PERSISTENCE OF PLANAR SPIRAL WAVES UNDER DOMAIN TRUNCATION NEAR THE CORE

DISSERTATION

Presented in Partial Fulfillment of the Requirements for

the Degree Doctor of Philosophy in the

Graduate School of The Ohio State University

By

Man Tsoi, B.S., M.Phil.

* * * * *

The Ohio State University

2006

Dissertation Committee:

Prof. B. Sandstede, Co-Adviser

Prof. Y. Lou, Co-Adviser

Prof. E. Overman

Prof. D. Terman

Approved by

Co-Adviser

Co-Adviser

GRADUATE PROGRAM IN MATHEMATICS

ABSTRACT

Planar spiral waves are rotating waves that rotate with constant angular velocity and behave like Archimedean spirals in their far field. Such spirals arise in many experiments as well as in reaction- diffusion models. In this thesis, the persistence of planar spiral waves is investigated upon excising a small hole from the domain near the core region of the spiral.

Before treating 2D patterns, we investigate the persistence of 1D pulses upon truncating the real line to large but bounded intervals, supplemented by boundary conditions at their end points. Under appropriate transversality assumptions on the boundary conditions, the persistence of pulses is established and their stability with respect to the reaction-diffusion system on the bounded interval is determined. It turns out that the stability properties of the truncated pulses depends on the choice of boundary conditions. These results are then applied to the front of the Nagumo equation and the fast pulse of the FitzHugh-Nagumo equation.

In the second part of the thesis, we analyse the persistence of 2D spiral waves by posing the elliptic partial differential equation as an ill-posed dynamical system in which the radius serves as the time- like variable. In this setting, the approach via Lyapunov-Schmidt reduction and Lin's method utilized in the one-dimensional case carries over to systems posed on the plane. Upon establishing suitable a-priori estimates for the dynamics on the center eigenspace, we prove the persistence of core-region spirals that satisfy the boundary

conditions, and afterwards match with far-field spirals to obtain a unique planar spiral that obeys the boundary conditions.

ACKNOWLEDGMENTS

I am greatly indebted to my advisor Prof. Björn Sandstede for his advice over the past six years. Without his guidance and inspiration on the subject, this thesis would not have been possible. I am grateful for his time, patience and encouragement to help me at difficult times.

I also thank Anna Ghazaryan and Vahagn Manukian for some simulating discussion over the subject and Ping-Shun Chan for proofreading part of this thesis.

Finally, I wish to thank my parents for their understanding, continuous support and encouragement.

VITA

1997	B.S. in Mathematics, Chinese University of Hong Kong
1999	M.Phil. in Mathematics, Chinese University of Hong Kong
1999-present	PhD student at Ohio State University

FIELDS OF STUDY

Major Field: Mathematics

TABLE OF CONTENTS

Page

LIST OF FIGURES

Figure **Page**

CHAPTER 1

INTRODUCTION

Historical Perspective

Spiral waves provide one of the most striking examples of pattern formation in non-linear active media. In 1946, Wiener and Rosenblueth [59] proposed a simple model that allows one to analyze various regimes of excitation propagation in a homogeneous neuron network or in heart tissue. In the framework of their model, they described the phenomenon of a spiral wave circulating around a hole in an excitable medium. In the late 1960's, Zaikin and Zhabotinsky [64] observed target patterns in the Belousov-Zhabotinsky (BZ) reaction in a thin unstirred layer of reacting solution of ferroin and malonic acid. This observation was regarded as the first recorded spatio-temporal pattern formation in chemical systems. Later Winfree [60] found self-sustained rotating spiral waves in thin layers of BZ reagent (See [7] for more experiment results on the BZ reaction). Ever since, these examples of pattern formation have fascinated researchers in applied mathematics, physics, chemistry, cardiology, and several branches of biology. Spiral waves also appear in several other chemical and biological systems, for instance, during the oxidation of carbon-monoxide on platinum surfaces [38], during the aggregation stage of the slime mode *Dictyostelium discoideum* [32] or the intracellular calcium wave signaling in immature *Xenopus* oocytes [1].

Spiral waves are mainly, though not exclusively [41], considered as spatio-temporal patterns generated by reaction-diffusion systems in excitable media (For the study of defects in the oscillatory media, see e.g. [51]). The complete scheme in the application is usually complicated, but qualitatively it can be broken down into much simpler building blocks. For instance, the BZ reaction involves more than 20 intermediate stages, but it can essentially be described by the following 2-component model [58]:

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_u \frac{\partial^2 u}{\partial x^2} + u(1-u) - \frac{v(u-a)}{u+a}, \\ \frac{\partial v}{\partial t} &= D_u \frac{\partial^2 v}{\partial x^2} - \frac{1}{\tau}(v - bu). \end{aligned}$$

Here u, v represent the normalized concentration of the activator and inhibitor, and $a \ll 1$, b and $\tau \gg 1$ are positive control parameters.

The Existence of Spiral Waves

The existence problem has been a long-standing issue in the study of spiral waves. Cohen, Neu and Rosales [12] were the first to construct a spiral wave solution for reaction-diffusion equations, and over the past three decades, Greenberg [19], Kopell and Howard [30], Keener and Tyson [28], Karma [26], Sandstede and Scheel [46, 47, 48, 49, 50, 52, 55], and among others, have investigated the existence and stability of spiral waves in reaction-diffusion systems. There are two approaches (Section 4, [16]) to this problem: One is *the kinematic theory of spirals*, that is, a direct geometric description of the spatio-temporal dynamics of the appearing sharp wave fronts; another one uses the mathematical description derived from experimentally observed spirals. For example, in this thesis, we assume that the spirals are rigidly rotating and of Archimedean shape. These features can be observed in many spiral waves.

Spiral Instabilities

One of the most interesting parts in the studies of pattern formation is the investigation of different states and the transition from one to another in different parameter regimes. For generic spiral waves, there are two types of transitions or instabilities that have been studied intensively: Spirals may begin to meander or drift instead of rotating rigidly [62]; they may also breakup either in the core or the farfield. These phenomena have been observed both experimentally and numerically [2]. The following pictures are variant snapshots of numerical simulation of the FitzHugh-Nagumo equation, by Sandstede and Scheel, based on Barkley's code EZSpiral.

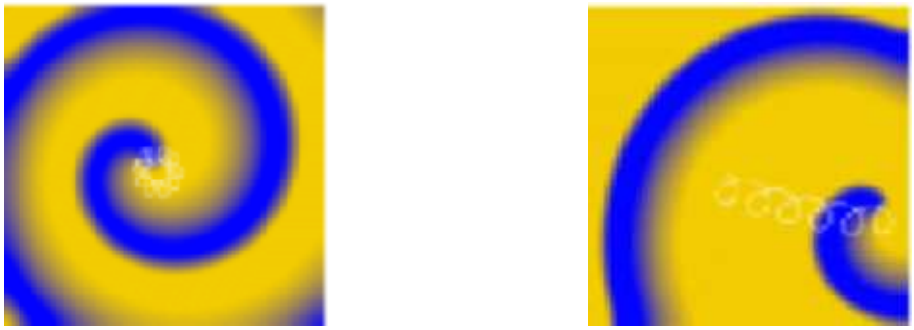

Figure 1.1: Meandering (left) and drifting (right) of the tip motion: The transition are near Hopf bifurcations of the original pattern. The drifting is a consequence of the resonance between the Hopf frequency and the frequency of the rigidly rotating spiral waves, see [52]. The superimposed spirals can be observed in the farfield for meandering cases.

Truncation Problem: Small Hole Near the Core

In this thesis, instead of investigating instabilities or bifurcations of spiral waves, we are interested in the persistence of spiral waves.

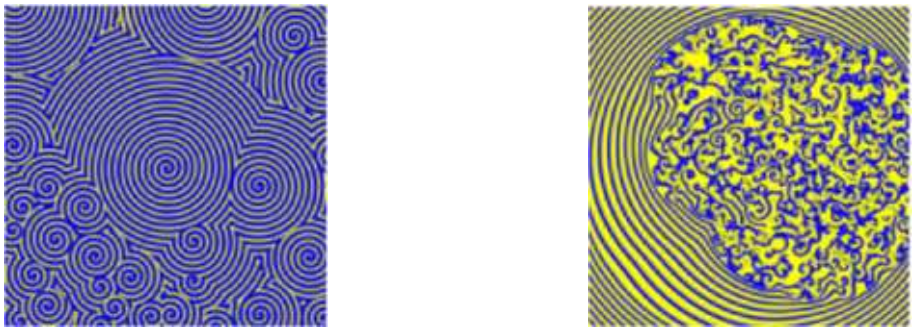

Figure 1.2: Farfield (left) and core (right) breakup: The instability is caused by the absolute spectrum of the spiral wave. The wavetrains transport the local perturbation to the farfield/core region. For the farfield (left), the parameters are in the regime of absolute instability. The subcritical nature of the instability amplifies the compression and expansion of the wavetrains until they collide and breakup, see [49, 50]. Thus, breakup can, to some extent, be described by instabilities of the asymptotic one-dimensional wave trains that then carry over to the full two-dimensional spiral. The similar idea can be applied in the studies on defects, see [51].

In early studies of spiral waves, it was realized that any spiraling pattern would necessarily entail a phase singularity at its core (For example, see [61]). It has also been known that the creation of spiral patterns does not require the presence of an inhomogeneity. For example, as observed in BZ reactions [60, 65], spiral waves are created by breaking the continuous front of an excitation wave, which does not require any impurities or gas bubbles. However, the presence of inhomogeneities will lead to a more complicated dynamics of the spirals. For example, in the low-voltage defibrillation of heart tissue, the repeated low-voltage perturbation with the same phase would direct the drift of spiraling pattern to the boundary of the cardiac tissue, at which the spiral can be extinguished due to the non-excitability of the boundary. Issues can now arise from the local inhomogeneities, which can interact with the spiraling pattern, and might prevent the spiral to drift away.

In this thesis, we consider two issues arising from the above consideration, namely, the existence of the spiral wave after excising a small hole near the core region, and the effect of the boundary condition imposed at the boundaries of the hole upon the dynamic of the spiral. The main result (Theorem 3.6.1) that we obtain is the following:

Theorem 1.0.1. *Consider the reaction-diffusion equation on the plane*

$$u_t = D\Delta u + f(u), \qquad u = u(x,t), x \in \mathbb{R}^2. \tag{1.1}$$

Assume that there is a generic spiral wave solution u_* of (1.1) with temporal frequency $\omega_* \neq 0$ and positive group velocity c_g. (The group velocity describes the change of temporal frequency with respect to the wave number, see (1.7) for the definition). Also, assume that u_* satisfies a certain transversality hypothesis (Hypothesis 3.1). Then the spiral is robust with respect to the Dirichlet or Neumann boundary condition at $r = \epsilon$ for ϵ small enough. More precisely, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, the following is true.

Consider the new domain

$$\Omega := \mathbb{R}^2 \backslash B_\epsilon(0)$$

with the boundary $\partial\Omega = \partial B_\epsilon(0)$. Then there exist two families of spiral waves $u_{Dir} = u_{Dir}(\epsilon)$ and $u_{Neu} = u_{Neu}(\epsilon)$ on Ω with frequency $\omega = \omega(\epsilon)$ such that u satisfies the Dirichlet boundary condition

$$u_{Dir}|_{\partial\Omega} = u_*(0)$$

or Neumann boundary condition

$$\frac{\partial u_{Neu}}{\partial n}|_{\partial\Omega} = 0$$

at $r = \epsilon$. Moreover, u_{Dir} and u_{Neu} depend smoothly on ϵ. Furthermore, up to some normalization (Hypothesis 3.4), we have the following expansion for ω near ω_*:

$$\begin{aligned} \omega &= \omega_* + \frac{\epsilon}{M \cdot \ln \epsilon}(u_*)_r(0) + O(\frac{\epsilon}{\ln^2 \epsilon}), \qquad \textit{(Dirichlet condition)}; \\ \omega &= \omega_* + \frac{\epsilon}{M}(u_*)_r(0) + O(\epsilon^2), \qquad \textit{(Neumann condition)}. \end{aligned}$$

Here M is an Melnikov-type integral with respect to ω, which is assumed to be non-zero (see Hypothesis 3.2).

Methodology: Lin's Method

Instead of embarking directly on an analysis of the two-dimensional case, we first study the corresponding one-dimensional situation to gain insight into the relevant issues. Afterwards, using radial dynamics we shall see that the main ideas carry over from the one-dimensional to the two-dimensional situation, even though the details will be far more complicated.

The scheme we use to study the one-dimensional situation is called *Lin's method*. In his paper [33], X.-B. Lin demonstrated that a functional-analytic framework can be applied to the problem of the bifurcation of periodic solutions near heteroclinic or homoclinic orbits of the following autonomous equation:

$$\dot{x} = f(x, \mu), \qquad x = x(t) \in \mathbb{R}^n. \tag{1.2}$$

Suppose that (1.2) possesses a homoclinic orbit $h(t)$ to a hyperbolic equilibrium p when $\mu = 0$ and the tangent spaces of stable and unstable manifold of p have a one-dimensional intersection along $h(t)$. Since p is hyperbolic, the linearized equation

$$\dot{x}(t) = D_x f(h(t), 0)x \tag{1.3}$$

has an exponential dichotomy (see Definition 2.4.1) on $[-L, 0]$, and $[0, L]$, respectively. That is, the phase space can be decomposed into invariant subspaces so that solutions in these subspaces decay exponentially in forward or backward time.

Let $\Sigma \ni h(0)$ be a $(n-1)$-dimensional hyperplane which is transverse to $h'(0)$. Since the tangent space $T_{h(0)}M^{cu}(p)$ of the center-unstable manifold and the tangent space $T_{h(0)}M^{cs}(p)$ of the center-stable manifold at $h(0)$ intersect along a one-dimensional subspace, there is a one-dimensional subspace $\Delta = (M^{cs}(p) + M^{cu}(p))^{\perp}$ of Σ, which is given by $span\{\psi(0)\}$. Here $\psi = \psi(t)$ is the unique, up to a scalar multiple, bounded nontrivial solution of the adjoint variational equation to (1.3):

$$\dot{y}(t) = -(D_x f(h(t), 0))^* y(t). \tag{1.4}$$

Suppose that we are interested in a $2L$-periodic solution $x(t) = x_{(L,\mu)}(t)$ of the equation (1.2), such that $x(t)$ is near $h(t)$ for $t \in [-L, L]$ with $L \gg 1$ and satisfies $x(-L) = x(L)$. With possible phase shift, we set (Figure 1.3)

$$x(0+) - x(0-) := \lim_{t \searrow 0} x(t) - \lim_{t \nearrow 0} x(t) \in \Sigma.$$

The mismatch $\lim_{t\searrow 0} x(t) - \lim_{t\nearrow 0} x(t)$ may not vanish due to the boundary condition $x(-L) = x(L)$. Therefore, if we consider that $x = x_{(\mu,L)}$ parametrized the stable manifold for $t \geq 0$ and the unstable manifold for $t \leq 0$, with a discontinuity at $t = 0$, then we can vary μ to match, at $t = 0$, two pieces of the solution $x(t)$, $t \geq 0$ or $t \leq 0$, for all L large in the following heuristic sense: First, according to the geometry of Σ and the existence of exponential dichotomies, we can decompose Σ into mutually orthogonal subspaces corresponding to b_+, b_- and Δ, in which Δ is one-dimensional and the solution in b_+ (b_-) decays exponentially in forward (backward) time (see Figure 1.4). Then we can match in the directions other than Δ by solving for b_+ and b_- in terms of μ and the parameter provided by the boundary condition $x(L) = x(-L)$. Lastly, we match two

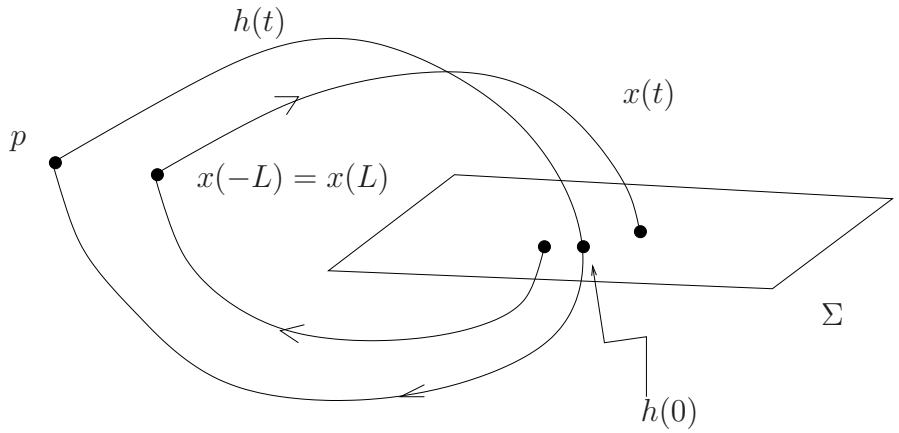

Figure 1.3: Lin's method: seeking periodic solution $x(t)$ accompanying the homoclinic solution $h(t)$.

solutions in the direction of Δ by adjusting μ. It turns out that the solvability of the last matching is equivalent to the solvability of the following equation

$$M\mu = \langle \psi(-L), h(L) \rangle - \langle \psi(L), q(-L) \rangle,$$

whose solvability is implied by the assumption $M := \int_{-\infty}^{\infty} \langle \psi(t), \partial_\mu f(h(t), 0) \rangle dt \neq 0$.

Throughout the above argument, the nonlinearity only contributes to the higher order terms, for which we formulate our version more rigorously in Lemma 2.6.1, 2.6.2 and 2.6.3, Theorem 2.7.1 and 2.7.2.

In Lin's method, the bifurcation function

$$G(L, \mu) = \langle \psi(L), h(-L) \rangle - \langle \psi(-L), h(L) \rangle + \mu \int_{-\infty}^{\infty} \langle \psi(t), D_\mu f(h(t), 0) \rangle dt + h.o.t.$$

generalizes the bifurcation function

$$G(\infty, \mu) = \mu \int_{-\infty}^{\infty} \langle \psi(t), D_\mu f(h(t), 0) \rangle dt + O(|\mu|^2).$$

for the homoclinic orbit.

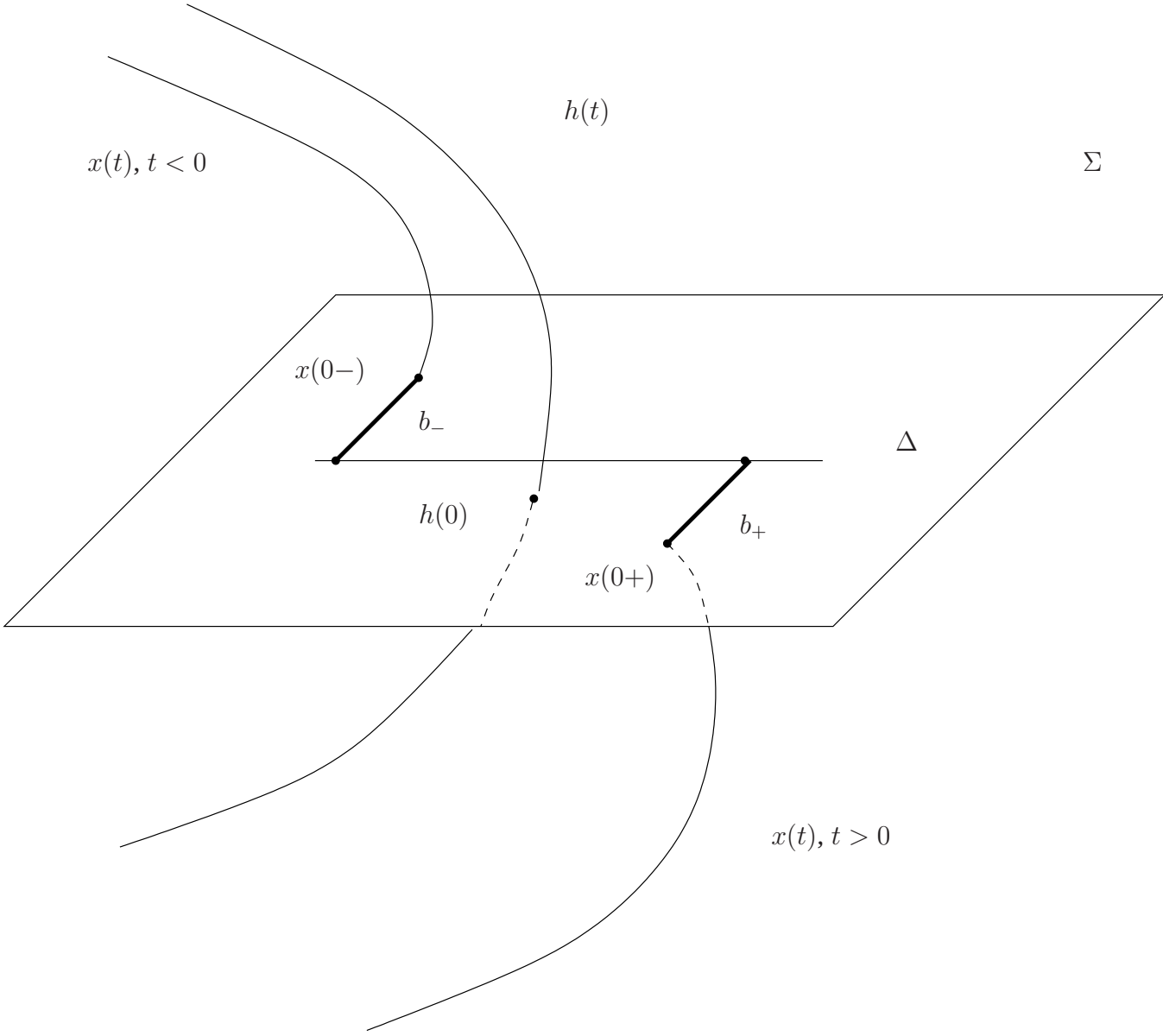

Figure 1.4: Lin's method: near the matching section Σ. Here b_+ and b_- may be linearly independent, despite the way they are depicted. We need to match $x(t)$, $t > 0$ and $t < 0$ at $t = 0$ in the directions of b_-, b_+ and Δ.

Later Lin's method was further developed by various authors, in the study of multiple pulses, homoclinic bifurcations, see, for example, [8, 44, 45, 43, 46, 53, 54], etc..

In Chapter 1, we consider the truncation problem of a homoclinic solution h on $\mathbb{R}$ to a solution on a large but finite interval $[T_-, T_+]$, with $-T_-$, $T_+ \gg 1$ and the boundary conditions imposed at $T_\pm$ are *separated*. We prove that the truncation boundary value problem is solvable (Theorem 2.7.1) if the boundary condition satisfies a certain transversality assumption (Hypothesis 2.4). Furthermore, the truncation destroys the translational invariance, then an eigenvalue $\lambda = \lambda_L \sim 0$ is created to replace the critical eigenvalue $\lambda = 0$. The sign of the eigenvalue is determined by the leading order terms $\frac{1}{M}(\langle \psi(T_-), P_-^{bc} h'(T_-)\rangle - \langle \psi(T_+), P_-^{bc} h'(T_+)\rangle)$, where

$$M = \int_{-\infty}^{\infty} \langle \psi(t), D_\mu f(h(t), 0)\rangle dt$$

is the Melnikov integral associated with μ (Theorem 2.7.2). Since ψ and h' decay exponentially, the eigenvalue λ_L is actually exponentially small in L. For a finer estimate with an exponentially weighted norm, see the super-convergence result in [43]. Afterward, we also show the application of our method to the truncated boundary value problem of the fast pulse for the FitzHugh-Nagumo equation and the front for the Nagumo equation.

Methodology: Radial Dynamics

The main idea is to reformulate the elliptic PDE's that governs the existence and the linear stability of the spiral wave as an ill-posed dynamical system in the radial direction. The idea of posing elliptic problems as ill-posed dynamical system can be traced back as early as [29].

The spiral wave we consider is *Archimedean*, that is, asymptotically periodic along the radial direction in the plane. More precisely, far away from the rotation center, along the radial direction, the spiral wave converges to a one-dimensional wave train u_∞ with period 2π, wave number k_∞, and temporal frequency ω_∞ (see (3.9)). Furthermore, consider the

formal limit $r \to \infty$, we find that the wave train satisfies the travelling wave equation for the one-dimensional reaction-diffusion equation. Therefore we refer the wave train to the *asymptotic wave train* of the spiral wave solution.

Spiral waves can then be captured as heteroclinic orbits connecting homogeneous steady states at the core at $r = 0$ and the asymptotic wave trains at $r = \infty$. This viewpoint, which we refer to as *spatial dynamics*, was first used in [55].

Now if the spiral wave is truncated near the core region, correspondingly we would have a one-dimensional truncation problem (along the radial direction) for the travelling wave.

Consider that the linearization of the reaction-diffusion equation at a spiral in a co-rotating frame

$$u_t = D\Delta u - \omega_* \partial_\psi u + f'(u_*(r,\psi))u := \mathcal{L}_* u.$$

In spirit of relating the spiral wave with its asymptotic wave train u_∞, we consider the one-dimensional reaction-diffusion equation

$$u_t = Du_{xx} + f(u), \qquad x \in \mathbb{R}, u \in \mathbb{R}^N. \tag{1.5}$$

Upon using the co-moving coordinates $(t, \xi = kx - \omega)$, the spatial operator in the reaction-diffusion equation (1.5) possesses the following linearization about u_∞:

$$\mathcal{L} := k^2 D\partial_{\xi\xi} + \omega\partial_\xi + f'(u_\infty(\xi)),$$

which we consider as an unbounded operator on $L^2(\mathbb{R}, \mathbb{C}^N)$. Write the associated eigenvalue problem $\mathcal{L}u = \lambda u$ as the ordinary differential equation

$$\begin{aligned} ku_\xi &= v \\ kv_\xi &= -D^{-1}[cv + f'(u_\infty(\xi))u - \lambda u]. \end{aligned} \tag{1.6}$$

Note that the coefficients are 2π-periodic. Let $\Phi(\lambda)$ be the associated period map, mapping an initial value to the solution evaluated at $\xi = 2\pi$. By Floquet theory (for example, Theorem 6.1 in [21]), the ODE (1.6) has a bounded solution if and only if the Evans function

$$E(\lambda, \nu) := \det[\Phi(\lambda) - e^{2\pi\nu/k}] = 0$$

for some $\nu \in i\mathbb{R}$. Note that $E(0, 0) = 0$.

We assume that

$$\partial_\lambda E(0, 0) \neq 0,$$

which guarantees that the generalized kernel of $\mathcal{L}$ is one-dimensional and spanned by u'_∞. The kernel of the formal adjoint operator $\mathcal{L}^{ad}$ is spanned by a function u_{ad} with $\langle u_{ad}, u'_\infty \rangle \neq 0$. Now apply Lyapunov-Schmidt reduction, then we obtain the reduced equation $h(\omega, k) = 0$. The derivative $\partial_\omega h(\omega_\infty, k_\infty) = \langle u_{ad}, u'_\infty \rangle \neq 0$, then we can solve $h(\omega, k) = 0$ for ω as a function of k by using the implicit function theorem and denote the solution by $\omega(k)$.

The *group velocity* c_g is defined by the derivative of the nonlinear dispersion relation $\omega = \omega(k)$ with respect to the wave number k at $k = k_\infty$:

$$c_g := \frac{\partial \omega}{\partial k}(k_\infty). \tag{1.7}$$

Definition 1.0.1. *We say that the spiral emits the wave train if the group velocity c_g of the asymptotic wave train is positive.*

Then we restrict our studies to a generic class of Archimedean spiral waves. Define the exponentially weighted spaces

$$L^2_\gamma := \{u \in L^2_{loc}(\mathbb{R}^2) : \|u\|^2_{L^2_\gamma} = \int_{\mathbb{R}^2} |e^{\gamma|x|}u(x)|^2 < \infty\},$$

with weight $\gamma \in \mathbb{R}$ in the radial direction r.

Definition 1.0.2. *An Archimedean spiral wave $u_* = u_*(r,\psi)$ is transverse if it emits a spectrally stable wave train and the generalized kernel of the linearization $\mathcal{L}_*$ about u_* in L^2_γ is one-dimensional for some small $\gamma > 0$.*

Then for reaction-diffusion systems with analytic kinetics, Sandstede and Scheel [49] proved that transverse Archimedean spirals are robust with respect to perturbations of the diffusion coefficients and/or the reaction kinetics. Also they proved that if the kinetics depends smoothly on a parameter μ, then transverse Archimedean spirals are also robust [47].

Approach for the Main Result

In Chapter 2, as we turn to the problem of truncation of the planar spiral wave in the core region, extra complications appear. First, observe that the Laplacian on the plane

$$\Delta_2 := \partial_{x_1x_1} + \partial_{x_2x_2} = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\psi\psi}$$

is singular at $r = 0$ in the polar coordinates. This motivates us to introduce a new time scale $\tau = \ln r$ in order to perform a blow-up analysis near $r = 0$ [47, 55].

Secondly, one extra feature we can observe for the rescaled linear equation is the presence of extra center directions: Taking the formal limit $\tau \to -\infty$ in the equation

$$\begin{aligned} u_\tau &= w \\ w_\tau &= -\partial_{\psi\psi}u - e^{2\tau}D^{-1}(\omega_*\partial_\psi u + f'(u_*(\tau,\psi))u) \\ &\quad -e^{2\tau}D^{-1}[(\omega-\omega_*)\partial_\psi(u_*+u) + f(u_*+u) - f(u) - f'(u_*)u], \end{aligned} \tag{1.8}$$

we have the asymptotic equation

$$\mathcal{U}_\tau = \mathcal{A}^-_\infty \mathcal{U}, \tag{1.9}$$

where

$$\mathcal{A}^-_\infty = \begin{pmatrix} 0 & 1 \\ -\partial_{\psi\psi} & 0 \end{pmatrix}$$

which can be solved by using Fourier series in $X^0 = H^1(\mathbb{S}^1, \mathbb{C}^N) \times L^2(\mathbb{S}^1, \mathbb{C}^N)$. Observe that in the solution space, there is a $2N$-dimensional subspace E^c_- on which solutions do not decay in either forward or backward time: One N-dimensional subspace E^{ker}_- (see Section 3.3.1) corresponds to steady states at $\tau = -\infty$ and its N-dimensional orthogonal complement E^{gker}_- corresponds to the space of the linearly growing solutions of the linearized equation. By applying standard perturbation tools (for example, the results in [40]), we learn that the equation (1.8) exhibits similar dynamical behavior on the ray $\tau \in (-\infty, s_*)$, for any fixed constant s_*. Thus, those corresponding $\{E^c_-(\tau) : \tau \in (-\infty, s_*)\}$ need to be accounted near the core.

In order to prove the existence of the truncated core spiral, we consider the core spiral, along with its derivative with respect to τ, as a solution of the variation-of-constants formula:

$$\begin{aligned} \mathcal{U}(\tau) &= \Phi^c_-(\tau, R_*) \begin{pmatrix} w^{ker}_- \\ w^{gker}_- \end{pmatrix} + \Phi^{ss}_-(\tau, -L) w^{ss}_- + \Phi^{uu}_-(\tau, R_*) w^{uu}_- \\ &\quad + \int_{-L}^{\tau} [\Phi^{ss}_-(\tau, s) + \Phi^c_-(\tau, s)] e^{2s} \mathcal{G}(\mathcal{U}, s) ds + \int_{R_*}^{\tau} \Phi^{uu}_-(\tau, s) e^{2s} \mathcal{G}(\mathcal{U}, s) ds. \end{aligned} \tag{1.10}$$

with appropriate boundary condition (for more details, see Section 3.2.3)

$$\mathcal{U}(-L) \in E^{bc}_- + d.$$

Here Φ^i_- are various evolution operators for $i = ss$, uu and c. The parameters w^i_- are taken either at $\tau = -L$ or $\tau = R_*$, $i = ker$, $gker$, ss and uu. Among them, (w^{ker}_-, w^{gker}_-) are the center components taken at $\tau = R_*$. E^{bc}_- is the boundary condition subspace and d specified as a boundary condition parameter.

Observe that this fixed point equation (1.10) cannot be solved in $(w^{ker}_-, w^{gker}_-, w^{ss}_-, w^{uu}_-, d)$ by using the Implicit Function Theorem, for the right hand side of (1.10) is not C^1-bounded in the initial condition w^{gker}_- corresponding to the linear growing solutions. The key to

overcome this difficulty is to develop a-priori estimates of w_-^{gker} in terms of the other parameters, exploiting transversality conditions which we assume to hold.

After obtaining the existence result of spiral waves in the core region, we would like to match the core region spiral waves with the farfield spiral. During the matching, we find that if we consider the temporal frequency ω as an extra parameter, then the other parameters w_-^{ker}, w_-^{uu} and w_+^{ss} (a parameter provided by the farfield spiral) can be expressed in terms of ω (Lemma 3.6.4). On the other hand, to match along the direction of the bounded nontrivial solution of the adjoint equation, we need to adjust ω. Thus, we see that $\mathcal{U} = (u, u_\tau)$ is parametrized by ω, and we also obtain the expansion of ω near ω_* (Theorem 3.6.1).

CHAPTER 2

TRUNCATION OF ONE-DIMENSIONAL PULSES

2.1 Motivation

Consider the following one-dimensional *reaction-diffusion* equation

$$U_t = DU_{xx} + cU_x + F(U), \qquad x \in [T_-, T_+], U \in \mathbb{R}^N, \tag{2.1}$$

in which the nonlinearity F is at least $\mathcal{C}^2$ and D is a diagonal matrix with positive entries. The N-dimensional vector U may describe a set of chemical concentrations, depending on time $t \in \mathbb{R}$ and the space variable x. The term cU_x models advection of chemicals.

We are interested in stationary solutions to (2.1) of the form

$$U(x,t) = U_*(x), \qquad x \in [T_-, T_+].$$

Thus, $U = U_*$ satisfies

$$0 = U_t = D\partial_{xx}U + c\partial_x U + F(U), \qquad x \in [T_-, T_+], U(x) \in \mathbb{R}^N. \tag{2.2}$$

Suppose that there exists a stationary solution, then we are also interested in the stability of such a solution. Consider the linearization of the right hand side of (2.2) around U_*

$$\mathcal{L} := D\partial_{xx} + c\partial_x + F_U(U_*).$$

Then the spectral stability of the stationary solution is determined by the following eigenvalue problem

$$\mathcal{L}U = \lambda U \tag{2.3}$$

posed on the Banach space $C^0_{unif}([T_-, T_+], \mathbb{R}^N)$ of bounded, uniformly continuous functions.

Note that the steady-state equation (2.2) and the eigenvalue problem (2.3) are both ordinary differential equations (ODE's), which can be rewritten as systems of first order ODE's. So the steady-state equation (2.2) reads

$$\frac{du}{dx} := \begin{pmatrix} U_x \\ V_x \end{pmatrix} = \begin{pmatrix} V \\ -D^{-1}(cV + F(U)) \end{pmatrix} =: f(u, c), \tag{2.4}$$

where $u = (U, V) \in \mathbb{R}^{2N}$, $x \in \mathbb{R}$. On the other hand, for the eigenvalue problem (2.3), we have

$$\begin{pmatrix} U_x \\ V_x \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\lambda - \partial_U F(U_*(x))) & -cD^{-1} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}. \tag{2.5}$$

Rewrite (2.5) to single out the coefficient λ. This yields

$$\begin{pmatrix} U_x \\ V_x \end{pmatrix} = \left[\begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}\partial_U F(U_*(x))) & -cD^{-1} \end{pmatrix} + \lambda B \right] \begin{pmatrix} U \\ V \end{pmatrix}, \tag{2.6}$$

with

$$B = \begin{pmatrix} 0 & 0 \\ D^{-1} & 0 \end{pmatrix}.$$

We say that λ is an *eigenvalue* of U_* if (2.6) has a nontrivial bounded solution.

In the following sections of this chapter, we concentrate on the ODE formulations (2.4) and (2.6).

2.2 Set-up

Consider a system of ODE's with a control parameter $\mu \in \mathbb{R}^p$:

$$\frac{du}{dx} = f(u, \mu), \qquad (u, \mu) \in \mathbb{R}^n \times \mathbb{R}^p, x \in \mathbb{R}, \tag{2.7}$$

where f is at least $\mathcal{C}^2$ such that $u \equiv 0$ is an equilibrium for any μ, that is, $f(0, \mu) = 0$ for all μ. For example, for the ODE's derived from the reaction-diffusion system, we have that $n = 2N$, where N is the number of species and the parameter μ is given by the advection parameter $c \in \mathbb{R}$.

In this chapter we consider the case that, for $\mu = 0$, the system (2.7) admits a *pulse* $h = h(x)$, which is a homoclinic orbit of (2.7) with $\lim_{x\to\pm\infty} h(x) = 0$. We also assume that h is stable for the PDE (2.1): Observe that due to the translation invariance, $\lambda = 0$ is always an eigenvalue for pulses. The stability of h means that the complement of $\{0\}$ in the spectrum of the linearization around h lies in the left half plane and is strictly bounded away from the imaginary axis.

Assuming the existence of h, we are interested, either for numerical computations or theoretical interest [8, 53, 54, 43, 46, 49], in the *truncated boundary value problem*: We want to find a solution $(u, \mu) = (u_T, \mu_T)$ of equation (2.7) on a large but bounded interval $[T_-, T_+]$, $T_- < 0 < T_+$ with $|T_-|$, $T_+ \gg 1$, that approximates h in some function space, say, $C^1([T_-, T_+], \mathbb{R}^n)$, and satisfies appropriate boundary conditions assigned at T_- and T_+. We are also interested in determining the stability of the resulting u_T on $[T_-, T_+]$, as a solution to the PDE (2.1) posed on $[T_-, T_+]$.

The boundary conditions we use here are *separated boundary conditions*, that is, u_T satisfies the boundary conditions if and only if $u_T(T_-)$ and $u_T(T_+)$ belong to two separately prescribed closed subspaces, respectively. The subspaces should satisfy certain transversality conditions, which will be stated later.

Note that (2.7) is *translation invariant*. That is, let $\tilde{h}(x) = h(x + x_0)$ for any fixed $x_0 \in \mathbb{R}$, then $\tilde{h}$ also satisfies (2.7). Consider the linearization

$$v' = f_u(h(x), 0)v, \quad x \in \mathbb{R}. \tag{2.8}$$

of the equation (2.7) about h. Then we have the following characterization of the translation invariance.

Lemma 2.2.1. *$h'(x)$ is a solution to the linearization (2.8).*

Proof. Differentiate the equation (2.7) with respect to x, then we have

$$(h')' = f_u(u, \mu)|_{u=h,\mu=0} \cdot h'.$$

□

Suppose that h arises as a steady state of the reaction-diffusion equation on the real line. Then Lemma 2.2.1 implies that $(U, V) = h'$ is an eigenfunction for the eigenvalue problem (2.6) with the eigenvalue $\lambda = 0$. Now upon the truncation of $\mathbb{R}$ to $[T_-, T_+]$, the translation invariance of (2.7) is destroyed; therefore, the critical eigenvalue $\lambda = 0$ for the pulse h on the real line is no longer an eigenvalue for u_T. On the other hand, if (u_T, μ_T) is close to $(h, 0)$, we expect that there are discrete eigenvalues of the eigenvalue problem (2.6) near $\lambda = 0$, for $(u, \mu) = (u_T, \mu_T)$. The number of the eigenvalues near $\lambda = 0$ should be equal to the multiplicity of the eigenvalue 0 for the pulse h. Therefore, more pictorially, the action of truncation moves the eigenvalues at 0 around and the stability/instability is determined by whether any eigenvalue moves into the open right half plane, see Figure 2.2.

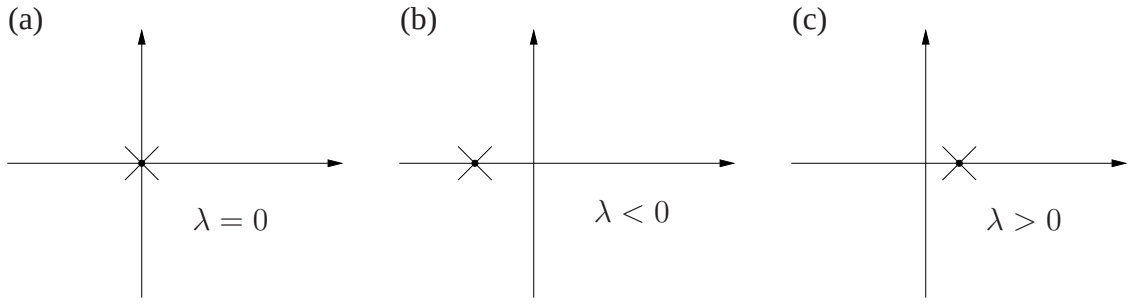

Figure 2.1: A single critical eigenvalue $\lambda = 0$ for the pulse h is depicted in (a). A truncation of the pulse to a large but finite interval with separated boundary conditions may move the eigenvalue to the left-half plane (in (b)) or the right-half plane (in (c)). In the latter case, the pulse is destabilized.

In comparison, suppose that, instead of affine separated boundary conditions, periodic boundary conditions are applied [46]. Notice that the translation invariance is preserved in this case: Consider that the solution is a solution of the reaction-diffusion equation, then the translation invariance is implied by the periodicity at the boundary points. Therefore, there will still be an eigenvalue at $\lambda = 0$. If the solution with periodic boundary conditions is extended to a *pulse train* on $\mathbb{R}$ [42, 16], then the resulting pulse train will have a small circle containing 0 as a part of the spectrum, and the tangency of the circle at 0 would determine the stability (Figure 2.2).

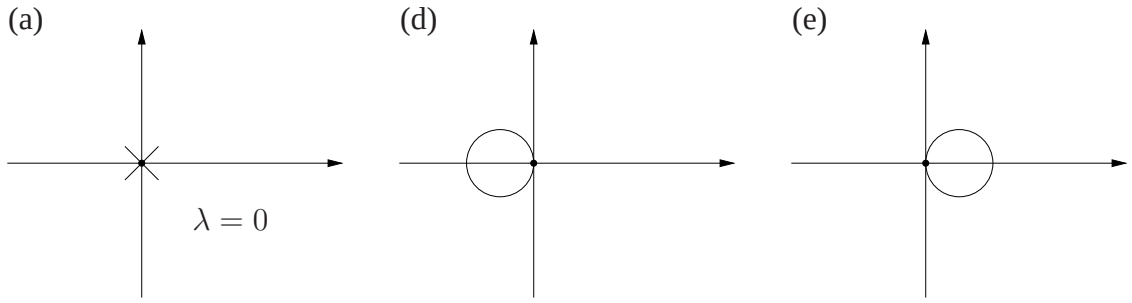

Figure 2.2: A single critical eigenvalue $\lambda = 0$ for the pulse h is depicted in (a). A wave train can be created by periodic extensions of a truncation of the pulse to a large but finite interval with periodic boundary conditions. The single critical eigenvalue will turn into a circle in (e)) and (f). In the latter case, the wave train is unstable.

2.3 Hypotheses

We now collect the assumptions under which we solve the truncated boundary value problem.

First, in order to conduct a perturbation analysis, we require some information on the spectrum of the linearization $f_u(u, \mu)$ about the asymptotic steady state $u = 0$ and $\mu = 0$. We assume that

Hypothesis 2.1. $f_u(0,0)$ *is hyperbolic.*

That is, there are η^s and $\eta^u > 0$, such that

$$spec(f_u(0,0)) \subset \{\lambda \in \mathbb{C} : \Re\lambda < -\eta^s \text{ or } \Re\lambda > \eta^u\}.$$

In other words, the spectrum of $f_u(0,0)$ is bounded away from the imaginary axis. Write

$$\eta := \min\{\eta^s, \eta^u\}.$$

Denote the spectral projections associated with the stable and unstable eigenvalues of $f_u(0,0)$ by P_0^s and P_0^u, with corresponding generalized eigenspaces E_0^s and E_0^u, respectively. Then hyperbolicity characterizes the dynamical behavior of the solutions to

$$\frac{du}{dx} = f_u(0,0)u$$

in the following sense: Any solution with initial condition in E_0^s exists and decays exponentially with rate η in forward time, while any solution with initial condition in E_0^u exists and decays exponentially with rate η in backward time.

Secondly, we require some transversality condition on the solution space of the equation (2.7). In the phase space, the pulse $h(x)$, $x \in \mathbb{R}$, is a homoclinic orbit to (2.7) at $\mu = 0$, which is in the intersection of the stable and unstable manifolds of the equilibrium $u = 0$. We assume that the stable and unstable manifold of $u = 0$ intersect as transversally as possible in the following sense:

Hypothesis 2.2. $h'(x)$ *is the only bounded solution to* (2.8)*, up to constant multiples.*

Geometrically, the hypothesis can be interpreted as that the intersection of the tangent spaces at $h(0)$ of the unstable and stable manifold at $u = 0$ is spanned by the vector $h'(0)$. In particular, $\lambda = 0$ has geometric multiplicity one as an eigenvalue of (2.6).

Let

$$w' = -f_u(h(x),0)^* w, \quad x \in \mathbb{R} \tag{2.9}$$

be *the adjoint variational equation* of (2.7), where $f_u(h(x),0)^*$ is the (formal) adjoint operator with respect to the standard scalar product on $\mathbb{R}^n$. By Hypothesis 2.2, the adjoint variational equation (2.9) admits, up to constant multiples, exactly one bounded solution $\psi(x)$ which decays exponentially as $|x| \to \infty$ (see, for example, [42]). Since the scalar product between solutions of the variational equation (2.8) and its adjoint (2.9) is independent of x (see Appendix), $\psi(x)$ is orthogonal to every solution of (2.8) that is bounded on either $\mathbb{R}^+$ or $\mathbb{R}^-$. In other words, $\psi(x)$ lies in the orthogonal complement of the tangent spaces to stable and unstable manifolds at the homoclinic orbit $h(x)$.

Thirdly, we introduce a more technical hypothesis: We assume that the following Melnikov-type integral is non-vanishing:

Hypothesis 2.3.

$$M := \int_{-\infty}^{\infty} \langle \psi(x), f_\mu(h(x),0) \rangle dx \neq 0.$$

Lastly, we consider the boundary conditions. In the spirit of approximation, an intuitive choice for the boundary condition would be that the solution is contained in some linear or nonlinear approximation of the invariant manifolds at $u = 0$, towards which the pulse converges [8, 43]. In this thesis, we concentrate on separated boundary conditions at $x = T_\pm$, that is, we require that there are two specified closed linear subspaces E_-^{bc} and E_+^{bc} of $\mathbb{R}^n$ and $P_\pm^{bc}$ projections onto a complement of $E_\pm^{bc}$, with null spaces $\mathcal{N}(P_\pm^{bc}) = E_\pm^{bc}$, such that the boundary points $u_T(T_\pm)$ satisfy:

$$u_T(T_-) \in E_-^{bc}, \qquad u_T(T_+) \in E_+^{bc}, \tag{2.10}$$

or in other words,

$$P_\pm^{bc} u_T(T_\pm) = 0. \tag{2.11}$$

Observe that the number of the boundary conditions at $T_\pm$ is equal to the codimension of $E^{bc}_\pm$.

We assume that

Hypothesis 2.4. $\mathbb{R}^n = E^{bc}_+ \oplus E^u_0 = E^{bc}_- \oplus E^s_0$.

Depending on the application, the choice of the complements of the ranges $\mathcal{R}(P^{bc}_\pm)$ can be an important issue. However, by Hypothesis 2.4, there is a canonical isomorphism $\mathcal{R}(P^{bc}_+) \cong E^u_0$ and $\mathcal{R}(P^{bc}_-) \cong E^s_0$. Therefore, we can choose the range of the projections

$$\mathcal{R}(P^{bc}_+) = E^u_0, \qquad \mathcal{R}(P^{bc}_-) = E^s_0. \tag{2.12}$$

We fix these choices throughout the chapter. Also consequently, we have

$$\dim E^{bc}_- + \dim E^{bc}_+ = n.$$

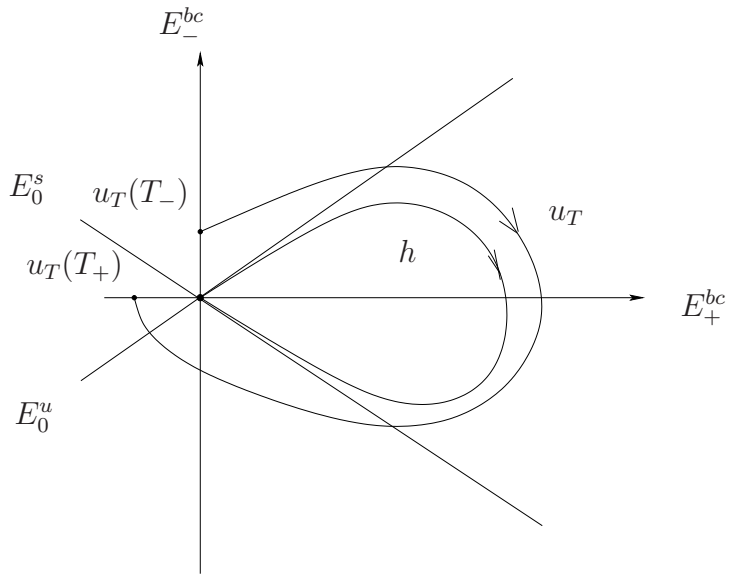

Figure 2.3: A generic picture for the inhomogeneous boundary condition: $u_T(T_-) \in E^s_0$, and $u_T(T_+) \in E^u_0$, in which u_T, accompanying the pulse h, is a solution to the truncated boundary value problem. Compare this to [43], where the ranges E^{bc}_- and E^{bc}_+ are taken to be E^u_0 and E^s_0, respectively.

2.4 The Variational Equation and Exponential Dichotomies

In this section we study the non-autonomous variational equation

$$\frac{dv}{dx} = f_u(h(x), 0)v, \quad x \in \mathbb{R}, \tag{2.13}$$

about the homoclinic orbit $h = h(x)$.

In connection with the variational equation we introduce the following concept:

Definition 2.4.1. (Exponential Dichotomy) *Let $J \subseteq \mathbb{R}$ be an unbounded interval ($J = \mathbb{R}^-, \mathbb{R}^+, \mathbb{R}$). The equation*

$$\frac{dv}{dx} = A(x)v, \quad v(x) \in \mathbb{R}^n, \tag{2.14}$$

with evolution operators $\Phi(x, y)$, $x, y \in J$, is said to have an exponential dichotomy on J if there are projections $P^s(x)$, defined for $x \in J$ and positive constants η^s, η^u and K with the following properties:

(Stability) *: Let $\Phi^s(x, y) := \Phi(x, y)P^s(y)$, then $\|\Phi^s(x, y)\| \leq Ke^{-\eta^s(x-y)}$, for $x, y \in J$ and $x \geq y$.*

(Instability) *: Let $P^u(x) := \mathrm{id} - P^s(x)$, and $\Phi^u(x, y) := \Phi(x, y)P^u(y)$, then $\|\Phi^u(y, x)\| \leq Ke^{-\eta^u(x-y)}$, for $x, y \in J$ and $x \geq y$.*

(Invariance) *: The projections $P^s(x)$ are continuous in $x \in J$ and commute with the evolutions*

$$P^s(x)\Phi(x, y) = \Phi(x, y)P^s(y).$$

The last property implies that

$$\Phi^s(x, y)v_0 \in \mathcal{R}(P^s(x)), \qquad x \geq y, x, y \in J$$

$$\Phi^u(x, y)v_0 \in \mathcal{N}(P^s(x)), \qquad x \leq y, x, y \in J$$

That is, (2.14) has an exponential dichotomy on J if we can decompose the phase space into a direct sum of *two* subspaces:

$$\mathbb{R}^n = \mathcal{R}(P^s(x_0)) \oplus \mathcal{N}(P^s(x_0)),$$

such that the set of initial conditions $u(x_0)$ leading to solutions $u(x)$ that decay *exponentially* in x for $x > x_0$, with x, $x_0 \in J$, is given by the range $\mathcal{R}(P^s(x_0))$ of the projection $P^s(x_0)$; The set of initial conditions $u(x_0)$ leading to solutions $u(x)$ that decay *exponentially* in x for $x < x_0$, with x, $x_0 \in J$, is given by the kernel $\mathcal{N}(P^s(x_0))$ of the projection $P^s(x_0)$.

Remark 2.4.1. *Observe that a solution decaying in backward (forward) time in $\mathbb{R}_+$ ($\mathbb{R}_-$) may not decay in backward (forward) time in $\mathbb{R}$, so generically the existence of exponential dichotomies on $J = \mathbb{R}^-$ and $J = \mathbb{R}^+$ does not imply the existence of exponential dichotomies on $J = \mathbb{R}$.*

One of the main properties of exponential dichotomies is that they persist under small perturbations of the equation. This property is often referred to as the robustness of exponential dichotomies. Here we have a version of the robustness theorem, given by Coppel[13]:

Theorem 2.4.1. *Let J be $\mathbb{R}^-$ or $\mathbb{R}^+$. Suppose that $A(\cdot) \in \mathcal{C}^0(I, \mathbb{C}^{n\times n})$ and the equation*

$$\frac{dv}{dx} = A(x)v, \tag{2.15}$$

has an exponential dichotomy on J with constants K, η^s and η^u as in Definition 2.4.1. There are positive constants δ_* and κ such that the following is true. If $B(\cdot) \in \mathcal{C}^0(I, \mathbb{C}^{n\times n})$ such that

$$\sup_{x\in J, |x|\geq M} |B(x)| < \frac{\delta}{\kappa}$$

for some $\delta < \delta_*$ and some $M \geq 0$, then a constant $\tilde{K} > 0$ exists such that the equation

$$\frac{dv}{dx} = (A(x) + B(x))v, \tag{2.16}$$

has an exponential dichotomy on J with constants $\tilde{K}$, $\eta^s - \delta$ and $\eta^u - \delta$. Moreover, the projections $P^s(x)$ and evolution $\Phi^s(t,s)$ associated with (2.16) are δ-close to those associated with (2.15) for all $t, s \in J$ with $|s|, |t| > M$.

Remark 2.4.2. *If the perturbation $B(x)$ in (2.16) converges to 0, then the projections of (2.16) converge to those of (2.15) in norm. For $J = \mathbb{R}_+$, if $\sup_\xi |B(\xi)|$ is sufficiently small, then the evolutions $\tilde{\Phi}^s(\xi,\zeta)$ and $\tilde{\Phi}^u(\xi,\zeta)$ of (2.16) can be found as the unique solution of the integral equation [40, 44, 42]*

$$\begin{aligned}
&\tilde{\Phi}^s(\xi,\zeta) - \Phi^s(\xi,\zeta) = \int_\xi^\infty \Phi^u(\xi,\tau)B(\tau)\tilde{\Phi}^s(\tau,\zeta)d\tau \\
&- \int_\zeta^\xi \Phi^s(\xi,\tau)B(\tau)\tilde{\Phi}^s(\tau,\zeta)d\tau + \int_0^\zeta \Phi^s(\xi,\tau)B(\tau)\tilde{\Phi}^u(\tau,\zeta)d\tau, \qquad 0 \le \zeta \le \xi \\
&\tilde{\Phi}^u(\xi,\zeta) - \Phi^u(\xi,\zeta) = \int_\xi^\zeta \Phi^u(\xi,\tau)B(\tau)\tilde{\Phi}^u(\tau,\zeta)d\tau \\
&- \int_0^\xi \Phi^s(\xi,\tau)B(\tau)\tilde{\Phi}^u(\tau,\zeta)d\tau + \int_\zeta^\infty \Phi^u(\xi,\tau)B(\tau)\tilde{\Phi}^s(\tau,\zeta)d\tau, \qquad 0 \le \xi \le \zeta
\end{aligned}$$

where $\Phi^s(\xi,\zeta)$ and $\Phi^u(\xi,\zeta)$ are evolutions of (2.15). Therefore, in Lemma 2.4.1, the evolutions of (2.16) converge to those of (2.15) if ξ and ζ are uniformly large.

Proposition 2.4.1. *The equation (2.13) has an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$.*

Proof. By Hypothesis 2.1, the asymptotic matrix $f_u(0,0)$ is hyperbolic, thus $\frac{dv}{dx} = f_u(0,0)v$ possesses an exponential dichotomy on both $J = \mathbb{R}^-$ and $J = \mathbb{R}^+$. Take

$$A(x) \equiv f_u(0,0), \qquad B(x) = f(h(x),0) - f(0,0).$$

Since f is of $\mathcal{C}^2$ and h converges to 0 as $x \to \pm\infty$, $B(x)$ converges to 0 as $|x| \to \infty$. Then by Theorem 2.4.1, the equation (2.13) has exponential dichotomies $P^s_+(x)$ on $\mathbb{R}_+$, and $P^s_-(x) = \mathrm{id} - P^u_-(x)$ on $\mathbb{R}_-$, with the rates $\eta^s - \delta$ and $\eta^u - \delta$, respectively, for any $0 < \delta \ll 1$. □

Remark 2.4.3. *Exponential dichotomies are not unique. Suppose that there exists an exponential dichotomy on $\mathbb{R}^+$. Then the range $\mathcal{R}(P^s(x))$ is uniquely determined, but not the kernel. Define [44]*

$$\Gamma : \mathcal{N}(P^s(0)) \to \mathcal{R}(P^s(0))$$

to be a linear map and the new projection

$$\tilde{P}^s(x) := P^s(x) - \Phi(x,0) \circ \Gamma \circ \Phi(0,x) P^u(x). \tag{2.17}$$

Then $\tilde{P}^s(x)$ is also an exponential dichotomy on $\mathbb{R}^+$.

Due to the presence of both $\mathbb{R}^-$- and $\mathbb{R}^+$-dichotomies for the equation (2.13), we consider the decomposition of the tangent space $T_{h(0)}\mathbb{R}^n$, which is canonically also $\mathbb{R}^n$, with respect to both $\mathbb{R}^-$- and $\mathbb{R}^+$-dichotomies.

Write

$$Y^c := \operatorname{span}\{h'(0)\}.$$

By Hypothesis 2.2, Y^c is the subspace of initial data that lead to bounded solutions of (2.13) on $\mathbb{R}$. Let Y^s and Y^u be the orthogonal complements of Y^c in the subspaces of $T_{h(0)}\mathbb{R}^n$ that consist of initial data that lead to solutions that decay in forward or backward time, respectively. Let $\psi(x)$ be the unique, up to scalars, nontrivial bounded solution of the adjoint variational equation (2.9). Write $Y^\perp := \operatorname{span}\{\psi(0)\}$. Then $Y^\perp \perp (Y^u \oplus Y^s \oplus Y^c)$.

Therefore we have a decomposition of the tangent space at $h(0)$ with respect to a chosen exponential dichotomy:

$$\mathbb{R}^n = Y^c \oplus Y^s \oplus Y^u \oplus Y^\perp, \tag{2.18}$$

in which

$$\begin{aligned} \mathcal{R}(P^s_+(0)) &= Y^c \oplus Y^s, \qquad \mathcal{R}(P^u_+(0)) = Y^u \oplus Y^\perp, \\ \mathcal{R}(P^u_-(0)) &= Y^c \oplus Y^u, \qquad \mathcal{R}(P^s_-(0)) = Y^s \oplus Y^\perp. \end{aligned} \tag{2.19}$$

Note that Remark 2.4.3 allows us to choose the ranges $\mathcal{R}(P_+^u(0))$ and $\mathcal{R}(P_-^s(0))$.

Geometrically, in the phase space $\mathbb{R}^n$ of the equation (2.13), the ranges of the projections $P_+^s(0)$ and $P_-^u(0)$ are the tangent spaces to stable and unstable manifolds of the origin at the homoclinic orbit $h(0)$:

$$\mathcal{R}(P_+^s(0)) = T_{h(0)}W^s(0), \qquad \mathcal{R}(P_-^u(0)) = T_{h(0)}W^u(0),$$
$$Y^c = T_{h(0)}W^s(0) \cap T_{h(0)}W^u(0).$$

Remark 2.4.4. *From now on, we use the subscript $-$ and $+$ to indicate whether $x < 0$ or $x > 0$. So $\Phi_+^u(t,s)$ and $\Phi_+^s(s,t)$ are the same as $\Phi^u(t,s)$ and $\Phi^s(s,t)$, respectively, but only for $s \geq t \geq 0$, and $\Phi_-^u(t,s)$ and $\Phi_-^s(s,t)$ for $\Phi^u(t,s)$ and $\Phi^s(t,s)$ for $t \leq s \leq 0$.*

2.5 Formulation of the Existence and Eigenvalue Problem

By the existence problem, we mean the following truncated boundary value problem:

$$u' = f(u,\mu), \qquad x \in [T_-, T_+], \tag{2.20}$$

with

$$P_+^{bc}u(T_+) = 0, \qquad P_-^{bc}u(T_-) = 0, \tag{2.21}$$

for $u \in \mathcal{C}^1([T_-, T_+], \mathbb{R}^n)$.

Now suppose that we have a solution (u_T, μ_T) of (2.20), then the eigenvalue problem, given by (2.6), is formulated as

$$v' = (f_u(u_T, \mu_T) + \lambda B)v, \qquad x \in [T_-, T_+], \tag{2.22}$$

with the boundary conditions

$$P_+^{bc}v(T_\pm) = 0, \tag{2.23}$$

for $v \in \mathcal{C}^1([T_-, T_+], \mathbb{R}^n)$.

In the following we concentrate first on the eigenvalue problem (2.22) and (2.23). (We can use the same approach for the existence problem with obvious modifications which are explained later.) The philosophy of solving such the truncated boundary value problem is as follows: We treat the truncated boundary value problem as two sub-problems defined on $[T_-, 0]$ and $[0, T_+]$ respectively. On either $[0, T_+]$ or $[T_-, 0]$, with the existence of exponential dichotomies, we can solve the the corresponding eigenvalue problem (2.22). Afterwards, match the two pieces of solutions at $x = 0$. During the matching, the sign of λ will be brought out, which then determines the (in)stability of u_T.

So we write the above eigenvalue problem (2.22) and (2.23) in the equivalent form

$$\begin{aligned} v_-' &= (f_u(u_T, \mu_T) + \lambda B)v_-, \qquad x \in (T_-, 0), \\ v_+' &= (f_u(u_T, \mu_T) + \lambda B)v_+, \qquad x \in (0, T_+), \\ v_-(0) &= v_+(0), \\ P_-^{bc} v(T_-) &= 0, \\ P_+^{bc} v(T_+) &= 0. \end{aligned} \tag{2.24}$$

Since (u_T', μ_T) solves (2.22) and (2.23), write the perturbed solution $v_\pm$ as

$$v_\pm(x) = u_T'(x) + w_\pm(x). \tag{2.25}$$

Considering that the solution $v_\pm$ is the perturbation for u_T', instead of h', the original coordinate system (2.18) should be modified accordingly. Define a projection $Q : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathcal{R}(Q) = \text{span}\{u_T'(0)\}, \qquad \mathcal{N}(Q) = Y^s \oplus Y^u \oplus Y^\perp.$$

Now if u_T is δ-close to the homoclinic orbit $h(x)$, $\mathcal{R}(Q)$ is close to the space Y^c, which is a complement to $\mathcal{N}(Q)$. Hence Q is well defined and its norm depends only on δ and not on u_T. Since (u_T', μ_T) satisfies the equation (2.22) for $\lambda = 0$, the equivalent formulation of

(2.22) for $w_\pm$ will be

$$\begin{aligned}
&(i) \qquad w'_\pm = f_u(h(x),0)w_\pm + (f_u(u_T,\mu_T) - f_u(h(x),0) + \lambda B)w_\pm + \lambda B u'_T(x),\\
&(ii) \qquad Qw_\pm(0) = 0,\\
&(iii) \qquad w_+(0) - w_-(0) \in Y^\perp,\\
&(iv) \qquad P^{bc}_\pm w_\pm(T_\pm) = -P^{bc}_\pm u'(T_\pm),
\end{aligned} \tag{2.26}$$

together with the condition

$$\xi := \langle \psi(0), w_+(0) - w_-(0)\rangle = 0. \tag{2.27}$$

Observe that $(ii) - (iii)$ in (2.26), along with (2.27), is equivalent to $v_-(0) = v_+(0)$.

As we suggest in $(2.26)(i)$, the linear term is split into two parts: the principal part $f_u(h(x),0)w_\pm$ and a perturbation term $(f_u(u_T,\mu_T) - f_u(h(x),0) + \lambda B)w_\pm$. Write

$$\begin{aligned}
D_\pm &:= -P^{bc}_\pm u'_T(T_\pm), \qquad (2.28)\\
H(x) &:= f_u(u_T,\mu_T) - f_u(h(x),0) + \lambda B, \qquad g(x) := \lambda B u'_T(x),\\
G_\pm(w_\pm, x) &:= H(x)w_\pm + g(x).
\end{aligned}$$

Also let $G := (G_-, G_+)$ and $D := (D_-, D_+) \in \mathcal{R}(P^{bc}_-) \oplus \mathcal{R}(P^{bc}_+) = E^s_0 \oplus E^u_0$.

Note that G defined in (2.28) is $w_\pm$-dependent. We shall first consider the system:

$$\begin{aligned}
&(i) \qquad w'_\pm = f_u(h(x),0)w_\pm + G_\pm(x),\\
&(ii) \qquad Qw_\pm(0) = 0,\\
&(iii) \qquad w_+(0) - w_-(0) \in Y^\perp,\\
&(iv) \qquad P^{bc}_\pm w_\pm(T_\pm) = D_\pm,
\end{aligned} \tag{2.29}$$

where $G_\pm$ is an arbitrary but fixed function, independent of $w_\pm$. After we solve $w_\pm = W(D,G)$ with $G = G(x)$ for some W, we will substitute back $G = G(w_\pm, x)$ and solve $w_\pm = W(D, G(w_\pm, \cdot))$ for $w_\pm$ using the Implicit Function Theorem.

For the existence problem, with the substitution $v_\pm = w_\pm + h$, we obtain the system

$$\begin{aligned} w'_\pm &= f(w_\pm + h, \mu) - f(h, 0) \\ &= f_u(h,0)w_\pm + f_\mu(h,0)\mu + O(|w_\pm|^2 + |w_\pm||\mu| + |\mu|^2)). \end{aligned}$$

Therefore,

$$D_\pm = -P^{bc}_\pm h(T_\pm), \qquad G_\pm(w_\pm, x) = \partial_\mu f(h,\mu)|_{\mu=0} \cdot \mu + O(|w_\pm|^2 + |w_\pm||\mu| + |\mu|^2).$$

2.6 Solving the Reduced Problem

Define the spaces

$$\begin{aligned} V_w : &= \mathcal{C}^0([T_-, 0]), \mathbb{R}^n) \oplus \mathcal{C}^0([0, T_+]), \mathbb{R}^n), \\ V_a : &= E^u_0 \oplus E^s_0, \\ V_b : &= Y^s \oplus Y^u. \end{aligned}$$

We claim that the general solution of (i)-(ii) of (2.29) is given by the variation-of-constant formula:

$$\begin{aligned} w_+(x) &= \Phi^u_+(x, T_+)a_+ + \Phi^s_+(x,0)b_+ + \int_0^x \Phi^s_+(x,y)G_+(y)dy \\ &\quad + \int_{T_+}^x \Phi^u_+(x,y)G_+(y)dy, \qquad (2.30) \\ w_-(x) &= \Phi^s_-(x, T_-)a_- + \Phi^u_-(x,0)b_- + \int_0^x \Phi^u_-(x,y)G_-(y)dy \\ &\quad + \int_{T_-}^x \Phi^s_-(x,y)G_-(y)dy. \end{aligned}$$

where the elements $a = (a_+, a_-) \in V_a$ and $b = (b_+, b_-) \in V_b$ are arbitrary.

Indeed, for w_+, the right hand side of (i) in (2.29) is

$$\begin{aligned} l.h.s. &= w_+' \\ &= (\Phi_+^u(x,T_+)a_+ + \Phi_+^s(x,0)b_+)' \\ &\quad + \left(\int_0^x \Phi_+^s(x,y)G_+(y)dy + \int_{T_+}^x \Phi_+^u(x,y)G_+(y)dy\right)' \\ &= f_u(h(x),0)(\Phi_+^u(x,T_+)a_+ + \Phi_+^s(x,0)b_+) + P_+^s(x)G_+(x) + P_+^u(x)G_+(x) \\ &\quad + \int_0^x f_u(h(x),0)\Phi_+^s(x,y)G_+(y)dy + \int_{T_+}^x f_u(h(x),0)\Phi_+^u(x,y)G_+(y)dy \\ &= f_u(h(x),0)w_+ + G_+(x) = r.h.s. \end{aligned}$$

Observe that $w_+(0) \in Y^u \oplus Y^s$, thus $Qw_+(0) = 0$. Similarly for $w_-(x)$.

Since both a and b are free in (2.30), we can choose b, in terms of a and G, to meet (2.29) (iii).

Lemma 2.6.1. *Let the linear operator defined by the right hand side of* (2.30) *be*

$$W_1 : V_a \times V_b \times V_w \to V_w.$$

There are constants C *and* $L_* > 0$ *such that the following is true for all* T_- *and* T_+ *with* $|T_-|, T_+ > L_*$. *There is a linear operator* $B_1 : V_a \times V_w \to V_b$ *such that* w *satisfies* (2.29) *(i)-(iii) if and only if*

$$b = (b_+, b_-) = B_1(a, G), \qquad w = W_1(a, B(a,G), G).$$

Furthermore, we have the estimates

$$|b_+| \le C(e^{-\eta^s|T_-|}|a_-| + |G_-|), \qquad |b_-| \le C(e^{-\eta^u T_+}|a_+| + |G_+|).$$

Write

$$L := \min\{|T_-|, T_+\}, \tag{2.31}$$

then

$$\begin{aligned} |B_1(a,G)| &\leq C(e^{-\eta L}|a|+|G|), \\ |W_1(a,b,G)| &\leq C(|a|+|b|+|G|), \\ |W_1(a,B_1(a,G),G)| &\leq C(|a|+|G|). \end{aligned} \tag{2.32}$$

Proof. Evaluating at $x=0$, we get

$$\begin{aligned} w_+(0)-w_-(0) &= b_+ - b_- + \Phi_+^u(0,T_+)a_+ - \Phi_-^s(0,T_-)a_- \\ &- \int_0^{T+} \Phi_+^u(0,y)G_+(y)dy - \int_{T_-}^0 \Phi_-^s(0,y)G_-(y)dy, \end{aligned} \tag{2.33}$$

with $\Phi_+^s(0,0)b_+ = b_+$ and $\Phi_-^u(0,0)b_- = b_-$.

To solve (2.29) (iii), it is sufficient to solve

$$P(Y^u, Y^s \oplus Y^c \oplus Y^\perp)(w_+ - w_-)(0) = 0,$$

$$P(Y^s, Y^c \oplus Y^u \oplus Y^\perp)(w_+ - w_-)(0) = 0.$$

Now project (2.33) onto $Y^s \oplus Y^u$. Note that since $(b_-, b_+) \in Y^s \oplus Y^u$,

$$P(Y^s, Y^c \oplus Y^u \oplus Y^\perp)b_+ = b_+, \qquad P(Y^u, Y^c \oplus Y^s \oplus Y^\perp)b_- = b_-.$$

Then $w_+(0) - w_-(0) \in Y^\perp$ holds if, and only if,

$$\begin{aligned} b_+ &= P(Y^s, Y^c \oplus Y^u \oplus Y^\perp)\left(\Phi_-^s(0,T_-)a_- + \int_{T_-}^0 \Phi_-^s(0,y)G_-(y)dy\right), \\ b_- &= P(Y^u, Y^c \oplus Y^s \oplus Y^\perp)\left(\Phi_+^u(0,T_+)a_+ + \int_0^{T_+} \Phi_+^u(0,y)G_+(y)dy\right). \end{aligned} \tag{2.34}$$

Observe that the right-hand side of the equations defines a bounded linear operator B_1 in (a,G) which satisfies the desired estimate. □

Next we solve (iv) in (2.29)

$$P_\pm^{bc} w_\pm(T_\pm) = D_\pm. \tag{2.35}$$

As the component a is free, we would like to choose a in terms of $G = G(x)$ and D such that (2.35) is satisfied. We formulate our claim as the following

Lemma 2.6.2. *There are constants C and $L_* > 0$ such that the following is true for every T_- and T_+ with $|T_-|, T_+ > L_*$. There are linear operators*

$$A_2 : \mathbb{R}^n \times V_w \to V_a, \qquad B_2 : \mathbb{R}^n \times V_w \to V_b, \qquad W_2 : \mathbb{R}^n \times V_w \to V_w.$$

such that w satisfies (2.29) if, and only if, w is given by

$$\begin{aligned} a &= A_2(D, G) \\ b &= B_2(D, G) := B_1(A_2(D, G), G) \\ w &= W_2(D, G) := W_1(A_2(D, G), B_2(D, G), G) \end{aligned}$$

where B_1 and W_1 are given in Lemma 2.6.1. Moreover,

$$a = (a_+, a_-) = A_2(D, G) = (D_+, D_-) + R_2(D, G), \tag{2.36}$$

for a certain bounded operator R_2, and we have the estimates

$$\begin{aligned} |R_2(D, G)| &\leq C(e^{-\eta L}|D| + |G|), \\ |B_2(D, G)| &\leq C(e^{-\eta L}|D| + |G|), \\ |W_2(D, G)| &\leq C(|D| + |G|). \end{aligned} \tag{2.37}$$

Remark 2.6.1. *The expression (2.36) only makes sense if $(D_-, D_+) \in E_0^s \oplus E_0^u$, which is true due to our choices (2.12) of the ranges of the projections $P_\pm^{bc}$.*

Proof. Evaluate (2.30) at $x = T_\pm$,

$$\begin{aligned} w_+(T_+) &= a_+ + (P_+^u(T_+) - P_0^u)a_+ + \Phi_+^s(T_+, 0)b_+ \\ &\quad + \int_0^{T_+} \Phi_+^s(T_+, y)G_+(y)dy, \end{aligned} \tag{2.38}$$

$$\begin{aligned} w_-(T_-) &= a_- + (P_-^s(T_-) - P_0^s)a_- + \Phi_-^u(T_-, 0)b_- \\ &\quad + \int_0^{T_-} \Phi_-^u(T_-, y)G_-(y)dy, \end{aligned} \tag{2.39}$$

with $P_0^u a_+ = a_+$ and $P_0^s a_- = a_-$. Here $b = (b_+, b_-) = B_1(a, G)$.

From Lemma 1.2 (ii) in [44], we obtain

$$|P^u_+(T_+) - P^u_0| \leq Ce^{-\eta^u T_+}, \qquad |P^s_-(T_-) - P^s_0| \leq Ce^{-\eta^s |T_-|},$$

and from Lemma 2.6.1,

$$|b| = |B_1(a, G)| \leq C(e^{-\eta L}|a| + |G|),$$

then from (2.35), we reach

$$\begin{aligned} D_+ = P^{bc}_+ D_+ &= P^{bc}_+(a_+ + O(e^{-\eta^u T_+}a_+) + O(e^{-\eta^s T_+}(e^{-\eta L}a_- + G_-)) + O(G_-)) \\ &= P^{bc}_+((1 + O(e^{-\eta L}))a_+ + O(e^{-2\eta L})a_- + O(G_-)), \end{aligned} \tag{2.40}$$

and

$$\begin{aligned} D_- = P^{bc}_- D_- &= P^{bc}_-(a_- + O(e^{-\eta^s |T_-|}a_-) + O(e^{-\eta^u |T_-|}(e^{-\eta L}a_+ + G_+)) + O(G_+)) \\ &= P^{bc}_-((1 + O(e^{-\eta L}))a_- + O(e^{-2\eta L})a_+ + O(G_+)). \end{aligned}$$

For sake of convenience, we concentrate on the equation for D_+. On $\mathbb{R}^n$, the projection P^{bc}_+ is not invertible due to the presence of the kernel. Recall that due to our choice of $\mathcal{R}(E^{bc}_+)$, the map $P^{bc}_+|_{E^u_0} = \mathrm{id}|_{E^u_0}$. Define the coefficient operator $K_+ := P^{bc}_+((1 + O(e^{-\eta^u L}))$ of a_+ in (2.40) . Then K_+ is also an isomorphism for large L, with $\|K^{-1}\| = 1 + O(e^{-\eta L})$.

Thus we can invert K_+:

$$a_+ = D_+ + O(e^{-\eta L})D_+ + O(e^{-2\eta L})a_- + O(G_-) \tag{2.41}$$

Similarly, we have

$$a_- = D_- + O(e^{-\eta L})D_- + O(e^{-2\eta L})a_+ + O(G_+) \tag{2.42}$$

Substitute a_+ from (2.41) into (2.42), we have

$$(1 + O(e^{-2\eta L}))a_- = D_- + O(e^{-\eta L})D_+ + O(G).$$

Then

$$a_- = D_- + O(e^{-\eta L})D_+ + O(e^{-2\eta L})D_- + O(1)G_+ + O(e^{-2\eta L})G_-,$$

and

$$a_+ = D_+ + O(e^{-\eta L})D_- + O(e^{-2\eta L})D_+ + O(1)G_- + O(e^{-2\eta L})G_+.$$

We conclude that

$$a = (a_+, a_-) = (D_+, D_-) + R_2(D, G) =: A_2(D, G), \tag{2.43}$$

where R_2 is a bounded operator with

$$|R_2(D, G)| \leq C(e^{-\eta L}|D| + |G|).$$

The other two estimates for B_2 and W_2 follow from the estimates in Lemma 2.6.1. □

Lastly, we need to solve (2.27) which reads

$$\xi := \langle \psi(0), w_+(0) - w_-(0) \rangle = 0.$$

We give an expansion of ξ in terms of D and G.

Lemma 2.6.3. *There are constants C and $L_* > 0$ such that the following is true for all T_- and T_+ with $|T_-|, T_+ > L_*$. Let $w = W_2(D, G)$ be given in Lemma 2.6.2. Then*

$$\begin{aligned} \xi &= \langle \psi(T_+), D_+ \rangle - \langle \psi(T_-), D_- \rangle - \int_0^{T_+} \langle \psi(x), G_+(x) \rangle dx \\ &\quad - \int_{T_-}^0 \langle \psi(x), G_-(x) \rangle dx + R(D, G), \end{aligned} \tag{2.44}$$

with

$$|R(D, G)| \leq Ce^{-\eta L}(e^{-\eta L}|D| + |G|).$$

Proof. Observe that ξ is linear in D and G and, by the decomposition (2.18), $\langle \psi(0), b_\pm \rangle = 0$. We obtain

$$\begin{aligned}
& \langle \psi(0), w_+(0) - w_-(0) \rangle \\
= \; & \langle \psi(0), \Phi_+^u(0, T_+) a_+ \rangle - \langle \psi(0), \Phi_-^s(0, T_+) a_- \rangle \\
& - \int_0^{T_+} \langle \psi(0), \Phi_+^u(0, x) G_+(x) \rangle dx - \int_{T_-}^0 \langle \psi(0), \Phi_-^s(0, x) G_-(x) \rangle dx \\
= \; & \langle \psi(T_+), a_+ \rangle - \langle \psi(T_-), a_- \rangle - \int_0^{T_+} \langle \psi(x), G_+(x) \rangle dx - \int_{T_-}^0 \langle \psi(x), G_-(x) \rangle dx.
\end{aligned}$$

Here we use that the evolution operator of the adjoint variational equation is given by $((\Phi_\pm^{u,s}(x, y))^{-1})^*$, see Proposition A.1.1, (2) in Appendix A.1. Substitute (2.43) into the last expression, then

$$\begin{aligned}
\xi \; = \; & \langle \psi(T_+), D_+ \rangle - \langle \psi(T_-), D_- \rangle + \mathcal{R}(D, G) \\
& - \int_0^{T_+} \langle \psi(x), G_+(x) \rangle dx - \int_{T_-}^0 \langle \psi(x), G_-(x) \rangle dx
\end{aligned}$$

with $|\mathcal{R}(D, G)| \leq C e^{-\eta L}(e^{-\eta L}|D| + |G|)$. Here we use the fact that $|\psi(x)| \leq C e^{-\eta |x|}$ which follows from Proposition A.1.1, (2) in Appendix A.1 and the fact that $f_u(0, 0)$ is hyperbolic. □

2.7 Solving of the Existence and Eigenvalue Problems

The existence problem of the truncated boundary value problem had been investigated by various authors, both numerically and analytically [8, 43, 53, 54, 43]. Recall that the boundary projection operators P_+^{bc} and P_-^{bc} in the following theorems are chosen subject to (2.12).

Theorem 2.7.1. *Assume that the hypotheses (2.1) - (2.4) are satisfied. There are constants $C > 0$ and $L_* \geq 0$, such that the following is true. Define $\delta := \frac{1}{L_*}$. For all T_- and T_+ with $|T_-|, T_+ > L_*$ and μ with $|\mu| < \delta$, there exists a solution (u_T, μ_T) of the equation (2.20)*

which satisfies the boundary conditions (2.21) and

$$\sup_{x\in(T_-,T_+)} |u_T(x) - h(x)| < C\delta, \qquad |\mu_T| < C\delta. \tag{2.45}$$

Moreover, we have the estimates:

$$|u_T(T_-)| + |u_T(T_+)| + \sup_{x\in[T_-,T_+]} |u_T(x) - h(x)| \le Ce^{-\eta L}, \tag{2.46}$$

$$|\mu_T| \le Ce^{-2\eta L}. \tag{2.47}$$

In particular,

$$|P_+^{bc}h(T_+)| + |P_-^{bc}h(T_-)| \le Ce^{-\eta L}. \tag{2.48}$$

Remark 2.7.1. *The uniqueness of the solution above can be obtained by imposing an additional phase condition [43].*

Proof. From Lemma 2.6.2, a substitution for G gives the following fixed point equation:

$$w = W_2(D, G) = W_2(D, \partial_\mu|_{\mu=0} f(h(x), \mu)\cdot\mu + O(|w_\pm|^2 + |w_\pm||\mu| + |\mu|^2)),$$

in V_w, or equivalently,

$$\begin{aligned} w &= W_2(0, O((|w_\pm|^2 + |w_\pm||\mu|)) + W_2(D, \partial_\mu f(h, 0)\cdot\mu + O(|\mu|^2)) \\ &=: (W_3(\mu))(w) + W_4(D, \mu). \end{aligned} \tag{2.49}$$

Since $|\partial_w(W_3(\mu))(0)| \le C|\mu|$, we can solve (2.49) for w:

$$w = (\mathrm{id} - W_3(\mu))^{-1}W_4(D, \mu) =: W_5(D, \mu),$$

for $|\mu|$ uniformly small. Then $|W_5| \le C(|D| + |\mu|)$.

The expansion for the jump ξ in Lemma 2.6.3 gives the equation:

$$\xi = \langle\psi(T_+), D_+\rangle - \langle\psi(T_-), D_-\rangle - \int_{T_-}^{T_+} \langle\psi(x), G_\pm(x)\rangle dx + \mathcal{R}(D, G) = 0,$$

which we need to solve. For the existence problem, we have

$$D_\pm = -P_\pm^{bc}h(T_\pm), \qquad G_\pm(w_\pm, x) = \partial_\mu f(h(x), 0)\cdot\mu + O(|w_\pm|^2 + |w_\pm||\mu| + |\mu|^2). \tag{2.50}$$

It remains to solve

$$\xi = \xi(\mu, G(\mu, W_5(D, \mu)) = 0,$$

with

$$\partial_\mu \xi|_{(\mu,D)=(0,0)} = -M + \partial_G \mathcal{R}(D, G)\partial_\mu G|_{(\mu,D)=(0,0)} = -M + O(e^{-\eta L}).$$

Therefore there exists $L_* > 0$ such that for all $|T_-|$, $T_+ > L_*$ and $|\mu| < \frac{1}{L_*}$, $\partial_\mu \xi$ is non-vanishing. By the Implicit Function Theorem, we can solve $\xi = 0$ for $\mu = \mu(D)$ in terms of D . Thus, we obtain the existence, and the estimate (2.45) holds.

In order to establish the estimates (2.46) and (2.47), we need to use the exponentially weighted norm on $C^0([T_-, T_+], \mathbb{R}^n)$. This can be done as in [33, 43]. We omit the details.

□

In the following lemma, we will develop a finer estimate of $u_T(T_\pm)$ by giving an expansion of $u_T(T_\pm)$ up to $O(e^{-2\eta L})$.

Lemma 2.7.1. *Let u_T be the solution of the truncated boundary value problem in Theorem 2.7.1. Then*

$$\begin{aligned} u_T(T_+) &= (\mathrm{id} - P_+^{bc})h(T_+) + O(e^{-2\eta L}), \\ u_T(T_-) &= (\mathrm{id} - P_-^{bc})h(T_-) + O(e^{-2\eta L}). \end{aligned}$$

Proof. Recall equations (2.38), (2.39) and Lemmata 2.6.1, 2.6.2,

$$\begin{aligned} u_T(T_+) - h(T_+) &= w_+(T_+) \\ &= a_+ + O(e^{-\eta L})a_+ + O(e^{-\eta L}(e^{-\eta L}a + |G|)) + O(1)G \\ &= D_+ + O(e^{-\eta L})D_+ + O(1)G \end{aligned}$$

As D and G are defined as in (2.50), we obtain that

$$\begin{aligned} u_T(T_+) &= h(T_+) - P_+^{bc}h(T_+) + O(e^{-\eta L})h(T_+) + O(1)\mu + O(|u_T - h|^2) \\ &= (\mathrm{id} - P_+^{bc})h(T_+) + O(e^{-2\eta L}). \end{aligned}$$

In the last equation, we use the estimates (2.46) and (2.47). The proof for $u_T(T_-)$ is similar.

□

Next, we consider the eigenvalue problem (2.22) with (2.23) for which we impose the following assumption:

Hypothesis 2.5.

$$\tilde{M} := \int_{-\infty}^{\infty} \langle \psi(x), Bh'(x) \rangle dx \neq 0.$$

Recall that, for the eigenvalue problem,

$$\begin{aligned} D_\pm &= -P_\pm^{bc} u_T'(T_\pm), \\ G_\pm(w, x) &= H(\lambda)w_\pm + g \\ &= (f_u(u_T, \mu_T) - f_u(h(x), 0) + \lambda B)w_\pm(x) + \lambda B u_T'(x). \end{aligned}$$

Theorem 2.7.2. *Assume that the hypotheses (2.1) - (2.4) and (2.5) are satisfied. There are constants $C, \delta > 0$ such that the following is true. Suppose that (u_T, μ_T) is the solution to the truncated boundary value problem (2.20) with boundary condition (2.21), such that*

$$\sup_{x \in (T_-, T_+)} |u_T(x) - h(x)| < \delta, \qquad |\mu_T| < \delta, \qquad L := \min\{|T_-|, T_+\} > \frac{1}{\delta}.$$

There exists bounded nontrivial $v = v(\lambda)$ and λ , with $|\lambda| < \delta$, which satisfy the equation (2.22) and the boundary conditions (2.23) if, and only if, $E(\lambda) = 0$, where

$$\begin{aligned} E(\lambda) &= \langle \psi(T_-), P_-^{bc} A(\mathrm{id} - P_-^{bc})h(T_-) \rangle - \langle \psi(T_+), P_+^{bc} A(\mathrm{id} - P_+^{bc})h(T_+) \rangle \\ &\quad -\lambda \int_{-\infty}^{\infty} \langle \psi(x), Bh'(x) \rangle dx + \tilde{\mathcal{R}}(\lambda) \end{aligned} \tag{2.51}$$

where $A := f_u(0,0)$ *and* $\tilde{\mathcal{R}}$ *satisfies the error estimate*

$$|\tilde{\mathcal{R}}(\lambda)| \leq C(e^{-\eta L}|\lambda| + |\lambda|^2 + e^{-3\eta L})$$

Proof. From Lemma 2.6.2, we have the solution operator $W_2 = W_2(D, G)$ for w. Substituting G gives the following fixed point equation:

$$w = W_2(D, G) = W_2(D, H(\lambda)w + \lambda B u_T'(x)).$$

Thus,

$$w = W_2(0, H(\lambda)w) + W_2(D, g) =: (W_3(\lambda))(w) + W_2(D, \lambda B u_T'(x)).$$

Observe that $|H(\lambda)| \leq C\delta$, for $|u_T - h|_\infty$, $|\mu_T|$ and $|\lambda| < \delta$, therefore $|W_3(\lambda)| \leq C\delta$. Since C is independent of δ, choose δ small enough and we can solve for ω

$$w = (\mathrm{id} - W_3(\lambda))^{-1} W_2(D, \lambda B u_T'(x)) =: W_4(D, \lambda) \tag{2.52}$$

with the estimate

$$|W_4(D, \lambda)| \leq C(|D| + |\lambda|).$$

Substitute back $w = W_4(D, \lambda)$ into the expansion (2.44), we obtain

$$\begin{aligned} \xi &= \langle \psi(T_+), D_+ \rangle - \langle \psi(T_-), D_- \rangle - \lambda \int_{T_-}^{T_+} \langle \psi(x), B u_T'(x) \rangle dx \\ &\quad - \int_{T_-}^{T_+} \langle \psi(x), H(\lambda) W_4(D, \lambda)(x) \rangle dx + \tilde{\mathcal{R}}(D, \lambda) \end{aligned} \tag{2.53}$$

where

$$\tilde{\mathcal{R}}(D, \lambda) := \mathcal{R}(D, G(D, \lambda)) = \mathcal{R}(D, H(\lambda) W_4(D, \lambda) + g). \tag{2.54}$$

Comparing (2.53) with our desired result (2.51), we are interested in estimating the term

$$\int_{T_-}^{T_+} \langle \psi(x), H(\lambda) W_4(D, \lambda)(x) \rangle dx,$$

the differences

$$\left| \int_{T_-}^{T_+} \langle \psi(x), B u_T'(x) \rangle dx - \int_{-\infty}^{\infty} \langle \psi(x), B h'(x) \rangle dx \right|,$$

and

$$|\langle\psi(T_+), P_+^{bc}(A(\mathrm{id}-P_+^{bc})h(T_+)-u'(T_+))\rangle|+|\langle\psi(T_-), P_-^{bc}(A(\mathrm{id}-P_-^{bc})h(T_-)-u'(T_-))\rangle|.$$

We collect these estimates and the estimate for D in the following lemma.

Lemma 2.7.2.

(i) $|D| \leq Ce^{-\eta L}$;

(ii) $\left|\int_{T_-}^{T_+}\langle\psi(x), H(\lambda)W_4(D,\lambda)(x)\rangle dx\right| \leq C(e^{-3\eta L}+e^{-\eta L}|\lambda|+|\lambda|^2)$;

(iii) $\left|\int_{T_-}^{T_+}\langle\psi(x), Bu_T'(x)\rangle dx - \int_{-\infty}^{\infty}\langle\psi(x), Bh'(x)\rangle dx\right| \leq Ce^{-\eta L}$

(iv) $|\langle\psi(T_+), P_+^{bc}A(\mathrm{id}-P_+^{bc})h(T_+) - P_+^{bc}u'(T_+)\rangle|$

$+|\langle\psi(T_-), P_-^{bc}A(\mathrm{id}-P_-^{bc})h(T_-) - P_-^{bc}u'(T_-)\rangle| \leq Ce^{-3\eta L}$.

Proof. (i) By (2.46) and (2.47), and since $f(0,0)=0$,

$$|D_\pm| = |P_\pm^{bc}(u_T'(T_\pm))| \leq C|f(u_T(T_\pm),\mu_T)| \leq C(|u_T(T_\pm)|+|\mu_T|) \leq Ce^{-\eta L}.$$

(ii) In

$$\int_{T_-}^{T_+}\langle\psi(x), H(\lambda)W_4(D,\lambda)(x)\rangle dx, \tag{2.55}$$

the first order term in λ is given by

$$\lambda\cdot\int_{T_-}^{T_+}\langle\psi(x), \partial_\lambda(H(\lambda)W_4(D,\lambda))(x)|_{\lambda=0}\rangle dx. \tag{2.56}$$

Observe that

$$\partial_\lambda(H(\lambda)W_4(D,\lambda))|_{\lambda=0} = B\cdot W_4(D,0) + (f_u(u_T,\mu_T)-f_u(h(x),0))\cdot\partial_\lambda W_4(D,0).$$

From (2.30), we see that the integral terms vanish for $w = W_1(a, B(a,0), 0)$ so that, by definition (2.52) of W_4, we have

$$|W_4(D,0)(x)| \leq Ce^{-\eta^s(|T_-|+x)}|D|, \qquad x\in[T_-,0];$$

$$|W_4(D,0)(x)| \leq Ce^{-\eta^u(T_+-x)}|D|, \qquad x\in[0,T_+].$$

Therefore

$$\left|\int_0^{T_+}\langle\psi(x),B\cdot W_4(D,0)(x)\rangle dx\right|\le C\int_0^{T_+}e^{-\eta^u(T_+-x)}e^{-\eta x}|D|dx\le Ce^{-\eta L}|D|. \tag{2.57}$$

and, similarly,

$$|\int_{T_-}^{0}\langle\psi(x),B\cdot W_4(D,0)(x)\rangle dx|\le C\int_{T_-}^{0}e^{-\eta^s|T_--x|}e^{-\eta|x|}|D|dx\le Ce^{-\eta L}|D|. \tag{2.58}$$

Since $f\in\mathcal{C}^2$,

$$\begin{aligned}&|f_u(u_T,\mu_T)-f_u(h,0)| \qquad (2.59)\\ \le\ & |f_u(u_T,\mu_T)-f_u(u_T,0)|+|f_u(u_T,0)-f_u(h,0)|\\ \le\ & C(|\mu_T|+\sup_{x\in[T_-,T_+]}|u_T(x)-h(x)|)\le Ce^{-\eta L}.\end{aligned}$$

Also, $|\partial_\lambda W_4(D,0)|\le C$. Therefore, (2.56) is bounded by $Ce^{-\eta L}|\lambda|$. With a similar argument, we find that the constant term in λ of (2.55) is

$$\int_{T_-}^{T_+}\langle\psi(x),H(0)W_4(D,0)(x)\rangle dx=O(e^{-2\eta L})|D|.$$

Substitute Part (i) $D_\pm=-P_\pm^{bc}u_T'(T_\pm)=O(e^{-\eta L})$ and the estimates (2.57), (2.58), (2.59) into (2.56), then we have (ii).

(iii)

$$\begin{aligned}&|\int_{T_-}^{T_+}\langle\psi(x),Bu_T'(x)\rangle dx-\int_{-\infty}^{\infty}\langle\psi(x),Bh'(x)\rangle dx|\\ \le\ & |\int_{-\infty}^{T_-}\langle\psi(x),Bh'(x)\rangle dx|+|\int_{T_+}^{\infty}\langle\psi(x),Bh'(x)\rangle dx|\\ &+|\int_{T_-}^{T_+}\langle\psi(x),B(u_T'(x)-h'(x))\rangle dx|\\ \le\ & C(e^{-2\eta L}+\sup_{x\in[T_-,T_+]}|u_T(x)-h(x)|)\le Ce^{-\eta L}.\end{aligned}$$

(iv) Since $f(0,0)=0$,

$$\begin{aligned} u_T' &= f(u_T,\mu_T) = f_u(0,\mu)u_T + O(|u_T|^2) + f_\mu(0,\mu_T)\mu_T + O(|\mu_T|^2) \\ &= f_u(0,0)u_T + O(|\mu_T||u_T| + |u_T|^2 + |\mu_T|) \\ &= Au_T + O(e^{-2\eta L}). \end{aligned}$$

By Lemma 2.7.1,

$$\begin{aligned} P_+^{bc}u_T'(T_+) &= P_+^{bc}A(\mathrm{id} - P_+^{bc})h(T_+) + O(e^{-2\eta L}), \\ P_-^{bc}u_T'(T_-) &= P_-^{bc}A(\mathrm{id} - P_-^{bc})h(T_-) + O(e^{-2\eta L}). \end{aligned}$$

Therefore,

$$\begin{aligned} &|\langle \psi(T_+), P_+^{bc}A(\mathrm{id} - P_+^{bc})h(T_+) - P_+^{bc}u'(T_+)\rangle| \\ &+|\langle \psi(T_-), P_-^{bc}A(\mathrm{id} - P_-^{bc})h(T_-) - P_-^{bc}u'(T_-)\rangle| \le Ce^{-3\eta L}. \end{aligned}$$

□

Since

$$\begin{aligned} |G(D,\lambda)| &= |(f_u(u_T,\mu_T) - f_u(h(x),0) + \lambda B)W_4(D,\lambda) + \lambda B u_T'(x)| \\ &\le C((e^{-\eta L} + |\lambda|)^2 + e^{-\eta L}|\lambda|) \le C(e^{-2\eta L} + e^{-\eta L}|\lambda| + |\lambda|^2), \end{aligned}$$

and

$$|\tilde{\mathcal{R}}(D,\lambda)| \le C(e^{-\eta L}(e^{-\eta L}|D| + |G(D,\lambda)|)) \le C(e^{-3\eta L} + e^{-2\eta L}|\lambda| + e^{-\eta L}|\lambda|^2),$$

we have

$$\begin{aligned} \xi &= \langle \psi(T_+), -P_+^{bc}A(\mathrm{id} - P_+^{bc})h(T_+)\rangle - \langle \psi(T_-), -P_-^{bc}A(\mathrm{id} - P_-^{bc})h(T_-)\rangle \\ &\quad +O(e^{-3\eta L}) - \lambda(\int_{-\infty}^{\infty}\langle \psi(x), Bh'(x)\rangle dx + O(e^{-\eta L})) + O(|\lambda|^2). \end{aligned} \tag{2.60}$$

Combining the error terms in (2.60), we have the result (2.51). □

Due to Theorem 2.7.2, with the Melnikov assumption, we have $\partial_\lambda \xi(D,0) \neq 0$ for all $|D|$ uniformly small. Thus, we can invoke the Implicit Function Theorem to solve for $\lambda = \lambda(D)$.

2.8 Applications

In this section, we will apply the results to the eigenvalue problem associated with the FitzHugh-Nagumo and the Nagumo equation. In both cases, we will investigate the given pulse h and verify that the hypotheses 2.1 - 2.4 and 2.5 are satisfied. By Theorem 2.7.1, there is then a unique truncated pulse u_T on $[T_-, T_+]$. By Theorem 2.7.2, up to higher order terms $\tilde{\mathcal{R}}(\lambda) = O(e^{-3\eta L} + e^{-\eta L}|\lambda| + |\lambda|^2)$, the persisting eigenvalue near $\lambda = 0$ is given by

$$\begin{aligned} \lambda \int_{-\infty}^{\infty} \langle \psi(x), Bh'(x) \rangle dx &= \langle \psi(T_-), P_-^{bc} A(\mathrm{id} - P_-^{bc}) h(T_-) \rangle \\ &\quad - \langle \psi(T_+), P_+^{bc} A(\mathrm{id} - P_+^{bc}) h(T_+) \rangle. \end{aligned} \tag{2.61}$$

If the sign of the Melnikov-type integral $\tilde{M}$ in Hypothesis 2.5 is determined, then the sign of λ is determined by the term $\langle \psi(T_-), P_-^{bc} u'(T_-) \rangle - \langle \psi(T_+), P_+^{bc} u'(T_+) \rangle$. The significance of this is that we can investigate some local dynamic features of the equilibrium $u = 0$ of the ODE (2.20), or equilibria for heteroclinic orbits, to study the dynamic behavior near the steady state u_T of the PDE (2.1).

The following lemma will help to simplify the calculations.

Lemma 2.8.1. *Suppose that the linear operator $A : E_0^s \oplus E_0^u \to E_0^s \oplus E_0^u$ has the following matrix representation*

$$A = \begin{pmatrix} -A^s & 0 \\ 0 & A^u \end{pmatrix},$$

in which $\mathbb{R}^n = E_0^s \oplus E_0^u$. Then $P_\pm^{bc} A(\mathrm{id} - P_\pm^{bc})$ have the following matrix representations:

$$\begin{aligned} P_+^{bc} A(\mathrm{id} - P_+^{bc}) &= \begin{pmatrix} 0 & 0 \\ B_+ A^s + A^u B_+ & 0 \end{pmatrix} \\ P_-^{bc} A(\mathrm{id} - P_-^{bc}) &= \begin{pmatrix} 0 & -A^s B_- - B_- A^u \\ 0 & 0 \end{pmatrix} \end{aligned}$$

for some linear bounded $B_- : E_0^u \cong \mathbb{R}^k \to E_0^s \cong \mathbb{R}^{n-k}$ *and* $B_+ : E_0^s \cong \mathbb{R}^{n-k} \to E_0^u \cong \mathbb{R}^k$ *such that*

$$\begin{aligned} E_-^{bc} &= \textit{graph } B_- = \{a + B_- a; a \in E_0^u\} \\ E_+^{bc} &= \textit{graph } B_+ = \{a + B_+ a; a \in E_0^s\} \end{aligned}$$

Remark 2.8.1. *From Lemma A.2.1 and Hypothesis 2.4, such a* B_+ *and* B_- *exist.*

Proof. By the hypothesis 2.4, $\mathbb{R}^n = E_+^{bc} \oplus E_0^u$. Taking $E = E_+^{bc}$ and $A = E_0^u$ in Lemma A.2.1, we obtain that

$$E_+^{bc} = \{a^s + B_+ a^s : a^s \in E_0^s\}, \qquad \text{for some linear bounded } B_+ : E_0^s \to E_0^u.$$

Since $P_+^{bc}|_{E_0^u} = \mathrm{id}_{E_0^u}$ and $P_+^{bc}|_{E^{bc}} = 0$, P_+^{bc} has the following representation

$$P_+^{bc} : E_0^s \oplus E_0^u \to E_0^s \oplus E_0^u, \qquad P_+^{bc} = \begin{pmatrix} 0 & 0 \\ -B_+ & 1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} P_+^{bc} A(\mathrm{id} - P_+^{bc}) &= \begin{pmatrix} 0 & 0 \\ -B_+ & 1 \end{pmatrix} \begin{pmatrix} -A^s & 0 \\ 0 & A^u \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -B_+ & 1 \end{pmatrix} \right] \\ &= \begin{pmatrix} 0 & 0 \\ B_+ A^s & A^u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B_+ & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ B_+ A^s + A^u B_+ & 0 \end{pmatrix} \end{aligned}$$

Similarly, we have

$$P_-^{bc} : E_0^s \oplus E_0^u \to E_0^s \oplus E_0^u, \qquad P_-^{bc} = \begin{pmatrix} 1 & -B_- \\ 0 & 0 \end{pmatrix},$$

and

$$P_-^{bc} A(\mathrm{id} - P_-^{bc}) = \begin{pmatrix} 0 & -A^s B_- - B_- A^u \\ 0 & 0 \end{pmatrix},$$

for some linear bounded $B_- : E_0^u \cong \mathbb{R}^k \to E_0^s \cong \mathbb{R}^{n-k}$. □

2.8.1 Fronts of the Nagumo Equation

We consider the *Nagumo equation*:

$$u_t = u_{xx} + u(1-u)(u-a), \quad x \in \mathbb{R}. \tag{2.62}$$

For each $a \in (0, \frac{1}{2})$, there exists a unique front $h_a(\xi)$, up to x-translations, with unique speed $c = c_a$, that connects $u = 0$ to $u = 1$. Equation (2.62) is related to the FitzHugh-Nagumo model for nerve action potentials. Here $u = 0$ corresponds to the resting state and $u = 1$ to the excited state of the nerve, and they are both stable.

More generally, we may consider the *bistable Nagumo equation* given by:

$$\begin{aligned} &u_t = u_{xx} + f(u), \\ &f(0) = f(1) = f(a) = 0, \\ &f < 0 \text{ on } (0, a); \quad f > 0 \text{ on } (a, 1) \\ &f'(0), f'(1) < 0; \quad F(1) = \int_0^1 f(x)dx > 0. \end{aligned} \tag{2.63}$$

Observe that $f(u)$ is cubic-like. Again, there is a unique front connecting the stable states $u = 0$ and $u = 1$. Furthermore, we have a global stability result: one can prove, using the comparison principle, that for a large class of initial data, solutions converge uniformly to a translation of the front with an exponential rate [17].

The Nagumo equation (2.62) possesses the following travelling fronts connecting $(0, 0)$ and $(1, 0)$:

$$(u, u_\xi) = h_a(\xi) := (\sigma(\xi), \sigma'(\xi)), \qquad \sigma(\xi) = \frac{e^{\frac{\xi}{\sqrt{2}}}}{e^{\frac{\xi}{\sqrt{2}}} + 1}. \tag{2.64}$$

with wave speed $c_a = \sqrt{2}(\frac{1}{2} - a)$. We have the following expansion of the homoclinic orbit $h_a(\xi)$:

$$h_a(\xi) = \begin{cases} (1, 0) + (1, -\frac{\sqrt{2}}{2})e^{-\frac{\xi}{\sqrt{2}}} + O(e^{(-\frac{1}{\sqrt{2}} - \nu)\xi}), & \xi \to \infty, \\ (1, \frac{\sqrt{2}}{2})e^{\frac{\xi}{\sqrt{2}}} + O(e^{(\frac{1}{\sqrt{2}} + \nu)\xi}), & \xi \to -\infty. \end{cases}$$

Furthermore, $\psi(\xi) = (\sigma''(\xi), -\sigma'(\xi))$ is a nontrivial bounded solution to the adjoint equation.

The eigenvalue problem associated with (2.62) is

$$\mathcal{L}u = u_{\xi\xi} - c_a u_\xi + f_u(h_a)u = \lambda u,$$

or, written as a system of ODEs,

$$\begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2(a+1)\sigma(\xi) - 3\sigma^2(\xi) - a & c_a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

where the coefficient matrix is hyperbolic for $\lambda = 0$ when $\xi \to \pm\infty$.

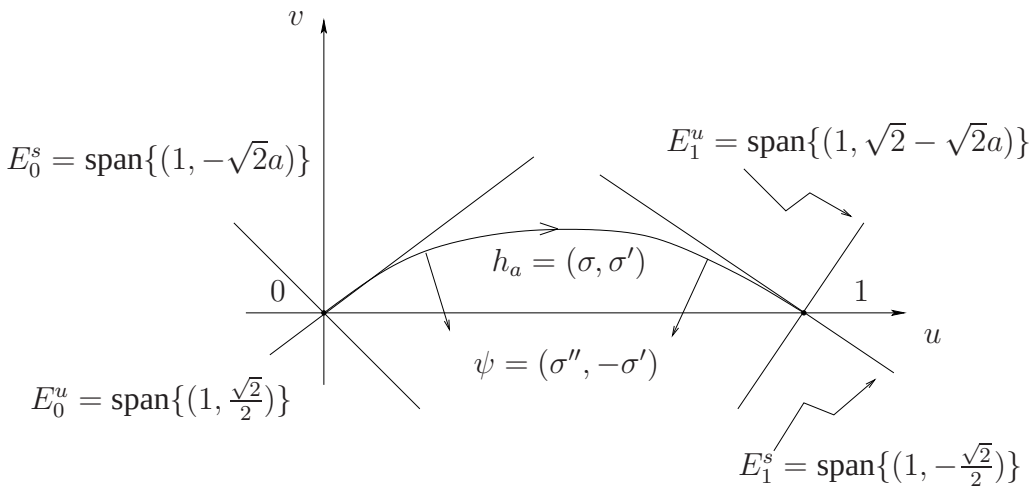

Figure 2.4: The front of the Nagumo equation. The eigenspaces can be calculated by invoking phase space analysis at $(0, 0)$ and $(1, 0)$.

Theorem 2.8.1. *Suppose that*

$$E_-^{bc} \oplus E_0^s = \mathbb{R}^2, \qquad E_+^{bc} \oplus E_1^u = \mathbb{R}^2,$$

where

$$E_0^s = span\{(1, -\sqrt{2})a\}, \qquad E_1^u = span\{(1, \sqrt{2} - \sqrt{2}a)\}.$$

Then we have (Figure 2.5):

(1) If $E_-^{bc} \subset I$, $E_+^{bc} \subset III$, then the truncated front is stable.

(2) If $E_-^{bc} \subset II$, $E_+^{bc} \subset IV$, then the truncated front is unstable.

In particular, if both boundary conditions are Dirichlet, then the truncated front is stable; if both conditions are Neumann, then the truncated front is unstable.

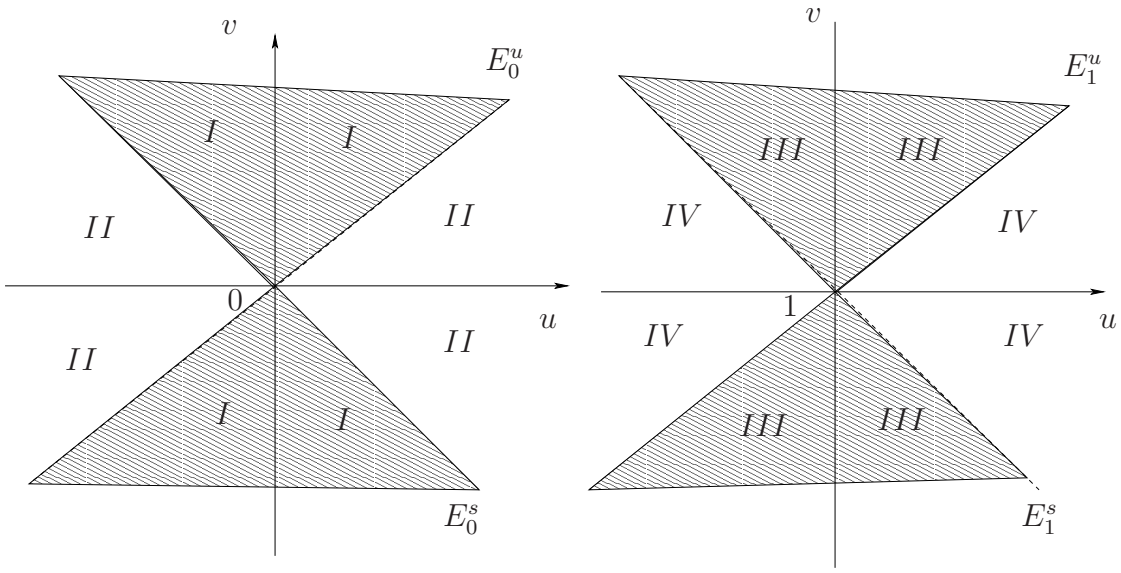

Figure 2.5: Regions for E_-^{bc} (left) and E_+^{bc} (right)

Proof. In comparison to equation (2.61),

$$\lambda = \frac{1}{\tilde{M}} \left(\langle \psi(T_-), P_-^{bc} A_- (\mathrm{id} - P_-^{bc}) h_a(T_-) \rangle - \langle \psi(T_+), P_+^{bc} A_+ (\mathrm{id} - P_+^{bc}) h_a(T_+) \rangle \right) + \tilde{R}(\lambda)$$

holds in the front case, in which

$$\tilde{M} = \int_{-\infty}^{\infty} \langle \psi(x), B h_a'(x) \rangle dx, \qquad A_- = f_u((0,0),0), \qquad A_+ = f_u((1,0),0),$$

and

$$\tilde{R}(\lambda) = O(e^{-3\eta L} + e^{-\eta L}|\lambda| + |\lambda|^2).$$

Since $h_a' = (\sigma', \sigma'')$ and ψ is chosen to be $(\sigma'', -\sigma')$, the Melnikov integral becomes

$$\int_{-\infty}^{\infty} \langle \psi(\xi), B h_a'(\xi) \rangle d\xi = \int_{-\infty}^{\infty} -(\sigma'(\xi))^2 d\xi < 0, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

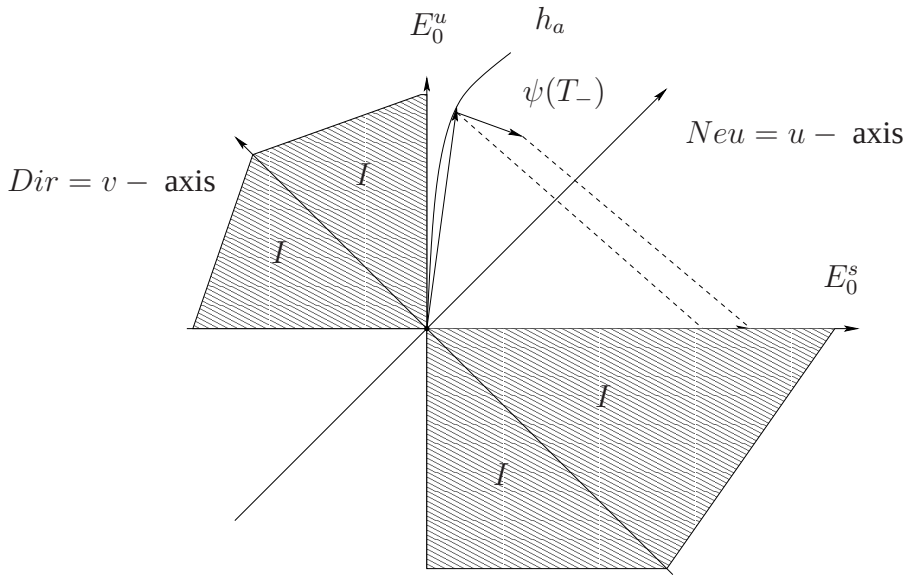

Figure 2.6: Dirichlet boundary condition at T_-: Observe that in comparison with Figure 2.5, in this picture, we rotate the coordinate system to obtain a better view. For large $|T_-|$, $h_a(T_-)$ is almost parallel to E_0^u and $\psi(T_-)$ to E_0^s.

In the following we would like to prove the theorem in the case of Dirichlet conditions at both end points. The proof for the general case is almost identical.

In Figure 2.6, choose the orientations of E_0^u and E_0^s, such that in $\mathbb{R}^2 = E_0^s \oplus E_0^u$,

$$\langle (0,1), h_a(T_-)\rangle > 0, \qquad \langle (1,0), \psi(T_-)\rangle > 0.$$

Then $Dir = E_-^{bc}$ for the Dirichlet condition at T_-, can be written as a graph $span\{(1, B_-)\}$ over E_0^u, with $B_- < 0$; Similarly, in Figure 2.7, choose the orientations of E_1^s and E_1^u, such that in $\mathbb{R}^2 = E_1^s \oplus E_1^s$,

$$\langle (1,0), -h_a(T_+)\rangle > 0 \qquad \langle (0,1), \psi(T_+)\rangle > 0.$$

Then $Dir = E_+^{bc}$ for the Dirichlet condition at T_+, can be written as a graph $span\{(1, B_+)\}$ over E_1^s, with $B_+ < 0$.

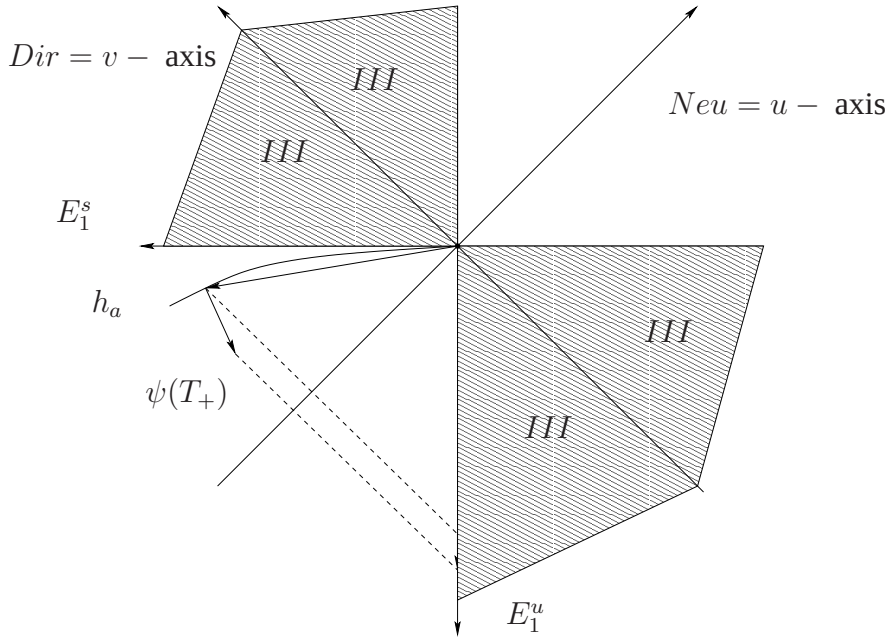

Figure 2.7: Dirichlet boundary condition at T_+: Similar to Figure 2.6.

Since

$$A_- = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2}a \end{pmatrix}, \qquad A_- = \begin{pmatrix} -\sqrt{2}a & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

and

$$\begin{aligned} P_-^{bc} A_-(\mathrm{id} - P_-^{bc}) &= \begin{pmatrix} 0 & -(\frac{1}{\sqrt{2}} + \sqrt{2}a)B_- \\ 0 & 0 \end{pmatrix}, \\ P_+^{bc} A_+(\mathrm{id} - P_+^{bc}) &= \begin{pmatrix} 0 & 0 \\ (\frac{1}{\sqrt{2}} + \sqrt{2}a)B_+ & 0 \end{pmatrix}, \end{aligned}$$

Therefore,

$$\langle \psi(T_-), P_-^{bc} A_-(\mathrm{id} - P_-^{bc}) h_a(T_-) \rangle < 0 \tag{2.65}$$

$$\langle \psi(T_+), P_+^{bc} A_+(\mathrm{id} - P_-^{bc}) h_a(T_-) \rangle > 0.$$

Therefore, $\lambda < 0$.

In general, if $E_-^{bc} \subset I$ and $E_+^{bc} \subset III$, then (2.65) holds and $\lambda < 0$.

For the Neumann boundary conditions we have

$$\langle \psi(T_-), P_-^{bc} A_-(\mathrm{id} - P_-^{bc}) h(T_-) \rangle > 0 \tag{2.66}$$

$$\langle \psi(T_+), P_+^{bc} A_+(\mathrm{id} - P_-^{bc}) h(T_-) \rangle < 0.$$

Then $\lambda > 0$, which also holds for the general case $E_-^{bc} \subset II$ and $E_+^{bc} \subset IV$.

□

2.8.2 Pulses of the FitzHugh-Nagumo Equation

Consider the FitzHugh-Nagumo equation (FHN) [37]:

$$\begin{aligned} u_t &= u_{xx} + f(u) - w \\ w_t &= \delta(u - \gamma w), \end{aligned} \tag{2.67}$$

in which we assume that $f(u) = u(1-u)(u-a)$, with $a \in (0, \frac{1}{2})$. Usually $\gamma > 0$ is small and $0 < \delta \ll 1$. The FHN (2.67) is a simplified model of the Hodgkin-Huxley equation, which was first used as a model for the propagation of electric signals along the giant nerve axon of squids. Instead of *space-clamped* dynamics [37], we consider the system with spatial diffusion in u without externally applied current.

In the co-moving frame $(\xi = x + ct, t)$, where $c > 0$, (2.67) becomes:

$$\begin{aligned} u_t &= u_{\xi\xi} - c u_\xi + f(u) - w, \\ w_t &= -c u_\xi + \delta(u - \gamma w). \end{aligned} \tag{2.68}$$

A travelling wave solution with speed c is a stationary solution $u_t = w_t = 0$ of (2.68). Let $\cdot = \frac{d}{d\xi}$, then a travelling wave is a bounded solutions of the following ODEs:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= cv - f(u) + w, \\ \dot{w} &= \epsilon(u - \gamma w) \end{aligned} \tag{2.69}$$

with $0 < \epsilon := \frac{\delta}{c} \ll 1$.

There are two types of pulse solutions we are interested in: slow and fast pulses. Slow pulses are perturbed solutions of the fast orbits in the fast system. Fast pulses, on the other hand, are perturbations of singular orbits consisting two pieces of slow manifolds and the connections between them [31]. "Fast" (pulse) and "slow" (pulse) are referred to the different time scales in the corresponding systems. For the existence of such slow and fast pulses parametrized by the propagation speed c, see [22]. Also see [31] for more references and a brief survey. Typically, slow pulses are unstable, and fast pulses are exponentially stable for small $\gamma > 0$ and marginally stable for $\gamma = 0$ [63]. Therefore, we investigate the truncated boundary value problem of the fast pulses.

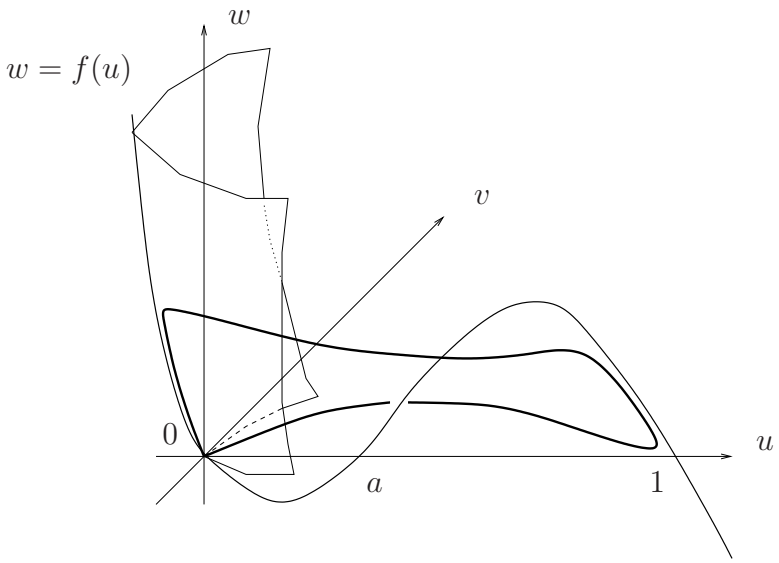

Figure 2.8: The fast pulse of the FitzHugh-Nagumo equation

Before stating our result, we discuss the spectral properties of the fast pulse. The linearized stability of the fast pulse (u, w) of (2.68) is determined by the spectrum of the

linear operator

$$\mathcal{L}(\tilde{u},\tilde{w}) = \begin{pmatrix} \tilde{u}_{\xi\xi} - c\tilde{u}_{\xi} + f_u(u)\tilde{u} - \tilde{w} \\ -c\tilde{w}_{\xi} + \delta(\tilde{u} - \gamma\tilde{w}) \end{pmatrix}.$$

The associated eigenvalue problem to (2.68) for λ near 0 is given by:

$$V' = \left.\begin{pmatrix} 0 & 1 & 0 \\ -f'(u) & c & 1 \\ \epsilon & 0 & -\epsilon\gamma \end{pmatrix}\right|_{(u,v,w)(\xi)=h(\xi)} \qquad V + \lambda BV := F_U V + \lambda BV,$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{c} \end{pmatrix}$$

where $h(\xi) = (u,v,w)(\xi)$ is the fast pulse solution. The spectrum of the asymptotic linearization

$$V' = F_U(\mathbf{0})V = \begin{pmatrix} 0 & 1 & 0 \\ a & c & 1 \\ \epsilon & 0 & -\epsilon\gamma \end{pmatrix} V \tag{2.70}$$

about the equilibrium $(u,v,w) = (0,0,0) = \mathbf{0}$ consists of three eigenvalues

$$\zeta^{ss} < \zeta^{s} = O(\epsilon) < 0 < \zeta^{u}, \qquad \zeta^{u} > |\zeta^{ss}| > |\zeta^{s}|. \tag{2.71}$$

Remark 2.8.2. *Here we would like to emphasize that the set of ζ^i, $i = s, ss, u$, as the eigenvalues of the linearization about the equilibrium, is a local dynamic feature of the travelling wave ODE, while λ, as the eigenvalue of the linearization about the pulse, describes the stability/instability of the truncated pulse as a solution to the FitzHugh-Nagumo PDE.*

The parameters c and a can be expressed as follows:

$$\begin{aligned} c &= \zeta^{ss} + \zeta^{s} + \zeta^{u} + \epsilon\gamma, \\ a &= -(\zeta^{ss}\zeta^{s} + \zeta^{u}\zeta^{s} + \zeta^{u}\zeta^{ss} + \epsilon\gamma c) \end{aligned}$$

in terms of eigenvalues. Also we can find the eigenvectors

$$\begin{aligned} V^s &= \tau_s \tilde{V}^s, \qquad \tilde{V}^s := \begin{pmatrix} 1 \\ \zeta^s \\ (\zeta^{ss}+\epsilon\gamma)(\zeta^u+\epsilon\gamma) \end{pmatrix}, \\ V^u &= \tau_u \tilde{V}^u, \qquad \tilde{V}^u := \begin{pmatrix} 1 \\ \zeta^u \\ (\zeta^{ss}+\epsilon\gamma)(\zeta^s+\epsilon\gamma), \end{pmatrix}, \\ V^{ss} &= \tau_{ss} \tilde{V}^{ss}, \qquad \tilde{V}^{ss} := \begin{pmatrix} 1 \\ \zeta^{ss} \\ (\zeta^s+\epsilon\gamma)(\zeta^u+\epsilon\gamma) \end{pmatrix}, \end{aligned}$$

corresponding to ζ^s, ζ^u and ζ^{ss} respectively, such that the pulse satisfies [11]:

$$h(\xi) = \begin{cases} V^s e^{\zeta^s \xi} + O(e^{(\zeta^s - \nu)\xi}), & \xi \to \infty, \\ V^u e^{\zeta^u \xi} + O(e^{(\zeta^u + \nu)\xi}), & \xi \to -\infty, \end{cases} \tag{2.72}$$

for some $\nu > 0$. Here $\tau_i > 0$, for $i = u$, s and ss [31].

Furthermore, there are eigenvectors

$$\tilde{W}^s := \begin{pmatrix} -c\epsilon + \zeta^s \epsilon \\ \epsilon \\ (\zeta^{ss}+\epsilon\gamma)(\zeta^u+\epsilon\gamma) \end{pmatrix}, \qquad \tilde{W}^u := \begin{pmatrix} -c\epsilon + \zeta^u \epsilon \\ \epsilon \\ (\zeta^{ss}+\epsilon\gamma)(\zeta^s+\epsilon\gamma) \end{pmatrix}, \tag{2.73}$$

to eigenvalues ζ^s and ζ^u, respectively, for the transpose $F_U^*(\mathbf{0})$ of the matrix $F_U(\mathbf{0})$.

Lemma 2.8.2. *For fixed $a \in (0, \frac{1}{2})$, there exists $\epsilon_* = \epsilon_*(a) > 0$ such that for all $0 < \epsilon < \epsilon_*$,*

$$\zeta^s < -\epsilon\gamma.$$

Proof. The characteristic polynomial of the asymptotic matrix in (2.70) is given by:

$$p(x) = -x^3 + (c - \epsilon\gamma)x^2 + (a + c\epsilon\gamma)x + a\epsilon\gamma + \epsilon.$$

Then $p(-\epsilon\gamma) = \epsilon > 0$. Choose $0 < \epsilon_* \ll a$, such that $\zeta^{ss} < -\epsilon\gamma$, which implies $\zeta^s < -\epsilon\gamma$. □

Observe that

$$\langle \tilde{V}^u, \tilde{W}^u \rangle = (\zeta^{ss}+\epsilon\gamma)^2(\zeta^s+\epsilon\gamma)^2 + \epsilon(\zeta^u - \zeta^{ss} - (\zeta^s+\epsilon\gamma)) > 0.$$

$$\langle \tilde{V}^s, \tilde{W}^s \rangle = (\zeta^{ss}\zeta^u)^2 + O(\epsilon) > 0.$$

On the other hand, choose $W^u \in span\{\tilde{W}^u\}$, such that

$$\langle V^u, W^u \rangle = 1, \tag{2.74}$$

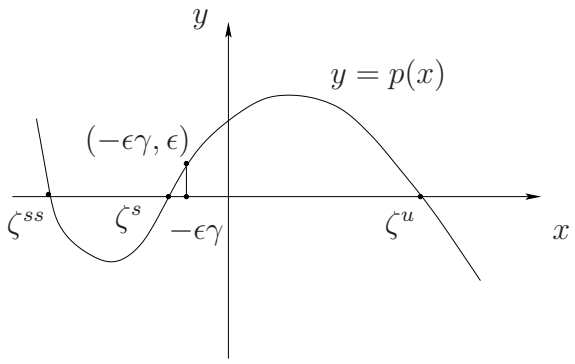

Figure 2.9: The graph of p

then there exists a $W^s \in span\{\tilde{W}^s\}$ with $\langle W^s, V^s\rangle > 0$ so that the bounded nontrivial solution $\psi(\xi)$ to the adjoint variational equation satisfies [31]

$$\psi(\xi) = \begin{cases} W^u e^{-\zeta^u \xi} + O(e^{-(\zeta^u+\nu)\xi}), & \xi \to \infty, \\ W^s e^{-\zeta^s \xi} + O(e^{-(\zeta^s-\nu)\xi}), & \xi \to -\infty. \end{cases} \tag{2.75}$$

With this choice of W^u (and consequently of W^s), the Melnikov integral becomes [45]

$$M = \int_{-\infty}^{\infty} \langle \psi(\xi), Bh'(\xi)\rangle d\xi = \int_{-\infty}^{\infty} \langle \psi(\xi), F_c(h(\xi), c)\rangle d\xi > 0.$$

We summarize the existence and stability result of the truncated fast pulse in the following:

Theorem 2.8.2. *Let the fast pulse $h = h(\xi)$ be the stationary solution of the FHN equation (2.67). Consider the following truncated boundary value problem*

$$\begin{aligned} &u_t = u_{\xi\xi} - cu_\xi + f(u) - w \\ &w_t = \delta(u - \gamma w), \qquad \xi \in (T_-, T_+), \\ &(u, v, w)(t, T_-) \in E_-^{bc}, \qquad (u, v, w)(t, T_+) \in E_+^{bc}. \end{aligned} \tag{2.76}$$

in which $f(u) = u(1-u)(u-a)$ and $a \in (0, \frac{1}{2})$. Assume that Hypothesis 2.4 is satisfied. Then for any fixed $a \in (0, \frac{1}{2})$, there exists a positive $\epsilon_* = \epsilon_*(a)$ such that the following

holds. For all $0 < \epsilon < \epsilon_*$, there exists $L_* = L_*(\epsilon) \gg 1$ such that for any $[T_-, T_+]$ with $T_- < 0 < T_+$ and $|T_-|$, $T_+ > L_*$, there exists a stationary solution of (2.76) near the fast pulse h. Moreover, the PDE stability of the truncated solution can be described as the following: Let

$$V^s := \lim_{\xi\to\infty} e^{-\zeta^s\xi} h(\xi), \qquad V^u := \lim_{\xi\to-\infty} e^{-\zeta^u\xi} h(\xi),$$

and

$$W^s := \lim_{\xi\to-\infty} e^{\zeta^s\xi} \psi(\xi), \qquad W^u := \lim_{\xi\to\infty} e^{\zeta^u\xi} \psi(\xi),$$

with the normalization (2.74). Consider

$$\begin{aligned} I &= I(T_-, T_+) \\ &:= e^{(\zeta^u-\zeta^s)T_-}\langle W^s, P_-^{bc} A(\mathrm{id} - P_-^{bc})V^u\rangle - e^{-(\zeta^u-\zeta^s)T_+}\langle W^u, P_+^{bc} A(\mathrm{id} - P_+^{bc})V^s\rangle. \end{aligned}$$

Then if $I < 0$, the truncated pulse is stable; if $I > 0$, then the truncated pulse is unstable.

Proof. The proof directly follows from Theorem 2.7.1, 2.7.2 and the estimates (2.72) and (2.75) for h and ψ, respectively, as $\xi \to \pm\infty$. □

Now we apply our result to Dirichlet and Neumann conditions.

Example 2.8.1. *Suppose that the boundary conditions are as follows:*

1. *Dirichlet at both ends:*

$$E_-^{bc} = v\text{-axis} = \{(u, w) = (0, 0)\}, \qquad E_+^{bc} = (v, w)\text{-plane} = \{u = 0\}.$$

In the coordinates of $\langle\tilde{V}^{ss}\rangle \oplus \langle\tilde{V}^s\rangle \oplus \langle\tilde{V}^u\rangle = E_0^s \oplus E_0^u \cong \mathbb{R}^3$,

$$V^{ss} = \begin{pmatrix} \tau_{ss} \\ 0 \\ 0 \end{pmatrix}, \qquad V^s = \begin{pmatrix} 0 \\ \tau_s \\ 0 \end{pmatrix} \qquad V^u = \begin{pmatrix} 0 \\ 0 \\ \tau_u \end{pmatrix}.$$

Recall that from Lemma A.2.1, there exists a bounded linear operator $B_- : E_0^u = \langle\tilde{V}^u\rangle \to E_0^s = \langle\tilde{V}^{ss}\rangle \oplus \langle\tilde{V}^s\rangle$, $B_- = \begin{pmatrix} B_-^1 \\ B_-^2 \end{pmatrix}$ such that $\begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} \in E_-^{bc}$ has coordinates

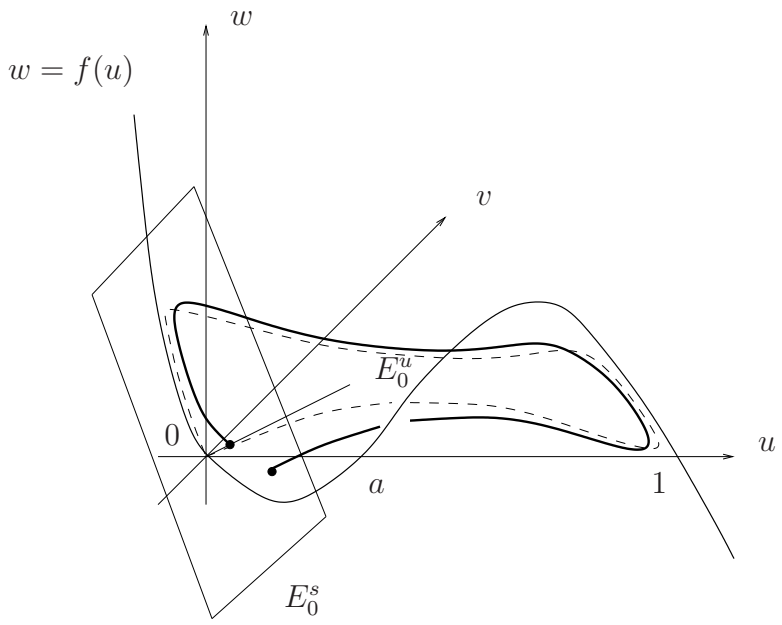

Figure 2.10: A truncated pulse u_T for the FitzHugh-Nagumo equation and in this special case, the boundary points $u_T(T_-) \in E_0^s$ and $u_T(T_+) \in E_0^u$. For a generic picture, the boundary points can move freely in the corresponding affine subspaces associated with $E_\pm^{bc}$, respectively.

$\begin{pmatrix} B_-^1 a \\ B_-^2 a \\ a \end{pmatrix}$ *in* $\tilde{V}^{ss} \oplus \tilde{V}^s \oplus \tilde{V}^u$. *Then*

$$\begin{pmatrix} \tilde{V}^{ss} & \tilde{V}^s & \tilde{V}^u \end{pmatrix} \begin{pmatrix} B_-^1 a \\ B_-^2 a \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}. \tag{2.77}$$

Solving (2.77) yields

$$\begin{pmatrix} B_-^1 a \\ B_-^2 a \\ a \end{pmatrix} = v \begin{pmatrix} (\zeta^{ss} - \zeta^u)(\zeta^s + \epsilon\gamma)(\zeta^u + \zeta^{ss} + \epsilon\gamma) \\ (\zeta^u - \zeta^{ss})(\zeta^s + \epsilon\gamma) \\ \zeta^u - \zeta^{ss} \end{pmatrix}.$$

Then

$$B_- = \begin{pmatrix} B_-^1 \\ B_-^2 \end{pmatrix} = \begin{pmatrix} -(\zeta^s + \epsilon\gamma)(\zeta^u + \zeta^{ss} + \epsilon\gamma) \\ \zeta^s + \epsilon\gamma \end{pmatrix}.$$

Similarly, there exists a bounded linear operator $B_+ : E_0^s = \langle \tilde{V}^{ss} \rangle \oplus \langle \tilde{V}^s \rangle \to E_0^u = \langle \tilde{V}^u \rangle$, $B_+ = \begin{pmatrix} B_+^1 & B_+^2 \end{pmatrix}$ *such that* $\begin{pmatrix} 0 \\ v \\ w \end{pmatrix} \in E_-^{bc}$ *has coordinates* $\begin{pmatrix} a_1 \\ a_2 \\ B_+^1 a_1 + B_+^2 a_2 \end{pmatrix}$

in $\tilde{V}^{ss} \oplus \tilde{V}^{s} \oplus \tilde{V}^{u}$. *Then*

$$\begin{pmatrix} \tilde{V}^{ss} & \tilde{V}^{s} & \tilde{V}^{u} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ B_+^1 a_1 + B_+^2 a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ w \end{pmatrix}. \tag{2.78}$$

Since

$$\det \begin{pmatrix} \tilde{V}^{ss} & \tilde{V}^{s} & \tilde{V}^{u} \end{pmatrix} = (\zeta^s - \zeta^u)(\zeta^s - \zeta^{ss})(\zeta^u - \zeta^{ss}) < 0,$$

$\begin{pmatrix} \tilde{V}^{ss} & \tilde{V}^{s} & \tilde{V}^{u} \end{pmatrix}$ *is non-singular. Then for any* (a_1, a_2)*, there is a unique pair* (v, w) *that satisfies (2.78). Taking any pair* $(a_1, a_2) = (0, a_2)$ *with* $a_2 \neq 0$ *in (2.78) yields* $B_+^2 = -1$ *and taking any pair* $(a_1, a_2) = (a_1, 0)$ *with* $a_1 \neq 0$ *in (2.78) yields* $B_+^1 = -1$. *Therefore,*

$$B_+ = \begin{pmatrix} B_+^1 & B_+^2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \end{pmatrix}.$$

Then in the coordinates of $\langle \tilde{V}^{ss} \rangle \oplus \langle \tilde{V}^{s} \rangle \oplus \langle \tilde{V}^{u} \rangle = E_0^s \oplus E_0^u \cong \mathbb{R}^3$*,*

$$\begin{aligned} P_-^{bc} A(\mathrm{id} - P_-^{bc}) V^u &= \begin{pmatrix} 0 & 0 & (\zeta^{ss} - \zeta^u) B_-^1 \\ 0 & 0 & (\zeta^s - \zeta^u) B_-^2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \tau_u \end{pmatrix} \\ &= \tau_u \begin{pmatrix} (\zeta^u - \zeta^{ss})(\zeta^s + \epsilon\gamma)(\zeta^u + \zeta^{ss} + \epsilon\gamma) \\ (\zeta^s - \zeta^u)(\zeta^s + \epsilon\gamma) \\ 0 \end{pmatrix} \\ P_+^{bc} A(\mathrm{id} - P_+^{bc}) V^s &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\zeta^u - \zeta^{ss}) B_+^1 & (\zeta^u - \zeta^s) B_+^2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \tau_s \\ 0 \end{pmatrix} \\ &= \tau_s \begin{pmatrix} 0 \\ 0 \\ \zeta^s - \zeta^u \end{pmatrix} \end{aligned}$$

Since $\langle V^i, W^j \rangle = 0$ *if* $i \neq j$*,* $i, j = s, ss$ *or* u *and* $\langle V^i, W^i \rangle > 0$*, if* $i = s$ *or* u*,*

$$\begin{aligned} \langle P_-^{bc} A(\mathrm{id} - P_-^{bc}) V^u, W^s \rangle &= \frac{\tau_u}{\tau_s} (\zeta^s - \zeta^u) B_-^2 \langle V^s, W^s \rangle \\ &= \frac{\tau_u}{\tau_s} (\zeta^s - \zeta^u)(\zeta^s + \epsilon\gamma) \langle V^s, W^s \rangle > 0 \\ \langle P_+^{bc} A(\mathrm{id} - P_+^{bc}) V^s, W^u \rangle &= \frac{\tau_s}{\tau_u} (\zeta^u - \zeta^s) B_+^2 \langle V^u, W^u \rangle \\ &= \frac{\tau_s}{\tau_u} (\zeta^s - \zeta^u) \langle V^u, W^u \rangle < 0 \end{aligned}$$

Thus, $I > 0$ *and the truncated pulse is unstable.*

The calculations in the following examples are similar.

2. *Dirichlet at $\xi = T_-$ and Neumann at T_+:*

$$E_-^{bc} = w\text{-axis} = \{(u,v) = (0,0)\}, \qquad E_+^{bc} = (u,w)\text{-plane} = \{v = 0\}.$$

A calculation shows that

$$B_- = \begin{pmatrix} B_-^1 \\ B_-^2 \end{pmatrix} = \begin{pmatrix} -(\zeta^u + \epsilon\gamma)(\zeta^s + \zeta^{ss} + \epsilon\gamma) \\ \zeta^u + \epsilon\gamma \end{pmatrix},$$

and

$$B_+ = \begin{pmatrix} B_+^1 & B_+^2 \end{pmatrix} = \begin{pmatrix} -\frac{\zeta^{ss}}{\zeta^u} & -\frac{\zeta^s}{\zeta^u} \end{pmatrix}.$$

Then

$$\begin{aligned} \langle P_-^{bc} A(\mathrm{id} - P_-^{bc})V^u, W^s\rangle &= \frac{\tau_u}{\tau_s}(\zeta^s - \zeta^u)B_-^2\langle V^s, W^s\rangle \\ &= \frac{\tau_u}{\tau_s}(\zeta^s - \zeta^u)(\zeta^u + \epsilon\gamma)\langle V^s, W^s\rangle < 0, \\ \langle P_+^{bc} A(\mathrm{id} - P_+^{bc})V^s, W^u\rangle &= \frac{\tau_s}{\tau_u}(\zeta^u - \zeta^s)B_+^2\langle V^u, W^u\rangle \\ &= -\frac{\tau_s}{\tau_u}(\zeta^u - \zeta^s)\frac{\zeta^s}{\zeta^u}\langle V^u, W^u\rangle > 0. \end{aligned}$$

We conclude that $I < 0$, and the truncated pulse is stable.

3. *Neumann at both ends:*

$$\begin{aligned} E_-^{bc} &= \{(\dot{u}, \dot{w}) = (0,0)\} = \{(u,v,w) | v = 0, u = \gamma w\}, \\ E_+^{bc} &= (u,w)\text{-plane} = \{v = 0\}. \end{aligned}$$

We have

$$B_-^2 = \frac{\gamma(\zeta^{ss} + \epsilon\gamma)(\zeta^u - \zeta^s) - (\zeta^u + \epsilon\gamma)(\zeta^{ss} - \zeta^s)}{\gamma(\zeta^u - \zeta^s) + (\zeta^s - \zeta^{ss})},$$

and

$$B_+ = \begin{pmatrix} B_+^1 & B_+^2 \end{pmatrix} = \begin{pmatrix} -\frac{\zeta^{ss}}{\zeta^u} & -\frac{\zeta^s}{\zeta^u} \end{pmatrix}.$$

Then

$$\begin{aligned}
&\langle P_-^{bc} A(\mathrm{id} - P_-^{bc})V^u, W^s\rangle = \frac{\tau_u}{\tau_s}(\zeta^s - \zeta^u)B_-^2\langle V^s, W^s\rangle \\
= \;& \frac{\tau_u}{\tau_s}(\zeta^s - \zeta^u)\frac{\gamma(\zeta^{ss} + \epsilon\gamma)(\zeta^u - \zeta^s) - (\zeta^u + \epsilon\gamma)(\zeta^{ss} - \zeta^s)}{\gamma(\zeta^u - \zeta^s) + (\zeta^s - \zeta^{ss})}\langle V^s, W^s\rangle < 0, \\
&\langle P_+^{bc} A(\mathrm{id} - P_+^{bc})V^s, W^u\rangle = \frac{\tau_s}{\tau_u}(\zeta^u - \zeta^s)B_+^2\langle V^u, W^u\rangle \\
= \;& -\frac{\tau_s}{\tau_u}(\zeta^u - \zeta^s)\frac{\zeta^s}{\zeta^u}\langle V^u, W^u\rangle > 0,
\end{aligned}$$

for $0 < \gamma \ll 1$.

We conclude that $I < 0$*, and the truncated pulse is stable.*

4. *Dirichlet at* $\xi = T_-$ *and Neumann at* T_+*:*

$$E_-^{bc} = v\text{-axis} = \{(u, w) = (0, 0)\}, \qquad E_+^{bc} = (u, w)\text{-plane} = \{v = 0\}.$$

A calculation shows that

$$B_- = \begin{pmatrix} B_-^1 \\ B_-^2 \end{pmatrix} = \begin{pmatrix} -(\zeta^s + \epsilon\gamma)(\zeta^u + \zeta^{ss} + \epsilon\gamma) \\ \zeta^s + \epsilon\gamma \end{pmatrix},$$

and

$$B_+ = \begin{pmatrix} B_+^1 & B_+^2 \end{pmatrix} = \begin{pmatrix} -\frac{\zeta^{ss}}{\zeta^u} & -\frac{\zeta^s}{\zeta^u} \end{pmatrix}.$$

Then

$$\begin{aligned}
\langle P_-^{bc} A(\mathrm{id} - P_-^{bc})V^u, W^s\rangle &= \frac{\tau_u}{\tau_s}(\zeta^s - \zeta^u)B_-^2\langle V^s, W^s\rangle \\
&= \frac{\tau_u}{\tau_s}(\zeta^s - \zeta^u)(\zeta^s + \epsilon)\gamma\langle V^s, W^s\rangle > 0, \\
\langle P_+^{bc} A(\mathrm{id} - P_+^{bc})V^s, W^u\rangle &= \frac{\tau_s}{\tau_u}(\zeta^u - \zeta^s)B_+^2\langle V^u, W^u\rangle \\
&= -\frac{\tau_s}{\tau_u}(\zeta^u - \zeta^s)\frac{\zeta^s}{\zeta^u}\langle V^u, W^u\rangle > 0.
\end{aligned}$$

Therefore,

$$I = c_1 e^{(\zeta^u - \zeta^s)T_-} - c_2 e^{-(\zeta^u - \zeta^s)T_+}, \qquad c_1, c_2 > 0.$$

If we fix T_+ and take T_- such that $|T_-|$ is large, then the truncated pulse will be stabilized; On the other hand, if we fix T_-, we can find T_+ large such that the truncated pulse is destabilized.

CHAPTER 3

TRUNCATION OF SPIRAL WAVES

3.1 Introduction

In this chapter, we investigate planar spiral waves that arise in reaction-diffusion equations with generic kinetic. More precisely, we study the persistence of spiral waves upon boring a small hole in the plane near the core region.

In polar coordinates (r, φ), a rigidly rotating spiral wave $u_*(r, \varphi)$ is an equilibrium to the governing reaction-diffusion equation

$$0 = \Delta u + \omega u_\varphi + f(u),$$

in the co-rotating frame. We take the view that $u_*(r, \varphi)$ is a function of the angle φ for each fixed value of the radius r. Our viewpoint therefore is to treat r as an evolution variable . In particular, spiral waves converge to constant functions of φ as $r \to 0$. On the other hand, Archimedian spiral waves approach asymptotic wavetrains, with possible minor angular modifications, as $r \to \infty$. In this sense, Archimedian spiral waves can be regarded as heteroclinic orbits in some function space of functions depending on φ that connect an equilibrium (constant function) in the core region $r = 0$ to a periodic orbit (wave train) in the far field $r \to \infty$.

Observe that the Laplacian operator

$$\Delta_2 := \partial_{x_1 x_1} + \partial_{x_2 x_2} = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\psi\psi}$$

is singular at $r = 0$ in the polar coordinates. This motivates a new rescaled variable $s = \ln r$ in order to perform a blow-up analysis near $r = 0$ [47, 55].

In this new radial time, there exist exponential dichotomies for the linear equation on any ray $(-\infty, s_*]$, for each fixed $s_* \in \mathbb{R}$. In the farfield, the perturbation term is not bounded, but upon some change of coordinates, we can also have exponential dichotomies on the ray $[R_*, +\infty)$ for some large R_*, see [47]. Then we can solve the linearized equations on the interval $s = \ln r \in [-L, \ln R_*]$ and $r \in [R_*, +\infty)$, where the choice of R_* is subject to the existence of the exponential dichotomies in the farfield. After applying the variation-of-constants formula, we can formally express the solution to the nonlinear equation as a fixed-point equation on the rays $(-\infty, R_*]$ and $[R_*, +\infty)$, respectively.

One of the issues concerning the fixed-point equation is that on the ray $(-\infty, R_*]$, besides the exponentially decaying solutions, we also have two N-dimensional (over $\mathbb{C}$) subspaces, consisting of initial conditions that correspond to constant and logarithmically growing solutions as $r \to 0$, respectively. When the boundary conditions satisfy a certain transversality condition, we can solve for the component of the initial conditions corresponding to logarithmically growing solutions and obtain appropriate a-priori estimate from which we can conclude that the general solution is C^1-bounded near $(u_*, \partial_r u_*)$. Then we can invoke the Implicit Function Theorem to establish the existence result of the core region spiral.

Once we have the existence of the core spiral and the farfield spiral, then the existence problem is reduced to matching core region and farfield spiral. We shall see that the two solutions can be matched at $r = R_*$ provided we adjust the temporal frequency ω appropriately.

3.2 Set-up

Consider the reaction-diffusion equation on the plane

$$u_t = D\Delta u + f(u), \qquad x \in \mathbb{R}^2, u \in \mathbb{R}^N. \tag{3.1}$$

Here D is a diagonal matrix with positive entries, and f is at least $\mathcal{C}^2$.

Suppose that the reaction-diffusion equation (3.1) admits a *rigidly rotating wave* solution

$$u(r, \varphi, t) = u_*(r, \psi), \qquad \psi = \varphi - \omega_* t$$

with nonzero temporal frequency ω_*. In the co-rotating frame (r, ψ), $u_*(r, \psi)$ is an equilibrium of

$$u_t = D\Delta u + \omega\partial_\psi u + f(u), \qquad x \in \mathbb{R}^2, u \in \mathbb{R}^N \tag{3.2}$$

for $\omega = \omega_*$. That is, $u_*(r, \psi)$ is a solution of the elliptic PDE

$$D\Delta u + \omega\partial_\psi u + f(u) = 0, \qquad x \in \mathbb{R}^2, u \in \mathbb{R}^N, \tag{3.3}$$

for $\omega = \omega_*$.

In polar coordinates, the equation (3.3) becomes

$$D\left(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\psi\psi}\right)u + \omega\partial_\psi u + f(u) = 0. \tag{3.4}$$

If we treat r as the time, then we can rewrite (3.4) as a system of ODE's on a Banach space of 2π-periodic functions in ψ. More precisely, we obtain

$$\begin{aligned} u_r &= v \\ v_r &= -\frac{1}{r}v - \frac{1}{r^2}\partial_{\psi\psi}u - D^{-1}[\omega\partial_\psi u + f(u)] \end{aligned} \tag{3.5}$$

defined on the Banach space $X = H^1(\mathbb{S}^1, \mathbb{C}^N) \times L^2(\mathbb{S}^1, \mathbb{C}^N) \ni \mathcal{U} = (u, v)$.

We are interested in solutions u of (3.4) near u_* for ω near ω_*. Thus, we write $u = u_* + \tilde{u}$, so that $\tilde{u}$ needs to satisfy

$$\begin{aligned} \tilde{u}_r &= \tilde{v} \\ \tilde{v}_r &= -\frac{1}{r}\tilde{v} - \frac{1}{r^2}\partial_{\psi\psi}\tilde{u} - D^{-1}[\omega_*\partial_\psi\tilde{u} + f'(u_*(r,\psi))\tilde{u}] \\ &\quad -D^{-1}[(\omega-\omega_*)\partial_\psi(u_*+\tilde{u}) + f(u_*+\tilde{u}) - f(\tilde{u}) - f'(u_*)\tilde{u}]. \end{aligned} \tag{3.6}$$

For convenience, we now drop the tilde, and consider

$$\begin{aligned} u_r &= v \\ v_r &= -\frac{1}{r}v - \frac{1}{r^2}\partial_{\psi\psi}u - D^{-1}[\omega_*\partial_\psi u + f'(u_*(r,\psi))u] \\ &\quad -D^{-1}[(\omega-\omega_*)\partial_\psi(u_*+u) + f(u_*+u) - f(u_*) - f'(u_*)u]. \end{aligned} \tag{3.7}$$

Observe that

$$-D^{-1}[(\omega-\omega_*)\partial_\psi(u_*+u) + f(u_*+u) - f(u_*) - f'(u_*)u] = O(|\omega-\omega_*| + |u|^2).$$

Therefore, we can treat equation (3.7) as a perturbation to

$$\begin{aligned} u_r &= v \\ v_r &= -\frac{1}{r}v - \frac{1}{r^2}\partial_{\psi\psi}u - D^{-1}[\omega_*\partial_\psi u + f'(u_*(r,\psi))u], \end{aligned} \tag{3.8}$$

which is the linearization of (3.5) about u_*. As in Chapter 1, we would like to study (3.7) and (3.8) for $r \in (0, R_*]$ and $r \in [R_*, \infty)$, which we refer to as the *core* and *farfield* regions respectively. First, however, we explain our assumptions on u_*.

3.2.1 Main Hypotheses

We introduce the hypotheses on the underlying spiral u_*.

First, we only consider spiral waves that are rotating, Archimedian and transverse:

Definition 3.2.1. *A rotating wave $u_*(r, \varphi - \omega_* t)$ is an Archimedean spiral wave if there exists a smooth 2π-periodic function $u_\infty(\vartheta)$, a smooth function $\theta(r)$ with $\theta'(r) \to 0$ as*

$r \to \infty$, *and a non-zero constant* k_* *such that*

$$|u_*(r, \cdot - \omega_* t) - u_\infty(k_* r + \theta(r) + \cdot - \omega_* t)|_{C^1(\mathbb{S}^1)} \to 0 \qquad \textit{as} \qquad r \to \infty. \tag{3.9}$$

In other words, Archimedean spiral waves can be approximated asymptotically, along the radial direction, by wave trains.

Remark 3.2.1. *Note that the condition* $\theta'(r) \to 0$ *as* $r \to \infty$ *on the derivative of angular modification implies that* $\frac{\theta(r)}{r} \to 0$ *as* $r \to \infty$. *One typical example is* $\theta(r) = \ln r$*: We allow* θ *to be slowly growing but slower than the radius.*

Recall that an Archimedean spiral wave $u_* = u_*(r, \psi)$ is transverse if it emits a spectrally stable wave train (see (1.7) and 1.0.1) and the generalized kernel of the linearization $\mathcal{L}_*$ about u_* in L^2_γ is one-dimensional for some small $\gamma > 0$.

If we regard the spiral wave u_* as a heteroclinic orbit connecting the steady state $u_*(0)$ at the rotation center and the wave trains at the farfield, then $(u_*, \partial_r u_*)$ is contained in the intersection of the center-stable manifold $M^{cs}_+(\omega_*)$ of wave trains in the farfield and the center-unstable manifold $M^{cu}_-(\omega_*)$ of the asymptotic steady states at the core. Due to the rotational invariance of the spiral wave, the intersection of the tangent spaces of the manifolds contains the subspace spanned by $(\partial_\psi u_*, \partial_\psi \partial_r u_*)$ due to the following lemma:

Lemma 3.2.1. *Suppose that* $u_*(r, \psi)$ *satisfies* (3.4) *with* $\omega = \omega_*$*, then* $\partial_\psi u_*(r, \psi)$ *satisfies*

$$D(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\psi\psi})u + \omega_* \partial_\psi u + f'(u_*)u = 0.$$

Proof. Differentiate (3.4) with respect to ψ. □

Therefore $(\partial_\psi u_*, \partial_\psi \partial_r u_*)$ satisfies (3.8). Now we would like to have the intersection of $M^{cs}_+(\omega_*)$ and $M^{cs}_+(\omega_*)$ as transverse as possible by requiring that the intersection is spanned by $(\partial_\psi u_*, \partial_\psi \partial_r u_*)$:

Hypothesis 3.1. *The subspace of bounded solutions to the variational equation* (3.8) *is spanned by* $(\partial_\psi u_*, \partial_\psi \partial_r u_*)$.

One useful consequence of Hypothesis 3.1 is related to the adjoint variational equation. Similar to the one-dimensional case, due to the invariance of the inner product $(A.1.1)(1)$, bounded solutions of the adjoint equation are orthogonal to every solution of the variational equation that is bounded as either $r \to 0$ or $r \to +\infty$. Therefore, bounded solution of the adjoint variational equation must lie in the orthogonal complement of the tangent space of the unstable manifold at the core and the stable manifold of the farfield.

We also assume that the Melnikov integral associated with $\tilde{\mathcal{G}}$ is non-vanishing. That is,

Hypothesis 3.2. $M := \int_{-\infty}^{+\infty} \langle \psi(\tau), D_\omega \tilde{\mathcal{G}}(\mathcal{U}_*, \omega_*)(\tau) \rangle d\tau \neq 0,$

in which

$$\tilde{\mathcal{G}}(\mathcal{U}, \omega)(\tau) = \begin{cases} e^{2\tau}\mathcal{G}(\mathcal{U}_-, \omega), & \mathcal{U}_- = (u, u_\tau), -L \leq \tau = \log r \leq \log R_* \\ \mathcal{G}(\mathcal{U}_+, \omega), & \mathcal{U}_+ = (u, u_r), \tau = r > R_* \end{cases},$$

and

$$\mathcal{G}(\mathcal{U}, \omega) = \begin{pmatrix} 0 \\ -D^{-1}[(\omega - \omega_*)\partial_\psi(u_* + u) + f(u_* + u) - f(u_*) - f'(u_*)u] \end{pmatrix}.$$

3.2.2 Main Result

With the assumptions in Section 3.2.1, our main result is stated as the following:

Theorem 3.2.1. *Suppose that there is a spiral wave solution* u_* *of (3.1) with temporal frequency* $\omega_* \neq 0$ *and positive group velocity* c_g*. Assume that* u_* *satisfies Hypotheses 3.1 and 3.2. Then the spiral is robust with respect to the Dirichlet or Neumann boundary condition at* $r = \epsilon$ *for* ϵ *small enough. More precisely, there exists* $\epsilon_0 > 0$ *such that for all* $0 < \epsilon < \epsilon_0$*, the following is true.*

Consider the new domain

$$\Omega := \mathbb{R}^2 \backslash B_\epsilon(0)$$

with the boundary $\partial\Omega = \partial B_\epsilon(0)$. *Then there exist two families of spiral waves* $u_{Dir} = u_{Dir}(\epsilon)$ *and* $u_{Neu} = u_{Neu}(\epsilon)$ *on* Ω *with frequency* $\omega = \omega(\epsilon)$ *such that* u *satisfies the Dirichlet boundary condition*

$$u_{Dir}|_{\partial\Omega} = u_*(0)$$

or Neumann boundary condition

$$\frac{\partial u_{Neu}}{\partial n}|_{\partial\Omega} = 0$$

at $r = \epsilon$. *Moreover,* u_{Dir} *and* u_{Neu} *depend smoothly on* ϵ. *Furthermore, up to some normalization (Hypothesis 3.4, which will be explained later), we have the following expansion for* ω *near* ω_**:*

$$\begin{aligned} \omega &= \omega_* + \frac{\epsilon}{M \cdot \ln \epsilon}(u_*)_r(0) + O(\frac{\epsilon}{\ln^2 \epsilon}), \qquad \textit{(Dirichlet condition)}; \\ \omega &= \omega_* + \frac{\epsilon}{M}(u_*)_r(0) + O(\epsilon^2), \qquad \textit{(Neumann condition)}. \end{aligned}$$

Here M *is the Melnikov integral with respect to* ω, *which is assumed to be non-zero by Hypothesis 3.2.*

The proof is divided into two parts: First, we prove that there exist a unique core region spiral subject to the boundary condition. By Proposition 7.2 in [47], there exists a farfield spiral. Then we match the farfield spiral with the core region spiral at some $\tau = R_*$. The group velocity assumption is only needed for the farfield spiral. Also in the proof, we see that the Dirichlet boundary conditions can be generalized to arbitrary (T1) boundary conditions.

3.2.3 Boundary Conditions

Recall that u_* is the presumed spiral wave solution, while $\tilde{u} = u_* + u$ is a perturbed solution, corresponding to another equilibrium of the reaction-diffusion equation near u_* where $u \in H^1(\mathbb{S}^1, \mathbb{C}^N)$ is an appropriate, small perturbation.

For the truncated boundary value problem, we need to impose boundary condition on $\tilde{u}$ at the circle $r = r_0$. For example,

$$\tilde{u}(r_0) = \tilde{d} \qquad \text{(Dirichlet boundary condition)},$$

$$\partial_r \tilde{u}(r_0) = \tilde{d}, \qquad \text{(Neumann boundary condition)},$$

or

$$(\alpha\tilde{u} + \partial_r \tilde{u})(r_0) = \tilde{d} \qquad \alpha \in \mathbb{R}, \alpha \neq 0 \qquad \text{(Robin boundary condition)}.$$

Then consider

$$\tilde{\mathcal{U}} := (\tilde{u}, \tilde{u}_r) = \mathcal{U}_* + \mathcal{U} = (u + u_*, u_r + \partial_r u_*).$$

Correspondingly, the boundary condition for $\tilde{\mathcal{U}}$ is an affine boundary condition at $r = r_0$. That is,

$$\tilde{\mathcal{U}}(r_0) = \mathcal{U}_*(r_0) + \mathcal{U}(r_0) \in E^{bc}_- + \tilde{d}.$$

Here E^{bc}_- is a closed linear subspace of $X = H^1(\mathbf{S}^1, \mathbb{C}^N) \times L^2(\mathbf{S}^1, \mathbb{C}^N)$ and $\tilde{d} = \begin{pmatrix} \tilde{d}_1 \\ 0 \end{pmatrix}$ in the orthogonal complement $(E^{bc}_-)^\perp$ is fixed.

For example, by Dirichlet or Neumann condition, we mean the following:

(Dirichlet) : $\tilde{\mathcal{U}}(r)|_{r=r_0} \in E^{bc}_- + \begin{pmatrix} \tilde{d}_1 \\ 0 \end{pmatrix}$, where $E^{bc}_- = \{\begin{pmatrix} 0 \\ \xi \end{pmatrix} : \xi \in L^2(\mathbf{S}^1, \mathbb{C}^N)\}$;

(Neumann) : $\tilde{\mathcal{U}}(r)|_{r=r_0} \in E^{bc}_-$, where $E^{bc}_- = \{\begin{pmatrix} \xi \\ 0 \end{pmatrix} : \xi \in H^1(\mathbf{S}^1, \mathbb{C}^N)\}$.

Note that since the boundary condition is for the perturbed solution $\tilde{\mathcal{U}}$, and with respect to the original radial time r, we need to translate the boundary condition for $\tilde{\mathcal{U}}$ at $r = r_0$

into the boundary condition for the perturbation $\mathcal{U}$ at $\tau = \ln r_0 =: -L$ in the logarithmic rescaled time. In this section, write

$$\tilde{u}(r) = \tilde{u}(e^\tau) =: \tilde{\mathsf{u}}(\tau).$$

Similarly, $u(r) =: \mathsf{u}(\tau)$ and $u_*(r) =: \mathsf{u}_*(\tau)$. Observe that if $\tilde{\mathcal{U}}(r_0) \in E^{bc}_- + \tilde{d}$, then in τ, we have

$$\begin{pmatrix} \tilde{u}(r_0) \\ \tilde{u}_r(r_0) \end{pmatrix} = \begin{pmatrix} \tilde{\mathsf{u}}(-L) \\ e^L \tilde{\mathsf{u}}_\tau(-L) \end{pmatrix} \in E^{bc}_- + \tilde{d}.$$

That is,

$$\begin{pmatrix} \tilde{\mathsf{u}}(-L) \\ \tilde{\mathsf{u}}_\tau(-L) \end{pmatrix} \in E^{bc}_- + \begin{pmatrix} \tilde{d}_1 \\ 0 \end{pmatrix}.$$

Therefore, with $u = \tilde{u} - u_*$,

$$\begin{pmatrix} \mathsf{u}(-L) \\ \mathsf{u}_\tau(-L) \end{pmatrix} \in E^{bc}_- + \begin{pmatrix} \tilde{d}_1 \\ 0 \end{pmatrix} - \begin{pmatrix} \mathsf{u}_*(-L) \\ (\mathsf{u}_*)_\tau(-L) \end{pmatrix}. \tag{3.10}$$

In short, we have

$$\begin{pmatrix} \mathsf{u}(-L) \\ \mathsf{u}_\tau(-L) \end{pmatrix} \in E^{bc}_- + d. \tag{3.11}$$

where

$$d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} := \begin{pmatrix} \tilde{d}_1 - u_*(r_0) \\ -e^{-L}\partial_r u_*(r_0) \end{pmatrix}. \tag{3.12}$$

In the last equality, we use that

$$(\mathsf{u}_*)_\tau(-L) = \partial_r u_*(r_0) \cdot \frac{dr}{d\tau} = e^{-L}\partial_r u_*(r_0).$$

In case of Dirichlet and Neumann conditions, we choose different d's as in the followings:

(Dirichlet)' :

$$\mathcal{U}(-L) \in E^{bc}_- + d,$$

with

$$d = \begin{pmatrix} \tilde{d}_1 - u_*(r_0), \\ 0 \end{pmatrix}, \qquad E^{bc}_- = \{ \begin{pmatrix} 0 \\ \xi \end{pmatrix}, \xi \in L^2(\mathbf{S}^1, \mathbb{C}^N) \}.$$

(Neumann)' :

$$\mathcal{U}(-L) \in E_-^{bc} + d,$$

with

$$d = \begin{pmatrix} 0 \\ -e^{-L}\partial_r u_*(r_0)) \end{pmatrix}, \qquad E_-^{bc} = \{\begin{pmatrix} \xi \\ 0 \end{pmatrix}, \xi \in H^1(\mathbf{S}^1, \mathbb{C}^N)\}.$$

Motivated by Dirichlet conditions, we consider the following transversality assumption:

(T1): $E_-^{bc} \oplus E_-^{ss} \oplus E_-^{ker} = X.$

Remark 3.2.2. *Although we need to solve the boundary condition at $-L$, we still assume the transversality of the boundary space and the corresponding asymptotic subspaces at $\tau = -\infty$. The reason for this is two-fold: Firstly, it is easier to verify this transversality assumption since we have the explicit expression for the asymptotic subspaces; secondly, these conditions allow us to solve for any large L.*

Lemma 3.2.2. *The Dirichlet condition satisfies the assumption (T1).*

Proof. For $u = \sum_k u_k e^{ik\psi} \in H^1(\mathbb{S}^1, \mathbb{C}^N)$, $\partial_\psi u = \sum_k k u_k e^{ik\psi} \in L^2(\mathbb{S}^1, \mathbb{C}^N)$.

$$\begin{aligned} X \ni \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \sum_k u_k e^{ik\psi} \\ \sum_k v_k e^{ik\psi} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \sum_k (v_k + k u_k) e^{ik\psi} \end{pmatrix} + \begin{pmatrix} \sum_{k\neq 0} u_k e^{ik\psi} \\ \sum_{k\neq 0} -k u_k e^{ik\psi} \end{pmatrix} + \begin{pmatrix} u_0 \\ 0 \end{pmatrix} \\ &\in E_-^{bc} \oplus E_-^{ss} \oplus E_-^{ker} \end{aligned}$$

□

Remark 3.2.3. *Neumann boundary conditions do not satisfy $(T1)$, due to the fact that*

$$E_-^{ker} \cap E_-^{bc} = E_-^{ker}. \tag{3.13}$$

Recall that the linear boundary value space $E_-^{bc} \subset X$ is closed. Thus, the graph lemma (Lemma A.2.1) gives an equivalent formulation for boundary conditions that satisfy the

transversality assumption $(T1)$:

$$E_-^{bc} = \text{ graph } W = \{v_-^{cu} + W v_-^{cu}; v_-^{cu} = e_-^{uu} + e_-^{gker} \in E_-^{uu} \oplus E_-^{gker}\}.$$

On the other hand, Neumann boundary conditions satisfy the following modified transversality assumption:

(T2): $E_-^{bc} \oplus E_-^{ss} \oplus E_-^{gker} = X.$

Lemma 3.2.3. *Neumann conditions satisfy (T2).*

Proof.

$$\begin{aligned} X \ni \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \sum_k u_k e^{ik\psi} \\ \sum_k v_k e^{ik\psi} \end{pmatrix} \\ &= \left[\begin{pmatrix} u_0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sum_{k\neq 0}(u_k + \frac{1}{k}v_k)e^{ik\psi} \\ 0 \end{pmatrix}\right] + \begin{pmatrix} \sum_{k\neq 0} -\frac{1}{k}v_k e^{ik\psi} \\ \sum_{k\neq 0} v_k e^{ik\psi} \end{pmatrix} + \begin{pmatrix} 0 \\ v_0 \end{pmatrix} \\ &\in E_-^{bc} \oplus E_-^{ss} \oplus E_-^{gker} \end{aligned}$$

□

For sake of convenience, we introduce the following

Definition 3.2.2. *A boundary condition is $T1$ if it satisfies the transversality assumption (T1).*

3.3 In the Core Region

In this section, we study equation (3.7) in the core area. Consider the logarithmic time $s = \ln r$, or $r = e^s$, then $r \to 0$ corresponds to $s \to -\infty$. Thus we obtain the following system which is equivalent to (3.7) in the core region $-\infty < s \leq s_*$:

$$\begin{aligned} u_s &=: w \\ w_s &= -\partial_{\psi\psi} u - e^{2s} D^{-1}(\omega_* \partial_\psi u + f'(u_*(\tau,\psi))u) \\ &\quad -e^{2s} D^{-1}[(\omega - \omega_*)\partial_\psi(u_* + u) + f(u_* + u) - f(u) - f'(u_*)u] \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} u_s &= e^s u_r =: w \\ w_s &= (e^s v)_s = e^s v + e^{2s} v_r \end{aligned}$$

Thus, the linearized equation (3.8) becomes

$$\begin{aligned} u_s &= w \\ w_s &= -\partial_{\psi\psi} u - e^{2s} D^{-1}(\omega_* \partial_\psi u + f'(u_*(e^s, \psi))u). \end{aligned} \tag{3.15}$$

Remark 3.3.1. *For sake of consistency, instead of using $(u(r), v(r))$ and $(u(s), w(s))$ for different time scales, we use (u, v) for $(u(\tau), u_\tau(\tau))$, with the spatial time τ as defined as in [55]:*

$$\tau = \begin{cases} \log r & r \leq \bar{r}, \\ r & r \geq 2\bar{r} \end{cases}$$

and any smooth interpolation for $\tau \in (\bar{r}, 2\bar{r})$.

We can write the linear equation (3.15) as an abstract differential equation on $X = H^1(\mathbb{S}^1, \mathbb{C}^N) \times L^2(\mathbb{S}^1, \mathbb{C}^N)$ in the more compact form:

$$\mathcal{U}_\tau = \mathcal{A}_{core}(\tau)\mathcal{U}, \qquad \mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.16}$$

which can also be written as

$$\mathcal{U}_\tau = \mathcal{A}_\infty^- \mathcal{U} + e^{2\tau} \begin{pmatrix} 0 & 0 \\ -D^{-1}(\omega_* \partial_\psi + f'(u_*(e^\tau, \psi)))u & 0 \end{pmatrix} \mathcal{U}, \tag{3.17}$$

where

$$\mathcal{A}_\infty^- = \begin{pmatrix} 0 & 1 \\ -\partial_{\psi\psi} & 0 \end{pmatrix}.$$

Observe that the principal part has constant coefficients, and that the principal equation

$$\mathcal{U}_\tau = \mathcal{A}_\infty^- \mathcal{U}, \qquad \mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.18}$$

coincides with the asymptotic equation obtained by formally taking $\tau = -\infty$. Since $\partial_\psi|_{H^1(\mathbb{S}^1, \mathbb{C}^N)} : H^1(\mathbb{S}^1, \mathbb{C}^N) \to L^2(\mathbb{S}^1, \mathbb{C}^N)$ is bounded, the perturbation term decays to 0 exponentially in norm.

3.3.1 Exponential Dichotomies

First, we solve the asymptotic equation (3.18) in X by using Fourier series. Write $u = \sum u_k e^{ik\psi}$. The equation for u_k is

$$(u_k)_{\tau\tau} = k^2 u_k.$$

which has the solutions $u_{k,1} = e^{k\tau}$, $u_{k,2} = e^{-k\tau}$ for $k \neq 0$ and $u_{0,1} = 1$ and $u_{0,2} = \tau$ for $k = 0$. Therefore we can decompose X into the strong stable, strong unstable and center subspaces:

$$\begin{aligned} E^{ss}_- &= span\{(ue^{ik\psi}, -kue^{ik\psi}); k \neq 0, u \in \mathbb{C}^N\} =: \sum_{k\neq 0} E^{ss}_k, \\ E^{uu}_- &= span\{(ue^{ik\psi}, kue^{ik\psi}); k \neq 0, u \in \mathbb{C}^N\} =: \sum_{k\neq 0} E^{uu}_k, \\ E^{c}_- &= span\{(u, v) \in \mathbb{C}^N \times \mathbb{C}^N\} =: E^{ker}_- \oplus E^{gker}_-, \end{aligned} \tag{3.19}$$

with

$$E^{ker}_- = \{(u, 0); u \in \mathbb{C}^N\}, \qquad E^{gker}_- = \{(0, v); v \in \mathbb{C}^N\}. \tag{3.20}$$

Then solutions of (3.18) with initial conditions in E^{ss}_- exist and decay exponentially with rate 1 in forward time; solutions with initial conditions in E^{uu}_- exist and decay exponentially with rate 1 in backward time. Lastly, solutions with the initial conditions in E^c_- are given by $(u + \tau v, v)$. For $v \neq 0$, they grow linearly while they are constant in τ for $v = 0$. In conclusion, the asymptotic linear equation admits an infinite-dimensional strong stable subspace E^{ss}_-, an infinite-dimensional strong unstable subspace E^{uu}_- and a $2N$-dimensional center subspace E^c_-.

We return to the variational equation (3.16) and introduce the following definition of *exponential dichotomy*, which is similar to Definition 2.4.1 in the previous chapter.

Definition 3.3.1. *[48] Let J be $\mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$ and X a Banach space. A non-autonomous linear equation*

$$\frac{d}{d\tau}\mathcal{U} = \mathcal{A}(\tau)\mathcal{U}, \qquad \mathcal{U}(\cdot) \in X,$$

is said to have an exponential dichotomy on J if there exist positive constants K, η and a strongly continuous family of projections $P^s : J \to L(X)$ such that the following is true:

- *Stability: There exist operators $\Phi^s_-(s,\sigma)$ defined for $s \geq \sigma$ with s, $\sigma \in J$ and differentiable in (s,σ) for $s > \sigma$, such that $\Phi^s_-(s,\sigma)\mathcal{U}_0$ is a solution of (3.14) for each $\mathcal{U}_0 \in X$, which decays exponentially in forward time:*

$$|\Phi^s_-(s,\sigma)\mathcal{U}_0|_X \leq Ke^{-\eta|s-\sigma|}|\mathcal{U}_0|_X, \qquad \sigma \leq s, s, \sigma \in J.$$

- *Instability: There exist operators $\Phi^u_-(s,\sigma)$ for $s \leq \sigma$ with s, $\sigma \in J$ and differentiable in (s,σ) for $s < \sigma$, such that $\Phi^u(s,\sigma)\mathcal{U}_0$ is a solution of (3.14) for each $\mathcal{U}_0 \in X$, which decays exponentially in backward time:*

$$|\Phi^u_-(s,\sigma)\mathcal{U}_0|_X \leq Ke^{-\eta|s-\sigma|}|\mathcal{U}_0|_X, \qquad s \leq \sigma, s, \sigma \in J.$$

- *Compatibility: $\Phi^i(\sigma,\sigma) = P^i(\sigma)$ for $i = u$, s for all $\sigma \in J$, and projections are bounded in norm uniformly in σ.*

- *Invariance: The solutions $\Phi^s(s,\sigma)\mathcal{U}_0$ and $\Phi^u(s,\sigma)\mathcal{U}_0$ satisfy*

$$\Phi^s(s,\sigma)\mathcal{U}_0 \in \mathcal{R}(P^s(s)) \quad \textit{for all} \quad s \geq \sigma, s, \sigma \in J;$$

$$\Phi^u(s,\sigma)\mathcal{U}_0 \in \mathcal{N}(P^s(s)) \quad \textit{for all} \quad s \leq \sigma, s, \sigma \in J.$$

Note that the asymptotic equation (3.18) has an exponential dichotomy, since the projections onto the subspaces defined in (3.19) are bounded by Lemma A.3.2. The following theorem states that solutions of the perturbed linear equation (3.17) behave in a similar fashion as those of the asymptotic equation (3.18).

Theorem 3.3.1. *[47] For any fixed $s_* \in \mathbb{R}$, the following is true. There exist a constant $C > 0$ and strongly continuous families $P_-^{uu}(s)$, $P_-^{ss}(s)$ and $P_-^{c}(s)$ of complementary projections, all defined for $-\infty < s < s_*$, as well as linear evolution operator $\Phi_-^{ss}(s,\sigma)$, $\Phi_-^{uu}(\sigma,s)$ and $\Phi_-^{c}(s,\sigma)$ of equation (3.17) which are strongly continuous in (s,σ) for $-\infty < \sigma \leq s \leq s_*$ and strongly differentiable in (s,σ) for $-\infty < \sigma < s < s_*$ such that the following is true :*

- *Compatibility: $\Phi_-^{i}(\sigma,\sigma) = P_-^{i}(\sigma)$ for all σ and $i = uu$, ss and c, and the projections are bounded in norm in s uniformly.*

- *Stability: For any $\mathcal{U}_0 \in X$, $\Phi_-^{uu}(s,\sigma)\mathcal{U}_0$ is a solution of (3.14) which decays exponentially in backward time:*

$$|\Phi_-^{uu}(s,\sigma)\mathcal{U}_0|_X \leq Ce^{-|s-\sigma|}|\mathcal{U}_0|_X, \qquad s \leq \sigma \leq s_*.$$

- *Instability: For any $\mathcal{U}_0 \in X$, $\Phi_-^{ss}(s,\sigma)\mathcal{U}_0$ is a solution of (3.14) which decays exponentially in forward time:*

$$|\Phi_-^{ss}(s,\sigma)\mathcal{U}_0|_X \leq Ce^{-|s-\sigma|}|\mathcal{U}_0|_X, \qquad \sigma \leq s \leq s_*.$$

- *Central directions: The range of the center projection $P_-^{c}(s)$ is finite-dimensional, and $\dim(\mathcal{R}(P_-^{c}(s)) = 2N$. For any $\mathcal{U}_0 \in \mathcal{R}(P_-^{c}(s))$, $\Phi_-^{c}(s,\sigma)\mathcal{U}_0$ is a solution that grows at most linearly.*

$$|\Phi_-^{c}(s,\sigma)\mathcal{U}_0|_X \leq C(1+|s-\sigma|)|\mathcal{U}_0|_X, \qquad s,\sigma \leq s_*.$$

  Moreover, we can decompose

$$\begin{aligned} P_-^{c}(s) &= P_-^{ker}(s) + P_-^{gker}(s), \\ \|P_-^{ker}(s)\| &+ \|P_-^{gker}(s)\| \leq C(|s|+1). \end{aligned}$$

For any $\mathcal{U}_0 \in X$, let $u_-^{ker} = P_-^{ker}(\sigma)\mathcal{U}_0$ and $u_-^{gker} = P_-^{gker}(\sigma)\mathcal{U}_0$, then

$$\begin{aligned} |\Phi_-^c(s,\sigma)u_-^{ker} - P_{-\infty}^{ker}u_-^{ker}|_X &\leq O(e^{-2\min\{|s|,|\sigma|\}})|u_-^{ker}|_X, \\ |\Phi_-^c(s,\sigma)u_-^{gker} - e^{A_{-\infty}^c(s-\sigma)}P_{-\infty}^{gker}u_-^{gker}|_X &\leq O(|s-\sigma|e^{-2\min\{|s|,|\sigma|\}})|u_-^{gker}|_X. \end{aligned}$$

Proof. For the proof, please see [47]. □

In Theorem 3.3.1, for any fixed σ the error terms for the evolution $\Phi_-^c(s,\sigma)$ of the center components are of order $O(u_-^{ker})$ and $O(|s|u_-^{gker})$, respectively. The following lemma shows that the error term actually decay exponentially when both s and σ are large.

Lemma 3.3.1. *Consider a non-autonomous linear system*

$$\frac{d}{dt}\mathcal{U} = (A + B(t))\mathcal{U} \tag{3.21}$$

defined on a Banach space X, where the linear operator $A : X \to X$ satisfies

$$\Re(spec(A)) \subseteq (-\infty, -1+\delta) \cup \{0\} \cup (1-\delta, \infty)$$

for some positive $\delta \ll 1$ and has the following decomposition

$$A = A|_{E^{ss}} \oplus A|_{E^c} \oplus A|_{E^{uu}},$$

in which $X = E^{ss} \oplus E^c \oplus E^{uu}$ is the spectral decomposition with respect to A with E^{ss}, E^c and E^{uu} the stable, center, and unstable generalized eigenspaces, respectively, and $B(t)$ is a small perturbation that decays exponentially, that is,

$$\|B(t)\| \leq Ce^{-2|t|},$$

uniformly for t large. Then there exist $L_* > 0$ and $0 < \epsilon \ll 1$, such that for any $L \geq L_*$ and $t \leq s \leq -L$, the following is true: There exists a unique evolution $\Phi_-^c(t,s)$ of (3.21) on the center subspace with

$$\Phi_-^c(t,s) = e^{A^c(t-s)} + O(e^{(2-\epsilon)t+\epsilon s} + e^{t+s}) = e^{A^c(t-s)} + O(e^{-L}). \tag{3.22}$$

Proof. The proof is an application of Banach fixed point theorem. Write (3.21) as

$$\frac{d}{dt}\mathcal{U} = A\mathcal{U} + B(t)\mathcal{U}.$$

$\Phi^c_-(t,s)$ can be obtained as a solution to

$$\begin{aligned}\Phi^c_-(t,s) = \quad & e^{A^c(t-s)} + \int_s^t e^{A^{uu}(t-\tau)}B(\tau)\Phi^c_-(\tau,s),s)d\tau + \\ & \int_{-\infty}^t e^{(A^c+A^{ss})(t-\tau)}B(\tau)\Phi^c_-(\tau,s),s)d\tau.\end{aligned} \tag{3.23}$$

We claim that the sum of two integral terms defines a contraction in $\Phi^c_-(t,s)$ for $t \le s$ and t and s uniformly large. First, we choose a constant $0 < \epsilon \ll 1$ such that

$$|e^{A^c(t-s)}|_X \le Ce^{\epsilon|t-s|}, \qquad t \le s \le -L_*.$$

Then we introduce the following norm for $\Phi^c_-(\cdot,s)$:

$$\|\Phi^c_-(\cdot,s)\|_\epsilon = \sup_{t\le s\le -L_*} e^{\epsilon(t-s)}|\Phi^c_-(t,s)|_X.$$

The estimates of the two integral terms yield

$$\begin{aligned}& \|\int_s^\cdot e^{A^{uu}(\cdot-\tau)}B(\tau)\Phi^c_-(\tau,s)d\tau\|_\epsilon \\ = \quad & \sup_{t\le s\le -L_*} e^{\epsilon(t-s)}|\int_s^t e^{A^{uu}(t-\tau)}B(\tau)\Phi^c_-(\tau,s)d\tau|_X \\ = \quad & \sup_{t\le s\le -L_*} |\int_s^t e^{A^{uu}(t-\tau)}B(\tau)e^{\epsilon(t-\tau)}e^{\epsilon(\tau-s)}\Phi^c_-(\tau,s)d\tau|_X \\ \le \quad & \sup_{t\le s\le -L_*} |\int_s^t e^{A^{uu}(t-\tau)}B(\tau)e^{\epsilon(t-\tau)}d\tau| \cdot \sup_{t\le s\le -L_*}\sup_{t\le\tau\le s} e^{\epsilon(\tau-s)}|\Phi^c_-(\tau,s)|_X \\ \le \quad & \sup_{t\le s\le -L_*} C|\int_s^t e^{t-s}e^{2\tau}e^{\epsilon(t-\tau)}d\tau| \cdot \|\Phi^c_-(\cdot,s)\|_\epsilon \\ \le \quad & \sup_{t\le s\le -L_*} C(e^{3t-s} - e^{(1+\epsilon)t+(1-\epsilon)s})\|\Phi^c_-(\cdot,s)\|_\epsilon \\ \le \quad & Ce^{-2L_*}\|\Phi^c_-(\cdot,s)\|_\epsilon,\end{aligned}$$

in which C is independent of L_*, and write $A^{cs} := A^c|_{E^c} \oplus A^{ss}|_{E^{ss}}$,

$$\begin{aligned}
& \| \int_{\cdot}^{-\infty} e^{A^{cs}(\cdot-\tau)} B(\tau)\Phi_-^c(\tau,s)d\tau\|_\epsilon \\
= & \sup_{t\le s\le -L_*} e^{\epsilon(t-s)} | \int_t^{-\infty} e^{A^{cs}(t-\tau)} B(\tau)\Phi_-^c(\tau,s)d\tau|_X \\
= & \sup_{t\le s\le -L_*} | \int_t^{-\infty} e^{A^{cs}(t-\tau)} B(\tau) e^{\epsilon(t-\tau)} e^{\epsilon(\tau-s)} \Phi_-^c(\tau,s)d\tau|_X \\
\le & \sup_{t\le s\le -L_*} | \int_t^{-\infty} e^{A^{cs}(t-\tau)} e^{\epsilon(t-\tau)} B(\tau)d\tau| \cdot \sup_{t\le s\le -L_*} \sup_{t\le\tau\le s} e^{\epsilon(\tau-s)} |\Phi_-^c(\tau,s)|_X \\
\le & \sup_{t\le s\le -L_*} C| \int_t^{-\infty} e^{2\tau} d\tau| \cdot \|\Phi_-^c(\cdot,s)\|_\epsilon \\
\le & \sup_{t\le s\le -L_*} Ce^{2t} \|\Phi_-^c(\cdot,s)\|_\epsilon \\
\le & Ce^{-2L_*} \|\Phi_-^c(\cdot,s)\|_\epsilon,
\end{aligned}$$

in which C is independent of L_*.

Choose $L_* \gg 1$ such that $Ce^{-2L_*} < \frac{\kappa}{2}$, for $0 < \kappa < 1$. Hence the right hand side of the equation (3.23) defines a uniform contraction on $(C^0((-\infty, -L_*), X), \|\cdot\|_\epsilon)$. By Banach fixed point theorem, there exists a unique solution $\Phi_-^c(\cdot, s)$ which satisfies (3.23).

Therefore, for $t \le s \le -L_*$,

$$|\Phi_-^c(t,s)|_X \le (1-\kappa)|e^{A^c(t-s)}|_X \le Ce^{\epsilon|t-s|}.$$

Then

$$\begin{aligned}
& \left| \int_s^t e^{A^{uu}(t-\tau)} B(\tau)\Phi_-^c(\tau,s)d\tau \right|_X \le C \left| \int_s^t e^{-(\tau-t)} e^{2\tau} e^{\epsilon(s-\tau)} d\tau \right|_X \\
= & \quad Ce^{t+\epsilon s} \int_s^t e^{(1-\epsilon)\tau} d\tau = C(e^{(2-\epsilon)t+\epsilon s} + e^{t+s}),
\end{aligned}$$

and

$$\left| \int_t^{-\infty} e^{A^{cs}(t-\tau)} B(\tau)\Phi_-^c(\tau,s)d\tau \right|_X \le C \int_t^{-\infty} e^{\epsilon(t-\tau)} e^{2\tau} e^{\epsilon(\tau-s)} d\tau \le Ce^{2t-\epsilon(s-t)},$$

which yield the estimate (3.22).

□

3.4 Exponential Dichotomies in the Farfield Region

In this section, we briefly describe the problem of existence of exponential dichotomies in the farfield and the related result given in [47]. Recall that in the farfield the governing equation is

$$\begin{aligned} u_r &= v \\ v_r &= -\frac{1}{r}v - \frac{1}{r^2}\partial_{\psi\psi}u - D^{-1}[\omega_*\partial_\psi u + f'(u_*(r,\psi))u] \\ &\quad -D^{-1}[(\omega-\omega_*)\partial_\psi(u_*+u) + f(u_*+u) - f(u) - f'(u_*)u] \end{aligned} \tag{3.24}$$

and the linearization of the governing equation (3.24) is:

$$\begin{aligned} u_r &= v \\ v_r &= -\frac{1}{r}v - \frac{1}{r^2}\partial_{\psi\psi}u - D^{-1}[\omega_*\partial_\psi u + f'(u_*(r,\psi))u]. \end{aligned} \tag{3.25}$$

Casting (3.25) in the Archimedean coordinates (r,ϑ), we have

$$\begin{aligned} u_r &= -(k_* + \theta'(r))\partial_\vartheta u + v \\ v_r &= -(k_* + \theta'(r))\partial_\vartheta v - \frac{1}{r}v - \frac{1}{r^2}\partial_{\vartheta\vartheta}u + D^{-1}(\omega_*\partial_\vartheta u + f'(u_*(r,\vartheta - k_* r - \theta(r))u) \end{aligned} \tag{3.26}$$

on the Banach space $X = H^1(\mathbb{S}^1,\mathbb{C}^N)\times L^2(\mathbb{S}^1,\mathbb{C}^N) \ni \mathcal{U} = (u,v)$.

Since the Archimedean spiral u_* is approximated by the radial wave train $u_\infty = u_\infty(\vartheta)$ in the farfield, we are interested in comparing solutions of (3.24) with solutions of the asymptotic equation

$$\begin{aligned} u_r &= v \\ v_r &= -D^{-1}[\omega_*\partial_\psi u + f'(u_\infty(k_* r + \psi))u]. \end{aligned} \tag{3.27}$$

for the wave train u_∞ with $\mathcal{U} = (u, v) \in H^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{C}^N) \times L^2(\mathbb{S}^1, \mathbb{C}^N)$. Again, we introduce the Archimedean coordinate $\vartheta = k_* r + \theta(r) + \psi$, then the equation (3.27) becomes

$$\begin{aligned} u_r &= -k_* \partial_\vartheta u + v \\ v_r &= -k_* \partial_\vartheta v - D^{-1}[\omega_* \partial_\psi u + f'(u_\infty(\vartheta))u]. \end{aligned} \tag{3.28}$$

Then we have the following theorem from [47]:

Theorem 3.4.1. *Assume that the asymptotic equation (3.28) has an exponential dichotomy. For every $\epsilon > 0$, there is an $R > 0$ such that the equation (3.26) has an exponential dichotomy on $[R, \infty)$ so that the projections of (3.28) and (3.26) are ϵ-close in norm for $r \geq R$.*

3.5 The Nonlinear Equation in the Core Region

3.5.1 Reduction

We now turn to the nonlinear equation:

$$\mathcal{U}_\tau = \mathcal{A}_{core}(\tau)\mathcal{U} + e^{2\tau}\mathcal{G}(\mathcal{U}(\tau), \omega), \qquad \mathcal{U} = (u, v) \in H^1(\mathbb{S}^1, \mathbb{C}^N) \times L^2(\mathbb{S}^1, \mathbb{C}^N) \tag{3.29}$$

in the core region, where

$$\mathcal{A}_{core}(\tau) = \begin{pmatrix} 0 & 1 \\ -\partial_{\psi\psi} - D^{-1}e^{2\tau}(\omega_* \partial_\psi + f'(u_*(r, \psi))) & 0 \end{pmatrix}$$

and

$$\mathcal{G}(\mathcal{U}, \omega) = \begin{pmatrix} 0 \\ -D^{-1}[(\omega - \omega_*)\partial_\psi(u_* + u) + f(u_* + u) - f(u_*) - f'(u_*)u] \end{pmatrix}$$

In order to solve the nonlinear equation, we describe its solutions via the variation-of-constants formula. Theorem 3.4.1 states that the farfield exponential dichotomies can only exist for some $R_* \gg 1$. On the other hand, exponential dichotomies exist in the core region $(-\infty, R_*]$ for any $R_* \in \mathbb{R}$, therefore we have the decomposition of $X =$

$H^1(\mathbb{S}^1,\mathbb{C}^N)\times L^2(\mathbb{S}^1,\mathbb{C}^N)$ near $\mathcal{U}_*(R_*)$ as:

$$X = Y_-^c \oplus Y_-^u \oplus Y_-^s = Y_-^{ker} \oplus Y_-^{gker} \oplus Y_-^u \oplus Y_-^s \tag{3.30}$$

Here Y_-^u, Y_-^s, Y_-^{ker} and Y_-^{gker} are subspaces of the tangent space $T_{\mathcal{U}_*(R_*)}X$, consisting of initial conditions that lead to exponentially decaying solutions in backward time, exponentially decaying solutions in forward time, almost constant solutions and linearly growing solutions over $\tau\in(-\infty,R_*]$, respectively. Moreover, Y_-^u, Y_-^s, Y_-^{ker} and Y_-^{gker} are given by the ranges of projections P_-^{uu}, $P_-^{ss}(R_*)$, P_-^{ker} and P_-^{gker} of a core region exponential dichotomy, respectively. We prescribe the strong unstable component w_-^{uu} and both center components w_-^{ker} and w_-^{gker} at $\tau=R_*$ and the strong stable component w_-^{ss} at $\tau=-L$.

In conclusion, we consider the variation-of-constants formula over the interval $[-L,R_*]$:

$$\begin{aligned}\mathcal{U}(\tau) &= \Phi_-^c(\tau,R_*)\begin{pmatrix} w_-^{ker}\\ w_-^{gker}\end{pmatrix} + \Phi_-^{ss}(\tau,-L)w_-^{ss} + \Phi_-^{uu}(\tau,R_*)w_-^{uu} \\ &\quad + \int_{-L}^{\tau}[\Phi_-^{ss}(\tau,s)+\Phi_-^c(\tau,s)]e^{2s}\mathcal{G}(\mathcal{U}(s),s)ds + \int_{R_*}^{\tau}\Phi_-^{uu}(\tau,s)e^{2s}\mathcal{G}(\mathcal{U}(s),s)ds.\end{aligned} \tag{3.31}$$

with

$$w_-^{ss}\in E_-^{ss},\qquad w_-^{uu}\in Y_-^u,\qquad \begin{pmatrix} w_-^{ker}\\ w_-^{gker}\end{pmatrix}\in Y_-^{ker}\oplus Y_-^{gker}.$$

Observe that the variation-of-constants formula (3.31) is a fixed-point equation. The obstacle that prevents us from applying the Implicit Function Theorem is the presence of the parameter w_-^{gker}, whose evolution gives linearly growing solution with respect to L.

In order to resolve this difficulty, we assume temporarily that the nonlinearity $\mathcal{G}(\mathcal{U}(\tau),\omega,\tau)$ depends only on τ, i.e. $\mathcal{G}(\mathcal{U}(\tau),\omega,\tau)=\mathcal{G}(\tau)$ where $\mathcal{G}\in\mathcal{C}^0([-L,R_*],X)$ so that

$$\|\mathcal{G}\|_\infty := \sup_{\tau\in[-L,R_*]}\|\mathcal{G}(\tau)\|_X<\infty. \tag{3.32}$$

Equation (3.31) then gives solutions to $\mathcal{U}_\tau = A_{core}(\tau)\mathcal{U}+\mathcal{G}(\tau)$. Evaluating $\mathcal{U}$ at $\tau=-L$ and substituting into the boundary condition imposed at $\tau=-L$, we can solve for both

w_-^{ss} and w_-^{gker} in terms of w_-^{ker}, w_-^{uu}, $\mathcal{G}$. Upon solving, we shall see that $\mathcal{U}$ is uniformly bounded in w_-^{ker}, w_-^{uu}, $\mathcal{G}$ and d. Now if $\mathcal{G}(\mathcal{U},\tau)$ is C^1-small with respect to $\mathcal{U}$ at $\mathcal{U}=0$, then we can substitute back $\mathcal{G}(\mathcal{U},\tau)$ and invoke the Implicit Function Theorem to solve the full nonlinear equation.

3.5.2 Solving Boundary Condition

Under the assumption (3.32) , the general solution (3.31) for the perturbation reads:

$$\begin{aligned}\mathcal{U}(\tau) &= \Phi_-^c(\tau,R_*)\begin{pmatrix} w_-^{ker} \\ w_-^{gker}\end{pmatrix} + \Phi_-^{ss}(\tau,-L)w_-^{ss} + \Phi_-^{uu}(\tau,R_*)w_-^{uu} \\ &\quad + \int_{-L}^{\tau}[\Phi_-^{ss}(\tau,s)+\Phi_-^c(\tau,s)]e^{2s}\mathcal{G}(s)ds + \int_{R_*}^{\tau}\Phi_-^{uu}(\tau,s)e^{2s}\mathcal{G}(s)ds.\end{aligned} \tag{3.33}$$

Taking $\tau=-L$ in the variation-of-constants equation (3.33), we obtain

$$\begin{aligned}\mathcal{U}(-L) &= \Phi_-^c(-L,R_*)\begin{pmatrix} w_-^{ker} \\ w_-^{gker}\end{pmatrix} + P_-^{ss}(-L)w_-^{ss} + \Phi_-^{uu}(-L,R_*)w_-^{uu} \\ &\quad + \int_{R_*}^{-L}\Phi_-^{uu}(-L,s)e^{2s}\mathcal{G}(s)ds.\end{aligned}$$

We expand $\Phi_-^c(-L,R_*)$ in terms of L: Consider that the evolution $\Phi_-^c(-L_*,R_*)$ is an isomorphism Υ between $\mathbb{C}^{2N}$'s given by

$$\Upsilon : E_-^c(R_*)\to E_-^c(-L_*), \qquad \begin{pmatrix} w_-^{ker} \\ w_-^{gker}\end{pmatrix} \mapsto \begin{pmatrix} w_-^{\tilde{k}er} \\ w_-^{\tilde{g}ker}\end{pmatrix} = \Phi_-^c(-L,R_*)\begin{pmatrix} w_-^{ker} \\ w_-^{gker}\end{pmatrix}$$

with $E_-^{(g)ker}(s) := Rg(P_-^{(g)ker}(s))$, $E_-^c(s) := Rg(P_-^c(s))$, and $E_-^c(s) = E_-^{ker}(s)\oplus E_-^{gker}(s)$.

Since Υ is an isomorphism, choose the coordinates in $Rg(P_-^c(-L_*))$ such that we have the following matrix representation of Υ:

$$\begin{pmatrix} w_-^{\tilde{k}er} \\ w_-^{\tilde{g}ker}\end{pmatrix} = \begin{pmatrix} \gamma_{11} & 0 \\ 0 & \gamma_{22}\end{pmatrix}\begin{pmatrix} w_-^{ker} \\ w_-^{gker}\end{pmatrix} \tag{3.34}$$

with both γ_{11} and γ_{22} invertible. From Lemma 3.3.1, we have

$$\Phi_-^c(-L,-L_*)\begin{pmatrix} w_-^{\tilde{k}er} \\ w_-^{\tilde{g}ker}\end{pmatrix} = e^{A_\infty^c(-L+L_*)}\begin{pmatrix} w_-^{\tilde{k}er} \\ w_-^{\tilde{g}ker}\end{pmatrix} + \begin{pmatrix} w_-^{\tilde{k}er} \\ w_-^{\tilde{g}ker}\end{pmatrix}O(e^{-L}),$$

for all $L \geq L_*$. Then

$$\Phi_-^c(-L, R_*) \begin{pmatrix} w_-^{ker} \\ w_-^{gker} \end{pmatrix} = \Phi_-^c(-L, -L_*) \circ \Phi_-^c(-L_*, R_*) \begin{pmatrix} w_-^{ker} \\ w_-^{gker} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{11} w_-^{ker} + (-L + L_*)\gamma_{22} w_-^{gker} \\ \gamma_{22} w_-^{gker} \end{pmatrix} + \begin{pmatrix} w_-^{ker} \\ w_-^{gker} \end{pmatrix} O(e^{-L})$$

Observe that setting $t = s = -L$ in (3.22) yields $P_-^c(-L) = P_-^c + O(e^{-L})$, $L \geq L_*$. Thus,

$$\begin{pmatrix} \gamma_{11} w_-^{ker} + (-L + L_*)\gamma_{22} w_-^{gker} \\ \gamma_{22} w_-^{gker} \end{pmatrix} = P_-^c \begin{pmatrix} \gamma_{11} w_-^{ker} + (-L + L_*)\gamma_{22} w_-^{gker} \\ \gamma_{22} w_-^{gker} \end{pmatrix} + O(e^{-L}).$$

Therefore, up to exponentially small terms, $\Phi_-^c(-L, R_*) \begin{pmatrix} w_-^{ker} \\ w_-^{gker} \end{pmatrix}$ is in $E_-^c = E_-^{ker} \oplus E_-^{gker}$.

In conclusion, we have

$$\begin{aligned} \mathcal{U}(-L) &= \Phi_-^{uu}(-L, R_*) w_-^{uu} + P_-^{ss}(-L) w_-^{ss} + \int_{R_*}^{-L} \Phi_-^{uu}(-L, \tau) e^{2\tau} \mathcal{G}(\tau) d\tau \quad (3.35) \\ &+ \begin{pmatrix} \gamma_{11} w_-^{ker} + (-L + L_*)\gamma_{22} w_-^{gker} \\ \gamma_{22} w_-^{gker} \end{pmatrix} + \begin{pmatrix} w_-^{ker} \\ w_-^{gker} \end{pmatrix} O(e^{-L}) \end{aligned}$$

Remark 3.5.1. *We emphasize that the components of*

$$\begin{pmatrix} \gamma_{11} w_-^{ker} + (-L + L_*)\gamma_{22} w_-^{gker} \\ \gamma_{22} w_-^{gker} \end{pmatrix}$$

are in $E_-^{ker} \oplus E_-^{gker}$.

Theorem 3.5.1. *Let the linear operator defined on the right hand side of* (3.35) *be*

$$\tilde{\mathcal{L}}(w_-^{ker}, w_-^{gker}, w_-^{ss}, w_-^{uu}, \mathcal{G})(-L).$$

There are constants C *and* $L_* > 0$ *such that the following is true for every* $L \geq L_*$*. There are linear operators* $W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)$ *and* $W^{ss}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)$ *such that*

$$\begin{aligned} w_-^{gker} &= W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d), \\ w_-^{ss} &= W^{ss}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d), \end{aligned}$$

if and only if

$$\mathcal{L}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d) := \tilde{\mathcal{L}}(w_-^{ker}, W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d), W^{ss}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d), w_-^{uu}, \mathcal{G})$$

satisfies either $T1$ boundary condition or Neumann boundary condition. Furthermore, we have the estimate

$$\begin{aligned} |W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)| &\leq \frac{C}{L}(|w_-^{ker}| + e^{-L}|w_-^{uu}| + e^{-L}\|\mathcal{G}\|_\infty + |d|), \quad (3.36) \\ |W^{ss}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)| &\leq C(e^{-L}|w_-^{ker}| + e^{-L}|w_-^{uu}| + e^{-L}\|\mathcal{G}\|_\infty + |d|). \quad (3.37) \end{aligned}$$

for $T1$ boundary conditions. For Neumann boundary conditions,

$$\begin{aligned} |W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G})| &\leq Ce^{-L}(|w_-^{ker}| + |w_-^{uu}| + \|\mathcal{G}\|_\infty + |\partial_r u_*(r_0)|), \quad (3.38) \\ |W^{ss}(w_-^{ker}, w_-^{uu}, \mathcal{G})| &\leq Ce^{-L}(|w_-^{ker}| + |w_-^{uu}| + \|\mathcal{G}\|_\infty + |\partial_r u_*(r_0)|). \quad (3.39) \end{aligned}$$

Moreover, in both cases, if $\mathcal{U}$ satisfies either $T1$ or Neumann boundary condition and the variation-of-constants formula for $\mathcal{G} = \mathcal{G}(\tau)$, then we have the estimate

$$\|\mathcal{U}\|_\infty := \sup_{\tau\in[-L,R_*]} \|\mathcal{U}(\tau)\|_{X_\tau} = \|\mathcal{L}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)\|_\infty \leq C(|w_-^{ker}|+|w_-^{uu}|+\|\mathcal{G}\|_\infty+|d|)). \quad (3.40)$$

Proof. We prove the theorem for $T1$ boundary conditions first. Any element that satisfies the boundary condition is of the form $d + e_-^{bc}$, with

$$e_-^{bc} = e^{uu} + e^{ker} + e^{ss} + e^{gker} \in E_-^{uu} \oplus E_-^{ker} \oplus E_-^{ss} \oplus E_-^{gker}.$$

By Lemma A.2.1, we have

$$e^{ss} + e^{ker} = W(e^{uu} + e^{gker})$$

for some linear bounded W. Therefore,

$$\begin{aligned} d + e^{bc} &= (P_-^{uu}d + e^{uu}) + (P_-^{ss}d + e^{ss}) + (P_-^{gker}d + e^{gker}) + (P_-^{ker}d + e^{ker}) \\ &= (P_-^{uu}d + e^{uu}) + (P_-^{ss}d + P_-^{ss}W(e^{uu} + e^{gker})) \quad (3.41) \\ &\quad +(P_-^{gker}d + e^{gker}) + (P_-^{ker}d + P_-^{ker}W(e^{uu} + e^{gker})). \end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{U}(-L) &= \Phi_-^{uu}(-L,R_*)w_-^{uu}+P_-^{ss}(-L)w_-^{ss}+\int_{R_*}^{-L}\Phi_-^{uu}(-L,\tau)e^{2\tau}\mathcal{G}(\tau)d\tau \quad (3.42)\\ &+\begin{pmatrix}\gamma_{11}w_-^{ker}+(-L+L_*)\gamma_{22}w_-^{gker}\\ \gamma_{22}w_-^{gker}\end{pmatrix}+\begin{pmatrix}w_-^{ker}\\ w_-^{gker}\end{pmatrix}O(e^{-L}).\end{aligned}$$

We have the following estimates for the summands in the above expression:

$$P_-^{ss}(-L)w_-^{ss}=w_-^{ss}+(P_-^{ss}(-L)-P_-^{ss})w_-^{ss}=(1+O(e^{-2L}))w_-^{ss}, \quad (3.43)$$

where we refer to Corollary 2 in [40] for the second equality;

$$\Phi_-^{uu}(-L,R_*)w_-^{uu}=O(e^{-L})w_-^{uu}; \quad (3.44)$$

$$\|\int_{R_*}^{-L}\Phi_-^{uu}(-L,\tau)e^{2\tau}\mathcal{G}(\tau)d\tau\|\le C\int_{R_*}^{-L}e^{-L+\tau}d\tau\cdot\|\mathcal{G}\|_\infty\le Ce^{-L}\|\mathcal{G}\|_\infty. \quad (3.45)$$

Comparing the strong unstable component in (3.41) and (3.42), we obtain

$$e^{uu}=O(e^{-L})w_-^{uu}+O(e^{-L})\mathcal{G}+O(e^{-2L})w_-^{ss}+\begin{pmatrix}w_-^{ker}\\ w_-^{gker}\end{pmatrix}O(e^{-L})-P_-^{uu}d, \quad (3.46)$$

Similarly,

$$\begin{aligned}e^{ss} &= P_-^{ss}W(e^{uu}+e^{gker}) \quad (3.47)\\ &= O(e^{-3L})w_-^{uu}+O(e^{-3L})\mathcal{G}+(1+O(e^{-2L}))w_-^{ss}+\begin{pmatrix}w_-^{ker}\\ w_-^{gker}\end{pmatrix}O(e^{-L})-P_-^{ss}d\\ e^{gker} &= O(e^{-3L})w_-^{uu}+O(e^{-3L})\mathcal{G}+O(e^{-2L})w_-^{ss}+\gamma_{22}w_-^{gker}\mathrm{id}_{E_-^{gker}} \quad (3.48)\\ &+\begin{pmatrix}w_-^{ker}\\ w_-^{gker}\end{pmatrix}O(e^{-L})-P_-^{gker}d.\\ e^{ker} &= P_-^{ker}W(e^{uu}+e^{gker}) \quad (3.49)\\ &= O(e^{-3L})w_-^{uu}+O(e^{-3L})\mathcal{G}+O(e^{-2L})w_-^{ss}+\gamma_{22}w_-^{gker}\mathrm{id}_{E_-^{gker}}+\\ &[\gamma_{11}w_-^{ker}+(-L+L_*)\gamma_{22}w_-^{gker}]\mathrm{id}_{E_-^{ker}}+\begin{pmatrix}w_-^{ker}\\ w_-^{gker}\end{pmatrix}O(e^{-L})-P_-^{ker}d.\end{aligned}$$

In the following, we would like to substitute (3.46) and (3.48) into (3.49) and solve for w_-^{gker} in terms of w_-^{ker}, w_-^{uu}, $\mathcal{G}$ and d.

Define the linear operator $T_L : E_-^{gker} \cong \mathbb{C}^N \to X$ by

$$T_L w_-^{gker} := (-(L - L_*)\gamma_{22}\mathrm{id}_{E_-^{ker}} + P_-^{ker} W \gamma_{22}\mathrm{id}_{E_-^{gker}} + O(e^{-L}))w_-^{gker}. \tag{3.50}$$

Lemma 3.5.1. *There are constants $\tilde{L} > L_*$ and $C > 0$ such that T_L is invertible for any $L \geq \tilde{L}$, and the inverse is uniformly bounded with $\|T_L^{-1}\| \leq \frac{C}{L}$.*

Proof. Write

$$T_L = L\left[-\left(1 - \frac{L_*}{L}\right)\gamma_{22}\mathrm{id}_{E_-^{ker}} + \frac{1}{L}P_-^{ker} W \gamma_{22}\mathrm{id}_{E_-^{gker}} + O\left(\frac{1}{L}e^{-L}\right)\right].$$

Since $\mathrm{id}_{E_-^{ker}}$ is invertible, so is $\mathrm{id}_{E_-^{ker}}(1 + O(\frac{1}{L})) + O(\frac{1}{L})$for large L. Then T_L is an isomorphism and $\|T_L^{-1}\| = \frac{1}{L}(\mathrm{id}_{E_-^{ker}}(1 + O(\frac{1}{L})) + O(\frac{1}{L}))^{-1} = O(\frac{1}{L})$ uniformly for large L. □

Therefore,

$$\begin{aligned} w_-^{gker} &= T_L^{-1}(\gamma_{11} w_-^{ker} + O(e^{-L})\mathcal{R}_2(w_-^{ker}, w_-^{uu}, \mathcal{G}) - P_-^{ker} d) \\ &=: W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d) \end{aligned} \tag{3.51}$$

with

$$|w_-^{gker}| \leq \frac{C}{L}(|w_-^{ker}| + e^{-L}|w_-^{uu}| + e^{-L}\|\mathcal{G}\|_\infty + |d|). \tag{3.52}$$

Solving (3.47) for the strong stable component w_-^{ss} yields

$$\begin{aligned} w_-^{ss} &= (1 + O(e^{-2L}))(P_-^{ss} d + P_-^{ss} W(e^{uu} + e^{gker}) + O(e^{-3L})w_-^{uu} + O(e^{-L})\mathcal{G} \\ &\quad + O(e^{-L})w_-^{ker} + O(e^{-L})W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)) \qquad (3.53) \\ &= (1 + O(e^{-2L}))((P_-^{ss} - P_-^{ss} W P_-^{uu})d + O(e^{-L})d + O(e^{-L})\mathcal{R}_4(w_-^{ker}, w_-^{uu}, \mathcal{G})). \\ &=: W^{ss}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d) \end{aligned}$$

with

$$|w_-^{ss}| \leq C(e^{-L}|w_-^{ker}| + e^{-L}|w_-^{uu}| + e^{-L}\|\mathcal{G}\|_\infty + |d|). \tag{3.54}$$

Then we have the desired estimates (3.36) and (3.37).

Thus, we obtain the following

$$\begin{aligned}
\mathcal{U}(\tau) &= \Phi_-^c(\tau, R_*) \begin{pmatrix} w_-^{ker} \\ W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}(\tau), d) \end{pmatrix} \\
&+\Phi_-^{ss}(\tau, -L) W^{ss}(w_-^{ker}, w_-^{uu}, \mathcal{G}(\tau), d) + \int_{R_*}^{\tau} \Phi_-^{uu}(\tau, s) e^{2s} \mathcal{G}(s) ds \\
&+ \int_{-L}^{\tau} [\Phi_-^{ss}(\tau, s) + \Phi_-^c(\tau, s)] e^{2s} \mathcal{G}(s) ds + \Phi_-^{uu}(\tau, R_*) w_-^{uu},
\end{aligned}$$

in which

$$\begin{aligned}
& |\Phi_-^c(\tau, R_*) \begin{pmatrix} 0 \\ W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d) \end{pmatrix}| \qquad (3.55) \\
\leq\ & (\|e^{A_{-\infty}^c(\tau - R_*)} P_-^{gker}\| + O(e^{-L}) |W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)| \\
\leq\ & |\begin{pmatrix} 1 & \tau - R_* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ P_-^{gker} W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d) \end{pmatrix}| \\
& +O(e^{-L}) |W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)| \\
\leq\ & CL |P_-^{gker} W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)|_{H^1} + |W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)|_{L^2} \\
\leq\ & C(|w_-^{ker}| + e^{-L} |w_-^{uu}| + e^{-L} \|\mathcal{G}\|_\infty + |d|),
\end{aligned}$$

and

$$\begin{aligned}
& \|\Phi_-^{ss}(\tau, -L) W^{ss}(w_-^{ker}, w_-^{uu}, \mathcal{G}(\tau), d)\| \qquad (3.56) \\
\leq\ & C(e^{-L} |w_-^{ker}| + e^{-L} |w_-^{uu}| + e^{-L} \|\mathcal{G}\|_\infty + |d|) \\
\leq\ & C(|w_-^{ker}| + |w_-^{uu}| + \|\mathcal{G}\|_\infty + |d|).
\end{aligned}$$

Therefore, (3.40) follows from the estimates (3.43)-(3.45) , (3.55) and (3.56).

This finishes the proof for $T1$ boundary conditions. Next, we turn to Neumann boundary conditions. The only difference for Neumann boundary conditions lies in the evaluation of the w_-^{gker} component. Since the Neumann condition satisfies the transversality condition

(T2), we have:

$$\begin{aligned} d + e_-^{bc} &= (P_-^{uu} d + e^{uu}) + (P_-^{ss} d + e^{ss}) + (P_-^{gker} d + v^{gker}) + (P_-^{ker} d + e^{ker}) \\ &= (P_-^{uu} d + e^{uu}) + (P_-^{ss} d + P_-^{ss} W(v^{uu} + e^{ker})) \\ &\quad + (P_-^{ker} d + e^{ker}) + (P_-^{gker} d + P_-^{gker} W(v^{uu} + e^{ker})), \end{aligned} \tag{3.57}$$

where $d = \begin{pmatrix} 0 \\ e^{-L}(\partial_r u_*(r_0)) \end{pmatrix}$. Comparing the components of (3.42) and (3.57), we obtain similar expressions as for Dirichlet conditions. Observe that in the case of Neumann conditions, the graph of W corresponds to $u_\tau(-L) = 0$ and therefore, $W|_{E_-^{ker}} = 0$. Hence, we have

$$\begin{aligned} &(\gamma_{22} \mathrm{id}_{E_-^{gker}} + R_3 O(e^{-L})) w_-^{gker} = P_-^{gker} d + P_-^{gker} W(\Phi_-^{uu}(-L, R_*) w_-^{uu} \\ &+ \int_{R_*}^{-L} \Phi_-^{uu}(-L, \tau) e^{2\tau} \mathcal{G}(\tau) d\tau - P_-^{uu} d - R_4(w_-^{ker}) O(e^{-L}) + R_5(w_-^{uu}, \mathcal{G}) O(e^{-3L}) \end{aligned}$$

with R_3, R_4 and R_5 bounded uniformly in L and R_4 and R_5 linear. For L large, we can solve for w_-^{gker}:

$$\begin{aligned} w_-^{gker} &= (\gamma_{22} \mathrm{id}_{E_-^{gker}} + O(e^{-L}))^{-1} \{ O(e^{-L}) \partial_r u_*(r_0) + O(e^{-L}) w_-^{uu} \\ &\qquad + O(e^{-L}) \mathcal{G} + O(e^{-L}) w_-^{ker} \} \\ &=: W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}) \end{aligned}$$

with

$$\|W^{gker}\| \leq C e^{-L} (|w_-^{ker}| + |w_-^{uu}| + \|\mathcal{G}\|_\infty + |\partial_r u_*(0)|),$$

and

$$\begin{aligned} w_-^{ss} &= (1 + O(e^{-2L})) \{ O(e^{-L}) \partial_r u_*(r_0) + O(e^{-L}) w_-^{uu} + O(e^{-L}) \mathcal{G} + O(e^{-L}) w_-^{ker} \} \\ &=: W^{ss}(w_-^{ker}, w_-^{uu}, \mathcal{G}), \end{aligned}$$

with

$$\|W^{ss}\| \leq C e^{-L} (|w_-^{ker}| + |w_-^{uu}| + \|\mathcal{G}\|_\infty + |\partial_r u_*(0)|).$$

That concludes the proof of Theorem 3.5.1.

□

3.5.3 The Substitution

Let V_- be $C^0([-L, R_*], X)$ equipped with sup-norm

$$\|\mathcal{U}\|_\infty = \sup_{\tau\in[-L,R_*]} \|\mathcal{U}(\tau)\|.$$

Now we return to the original full nonlinear equation (3.31) on V_- together with the boundary condition, that is, we substitute $\mathcal{G} = \mathcal{G}(\mathcal{U}, \omega)$ for $\mathcal{G}$ in $\mathcal{U} = \mathcal{L}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)$. Thus, we consider the following fixed point equation:

$$\mathcal{U} = \mathcal{L}(w_-^{ker}, w_-^{uu}, \mathcal{G}(\mathcal{U}, \omega), d)$$

on V_- with parameters $(w_-^{ker}, w_-^{uu}, \omega, d)$. Explicitly, we have

$$\begin{aligned}\mathcal{U}(\tau) &= \Phi_-^c(\tau, R_*) \begin{pmatrix} w_-^{ker} \\ W^{gker}(w_-^{ker}, w_-^{uu}, \mathcal{G}(\mathcal{U}, \omega)(\tau), d) \end{pmatrix} + \Phi_-^{uu}(\tau, R_*) w_-^{uu} \\ &+\Phi_-^{ss}(\tau, -L) W^{ss}(w_-^{ker}, w_-^{uu}, \mathcal{G}(\mathcal{U}, \omega)(\tau), d) + \int_{R_*}^{\tau} \Phi_-^{uu}(\tau, s) e^{2s} \mathcal{G}(\mathcal{U}, \omega)(s) ds \\ &+ \int_{-L}^{\tau} [\Phi_-^{ss}(\tau, s) + \Phi_-^c(\tau, s)] e^{2s} \mathcal{G}(\mathcal{U}, \omega)(s) ds. \end{aligned} \tag{3.58}$$

in which $\mathcal{G} : V_- \times \mathbb{R} \to V_-$ and

$$\mathcal{G}(\mathcal{U}, \omega) = \begin{pmatrix} 0 \\ -D^{-1}[(\omega - \omega_*)\partial_\psi(u_* + u) + f(u_* + u) - f(u_*) - f'(u_*)u] \end{pmatrix}.$$

In order to invoke the Implicit Function Theorem, see, e.g. [10], we need C^1-smallness of $\mathcal{G}(\mathcal{U}, \omega)$ with respect to $\mathcal{U}$. We formulate this property as the following

Lemma 3.5.2. *$\mathcal{G}$ is continuously differentiable and, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any $\mathcal{U}$ and ω with $\|\mathcal{U}\|_\infty + |\omega - \omega_*| < \delta$, we have*

$$\|\mathcal{G}(\cdot, \omega)\|_{C^1} = \|\mathcal{G}(\cdot, \omega)\|_\infty + \|D_{\mathcal{U}}\mathcal{G}(\cdot, \omega)\|_\infty < \epsilon.$$

Proof. Write

$$\mathcal{G}(\mathcal{U}, \omega) = (\omega - \omega_*) \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u_* \end{pmatrix} + \mathcal{N}_1(\omega)\mathcal{U} + \mathcal{N}_2(\mathcal{U}),$$

in which

$$\begin{aligned}\mathcal{N}_1(\omega)\mathcal{U} &= (\omega-\omega_*)\begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u \end{pmatrix},\\ \mathcal{N}_2(\mathcal{U}) &= \begin{pmatrix} 0 \\ -D^{-1}(f(u_*+u)-f(u_*)-f'(u_*)u) \end{pmatrix}.\end{aligned}$$

We claim that the linear operator $\mathcal{N}_1(\omega)$ is bounded and hence continuous:

$$\begin{aligned}\|\mathcal{N}_1(\omega)\|_\infty &= \sup_{\|\mathcal{U}\|<\delta}\|\mathcal{N}_1(\omega)\mathcal{U}\| = \sup_{\tau\in[-L,R_*]}|(\mathcal{N}_1(\omega)\mathcal{U})(\tau)|_X\\ &= \sup_{\tau\in[-L,R_*]}|-D^{-1}(\omega-\omega_*)\partial_\psi u|_{L^2}\\ &\le |D^{-1}||\omega-\omega_*|\sup_{\tau\in[-L,R_*]}(|u|_{L^2}^2+|\partial_\psi u|_{L^2}^2)^{\frac{1}{2}}\\ &\le C|\omega-\omega_*|\|\mathcal{U}\|_\infty.\end{aligned}$$

For any $\mathcal{W}=\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\in V_-\backslash 0$, define

$$T\mathcal{W} := \begin{pmatrix} 0 \\ (f'(u_*+u)-f'(u_*))w_1 \end{pmatrix}.$$

Then

$$\begin{aligned}&\|\mathcal{G}(\mathcal{U}+\mathcal{W},\omega)-\mathcal{G}(\mathcal{U},\omega)-(\mathcal{N}_1(\omega)+T)\mathcal{W}\|\cdot|\mathcal{W}|^{-1}\\ \le\ & |D^{-1}|\sup_{\tau\in[-L,R_*]}|f(u_*+u+w_1)-f'(u_*+u)w_1-f(u_*+u)|_{L^2}|\mathcal{W}|^{-1}\\ \le\ & |D^{-1}|\sup_{\tau\in[-L,R_*]}\sup_{\xi\in[0,1]}|f'(u_*+u+\xi w_1)-f'(u_*+u)|_{L^2}\to 0, \qquad \text{as } \mathcal{W}\to 0.\end{aligned}$$

Therefore, $D_{\mathcal{U}}\mathcal{G}(\cdot,\omega)=\mathcal{N}_1(\omega)+T$. In particular, $D_{\mathcal{U}}\mathcal{G}(0,\omega)=\mathcal{N}_1(\omega)$. Similarly, define $\mathcal{M}(\mathcal{U}):\mathbb{R}\to V_-$ by

$$\mathcal{M}(\mathcal{U})\omega = \begin{pmatrix} 0 \\ -D^{-1}\omega\partial_\psi(u+u_*) \end{pmatrix}.$$

We have that

$$\|\mathcal{M}(\mathcal{U})\|\le C|\partial_\psi(u+u_*)|_{L^2}<\tilde{C}<\infty,$$

with

$$\mathcal{G}(\mathcal{U},\omega+\hat{\omega})-\mathcal{G}(\mathcal{U},\omega)-\mathcal{M}(\mathcal{U})\hat{\omega}\equiv 0.$$

Therefore, $D_\omega \mathcal{G}(\mathcal{U}, \cdot) = \mathcal{M}(\mathcal{U})$. Therefore $\mathcal{G}$ is continuously differentiable. Observe that

$$\mathcal{G}(\mathcal{U}, \omega) = O(|\omega - \omega_*|) + O(|u|^2), \qquad D_{\mathcal{U}}\mathcal{G}(\mathcal{U}, \omega) = O(|\omega - \omega_*| + \|\mathcal{U}\|_\infty),$$

the second part of the lemma follows. □

Theorem 3.5.2. *There exist δ, $\rho > 0$, such that for any $L > \frac{1}{\delta}$, $(w^{ker}, w^{uu}_-) \in Y^{ker}_- \oplus Y^u_-$, $\omega \in \mathbb{R}$ and $d \in (E^{bc}_-)^\perp$ with $|w^{ker}_-|+|w^{uu}_-|+|\omega-\omega_*|+|d| < \delta$, there exists a unique solution $\mathcal{U} = \mathcal{F}(w^{ker}_-, w^{uu}_-, \omega, d; L) \in B_\rho(0)$ of the nonlinear equation (3.58) which satisfies (T1) or Neumann boundary condition. Moreover, $\mathcal{F}$ depends smoothly on $(w^{ker}, w^{uu}_-, \omega, d)$ with*

$$\|\mathcal{F}\|_\infty \leq C(|w^{ker}_-| + |w^{uu}_-| + |\omega - \omega_*| + |d|). \tag{3.59}$$

Proof. Write the fixed point equation as

$$\hat{\mathcal{L}}(\mathcal{U}, w^{ker}_-, w^{uu}_-, \omega, d) = \mathcal{U} - \mathcal{L}(w^{ker}_-, w^{uu}_-, \mathcal{G}(\mathcal{U}, \omega), d) = 0,$$

in which

$$\hat{\mathcal{L}} : C^0([-L, R_*), X) \times (Y^{ker}_- \times Y^u_- \times \mathbb{R} \times (E^{bc}_-)^\perp) \to C^0([-L, R_*), X),$$

and

$$\mathcal{L} : Y^{ker}_- \times Y^u_- \times C^0([-L, R_*), X) \times (E^{bc}_-)^\perp \to C^0([-L, R_*), X)$$

defined as

$$\mathcal{L}(w^{ker}_-, w^{uu}_-, \mathcal{G}, d) := \tilde{\mathcal{L}}(w^{ker}_-, W^{gker}(w^{ker}_-, w^{uu}_-, \mathcal{G}, d), W^{ss}(w^{ker}_-, w^{uu}_-, \mathcal{G}, d), w^{uu}_-, \mathcal{G}).$$

First, observe that $\hat{\mathcal{L}}(0, 0, 0, 0, \omega_*, 0) = 0$. Then we need to prove that $\hat{\mathcal{L}}$ is C^1-bounded in $(w^{ker}_-, w^{uu}_-, \mathcal{U}, \omega, d)$ near $(\mathcal{U}, w^{ker}_-, w^{uu}_-, \omega, d) = (0, 0, 0, 0, \omega_*, 0)$ for L large: From Theorem 3.5.1 and Lemma 3.5.2, there exists a $\delta > 0$, such that for all $L > \frac{1}{\delta}$ and

$|w_-^{ker}| + |w_-^{uu}| + |\omega - \omega_*| + \|\mathcal{U}\|_\infty + |d| < \delta$,

$$
\begin{aligned}
& \|\hat{\mathcal{L}}(\mathcal{U}, w_-^{ker}, w_-^{uu}, \omega, d)|_\infty \\
\leq\ & \|\mathcal{U}\|_\infty + \|\mathcal{L}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)|_\infty \\
\leq\ & \|\mathcal{U}\|_\infty + C(|w_-^{ker} + |w_-^{uu}| + \|\mathcal{G}\|_\infty + |d|) \leq C((|w_-^{ker} + |w_-^{uu}| + \|\mathcal{G}(\cdot,\omega)\|_\infty + |d|) \\
\leq\ & \|\mathcal{U}\|_\infty + C(|w_-^{ker}| + |w_-^{uu}| + |\omega - \omega_*| + |d| + \|\mathcal{U}\|_\infty^2 + |\omega - \omega|^2) \\
\leq\ & C(|w_-^{ker}| + |w_-^{uu}| + \|\mathcal{U}\|_\infty + |\omega - \omega_*| + |d|)
\end{aligned}
$$

Secondly, we claim that $D_{\mathcal{U}}\hat{\mathcal{L}}(w_-^{ker}, w_-^{uu}, \mathcal{U}, \omega, d)$ has a bounded inverse near $(w_-^{ker}, w_-^{uu}, \mathcal{U}, \omega, d) = (0, 0, 0, 0, \omega_*, 0)$: For any $\epsilon > 0$, by Theorem 3.5.1, there exist constants C, $\delta > 0$ such that

$$\|D_{\mathcal{G}}\mathcal{L}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)\| < C, \qquad \mathcal{G} = \mathcal{G}(\tau) \in C^0([-L, R_*], X), \tag{3.60}$$

and $\|\mathcal{G}(\cdot,\omega)\|_{C^1} < \epsilon$, for $|w_-^{ker}| + |w_-^{uu}| + \|\mathcal{U}\|_\infty + |\omega - \omega_*| + |d| < \delta$. In particular, $\mathcal{G}(\cdot,\omega) \in C^0([-L, R_*], X)$. Then by Lemma 3.5.2,

$$
\begin{aligned}
\|D_{\mathcal{U}}\mathcal{L}(w_-^{ker}, w_-^{uu}, \mathcal{G}(\mathcal{U},\omega), d)\|_\infty &\leq \|D_{\mathcal{G}}\mathcal{L}(w_-^{ker}, w_-^{uu}, \mathcal{G}, d)\|_\infty \cdot \|D_{\mathcal{U}}\mathcal{G}(\mathcal{U},\omega)\|_\infty \\
&< C\epsilon
\end{aligned}
$$

for $|w_-^{ker}| + |w_-^{uu}| + \|\mathcal{U}\|_\infty + |\omega - \omega_*| + |d| < \delta$. Then in particular, $D_{\mathcal{U}}\hat{\mathcal{L}}(0, 0, 0, \omega_*, 0)$ has a bounded inverse. Hence we can then apply the Implicit Function Theorem to solve for $\mathcal{U}$ in terms of w_-^{ker}, w^{uu}, ω and d and obtain the estimate (3.59). □

3.6 Matching

In this section, we denote the two perturbed solutions in the core region and the farfield, by $\mathcal{U}_* + \mathcal{U}_-$ and $\mathcal{U}_* + \mathcal{U}_+$. Here $\mathcal{U} = \mathcal{U}_+$ for $\tau \geq R_*$ and $\mathcal{U} = \mathcal{U}_-$ for $\tau \leq R_*$. Our goal is to match these solutions at $\tau = R_*$.

We organize this section as follows: In Section 3.6.1 and 3.6.2, we set up an appropriate coordinate system at $\tau = R_*$ in which we can match $\mathcal{U}_-$ and $\mathcal{U}_+$ near $\mathcal{U}_*(R_*)$ in Section 3.6.3.

3.6.1 Exponential Dichotomies and Fredholm Properties

In the previous sections, we have that the existence of exponential dichotomies on the ray $(-\infty, R_*]$ for some $R_* \gg 1$. In this section, we prove certain Fredholm properties implied by exponential dichotomies.

Due to Hypothesis 3.1, when $\omega = \omega_*$, the tangent space $T_{\mathcal{U}(R_*)}M_+^{cs}(u_*, \omega)$ of the local center-stable manifold $M_+^{cs}(u_*, \omega)$ of the wave trains and the tangent space of the local center-unstable manifold $M_-^{cu}(u_*, \omega)$ of the asymptotic states at $\mathcal{U}_*(R_*)$ intersect along a one-dimensional subspace generated by $\partial_\psi \mathcal{U}_*(R_*)$. (From now on, we include the angular velocity ω as an extra parameter.) Denote the one-dimensional subspace by I. From the following lemma, we conclude that the complement of the sum $T_{\mathcal{U}(R_*)}M_+^{cs}(u_*, \omega_*) + T_{\mathcal{U}(R_*)}M_-^{cu}(u_*, \omega_*)$ is one-dimensional and given by the unique nontrivial bounded solution of the adjoint variational equation.

Lemma 3.6.1. *([47]) The map*

$$\iota : Rg(P_-^{ss}(R_*) + P_-^{gker}(R_*)) \times Rg(P_+^{uu}(R_*)) \to X, \qquad (u, v) \mapsto u + v,$$

is Fredholm, and its index is 0.

Proof. The tool we shall use here is a bordering lemma (e.g., Lemma 3.5 in [51]). Note that the map is defined differently from the map ι_{spiral} in [47], in which the domain of ι contains different center directions. From [47],

$$\iota_{spiral} : Rg(P_-^{ss}(R_*) + P_-^{ker}(R_*)) \times Rg(P_+^{uu}(R_*)) \to X, \qquad (u, v) \mapsto u + v,$$

is Fredholm with index 0.

Define

$$\tilde{\iota} : Rg(P_-^{ss}(R_*) + P_-^c(R_*)) \times Rg(P_+^{uu}(R_*)) \to X, \qquad (u, v) \mapsto u + v.$$

This is an enlarged map by attaching the other center direction $Rg(P_-^{gker}(R_*))$ to the map ι_{spiral}. From the bordering lemma, $\tilde{\iota}$ is Fredholm with index N.

On the other hand, $\tilde{\iota}$ can be obtained by attaching $Rg(P_-^{ker}(R_*))$ to the map ι. Observe that $Rg(P_-^{ker}(R_*))$ is transverse to both $Rg(P_-^{ss}(R_*))$ and $Rg(P_-^{gker}(R_*))$. Suppose that

$$\dim(Rg(P_-^{ker}(R_*) \cap Rg(P_+^{uu}(R_*)) = k,$$

then

$$\dim coker(\iota) - (N - k) = \dim coker(\tilde{\iota}), \qquad \dim \ker(\iota) + k = \dim \ker(\tilde{\iota}).$$

Thus we conclude that ι is Fredholm with index 0. □

3.6.2 Frames of Matching

In order to match, we decompose the tangent space of X at $\mathcal{U}_*(R_*)$ into subspaces and match in each subspace.

From the decomposition (3.30),

$$X = Y_-^{ker} \oplus Y_-^u \oplus (Y_-^{gker} \oplus Y_-^s),$$

Also at $\omega = \omega_*$, the tangent space of M_-^{cu} at $\mathcal{U}_*(R_*)$ is given by

$$T_{\mathcal{U}_*(R_*)} M_-^{cu}(u_*, \omega_*) = Y_-^{ker} \oplus I \oplus Y_-^u.$$

From the farfield, we have

$$X = X_+^u \oplus X_+^s,$$

in which X_+^s and X_+^u are the subspaces of initial conditions that lead to exponentially decaying solutions in forward and backward time, respectively. The tangent space of $M_+^{cs}(u_*, \omega_*)$ is given by [47]

$$T_{\mathcal{U}_*(R_*)} M_+^{cs}(u_*, \omega_*) = I \oplus X_+^s.$$

Combining these two frames together, we have the following decomposition:

$$X = I \oplus Y_-^{ker} \oplus Y_-^u \oplus X_+^s \oplus Z.$$

in which Z is spanned by the unique (up to a scalar) nontrivial bounded solution of the adjoint variational equation. The transversality of Y_-^{ker}(or Y_-^{uu}) with X_+^s comes from Hypothesis 3.1, since otherwise the variational equation about the spiral u_* admits a second linearly independent bounded nontrivial solution in addition to $\partial_\psi \mathcal{U}(R_*)$. We know from the Fredholm property in the last section that $\dim Z = 1$.

Definition 3.6.1. *Define*

$$P_+ : X \to Y_-^{ker} \oplus Y_-^u \oplus Z$$

be the orthogonal projection onto $Y_-^{ker} \oplus Y_-^u \oplus Z$ with kernel X_+^s and

$$P_- : X \to X_+^s \oplus Z$$

be the orthogonal projection onto $X_+^s \oplus Z$ with kernel $Y_-^{ker} \oplus Y_-^u$.

3.6.3 Matching

To match $\mathcal{U}_-$ and $\mathcal{U}_+$ at R_* means to find solution $(w_-^{ker}, w_-^{uu}, w_+^{ss}, \omega, d)$ of the following equation:

$$\mathcal{H}(w_-^{ker}, w_-^{uu}, w_+^{ss}, \omega, d) = 0,$$

where $\mathcal{H} : (Y_-^{ker} \oplus Y_-^u \oplus X_+^s) \times \mathbb{R} \times (E_-^{bc})^\perp \to X/I$,

$$(w_-^{ker}, w_-^{uu}, w_+^{ss}, \omega, d) \overset{\mathcal{H}}{\mapsto} \begin{pmatrix} P(Y_-^{ker}, X_+^s \oplus Z \oplus Y_-^u \oplus I)(\mathcal{U}_-(R_*) - \mathcal{U}_+(R_*)) \\ P(Y_-^u, X_+^s \oplus Z \oplus Y_-^{ker} \oplus I)(\mathcal{U}_-(R_*) - \mathcal{U}_+(R_*)) \\ P(X_+^s, Y_-^{ker} \oplus Z \oplus Y_-^u \oplus I)(\mathcal{U}_-(R_*) - \mathcal{U}_+(R_*)) \\ P(Z, X_+^s \oplus Y_-^{ker} \oplus Y_-^u \oplus I)(\mathcal{U}_-(R_*) - \mathcal{U}_+(R_*)) \end{pmatrix}. \tag{3.61}$$

For sake of convenience, write the projections in (3.61) as P_i, $i = 1, 2, 3$ and 4 in a sequential order. Note that we do not need to match in the I-direction. Indeed, since the reaction-diffusion equation on $\mathbb{R}^2 \backslash B_\epsilon(0)$ is equivariant with respect to rotations in $\mathbb{S}^1$. Due to the

$\mathbb{S}^1$-symmetry of the underlying PDE with respect to rotations, $u(r,\theta+\psi)$ is a solution if and only if $u(r,\psi)$ is. Therefore, we can always match in the direction of $I = span\{\partial_\psi \mathcal{U}_*(R_*)\}$ generated by $\mathbb{S}^1$ by rotating $\mathcal{U}_-(R_*)$ and $\mathcal{U}_+(R_*)$ appropriately.

From [47], we have the following variation-of-constants formula for $\mathcal{U}_+$:

$$\begin{aligned}\mathcal{U}_+(r) &= \Phi_+^{ss}(r,R_*)w_+^{ss} + (\omega-\omega_*)\int_{R_*}^{r}\Phi_+^{ss}(r,\tau)\begin{pmatrix}0\\ -D^{-1}\partial_\psi(u_*+u)\end{pmatrix}d\tau \\ &\quad +(\omega-\omega_*)\int_{\infty}^{r}\Phi_+^{uu}(r,\tau)\begin{pmatrix}0\\ -D^{-1}\partial_\psi(u_*+u)\end{pmatrix}d\tau \\ &\quad +\int_{R_*}^{r}\Phi_+^{ss}(r,\tau)\mathcal{N}_2(\mathcal{U}_+(\tau))d\tau + \int_{\infty}^{r}\Phi_+^{uu}(r,\tau)\mathcal{N}_2(\mathcal{U}_+(\tau))d\tau,\end{aligned} \tag{3.62}$$

where $\mathcal{N}_2(\mathcal{U}) = O(|\mathcal{U}|^2)$. Observe that (3.62) is parameterized by ω and w_+^{ss} and the estimate

$$\|\mathcal{U}_+\|_\infty \le C(|w_+^{ss}| + |\omega-\omega_*|)$$

holds for $\mathcal{U}_+$.

Evaluating both variation-of-constants formulae (3.58) and (3.62) at R_*, we have

$$\begin{aligned}\mathcal{U}_-(R_*) &= \begin{pmatrix}w_-^{ker}\\ w_-^{gker}\end{pmatrix} + w_-^{uu} + \Phi_-^{ss}(R_*,-L)w_-^{ss} \\ &\quad +(\omega-\omega_*)\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{c}(R_*,\tau)]e^{2\tau}\begin{pmatrix}0\\ -D^{-1}\partial_\psi(u_*+u)\end{pmatrix}d\tau \\ &\quad +\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{c}(R_*,\tau)]e^{2\tau}\mathcal{N}_2(\mathcal{U}_-(\tau))d\tau\end{aligned} \tag{3.63}$$

and

$$\begin{aligned}\mathcal{U}_+(R_*) &= w_+^{ss} + (\omega-\omega_*)\int_{\infty}^{R_*}\Phi_+^{uu}(R_*,\tau)\begin{pmatrix}0\\ -D^{-1}\partial_\psi(u_*+u)\end{pmatrix}d\tau \\ &\quad +\int_{\infty}^{R_*}\Phi_+^{uu}(R_*,\tau)\mathcal{N}_2(\mathcal{U}_+)d\tau.\end{aligned} \tag{3.64}$$

Recall that in Hypothesis 3.1, we assume that $M^{cu}(u_*,\omega)$ intersects $M_+^{cs}(u_*,\omega)$ transversely at $\omega=\omega_*$. We claim in the following lemma that

$$M_-^{bc}(w_-^{ker},w_-^{uu},\omega_*,d)(L) := \mathcal{U}_* + \mathcal{U}_-(w^{ker},w_-^{uu},\omega_*,d)(L)$$

is of $O(\frac{1}{L})$ or $O(e^{-L})$-close to $M_-^{cu}(u_*,\omega_*)$ at $\tau = R_*$, depending on whether (T1) or (T2) are met. The distance is given by $P_-\mathcal{U}_-(R_*)$ (see Definition (3.6.1)).

Lemma 3.6.2. *There exists L_* such that for all $L > L_*$,*

$$\|P_-\mathcal{U}_-(R_*)\| \leq \frac{C}{L}(|w_-^{ker}| + e^{-L}|w_-^{uu}| + |d|)$$

for $T1$ boundary conditions, and

$$\|P_-\mathcal{U}_-(R_*)\| \leq Ce^{-L}(|w_-^{ker}| + |w_-^{uu}| + |d|)$$

for Neumann boundary conditions.

Proof. Recall that from (3.63),

$$\begin{aligned}
\mathcal{U}_-(R_*) &= \begin{pmatrix} w_-^{ker} \\ w_-^{gker} \end{pmatrix} + w_-^{uu} + \Phi_-^{ss}(R_*,-L)w_-^{ss} \\
&\quad +(\omega-\omega_*)\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{c}(R_*,\tau)]e^{2\tau}\begin{pmatrix} 0 \\ -D^{-1}\partial_\psi(u_*+u) \end{pmatrix}d\tau \\
&\quad +\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{c}(R_*,\tau)]e^{2\tau}\mathcal{N}_2(\mathcal{U}_-(\tau))d\tau
\end{aligned}$$

The estimate for the integral gives

$$\begin{aligned}
&|\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{c}(R_*,\tau)]e^{2\tau}\mathcal{N}_2(\mathcal{U}_-(\tau))d\tau|_X \\
\leq\ & C\int_{-L}^{R_*}(e^{\tau-R_*}+e^{\epsilon(R_*-\tau)})e^{2\tau}d\tau\cdot\|\mathcal{N}_2(\mathcal{U}_-)\|_\infty \\
\leq\ & C(|w_-^{ker}|+|w_-^{uu}|+|\omega-\omega_*|+|d|)^2.
\end{aligned}$$

with $0<\epsilon\ll 1$.

Setting $\omega=\omega_*$, we see that the relevant linear part of $P_-\mathcal{U}_-(R_*)$ is given by

$$\begin{aligned}
&|w_-^{gker}+\Phi_-^{ss}(R_*,-L)w_-^{ss}|_X \\
\leq\ & \frac{C}{L}(|w_-^{ker}|+e^{-L}|w_-^{uu}|+e^{-L}\|\mathcal{F}(w_-^{ker},w_-^{uu},\omega,d)\|_\infty^2+|d|) \\
& +Ce^{-L}(\frac{C}{L}|w_-^{ker}|+e^{-L}|w_-^{uu}|+e^{-L}\|\mathcal{F}(w_-^{ker},w_-^{uu},\omega,d)\|_\infty^2+|d|) \\
\leq\ & \frac{C}{L}(|w_-^{ker}|+e^{-L}|w_-^{uu}|+|d|+(|w_-^{ker}|+|w_-^{uu}|+|d|)^2)
\end{aligned}$$

for $T1$ boundary conditions; and

$$\begin{aligned} & |w_-^{gker} + \Phi_-^{ss}(R_*, -L)w_-^{ss}|_X \\ \leq\ & Ce^{-L}(|w_-^{ker}| + |w_-^{uu}| + \|\mathcal{F}(w_-^{ker}, w_-^{uu}, \omega_*, d)\|_\infty^2 + |d|) \\ & + Ce^{-2L}(|w_-^{ker}| + |w_-^{uu}| + e^{-L}\|\mathcal{F}(w_-^{ker}, w_-^{uu}, \omega_*, d)\|_\infty^2 + |d|) \\ \leq\ & Ce^{-L}(|w_-^{ker}| + |w_-^{uu}| + |d| + (|w_-^{ker}| + |w_-^{uu}| + |d|)^2) \end{aligned}$$

for Neumann conditions. Also we have similar estimate for the derivatives with respect to w_-^{ker} and w_-^{uu}.

□

Define

$$\tilde{\mathcal{G}}(\mathcal{U}, \omega)(\tau) = \begin{cases} e^{2\tau}\mathcal{G}(\mathcal{U}_-, \omega), & \mathcal{U}_- = (u, u_\tau), -L \leq \tau = \log r \leq \log R_* \\ \mathcal{G}(\mathcal{U}_+, \omega), & \mathcal{U}_+ = (u, u_r), \tau = r > R_* \end{cases} . \tag{3.65}$$

in which

$$\mathcal{G}(\mathcal{U}, \omega) = \begin{pmatrix} 0 \\ -D^{-1}[(\omega - \omega_*)\partial_\psi(u_* + u) + f(u_* + u) - f(u_*) - f'(u_*)u] \end{pmatrix}.$$

Lemma 3.6.3. *There exist* $(B_-^{ker}, B_-^{uu}, B_+^{ss}) \in Y_-^{ker} \oplus Y_-^u \oplus X_+^s$, $C_+^{ss} : (E_-^{bc})^\perp \to X$ *and operators* R_{ker}, R_{uu} *and* R_{ss} *such that*

$$P_i(\mathcal{U}_-(R_*) - \mathcal{U}_+(R_*)) = 0, \qquad i = 1, 2, 3$$

is equivalent to

$$\begin{aligned} w_-^{ker} &= (\omega - \omega_*)B_-^{ker} + \mathcal{R}_{ker}(w_-^{ker}, w_-^{uu}, w_+^{ss}, d, \omega - \omega_*), \\ w_-^{uu} &= (\omega - \omega_*)B_-^{uu} + \mathcal{R}_{uu}(w_-^{ker}, w_-^{uu}, w_+^{ss}, d, \omega - \omega_*), \\ w_+^{ss} &= (\omega - \omega_*)B_+^{ss} + C_+^{ss}d + \mathcal{R}_{ss}(w_-^{ker}, w_-^{uu}, w_+^{ss}, d, \omega - \omega_*), \end{aligned} \tag{3.66}$$

where

$$\mathcal{R}_i = O(|w_-^{ker}|^2 + |w_-^{uu}|^2 + |w_+^{ss}|^2 + |\omega - \omega_*|^2 + |d|^2), \qquad i = ker, uu, ss.$$

In addition, C_+^{ss} satisfies the following estimate:

$$\begin{aligned} \|C_+^{ss}\| &\leq \frac{C}{L}, \qquad \textit{for } T1 \textit{ boundary conditions} ; \\ \|C_+^{ss}\| &\leq Ce^{-L}, \qquad \textit{for Neumann conditions} . \end{aligned}$$

Proof. Let

$$P_1(\mathcal{U}_-(R_*) - \mathcal{U}_+(R_*) = 0,$$

then we reach

$$\begin{aligned} 0 &= w_-^{ker} + (\omega - \omega_*) \int_{-L}^{R_*} \Phi_-^{ker}(R_*, \tau) e^{2\tau} \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi(u_* + u) \end{pmatrix} d\tau \\ &\quad -(\omega - \omega_*) P_1 \int_{\infty}^{R_*} \Phi_+^{uu}(R_*, \tau) \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi(u_* + u) \end{pmatrix} d\tau \\ &\quad + \int_{-L}^{R_*} \Phi_-^{ker}(R_*, \tau) e^{2\tau} \mathcal{N}_2(\mathcal{U}_-(\tau)) d\tau - P_1 \int_{\infty}^{R_*} \Phi_+^{uu}(R_*, \tau) \mathcal{N}_2(\mathcal{U}_+(\tau)) d\tau, \end{aligned}$$

which yields

$$w_-^{ker} = -(\omega - \omega_*) B_-^{ker} + R_{ker}(w_-^{ker}, w_-^{uu}, w_+^{ss}, d, \omega - \omega_*), \tag{3.67}$$

in which

$$\begin{aligned} B_-^{ker} &:= P_1 \left(\int_{-L}^{R_*} \Phi(R_*, \tau) e^{2\tau} \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u_* \end{pmatrix} d\tau \right. \\ &\qquad \left. - \int_{\infty}^{R_*} \Phi(R_*, \tau) \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u_* \end{pmatrix} d\tau \right) \\ &= P_1 \int_{-L}^{\infty} \Phi(R_*, \tau) D_\omega \tilde{\mathcal{G}}(\mathcal{U}_*, \omega_*) d\tau, \end{aligned}$$

and

$$\begin{aligned} & R_{ker}(w_-^{ker}, w_-^{uu}, w_+^{ss}, d, \omega - \omega_*) \\ := \; & -(\omega - \omega_*) \left(\int_{-L}^{R_*} \Phi_-^{ker}(R_*, \tau) e^{2\tau} \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u \end{pmatrix} d\tau \right. \\ & \left. -P_1 \int_{\infty}^{R_*} \Phi_+^{uu}(R_*, \tau) \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u \end{pmatrix} d\tau \right) \\ & - \int_{-L}^{R_*} \Phi_-^{ker}(R_*, \tau) e^{2\tau} \mathcal{N}_2(\mathcal{U}_-(\tau)) d\tau + P_1 \int_{\infty}^{R_*} \Phi_+^{uu}(R_*, \tau) \mathcal{N}_2(\mathcal{U}_+(\tau)) d\tau, \end{aligned}$$

where

$$P_1 = P(Y_-^{ker}, X_+^s \oplus Z \oplus Y_-^u \oplus I),$$

$$\Phi(R_*,\tau) = \left\{ \begin{array}{ll} \Phi_+(R_*,\tau) = (\Phi_+^{ss} + \Phi_+^{uu})(R_*,\tau), & \tau \geq R_* \\ \Phi_-(R_*,\tau) = (\Phi_-^{ss} + \Phi_-^{uu} + \Phi_-^{ker} + \Phi_-^{gker})(R_*,\tau), & \tau \leq R_* \end{array} \right. .$$

Since

$$\| \left(\begin{array}{c} 0 \\ -D^{-1}\partial_\psi u \end{array} \right) \|_X \leq |D^{-1}| \|\mathcal{U}\|_X,$$

and

$$e^{2\tau}\|\Phi_-^{ker}(R_*,\tau)\|_{L(X)}, \|\Phi_-^{uu}(R_*,\tau)\|_{L(X)} = O(e^\tau), \qquad \text{as } \tau \to -\infty,$$

$$|R_{ker}(w_-^{ker}, w_-^{uu}, w_+^{ss}, d, \omega - \omega_*)| \leq C(|\omega - \omega_*| \|\mathcal{U}\|_\infty + \|\mathcal{N}_2(\mathcal{U})\|_\infty)$$

$$\leq \quad C(\|\mathcal{U}\|_\infty^2 + |\omega - \omega_*|^2|) \leq C(|w_-^{ker}|^2 + |w_-^{uu}|^2 + +|w_+^{ss}|^2 + |\omega - \omega_*|^2 + |d|^2).$$

Similarly, $P_2(\mathcal{U}_-(R_*) - \mathcal{U}_+(R_*) = 0$ yields

$$\begin{aligned} 0 \quad = \quad & w_-^{uu} - (\omega - \omega_*) P_2 \int_\infty^{R_*} \Phi_+^{uu}(R_*,\tau) \left(\begin{array}{c} 0 \\ -D^{-1}\partial_\psi(u_* + u) \end{array} \right) d\tau \\ & + P_2 \int_\infty^{R_*} \Phi_+^{uu}(R_*,\tau) \mathcal{N}_2(\mathcal{U}_+) d\tau. \end{aligned}$$

Then

$$w_-^{uu} = (\omega - \omega_*) B_-^{uu} + R_{uu}(w_-^{ker}, w_-^{uu}, w_+^{ss}, d, \omega - \omega_*),$$

with

$$B_-^{uu} := P_2 \int_\infty^{R_*} \Phi(R_*,\tau) e^{2\tau} \left(\begin{array}{c} 0 \\ -D^{-1}\partial_\psi u_* \end{array} \right) d\tau P_2 \int_\infty^{\infty} \Phi(R_*,\tau) D_\omega \tilde{\mathcal{G}}(\mathcal{U}_*, \omega_*) d\tau,$$

and

$$\begin{aligned} & R_{uu}(w_-^{ker}, w_-^{uu}, w_+^{ss}, d, \omega - \omega_*) \\ := \quad & (\omega - \omega_*) \int_\infty^{R_*} \Phi_+^{uu}(R_*,\tau) \left(\begin{array}{c} 0 \\ -D^{-1}\partial_\psi u \end{array} \right) d\tau - P_2 \int_\infty^{R_*} \Phi_+^{uu}(R_*,\tau) \mathcal{N}_2(\mathcal{U}_+(\tau)) d\tau, \end{aligned}$$

where

$$P_2 = P(Y_-^u, X_+^s \oplus Z \oplus Y_-^{ker} \oplus I),$$

$$\Phi(R_*,\tau) = \left\{ \begin{array}{ll} \Phi_+(R_*,\tau) = (\Phi_+^{ss} + \Phi_+^{uu})(R_*,\tau), & \tau \geq R_* \\ \Phi_-(R_*,\tau) = (\Phi_-^{ss} + \Phi_-^{uu} + \Phi_-^{ker} + \Phi_-^{gker})(R_*,\tau), & \tau \leq R_* \end{array} \right. ,$$

Since

$$\| \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u \end{pmatrix} \|_X \le |D^{-1}| \|\mathcal{U}\|_X,$$

and

$$\|\Phi_+^{uu}(R_*,\tau)\|_{L(X)} = O(e^{-\tau}), \qquad \text{as } \tau \to \infty,$$

$$|R_{uu}(w_-^{ker}, w_-^{uu}, w_+^{ss}, d, \omega-\omega_*)| \le C(|\omega-\omega_*| \|\mathcal{U}\|_\infty + \|\mathcal{N}_2(\mathcal{U})\|_\infty)$$

$$\le \quad C(\|\mathcal{U}\|_\infty^2 + |\omega-\omega_*|^2|) \le C(|w_-^{ker}|^2 + |w_-^{uu}|^2 + +|w_+^{ss}|^2 + |\omega-\omega_*|^2 + |d|^2).$$

Lastly, $P_3(\mathcal{U}_-(R_*) - \mathcal{U}_+(R_*) = 0$ yields

$$\begin{aligned} 0 \;=\; & w_+^{ss} - P_3(\Phi_-^{ss}(R_*,-L)w_-^{ss} + w_-^{gker}) \\ & -(\omega-\omega_*)P_3 \int_{-L}^{R_*} [\Phi_-^{ss}(R_*,\tau) + \Phi_-^{gker}(R_*,\tau)] e^{2\tau} \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi(u_*+u) \end{pmatrix} d\tau \\ & -P_3 \int_{-L}^{R_*} [\Phi_-^{ss}(R_*,\tau) + \Phi_-^{gker}(R_*,\tau)] e^{2\tau} \mathcal{N}_2(\mathcal{U}_+) d\tau. \end{aligned}$$

Then

$$\begin{aligned} w_+^{ss} \;=\; & P_3(\Phi_-^{ss}(R_*,-L)w_-^{ss} + w_-^{gker}) \\ + \; & (\omega-\omega_*)P_3 \int_{-L}^{R_*} [\Phi_-^{ss}(R_*,\tau) + \Phi_-^{gker}(R_*,\tau)] e^{2\tau} \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi(u_*+u) \end{pmatrix} d\tau \\ & +P_3 \int_{-L}^{R_*} [\Phi_-^{ss}(R_*,\tau) + \Phi_-^{gker}(R_*,\tau)] e^{2\tau} \mathcal{N}_2(\mathcal{U}_+) d\tau \\ = \; & P_3(\Phi_-^{ss}(R_*,-L)W^{ss}(w_-^{ker}, w_-^{uu}, \omega, d) + W^{gker}(w_-^{ker}, w_-^{uu}, \omega, d)) \\ + \; & (\omega-\omega_*)P_3 \int_{-L}^{R_*} [\Phi_-^{ss}(R_*,\tau) + \Phi_-^{gker}(R_*,\tau)] e^{2\tau} \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u_* \end{pmatrix} d\tau \\ + \; & P_3 \int_{-L}^{R_*} [\Phi_-^{ss}(R_*,\tau) + \Phi_-^{gker}(R_*,\tau)] e^{2\tau} \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u \end{pmatrix} d\tau \\ + \; & P_3 \int_{-L}^{R_*} [\Phi_-^{ss}(R_*,\tau) + \Phi_-^{gker}(R_*,\tau)] e^{2\tau} \mathcal{N}_2(\mathcal{U}_+) d\tau \end{aligned}$$

Define

$$\begin{aligned}
B_+^{ss}(\omega-\omega_*) &:= P_3[W^{gker}(B_-^{ker}(\omega-\omega_*), B_-^{uu}(\omega-\omega_*), \omega-\omega_*, 0) \\
&+ \Phi_-^{ss}(R_*,-L)W^{ss}(B_-^{ker}(u_*)(\omega-\omega_*), B_-^{uu}(\omega-\omega_*), \omega-\omega_*, 0)] \\
&+ (\omega-\omega_*)\cdot P_3\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{gker}(R_*,\tau)]e^{2\tau}\begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u_* \end{pmatrix} d\tau, \\
& R_{ss}(w_-^{ker}, w_-^{uu}, w_+^{ss}, d, \omega-\omega_*) \\
&= P_3\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{gker}(R_*,\tau)]e^{2\tau}\begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u \end{pmatrix} d\tau \\
&+ P_3\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{gker}(R_*,\tau)]e^{2\tau}\mathcal{N}_2(\mathcal{U}_+)d\tau
\end{aligned}$$

and

$$C_+^{ss}d := P_3(W^{gker}(0,0,0,d)+\Phi_-^{ss}(R_*,-L)W_-^{ss}(0,0,0,d)).$$

Since

$$\|\begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u \end{pmatrix}\|_X \le |D^{-1}|\|\mathcal{U}\|_X,$$

and

$$\|\Phi_-^{ss}(R_*,\tau)\|_{L(X)}, e^{2\tau}\|\Phi_-^{gker}(R_*,\tau)\|_{L(X)} = O(e^{-\tau}), \qquad \text{as } \tau\to-\infty,$$

$$|R_{ss}(w_-^{ker}, w_-^{uu}, w_+^{ss}, d, \omega-\omega_*)| \le C(|\omega-\omega_*|\|\mathcal{U}\|_\infty + \|\mathcal{N}_2(\mathcal{U})\|_\infty)$$

$$\le C(\|\mathcal{U}\|_\infty^2 + |\omega-\omega_*|^2|) \le C(|w_-^{ker}|^2+|w_-^{uu}|^2++|w_+^{ss}|^2+|\omega-\omega_*|^2+|d|^2).$$

The estimate of C_+^{ss} follows from the estimates of W^{gker} and W^{ss} in Theorem 3.5.1.

□

We now assert that we can solve can (3.66) for w_-^{ker}, w_-^{uu} and w_+^{ss}.

Lemma 3.6.4. *There exists $\delta>0$ such that for any $L>1/\delta$ and any (ω,d) with $|d|+|\omega-\omega_*|<\delta$, there exists a unique solution $(w_-^{ker}, w_-^{uu}, w_+^{ss})$ of (3.66), that depends smoothly on*

(ω, d) *and*

$$\begin{aligned} w_-^{ker} &= (\omega-\omega_*)\tilde{B}_-^{ker}(\omega-\omega_*, d) + O(|d|^2), \\ w_-^{uu} &= (\omega-\omega_*)\tilde{B}_-^{uu}(\omega-\omega_*, d) + O(|d|^2), \\ w_+^{ss} &= (\omega-\omega_*)\tilde{B}_+^{ss}(\omega-\omega_*, d) + \tilde{C}_+^{ss} d + O(|d|^2), \end{aligned}$$

Proof. Apply the Implicit Function Theorem to equation (3.66). □

The following lemmata (Lemma 3.6.5, 3.6.2 and 3.6.6) are for matching in Z.

Lemma 3.6.5. *For any fixed nontrivial bounded solution ψ of the adjoint variational equation, there exist $\psi_- \in \mathbb{R}^{2N} \cong E_-^c$ and $L_* \geq 0$ such that for any $L \geq L_*$,*

$$\psi(-L) = \psi_- + O(e^{-L}).$$

Proof. Recall that from Lemma 3.3.1, for any $\mathsf{w}_-^{ker} \in E_-^{ker}(-L)$, there exists a unique $w_-^{ker}(-L_*) \in E_-^{ker}(-L_*)$ such that

$$\begin{pmatrix} \mathsf{w}_-^{ker} \\ 0 \end{pmatrix} = \Phi^c(-L, -L_*) \begin{pmatrix} w_-^{ker}(-L_*) \\ 0 \end{pmatrix} + O(e^{-L}) = \begin{pmatrix} w_-^{ker}(-L_*) \\ 0 \end{pmatrix} + O(e^{-L});$$

and for any $\mathsf{w}_-^{gker} \in E_-^{gker}(-L)$, there exists a unique $w_-^{gker}(-L_*) \in E_-^{gker}(-L_*)$ such that

$$\begin{aligned} \begin{pmatrix} 0 \\ \mathsf{w}_-^{gker} \end{pmatrix} &= \Phi^c(-L, -L_*) \begin{pmatrix} 0 \\ w_-^{gker}(-L_*) \end{pmatrix} + O(e^{-L}) \\ &= \begin{pmatrix} (-L+L_*) w_-^{gker}(-L_*) \\ w_-^{gker}(-L_*) \end{pmatrix} + O(e^{-L}). \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \psi(-L_*), \begin{pmatrix} \mathsf{w}_-^{ker} \\ 0 \end{pmatrix} \rangle &= \langle \psi(-L_*), \begin{pmatrix} w_-^{ker}(-L_*) \\ 0 \end{pmatrix} + O(e^{-L}) \rangle \\ = \quad \langle \psi(R_*), \Phi^c(R_*, -L_*) w_-^{ker}(-L_*) \rangle &+ \langle \psi(-L_*), O(e^{-L}) \rangle = O(e^{-L}) \end{aligned}$$

with $\psi(R_*) \perp Y_-^{ker}$ at $\tau = R_*$. Also,

$$\langle \psi(-L), \begin{pmatrix} \mathsf{w}_-^{ker} \\ 0 \end{pmatrix} \rangle = \langle \psi(-L_*), \begin{pmatrix} w_-^{ker}(-L_*) \\ 0 \end{pmatrix} \rangle.$$

Then

$$\langle \psi(-L_*), \begin{pmatrix} \mathsf{w}_-^{ker} \\ 0 \end{pmatrix} \rangle = \langle \psi(-L), \begin{pmatrix} \mathsf{w}_-^{ker} \\ 0 \end{pmatrix} \rangle + O(e^{-L}) = O(e^{-L}).$$

Similarly,

$$\langle \psi(-L_*), \begin{pmatrix} 0 \\ \mathsf{w}_-^{gker} \end{pmatrix} \rangle = \langle \psi(-L_*), \begin{pmatrix} (-L+L_*)w_-^{gker}(-L_*) \\ w_-^{gker}(-L_*) \end{pmatrix} + O(e^{-L}) \rangle$$

$$= \langle \psi(-L_*), \begin{pmatrix} 0 \\ w_-^{gker}(-L_*) \end{pmatrix} + O(e^{-L}) \rangle$$

On the other hand,

$$\langle \psi(-L), \begin{pmatrix} 0 \\ \mathsf{w}_-^{gker} \end{pmatrix} \rangle = \langle \psi(-L_*), \begin{pmatrix} 0 \\ w_-^{gker}(-L_*) \end{pmatrix} \rangle.$$

Then

$$\langle \psi(-L_*), \begin{pmatrix} 0 \\ \mathsf{w}_-^{gker} \end{pmatrix} \rangle = \langle \psi(-L), \begin{pmatrix} 0 \\ \mathsf{w}_-^{gker} \end{pmatrix} \rangle + O(e^{-L}).$$

In conclusion, we have

$$P_-^c(-L)\psi(-L) = P_-^c(-L)\psi(-L_*) + O(e^{-L}) = P_-^{gker}(-L)\psi(-L_*) + O(e^{-L}).$$

For any $\mathsf{w}_-^{ss} \in E_-^{ss}(-L)$ with $\|\mathsf{w}_-^{ss}\|_X = 1$,

$$\langle P_-^h(-L)\psi(-L), \mathsf{w}_-^{ss} \rangle = \langle \psi(-L), \mathsf{w}_-^{ss} \rangle \tag{3.68}$$

$$= \langle \psi(R_*), \phi_-^{ss}(R_*, -L)\mathsf{w}_-^{ss} \rangle = O(e^{-L})\|\psi(R_*)\| = O(e^{-L}).$$

For any $\mathsf{w}_-^{uu} \in E_-^{uu}(-L)$, there exists $\tilde{\mathsf{w}}_-^{uu} \in Y_-^u$ such that $\mathsf{w}_-^{uu} = \Phi_-^{uu}(-L, R_*)\tilde{\mathsf{w}}_-^{uu}$.

Therefore,

$$\langle P_-^h(-L)\psi(-L), \mathsf{w}_-^{uu} \rangle = \langle \psi(R_*), \tilde{\mathsf{w}}_-^{uu} \rangle = 0. \tag{3.69}$$

From (3.68) and (3.69), we conclude that $P_-^h(-L)\psi(-L) = O(e^{-L})$.

Define $\psi_- := P_-^{gker}\psi(-L_*)$, then by Lemma 3.3.1,

$$\begin{aligned}\psi(-L) &= P_-^{gker}(-L)\psi(-L) + P_-^h(-L)\psi(-L)\\ &= [(P_-^{gker} + O(e^{-L}))\psi(-L_*) + O(e^{-L})] + P_-^h(-L)\psi(-L)\\ &= P_-^{gker}\psi(-L_*) + O(e^{-L})\\ &= \psi_- + O(e^{-L})\end{aligned}$$

□

Recall that from Definition 3.6.1,

$$\langle \psi(R_*), \mathcal{U}_-(R_*)\rangle = \langle \psi(R_*), P_-(\mathcal{U}_-(R_*)\rangle.$$

when $\omega = \omega_*$. Therefore, as a by-product, Lemma 3.6.2 gives an estimate of $\langle \psi(R_*), \mathcal{U}_-(R_*)\rangle$ with $\omega = \omega_*$. For ω near ω_*, we derive a more precise expansion for the mismatch $\langle \psi(R_*), \mathcal{U}_-(R_*) - \mathcal{U}_+(R_*)\rangle$.

We assume that the Melnikov integral associated with $\tilde{\mathcal{G}}$ is non-vanishing. That is,

Hypothesis 3.3. $M := \int_{-\infty}^{+\infty} \langle \psi(\tau), D_\omega \tilde{\mathcal{G}}(\mathcal{U}_*, \omega_*)(\tau)\rangle d\tau \neq 0.$

The Melnikov integral M is the jump of $M_-^{cu}(u_*, \omega)$ and $M_+^{cs}(u_*, \omega)$ in the direction of the unique bounded solution ψ of the adjoint at $\tau = R_*$, which describes the transversality of the intersection of $M_-^{cu}(u_*, \omega)$ and $M_+^{cs}(u_*, \omega)$ with respect to ω.

Lemma 3.6.6. *The mismatch in the Z-direction is given by*

$$\begin{aligned}\xi &:= \langle \psi(R_*), \mathcal{U}_-(R_*) - \mathcal{U}_+(R_*)\rangle\\ &= \langle \psi(R_*), w_-^{gker}\rangle + \langle \psi(R_*), \Phi(R_*, -L)w_-^{ss}\rangle + (M + O(e^{-L}))(\omega - \omega_*)\\ &\quad + \mathcal{R}_Z(\omega - \omega_*, d)\end{aligned}$$

in which the remainder $\mathcal{R}_Z$ satisfies the following estimate:

$$\|\mathcal{R}_Z\| = O(|\omega - \omega_*|^2 + |d|^2).$$

Proof. We collect the principal term of $\omega-\omega_*$ in $(\mathcal{U}_-(R_*)-\mathcal{U}_+(R_*))$ as the following

$$\begin{aligned} J_l &= (\omega-\omega_*)\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{c}(R_*,\tau)]e^{2\tau}\begin{pmatrix}0\\ -D^{-1}\partial_\psi u_*\end{pmatrix}d\tau \\ &\quad -(\omega-\omega_*)\int_{+\infty}^{R_*}\Phi_+^{uu}(R_*,\tau)\begin{pmatrix}0\\ -D^{-1}\partial_\psi u_*\end{pmatrix}d\tau, \end{aligned}$$

The higher order term J_{nl} given in the following arises from the dependence of w_-^{gker}, w_-^{ss} and u on $\omega-\omega_*$:

$$\begin{aligned} J_{nl} &:= ((\omega-\omega_*)\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{c}(R_*,\tau)]e^{2\tau}\begin{pmatrix}0\\ -D^{-1}\partial_\psi u\end{pmatrix}d\tau \\ &\quad +\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{c}(R_*,\tau)]e^{2\tau}\mathcal{N}_2(\mathcal{U}_-(\tau))d\tau) \\ &\quad -((\omega-\omega_*)\int_{+\infty}^{R_*}\Phi_+^{uu}(R_*,\tau)\begin{pmatrix}0\\ -D^{-1}\partial_\psi u\end{pmatrix}d\tau \\ &\quad +\int_{\infty}^{R_*}\Phi_+^{uu}(R_*,\tau)\mathcal{N}_2(\mathcal{U}_+(\tau))d\tau) \end{aligned}$$

We project J_l on the direction of Z, which yields

$$\begin{aligned} &(\omega-\omega_*)\langle\psi(R_*),\int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{c}(R_*,\tau)]e^{2\tau}\begin{pmatrix}0\\ -D^{-1}\partial_\psi u_*\end{pmatrix}d\tau \\ &\qquad -\int_{+\infty}^{R_*}\Phi_+^{uu}(R_*,\tau)\begin{pmatrix}0\\ -D^{-1}\partial_\psi u_*\end{pmatrix}d\tau\rangle \\ =\quad &(\omega-\omega_*)\int_{-L}^{R_*}\langle\psi(R_*),[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{c}(R_*,\tau)]e^{2\tau}\begin{pmatrix}0\\ -D^{-1}\partial_\psi u_*\end{pmatrix}\rangle d\tau \\ &-(\omega-\omega_*)\int_{+\infty}^{R_*}\langle\psi(R_*),\Phi_+^{uu}(R_*,\tau)\begin{pmatrix}0\\ -D^{-1}\partial_\psi u_*\end{pmatrix}\rangle d\tau \\ =\quad &(\omega-\omega_*)\int_{-L}^{R_*}\langle[\Phi_-^{ss}(R_*,\tau)+\Phi_-^{gker}(R_*,\tau)]^*\psi(R_*),e^{2\tau}\begin{pmatrix}0\\ -D^{-1}\partial_\psi u_*\end{pmatrix}\rangle d\tau \\ &-(\omega-\omega_*)\int_{+\infty}^{R_*}\langle\Phi_+^{uu}(R_*,\tau)^*\psi(R_*),\begin{pmatrix}0\\ -D^{-1}\partial_\psi u_*\end{pmatrix}\rangle d\tau \\ =\quad &(\omega-\omega_*)(\int_{-L}^{R_*}\langle\psi(\tau),e^{2\tau}\begin{pmatrix}0\\ -D^{-1}\partial_\psi u_*\end{pmatrix}\rangle d\tau-\int_{+\infty}^{R_*}\langle\psi(\tau),\begin{pmatrix}0\\ -D^{-1}\partial_\psi u_*\end{pmatrix}\rangle d\tau) \\ =\quad &(\omega-\omega_*)\int_{-L}^{+\infty}\langle\psi(\tau),D_\omega\tilde{\mathcal{G}}(\mathcal{U}(\omega),\omega)|_{\omega=\omega_*,\mathcal{U}=0}(\tau)\rangle d\tau. \end{aligned}$$

Observe that

$$
\begin{aligned}
& \Big| \int_{-\infty}^{-L} \langle \psi(\tau), D_\omega \tilde{\mathcal{G}}(\mathcal{U}, \omega)|_{\omega=\omega_*, \mathcal{U}=0}(\tau) \rangle d\tau \\
\leq \;& \|\psi\|_\infty \cdot \int_{-\infty}^{-L} e^{2\tau} \|D_\omega \mathcal{G}(\mathcal{U}, \omega)|_{\omega=\omega_*, \mathcal{U}=0}\|_\infty d\tau \qquad (3.70) \\
= \;& O(e^{-L}) \|D_\omega \tilde{\mathcal{G}}(\mathcal{U}(\omega), \omega)|_{\omega=\omega_*, \mathcal{U}=0}\|_\infty. \qquad (3.71)
\end{aligned}
$$

Next, we would like to project J_{nl} along the direction of Z: From Theorem 3.5.2 and Lemma 3.6.3,

$$
\begin{aligned}
& |\langle \psi(R_*), (\omega - \omega_*) \int_\infty^{R_*} \Phi_+^{uu}(R_*, \tau) \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u \end{pmatrix} d\tau \rangle| \\
\leq \;& C|\omega - \omega_*| \|\psi(R_*)\| (|w_+^{ss}| + |\omega - \omega_*| + |d|) \int_\infty^{R_*} e^{R_* - \tau} d\tau \\
\leq \;& C(|\omega - \omega_*|^2 + |w_+^{ss}|^2 + |d|^2) \\
\leq \;& C(|\omega - \omega_*|^2 + |d|^2); \\
& |\langle \psi(R_*), (\omega - \omega_*) \int_{-L}^{R_*} [\Phi_-^{ss}(R_*, \tau) + \Phi_-^{c}(R_*, \tau)] e^{2\tau} \begin{pmatrix} 0 \\ -D^{-1}\partial_\psi u \end{pmatrix} d\tau \rangle| \\
\leq \;& C|\omega - \omega_*| \|\psi(R_*)\| (|w_-^{ker}| + |w_-^{uu}| + |\omega - \omega_*| + |d|) \int_{-L}^{R_*} [e^{\tau - R_*} + (\tau - R_*) e^{2\tau}] d\tau \\
\leq \;& C(|\omega - \omega_*|^2 + |w_-^{ker}|^2 + |w_+^{uu}|^2 + |d|^2) \\
\leq \;& C(|\omega - \omega_*|^2 + |d|^2); \\
& |\langle \psi(R_*), \int_\infty^{R_*} \Phi_+^{uu}(R_*, \tau) \mathcal{N}_2(\mathcal{U}_+(\tau)) d\tau \rangle| \\
\leq \;& C\|\psi(R_*)\| (|w_+^{ss}|^2 + |\omega - \omega_*|^2 + |d|^2) \int_\infty^{R_*} e^{R_* - \tau} d\tau \\
\leq \;& C(|\omega - \omega_*|^2 + |w_+^{ss}|^2 + |d|^2) \\
\leq \;& C(|\omega - \omega_*|^2 + |d|^2);
\end{aligned}
$$

$$
\begin{aligned}
&|\langle\psi(R_*), \int_{-L}^{R_*}[\Phi_-^{ss}(R_*,\tau)+\Phi_-^c(R_*,\tau)]e^{2\tau}\mathcal{N}_2(\mathcal{U}_-(\tau))d\tau\rangle| \\
\leq\ & C\|\psi(R_*)\|(|w_-^{ker}|^2+|w_-^{uu}|^2+|\omega-\omega_*|^2+|d|^2)\int_{-L}^{R_*}[e^{\tau-R_*}+(\tau-R_*)e^{2\tau}]d\tau \\
\leq\ & C(|\omega-\omega_*|^2+|w_-^{ker}|^2+|w_+^{uu}|^2+|d|^2) \\
\leq\ & C(|\omega-\omega_*|^2+|d|^2).
\end{aligned}
$$

In conclusion, we have

$$
\begin{aligned}
\xi &= \langle\psi(R_*),\mathcal{U}_-(R_*)-\mathcal{U}_+(R_*)\rangle \\
&= \langle\psi(R_*),w_-^{gker}\rangle+\langle\psi(R_*),\Phi_-^{ss}(R_*,-L)w_-^{ss}\rangle \\
&\qquad +(M+O(e^{-L}))(\omega-\omega_*)+\langle\psi(R_*),J_{nl}\rangle \\
&= \langle\psi(R_*),w_-^{gker}\rangle+\langle\psi(R_*),\Phi_-^{ss}(R_*,-L)w_-^{ss}\rangle \\
&\qquad +(M+O(e^{-L}))(\omega-\omega_*)+\mathcal{R}_Z(\omega-\omega_*,d)
\end{aligned}
$$

with

$$
\|\mathcal{R}_Z\| = O(|\omega-\omega_*|^2+|d|^2).
$$

□

Before we state our main result, we consider a generic assumption which we will use for the derivation of the ω-expansion: We assume that

Hypothesis 3.4. $\psi_- \neq 0$.

Since $\psi_- \in \{0\}\times\mathbb{R}^N \subset E_-^c$,

$$
\langle P_-^{gker}\psi, \begin{pmatrix} 0 \\ \gamma_{22}P_-^{ker}\mathrm{id}_{E_-^c} \end{pmatrix}\rangle \neq 0.
$$

with the inner product taken in E_-^c. Therefore, if Hypothesis 3.4 is satisfied, then we can always normalize ψ such that

$$
\langle\psi, \begin{pmatrix} 0 \\ \gamma_{22}P_-^{ker}\mathrm{id}_{E_-^c} \end{pmatrix}\rangle = 1. \tag{3.72}
$$

Then we state our main result as the following:

Theorem 3.6.1 (T1 and Neumann boundary conditions). *Assume that Hypotheses 3.1, 3.2 and 3.4 are satisfied. There exists $\delta_1, \delta_2 > 0$, such that for any $L > 1/\delta_1$ and any $(w_-^{ker}, w_-^{uu}, w_+^{ss}, \omega)$ with $|w_-^{ker}| + |w_-^{uu}| + |w_+^{ss}| + |\omega - \omega_*| < \delta_1$, then there exists a unique solution*

$$\mathcal{U} = \mathcal{U}(w_-^{ker}(d), w_-^{uu}(d), w_+^{ss}(d), \omega(d), d) \in B_{\delta_2}(0) \subset H^1(\mathbb{S}^1, \mathbb{C}^N) \times L^2(\mathbb{S}^1, \mathbb{C}^N)$$

of the truncated boundary value problem in which the boundary condition reads

$$\mathcal{U}(-L) \in E_-^{bc} + d \tag{3.73}$$

where either

$$d = \begin{pmatrix} d_1 \\ 0 \end{pmatrix} := \begin{pmatrix} u_*(0) - u_*(e^{-L}) \\ 0 \end{pmatrix} \qquad \textit{(T1 condition)}, \tag{3.74}$$

or

$$d = \begin{pmatrix} 0 \\ d_2 \end{pmatrix} := \begin{pmatrix} 0 \\ -e^{-L}(u_*)_r(e^{-L})) \end{pmatrix} \qquad \textit{(Neumann condition)} \tag{3.75}$$

at $-L$. Moreover, $(w_-^{ker}, w_-^{uu}, w_+^{ss}, \omega)$ depends smoothly on d and

$$|w_-^{ker}| + |w_-^{uu}| + |w_+^{ss}| + |\omega - \omega_*| \leq \frac{Ce^{-L}}{L}|\partial_r u_*(0)|, \qquad \textit{(T1 condition)};$$

$$|w_-^{ker}| + |w_-^{uu}| + |w_+^{ss}| + |\omega - \omega_*| \leq Ce^{-L}|\partial_r u_*(0)|. \qquad \textit{(Neumann condition)}.$$

Furthermore, using the normalization (3.72), then ω has the expansion

$$\omega = \omega_* + \frac{1}{M}\frac{u_*(0) - u_*(e^{-L})}{L} + \mathcal{R}, \qquad \textit{(T1 condition)},$$

near ω_* where $\mathcal{R} = O(\frac{e^{-L}}{L^2})|\partial_r u_*(0)| + O(\frac{e^{-2L}}{L})|\partial_r u_*(0))|^2$, and

$$\omega = \omega_* + \frac{1}{M}e^{-L}\partial_r u_*(e^{-L}) + \mathcal{R}, \qquad \textit{(Neumann condition)}.$$

where $\mathcal{R} = O(e^{-2L}))|\partial_r u_*(e^{-2L})|$.

Proof. From Lemma 3.6.3, we can solve w_-^{ker}, w_-^{uu} and w_+^{ss} in terms of $\omega-\omega_*$ and d. Next, we need to investigate the expression $\xi := \langle \psi(R_*), \mathcal{U}_-(R_*) - \mathcal{U}_+(R_*)\rangle$ provided in Lemma 3.6.6 in the case of Dirichlet and Neumann conditions.

First, we consider the case of Dirichlet conditions. Recall that T_L defined in (3.50) is linear,

$$\begin{aligned}
&\langle \psi(R_*), w_-^{gker}\rangle \\
=\ & \langle \psi(-L), \Phi_-^c(-L,R_*) w_-^{gker}\rangle \\
=\ & \langle \psi(-L), \begin{pmatrix} (-L+L_*)\gamma_{22} w_-^{gker} \\ \gamma_{22} w_-^{gker}\end{pmatrix} + O(e^{-L}) w_-^{gker}\rangle \\
=\ & \langle \psi(-L), \begin{pmatrix} 0 \\ \gamma_{22} w_-^{gker}\end{pmatrix}\rangle + (-L+L_*)\langle \psi(-L), \begin{pmatrix} \gamma_{22} w_-^{gker} \\ 0\end{pmatrix}\rangle + O(e^{-L}) w_-^{gker} \\
=\ & \langle \psi(-L), \begin{pmatrix} 0 \\ \gamma_{22} w_-^{gker}\end{pmatrix}\rangle + O(Le^{-L}) w^{gker} + O(e^{-L}) w_-^{gker} \\
=\ & \langle \psi(-L), \begin{pmatrix} 0 \\ \gamma_{22} T_L^{-1}(\gamma_{11} w_-^{ker} - P_-^{ker} d + O(e^{-L})\mathcal{R}_2(w_-^{ker}, w_-^{uu}, \omega-\omega_*, d)\end{pmatrix}\rangle \\
& \quad + O(Le^{-L}) w_-^{gker} \\
=\ & \langle \psi(-L), \begin{pmatrix} 0 \\ \gamma_{22} T_L^{-1}(\gamma_{11} w_-^{ker})\end{pmatrix}\rangle - \langle \psi(-L), \begin{pmatrix} 0 \\ \gamma_{22} T_L^{-1} P_-^{ker} d\end{pmatrix}\rangle \\
& + O(e^{-L})\langle \psi(-L), \begin{pmatrix} 0 \\ \mathcal{R}_2(w_-^{ker}, w_-^{uu}, \omega-\omega_*, d)\end{pmatrix}\rangle + O(e^{-L})(|\omega-\omega_*| + |d|)
\end{aligned}$$

From Lemma 3.6.3, $w_-^{ker} = (\omega-\omega_*)B_-^{ker} + \mathcal{R}_{ker}(w_-^{ker}, w_-^{uu}, w_+^{ss}, \omega-\omega_*, d)$. Then

$$\begin{aligned}
& \langle \psi(-L), \begin{pmatrix} 0 \\ \gamma_{22} T_L^{-1}(\gamma_{11} w_-^{ker})\end{pmatrix}\rangle \\
=\ & \langle \psi_- + O(e^{-L}), \begin{pmatrix} 0 \\ \frac{1}{L}(1+O(\frac{1}{L}))\gamma_{22}[\gamma_{11}(\omega-\omega_*)B_-^{ker} + \mathcal{R}_{ker}]\end{pmatrix}\rangle \\
=\ & \frac{1}{L}(\omega-\omega_*)\langle \psi_-, \begin{pmatrix} 0 \\ \gamma_{22}\gamma_{11} B_-^{ker}\end{pmatrix}\rangle + \mathcal{R}^{\psi,1}(\omega-\omega_*, d)
\end{aligned}$$

in which $\mathcal{R}_{ker} = \mathcal{R}_{ker}(w_-^{ker}, w_-^{uu}, w_+^{ss}, \omega-\omega_*, d)$ and the higher order terms

$$\begin{aligned}
\mathcal{R}^{\psi,1}(\omega-\omega_*, d) \ =\ & O(\frac{1}{L^2})(\omega-\omega_*)\langle \psi_-, \begin{pmatrix} 0 \\ \gamma_{22}\gamma_{11} B_-^{ker}\end{pmatrix}\rangle + O(\frac{|\omega-\omega_*|^2 + |d|^2}{L}) \\
& + O(\frac{e^{-L}}{L})(\omega-\omega_*)\langle \mathrm{id}_{E_-^c}, \begin{pmatrix} 0 \\ \gamma_{22}\gamma_{11} B_-^{ker}\end{pmatrix}\rangle.
\end{aligned}$$

Similarly,

$$\langle\psi(-L),\begin{pmatrix}0\\ \gamma_{22}T_L^{-1}P_-^{ker}d\end{pmatrix}\rangle=\frac{1}{L}\langle\psi_-,\begin{pmatrix}0\\ \gamma_{22}P_-^{ker}d\end{pmatrix}\rangle+\mathcal{R}^{\psi,2}(d),$$

in which the higher order term

$$\mathcal{R}^{\psi,2}(d)=O(\frac{1}{L^2})\langle\psi_-,\begin{pmatrix}0\\ \gamma_{22}P_-^{ker}d\end{pmatrix}\rangle.$$

On the other hand, from Theorem 3.5.1, we have

$$\begin{aligned}
&|\langle\psi(R_*),\Phi_-^{ss}(R_*,-L)w_-^{ss}\rangle|\le Ce^{-L}(e^{-L}|w_-^{ker}|+e^{-L}|w_-^{uu}|+e^{-L}\|\mathcal{G}\|_\infty+|d|)\\
\le\ & Ce^{-L}(e^{-L}|w_-^{ker}|+e^{-L}|w_-^{uu}|+e^{-L}(|\omega-\omega_*|+|w_-^{ker}|^2+|w_-^{uu}|^2+|\omega-\omega_*|^2\\
&+|d|^2)+|d|)\\
\le\ & C(e^{-2L}|\omega-\omega_*|+e^{-L}|d|+e^{-2L}|\omega-\omega_*|^2+e^{-2L}|d|^2)
\end{aligned}$$

Therefore, setting $\xi=0$ yields

$$\begin{aligned}
0\ =\ & \left(M+O(e^{-L})+\frac{1}{L}\langle\psi_-,\begin{pmatrix}0\\ \gamma_{22}\gamma_{11}B_-^{ker}\end{pmatrix}\rangle\right)(\omega-\omega_*)-\frac{1}{L}\langle\psi_-,\begin{pmatrix}0\\ \gamma_{22}P_-^{ker}d\end{pmatrix}\rangle\\
&+\tilde{R}_Z(\omega-\omega_*,d).
\end{aligned}$$

with

$$\tilde{R}_Z(\omega-\omega_*,d)=O(\frac{1}{L^2})(|\omega-\omega_*|+|d|)+O(\frac{1}{L})(|\omega-\omega_*|^2+|d|^2).$$

Then with $d=\begin{pmatrix}d_1\\0\end{pmatrix}:=\begin{pmatrix}u_*(0)-u_*(e^{-L})\\0\end{pmatrix}$, we reach

$$\begin{aligned}
\omega\ =\ & \omega_*+\frac{1}{L}\frac{\left\langle\psi_-,\begin{pmatrix}0\\ \gamma_{22}P_-^{ker}\begin{pmatrix}u_*(0)-u_*(e^{-L})\\0\end{pmatrix}\end{pmatrix}\right\rangle}{M+O(\frac{1}{L})}\\
&+O(\frac{1}{L^2})(|\omega-\omega_*|+|d|)+O(\frac{1}{L})(|\omega-\omega_*|^2+|d|^2).\\
=\ & \omega+\frac{\left\langle\psi_-,\begin{pmatrix}0\\ \gamma_{22}P_-^{ker}\mathrm{id}_{E_-^c}\end{pmatrix}\right\rangle}{M\cdot L}(u_*(0)-u_*(e^{-L}))\qquad(3.76)\\
&+O(\frac{1}{L^2})(|\omega-\omega_*|+|d|)+O(\frac{1}{L})(|\omega-\omega_*|^2+|d|^2)\\
=\ & \omega+\frac{u_*(0)-u_*(e^{-L})}{M\cdot L}+O(\frac{1}{L^2})(|\omega-\omega_*|+|d|)+O(\frac{1}{L})(|\omega-\omega_*|^2+|d|^2)
\end{aligned}$$

For Neumann conditions, with the identical argument except the estimate for w_-^{gker} and w_-^{ss}, we have that

$$|\omega - \omega_*| \leq Ce^{-L}|\partial_r u_*(e^{-L})| \leq Ce^{-L}|\partial_r u_*(0)|. \tag{3.77}$$

Then the estimates for w_-^{ker}, w_-^{uu} and w_+^{ss} follow from Lemma 3.6.1 and (3.77).

□

APPENDIX A

SUPPLEMENTARY NOTES

A.1 Adjoint equation

Suppose that $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space and consider on X the linear equation

$$\frac{d}{dt}V = A(t)V, \qquad V \in X, \tag{A-1}$$

in which $A(t)$ is closed and densely defined. Then there is a unique maximal operator $A(t)^*$ adjoint to $A(t)$, see [27]. Define the adjoint variational equation with respect to the inner product to be

$$\frac{d}{dt}W = -(A(t))^*W, \qquad W \in X. \tag{A-2}$$

Suppose that $(A-1)$ and $(A-2)$ possess evolution operators $\Phi(s,t)$ and $\Psi(s,t)$ respectively.

We collect useful properties about the adjoint equation in the following proposition.

Proposition A.1.1. *(1) Suppose that $V(t)$ and $W(t)$ satisfy $(A-1)$ and $(A-2)$ respectively. Then $\langle W(t), V(t)\rangle$ is constant with respect to t.*

(2) $\Psi(t,s) = (\Phi(s,t))^*$.

Proof. For (1), we have

$$\begin{aligned}\frac{d}{dt}\langle W(t), V(t)\rangle &= \langle \dot{W}(t), V(t)\rangle + \langle W(t)\dot{V}(t)\rangle = \langle -A(t)^*W(t), V(t)\rangle + \langle W(t), A(t)V(t)\rangle \\ &= \langle W(t), -A(t)V(t)\rangle + \langle W(t), A(t)V(t)\rangle = 0.\end{aligned}$$

For (2), by definition, $\Phi(t,s)\circ\Phi(s,t) = id$. Differentiating yields

$$\begin{aligned}(D_t\Phi(t,s))\circ\Phi(s,t) + \Phi(t,s)\circ D_t\Phi(s,t) &= 0\\ A(t)\Phi(t,s)\circ\Phi(s,t) + \Phi(t,s)\circ D_t\Phi(s,t) &= 0\\ D_t\Phi(s,t) &= -\Phi(s,t)A(t)\end{aligned}$$

Taking adjoint on both side, we reach $D_t(\Phi(s,t))^* = -A(t)^*(\Phi(s,t))^*$. By the uniqueness, we have (2).

□

A.2 A Graph Lemma

Lemma A.2.1. *Suppose that $(X, \langle\cdot,\cdot\rangle)$ is a Hilbert space, and E, $A \subset X$ are closed linear subspaces. Then E satisfies*

$$X = E \oplus A$$

if and only if E can be written as a graph over the orthogonal complement $A^\perp$ of A, that is, if there exists a linear and bounded operator

$$W : A^\perp \to A$$

such that $E =$ graph $W = \{a^\perp + Wa^\perp; a \in A^\perp\}$.

Proof. Since $X = A\oplus A^\perp$, for any $e \in E \subset X$, there exists a unique pair $(a, a^\perp) \in A\oplus A^\perp$ such that

$$e = a + a^\perp. \tag{A-3}$$

Suppose that $X = E \oplus A$, then

$$a^\perp = \tilde{e} + \tilde{a} \tag{A-4}$$

for a unique pair $(\tilde{e}, \tilde{a}) \in E \oplus A$. Comparing (A-3) and (A-4) yields $\tilde{e} = e$, and $\tilde{a} = -a$. Define the projection $P : X \to A$ on A along $A^\perp$ by $Pa^\perp := -a$. Therefore,

$$e = a^\perp + a = a^\perp - Pa^\perp =: a^\perp + Wa^\perp.$$

The boundedness of P is given as an application of the closed graph theorem (see, e.g. [27]).

Conversely, if $E =$ graph $W = \{a^\perp + Wa^\perp; a \in A^\perp\}$ with $W : A^\perp \to A$ bounded, then

$$X \in x = a + a^\perp = (a - Wa^\perp) + (a^\perp + Wa^\perp) \in A \oplus E.$$

On the other hand, if $x = \tilde{a} + \tilde{e} = \tilde{a} + (\hat{a^\perp} + W\hat{a^\perp})$, for some $e \in E$, $\tilde{a} \in A$ and $\hat{a^\perp} \in A^\perp$, then

$$a - \tilde{a} - W\hat{a^\perp} = \hat{a^\perp} - a^\perp \in A \cap A^\perp = \{0\}.$$

Then $\hat{a^\perp} = a^\perp$ and $e = \tilde{e}$. Thus, $X = E \oplus A$.

□

A.3 Operator $\mathcal{A}_\infty^-$

Lemma A.3.1. *The equation*

$$\mathcal{U}_s = \mathcal{A}_\infty^- \mathcal{U}, \qquad \mathcal{U} = (u, w) \tag{A-5}$$

is closed on $X = H^1(\mathbb{S}, \mathbb{C}^N) \times L^2(\mathbb{S}, \mathbb{C}^N)$ *and with dense domain* $H^2(\mathbb{S}^1, \mathbb{C}^N) \times H^1(\mathbb{S}^1, \mathbb{C}^N)$.

Proof. $\mathcal{U}_s \in T_{\mathcal{U}} X \cong X$, the tangent space of X at $\mathcal{U}$. Then we have $w \in H^1(\mathbb{S}^1, \mathbb{C}^N)$ and $-\partial_{\psi\psi} u \in L^2(\mathbb{S}^1, \mathbb{C}^N)$. Also the boundedness of $\partial_\psi u$ is from the integration by parts and the fact that $\mathbb{S}^1$ has no boundary:

$$\langle \partial_\psi u, \partial_\psi u \rangle_{L^2(\mathbb{S}^1, \mathbb{C}^N)} = -\langle u, \partial_{\psi\psi} u \rangle_{L^2(\mathbb{S}^1, \mathbb{C}^N)} < \infty.$$

Hence $u \in H^2(\mathbb{S}^1, \mathbb{C}^N) \times H^1(\mathbb{S}^1, \mathbb{C}^N)$.

Suppose that $\{x_n\}$ is a sequence in X and $x_n \to x$ in X; also $\{\mathcal{A}_\infty^- x_n\}$ is a Cauchy sequence in X. That is, $\{x_n\}$ is $\mathcal{A}_\infty^-$-convergent to x. By definition, we need to prove that $x \in \mathcal{D}(\mathcal{A}_\infty^-)$, the domain of the operator $\mathcal{A}_\infty^-$ and $\mathcal{A}_\infty^- x = \lim \mathcal{A}_\infty^- x_n$.

$x \in \mathcal{D}(\mathcal{A}_\infty^-)$: First, since $\{\mathcal{A}_\infty^- x_n\}$ is Cauchy in X, it converges to some y in X. Write $x_n = \begin{pmatrix} x_{n,1} \\ x_{n,2} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ in X, we have

$$\|\mathcal{A}_\infty^- x_n - y\|_X^2 = \| - \partial_{\psi\psi} x_{n,1} - y_2\|_{L^2}^2 + \|x_{n,2} - y_1\|_{H^1}^2 \to 0.$$

Secondly, we also have that $x_n \to x$ in X, that gives

$$\|x_n - x\|_X^2 = \|x_{n,1} - x_1\|_{H^1}^2 + \|x_{n,2} - x_2\|_{L^2}^2 \to 0$$

Combining the above two facts, we have that $x_{n,1}$ converges in H^2 and $x_{n,2}$ converges in H^1. That is, x_n converges to $z \in H^2 \times H^1$. Now

$$\|x - z\|_X \le \|x_n - x\|_X + \|x_n - z\|_X \le \|x_n - x\|_X + \|x_n - z\|_{H^2 \times H^1} \to 0.$$

So $x = z$.

$\mathcal{A}_\infty^- x = \lim \mathcal{A}_\infty^- x_n$:

$$\mathcal{A}_\infty^- x_n = \begin{pmatrix} 0 & 1 \\ -\partial_{\psi\psi} & 0 \end{pmatrix} \begin{pmatrix} x_{n,1} \\ x_{n,2} \end{pmatrix} = \begin{pmatrix} x_{n,2} \\ -\partial_{\psi\psi} x_{n,1} \end{pmatrix} \to \begin{pmatrix} x_2 \\ \partial_{\psi\psi} x_1 \end{pmatrix} = \mathcal{A}_\infty^- x.$$

Note that the convergence in the previous expression is in $H^1 \times L^2$ and is due to the fact that $x_n \to x$ in $H^2 \times H^1$. □

Lemma A.3.2. *The asymptotic spectral projections P_-^{uu} and P_-^{ss} are bounded.*

Proof. In order to study the (un)stable projections P_-^{ss}(P_-^{uu}) and the center projection P_-^c, we first study the behavior of the projections onto a single mode.

Let us first introduce some notation. Let

$$P_k^{uu} : X = H^1 \times L^2 \to E_k^{uu} \subset E_k \subset X$$

to be the *unstable projection onto the k-th mode in* X^0 with the null space to be the complement of the k-th mode. Here the norm in X^0 is inherited from X and the null space is

$(E_k^{uu})^{\perp}$ in X. Then

$$P_-^{uu} = \oplus_{k\neq 0} P_k^{uu}|_{E_k}.$$

and P_k^{uu} maps E_k to $E_k^{uu} \subset E_k$. Similarly for P_k^{ss}; also let E_k to be the k-th mode of X, i.e. $E_k = span\{(ue^{ik\psi}, ve^{ik\psi}) \in X; u, v \in \mathbb{C}^N\}$.

For $(u_k e^{ik\psi}, ve^{ik\psi}) \in X$, $k \neq 0$, the affine subspace $\{(ue^{ik\psi}, (-k(u-u_k)+v_k)e^{ik\psi});$ $u, v \in \mathbb{C}^N\}$ is parallel to E_k^{ss} and passing through the point $(u_k e^{ik\psi}, v_k e^{ik\psi})$. It intersects E_k^{uu} at $\frac{1}{2}((u_k + \frac{1}{k}v_k)e^{ik\psi}, (ku_k + v_k)e^{ik\psi})$. Hence we have the following:

$$P_k^{uu}|_{E_k} : (u_k e^{ik\psi}, v_k e^{ik\psi}) \mapsto \frac{1}{2}((u_k + \frac{1}{k}v_k)e^{ik\psi}, (ku_k + v_k)e^{ik\psi}).$$

with the norm $\|(u_k e^{ik\psi}, v_k e^{ik\psi})\|_k = ((1+|k|^2)|u_k|^2 + |v_k|^2)^{\frac{1}{2}}$. Now since the projection P_-^{uu} is the direct sum of P_k^{uu}, $P_k^{uu}|_{E_m}$ is zero for $k \neq m$. Hence we can suppress the restriction to E_k if no confusion caused.

$$\begin{aligned} \|P_k^{uu}\| &= \sup_{\|(u_k e^{ik\psi}, v_k e^{ik\psi})\|_X \leq 1} \|P_k^{uu}(u_k e^{ik\psi}, v_k e^{ik\psi})\|_k \qquad \text{(A-6)} \\ &= \sup_{((1+|k|^2)|u_k|^2 + |v_k|^2) \leq 1} \frac{1}{2}((1+|k|^2)|u_k + \frac{1}{k}v_k|^2 + |ku_k + v_k|^2)^{\frac{1}{2}} \end{aligned}$$

The term $(1+|k|^2)|u_k + \frac{1}{k}v_k|^2 \leq (1+|k|^2)(\frac{1}{1+|k|^2} + \frac{1}{k})^2$ and $|ku_k + v_k| \leq \frac{|k|}{1+|k|^2} + 1$. Hence P_k^{uu} is bounded for fixed k.

Now let us consider the norm of P_-^{uu}. For $(u, v) \in X \backslash E_-^c$, write

$$(u, v) = (\sum_{k\neq 0} u_k e^{ik\psi}, \sum_{k\neq 0} v_k e^{ik\psi}),$$

then

$$\begin{aligned} \|P_-^{uu}\| &:= \sup_{\|(u,v)\|_X \leq 1} \|P_-^{uu}(u, v)\|_k \\ &\leq \sup_{\|(u,v)\|_X \leq 1} (\sum_{k\neq 0} \|P_k^{uu}(u_k e^{ik\psi}, v_k e^{ik\psi})\|_k^2)^{\frac{1}{2}} \\ &= \sup_{\|(u,v)\|_X \leq 1} \frac{1}{2} \sum_{k\neq 0} ((1+|k|^2)|u_k + \frac{1}{k}v_k|^2 + |ku_k + v_k|^2). \end{aligned}$$

Recall that $\|(u,v)\|_X \leq 1$ implies that $\sum_k((1+|k|^2)|u_k|^2+|v_k|^2) \leq 1$. Write

$$\begin{aligned} & \sum_{k\neq 0}((1+|k|^2)|u_k+\frac{1}{k}v_k|^2+|ku_k+v_k|^2) \\ \leq \quad & 2\sum_{k\neq 0}((1+|k|^2)(|u_k|^2+\frac{1}{k^2}|v_k|^2)+|k|^2|u_k|^2+|v_k|^2) \\ \leq \quad & 6\sum_{k\neq 0}((1+|k|^2)|u_k|^2+|v_k|^2) \leq 6. \end{aligned}$$

Hence P_-^{uu} is bounded as well. □

A.4 Equivariance of the reaction-diffusion equation

Consider the Euclidean symmetry group $SE(2) = \mathbb{S}^1 \times \mathbb{R}^2$ [15] with the multiplication

.

$$(\tilde{\theta},\tilde{a})\cdot(\theta,a) = (\tilde{\theta}+\theta, \tilde{a}+e^{-i\tilde{\theta}}a), \qquad \tilde{\theta},\theta \in \mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}, \tilde{a}, a \in \mathbb{R}^2 \cong \mathbb{C}.$$

Observe that $(SE(2),\cdot)$ is a non-abelian group with

$$(\theta,a)^{-1} = (-\theta, -e^{-i\theta}a).$$

Recall the reaction-diffusion equation on the plane

$$u_t = D\Delta u + f(u), \qquad x \in \mathbb{R}^2, u \in \mathbb{R}^N. \tag{A-7}$$

Suppose that $u = u(x,t)$ satisfies (A-7), then consider

$$((\theta,a)\circ u)(x,t) := \tilde{u}(x,t) = u(e^{-i\theta}(x-a),t), \qquad (\theta,a) \in \mathbb{S}^1 \times \mathbb{R}^2 \cong SE(2). \tag{A-8}$$

For sake of convenience, we would like to reformulate (A-7) as an ODE in some appropriate Banach space of functions:

$$\frac{\partial}{\partial t}U = \mathcal{F}U := D\Delta U + f(U) \tag{A-9}$$

for $U = U(t) = u(\cdot,t)$.

Lemma A.4.1. *Equation (A-9) is equivariant under the action of the Euclidean symmetry group* $SE(2) = \mathbb{S}^1 \times \mathbb{R}^2$.

Proof. We can verify that $(A-9)$ is $SE(2)$-*equivariant*, that is,

$$(\theta, a) \circ \mathcal{F}U = \mathcal{F}((\theta, a) \circ U) \qquad (\theta, a) \in SE(2). \tag{A-10}$$

It is sufficient to verify $(A-10)$ for the generators: translations $(0, a)$ and rotations $(\theta, 0)$ due to the linearity of the group $SE(2)$. For the Laplacian, the equivariance is implied by the translation and rotation invariance of the Laplacian:

$$\begin{aligned}
\Delta((0,a)\circ U)(x) &= \Delta(U(x-a)) = (\Delta U)(x-a) = ((0,a)\circ \Delta U)(x) \\
\Delta((\theta,0)\circ U)(x) &= \Delta(U(e^{-i\theta}x)) = tr(D^2(U(e^{-i\theta}x))) \\
&= tr((e^{-i\theta})^{-1}\cdot D^2U\cdot(e^{-i\theta}))(x) \\
&= tr((e^{-i\theta})^{T}\cdot D^2U\cdot(e^{-i\theta}))(x) \\
&= (tr(D^2U))(e^{-i\theta}x) \\
&= (\Delta U)(e^{-i\theta}x) = ((\theta,0)\circ \Delta U)(x)
\end{aligned}$$

The equivariance of f is implied by the fact that f is not explicitly depending on x.

Then if U solves (A-9), then

$$\partial_t((\theta, a) \circ U) = (\theta, a) \circ \partial_t U = (\theta, a) \circ \mathcal{F}U = \mathcal{F}((\theta, a) \circ U).$$

Therefore, $(\theta, a) \circ U$ solves (A-9). Then the result follows.

□

BIBLIOGRAPHY

[1] A. Atri , J. Amundson , D. Clapham , J. Sneyd, *A single-pool model for intracellular calcium oscillations and waves in the Xenopus laevis oocyte*, Biophys J. **65(4)** (1993), 1727-1739.

[2] M. Bär, M. Eiswirth, *Turbulence due to spiral breakup in a continuous excitable medium*, Phys. Rev. E **48** (1993), R1635-R1637.

[3] D. Barkley, *Linear stability analysis of spiral waves in excitable media*, Phys. Rev. Lett. **68** (1992), 2090-2093.

[4] D. Barkley, *Euclidean symmetry and the dynamics of rotating spiral waves*, Phys. Rev. Lett. **72** (1994), 164-167.

[5] M. Braune, H. Engel, *Compound rotation of spiral waves in a light-sensitive Belousov-Zhabotinsky medium*, Chem. Phys. Lett., **204(3,4)** (1993), 257-264.

[6] M. Braune, H. Engel, *Compound rotation of spiral waves in active media with periodically modulated excitability*, Chem. Phys. Lett. **211(6)** (1993), 534-540.

[7] A. L. Belmonte, Q. Ouyang, J.-M. Flesselles, *Experimental survey of spiral dynamics in the Belousov-Zhabotinsky reaction*, J. Phys. II (France) **7** (1997), 1425-1468.

[8] W.-J. Beyn, *The numerical computation of connecting orbits in dynamical systems*, IMA J. Nemer. Anal. **10(3)**(1990), 379-405.

[9] C. Chicone, *Ordinary differential equations with applications*, Springer, 1999.

[10] S-N. Chow, J.K. Hale, *Methods of bifurcation Theory*, Springer-Verlag, 1982.

[11] E. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, 1955.

[12] D.S. Cohen, J.C. Neu, R.R. Rosales, *Rotating spiral wave solutions of reaction-diffusion equations*, SIAM J. Appl. Math. **35** (1978), 536-547.

[13] W.A. Coppel, *Dichotomies in stability theory*, Lecture Notes in Mathematics, **629**, Springer-Verlag, Berlin, 1978.

[14] J. Deng and S. Nii, *Infinite dimensional Evans function theory for elliptic eigenvalue problems in a channel*, Preprint.

[15] B. Fiedler, B. Sandstede, A. Scheel, C. Wulff, *Bifurcation from relative equilibria of noncompact group actions: Skew product, meanders and drifts*, Doc. Math. J. DMV **1** (1996), 479-505.

[16] B. Fiedler, A. Scheel, *Spatio-temporal dynamics of reaction-diffusion patterns*, Trends in nonlinear analysis, Springer (2003), 23-152.

[17] P.C. Fife, J.B. McLeod,*The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Rat. Mech. Anal. **65(4)** (1977), 335-361.

[18] A. Friedman, *Partial Differential equations*, Holt, Rinehart and Winston, 1969.

[19] J.M. Greenberg, *Spiral waves for $\lambda-\omega$ Systems*, SIAM J. Applied Math. **39(2)** (1980), 301-309.

[20] J. Guckenheimer, P. Holmes, *Nonlinear oscillators, dynamical systems, and bifurcations of vector fields*, **AMS 42**, Springer, New York, 1983.

[21] P. Hartman, *Ordinary differential equations*, CIAM, SIAM, 2002.

[22] S. Hastings, *On the existence of homoclinic and periodic orbits for the FitzHugh-Nagumo equations*, Quart. J. Math. Oxford(2) **27** (1976), 123-134.

[23] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, **840**, Springer-Verlag, Berlin, 1981.

[24] A. Holden, *Nonlinearity in the heart*, *Physics World*, November 1998.

[25] G.R. Ivanitskii, V. I. Krinsky, A.N. Zaikin, A. M. Zhabontinsky, Sov. Sci. Rev. **2D**, 1980.

[26] A. Karma, *Universal limit of spiral wave propagation in excitable media*, Phys. Rev. Lett. **66** (1991), 2274-2277.

[27] T. Kato, *Perturbation theory of linear operators*, Springer, 1995.

[28] J. P. Keener, J. J. Tyson, *Spiral waves in the Belousov-Zhabotinskii reaction*, Phys. D **21** (1986), 307-324.

[29] K. Kirchgässner, *Wave solutions of reversible systems and application*, J. Differential Equations **45**(1982), 113-127.

[30] N. Kopell, L. N. Howard, *Target pattern and spiral solutions to reaction-diffusion equations with more than one space dimension*, Adv. in Appl. Math. **2** (1981), 417–449.

[31] M. Krupa, B. Sandstede and P. Szmolyan, *Fast and slow waves in the FitzHugh-Nagumo equation*, J. of Differential Equations **133** (1997), 49-97.

[32] H. Levine, I. Arason, L. Tsimring, T. V. Truong, *Positive genetic feedback governs cAMP spiral wave formation in Dictyostelium*, Proc. Natl. Acad. Sci. USA **93** (1996), 6382-6386.

[33] X.-B. Lin, *Using Melnikov's method to solve Silnikov's problem*, Proceeding of the Royal Society of Edinburgh **116A**, (1990), 295-325.

[34] A. Mielke, S.V. Zelik, *Infinite-dimensional trajectory attractors of elliptic boundary-value problems in cylindrical domains*, Russian Math. Surveys, 57:4 (2002), 753-784.

[35] A.S. Mikhailov, *Foundations of synergetics I, Distributed active system* , Springer-Verlag, 2000.

[36] G. Métivier and K. Zumbrun, *Large Viscous Boundary Layers for Noncharacteristic Nonlinear Hyperbolic Problems*, Memoirs of the American Mathematical Society, 2005.

[37] J.D. Murray, *Mathematical Biology*, 2nd Ed., Springer, 1991.

[38] S. Nettesheim, A. Von Oertzen, H.H. Rotermund, G. Ertl, *Reaction diffusion patterns in the catalytic CO-oxidation on Pt(110): Front propagation and spiral waves*, J. Chem. Phys. **98(12)** (1993), 9977-9985.

[39] *Computational biology of the heart*, edited by A.V. Panfilov and A.V. Holden, John Wiley and sons, 1997.

[40] D. Peterhof, B. Sandstede and A. Scheel, *Exponential dichotomies for solitary wave solutions of semilinear elliptic equations on infinite cylinders*, J. Diff. Eqns. **140** (1997), 266-308.

[41] J. Paullet, B. Ermentrout, W. Troy, *The existence of spiral waves in an oscillatory reaction-diffusion system*, SIAM J. of Applied Math. **54(5)** (1994), 1386-1401.

[42] B. Sandstede, *Stability of travelling waves*, Handbook of Dynamical Systems II, North-Holland (2002), 983-1055.

[43] B.Sandstede, *Convergence estimates for the numerical approximation of homoclinic solutions*, IMA Journal of Numerical Ananlysis, **17**, 1997.

[44] B. Sandstede, *Verzweigungstheorie homokliner Verdopplungen*, Doctoral thesis, University of Stuttgart, 1993.

[45] B. Sandstede, *Stability of multiple-pulse solutions*, Trans. Amer. math. Soc. **350** (1998), 429-472.

[46] B. Sandstede and A. Scheel, *On the stability of periodic travelling waves with large spatial period*, J. of Differential Equations **172** (2001), 134-188.

[47] B. Sandstede and A. Scheel, *Stability and instability of spiral waves*, Manuscript, 2002.

[48] B. Sandstede and A. Scheel, *On the structure of spectra of modulated travelling waves*, Math. Nachr.**232** (2000), 39-93.

[49] B. Sandstede and A. Scheel, *Absolute versus convective instability of spiral waves*, Phys. Rev.E. **62** (2000), 7708-7714.

[50] B. Sandstede, A. Scheel *Absolute and convective instabilities of waves on unbounded and large bounded domains*, Physica D **145** (2000), 233-277.

[51] B. Sandstede and A. Scheel, *Defects in oscillatory media: Toward a classification*, SIAM J. Applied Dynamical Systems **3(1)** (2004), 1-68.

[52] B. Sandstede, A. Scheel and C. Wulff, in *Pattern formation in continuous and coupled systems*, M. Golubitsky, D. Luss and S. Strogatz (eds.), Springer-Verlag. IMA Volumes in Mathematics and its Applications **115** (1999), 249-264.

[53] S. Schecter, *Numerical Computation of Saddle-Node Homoclinic Bifurcation Points*, SIAM J. Numer. Anal. **30** (1993), 1155-1178.

[54] S. Schecter, *Rate of Convergence of Numerical Approximation to Homoclinic Bifurcation Points*, IMA J. Numer. Anal. **15(1)** (1995), 23-60.

[55] A. Scheel, *Bifurcation to spiral waves in reaction-diffusion system*, SIAM Journal of Math. Anal. **29(6)** (1998), 1399-1418.

[56] A. Scheel, *Radially symmetric Patterns of reaction-diffusion systems*, Memoirs of American Mathematical Society, 2003.

[57] A.M. Turing, *The chemical basis of morphogenesis*, Phil. Trans. R. Soc. Lond. **237** (1952), 37-72.

[58] J.J. Tyson, P.C. Fife, *Target patterns in a realistic model of the Belousov-Zhabotinskii reaction*, J. Chem. Phys. **73(5)** (1980), 2224-2237.

[59] N. Wiener, A. Rosenblueth, Arch. Inst. Cadiol. Mex. **16** (1946), 205-265.

[60] A. T. Winfree, *Spiral waves of chemical activity*, Science **175(4022)** (1972), 634-636.

[61] A. T. Winfree, *Rotating solutions to reaction-diffusion equations in simply-connected media*, Proceedings of Conference on Mathematical Aspect of Chemical and biochemical Problems and Quantum Chemistry, SIAM-MAS Proceedings **8** (1974), 13-31.

[62] C. Wulff, *Theory of meandering and drifting spiral waves in reaction-diffusion systems*, Ph.D. dissertation, 1996.

[63] E. Yanagida, *Stability of the fast travelling pulse solution of the Fitzhugh-Nagumo equations*, J. Math. Biol. **22** (1985), 81-104.

[64] A. N. Zaikin, A. M. Zhabotinskii, *Concentration wave propagation in two-dimensional liquid-phase self-oscillating system*. Nature, **225**(1970), 535-537.

[65] A. N. Zaikin, A. M. Zhabotinskii, *Spatial effects in a self-oscillatory chemical system*, Oscillatory Processes of Biological and Chemical Systems **2** (1971), 314-317.