considering sliding mode observers, chattering phenomena is also problematic, because high frequency terms in the reconstructed observer state variables may also excite neglected system dynamics when these observer states are used for feedback purposes. Chattering phenomena appears as an obstacle for implementation of the theoretical sliding mode control techniques in practise and it should be overcome to retain the benefits of sliding mode in practical applications.

In the past decades, several studies have been conducted to reduce chattering. One of these approaches is to continuously approximate the original discontinuous control law in the vicinity of the sliding manifold by an interpolation so that high control activity around the manifold which is considered as the major reason of chattering is already taken care of [23]. For this purpose, a boundary layer is defined around the manifold and its attractiveness are guaranteed using the original discontinuous control law outside the layer. Similar ideas have been researched in many related papers [26], [22], [24]. However, this approach is nonadmissible for some applications especially where the only actuator elements are on-off type. Another method to cope with chattering is to use an asymptotic observer [2]. In this method, an asymptotic state observer is inserted into the closed loop which bypasses the undesired high frequency components acting as a low pass filter. Later on, these control law interpolation and the state observation ideas have been brought together and generalized under the title of prefiltering and postfiltering [24]. In this formulation, the prefiltering block generalizes the control law interpolation whereas postfiltering block generalizes the state observation idea. Frequency shaping technique of [7] also shows a filter-like

behavior in the closed loop and reduces chattering by suppressing high frequency sliding mode dynamics. It is possible to extend the list of these types of methods. However, for all of them, chattering reduction is achieved at the expense of robustness and there is always a tradeoff between the robustness of the system and the amount of chattering in the system.

Although the variable structure theory has first been built upon the continuous time system theory, the application of a variable structure control law inevitably requires a computer implementation. Designing a control law based on the continuous time VSS theory and then implementing the same control law in discrete time is not reasonable, because the sampling frequency cannot meet the theoretical requirements of a VSS. Continuous time sliding mode control techniques loose their robustness properties after discretization because of the sampled data nature of the system where the control law takes a new action only at the sampling instances and holds the same value during one sampling period. Discrete time sliding mode theory has been constructed with this motivation as an attempt to fill in the gap between the theory and the practical implementation requirements. However, most of the studies in this area have concentrated on the development of discrete time counterparts to the continuous time controllers and sliding mode conditions.

Sarptürk *et. al* have chosen a set of sliding mode conditions [36], [37],

$$[s_{k+1}^i - s_k^i] \, signs_k^i < 0$$

$$[s_{k+1}^i + s_k^i] \, signs_k^i \geq 0 \qquad (1.1)$$

for each hypersurface s^i. In the above set, the first condition suffices for a switched motion about the associated hyperplane whereas the second prevents oscillations of

trajectories with an increasing amplitude. Two conditions together produce upper and lower bounds on the control input. As to controller design, Utkin and Drakunov have introduced the discrete time equivalent control law which brings trajectories to the manifold at one step [33], [34]. Discrete time equivalent control law is the ideal solution of the discrete time sliding mode control and it does not lead to any chattering phenomena since it is a non-switching type of control. However, there are two problems with this control. First, it is not always possible to bring trajectories to the manifold from an arbitrary position at one step because of the discrete time nature of the system. Second, an uncertainty in the system makes the calculation of the equivalent control law impossible. In 1990, Furuta *et. al* proposed a sliding sector in which the closed loop system is stable without any control action instead of defining a sliding manifold. A discrete time VSS controller has also been designed to steer the state trajectories to the inside of the sliding sector. The idea of sliding sector has been developed further and also extended for continuous time systems [43], [44], [45], [46]. Kaynak and Denker incorporated the deviations from the sliding manifold into the discrete time controller design and come up with a non-switching type, robust and accurate control law in the presence of system uncertainties [38]. Zanasi *et. al* have modified the Discontinuous Integral Control (DIC) technique of [40] properly for discrete time systems [41], [42]. In their setting, the derivative of the controlled variable improves the disturbance estimation and therefore reduces chattering. In 1995, Su and Özgüner have analyzed the chattering problem in a sampled data system framework and proposed discrete time control laws to maintain the states in the sampling time order vicinity of the sliding manifold in the presence of unknown disturbances and parametric uncertainties [8], [50], [51]. The proposed

design methods have been investigated successfully for three types of sampled data systems: linear, nonlinear and stochastic systems.

1.2   Thesis Organization

This thesis is organized into five chapters. Chapter 2 discusses the discrete time sliding mode control theory from a robustness point of view. A twofold control action is considered. First, the trajectories are brought to the sampling time vicinity of the manifold in which the equivalent control is in the admissible control domain using the first control law. Once the state trajectories fall into this region, this control law is deactivated and the generalized equivalent control law is applied. A disturbance corrective term is inserted to the original equivalent control law for disturbance rejection. This term removes the disturbance effects from the generalized control variable completely in case of a disturbance with known dynamics. If the disturbance model is unknown, a fictitious model is fitted for it examining the deviations of the trajectories from the manifold and the robustness of the system is also satisfied with a sufficient accuracy. The robustness properties of the proposed control laws are tested on a truck-semitrailer system during highway maneuvers.

Chapter 3 focuses on continuous time sliding mode observers. Standard Luenberg-type observers are also reviewed to illustrate the finite time convergence characteristics of the sliding mode observers. The design procedure is summarized as a theorem under some structural assumptions for the system. The estimation of the unmeasurable states of a truck-semitrailer system during a maneuver illustrates the approach.

In Chapter 4, a discrete time counterpart to the continuous time sliding mode observer of Chapter 3 is developed. The resulting discrete time sliding mode observer

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consists of two layers. The first layer provides the discrete time equivalent control law for the second layer where the actual state observation is done. The proposed observer not only achieves the finite time state reconstruction aim but also decouples the error motion into two smaller order motions while in sliding mode. The same observer problem with Chapter 3 is considered and the simulation results verify the validity and the effectiveness of the approach.

Chapter 5 summarizes the theoretical analysis and points out some ideas for future directions. In Appendix A, the derivation of the linear continuous time truck-semitrailer model is given starting from a two dimensional, nonlinear bicycle model and a numerical model is also obtained using a default parameter set.

CHAPTER 2

DISCRETE TIME SLIDING MODE CONTROL

Variable structure systems (VSS) have been studied by many authors for several decades. However, most of the efforts have concentrated on the continuous time case. The continuous time sliding mode controller is robust to matched parametric uncertainties and external disturbances because of the motion on a prescribed manifold. However, due to practical considerations the trajectories cannot be kept exactly on the manifold, instead they oscillate around it. Even if, we could have an infinite switching frequency, there would be again some oscillations in the presence of uncertainties in the system. These high frequency oscillations have already been called chattering. There are several approaches to deal with chattering as mentioned in Chapter 1. However, full chattering rejection is impossible unless we design a VSS control law in discrete time. Before going into the discrete time sliding mode theory to alleviate the problem without loosing the robustness properties of the continuous time VSS control, let us illustrate a typical continuous time VSS controller design procedure for a linear time invariant system. Consider

$$\dot{x} = Ax + Bu \tag{2.1}$$

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where $x \in R^n$, $u \in R^m$, A, B are constant matrices of appropriate dimensions and the system is assumed to be controllable. Defining a sliding manifold

$$\mathcal{S} = \{x | s = Cx = 0\} \tag{2.2}$$

where $s \in R^m$, $C \in R^{m \times n}$, we obtain the following sliding mode dynamics.

$$\dot{s} = CAx + CBu \tag{2.3}$$

Choosing the control law

$$u = -(CB)^{-1}(CAx + K sgns) \tag{2.4}$$

and inserting (2.4) into (2.3) yield the sliding mode dynamics as follows:

$$\dot{s} = -K sgns \tag{2.5}$$

The generalized control variable s converges to the zero vector, choosing K as a diagonal, positive define $m \times m$ matrix. After sliding mode takes place

$$s = Cx = 0 \tag{2.6}$$

and the stability of the sliding mode itself is guaranteed with a proper choice of C. Outside the manifold, the trajectories should be steered to the manifold as soon as possible. This could be assured choosing each entry of the matrix K larger. However, this is not a good solution because the magnitude of the chattering terms is proportional to the magnitude of K as well.

In the rest of the chapter, we shall study the discrete time sliding mode design theory as a solution to the chattering phenomena and design a full discrete time controller rather than first deriving a continuous time control law in the continuous time

sliding mode framework and then discretizing it for a sampled data implementation. In section 2.1, we present the basics and the definitions of the theory which will be used throughout the chapter. Section 2.2 summarizes two sliding manifold design procedures. In section 2.3, we propose a discrete time sliding mode control law and state it as a theorem for an ideal system. Section 2.4 is dedicated to the improvement of the previously proposed control law to take care of external disturbances with known dynamics. A corrective input term is added to the control law to eliminate the effect of the disturbance in the generalized control variable. Finally, in section 2.5 practical implementation concerns of the proposed control laws are discussed and simulation results on a truck-semitrailer system in the presence of an external disturbance verify the theoretical results.

2.1 Problem Statement

The discrete time state space representation of the system given in (2.1) is obtained by passing the control input through a zero order hold as follows:

$$x_{k+1} = \Phi x_k + \Gamma u_k \tag{2.7}$$

where $\Phi = e^{AT}$, $\Gamma = \int_0^T e^{As} B ds$. It is assumed that the pair (Φ, Γ) is also controllable after discretization. The control objective is to steer the states of (2.7) to zero using discrete time sliding mode techniques. As in the continuous time case, the proposed control design consists of two steps. First, a connected subspace of the original n dimensional state space which has the same dimensionality with the control input is found so that if the system trajectories are kept on this subspace at each sampling instant, convergence of them to the origin is guaranteed. This subspace has been

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called sliding manifold before and for a discrete time system it is formulated by

$$\mathcal{S} = \{x_k | s_k = C(x_k) = 0\} \tag{2.8}$$

where x_k and s_k denote the value of the associated variable at $t = kT$. For a linear system, the sliding manifold is usually chosen as the linear combination of the states rather than an arbitrary function of them, i.e;

$$\mathcal{S} = \{x_k | s_k = Cx_k = 0\} \tag{2.9}$$

where $s \in R^m$ and C is an $m \times n$ constant matrix. If the state trajectories are kept on the manifold at each sampling instant, the system is said to be in discrete time sliding mode. Note that, unlike the continuous time sliding mode definition, states do not necessarily have to be on the manifold continuously, instead they are allowed to deviate from the manifold between two consecutive sampling instants in a sampled data implementation. Second, a discrete time control law which enforces the trajectories towards the manifold and keeps them on it is determined. If the manifold is chosen properly to assure the stability of the system in sliding mode, the overall problem turns into a regulation problem in the m dimensional generalized control variable s. Before discussing the discrete time sliding mode controller design, we first briefly mention two sliding manifold design procedures.

2.2 Constructing Sliding Manifold

In the previous section, the sliding manifold for a linear system has been defined as

$$\mathcal{S} = \{x_k | s_k = Cx_k = 0\} \tag{2.10}$$

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where $s \in R^m$ and C is $m \times n$ matrix. The dimension of the sliding manifold is the same as that of the control input so that when the system trajectories travel along the manifold, they satisfy some desired characteristics irrespective of the original system dynamics. In other words, the system dynamics are completely replaced by the reduced order sliding mode dynamics thereafter the system is in sliding mode. According to the definition of (2.10), the manifold of a linear system is parameterized by an $m \times n$ matrix C. Therefore, the manifold design problem hinges on choosing a suitable C. One of the ways of designing a manifold is eigenvalue assignment. In this method, the C matrix is calculated according to the desired eigenvalue locations of the system in sliding mode. The design procedure is as follows:

Using controllability of the pair (Φ, Γ), the original system representation given in (2.7) can be transformed into a controllable canonical form with an appropriate similarity transformation matrix P, i.e;

$$\begin{bmatrix} \bar{x}_{1,k+1} \\ \bar{x}_{2,k+1} \\ \vdots \\ \bar{x}_{n-1,k+1} \\ \bar{x}_{n,k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \end{bmatrix} \begin{bmatrix} \bar{x}_{1,k} \\ \bar{x}_{2,k} \\ \vdots \\ \bar{x}_{n-1,k} \\ \bar{x}_{n,k} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u_k \quad (2.11)$$

$$\bar{x}_{k+1} = \bar{\Phi} \bar{x}_k + \bar{\Gamma} u_k \quad (2.12)$$

where $\bar{x}_k = P x_k$ and a_i's are the coefficients of the characteristic equation of the system. The generalized control variable s can be written in terms of the new coordinates as follows:

$$\begin{aligned} s_k &= \bar{c}_1 \bar{x}_{1,k} + \bar{c}_2 \bar{x}_{2,k} + \cdots + \bar{c}_{n-1} \bar{x}_{n-1,k} + \bar{c}_n \bar{x}_{n,k} = \bar{C} \bar{x}_k \\ s_k &= [\bar{c}_1 + \bar{c}_2 q + \cdots + \bar{c}_{n-1} q^{n-2} + \bar{c}_n q^{n-1}] \bar{x}_{1,k} = E(q) \bar{x}_{1,k} \end{aligned} \quad (2.13)$$

In sliding mode, the motion of the state trajectories are characterized by the equation

$$E(q) \bar{x}_{1,k} = 0 \quad (2.14)$$

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Therefore, the eigenvalues of the system in sliding mode are the roots of the polynomial

$$E(q) = \bar{c}_1 + \bar{c}_2 q + \cdots + \bar{c}_{n-1} q^{n-2} + \bar{c}_n q^{n-1} = 0 \qquad (2.15)$$

and the sliding mode stability is guaranteed choosing the c_i's so as to place the roots of the polynomial $E(q)$ inside the unit disc. Besides the stability, the manifold can also be chosen to minimize a quadratic cost function and this is usually referred to as the optimal sliding mode. For an $(n-1)$ dimensional optimization problem, one can find the unique optimal state feedback solving the associated Riccatti equation. This determines the prospective sliding mode eigenvalues of the optimal system. Matching them with the roots of the polynomial (2.15), the optimal manifold design objective is achieved. When $\bar{C}$ has been found, C of the original representation follows from the transformation

$$C = \bar{C} P \qquad (2.16)$$

The previous design procedure can only be applied to single input systems. Therefore, a more general approach becomes necessary. A different sliding manifold design procedure which is applicable for both single-input and multi-input systems is given as follows:

Assume that the original system is transformed into the following canonical form

$$\begin{array}{rcl} x_{1,k+1} & = & \Phi_{11} x_{1,k} + \Phi_{12} x_{2,k} \\ x_{2,k+1} & = & \Phi_{21} x_{1,k} + \Phi_{22} x_{2,k} + B_2 u \end{array} \qquad (2.17)$$

with a transformation

$$Px = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] \qquad (2.18)$$

where $x_1 \in R^{n-m}$ and $x_2 \in R^m$. The eigenvalues of sliding mode are those of $(\Phi_{11} - \Phi_{12} K)$ and they can be assigned arbitrarily provided that the pair (Φ_{11}, Φ_{12})

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is controllable. However, this controllability requirement automatically holds since (Φ, Γ) pair is assumed to be controllable. Once K has been chosen so as to have the desired dynamics in sliding mode, C is calculated from

$$C = [K \ I_m]P \tag{2.19}$$

2.3 Sliding Mode Controller for a Discrete Time System

In this section, we design a discrete time sliding mode controller assuming that the sliding manifold has been chosen properly to satisfy the given design specifications (stability, optimality, $\cdots$ etc). If there are no external disturbances and parametric uncertainties in the system, the following control law called discrete time equivalent control which is the solution of $s_{k+1} = 0$ steers the state trajectories onto the manifold at the next step

$$u_k^{eq} = -(C\Gamma)^{-1}C\Phi x_k \tag{2.20}$$

provided that $C\Gamma$ is nonsingular. The equivalent control law is the ideal solution of the regulation problem of s and does not lead to a chattering problem since it does not contain a discontinuous term. However, there are two problems with this control. First, if there is uncertainty in the system the equivalent control will contain these unknown terms and cannot be calculated. Second, even if everything in the system is ideal, the control law tends to infinity in magnitude when the sampling time approaches zero. Therefore, unless the state trajectories are in the vicinity of the origin, the equivalent control law may not be in the admissible control domain. For the time being, let us assume that there are no external disturbances and parametric uncertainties in the system so that the first problem may be skipped. We will first

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study the second problem and the discussion of the first one will be postponed to section 2.4.

Incorporating $\Phi = e^{AT}$ into (2.20), the equivalent control is written as a linear combination of the states and the generalized control variable s as follows:

$$
\begin{aligned}
u_k^{eq} &= -(C\Gamma)^{-1}C\Phi x_k \\
&= -(C\Gamma)^{-1}Ce^{AT}x_k \\
&= -(C\Gamma)^{-1}C[I + AT + \frac{A^2T^2}{2!} + \cdots + \frac{A^kT^k}{k!} + \cdots]x_k \\
&= -(C\Gamma)^{-1}s_k - (C\Gamma)^{-1}C[AT + \frac{A^2T^2}{2!} + \cdots + \frac{A^kT^k}{k!} + \cdots]x_k \quad (2.21)
\end{aligned}
$$

Since Γ is in the order of T, the second term of (2.21) is in the order of the state vector and it represents the admissible part of the equivalent control. However, the first term has no elements to cancel the effect of Γ^{-1} unless s is in the sampling time order. Therefore, the overall expression may not be in the admissible control domain unless the trajectories are in the vicinity of the manifold. Note that, (2.20) without considering that Γ and Φ have some structural properties because of the discretization process would just give that the state trajectories should be in the vicinity of the origin. This would make the equivalent control unworthy because of being the state trajectories around the origin would be very unrealistic for the objective of steering them to the origin. However, our analysis has revealed that it is sufficient for trajectories to be in the vicinity of the manifold to make sure that the equivalent control is in the admissible domain. Therefore, another control law is necessary if the trajectories are far away from the manifold. The overall controller consists of two subcontrollers where only one of them takes action according to the location of the trajectories at any time. The second subcontroller should steer the

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trajectories towards the manifold, and whenever they fall into the boundary layer, the equivalent control is applied to induce sliding mode at the next step. When sliding mode takes place, trajectories converge to the origin along the manifold with a dynamic adopted by C as discussed before. To express the location of the trajectories, we define a boundary layer which intuitively describes a subregion of the original state space in which the equivalent control can be provided as follows:

$$\mathcal{B} = \{x_k | s_k = Cx_k \approx \mathcal{O}(T)\} \tag{2.22}$$

This boundary layer does not have an analytical expression, because it is hard to distinguish the points of the state space where the equivalent control is admissible from the others mathematically. This definition is introduced just for notational purposes and it will be used to say that the equivalent control belongs to the admissible control domain or not at a specific point of the state space.

2.3.1 Control Law off the Manifold

The control law off the manifold should be chosen so as to steer the state trajectories to the manifold from a bounded initial location without exceeding control limits. This can be achieved satisfying that the generalized control vector s and its first order forward difference have different signs, componentwise. To this end, the following control is chosen

$$u_k = -(C\Gamma)^{-1}[C\Phi x_k - s_k + Msgns_k] \tag{2.23}$$

where M is a positive definite diagonal matrix, Φ, Γ are the system parameters and C characterizes the manifold as before. Manipulating (2.23) with (2.7) produces

$$s_{k+1} = C\Phi x_k + C\Gamma u_k$$

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$$s_{k+1} = C\Phi x_k + C\Gamma(-C\Gamma)^{-1}[C\Phi x_k - s_k + Msgns_k]$$

$$s_{k+1} = s_k - Msgns_k \qquad (2.24)$$

and the control satisfies its objective

$$\nabla s_k = -Msgns_k \qquad (2.25)$$

Inserting $\Phi = e^{AT}$ into (2.23), the control law is written as a weighted combination of the state vector and the generalized control variable as follows:

$$
\begin{aligned}
u_k &= -(C\Gamma)^{-1}[Ce^{AT}x_k - s_k + Msgns_k] \\
&= -(C\Gamma)^{-1}\left(C[AT + \frac{A^2T^2}{2!} + \cdots + \frac{A^kT^k}{k!} + \cdots]x_k + Msgns_k\right) \\
&= -(C\Gamma)^{-1}C[AT + \frac{A^2T^2}{2!} + \cdots + \frac{A^kT^k}{k!} + \cdots]x_k - (C\Gamma)^{-1}Msgns_k \ (2.26)
\end{aligned}
$$

The first term of (2.26) is in the order of the state vector. Choosing M and C such that $M/\parallel C \parallel$ is in the sampling time order where $\parallel C \parallel$ denotes any matrix norm of C, the overall control is also guaranteed to be in the admissible control domain. When this control is applied to the system, each component of the s vector approaches to the manifold by a fixed amount determined by the associated entry of M. If each diagonal term of M is chosen in such a way that none of the entries of the s vector overshoots the boundary layer, the state trajectories converge eventually to the vicinity of the manifold from any bounded initial condition without exceeding the control limits. However, around the manifold each component of the s vector starts to oscillate and exact convergence onto the manifold cannot be guaranteed just with this control. Therefore, after the trajectories fall inside the boundary layer, this control is deactivated, equivalent control is applied and sliding mode takes place at the next step. If there are no disturbances or uncertainties in the system state trajectories do not leave the boundary layer and sliding mode exists at any time.

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2.3.2 Control Law around the Manifold

When the trajectories are in the vicinity of the manifold, the equivalent control is applicable and induces sliding mode at the next step. However, for the sake of generality, we propose an alternative control law

$$u_k = -(C\Gamma)^{-1}[C\Phi x_k - \Omega s_k] \tag{2.27}$$

where Ω is a stable diagonal matrix. Application of this control to the system yields the following sliding mode dynamics:

$$
\begin{aligned}
s_{k+1} &= C\Phi x_k + C\Gamma u_k \\
s_{k+1} &= C\Phi x_k + C\Gamma(-(C\Gamma)^{-1}[C\Phi x_k - \Omega s_k]) \\
s_{k+1} &= \Omega s_k \\
\| s_k \| &\rightarrow 0
\end{aligned}
\tag{2.28}
$$

The convergence rate of the states onto the sliding manifold is determined by the eigenvalues of Ω. Any stable Ω would satisfy the convergence, however we choose it also a diagonal matrix just to make sure that the magnitude of the each component of s also decreases monotonically. This prevents some of the s vector components to make an overshoot before converging to zero. Note that, if Ω were a zero matrix the control law would become exactly the equivalent control. In theory, the sliding mode control law starts to stabilize the system in sliding mode. Therefore, Ω should be chosen as zero to guarantee sliding mode in finite time. However, the sampled data implementation of the discrete time control law keeps the trajectories on the manifold only at each sampling instant and they can deviate from it between two consecutive sampling instances anyhow. Exponential converge and this type of intersampling

behavior do not differ so much in practise and therefore a small Ω also performs the objective.

To sum up, the proposed discrete time sliding mode controller is twofold. First, the trajectories are steered towards the manifold without exceeding control limits using (2.23). After they reach the vicinity of the manifold, the first subcontroller is deactivated and (2.27) guarantees sliding mode. Theorem 1 summarizes the complete control law.

Theorem 1 *The control law*

$$u_k = \begin{cases} -(C\Gamma)^{-1}[C\Phi x_k - s_k + M sgn(s_k)] & \text{if } s_k \notin \mathcal{B} \\ -(C\Gamma)^{-1}[C\Phi x_k - \Omega s_k] & \text{if } s_k \in \mathcal{B} \end{cases} \tag{2.29}$$

guarantees the reachability and the existence of sliding mode satisfying the sliding mode conditions:

$$\begin{array}{ll} s_{k+1} - s_k = -M sgn(s_k) & \text{if } s_k \notin \mathcal{B} \\ s_{k+1} = \Omega s_k & \text{if } s_k \in \mathcal{B} \end{array} \tag{2.30}$$

choosing Ω as a stable diagonal matrix, and M as a diagonal positive definite matrix whose entries are in the sampling time order.

2.4 Robustness of the Controller

In this section, we study the robustness issues of the discrete time sliding mode control theory and improve the previously proposed control law to satisfy the robustness of the system in the presence of uncertainties in the system. Consideration will be given to the following uncertain discrete time system

$$x_{k+1} = \Phi x_k + \Gamma u_k + F d_k \tag{2.31}$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the input vector, $d \in R^l$ is the disturbance vector and Φ, Γ, F are constant matrices of appropriate dimensions. The pair (Φ, Γ)

is assumed to be controllable and the disturbance matching condition holds, i.e;

$$rank[\Gamma \mid F] = rank[\Gamma] \qquad (2.32)$$

First, we consider the case where the trajectories are away from the manifold. Applying (2.23) to (2.31), the following sliding mode dynamic is produced:

$$s_{k+1} = s_k - Msgns_k + D_k \qquad (2.33)$$

where $D_k \triangleq CFd_k$ If each entry of M is chosen as

$$M_i \geq \parallel D_k \parallel_\infty \qquad (2.34)$$

the forward difference of s_k and s_k itself are guaranteed to have different signs, componentwise, each s_k component decreases monotonically and the state trajectories are drawn to the sampling time order vicinity of the manifold in finite time. For implementation purposes, it has been previously found that each entry of M should be in the sampling time order. This constraints the robustness of the controller only to disturbances where D_k is also in the sampling time order. At first glance, this seems very unrealistic. However, F is in the sampling time order since it is also obtained by discretizing its continuous time counterpart and so $D_k = CFd_k$ is. Since the control law off the manifold is itself robust to external disturbances, in the rest of the chapter we assume that the initial condition is in the vicinity of the manifold without loss of generality. It is further assumed that the disturbance does not throw the trajectories to a point where the prospective robust control law is not in the admissible control domain so that the initial condition assumption is also valid at any time.

If the ideal equivalent control were applied to the system, one would get

$$s_{k+1} = CFd_k \qquad (2.35)$$

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and a linear transformed version of the disturbance would appear in the generalized control variable s. However, the control objective has already been reduced to the problem of regulating s at zero. If s can be kept at zero at each sampling instant, state variables eventually converge zero no matter what the disturbance is. Therefore, the question of whether one can eliminate the effect of the disturbance by somehow filtering it from s becomes the major concern. Before giving an answer to this question, we first assume that the disturbance dynamics are known, i.e;

$$E(q^{-1})D_k^i = 0 \tag{2.36}$$

where

$$E(q^{-1}) = e_0 + e_1 q^{-1} + \cdots + e_r q^{-r} \tag{2.37}$$

and D_k^i represents the ith entry of D_k. The proposed control law has the following form:

$$u_k = -(C\Gamma)^{-1}[C\Phi x_k - F_k] \tag{2.38}$$

where $F_k \in R^m$ is inserted to the control law to filter out the disturbance and it is given by

$$F_k = \frac{q}{E(q^{-1})}[I_m E(q^{-1}) - e_0(I_m - \Omega q^{-1})]s_k \tag{2.39}$$

where Ω is a stable diagonal matrix as before. Note that the orders of the denominator and the numerator polynomials are the same and equal to the disturbance degree. Applying (2.38) to the system provides

$$s_{k+1} = F_k + D_k \tag{2.40}$$

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Since Ω is diagonal, the transfer matrix from s to F is also diagonal and each component of F_k is associated with only one of the entries of s as follows:

$$F_k^i = \frac{(e_1 + e_0\Omega^i) + e_2 q^{-1} + \cdots + e_r q^{-r+1}}{E(q^{-1})} s_k^i \qquad (2.41)$$

Multiplying each side of (2.42)

$$s_{k+1}^i = F_k^i + D_k^i \qquad (2.42)$$

by $E(q^{-1})$ produces

$$E(q^{-1})s_{k+1}^i = [(e_1 + e_0\Omega^i) + e_2 q^{-1} + \cdots + e_r q^{-r+1}]s_k^i \qquad (2.43)$$

$$e_0 s_{k+1}^i + e_1 s_k^i + \cdots + e_r s_{k+1-r}^i = e_0\Omega^i s_k^i + e_1 s_k^i + e_2 s_{k-1}^i + \cdots + e_r s_{k+1-r}^i \qquad (2.44)$$

Some of the terms at the right and the left sides of the above expression are the same. Before r steps, first $(r - 1)$ values of F are arbitrarily assigned and do not contain information about the disturbance. Therefore, one has to shift both sides of (2.44 $(r - 1)$) steps forward in time before canceling these terms. With this analysis, we obtain

$$s_{k+1+r}^i = \Omega^i s_{k+r}^i \qquad (2.45)$$

and $s_k \to 0$ choosing each $|\Omega^i| < 1$. Note that, for the first r steps s_k behaves arbitrarily but its convergence to zero is guaranteed after r steps. Choosing each Ω^i as zero, sliding mode can be induced at the $(r + 1)^{th}$ step and trajectories converge to the origin through the manifold irrespective of the disturbance.

So far, it has been assumed that disturbance dynamics are known exactly. However, in most of the practical situations, the known information about the disturbance is very little or even none. So, the robustness issue of the controller to an arbitrary disturbance becomes the major concern. The rest of the section discusses the robustness of the system for these type of disturbances within a deterministic framework.

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Disturbance d_k, or alternatively $D_k = CFd_k$, is not known at the present time. However, the time history of D_k can be deduced from the equation

$$D_{k-r} = s_{k+1-r} - C\Phi x_{k-r} - C\Gamma u_{k-r} \qquad (2.46)$$

for any $r > 0$ since all the terms at the right hand side are known. The previous values of D_k give an intuition about the structure of the disturbance and this intuition can be used to fit a disturbance model and to estimate the present value of it. Note that as the number of past values used for model fitting increases, the accuracy of the model increases as well. However, this also makes the controller more complicated. As a starting point, we assume that the disturbance does not vary too much between two consecutive sampling instances so that its present value may be taken equal to the one step previous value. This is the exact solution for a constant disturbance. However even if the disturbance is not constant, it is predicted within an accuracy of the sampling time order provided that the disturbance is first order differentiable. By the nth order differentiability of a time sequence for an integer n, we mean that the nth order forward or backward difference of the associated sequence is in the sampling time order vicinity of zero.

Manipulating (2.38) with $D_k = D_{k-1}$ yields the following sliding mode dynamics:

$$s_{k+1} = F_k + s_k - F_{k-1} \qquad (2.47)$$

Since the objective is to satisfy $s_{k+1} = \Omega s_k$ where Ω is a stable diagonal matrix, F should satisfy

$$F_k = F_{k-1} - (I_m - \Omega)s_k \qquad (2.48)$$

(2.48) represents a first order difference equation for F_k and it can be used to update u_k dynamically. Substituting (2.48) into (2.38), the final control law for constant or

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slowly varying disturbances is found to be

$$u_k = -(C\Gamma)^{-1}[C\Phi x_k - (I_m - \Omega)\sum s_k] \tag{2.49}$$

This control increases robustness of the controller to smooth disturbances. However, the estimation accuracy may be increased further using more than one past value of the disturbance. Assuming that the first order difference of the disturbance does not vary too much between two consecutive steps, the following relations for D_k are obtained.

$$
\begin{aligned}
D_k - D_{k-1} &\approx D_{k-1} - D_{k-2} \\
D_k &\approx 2D_{k-1} - D_{k-2}
\end{aligned}
\tag{2.50}
$$

The increase in the estimation accuracy is obvious. The control law which has been derived under the constant disturbance assumption gives the exact estimation only for constant disturbances. However, the final estimation works perfect not only for constant disturbances but also those whose first order differences are constant as well. To increase the accuracy, it can be further assumed that the second order difference of the disturbance does not change too much, i.e;

$$
\begin{aligned}
D_k - 2D_{k-1} + D_{k-2} &\approx D_{k-1} - 2D_{k-2} + D_{k-3} \\
D_k &\approx 3D_{k-1} - 3D_{k-2} + D_{k-3}
\end{aligned}
$$

This equivalent to saying that

$$\sum_{j=0}^{3} \mathcal{C}(3,j)(-1)^j D_{k-j} = 0 \quad \Leftrightarrow \quad \sum_{j=0}^{3} \mathcal{C}(3,j)(-1)^j q^{-j} D_k = 0 \tag{2.51}$$

where $\mathcal{C}(n,m)$ is given by

$$\mathcal{C}(n,m) = \frac{n!}{(n-m)!m!} \tag{2.52}$$

If this process is continued, the following disturbance model is eventually obtained

$$\sum_{j=0}^{r} \mathcal{C}(r,j)(-1)^j q^{-j} D_k = 0 \tag{2.53}$$

at the rth step for an integer $r > 0$.

Choosing the control law again in the form of (2.38), the following relations are produced.

$$\left.\begin{aligned} s_{k+1} &= F_k + D_k \\ s_k &= F_{k-1} + D_{k-1} \\ &\vdots \\ s_{k+1-r} &= F_{k-r} + D_{k-r} \end{aligned}\right\} \Rightarrow \triangle^r(q^{-1})s_{k+1} = \triangle^r(q^{-1})F_k + \triangle^r(q^{-1})D_k \tag{2.54}$$

where

$$\triangle^r(q^{-1}) = \sum_{j=0}^{r} \mathcal{C}(r,j)(-1)^j q^{-j} = (1 - q^{-1})^r \tag{2.55}$$

Inserting the desired s dynamic $s_{k+1} = \Omega s_k$ into (2.54) and using $\triangle^r D_k = 0$ provide

$$\Omega s_k + \sum_{j=1}^{r} \mathcal{C}(r,j)(-1)^j s^{k+1-j} = \sum_{j=0}^{r} \mathcal{C}(r,j)(-1)^j F^{k-j} \tag{2.56}$$

$$\left[\Omega + I_m \sum_{j=1}^{r} \mathcal{C}(r,j)(-1)^j q^{-(j-1)}\right] s_k = \left[\sum_{j=0}^{r} \mathcal{C}(r,j)(-1)^j q^{-j}\right] F_k \tag{2.57}$$

and the ith entry of F_k can be related to that of s_k as follows:

$$F_k^i = \frac{\Omega^i + q \sum_{j=1}^{r} \mathcal{C}(r,j)(-1)^j q^{-j}}{\triangle^r(q^{-1})} s_k^i \tag{2.58}$$

Note that, during the derivation of the filter equation from s to F, some properties of the binomial expansion have been used. However, the same result would also be obtained replacing $E(q^{-1})$ by $\triangle^r(q^{-1})$ in (2.39), i.e;

$$\begin{aligned} F_k &= \frac{q}{\triangle^r(q^{-1})}[\triangle^r(q^{-1}) - (I_m - \Omega q^{-1})]s_k \\ &= \frac{1}{\triangle^r(q^{-1})}[\Omega + qI_m \sum_{j=1}^{r} \mathcal{C}(r,j)(-1)^j q^{-j}]s_k \end{aligned} \tag{2.59}$$

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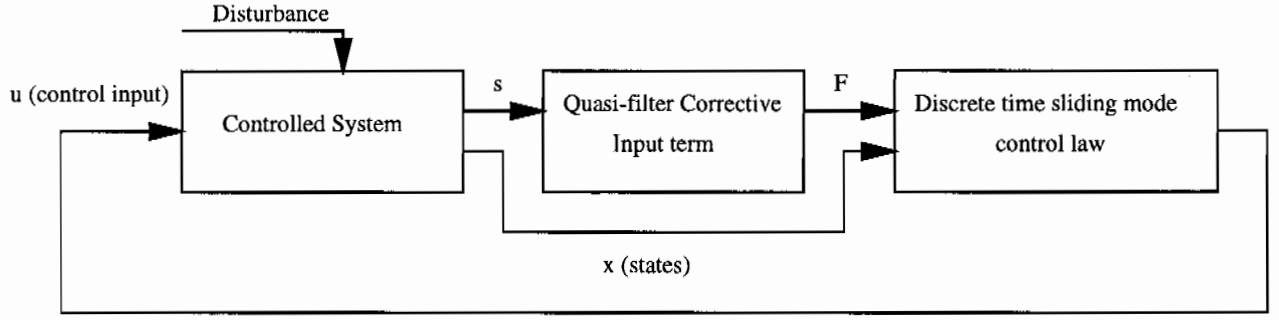
Figure 2.1: Block diagram for the closed loop system

Note that, the orders of the denominator and the numerator polynomials are the same and therefore the filter equation is realizable. With this control law, the trajectories are kept around the manifold with an accuracy in the order of T^r and slide towards to the origin while in sliding mode.

The complete control structure is shown in Figure 2.1 and Theorem 2 summarizes the robust discrete time sliding mode control laws.

Theorem 2 *The control law*

$$u_k = -(C\Gamma)^{-1}[C\Phi x_k - F_k] \tag{2.60}$$

satisfies robustness of the controller to external disturbances which are at least r^{th} order differentiable choosing each component of F_k

$$F_k^i = \frac{q[E(q^{-1}) - e_0(1 - \Omega^i q^{-1})]}{E(q^{-1})} s_k^i$$

if $E(q^{-1})D_k^i = 0$,

$$F_k^i = -\frac{(1 - \Omega^i)}{(1 - q^{-1})} s_k^i$$

if $D_k = D_{k-1}$, and finally

$$F_k^i = \frac{\Omega^i + q\sum_{j=1}^{r} \mathcal{C}(r,j)(-1)^j q^{-j}}{\triangle^r(q^{-1})} s_k^i$$

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if the disturbance dynamic is unknown, but modeled as if $\triangle^r(q^{-1})D_k^i = 0$ where $|\Omega^i| < 1$ and $D_k = CF d_k$.

2.5 Simulations

In this section, the robustness of the proposed discrete time sliding mode controllers is tested on a truck-semitrailer system. The following linear truck-semitrailer model is used for simulation purposes and its derivation is given in Appendix A.

$$\dot{x} = \begin{bmatrix} -14.5045 & 12.2412 & 6.5102 & 10.2890 \\ -11.3014 & -53.7688 & 3.2160 & 5.0828 \\ 2.6833 & 5.6393 & -3.3453 & -5.2011 \\ 0 & -1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} -110.8316 \\ 118.5998 \\ 1.0463 \\ 0 \end{bmatrix} \delta \quad (2.61)$$

where

$$x = \begin{bmatrix} V_1 \\ r_1 \\ r_2 \\ \Psi \end{bmatrix}$$

V_1 : Lateral velocity of the tractor
r_1 : Yaw rate of the tractor
r_2 : Yaw rate of the semitrailer
Ψ : Articulation angle
δ : Steer angle on the front tires of the tractor

The problem considered here is to bring the articulated vehicle into a straight position after a maneuver or a turn under the presence of an uncertainty in the system. The angle on the front tires of the tractor is controlled using the information about the lateral velocity of the tractor, yaw rates of both the tractor and the semitrailer and the articulation angle for this purpose. Choosing the sampling time as 0.1 *sec.*, the following discrete time system parameters are calculated:

$$\Phi = \begin{bmatrix} 0.2016 & 0.0867 & 0.3236 & 0.4640 \\ -0.0540 & -0.0117 & -0.0080 & -0.0156 \\ 0.0620 & 0.1028 & 0.7508 & -0.3555 \\ 0.0141 & -0.0081 & 0.0853 & 0.9785 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -4.1689 \\ 3.0130 \\ 0.5830 \\ -0.1977 \end{bmatrix}$$

using $\Phi = e^{AT}$ and $\Gamma = \int_0^T e^{A\lambda} B d\lambda$. Figure 2.2 shows the external disturbance in
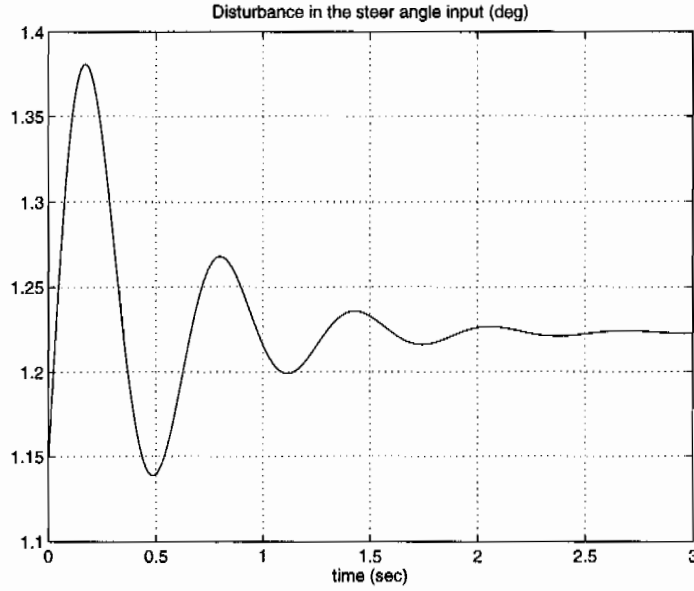
Figure 2.2: Disturbance in the steer angle

the system. It has three poles at $s = 0, -2 \pm 10j$ and is added to the input of the system. The overall uncertain system has the following representation

$$x_{k+1} = \Phi x_k + \Gamma u_k + F d_k \qquad (2.62)$$

where $F = \Gamma$, d_k represents the value of the disturbance at $t = kT$ and the objective is to steer x_k to the origin from an initial condition. Note that, the input coefficient matrix Γ is not in the sampling time order. Therefore, a prospective problem of finding a control law which may not be in the admissible control domain does not exist. The following simulations compare the two robust discrete time sliding mode controllers to the ideal equivalent control. For one of the robust controllers, the disturbance dynamic is assumed to be known exactly, whereas for the other, a dynamical model is fitted assuming that the second order difference of the disturbance is constant.

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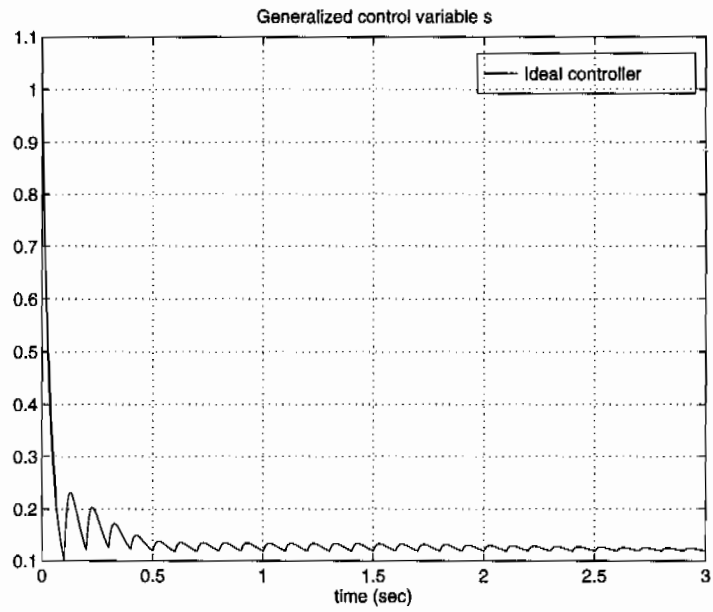
Figure 2.3: Generalized control variable with ideal controller
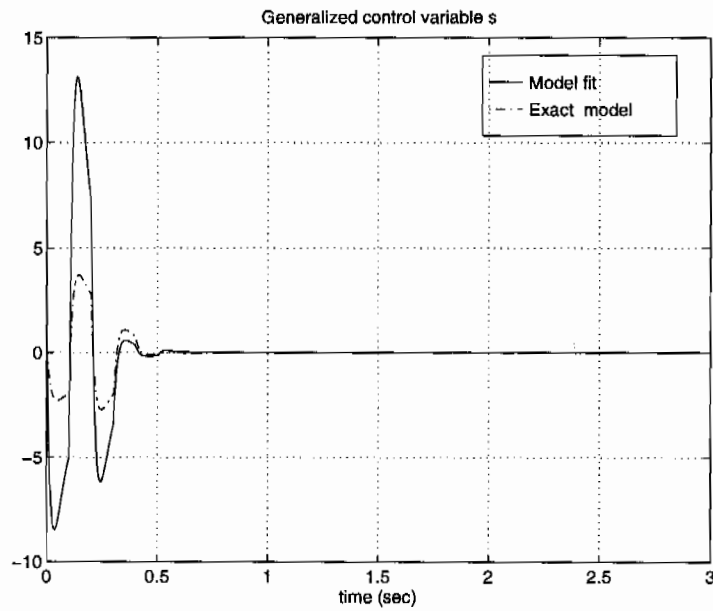
Figure 2.4: Generalized control variable with robust controllers

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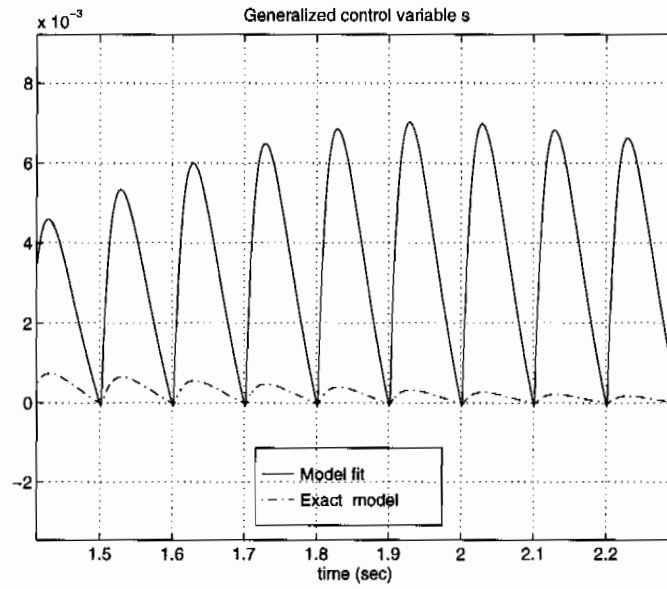
Figure 2.5: Generalized control variable with robust controllers
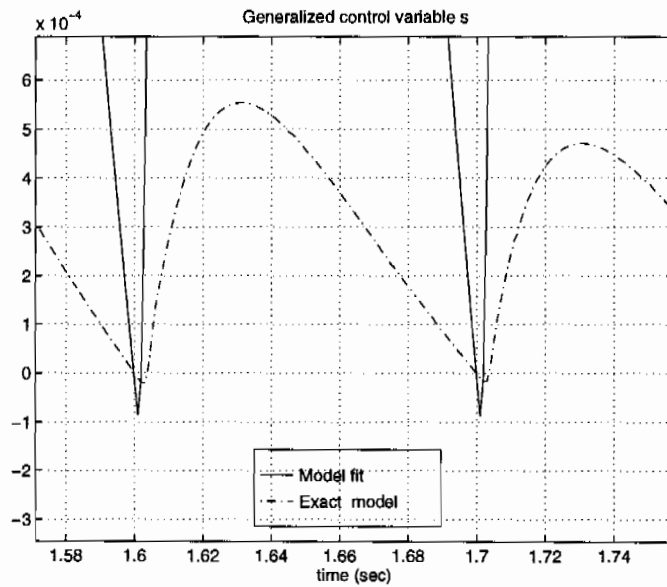
Figure 2.6: Generalized control variable with robust controllers

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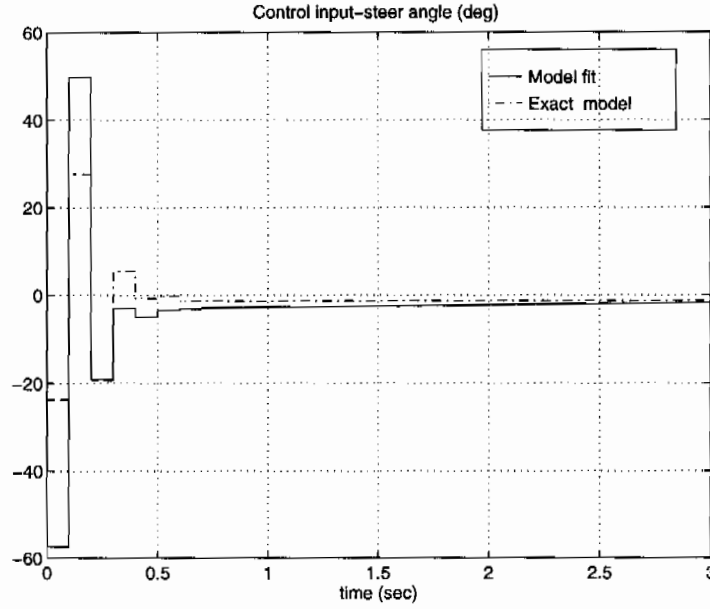
Figure 2.7: Control inputs with robust controllers

The ideal equivalent control law cannot regulate s to zero as shown in Figure 2.3. Figures 2.4, 2.5 and 2.6 all show the generalized control variable s in three different scales in case of application of the robust controllers. Both control laws successfully steer the trajectories to the manifold at each sampling instant. However, the trajectories deviate from the manifold between two consecutive sampling instances because of sampled data implementation of the control law, but the deviations are in the sampling time order. Figure 2.6 presents the positions of the trajectories at each sampling instant in a small scale. The superiority of the first robust control law which uses the exact disturbance dynamics over the other where the disturbance model is also estimated dynamically is obvious examining the approximity of the trajectories to the manifold. Figure 2.7 compares the control inputs for both controllers. Note that, the control law of known disturbance dynamics is smoother than the other,

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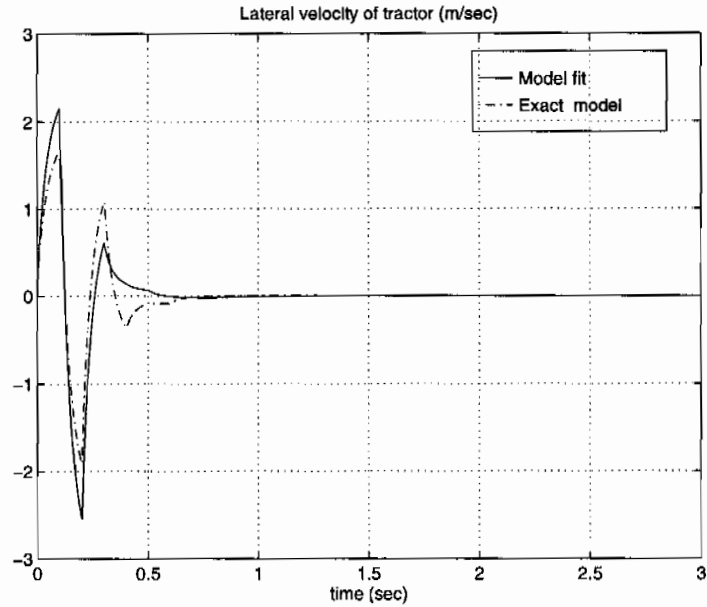
Figure 2.8: Tractor lateral velocity

because the former exactly learns the disturbance after a small transient whereas the latter estimates it within an accuracy.

To sum up, both controllers achieve the control objective successfully in the presence of a disturbance. The filter-like structure of the control laws eliminates the effect of the disturbance from the generalized control variable s almost completely. The state trajectories are confined to the vicinity of the manifold with a sufficiently high accuracy, discrete time sliding mode takes place and the trajectories eventually converge to the origin with a dynamic adopted by C. Note that for both cases, the desired eigenvalues of sliding mode has been located at $z = 0.5, 0.7, 0.8$ in discrete time and Ω has been chosen as a zero matrix to resemble the ideal equivalent control. Figures 2.8-2.11 show the states of the system for both control laws on top of each other.

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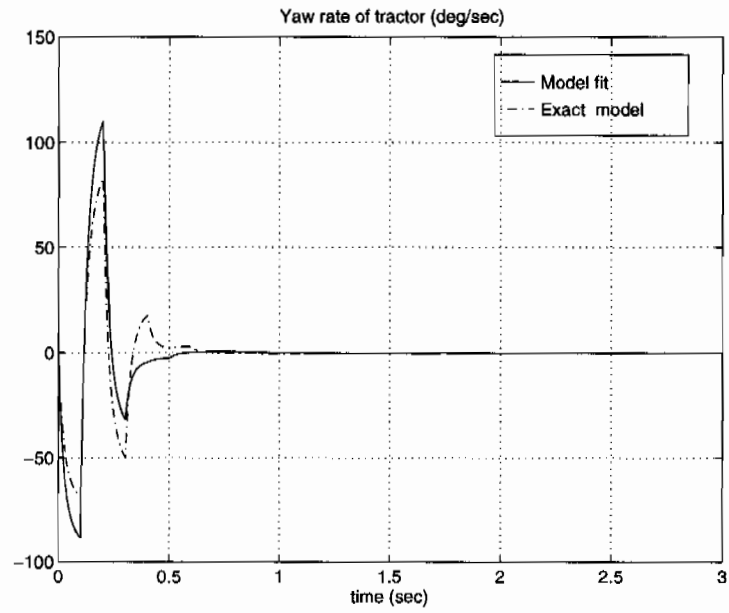
Figure 2.9: Tractor yaw rate
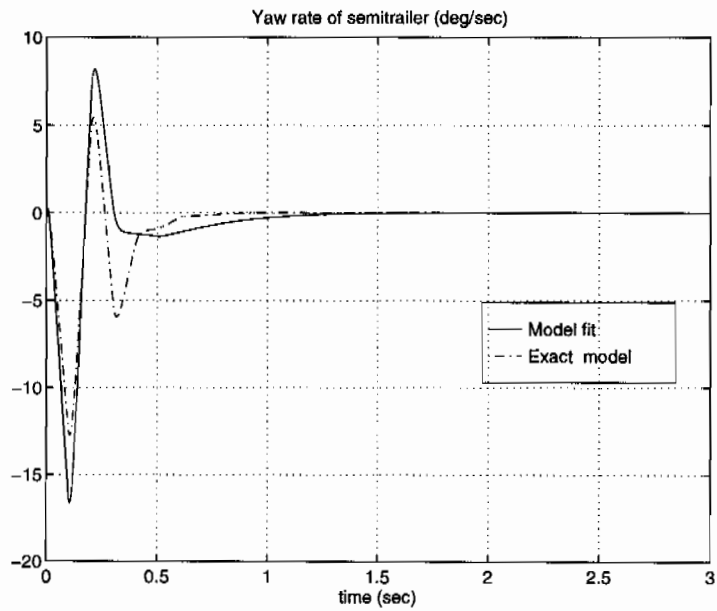
Figure 2.10: Semitrailer yaw rate
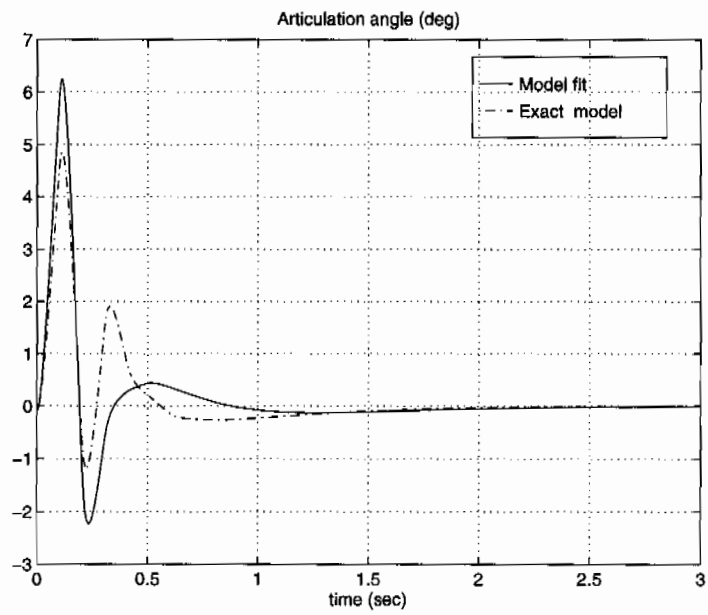
Figure 2.11: Articulation angle

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CHAPTER 3

CONTINUOUS TIME SLIDING MODE OBSERVERS

Many controllers in the multivariable system context are normally of the linear or nonlinear state feedback type and they call for the complete information about the the plant states if they are to be implemented. However, it is either impossible or inappropriate to measure all the state variables because of some practical considerations. Therefore, if we want to retain the benefits of state feedback type controllers, this problem has to be overcome. The usage of an observer is the most feasible solution to the problem. An observer is simply an auxiliary dynamic system driven by the available system inputs and the outputs which yields the original state vector at least asymptotically under some structural assumptions on the system in consideration. The reconstructed state vector can then be substituted for the real state vector in the original feedback control law. Figure 3.1 illustrates an open loop state reconstruction process.

The organization of the chapter is as follows: Section 3.1 reviews the fundamental asymptotic observers and constructs the basics of an observer design process. In section 3.2, first a reduced order observer is designed using the sliding mode theory and then the possibility of designing an observer which reconstructs the original state vector of the system in finite time is discussed. Equivalent control methodology is
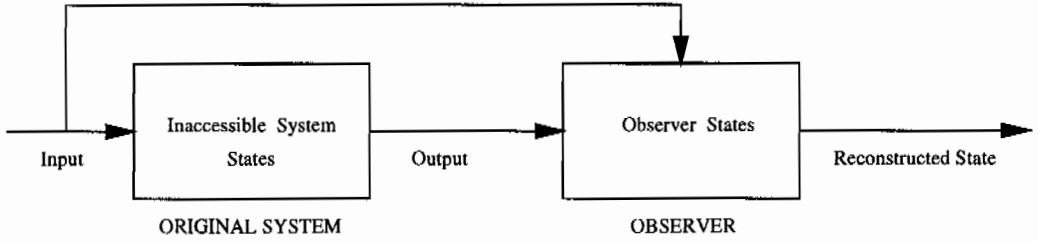
Figure 3.1: Open loop state observation

exploited for this purpose. Finally, the finite time convergence characteristics of the proposed continuous time sliding mode observers are tested on a linearized truck-semitrailer system during a lane change-type maneuver in section 3.3.

3.1 Asymptotic Observers

In this section, we review the classical observer design methods initiated by David G. Luenberger. Two design methods are picked up to summarize the frequently used observer design methods up to this time. Consider

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
\tag{3.1}
$$

where $x \in R^n$, $u \in R^p$, $y \in R^m$ and A, B, C are constant matrices of appropriate dimensions. The (A, C) pair is assumed to be observable. We further assume that C has full rank without loss of generality. The state vector of the system is unknown, however a reduced order linear combination of the states is available in the form of the output. The question is whether we can design a new dynamic system which presumably generates the original state vector just using the output and the input vectors of the system. This question is first answered by D.G. Luenberger. According

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to his method, an observer is built in the following form:

$$\dot{\hat{y}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{3.2}$$

Subtracting (3.2) from (3.1) yields

$$\dot{e}(t) = (A - LC)e(t) \tag{3.3}$$

where $e(t) = x(t) - \hat{x}(t)$ represents the difference between the actual and the observed value of the state vector. Since the system considered is observable, eigenvalues of the matrix $(A - LC)$ can be assigned arbitrarily selecting the free observer parameter properly such that $e(t) \to 0_{n \times 1}$ as $t \to \infty$ and the eigenvalues of $(A - LC)$ determine the convergence rate. In the literature, this observer is usually called full order Luenberger observer since the observer order is equal to the system order.

However, since there are m outputs of the system, a proper static transformation on y should yield m of the state variables directly. With this transformation the new state equation appears as follows:

$$\begin{array}{rcl} \dot{y} & = & A_{11}y + A_{12}x_1 + B_1u \\ \dot{x}_1 & = & A_{21}y + A_{22}x_1 + B_2u \end{array} \tag{3.4}$$

Because y is known, it does not need to be estimated and the original problem of estimating the n-dimensional state vector can be replaced by a reduced order problem of reconstructing only x_1. Let the observer related to x_1 be

$$\dot{\hat{x}}_1 = F\hat{x}_1 + G_1y + G_2u + L\dot{y} \tag{3.5}$$

Inserting the expression of $\dot{y}$ into (3.5) and subtracting (3.5) from (3.4) yield the following error dynamics

$$\dot{e}_1 = Fe_1 + (A_{21} - LA_{11} - G_1)y + (A_{22} - F - LA_{12})x_1 + (B_2 - G_2 - LB_1)u \tag{3.6}$$

where e_1 denotes the error in the variable x_1. Choosing

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- $F = A_{22} - LA_{12}$

- $G_1 = A_{21} - LA_{11}$

- $G_2 = B_2 - LB_1$

the error dynamics related to x_1 become independent of the states and the inputs of the system and can be guaranteed to decay to zero satisfying the relation

$$\dot{e}_1(t) = (A_{22} - LA_{12})e_1(t) \tag{3.7}$$

choosing the matrix $(A_{22} - LA_{12})$ is stable. This necessitates an observability assumption on the pair (A_{22}, A_{12}). However, because the original system is observable, this pair is also observable since the observability of the (A, C) pair is sufficient for this observability condition. Therefore eigenvalues of $(A_{22} - LA_{12})$ can arbitrarily be placed and $e_1 \rightarrow 0_{(n-m)\times 1}$ as $t \rightarrow \infty$. To get rid of the $\dot{y}$ term in the observer equation, a new variable is defined

$$\eta = \hat{x}_1 - Ly \tag{3.8}$$

which satisfies the following dynamical relation:

$$\dot{\eta} = (A_{22} - LA_{12})\eta + (A_{22}L - LA_{12}L + A_{12} - LA_{11})y + (B_2 - LB_1)u \tag{3.9}$$

and $\hat{x}_1 = \eta + Ly$ yields the final estimate. Note that the order of the observer has reduced to $(n - m)$ as desired and convergence takes place asymptotically.

3.2 Sliding Mode Observers Using Equivalent Control

In the previous section, we discussed two Luenberger-type observer design methods. In both of them, a dual system which has the same dynamics with those of the

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variable to be observed is formed with an additional linear term and the convergence is guaranteed selecting this auxiliary term properly. However, the state reconstruction takes place asymptotically. The eigenvalues of the associated error variable dynamics can be chosen very far away from the jw-axis to increase the decay rate, but asymptotic convergence characteristic of a Luenberger observer cannot be removed completely. Therefore, these observers will be called asymptotic observers from now on.

However, it would be more appropriate to reconstruct the state vector in finite time for feedback control purposes. In the rest of the chapter we discuss the sliding mode observer design theory using the equivalent control concept to satisfy this aim. We first start with the sliding mode realization of a reduced order asymptotic observer to construct the basics and the terminology of the sliding mode observer design theory that will be used throughout the chapter.

3.2.1   Sliding Mode Realization of a Reduced Order Observer

In this section, the problem of state estimation of a linear system through the information on the known variables is studied in the framework of the sliding mode theory. Consideration will be given again to the system

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
\tag{3.10}
$$

where $x \in R^n$, $u \in R^p$, $y \in R^m$ and A, B, C are constant matrices with proper dimensions. The system is observable and C has full rank without loss of generality as usual. For the problem of reconstructing the state vector at least asymptotically using only the input and the output vectors of the system, we propose an observer in

the following form

$$\dot{\hat{x}} = A\hat{x} + Bu + Lsgn(y - \hat{y}) \tag{3.11}$$

where *sgn* function is defined as columnwise, i.e;

$$sgn\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} sgn(x_1) \\ sgn(x_2) \\ \vdots \\ sgn(x_{n-1}) \\ sgn(x_n) \end{bmatrix}$$

with

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and $L \in R^{n \times m}$ is the free observer parameter to be chosen so as to satisfy the convergence of $\hat{x}$ to x. Taking the difference between (3.11) and (3.10) yields

$$\dot{e}_x = Ae_x - Lsgn \, e_y \tag{3.12}$$

where $e_x = x - \hat{x}$ and $e_y = y - \hat{y}$. Yet, the answer to the question of choosing L such that $e_x \to 0$ at least as $t \to \infty$ is not straightforward at this stage. To this end, the system given in (3.10) is passed through a static transformation to obtain directly the available m outputs of the system as a part of its n dimensional state vector. Using a transformation which has the form below

$$T = \begin{bmatrix} C \\ R \end{bmatrix}$$

where $R \in R^{(n-m) \times n}$ is arbitrary provided that T is invertible and defining $x_1 = Rx$, the following representation is obtained

$$\begin{array}{rcll} \dot{y} & = & A_{11}y + A_{12}x_1 + B_1u & m \\ \dot{x}_1 & = & A_{21}y + A_{22}x_1 + B_2u & n-m \end{array} \tag{3.13}$$

where

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad TB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

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A sliding mode observer which will somehow reconstruct the state vector of the new representation from y and u is described by

$$
\begin{aligned}
\dot{\hat{y}} &= A_{11}\hat{y} + A_{12}\hat{x}_1 + B_1 u + L_1 sgn\ e_y \\
\dot{\hat{x}}_1 &= A_{21}\hat{y} + A_{22}\hat{x}_1 + B_2 u + L_2 sgn\ e_y
\end{aligned}
\tag{3.14}
$$

where $L_1 \in R^{m \times m}$ and $L_2 \in R^{(n-m) \times m}$ are the parameters of the observer.

Representing the mismatch in the process by the state reconstruction error

$$
\begin{aligned}
e_y &= y - \hat{y} \\
e_{x_1} &= x_1 - \hat{x}_1
\end{aligned}
\tag{3.15}
$$

following error dynamics are obtained

$$
\begin{aligned}
\dot{e}_y &= A_{11}e_y + A_{12}e_{x_1} - L_1\ sgn\ e_y \\
\dot{e}_{x_1} &= A_{21}e_y + A_{22}e_{x_1} - L_2\ sgn\ e_y
\end{aligned}
\tag{3.16}
$$

and the state reconstruction problem turns into a regulation problem in e_y and e_{x_1}. This problem is solved using sliding mode theory at two steps as follows: First, the error trajectories are steered to a manifold defined by

$$
\mathcal{S} = \{e_x | e_y = y - \hat{y} = 0\}
\tag{3.17}
$$

Let

$$
V = \frac{1}{2}e_y^2 \geq 0
$$

be a Lyapunov function for the the variable e_y. Choosing L_1 as a diagonal matrix where each entry is greater than $\| A_{11}e_y + A_{12}e_{x_1} \|_\infty$, we get

$$
\frac{dV}{dt} = e_y \dot{e}_y = e_y(A_{11}e_y + A_{12}e_{x_1} - L_1\ sgn\ e_y) \leq 0
$$

along the error trajectories so that e_y decreases monotonically and eventually becomes zero. Note that the equality signs hold only when e_y is the zero vector, hence the system is asymptotically stable indeed and an L_1 which ensures the asymptotic

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stability can always be found for a bounded initial state. Therefore, sliding mode takes place on the manifold a finite time later. In sliding mode the equivalent value of the first observer auxiliary input is calculated as follows:

$$[L_1 \; sgn \; e_y]_{eq} = A_{12}e_{x_1} \qquad (3.18)$$

substituting $e_y = 0$ and $\dot{e}_y = 0$ into (3.16). In this setting, the equivalent control is used to describe the average of the discontinuous control law when the associated trajectories are on the manifold and it can be obtained by passing the discontinuous input term through a low pass filter so as to eliminate the high frequency oscillations resulted from imperfections without loosing the error dynamics. Inserting (3.18) into (3.16) produces the error motion in sliding mode:

$$\begin{aligned}
\dot{e}_{x_1} &= A_{21}e_y + A_{22}e_{x_1} - L_2 L_1^{-1}[L_1 \; sgn \; e_y]_{eq} \\
\dot{e}_{x_1} &= A_{22}e_{x_1} - L_2 L_1^{-1} A_{12}e_{x_1} \\
\dot{e}_{x_1} &= (A_{22} - L_2 L_1^{-1} A_{12})e_{x_1}
\end{aligned} \qquad (3.19)$$

If the original system is observable, the pair (A_{22}, A_{12}) is also observable and the eigenvalues of the matrix $(A_{22} - L_2 L_1^{-1} A_{12})$ can be assigned arbitrarily. With an appropriate L_2, e_{x_1} decays exponentially and observer states converge to the actual states. Therefore, the goal of state reconstruction using sliding modes is achieved. Note that, this is exactly the sliding mode realization of a reduced order asymptotic observer. A sliding mode is induced in the error associated with the output vector in finite time and after that, the remaining $(n - m)$ error variables die out exponentially. Since the original state vector is simply a linear combination of the m dimensional finite time converging and the $(n - m)$ dimensional exponentially converging state variables, the original state reconstruction takes place asymptotically. Theorem 1 summarizes the design procedure.

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Theorem 1 *The sliding mode observer*

$$\dot{\hat{x}} = A\hat{x} + Bu + Lsgn(y - \hat{y})$$

reconstructs the state vector of the system given in (3.10) asymptotically choosing

$$L = T^{-1}\begin{bmatrix} L_1 \\ L_2 L_1 \end{bmatrix},$$

$L_1 \in R^{m \times m}$ *is a positive definite diagonal matrix whose each diagonal term is greater than* $\| A_{11}e_y + A_{12}e_{x_1} \|_\infty$, $L_2 \in R^{(n-m) \times m}$ *is the matrix which places the eigenvalues of* $(A_{22} - L_2 A_{12})$ *at desired locations and* T *is the similarity transformation matrix given by*

$$T = \begin{bmatrix} C \\ R \end{bmatrix}$$

with an arbitrary $(n - m) \times n$ *matrix* R *as long as* T *is nonsingular. The* (A, C) *pair is assumed to be observable and the eigenvalues of the matrix* $(A_{22} - L_2 A_{12})$ *determine the reconstruction speed.*

3.2.2 Finite Time Converging Sliding Mode Observers

In the previous section, we discussed the sliding mode realization of a reduced order asymptotic observer using equivalent control concept. However, exact convergence took place again at infinity. In this section the design algorithm will be generalized to satisfy the finite time convergence of the all states.

We start from (3.13) which is repeated below for convenience.

$$\begin{array}{lll} \dot{y} & = & A_{11}y + A_{12}x_1 + B_1u \qquad m \\ \dot{x}_1 & = & A_{21}y + A_{22}x_1 + B_2u \qquad n-m \end{array} \tag{3.20}$$

The corresponding sliding mode observer for the first equation of (3.20) is again written as a replica of it with an additional nonlinear auxiliary input term as follows:

$$\dot{\hat{y}} = A_{11}\hat{y} + A_{12}\hat{x}_1 + B_1u + L_1 sgn(y - \hat{y}) \tag{3.21}$$

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where L_1 is a positive definite, diagonal $m \times m$ matrix. Subtracting (3.21) from the first equation of (3.20), one gets:

$$\dot{e}_y = A_{11}e_y + A_{12}e_{x_1} - L_1 \, sgn \, e_y \tag{3.22}$$

where $e_y = y - \hat{y}$. With a sufficiently large L_1, e_y and $\dot{e}_y$ are guaranteed to have different signs componentwise so that sliding mode takes place on the manifold defined by

$$\mathcal{S} = \{e_x | e_y = 0\} \tag{3.23}$$

as before. So far, the same design procedure of the preceding section has been followed. However, at this stage we define a new variable $y_1 = L_1^{-1} A_{12} x_1$ and combine the second equation of (3.20) with this final equation to form a new system as follows:

$$\begin{array}{rclc} \dot{x}_1 & = & A_{22}x_1 + A_{21}y + B_2 u & \quad n-m \\ y_1 & = & L_1^{-1} A_{12} x_1 & \quad m \end{array} \tag{3.24}$$

$m \times (n-m)$

This system has exactly the same form with the one which we have started with, where x_1 is the state of the system, y_1 is the output and y and u are the inputs. Furthermore, the dimension of the system has reduced to $(n - m)$ and the output vector of the new system is m dimensional provided that A_{12} has full rank.

Note that, we originally started with a larger dimensional system and reconstructed m of the observer states in finite time. Now, a valid question is whether we can also use the same procedure to observe some states of the observer associated with this reduced order system. If so, we will provide the finite time convergence of more than m states and hopefully come up with a full finite time converging observer sequentially applying this logic.

The answer of this question depends on some assumptions on the system. If

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- the pair $(A_{22}, L_1^{-1} A_{12})$ is observable

- the matrix $L_1^{-1} A_{12}$ has full rank

- y_1 is available

then we can apply the same procedure. The first assumption automatically holds if the original system is observable since observability of the pair (C, A) is sufficient for observability of the pair (A_{22}, A_{12}). The second assumption may or may not hold since A_{12} is a system matrix which cannot be adjusted. However, even if the rank condition is not satisfied, a new variable $y_1' = \overset{\bar{m} \times m}{K} y_1$ can be defined

$$
\begin{aligned}
y_1' &= K L_1^{-1} A_{12} x_1 \\
y_1' &= \bar{A}_{12} x_1
\end{aligned}
\tag{3.25}
$$

where $\mathrm{rank}(\bar{A}_{12}) = \bar{m}$ for $\bar{m} < m$ so that the final system is rewritten with an output vector of lower dimension. Therefore, the rank condition can also be assumed to be satisfied without loss of generality. The third assumption looks very unrealistic at first glance because y_1 is itself a linear combination of the unknown states of the system and hence it is not available directly. However, this information can be extracted from the system exploiting the equivalent control method. Before discussing the details of how this method solves the problem, we need to give the basics of it for a continuous time system.

In the literature, the equivalent control of a system is defined as the average value of the discontinuous control in sliding mode which is itself enough to keep the state trajectories on the manifold, thereafter they hit the manifold. Mathematically, it is the solution of the dynamic equation associated with the generalized control variable for the control input imposing the constraints of sliding mode. For example, for a

linear continuous time system let $\mathcal{S}$ denote the manifold

$$\mathcal{S} = \{x | s = Cx = 0\}. \tag{3.26}$$

The sliding mode dynamics are

$$\dot{s} = CAx + CBu \tag{3.27}$$

and the equivalent control is found equating $\dot{s}$ and s to zero as follows:

$$u_{eq} = -(CB)^{-1}CAx \tag{3.28}$$

Outside the manifold, the control law

$$u = -(CB)^{-1}(CAx + K \, sgn \, s) \tag{3.29}$$

is sufficient to steer the state trajectories and to keep them on the manifold with a positive K. With this control law, the trajectories converge to the manifold monotonically and as soon as they reach the manifold, the control law starts to switch. The switching frequency theoretically should be infinite to completely eliminate the oscillations around the manifold. However, in real life the switching frequency is limited by some practical constraints. So, there will be always some oscillations around the manifold. The magnitude of these oscillations is related to the magnitude of the discontinuous term whereas the oscillation frequency is mainly determined by the switching frequency. When the discontinuous control law is filtered by a low pass filter whose cutoff frequency is less than the switching frequency but higher than the maximum frequency of the system dynamics, the high frequency oscillations in the input are eliminated and the filter output yields the equivalent control. This property of the equivalent control can be used to extract some additional information from the

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system. When considering the prospective sliding mode observer, $L_1\, sgn\, e_y$ term is considered as the input which enforces sliding mode and its equivalent value should be found. Solving (3.22) for $L_1\, sgn\, e_y$ after replacing e_y and $\dot{e}_y$ by zero produces

$$\begin{aligned}
[L_1\, sgn\, e_y]_{eq} &= A_{12}e_{x_1} \\
&= L_1 e_{y_1}
\end{aligned} \qquad (3.30)$$

Therefore, even though y_1 is not directly available, a linear transformed version of the error in this variable can be obtained passing the first observer auxiliary input through a low pass filter to yield its equivalent as discussed before.

Since all the assumptions have been satisfied, the original problem can be replaced by a reduced order problem of reconstructing the state vector of the system given in (3.24). Using the same steps, (3.24) is first transformed into

$$\begin{aligned}
\dot{y}_1 &= A_{31}y_1 + A_{32}x_2 + A_{33}y + B_3u \\
\dot{x}_2 &= A_{41}y_1 + A_{42}x_2 + A_{43}y + B_4u
\end{aligned} \qquad (3.31)$$

with an appropriate similarity transformation matrix given by

$$T_1 = \begin{bmatrix} L_1^{-1}A_{12} \\ R_1 \end{bmatrix}$$

where $R_1 \in R^{(n-2m)\times n}$ is arbitrary as long as T_1 is invertible and $x_2 = R_1 x_1$. The sliding mode observer for the first equation of (3.31) is built as follows:

$$\dot{\hat{y}}_1 = A_{31}\hat{y}_1 + A_{32}\hat{x}_2 + A_{33}\hat{y} + B_3u + L_2\, sgn(y_1 - \hat{y}_1) \qquad (3.32)$$

Subtracting (3.32) from the first equation of (3.31) provides

$$\dot{e}_{y_1} = A_{31}e_{y_1} + A_{32}e_{x_2} + A_{33}e_y - L_2\, sgn\, e_{y_1} \qquad (3.33)$$

where

$$\begin{aligned}
e_{y_1} &= y_1 - \hat{y}_1 \quad \text{and} \\
e_{x_2} &= x_2 - \hat{x}_2
\end{aligned} \qquad (3.34)$$

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When sliding mode occurs on $\mathcal{S}$, e_y becomes a zero vector and (3.33) turns into

$$\dot{e}_{y_1} = A_{31}e_{y_1} + A_{32}e_{x_2} - L_2\,sgn\,e_{y_1} \tag{3.35}$$

At this stage, we define a new manifold $\mathcal{S}_1$

$$\mathcal{S}_1 = \{e_x|e_{y_1} = 0 \cap e_y = 0\} \tag{3.36}$$

and choose a proper L_2 to guarantee that e_{y_1} and $\dot{e}_{y_1}$ have different signs, componentwise. Sliding mode takes place after a finite time and m of the state variables of the system in (3.31) converges zero.

Since

$$\begin{aligned} L_2\,sgn\,e_{y_1} &= L_2\,sgn\,(L_1^{-1}A_{12}e_{x_1}) \\ &= L_2\,sgn\,(L_1^{-1}[L_1\,sgn\,e_y]_{eq}) \\ L_2\,sgn\,e_{y_1} &= L_2\,sgn\,([\,sgn\,e_y]_{eq}) \end{aligned} \tag{3.37}$$

final observer equation for the reduced order system is rewritten as follows:

$$\dot{\hat{y}}_1 = A_{31}\hat{y}_1 + A_{32}\hat{x}_2 + A_{33}\hat{y} + B_3u + L_2\,sgn\,([\,sgn\,e_y]_{eq}) \tag{3.38}$$

At the next step, we define a new variable $y_2 = L_2^{-1}A_{32}x_2$ and form a new system as follows:

$$\begin{aligned} \dot{x}_2 &= A_{42}x_2 + A_{43}y + A_{41}y_1 + B_4u \\ y_2 &= L_2^{-1}A_{32}x_2 \end{aligned} \tag{3.39}$$

Assuming that A_{32} has also full rank, one can transform (3.39) into

$$\begin{aligned} \dot{y}_2 &= A_{51}y_2 + A_{52}x_3 + A_{53}y + A_{54}y_1 + B_5u \\ \dot{x}_3 &= A_{61}y_2 + A_{62}x_3 + A_{63}y + A_{64}y_1 + B_6u \end{aligned} \tag{3.40}$$

and choose

$$\dot{\hat{y}}_2 = A_{51}\hat{y}_2 + A_{52}\hat{x}_3 + A_{53}\hat{y} + A_{54}\hat{y}_1 + B_5u + L_3\,sgn(y_2 - \hat{y}_2) \tag{3.41}$$

as the observer equation for the variable y_2. Subtracting (3.41) from the first equation of (3.40) provides

$$\dot{e}_{y_2} = A_{51}e_{y_2} + A_{52}e_{x_3} + A_{53}e_y + A_{54}e_{y_1} - L_3\,sgn\,e_{y_2} \tag{3.42}$$

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and in sliding mode, (3.42) turns into

$$\dot{e}_{y_2} = A_{51}e_{y_2} + A_{52}e_{x_3} - L_3 \, sgn \, e_{y_2} \tag{3.43}$$

replacing both e_y and e_{y_1} by zero. We then choose L_3 as a diagonal matrix with sufficiently large diagonal terms to steer the error trajectories to a new manifold defined by

$$\mathcal{S}_2 = \{e_x | e_{y_2} = 0 \, \cap e_{y_1} = 0 \, \cap \, e_y = 0\} \tag{3.44}$$

and sliding mode takes place after a finite time.

To generalize the procedure, let us assume that, we have the following reduced order system at k^{th} step.

$$
\begin{aligned}
\dot{y}_{k-1} &= A_{2k-1,1}y_{k-1} + A_{2k-1,2}x_k + A_{2k-1,3}y + A_{2k-1,4}y_1 + \cdots \\
&\quad + A_{2k-1,k+1}y_{k-2} + B_{2k-1}u \\
\dot{x}_k &= A_{2k,1}y_{k-1} + A_{2k,2}x_k + A_{2k,3}y + A_{2k,4}y_1 + \cdots \\
&\quad + A_{2k,k+1}y_{k-2} + B_{2k}u
\end{aligned}
\tag{3.45}
$$

The sliding mode observer for the first equation of (3.45) is chosen in the form

$$
\begin{aligned}
\dot{\hat{y}}_{k-1} &= A_{2k-1,1}\hat{y}_{k-1} + A_{2k-1,2}\hat{x}_k + A_{2k-1,3}\hat{y} + A_{2k-1,4}\hat{y}_1 + \cdots \\
&\quad + A_{2k-1,k+1}\hat{y}_{k-2} + B_{2k-1}u + L_k \, sgn \, e_{y_{k-1}}
\end{aligned}
\tag{3.46}
$$

and the error in y_{k-1} is found to be

$$
\begin{aligned}
\dot{e}_{y_{k-1}} &= A_{2k-1,1}e_{y_{k-1}} + A_{2k-1,2}e_{x_k} + A_{2k-1,3}e_y + A_{2k-1,4}e_{y_1} + \cdots \\
&\quad + A_{2k-1,k+1}e_{y_{k-2}} - \mathcal{L}^k(e_y)
\end{aligned}
\tag{3.47}
$$

where $\mathcal{L}_k(.)$ has been introduced as a new operator

$$\mathcal{L}^k(.) = L_k \underbrace{sgn(\cdots sgn[\, sign(.)_{eq}]_{eq})}_{k} \tag{3.48}$$

just for notational purposes. Defining a new manifold $\mathcal{S}_{k-1}$

$$\mathcal{S}_{k-1} = \{e_x | e_{y_{k-1}} = 0 \, \cap \, \mathcal{S}_{k-1} \subset \mathcal{S}_{k-2}\} \tag{3.49}$$

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and choosing a sufficiently large L_k, the error trajectories are steered to it. At the next step, a new variable $y_k = L_k^{-1} A_{2k-1,2} x_k$ is defined, a new system of a lower dimension than the one step before is formed, this system is transformed into a new state space realization and so on and so forth.

Note that this process can be hopefully continued to satisfy the finite time convergence of the all states. At each step, the dimension of the state vector of the new system becomes less than that of the system of one step before. Therefore, this process is presumed to terminate a finite step later. The termination of the procedure depends on the number of states, the number of outputs and the (A, B, C) triple. Figure 3.2 shows the structure of the observer and the possible termination ways can be summed in two cases.

- The number of states of the system is an integer multiple of the number of outputs and all $A_{2i-1,2}$'s have full rank. If this is the case, the procedure terminates at the kth step where k is n/m and n, m are the number of states and outputs of the system, respectively.

For this case, the observer equations designed at each step can be gathered together to form the complete observer as follows:

$$
\begin{aligned}
\dot{\hat{y}} &= A_{11}\hat{y} + L_1\hat{y}_1 + B_1 u + \mathcal{L}^1(e_y) \\
\dot{\hat{y}}_1 &= A_{31}\hat{y}_1 + L_2\hat{y}_2 + A_{33}\hat{y} + B_3 u + \mathcal{L}^2(e_y) \\
\dot{\hat{y}}_2 &= A_{51}\hat{y}_2 + L_3\hat{y}_3 + A_{53}\hat{y} + A_{54}\hat{y}_1 + B_5 u + \mathcal{L}^3(e_y) \\
&\vdots \\
\dot{\hat{y}}_{k-1} &= A_{2k-1,1}\hat{y}_{k-1} + A_{2k-1,3}\hat{y} + A_{2k-1,4}\hat{y}_1 + \cdots \\
&\quad + A_{2k-1,k+2}\hat{y}_{k-2} + B_{2k-1} u + \mathcal{L}^k(e_y)
\end{aligned}
\tag{3.50}
$$

where each L_i is an $m \times m$, diagonal matrix with positive entries and the degree of the $\hat{y}_i$'s is the same as that of the original output. Therefore, the observer and the

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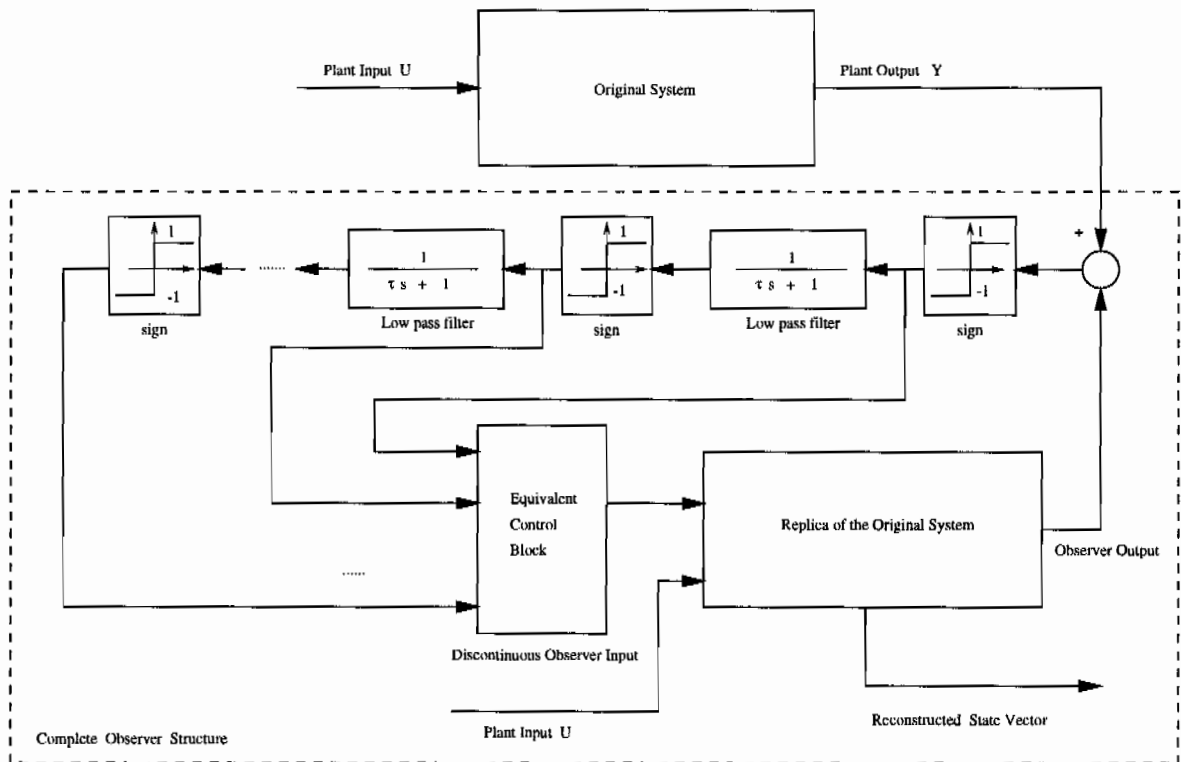
Figure 3.2: State reconstruction with a full sliding mode observer

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system orders are the same. (3.50) can also be written as a state space equation as follows:

$$\dot{\hat{\mathcal{Y}}} \;=\; \bar{A}\hat{\mathcal{Y}} + \bar{B}u + \mathcal{U} \tag{3.51}$$

where

$$\bar{A} \;=\; \begin{bmatrix} A_{11} & L_1 & 0 & 0 & \cdots & 0 & 0 \\ A_{33} & A_{31} & L_2 & 0 & \cdots & 0 & 0 \\ A_{53} & A_{54} & A_{51} & L_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ A_{2k-3,3} & A_{2k-3,4} & A_{2k-3,5} & \cdots & \cdots & A_{2k-3,1} & L_{k-1} \\ A_{2k-1,3} & A_{2k-1,4} & A_{2k-1,5} & \cdots & \cdots & A_{2k-1,k+2} & A_{2k-1,1} \end{bmatrix},$$

$$\bar{B} \;=\; \begin{bmatrix} B_1 \\ B_3 \\ B_5 \\ \vdots \\ B_{2k-3} \\ B_{2k-1} \end{bmatrix},$$

$$\text{and} \quad \hat{\mathcal{Y}} \;=\; \begin{bmatrix} \hat{y} \\ \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_{k-2} \\ \hat{y}_{k-1} \end{bmatrix}, \quad \mathcal{U} \;=\; \begin{bmatrix} \mathcal{L}^1(e_y) \\ \mathcal{L}^2(e_y) \\ \mathcal{L}^3(e_y) \\ \vdots \\ \mathcal{L}^{k-1}(e_y) \\ \mathcal{L}^k(e_y) \end{bmatrix}$$

However, the state variables of the observer of (3.51) are very different from those of the original system. If the original system is also brought to the same form with the observer

$$\dot{\mathcal{Y}} \;=\; \bar{A}\mathcal{Y} + \bar{B}u \tag{3.52}$$

where

$$\mathcal{Y} \;=\; \begin{bmatrix} y \\ y_1 \\ y_2 \\ \vdots \\ y_{k-2} \\ y_{k-1} \end{bmatrix}$$

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the error dynamics are obtained in the following matrix differential form

$$\dot{\mathcal{E}} \;=\; \bar{A}\mathcal{E} - \mathcal{U} \tag{3.53}$$

where

$$\mathcal{E} \;=\; \mathcal{Y} - \hat{\mathcal{Y}} \;=\; \begin{bmatrix} e_y \\ e_{y_1} \\ e_{y_2} \\ \vdots \\ e_{y_{k-2}} \\ e_{y_{k-1}} \end{bmatrix}$$

subtracting (3.51) from (3.52).

To sum up, the error vector $\mathcal{E}$ converges to zero selecting each L_i large enough to enforce sliding mode. Note that, initially the observer error vector is arbitrarily located in the transformed state space. If the first observer parameter L_1 is chosen properly, error trajectories are steered to the first manifold which has been defined as $\mathcal{S} = \{e_x | e_y = 0\}$ and sliding mode occurs on it. After that, the error trajectories are also steered to a subset of the previous manifold called the second manifold which has been defined as $\mathcal{S}_1 = \{e_x | e_y = 0 \cap e_{y_1} = 0\}$ passing the first observer input through a low pass filter to get e_{y_1} and choosing L_2 sufficiently large. As the process continues, error trajectories eventually converge to the origin of the state space traveling from one manifold to the next manifold which is a subset of the previous one.

However, the observer state variables differ from those of the system considerably. $\bar{A}$ and $\bar{B}$ should be computed for implementation purposes. Furthermore, the final reconstructed observer state vector will be a transformed version of the original state vector. However, it would be more appropriate to design the sliding mode observer in the same state space realization with the system. This can be achieved by retransforming every single observer equation into its original form before proceeding to the

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next step. It is a very straightforward process and Theorem 2 summarizes the result without going into further detail.

Theorem 2 *The continuous-time sliding mode observer for the system given in (3.10) has the following form:*

$$\dot{\hat{x}} = A\hat{x} + Bu + P_1^{-1} \left[P_2^{-1} \left[P_3^{-1} \left[\begin{array}{c} L_1 sgn(e_y) \\ L_2 \ sgn[\, sgn(e_y)_{eq}] \\ L_3 \ sgn(\, sgn[\, sign(e_y)_{eq}]_{eq}) \\ P_4^{-1} \quad \cdot \cdot \cdot \end{array} \right] \right] \right] \tag{3.54}$$

where $e_y = y - \hat{y}$;

$$P_1 = \left[\begin{array}{c} C \\ R_1 \end{array} \right] \qquad P_1 A P_1^{-1} = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right]$$

$$P_2 = \left[\begin{array}{c} L_1^{-1} A_{12} \\ R_2 \end{array} \right] \qquad P_2 A_{22} P_2^{-1} = \left[\begin{array}{cc} A_{31} & A_{32} \\ A_{41} & A_{42} \end{array} \right]$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$P_l = \left[\begin{array}{c} L_{l-1}^{-1} A_{2l-3,2} \\ R_l \end{array} \right]$$

where $A_{2k-1,1} \in R^{m \times m}$, $A_{2k-1,2} \in R^{m \times (n-km)}$, $A_{2k,1} \in R^{(n-km) \times m}$ *and* $A_{2k,2} \in R^{(n-km) \times (n-km)}$; L_k's *are positive definite diagonal matrices and* R_k's *are arbitrary with appropriate dimensions as long as* P_k's *are invertible. It has been assumed that the matrices* $L_{k-1}^{-1} A_{2k-3,2}$ *have full rank,* $n/m = l$ *is an integer and the system is observable.*

- Some $A_{2k-3,2}$'s do not have full rank. If the number of states of the given system is not an integer multiple of the number of outputs, this situation is inevitable. At each step the dimension of the new system is reduced by the degree of the output vector up to facing a rank drop, so that at one step the number of the output of the resulting system will definitely be more than the number of states

of the system which means that rank at that stage cannot be full anymore. The same situation can happen even if the number of states of the given system is an integer multiple of the number of outputs.

If this is the case, the same design procedure is applicable. However, one has to do some matrix operations at the level where rank drop has occurred to be able to proceed to the next step. The order of the final sliding mode observer will be less than the system order and Theorem 2 cannot be used. However, the original system can be transformed into a new form as in (3.52) and the observer design is performed based on this new realization.

3.3 Simulations

In this section, we test the performance of the proposed continuous time sliding mode observers on a truck-semitrailer system whose linear model has already been used in Chapter 2. The problem is to reconstruct the state vector of the system during a maneuver with a constant longitudinal velocity of 15 m/s using the information on the yaw rates of the tractor and the semitrailer. It is assumed that the sensor readings are the actual values and the model exactly represents the system.

We first start with the design of a reduced order observer using equivalent control. The number of outputs of the system for this case is two and there are four states of the system. Therefore, sliding mode is induced on the known 2 states and the errors in the remaining states decay asymptotically in sliding mode. In the simulations, the eigenvalues of the observer have been located at -2 and -4 and L has been chosen
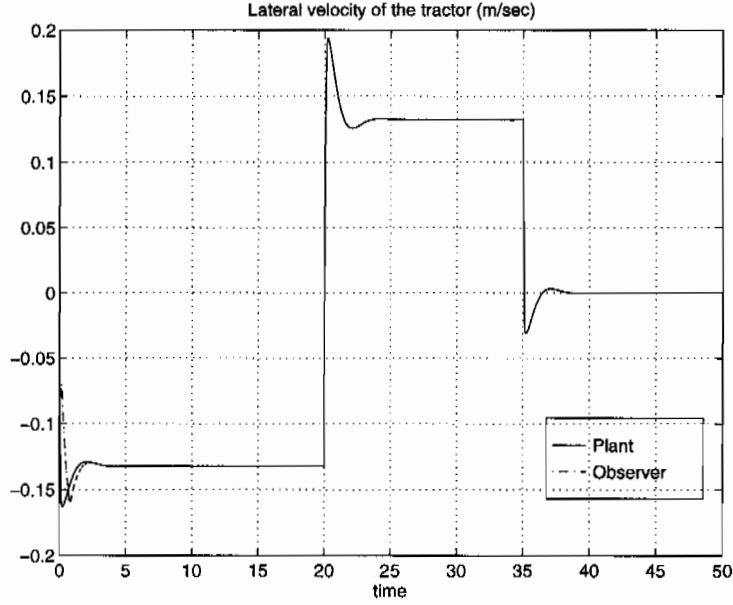
Figure 3.3: Tractor lateral velocity (Reduced order sliding mode)

as follows:

$$L = \begin{bmatrix} 0.1074 & -0.0653 \\ 0.1 & 0 \\ 0 & 0.1 \\ -0.0591 & -0.1588 \end{bmatrix}$$

Figures 3.3-3.6 show the reconstructed and the original state variables on top of each other. Note that, the observer states which correspond to the yaw rates of the tractor and the semitrailer converge immediately. After sliding mode takes place on these variables the remaining observer states related to the lateral velocity of the tractor and the articulation angle also converge to the actual values.

Second, we design a full sliding mode observer for the same scenario. The ratio between the number of states and the outputs of the system is two and there has been found no rank drop. Therefore, Theorem 2 is applicable and the design procedure terminates at two steps. Note that, the matrices P_1 and P_2 are not unique since they
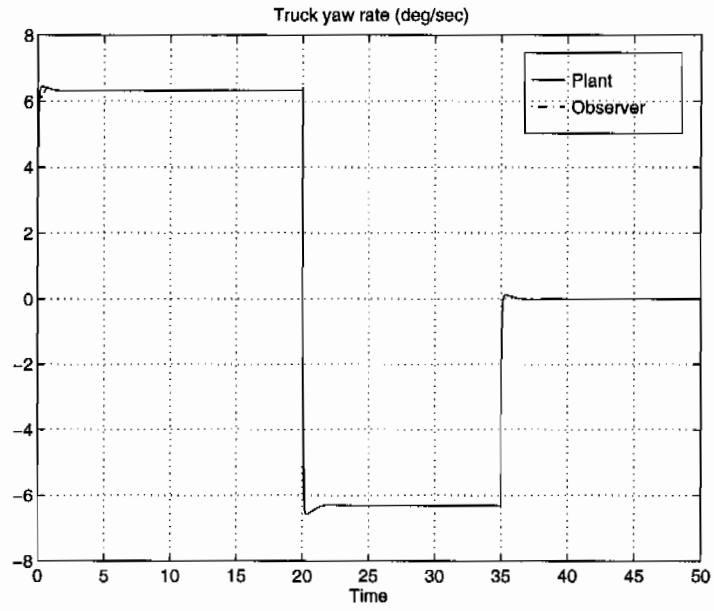
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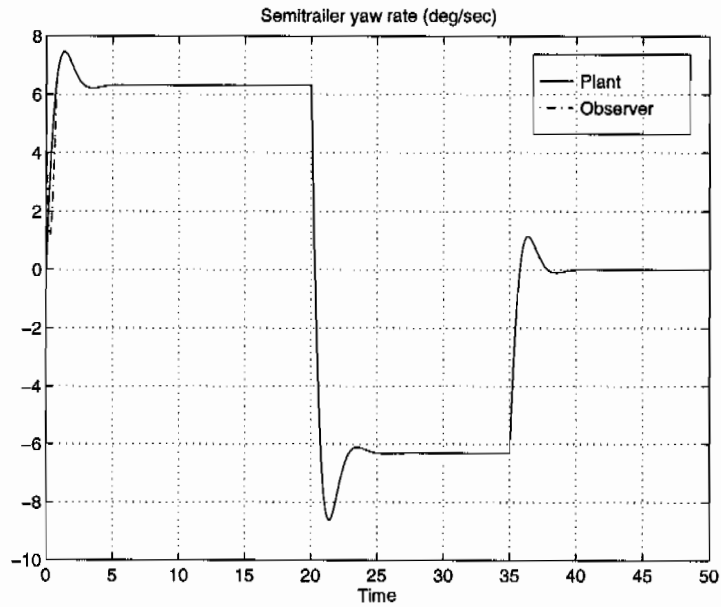
Figure 3.4: Tractor yaw rate (Reduced order sliding mode)

Figure 3.5: Semitrailer yaw rate (Reduced order sliding mode)

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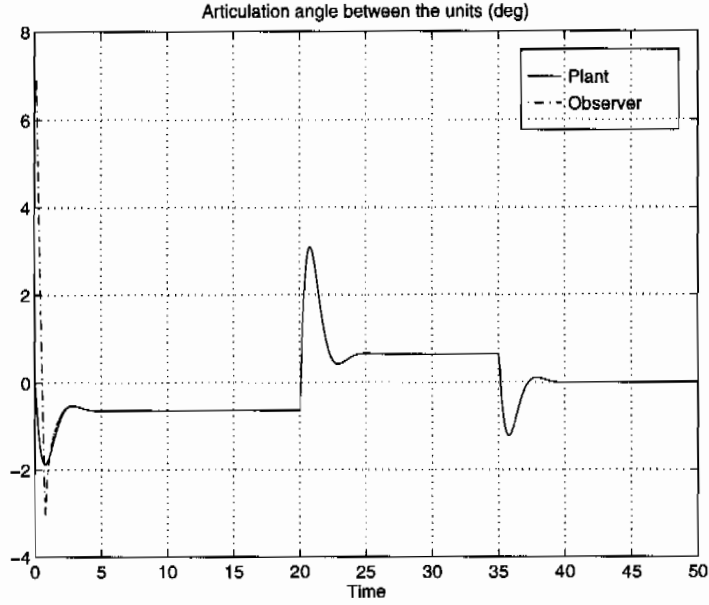
Figure 3.6: Articulation angle (Reduced order sliding mode)

both have a random part. In the simulations, we have chosen them as follows:

$$
P_1 \;=\; \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.0346 & 0.5297 & 0.0077 & 0.0688 \\ 0.0535 & 0.6771 & 0.3834 & 0.4175 \end{bmatrix} \;,\quad P_2 \;=\; \begin{bmatrix} -3.0632 & 0.5716 \\ 0.8585 & -0.2205 \end{bmatrix},
$$

$L_1 = 0.15 I_2$, $L_2 = 0.15 I_2$ and the time constant of the filter through which the discontinuous auxiliary observer input has been passed to obtain the equivalent value of it has been 5 *msec*.

In simulations, the block which represents the continuous time truck-semitrailer has been located between a sampler and a zero order hold and the discretized version of the continuous time sliding mode observer with a sampling time of 0.01 *sec.* has been implemented to resemble a real sampled data control system. Figures 3.7-3.10 show the observer and the system state variables on top of each other in case of application of the full sliding mode observer. The convergence takes place in finite

60
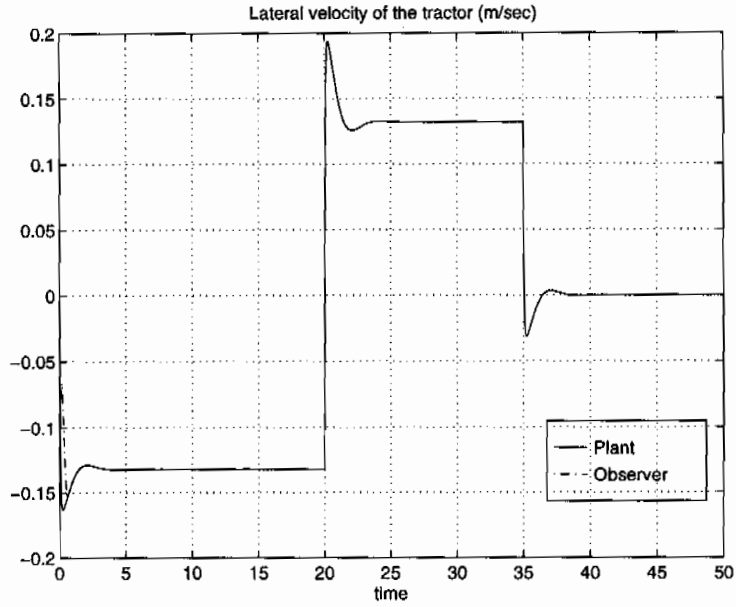
Figure 3.7: Tractor lateral velocity (Full sliding mode)

time and the convergence time is very small as expected. The zoomed version of Figure 3.7 related to truck lateral velocity has also been presented to illustrate the intersampling behavior of the observer motion around the manifold. The amplitude of these oscillations could be reduced choosing L_1 and L_2 smaller but in that case the convergence rate would decrease as well. With the sampled data implementation, observer takes corrective action only at the sampling instances and the motion is arbitrary between them.

The comparison of the two simulation sets reveals the superiority of the full sliding mode observer over the sliding mode implemented reduced order one in terms of their convergence characteristics. The observer states related to the yaw rates of the tractor and the semitrailer converge in finite time for both cases. However, the remaining states of the reduced order observer converge asymptotically. Since the smallest

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Figure 3.8: Tractor yaw rate (Full sliding mode)
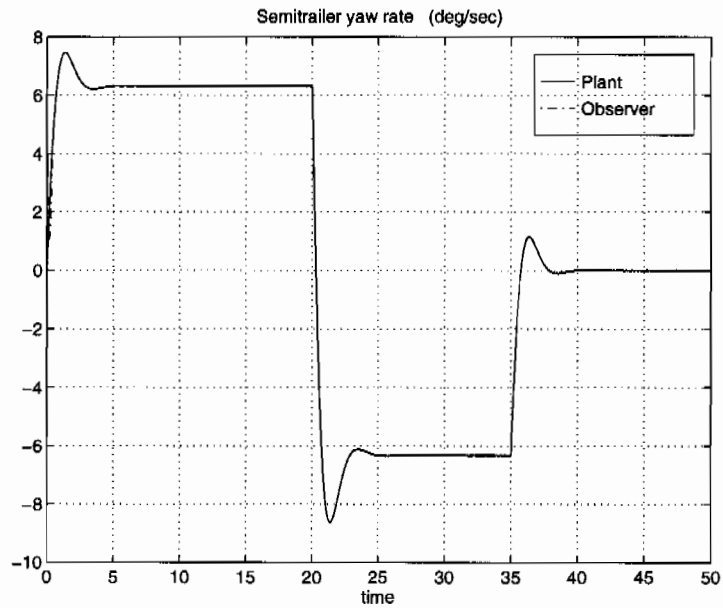
Figure 3.9: Semitrailer yaw rate (Full sliding mode)

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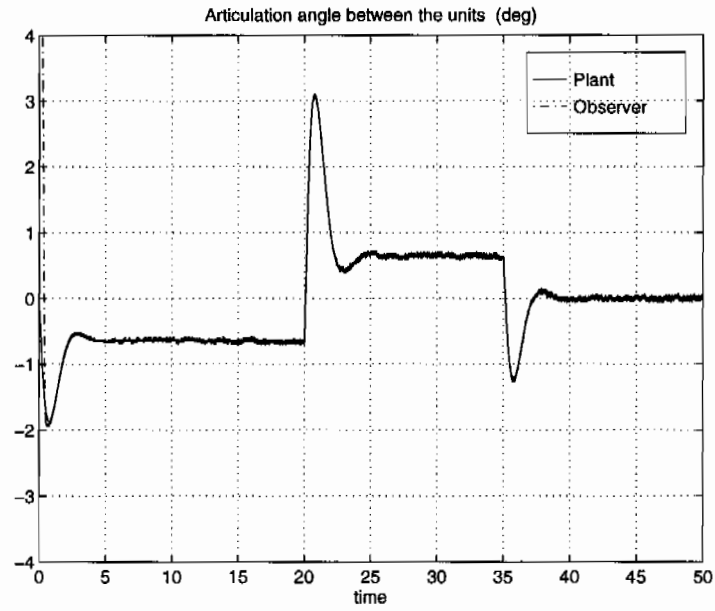
Figure 3.10: Articulation angle (Full sliding mode)

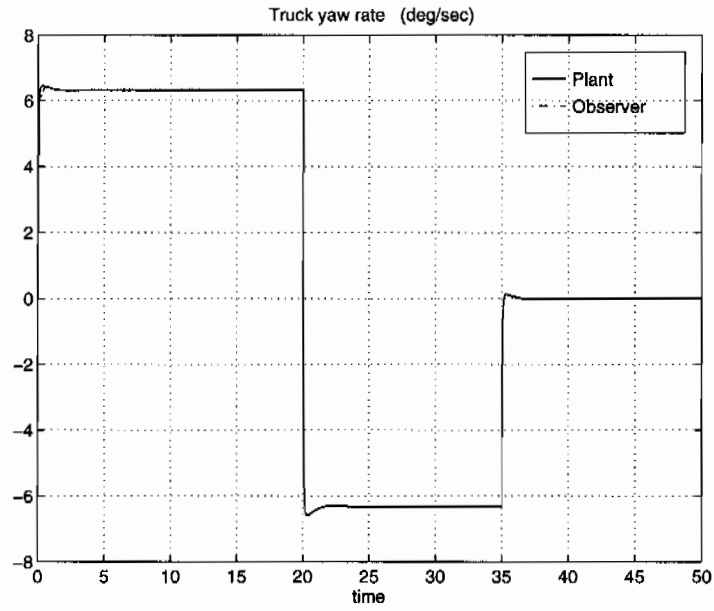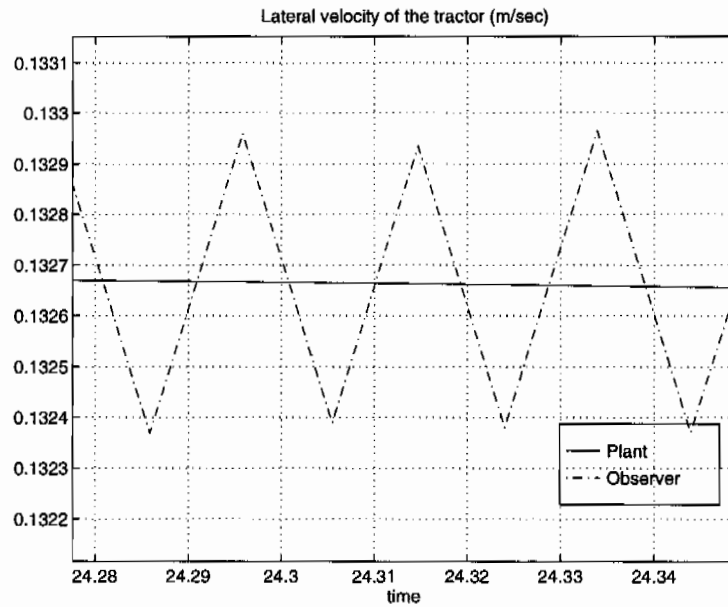Figure 3.11: Small scale tractor lateral velocity (Full sliding mode)

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observer eigenvalue is -2, the errors in these variables reach 2% of their initial values at most at $t = 2.5$ *sec.* so that just looking at the convergence times may not be enough to understand the asymptotic convergence situation. However, examining these states carefully, one can see that the error associated with each of those states first goes to zero and then escapes but finally becomes zero again and remains there. In case of the the full sliding mode observer these errors reach zero and thereafter they remain at that value. This is the most feasible way of exploring the asymptotic convergence or finite time converging characteristic of a variable.

In conclusion, the proposed full sliding mode observer has achieved the goal of state reconstruction in finite time, however at the expense of small chattering terms in the observer states because of the sampled data implementation. Sliding mode implemented reduced order observer has also reconstructed the original state vector in very short time, however the convergence has taken place asymptotically and its states contain small magnitude, high frequency oscillations as well.

CHAPTER 4

DISCRETE TIME SLIDING MODE OBSERVERS

In the previous chapter, continuous time observer design methods have been discussed. We started with the review of standard Luenberger-type asymptotic state observers and stated their asymptotic state convergence characteristics as their major weakness. The convergence speed would be increased placing the observer poles further left from the imaginary axis but this would decrease the bandwidth of the observer. This gave us the motivation for studying the possible ways of designing a finite time converging observer in the framework of the sliding mode theory. The original system has been transformed sequentially into proper forms and the complete observer structure showed a nested structure. The equivalent control methodology has been exploited from different perspectives and used to extract information on the unknown quantities. Simulations on a truck-semitrailer illustrated the effectiveness of the design procedure over the standard asymptotic observers. However, discrete time implementation of the continuous time observer caused the observer states to chatter around the real states because of the discontinuous terms in the observer equation. Theoretically, these high frequency oscillations can be removed with an infinite switching frequency. However, practical implementation issues of computer

control require the extension of the design procedure for a full discrete time observer using the discrete time sliding mode theory.

In this chapter, we will first summarize some standard discrete time observer design methods and build the terminology that will be used later in the discrete time sliding mode observer discussion. Then the continuous time sliding mode observer design procedure will be extended to discrete time systems and proposed observers will be tested on the estimation of the unmeasurable states of the truck-semitrailer system of the previous chapters in simulations as usual.

4.1 Standard Discrete Time Observers

In this section, discrete time versions of the continuous time Luenberger observers will be summarized. Consideration is given to the following discrete time system

$$
\begin{aligned}
x_{k+1} &= \Phi x_k + \Gamma u_k \\
y_k &= C x_k
\end{aligned}
\tag{4.1}
$$

where $x \in R^n$, $u \in R^p$, $y \in R^m$ and Φ, Γ, C are constant matrices of appropriate dimensions. The pair (Φ, C) is assumed to be observable. State variables of the system are unknown, but a linear combination of them as a vector of lower dimension in the form of the output is available. The problem is to form a dynamical system which admits the output and the input of the original system as its input and yields the estimated states such that its output tends to the system states at least asymptotically. We further assume that there is no parametric uncertainties or external disturbances and the model exactly represents the system.

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4.1.1 Full Order Asymptotic Observers

We build an observer

$$\hat{x}_{k+1} = F\hat{x}_k + G_1 y_k + G_2 u_k \tag{4.2}$$

and want to satisfy $\hat{x}_k \to x_k$ while $k \to \infty$ properly choosing the observer parameters F, G_1 and G_2. Defining

$$e_k = x_k - \hat{x}_k \tag{4.3}$$

the dynamical equation that the error variable e_k satisfies is given by

$$
\begin{aligned}
e_{k+1} &= \Phi x_k + \Gamma u_k - F\hat{x}_k - G_1 y_k - G_2 u_k \\
&= \Phi(e_k + \hat{x}_k) + \Gamma u_k - F\hat{x}_k - G_1 C(e_k + \hat{x}_k) - G_2 u_k \\
&= (\Phi - G_1 C)e_k + (\Phi - F - G_1 C)\hat{x}_k + (\Gamma - G_2)u_k
\end{aligned} \tag{4.4}
$$

Picking F and G_2

$$
\begin{aligned}
F &= \Phi - G_1 C \\
G_2 &= \Gamma
\end{aligned} \tag{4.5}
$$

the error dynamics become independent of $\hat{x}$ and u

$$e_{k+1} = (\Phi - G_1 C)e_k \tag{4.6}$$

Under the observability assumption for the pair (Φ, C), the eigenvalues of the matrix $(\Phi - G_1 C)$ can be located inside the unit disc and the convergence of e to the zero vector is guaranteed. After choosing G_1 according to the desired convergence dynamics, the other observer parameters F and G_2 are calculated from (4.5) and the final observer equation becomes

$$
\begin{aligned}
\hat{x}_{k+1} &= (\Phi - G_1 C)\hat{x}_k + G_1 y_k + G_2 u_k \\
&= \Phi\hat{x}_k + \Gamma u_k + G_1(y_k - C\hat{x}_k)
\end{aligned} \tag{4.7}
$$

and the observer structure is illustrated in Figure 4.1.
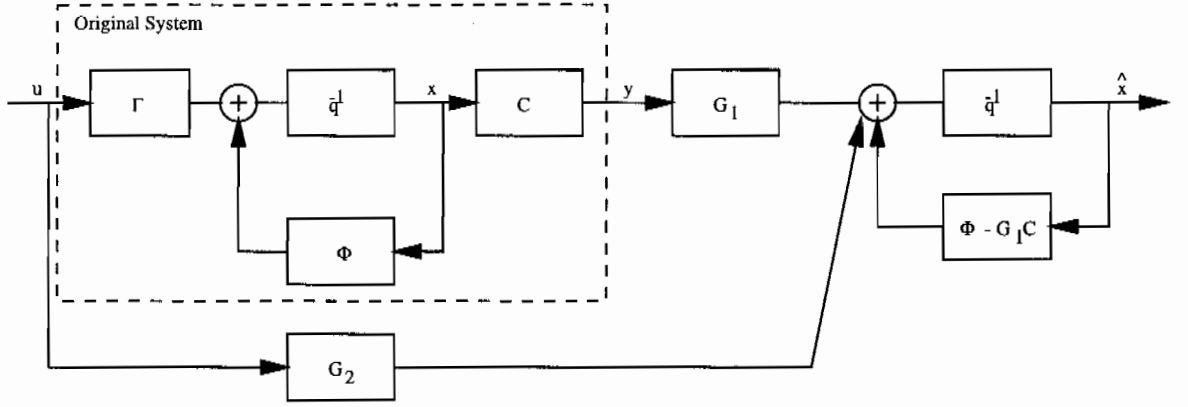
Figure 4.1: Structure of a Full Order Asymptotic Observer

4.1.2 Reduced Order Asymptotic Observers

The full order observer designed in the previous section is a dynamic system of order n. However, since the m linear combinations of the states are available in the form of the output, m states can be obtained directly by a static transformation on y. Therefore, a proper $(n-m)^{th}$ order observer would be enough for the full state observation. Defining a new vector variable $\eta \in R^{n-m}$, we obtain

$$\left[\begin{array}{c} y \\ \eta \end{array} \right] \; = \; \left[\begin{array}{c} C \\ T \end{array} \right] \; x \tag{4.8}$$

If the matrix $\left[\begin{array}{c} C \\ T \end{array} \right]$ is invertible and η is known then x can be obtained directly. Manipulating the definition for η with the system model given in (4.1), one obtains

$$\eta_{k+1} = T\Phi x_k + T\Gamma u_k \tag{4.9}$$

Let the corresponding observer equation for η be

$$\hat{\eta}_{k+1} = F\hat{\eta}_k + G_1 u_k + G_2 y_k \tag{4.10}$$

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where $F \in R^{(n-m)\times(n-m)}$, $G_1 \in R^{(n-m)\times p}$ and $G_2 \in R^{(n-m)\times m}$ are the observer parameters to be chosen so as to achieve the state estimation objective. Subtracting (4.10) from (4.9) yields the error dynamics as follows:

$$
\begin{array}{rcl}
e_{\eta,k+1} & = & T\Phi x_k + T\Gamma u_k - F(Tx_k - e_{\eta,k}) - G_1 u_k - G_2 C x_k \\
e_{\eta,k+1} & = & (T\Phi - FT - G_2 C)x_k + F e_{\eta,k} + (T\Gamma - G_1)u_k
\end{array}
\tag{4.11}
$$

We want to have $\eta \to 0$ as $k \to \infty$ independent of x and u. Picking

$$
\begin{array}{l}
G_1 = T\Gamma \\
T\Phi - FT = G_2 C,
\end{array}
\tag{4.12}
$$

the error dynamics become

$$
e_{\eta_{k+1}} = F e_{\eta,k}
\tag{4.13}
$$

If T is chosen as

$$
T = \sum_{i=1}^{n} t_i P_i^T
\tag{4.14}
$$

where

$$
\begin{bmatrix} P_1^T \\ P_2^T \\ \vdots \\ P_n^T \end{bmatrix} = Q^{-1} = [q_1 q_2 \cdots q_n],
$$

q_i's are the right eigenvectors of Φ, the following identity is obtained

$$
(\lambda_i I - F)t_i = G_2 C q_i
\tag{4.15}
$$

Since the observability of the pair (C, Φ) is sufficient for $Cq_i \neq 0$, $(\lambda_i I - F)$ is invertible if the poles of the system are different of those of the observer. Bearing this in mind and choosing

- F such that $\lambda\{F\} \cap \lambda\{\Phi\} = \emptyset$ and F is stable

- G_2 such that $\text{rank}(G_2) = \min(n - m, m)$

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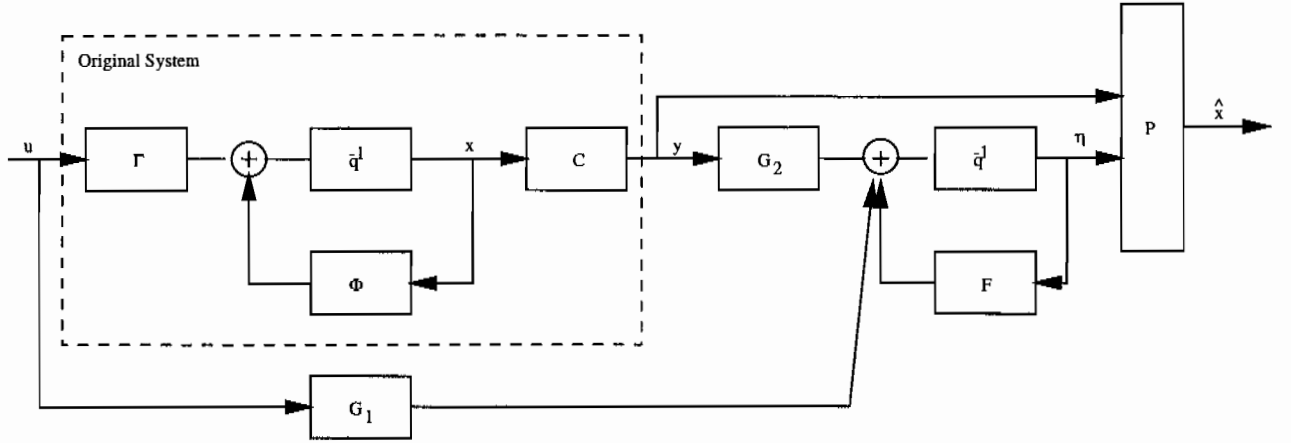
Figure 4.2: Structure of a Reduced Order Asymptotic Observer

- T and G_1 such that they satisfy the constraints in (4.12) and the matrix

$$\left[\begin{array}{c} C \\ T \end{array} \right]$$

has full rank,

e_η is guaranteed to converge zero and the convergence rate is determined by the eigenvalues of F. Using

$$\left[\begin{array}{c} C \\ T \end{array} \right] x = \left[\begin{array}{c} y \\ \eta \end{array} \right]$$

the estimated state vector is calculated by

$$x_k = \left[\begin{array}{c} C \\ T \end{array} \right]^{-1} \left[\begin{array}{c} y_k \\ \eta_k \end{array} \right] \tag{4.16}$$

Note that the order of the observer equation given in (4.10) is $(n - m)$ indeed and the observer structure is illustrated in Figure 4.2.

4.2 Discrete Time Sliding Mode Observers

In the previous section, we reviewed two standard discrete time observer design methods. In both methods, the observer model simply has been a replication of

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the original model with an extra error term and the state convergence has been provided assigning the poles of the closed loop error dynamics with a proper selection of the observer parameters. Therefore, these design methods have exactly the same reasoning with the standard asymptotic continuous time observers of the previous chapter. However, unlike the continuous time case, the finite time convergence of these discrete time observers can be guaranteed placing the eigenvalues of the matrix $(\Phi - G_1 C)$ at the origin. The solution to the difference equation of (4.6) is given by

$$e_{k+r} = (\Phi - G_1 C)^r e_k \tag{4.17}$$

However,

$$(\Phi - G_1 C)^r = 0_{n \times 1} \tag{4.18}$$

for $r > n$ if all the eigenvalues of $(\Phi - G_1 C)$ are zero and so e converges to the zero vector at finite steps independent of the initial error. This property of discrete time systems is called deadbeat response.

In this section, we study the sliding mode realization of a reduced order discrete time observer. As discussed in the previous chapter, a continuous time sliding mode observer steers the observer error trajectories to the origin in finite time. However, a standard discrete time observer can also do the same job because of the deadbeat response property of a discrete time system. Therefore the finite time state convergence characteristic of a sliding mode observer looses its importance for the discrete case at first glance. However, besides the convergence property, the proposed sliding mode observer of this section will also decompose the error motion into two independent reduced order motions. Furthermore, these motions will be in the form of a zero input

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response and the error dynamics will be decoupled from any other dynamical effects thereafter the system is in sliding mode.

We consider again the following discrete time system

$$
\begin{aligned}
x_{k+1} &= \Phi x_k + \Gamma u_k \\
y_k &= C x_k
\end{aligned}
\tag{4.19}
$$

where $x \in R^n$, $u \in R^p$, $y \in R^m$ and Φ, Γ, C are constant matrices with appropriate dimensions. The system is observable and C has full rank without loss of generality. The problem is to design a discrete time observer in the framework of the discrete time sliding mode theory using the p-dimensional input vector and the m-dimensional output vector.

Let the observer equation for the system given in (4.19) be in the form

$$
\hat{x}_{k+1} = \Phi \hat{x}_k + \Gamma u_k + \Upsilon v_k
\tag{4.20}
$$

where $\hat{x}_k$ represents the observer state vector at time $t = kT$, T is the sampling time and Υv_k is an arbitrary function of the known variables. Consider the error

$$
e_x = x - \hat{x}
\tag{4.21}
$$

which satisfies the following difference equation

$$
\begin{aligned}
e_{x,k+1} &= x_{k+1} - \hat{x}_{k+1} \\
e_{x,k+1} &= \Phi x_k + \Gamma u_k - \Phi \hat{x}_k - \Gamma u_k - \Upsilon v_k \\
e_{x,k+1} &= \Phi e_{x,k} - \Upsilon v_k
\end{aligned}
\tag{4.22}
$$

We would like the error vector e to become zero in finite time. Choosing the observer auxiliary input

$$
\Upsilon v_k = \Phi e_{x,k} = \Phi(x_k - \hat{x}_k)
$$

e_x would become zero and the estimated states would exactly match with the real plant states at the next step. However, the auxiliary observer input contains itself

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the plant state vector which is being tried to find. Therefore, this iterative problem needs to be solved using an alternative way.

First, the system given in (4.19) is transformed into the following form:

$$\begin{array}{llll} y_{k+1} &=& \Phi_{11}y_k + \Phi_{12}x_{1,k} + \Gamma_1 u_k & m \\ x_{1,k+1} &=& \Phi_{21}y_k + \Phi_{22}x_{1,k} + \Gamma_2 u_k & n\text{-}m \end{array} \qquad (4.23)$$

where

$$T\Phi T^{-1} = \left[\begin{array}{cc} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{array} \right] \ , \ \ T\Gamma = \left[\begin{array}{c} \Gamma_1 \\ \Gamma_2 \end{array} \right],$$

using a proper similarity transformation of the states given by

$$T = \left[\begin{array}{c} C \\ R \end{array} \right] \qquad \begin{bmatrix} y_k \\ y_{1,k} \end{bmatrix} = \begin{bmatrix} C \\ R \end{bmatrix} x_k = T x_k$$

where R is an arbitrary $(n-m) \times n$ matrix as long as T is invertible and $x_{1,k} = Rx_k$

The corresponding sliding mode observer using the new realization of the system is written as a replica of the system given in (4.23) with an additional innovative auxiliary input term as follows:

$$\begin{array}{llll} \hat{y}_{k+1} &=& \Phi_{11}\hat{y}_k + \Phi_{12}\hat{x}_{1,k} + \Gamma_1 u_k - v_k \\ \hat{x}_{1,k+1} &=& \Phi_{21}\hat{y}_k + \Phi_{22}\hat{x}_{1,k} + \Gamma_2 u_k + Lv_k \end{array} \qquad (4.24)$$

where $v_k \in R^m$ and $L \in R^{(n-m)\times m}$. The corresponding error dynamics can be found by subtracting (4.24) from (4.23) as follows:

$$\begin{array}{llll} e_{y,k+1} &=& \Phi_{11}e_{y,k} + \Phi_{12}e_{x_1,k} + v_k \\ e_{x_1,k+1} &=& \Phi_{21}e_{y,k} + \Phi_{22}e_{x_1,k} - Lv_k \end{array} \qquad (4.25)$$

where e_y and e_{x_1} are defined as

$$\begin{array}{llll} e_y &=& y - \hat{y} \\ e_{x_1} &=& x_1 - \hat{x}_1 \end{array}$$

Examining the equation set (4.25), we can modify the problem of state observation into an equivalent problem of finding an auxiliary observer input v_k in terms of the

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known quantities so that the observer error vector components e_y and e_{x_1} are steered to zero at finite steps.

As previously discussed, a general sliding mode design has two steps. First, a manifold is designed so that while the associated system trajectories travel along the manifold, the system satisfies desired design specifications. Second, a control law is found to steer and keep the trajectories on the sliding manifold. When considering sliding mode observer design, the manifold is defined as

$$\mathcal{S} = \{e_x | e_y = 0\} \tag{4.26}$$

which is simply a subregion of the n-dimensional state space composed by the regions from where the error trajectories possibly pass when the error in the output variable y is kept at zero. If the error trajectories are on the manifold at each sampling instance, we say that the observer is in discrete time sliding mode. Between the two sampling instances, trajectories are allowed to deviate from the manifold, however the deviations do not exceed the order of the sampling time. There are many possible ways of choosing the sliding manifold other than the one previously defined. For example, a linear combination of the error vector could also be chosen as a possible manifold, however the new manifold would be just a transformed version of it, because the degree of the manifold cannot be less than the degree of the variable that will be used to enforce the sliding mode. Therefore, the dimension of the sliding manifold cannot be less than m which is the output order of the original system and our choice is very reasonable. After designing the manifold, the selection of the auxiliary observer input so as to converge the trajectories to the manifold becomes the major concern. Before further going into detail, we need to explain the equivalent control concept for a discrete time system.

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As discussed before, the equivalent control is defined as the average value of the variable which keeps the system trajectories on the sliding manifold after sliding mode takes place. Therefore, the implementation of the equivalent control suffices for the velocity vector of trajectories to be orthogonal to the gradient vector of the manifold at any time provided that there is no parametric uncertainties or external disturbances in the system. When considering continuous time systems, the real control law is chosen as the equivalent control term with an additional discontinuous term to take into account the reaching phase and to increase the robustness of the system. As to discrete time sliding mode, the definition of the equivalent control is very similar to the continuous time case and it is defined as the control input which brings the trajectories to the sliding manifold at one step as discussed in Chapter 2. The control input term which is used in the equivalent control definition does not have to be the real input of the system, it might be just an auxiliary input as well. According to the chosen sliding manifold, the equivalent control is found by solving the auxiliary observer input with the constraint $e_{y,k+1} = 0$ as follows:

$$v_{k,eq} = -\Phi_{11}e_{y,k} - \Phi_{12}e_{x_1,k} \tag{4.27}$$

If the equivalent control is applied to the system, sliding mode occurs at the next step and e_{x_1} satisfies the following dynamical relation

$$\begin{aligned} e_{x_1,k+1} &= (\Phi_{22} + L\Phi_{12})e_{x_1,k} + (\Phi_{21} + L\Phi_{11})e_{y,k} \\ e_{x_1,k+1} &= (\Phi_{22} + L\Phi_{12})e_{x_1,k} \end{aligned} \tag{4.28}$$

and converges to the zero vector choosing a proper L such that the matrix $\Phi_{22} + L\Phi_{12}$ has the desired eigenvalues. Placing the eigenvalues of the matrix $\Phi_{22} + L\Phi_{12}$ at the desired locations requires observability of the pair (Φ_{22}, Φ_{12}). However, observability of the pair (Φ, Γ) is sufficient for the previous observability requirement as discussed

before. This is exactly a sliding mode realization of the standard reduced order asymptotic observer. After sliding mode takes place on the sliding manifold, the observer error in the variable x_1 also converges to zero and our control objective is achieved. Note that, unlike the discrete time sliding mode controller design where the control law has been chosen different in the reaching phase and in the vicinity of the manifold, we choose a single observer input throughout the state space because there is no restriction on the auxiliary observer input since it is just a fictitious input to the system.

However, the equivalent observer auxiliary input cannot be provided since it contains an unknown term, e_{x_1}. This is the main difference of the discrete time sliding mode observer design procedure from the continuous one. In the continuous case, we also do not know the values of some terms at the right hand side. However, since the control aim at that stage is to steer the states towards the sliding manifold, observer parameter L can be chosen large enough so as to suppress the effects of the unknown terms so that the distance of the trajectories from the sliding manifold decreases monotonically. As to the discrete time case, a discontinuous term is undesirable because of the finite switching frequency. Therefore, it becomes necessary to know all the terms at the right hand side to compute the auxiliary input which brings error trajectories to the sliding manifold.

To compute the equivalent auxiliary observer input, (4.25) can be combined into a single equation as follows:

$$
\begin{aligned}
e_{y,k+1} &= \Phi_{11}e_{y,k} + \Phi_{12}e_{x_1,k} + v_k \\
v_k &= e_{y,k+1} - \Phi_{11}e_{y,k} - \Phi_{12}e_{x_1,k} \\
e_{x_1,k+1} &= \Phi_{21}e_{y,k} + \Phi_{22}e_{x_1,k} - Le_{y,k+1} + L\Phi_{11}e_{y,k} + L\Phi_{12}e_{x_1,k} \\
e_{x_1,k+1} &= (\Phi_{22} + L\Phi_{12})e_{x_1,k} + (\Phi_{21} + L\Phi_{11})e_{y,k} - Le_{y,k+1} \\
e_{x_1,k} &= (\Phi_{22} + L\Phi_{12})e_{x_1,k-1} + (\Phi_{21} + L\Phi_{11})e_{y,k-1} - Le_{y,k}'
\end{aligned}
\tag{4.29}
$$

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and the unknown $e_{x_1,k}$ term in the equivalent observer input can be replaced by z_{k+1} which is given by

$$z_{k+1} = \bar{\Phi} z_k + \bar{\Gamma} \qquad (4.30)$$

where $\bar{\Phi} = \Phi_{22} + L\Phi_{12}$ and $\bar{\Gamma} = (\Phi_{21} + L\Phi_{11})e_{y,k-1} - Le_{y,k}$.

Note that, the dynamical equation of z_k is exactly the same as that of $e_{x_1,k}$ and the difference between z_{k+1} and $e_{x_1,k}$ goes to zero at most at $n - m$ steps independent of the initial error choosing all the eigenvalues of the matrix $(\Phi_{22} + L\Phi_{12})$ at the origin. Therefore, the equivalent observer input can be updated as:

$$\sqrt{\ (\Phi_{22} + L\phi_{12})\,z_k + (\phi_{21} + L\phi_{11})\,\ell_{y,k-1} - L\,\ell_{y,k}}$$

$$\begin{aligned}
\hat{v}_{k,eq} &= -\Phi_{11}e_{y,k} - \Phi_{12}z_{k+1} \\
\hat{v}_{k,eq} &= -(\Phi_{11} - \Phi_{12}L)e_{y,k} - \Phi_{12}(\Phi_{21} + L\Phi_{11})e_{y,k-1} - \Phi_{12}(\Phi_{22} + L\Phi_{12})z_k
\end{aligned} \qquad (4.31)$$

The estimated observer auxiliary input $\hat{v}_{k,eq}$ converges to the equivalent input v_{eq} after $(n - m)$ steps later placing all the eigenvalues of the matrix $(\Phi_{22} + L\Phi_{12})$ at the origin and the discrete time sliding mode takes place at the $(n - m + 1)^{th}$ step. In sliding mode, e_{x_1} which represents the rest of the composite error vector other than e_y converges zero satisfying the relation

$$\textit{after}\ (n\text{-}m+1)\,\textit{step}: \qquad e_{x_1,k+1} = (\Phi_{22} + L\Phi_{12})e_{x_1,k} \qquad (4.32)$$

Since all the eigenvalues of $(\Phi_{22} + L\Phi_{12})$ have already been assigned as zero, e_{x_1} also converges in finite time and our control objective is achieved. Note that, in sliding mode the total observer motion is decomposed into two independent motions and these motions show a zero input characteristic. Furthermore, the eigenvalues of these submotions are controlled by the same parameter.

To sum up, the complete observer structure consists of two layers as shown in Figure 4.3. The first layer is used to estimate the unknown error terms which are necessary in the equivalent input calculation and the second layer is the layer where
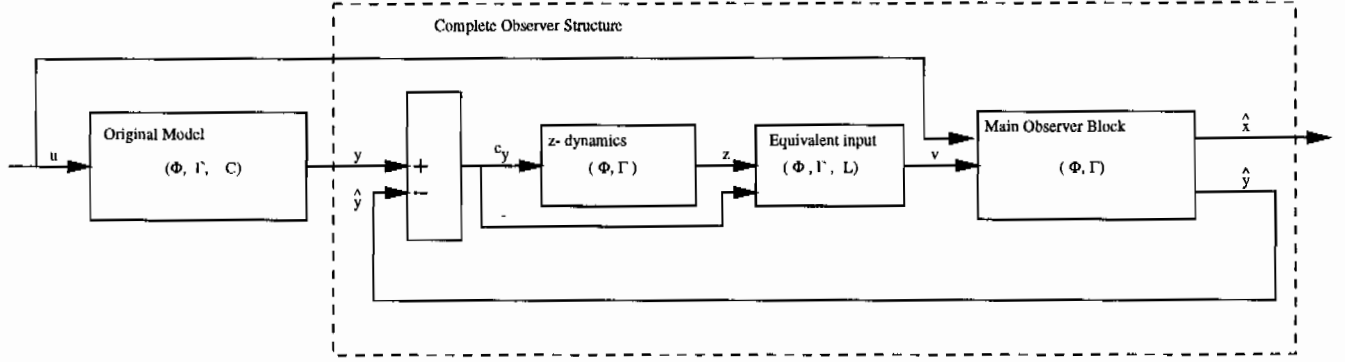
Figure 4.3: Structure of the Discrete Time Sliding Mode Observer

the state estimation is done. The second layer admits the output of the first layer which is the estimated observer auxiliary input as its input and yields the estimated states of the original system. Note that, the discrete time sliding mode observer design is simpler and more straightforward than the continuous time sliding mode observer design. The original system is only once transformed into another form and there is no filtering block as in the continuous time case, instead shifting the discrete time equations in time performs the dual task.

The following theorem summarizes the previous derivation for the design of a sliding mode observer in the original state space realization of an observable plant.

Theorem 1 *Given*

$$x_{k+1} = \Phi x_k + \Gamma u_k$$

$$y_k = C x_k$$

where $x \in R^n$, $u \in R^p$, $y \in R^m$, $rank(C) = m$ and the (A, C) pair is observable. The discrete-time sliding mode observer has the following form:

$$\hat{x}_{k+1} = \Phi \hat{x}_k + \Gamma u_k - T^{-1} \begin{bmatrix} I_m \\ -L \end{bmatrix} v_k$$

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where

$$v_k = -(\Phi_{11} - \Phi_{12}L)e_{y,k} - \Phi_{12}(\Phi_{21} + L\Phi_{11})e_{y,k-1} - \Phi_{12}(\Phi_{22} + L\Phi_{12})z_k; \qquad (4.31)$$

$$T\Phi T^{-1} = \left[\begin{array}{cc} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{array} \right];$$

where $\Phi_{11} \in R^{m\times m}$, $\Phi_{12} \in R^{m\times(n-m)}$, $\Phi_{21} \in R^{(n-m)\times m}$ and $\Phi_{22} \in R^{(n-m)\times(n-m)}$;

$$e_{y,k} = y_k - \hat{y}_k;$$

z_k is the state of the following error driven system

$$z_{k+1} = \bar{\Phi}z_k + \bar{\Gamma}$$

where $\bar{\Phi} = (\Phi_{22} + L\Phi_{12})$, $\bar{\Gamma} = (\Phi_{21} + L\Phi_{11})e_{y,k-1} - Le_{y,k}$;

T is the similarity transformation matrix;

$$T = \left[\begin{array}{c} C \\ R \end{array} \right]$$

where $R \in R^{(n-m)\times n}$ is arbitrary provided that T is invertible and L is the observer gain matrix which places all the eigenvalues of $(\Phi_{22} + L\Phi_{12})$ at the origin.

4.3 Discrete Time Sliding Mode Observers with Disturbance

In section 4.2, we have discussed a discrete time sliding mode observer based on an ideal system representation assuming that there are no disturbances and uncertainties in the system. In this section, this assumption is released and consideration is given to the following uncertain discrete time system:

$$
\begin{array}{rcl}
x_{k+1} & = & \Phi x_k + \Gamma u_k + F w_k \\
y_k & = & C x_k + D h_k
\end{array} \qquad (4.33)
$$

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where $x \in R^n$, $y \in R^m$, $u \in R^p$ are the state, the output and the input vectors, respectively as before and $w \in R^{l_1}$, $h \in R^{l_2}$ denote the external disturbances to the system. The pair (Φ, C) is assumed to be observable. It is further assumed that the disturbance dynamics are known exactly, i.e;

$$
\begin{aligned}
w_{k+1} &= \mathcal{W}w_k \\
h_{k+1} &= \mathcal{H}h_k
\end{aligned}
\tag{4.34}
$$

If we attack to solve the problem directly using the design procedure of section 4.2, the system is first transformed into:

$$
\begin{aligned}
y_{k+1} &= \Phi_{11}y_k + \Phi_{12}x_{1,k} + D_1h_k + F_1w_k + \Gamma_1u_k \\
x_{1,k+1} &= \Phi_{21}y_k + \Phi_{22}x_{1,k} + D_2h_k + F_2w_k + \Gamma_2u_k
\end{aligned}
\tag{4.35}
$$

The corresponding sliding mode observer is given by

$$
\begin{aligned}
\hat{y}_{k+1} &= \Phi_{11}\hat{y}_k + \Phi_{12}\hat{x}_{1,k} + \Gamma_1u_k - v_k \\
\hat{x}_{1,k+1} &= \Phi_{21}\hat{y}_k + \Phi_{22}\hat{x}_{1,k} + \Gamma_2u_k + Lv_k
\end{aligned}
\tag{4.36}
$$

Subtracting equation 4.36 from equation 4.35 yields the following error dynamics:

$$
\begin{aligned}
e_{y,k+1} &= \Phi_{11}e_{y,k} + \Phi_{12}e_{x_1,k} + D_1h_k + F_1w_k + v_k \\
e_{x_1,k+1} &= \Phi_{21}e_{y,k} + \Phi_{22}e_{x_1,k} + D_2h_k + F_2w_k - Lv_k
\end{aligned}
\tag{4.37}
$$

At this stage, v_k should be chosen so as to induce a sliding mode on the manifold $\mathcal{S} = \{e_x | e_y = 0\}$ according to the procedure used for an ideal system. However, the observer equations now contain unknown disturbance terms. The elimination of these disturbance terms by inserting a filter-like corrective term to the ideal equivalent control law as discussed in Chapter 2 for a discrete time sliding mode controller could be proposed as a solution. However, this can not solve the problem because the unknown system and the disturbance states appear at the same side of the equation and their dynamics cannot be separated from each other. However, since the disturbance dynamics are known, the variables w and h which represent the disturbance

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states can also be treated as unmeasurable system states. Incorporating them to the original system dynamics, the following augmented system is obtained:

$$y_{k+1} = \Phi_{11} y_k + \begin{bmatrix} \Phi_{12} & D_1 & F_1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ h_k \\ w_k \end{bmatrix} + \Gamma_1 u_k$$

$$\begin{bmatrix} x_{1,k+1} \\ h_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi_{22} & D_2 & F_2 \\ 0 & \mathcal{H} & 0 \\ 0 & 0 & \mathcal{W} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ h_k \\ w_k \end{bmatrix} + \begin{bmatrix} \Phi_{21} \\ 0 \\ 0 \end{bmatrix} y_k + \Gamma_1 u_k \qquad (4.38)$$

Assuming that, this augmented system is also observable, (4.38) is the counterpart to (4.23) and Theorem 1 is directly applicable. Note that, the order of the z dynamics increases by $l_1 + l_2$ which is the sum of the disturbance degrees, however the disturbance states w and h are also estimated in finite time as well.

4.4 Simulations

In this section, the performance of the proposed discrete time sliding mode observers is tested on a truck-semitrailer system. First, it will be assumed that there are no external disturbances and measurement errors in the system and Theorem 1 will be directly applied to the system. Second, the robustness of the proposed sliding mode observer is illustrated in the presence of disturbances with known dynamics and simulation results will be illustrated.

The state space representation of the linear model has already been given and used in the previous chapters and our design will be based it. The problem is to estimate the lateral velocity of the tractor and the articulation angle between the tractor and the semitrailer using the measurements of the yaw rates and the steer angle input as in the continuous time observer example. Before finding the observer parameters, we need to obtain the discrete time representation of the model using the following

identities

$$\Phi = e^{AT} \quad \Gamma = \int_0^T e^{A\lambda} B d\lambda \tag{4.39}$$

where T is the sampling time, (A, B) is the continuous time system parameters and the pair (Φ, Γ) represents the discrete time version of the parameters. Choosing the sampling time as 0.1 *sec.*, we get

$$\Phi = \begin{bmatrix} 0.2016 & 0.0867 & 0.3236 & 0.4640 \\ -0.0540 & -0.0117 & -0.0080 & -0.0156 \\ 0.0620 & 0.1028 & 0.7508 & -0.3555 \\ 0.0141 & -0.0081 & 0.0853 & 0.9785 \end{bmatrix} , \quad \Gamma = \begin{bmatrix} -4.1689 \\ 3.0130 \\ 0.5830 \\ -0.1977 \end{bmatrix}$$

The discrete time sliding mode observer parameters T and L can be found using these matrices. Note that T is not unique since it contains R which is arbitrary as long as T is invertible. Once T has been chosen, Φ_{ij}'s are automatically obtained and L is calculated so that the two eigenvalues of the matrix $\Phi_{22} + L\Phi_{12}$ are located at the origin. Following this logic, a possible set of observer parameters is selected as follows:

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.2190 & 0.6789 & 0.934 & 0.5194 \\ 0.0470 & 0.6793 & 0.3835 & 0.8310 \end{bmatrix} , \quad L = \begin{bmatrix} 2.1064 & 0.6581 \\ 2.2648 & 1.8354 \end{bmatrix}$$

The observer is used for the estimation of the previously mentioned unmeasurable states of the truck-semitrailer during a lane change maneuver. The vehicle is driven with a longitudinal speed of 15 *m/sec.* and the steer angle is adjusted during the maneuver so as to do a lane change. Figures 4.4-4.8 show the simulation results of the observer for the open loop system. In simulations, the model is chosen as a continuous time system between a zero order hold and a sampler whereas the observer equation is implemented as a discrete time system. Therefore, the overall system may be called a sampled data system. The convergence speed is very high as expected
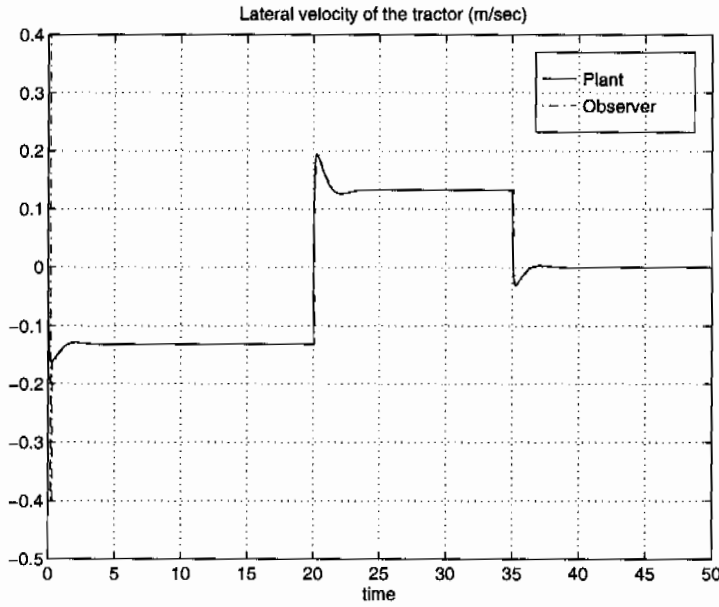
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Figure 4.4: Tractor lateral velocity

since the overall motion shows a deadbeat characteristic. We have also presented the zoomed version of Figure 4.4 associated with the real and the observed lateral velocities to examine the small scale behavior of the error motion in Figure 4.8. It has been observed that at each sampling instance the observed state exactly matches with the real one, however the motion is arbitrary and in the order of the sampling time between two consecutive sampling instances.

Note that no chattering phenomena has been observed in the estimated variables as opposed to continuous time case since the observer equation does not contain a discontinuous term and the system has no external disturbances and uncertainties. However, since the observer is eventually supposed to be implemented in discrete time even if it is designed in continuous time, the intersampling behaviors of the observer states will be out of control in both cases. But, continuous time observer
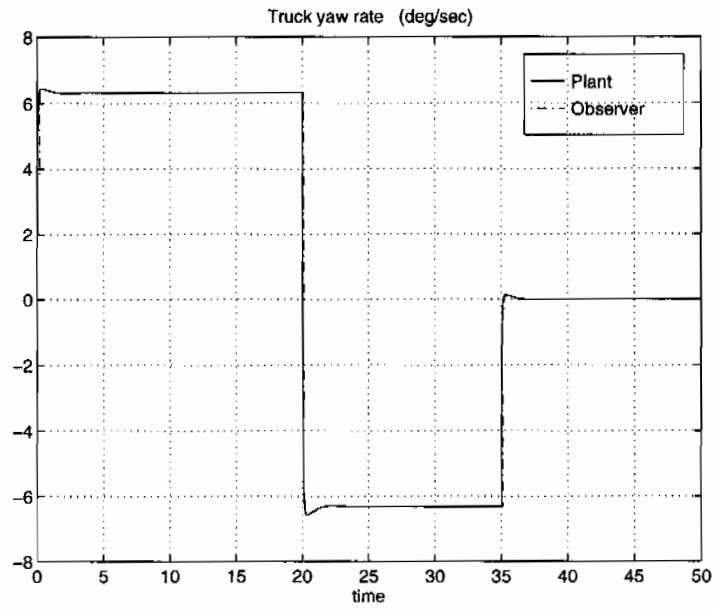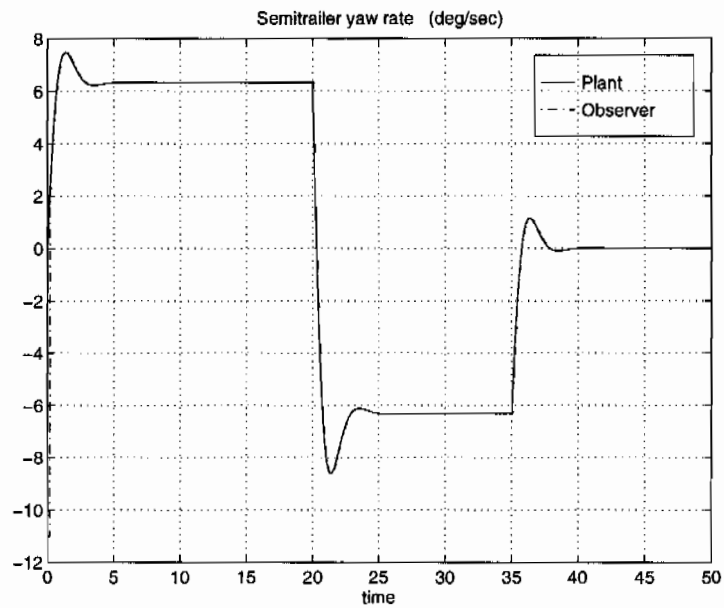
83

Figure 4.5: Tractor yaw rate

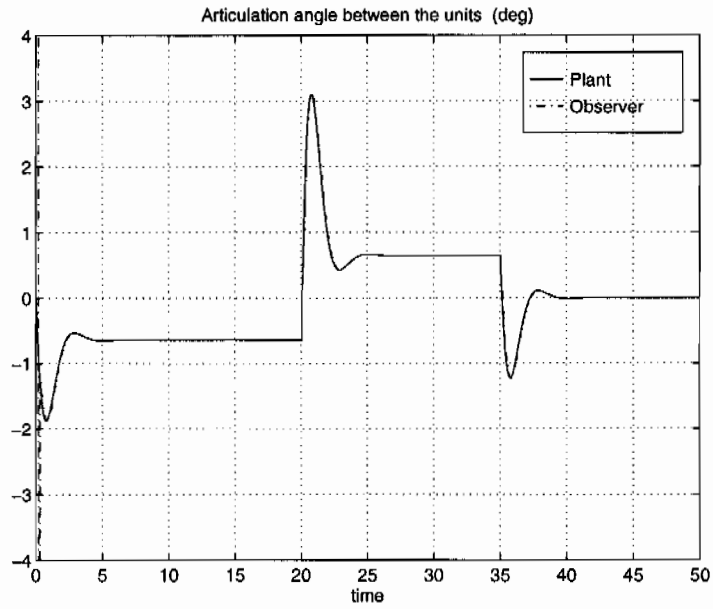Figure 4.6: Semitrailer yaw rate

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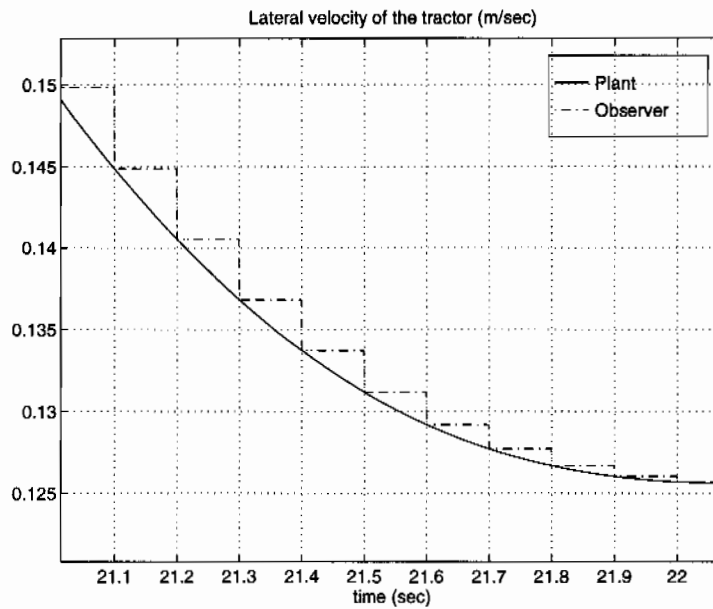Figure 4.7: Articulation angle
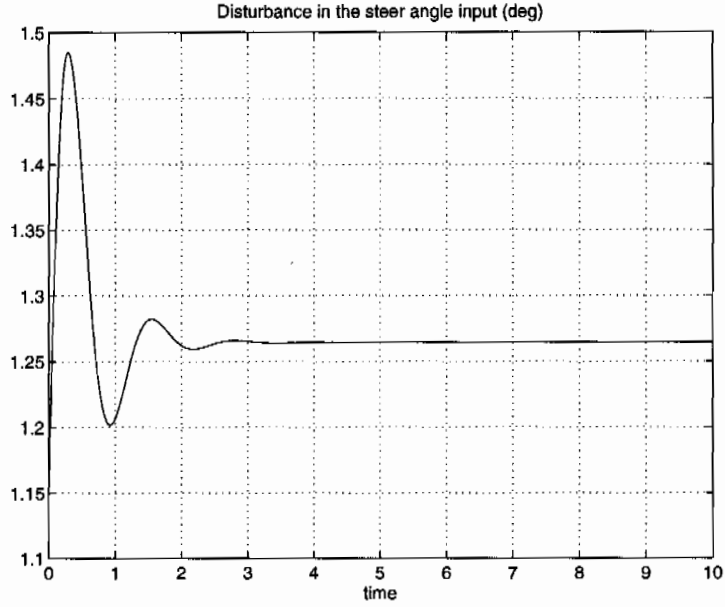

Figure 4.8: Small scale tractor lateral velocity

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Figure 4.9: Disturbance in the steer angle input

states are not exactly on the manifold even at each sampling instance as well, instead they oscillate around the manifold. Therefore, in this setting, chattering is used to describe the motion just at the sampling instances.

So far, it has been assumed that there are no external disturbances in the system. From now on, this assumption is released and it is considered that there may be an error in the steer angle input of the system. Therefore, the real input of the system and the input which is used for the prospective observer will not be the same. For simulation purposes, the poles of the disturbance are chosen at $s = 0, -2 \pm 5j$ in continuous time and its time plot is shown in Figure 4.9. Using the the location of the disturbance and the information on its poles, it can be modeled as follows:

$$\dot{w} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -29 & -4 \end{bmatrix} w \qquad (4.40)$$

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and

$$F = B \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad (4.41)$$

where B is the input coefficient matrix of the original continuous time system from which the discrete time system given in (4.1) has been obtained by discretization. Augmenting the discretized disturbance equation with the system and applying the same design procedure, we end up with a robust 7th-order discrete time sliding mode observer. It shows the same characteristics with the ideal discrete time sliding mode observer, however it also estimates the disturbance and its order is greater than the order of the system. Since, the same design procedure will be applied to the augmented system and the details of the procedure have already been illustrated for the ideal case, Figures 4.10- 4.14 are just attached to show the effectiveness of the proposed robust discrete time sliding mode observer without going into further discussion.
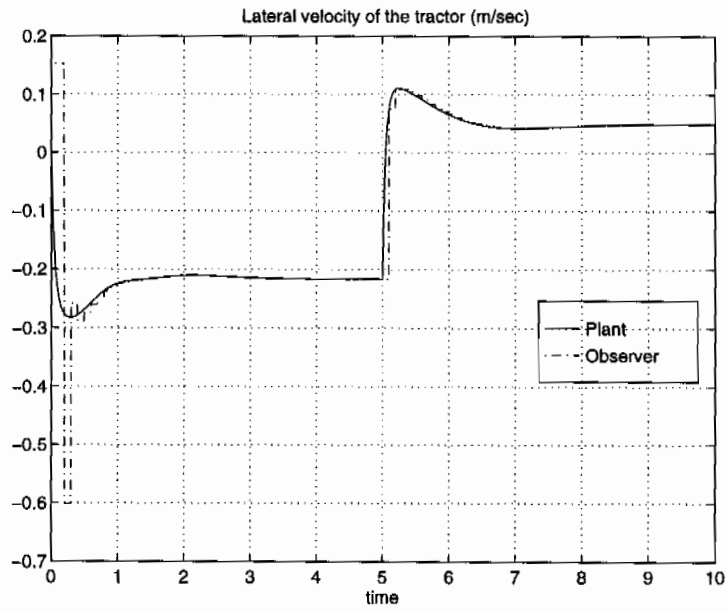
Figure 4.10: Tractor lateral velocity with disturbance
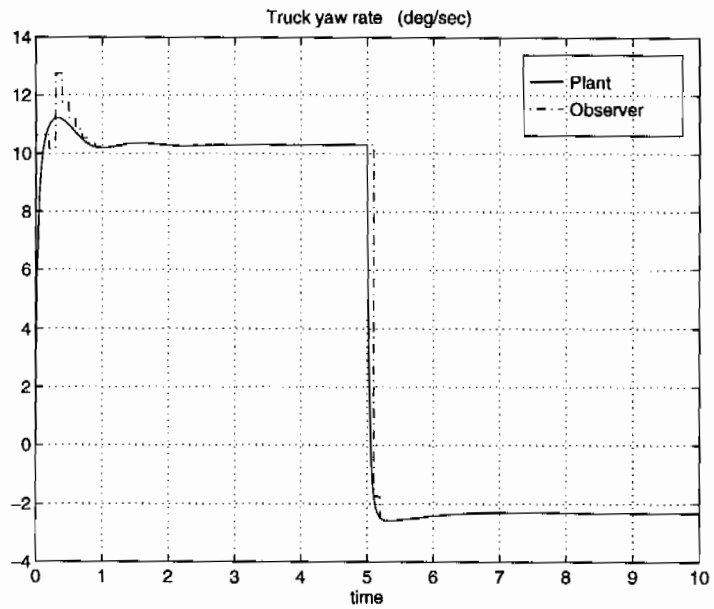

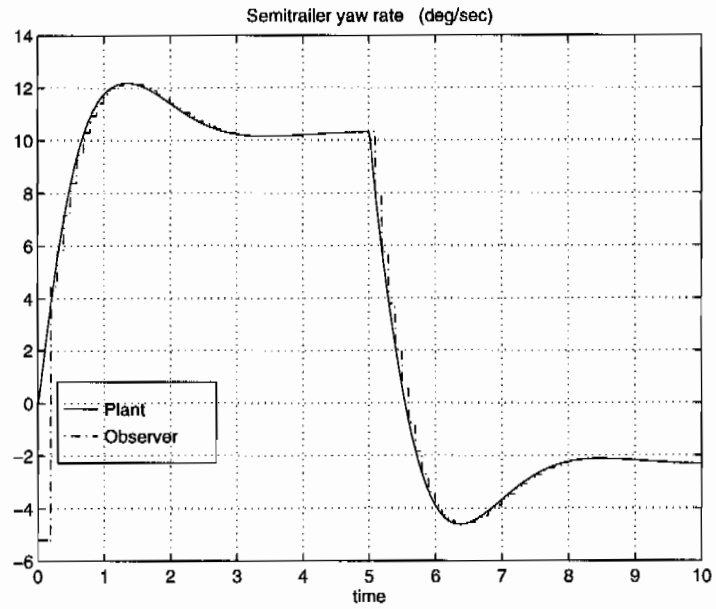Figure 4.11: Tractor yaw rate with disturbance
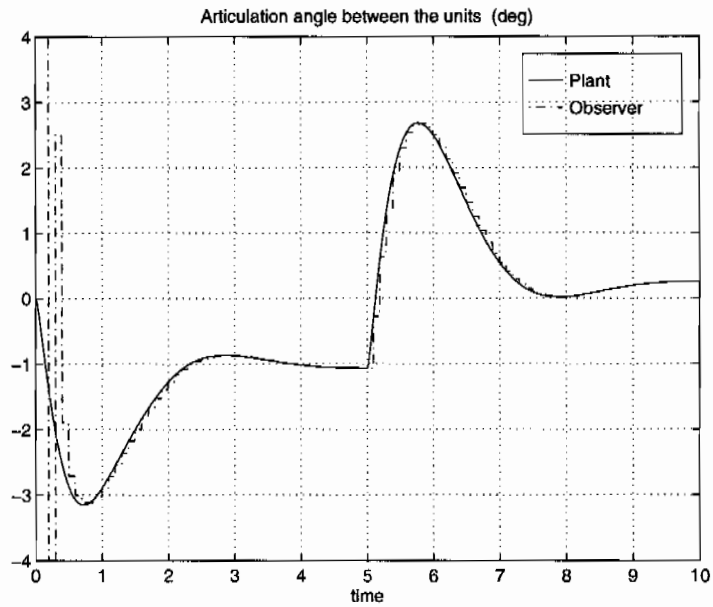
Figure 4.12: Semitrailer yaw rate with disturbance
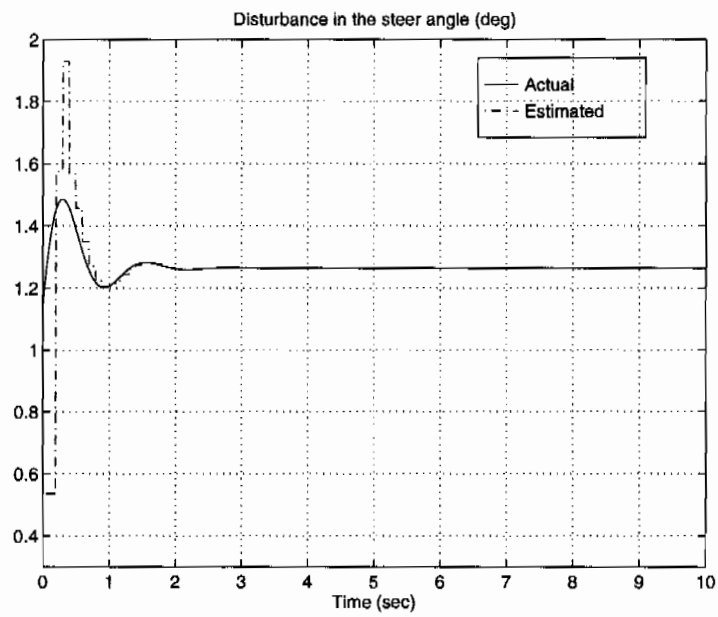

Figure 4.13: Articulation angle with disturbance

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Figure 4.14: Disturbance and its estimate

CHAPTER 5

CONCLUSION

5.1 Summary of Study

In this thesis, sliding mode controllers and observers are examined. Most of the study consider a discrete time system. Simulations on a truck-semitrailer system for typical highway maneuvers verify the theoretical results.

In Chapter 2 discrete time sliding mode controller design theory is discussed. A twofold control action is proposed for implementation purposes. The overall state space are fictitiously divided into two parts defining a region, called boundary layer, in the sampling time vicinity of the manifold. Inside the boundary layer, the equivalent control can induce sliding mode without exceeding control limits. While the trajectories are away from the manifold another control law is used to guarantee the attractiveness of the layer and to steer the trajectories towards the manifold. The equivalent control is generalized to increase the robustness of the system to external disturbances. The deviations in the generalized control variable resulting from disturbances are used to update the control law dynamically so as to suppress the disturbance. Composite control law is tested successfully on a truck-semitrailer system to bring the vehicle to a straight position after a turn or a maneuver in simulations.

In Chapter 3, continuous time sliding mode observers are examined for the reconstruction of the state vector in finite time. Equivalent control methodology is exploited for this purpose. Error trajectories are brought to the origin by making them pass through a finite number of manifolds successively. Each manifold is a subset of the previous manifold and shrinks to the origin in the final step. Equivalent value of the auxiliary input is obtained passing the input through a low pass filter to cut off the high frequency dynamics and this equivalent value is used to extract some information from the system at the next step. Observer equations designed at each step are gathered together to form the final observer equation and this final equation is retransformed back into the original state space form of the system for implementation concerns. The finite time converging characteristics of the proposed observer are illustrated on a truck-semitrailer system for the estimation of the unmeasurable states during a maneuver.

Chapter 4 concentrates on the discrete time sliding mode observers. Discrete time equivalent control concept is used to get a counterpart to the continuous time case. The proposed observer consists of two layers. The first layer produces the equivalent control for the second layer where the state estimation is done. Unlike the continuous time case, a new error driven auxiliary system is formed to obtain the equivalent control instead of filtering blocks. The observer design procedure is also extended to take care of disturbances with known dynamics. The same observer problem of Chapter 3 is considered to show the validity and the effectiveness of the theoretical analysis. Simulation results show the superiority of the discrete time sliding mode observer over the continuous one in terms of chattering phenomena. However, because of the discrete time nature of the system error trajectories cannot

be kept on the associated manifold between the sampling instances and therefore an arbitrary intersampling behavior appears.

In Appendix A, a linearized truck-semitrailer model is derived from a nonlinear bicycle model and the numerical model which has been used for simulation purposes throughout the thesis is also obtained.

5.2   Future Directions

The proposed controllers and observers satisfy their objectives for a LTI system. However, their robustness properties are confined to a limited class of external disturbances. The robustness can further be assured not only for a larger class of disturbances but also for parametric uncertainties in a probabilistic framework. Furthermore, the whole design theory can also be generalized to time varying and nonlinear systems.

APPENDIX A

AN ARTICULATED MODEL

In Appendix, the derivation of the linear truck-semitrailer model which has been used throughout the thesis for simulation purposes is presented. The articulated system considered is formed by a tractor and a semitrailer connected to each other by a fifth wheel. Our main motion for dealing with this system is not only the need for an efficient articulated vehicle model which shows the interacting dynamics and behaviors of the different modes of a generic vehicle for highway studies but also to use the resulting model in the verification of the theoretical results of the thesis.
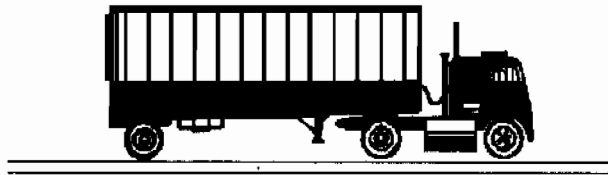

Figure A.1: Three-axle Truck-semitrailer Side View

Figure A.1 shows the side view of the three-axle truck-semitrailer. The vehicle can be treated as a planar model composed of several lumped masses connected by compliant linkages representing the suspensions. The tractor and the semitrailer

bodies supported by suspension systems at each axle are the primary masses which are referred to as "sprung masses". The sprung mass is considered rigid with mass properties concentrated at its center of gravity and moment of inertia about the center of gravity. The additional masses such as axle, brakes, wheels are referred to as "unsprung masses". The rocker arms connect the unsprung masses to sprung masses. The suspension actuators are located between the corners of the vehicle and the rocker arms.

The organization of the appendix is as follows: First, a nonlinear 5-state 2-D model is derived neglecting the vertical dynamics of the vehicle. The model covers the dynamics of the tractor and the semitrailer sprung masses having three degrees of freedom along the following variables: longitudinal velocity, lateral velocity and the angular velocity around the vertical axis, called yaw rate. Second, this nonlinear model is linearized under the small steer angle and the constant longitudinal velocity assumptions using the physical properties of the vehicle under these assumptions. The resulting linear model has four states and a single input. The parameters of the linear model are adjusted to match with the nonlinear model during a maneuver and finally the numerical model of the previous chapters is obtained.

A.1 Nonlinear Planar Truck-Semitrailer Model

In this section, a 2-D nonlinear truck-semitrailer model is derived. Figure A.2 shows the top view of the vehicle. Steer angle and the longitudinal force on the front tires of the tractor are the two inputs to the system.

Two local coordinate systems are fixed to the sprung mass CG's of the tractor and the semitrailer to describe the orientation of the vehicle and to write the differential
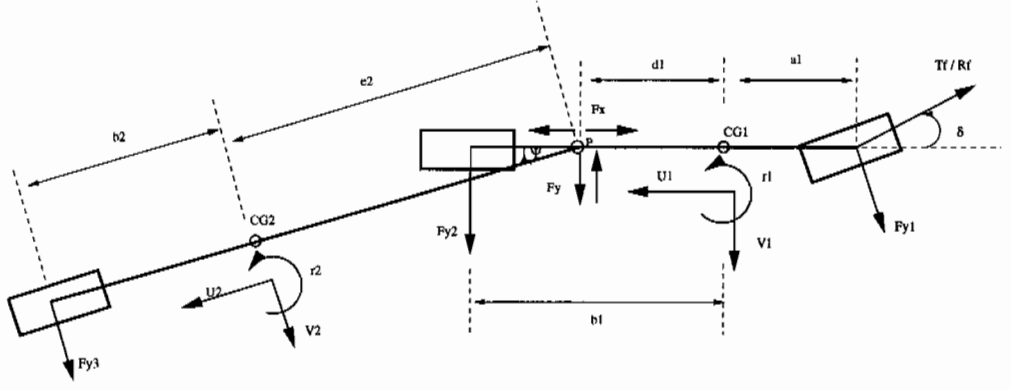
Figure A.2: Bicycle Truck-semitrailer Model

motion equations which represent the dynamics of the sprung masses. The longitudinal force on the front tires of the tractor is represented by T_f/R_f where T_f denotes the torque on the tractor front axle and R_f is the tire radius. δ represents the steer angle on the tractor front tires and it is referenced as positive in the counter-clockwise direction. Two more forces are attached to the tires to describe the force that will be exerted on the tires by the road during turns. The fifth wheel is assumed to transmit two linear action-reaction forces. The equations of motion of the tractor and the semitrailer can be derived independently.

A.1.1   Dynamic Equations of the Tractor

The motion equations of the tractor in the longitudinal and lateral directions and the rotational dynamics around the local vertical axis can be found by summing the forces and the moments using the free body diagram of the tractor shown in Figure A.3. Taking the summation of the forces in the local x axis yields (A.1)

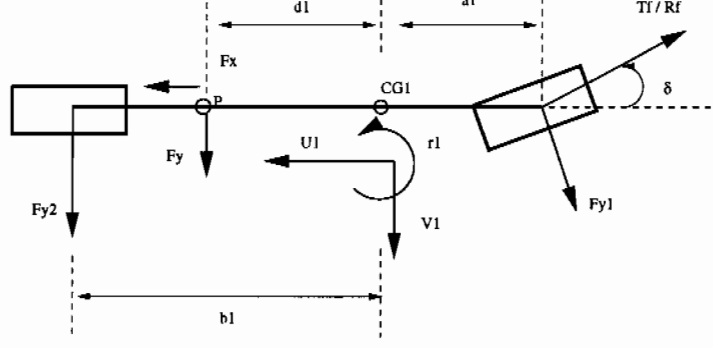$$m_1(\dot{U}_1 - V_1 r_1) = A_\rho U_1^2 - \frac{T_f}{R_f} + F_x,$$

(A.1)

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Figure A.3: Tractor Freebody Diagram

totaling the forces in the y direction produces (A.2)

$$m_1(\dot{V}_1 + U_1 r_1) = F_{y1} + F_{y2} - \frac{T_f}{R_f}\delta + F_y \tag{A.2}$$

and finally summing moments about the local z axis of the tractor provides (A.3)

$$I_{z1}\dot{r}_1 = -a_1 F_{y1} + b_1 F_{y2} + a_1 \frac{T_f}{R_f}\delta + d_1 F_y \tag{A.3}$$

where the meanings of the variables and the parameters are listed in Table A.1.

Note that, the direction of motion is chosen as the negative x direction. Through algebraic manipulations, the dynamical equations of the tractor can be gathered together:

$$\dot{U}_1 = V_1 r_1 + \frac{1}{m_1}(A_\rho U_1^2 - \frac{T_f}{R_f} + F_x)$$

$$\dot{V}_1 = -U_1 r_1 + \frac{1}{m_1}(F_{y1} + F_{y2} - \frac{T_f}{R_f}\delta + F_y)$$

$$\dot{r}_1 = \frac{1}{I_{z1}}(-a_1 F_{y1} + b_1 F_{y2} + a_1 \frac{T_f}{R_f}\delta + d_1 F_y) \tag{A.4}$$

A.1.2 Dynamic Equations of the Semitrailer

The top view of the semitrailer is shown in Figure A.4. Steer angles and longitudinal tire forces are all assumed to be zero. The direction of travel is again in the

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U_1	:	Longitudinal velocity of the tractor
V_1	:	Lateral velocity of the tractor
r_1	:	Yaw velocity of the tractor
δ	:	Steer angle on the front tires of the tractor
F_x	:	Longitudinal force on the pin point wrt. local coordinate system of the tractor
F_y	:	Lateral force on the pin point wrt. local coordinate system of the tractor
m_1	:	Mass of the tractor
I_{z1}	:	Moment of inertia of the tractor around the local z axis
F_{y1}	:	Lateral force on the front tires of the tractor
F_{y2}	:	Lateral force on the rear tires of the tractor
A_ρ	:	Aerodynamic drag force coefficient
a_1	:	Distance from the tractor CG to the front tires of the tractor
b_1	:	Distance from the tractor CG to the rear tires of the tractor
d_1	:	Distance from the tractor CG to the pin point

Table A.1: Parameters of the tractor

negative x direction. The coupling point of the tractor and the semitrailer is represented by P. The dynamical equations of the semitrailer are written summing forces and moments in the longitudinal, lateral directions and around the vertical axis, in turn.

Taking the summation of the forces in the y direction produces (A.5)

$$m_2(\dot{U}_2 - V_2 r_2) = A_\rho U_2^2 - F_x,$$ (A.5)

summing the forces in the x direction gives (A.6)

$$m_2(\dot{V}_2 + U_2 r_2) = F_{y3} - F_y + F_x \Psi$$ (A.6)

and finally totaling the moments about the local z axis of the semitrailer, one obtains:

$$I_{z2}\dot{r}_2 = e_2 F_y + b_2 F_{y3} - e_2 F_x \Psi$$ (A.7)

Table A.2 lists the meanings of the parameters which appear in (A.5) -(A.7).

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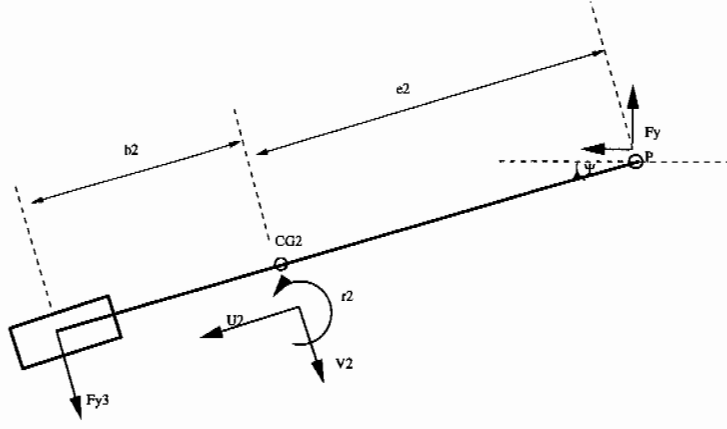
Figure A.4: Semitrailer Freebody Diagram

After some trivial manipulations, semitrailer dynamics can be summarized as follows:

$$\dot{U}_2 = V_2 r_2 + \frac{1}{m_2}(A_\rho U_2^2 - F_x)$$

$$\dot{V}_2 = -U_2 r_2 + \frac{1}{m_2}(F_{y3} - F_y + F_x \Psi)$$

$$\dot{r}_2 = \frac{1}{I_{z2}}(e_2 F_y + b_2 F_{y3} - e_2 F_x \Psi) \qquad (A.8)$$

A.1.3 Coupling Dynamics at the Pinpoint

The tractor and the semitrailer are coupled at the pinpoint as shown in Figure A.2. It is assumed that two linear forces are transmitted between the tractor and the semitrailer through the fifth wheel and the fifth wheel has no dynamical structure so that these forces are considered as action-reaction forces. The longitudinal and the lateral velocities of the semitrailer can be expressed in terms of the other variables exploiting the physical limitations imposed by the interconnection and using the invariance of the velocity at the pinpoint.

U_2	:	Longitudinal velocity of the semitrailer
V_2	:	Lateral velocity of the semitrailer
r_2	:	Yaw velocity of the semitrailer
Ψ	:	Articulation angle between the tractor and the semitrailer
F_x	:	Longitudinal force on the pin point wrt. local coordinate system of the tractor
F_y	:	Lateral force on the pin point wrt. local coordinate system of the tractor
m_2	:	Mass of the semitrailer
I_{z2}	:	Moment of inertia of the semitrailer around the local z axis
F_{y3}	:	Lateral force on the tires of the semitrailer
A_ρ	:	Aerodynamic drag force coefficient
b_2	:	Distance from the semitrailer CG to the rear tires of the semitrailer
e_2	:	Distance from the semitrailer CG to the pin point

Table A.2: Parameters of the semitrailer

The velocity vector $\mathbf{V}_1^p = (U_1^p, V_1^p)$ of the pinpoint is calculated in terms of the local coordinate system of the tractor as follows:

$$
\begin{aligned}
U_1^p &= U_1 \\
V_1^p &= V_1 + r_1 d_1
\end{aligned}
\qquad (A.9)
$$

from the equation $\mathbf{V}^p = \mathbf{V}_1 + \mathbf{w}_1 \times \mathbf{r}_1$ where $\mathbf{V}_1 = (U_1, V_1, 0)$ is the linear velocity vector of the tractor CG, $\mathbf{r}_1 = (0, 0, r_1)$ is the angular velocity vector and $\mathbf{r}_1 = (d_1, 0, 0)$ is the position vector of the pinpoint with respect to CG of the tractor.

Similarly, the velocity of the pinpoint can also be calculated in terms of the local coordinate system of the semitrailer

$$
\begin{aligned}
U_2^p &= U_2 \\
V_2^p &= V_2 - r_2 e_2
\end{aligned}
\qquad (A.10)
$$

using the equation $\mathbf{V}^p = \mathbf{V}_2 + \mathbf{w}_2 \times \mathbf{r}_2$ where $\mathbf{V}_2 = (U_2, V_2, 0)$ is the linear velocity vector of the semitrailer CG, $\mathbf{r}_2 = (0, 0, r_2)$ is the angular velocity vector and $\mathbf{r}_2 = (-e_2, 0, 0)$ is the position vector of the pinpoint with respect to CG of the semitrailer.

However, at the pinpoint, these two velocities calculated in terms of the tractor and the semitrailer local coordinate systems must be the same as shown in Figure A.5,
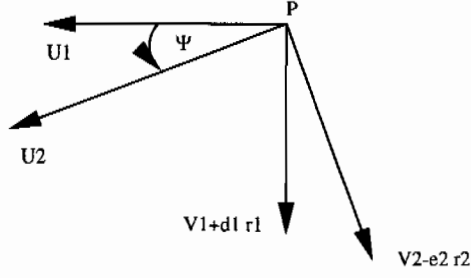
Figure A.5: Velocity Transformation

i.e;

$$
\begin{aligned}
U_2^p &= \quad U_1^p \cos \Psi + V_1^p \sin \Psi \\
V_2^p &= -U_1^p \sin \Psi + V_1^p \cos \Psi
\end{aligned}
\tag{A.11}
$$

where Ψ satisfies

$$
\dot{\Psi} = r_2 - r_1
\tag{A.12}
$$

Through some trivial manipulations, the lateral and the longitudinal velocity of the semitrailer are obtained in terms of those of the tractor as follows:

$$
\begin{aligned}
U_2 &= \quad U_1 \cos \Psi + (V_1 + d_1 r_1) \sin \Psi \\
V_2 &= -U_1 \sin \Psi + (V_1 + d_1 r_1) \cos \Psi + e_2 r_2
\end{aligned}
\tag{A.13}
$$

Since the longitudinal and the lateral velocities of the tractor and the articulation angle between the two bodies are enough to obtain the linear velocity vector of the semitrailer, two dynamical equations of the semitrailer associated with these variables become redundant and can be used to express the pinpoint forces in terms of the independent variables, i.e;

$$
\begin{aligned}
F_x &= -m_2(\dot{U}_2 - V_2 r_2) + A_\rho U_2^2 \\
F_y &= -m_2(\dot{V}_2 + U_2 r_2) + F_{y3} + F_x \Psi
\end{aligned}
\tag{A.14}
$$

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A.1.4 Lateral Tire Forces

The lateral tire forces are obtained by linearizing the dynamic equations of the tires at the steady state and approximated as

$$F_{yi} = k_i \alpha_i \qquad (A.15)$$

for $i = 1, \cdots, 3$ where α_i's denote the tire slip angles which represent the angles between the longitudinal direction and the velocity vector of the ith tire assuming that they are small and k_i's are constants of the tire models. Therefore, three lateral tire forces can be written as

$$F_{y1} = k_1 \left[\delta - \arctan \left(\frac{V_1 - a_1 r_1}{U_1} \right) \right]$$

$$F_{y2} = -k_2 \arctan \left(\frac{V_1 + b_1 r_1}{U_1} \right)$$

$$F_{y3} = -k_3 \arctan \left(\frac{V_2 + b_2 r_2}{U_2} \right) \qquad (A.16)$$

A.2 Linearized Truck-semitrailer Model

In this section, a linear model is derived for the truck-semitrailer system based on the nonlinear model of the previous section . The resulting system will have one input and four states. We impose the following assumptions:

- The longitudinal velocity of the tractor is constant:

$$U_1 = U$$

- The steer angle of the tractor is small enough so that the longitudinal velocity of the semitrailer is equal to that of the tractor:

$$U_2 \approx U_1 = U$$

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- The longitudinal force on the front tires of the tractor and the longitudinal pin point force do not change during the maneuver and they can be approximated as $2A_\rho U^2$ and $A_\rho U^2$, respectively as follows:

$$\dot{U}_2 = V_2 r_2 + \frac{1}{m_2}\left(A_{rho}U_2^2 - F_x\right) \approx 0$$

$$F_x \approx m_2 V_2 r_2 + A_{rho}U_2^2$$

$$\approx A_{rho}U_2^2$$

$$\approx A_{rho}U^2 \tag{A.17}$$

Similarly,

$$\dot{U}_1 = V_1 r_1 + \frac{1}{m_1}(A_\rho U_1^2 - \frac{T_f}{R_f} + F_x) \approx 0$$

$$\frac{T_f}{R_f} \approx m_1 V_1 r_1 + A_\rho U_1^2 + F_x$$

$$\approx A_\rho U_1^2 + F_x$$

$$\approx 2A_\rho U_1^2 \tag{A.18}$$

Under these assumptions, the linear truck-semitrailer model is derived inserting the lateral tire pinpoint force expressions into the dynamical equations associated with the lateral velocity of the tractor, yaw rate of the tractor, yaw rate of the semitrailer and the articulation angle as follows:

The pinpoint force in the lateral direction is given by

$$F_y = -m_2 \dot{V}_2 - m_2 U_2 r_2 + F_{y3} + F_x \Psi \tag{A.19}$$

However, the lateral velocity of the semitrailer and its derivative can be rewritten as

$$\begin{aligned} V_2 &= -U\Psi + V_1 + d_1 r_1 + e_2 r_2 \\ \dot{V}_2 &= -U\dot{\Psi} + \dot{V}_1 + d_1 \dot{r}_1 + e_2 \dot{r}_2 \end{aligned} \tag{A.20}$$

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Inserting (A.20) into (A.19) produces (A.21)

$$F_y = m_2 U \dot{\Psi} - m_2 \dot{V}_1 - m_2 d_1 \dot{r}_1 - m_2 e_2 \dot{r}_2 - m_2 U r_2 + F_{y3} + F_x \Psi \qquad (A.21)$$

assuming that the articulation angle is also small so that the small angle assumptions are valid, i.e; $\sin \Psi \approx \Psi$ and $\cos \Psi \approx 1$.

Putting the equations of lateral tire forces and the lateral pinpoint force into (A.2), (A.3), (A.7) and (A.12), we end up with the following equation set after some straightforward manipulations:

$$(1 + \frac{m_2}{m_1})\dot{V}_1 + \frac{m_2 d_1}{m_1}\dot{r}_1 + \frac{m_2 e_2}{m_1}\dot{r}_2 - \frac{m_2 U}{m_1}\dot{\Psi} =$$

$$-\frac{(k_1 + k_2 + k_3)}{m_1 U}V_1 - U r_1 - \frac{(-k_1 a_1 + k_2 b_1 + k_3 d_1)}{m_1 U}r_1 - \frac{m_2}{m_1}U r_2 - \frac{k_3}{m_1 U}(e_2 + b_2)r_2$$

$$\frac{k_3}{m_1}\Psi + \frac{F_x}{m_1}\Psi + \frac{1}{m_1}(k_1 - \frac{T_f}{R_f})\delta \qquad (A.22)$$

$$d_1 m_2 \dot{V}_1 + (I_{z1} + m_2 d_1^2)\dot{r}_1 + d_1 e_2 m_2 \dot{r}_2 - d_1 m_2 U \dot{\Psi} =$$

$$\frac{(a_1 k_1 - b_1 k_2 - d_1 k_3)}{U}V_1 - \frac{(a_1^2 k_1 + b_1^2 k_2 + d_1^2 k_3)}{U}r_1 - d_1 m_2 U r_2 - a_1(k_1\frac{T_f}{R_f})\delta$$

$$-\frac{d_1 k_3}{U}(e_2 + b_2 + (d_1 k_3 + d_1 F_x)\Psi \qquad (A.23)$$

$$e_2 m_2 \dot{V}_1 + (m_2 d_1 e_2)\dot{r}_1 + (I_{z2} + m_2 e_2^2)\dot{r}_2 - e_2 m_2 U \dot{\Psi} =$$

$$-\frac{k_3}{U}(b_2 + e_2)V_1 - \frac{k_3}{U}(b_2 + e_2)d_1 r_1 - (\frac{k_3}{U}(b_2 + e_2)^2 + m_2 e_2 U)r_2$$

$$+k_3(b_2 + e_2) \qquad (A.24)$$

$$\dot{\Psi} = r_2 - r_1 \qquad (A.25)$$

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From these equations, the linear truck-semitrailer model can be written in the following state space form:

$$\dot{x} = Ax + B\delta \qquad (A.26)$$

where x is the state vector which is given by

$$x = \begin{bmatrix} V_1 \\ r_1 \\ r_2 \\ \Psi \end{bmatrix}$$

V_1 : Lateral velocity of the tractor
r_1 : Yaw rate of the tractor
r_2 : Yaw rate of the semitrailer
Ψ : Articulation angle
δ : Steer angle on the front tires of the tractor
U : Longitudinal velocity of the vehicle

and its components are illustrated in Figure A.6.

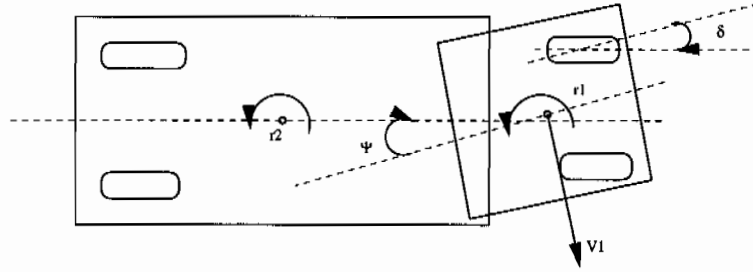Figure A.6: Representations of the State Variables

The matrices A and B are given by

$$A = A_1^{-1} A_2 \qquad\qquad B = A_1^{-1} B_1 \qquad \text{where}$$

$$A_1 = \begin{bmatrix} 1 + \frac{m_2}{m_1} & \frac{m_2 d_1}{m_1} & \frac{m_2 e_2}{m_1} & -\frac{m_2 U}{m_1} \\ d_1 m_2 & I_{z1} + m_2 d_1^2 & d_1 e_2 m_2 & -d_1 m_2 U \\ e_2 m_2 & e_2 d_1 m_2 & I_{z2} + m_2 e_2^2 & -e_2 m_2 U \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$A_2 = \begin{bmatrix} -\frac{k_1+k_2+k_3}{m_1 U} & -U - \frac{-k_1 a_1 + k_2 b_1 + k_3 d_1}{m_1 U} & -\frac{m_2}{m_1} U - \frac{k_3}{m_1 U}(e_2 + b_2) & \frac{k_3 + F_x}{m_1} \\[2ex] \frac{a_1 k_1 - b_1 k_2 - d_1 k_3}{U} & -\frac{a_1^2 k_1 + b_1^2 k_2 + d_1^2 k_3}{U} & -d_1 U m_2 - \frac{d_1 k_3}{U}(e_2 + b_2) & d_1(k_3 + F_x) \\[2ex] -\frac{k_3}{U}(b_2 + e_2) & -\frac{k_3}{U}(b_2 + e_2)d_1 & -\frac{k_3}{U}(b_2 + e_2)^2 - m_2 e_2 U & k_3(b_2 + e_2) \\[2ex] 0 & -1 & 1 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} \frac{1}{m_1}\left(k_1 - \frac{T_f}{R_f}\right) \\[2ex] -a_1 k_1 + a_1 \frac{T_f}{R_f} \\[2ex] 0 \\[2ex] 0 \end{bmatrix}$$

where all the parameters which appear in the preceding equations are as defined as before.

A.3   Numerical Model

In the previous sections, we first derived a nonlinear truck-semitrailer model and then linearized it to obtain a linear model which depends on several truck parameters related to its geometry and the longitudinal velocity of the tractor. However, we need to have a numerical model for the simulation testings of the several sliding mode controllers and observer design methods. To this end, the numerical linearized truck-semitrailer model is obtained in the following state space form:

$$\dot{x} = Ax + B\delta \tag{A.27}$$

Parameter	Meaning of the Parameter	Value
U	Longitudinal velocity of the truck-semitrailer	-15 m/sec
a_1	Distance btw. the tractor CG and the tractor front tires	1.7424 m
b_1	Distance btw. the tractor CG and the tractor rear tires	2.0676 m
d_1	Distance btw. the pinpoint and the tractor front tires	0.8611 m
m_1	Tractor mass	5450 kg
m_2	Semitrailer mass	27800 kg
b_2	Distance btw. the CG and the rear tires of the semitrailer	2.3886 m
e_2	Distance btw. the pinpoint to the semitrailer CG	7.3396 m
I_1	Tractor moment of inertia around the vertical axis	9500 $kg - m^2$
I_2	Semitrailer moment of inertia around the vertical axis	200000 $kg - m^2$
A_ρ	Aerodrag force coefficient	0.5 Ns^2/m^2
k_1	Front axle tire parameter of the tractor	-630000 N/rad
k_2	Rear axle tire parameter of the tractor	-1848000 N/rad
k_1	Rear axle tire parameter of the semitrailer	-600000 N/rad

Table A.3: Numerical parameters

where x is the state vector which is defined as before and

$$
A = \begin{bmatrix}
-14.5045 & 12.2412 & 6.5102 & 10.2890 \\
-11.3014 & -53.7688 & 3.2160 & 5.0828 \\
2.6833 & 5.6393 & -3.3453 & -5.2011 \\
0 & -1 & 1 & 0
\end{bmatrix}, \quad
B = \begin{bmatrix}
-110.8316 \\
118.5998 \\
1.0463 \\
0
\end{bmatrix}
$$

using default parameters of Table A.3.

This numerical model is also simulated on a typical scenario. The vehicle speed is kept at 15 m/sec. through the maneuver and the steer angle command shown in Figure A.7 is applied.

Note that, the area under the steer angle command is zero. Therefore, the average steering effect is also zero and the vehicle shifts to a parallel road with respect to the initial one at the end of the maneuver. Figure A.8 illustrates the trajectory which
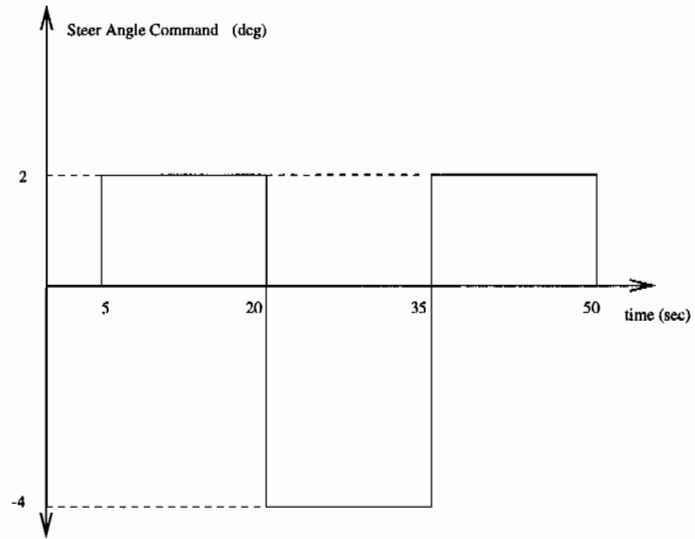
Figure A.7: Steer angle command to the vehicle

the tractor CG traces and Figures A.9-A.11 show the state variables of the truck-semitrailer system during the maneuver.
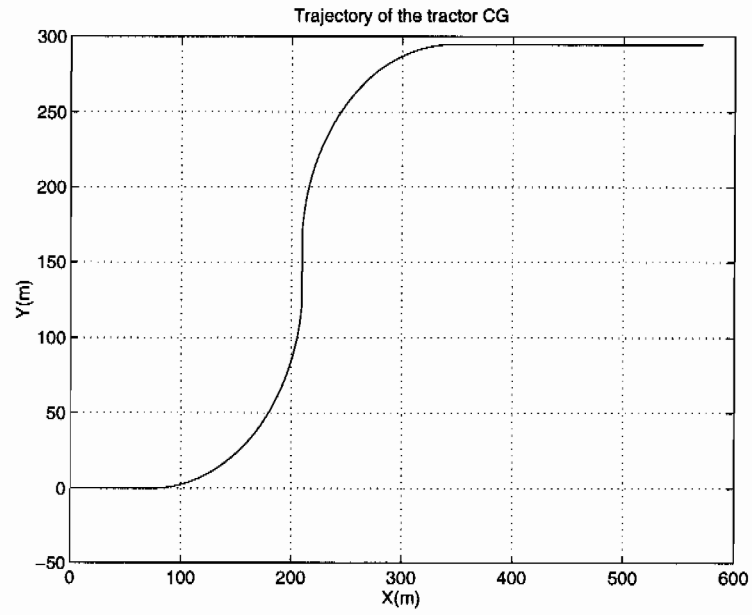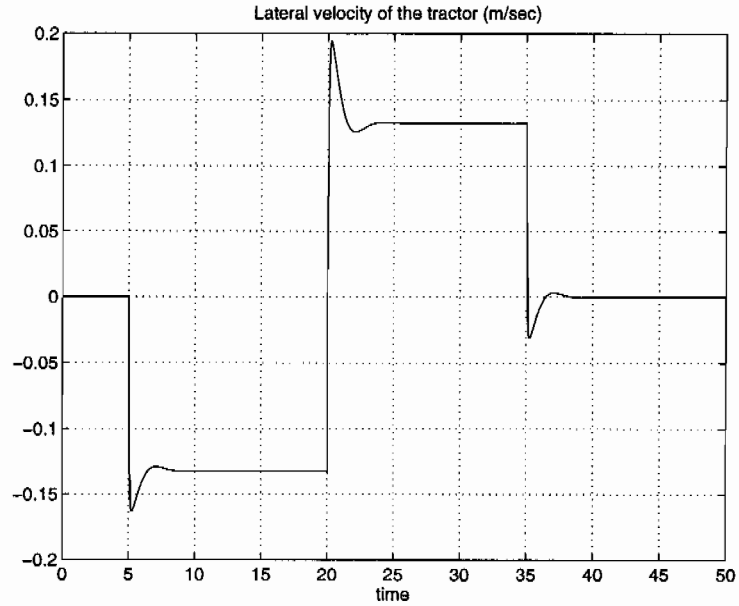
Figure A.8: CG trajectory of the tractor

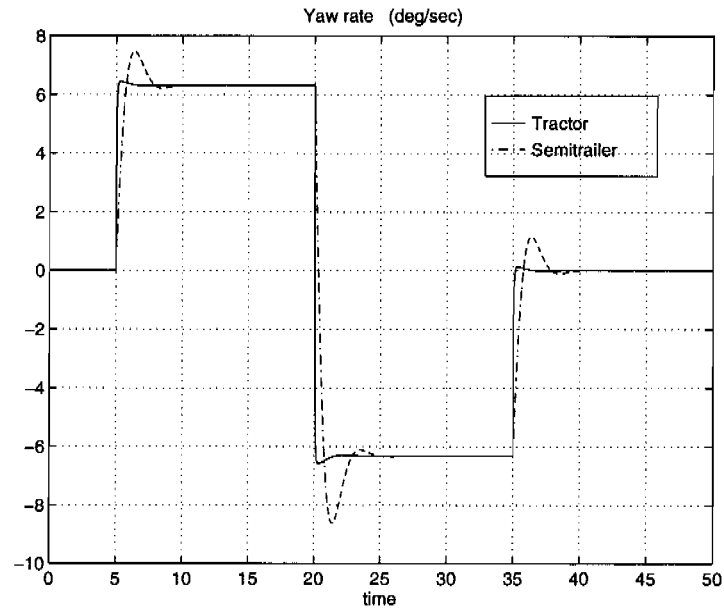Figure A.9: Tractor lateral velocity

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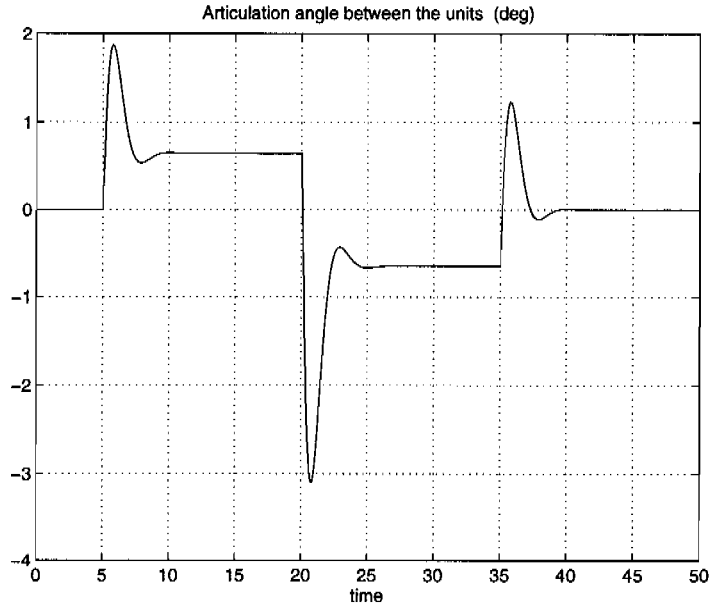Figure A.10: Yaw rates of tractor and semitrailer

Figure A.11: Articulation angle

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