DESIGNS AND METHODS FOR THE IDENTIFICATION OF ACTIVE LOCATION AND DISPERSION EFFECTS

DISSERTATION

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ABSTRACT

The identification of active factors, both in terms of location effects and dispersion effects, is a goal of experimental design. The research presented has two components: (i) the classification and ranking of designs for estimation of location effects, and (ii) the discovery of an improved method for the identification of dispersion effects in replicated experiments.

Combinatorially non-isomorphic projection designs (for qualitative factors) are often evaluated by the generalized wordlength pattern. In the first component of my research, I propose an alternative criterion. This criterion is based on the average squared correlations of complete sets of orthonormal contrasts and it will be shown that the criterion is independent of the contrasts selected. Examples demonstrate that the criterion is better able to rank order and distinguish projections from three-level orthogonal arrays than the generalized wordlength pattern.

This average squared correlations criterion is further extended to the case of distinguishing and ranking geometrically non-isomorphic designs. Examples will demonstrate the capability of the criterion to distinguish and rank order projection designs from three-level orthogonal arrays with quantitative factors.

In replicated experiments, the treatment replicates are used to compute a measure of the variability of response for that treatment. Using traditional methods of analysis, summary statistics are calculated for the within-treatment replicates and are used to test for a dispersion effect. The disadvantage of this approach is the loss of degrees of freedom. In the second component of my research, I present an alternative method of analysis. This method transforms each observation into an individual measure of variability. This method preserves all original degrees of freedom, and thereby increases power over the traditional method. A comparison of the new method to the traditional approach, based on computer simulation, is presented.

For Mark...

and Maxie, Marti, and Luci

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PUBLICATIONS

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FIELDS OF STUDY

Major Field: Statistics

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CHAPTER 1

INTRODUCTION

Why do researchers conduct experiments? One answer is that researchers want to identify and estimate *factors* that significantly affect a measurement, called a *response*, of interest. How can factors affect a response? There are two types of effects: location effects and dispersion effects. A *location effect* is the change in the mean of the response at different levels of the factor, while a *dispersion effect* is the change in the variability of the response at different levels of the factor. Figure 1.1 shows each of the two types of effect. Figure 1.1 (a) shows a location effect: the factor set at the high level produces higher response values than the factor set at the low level, but the variability of the response is the same for each setting. In contrast, in Figure 1.1 (b), the factor set at the high level produces a more variable response than the factor set at the low level, showing a dispersion effect of the factor; the mean of the response at each level of the factor is equal in (b).

Effects, both location and dispersion, can result from either a single factor or from a combination of factors. A *main effect* is the effect of a single factor average over all other factors; if a change in the level setting of a given factor has a significant impact on the response, the factor is said to have a main effect. In contrast, if the change in the effect on the response of settings of one factor is dependent upon the level setting

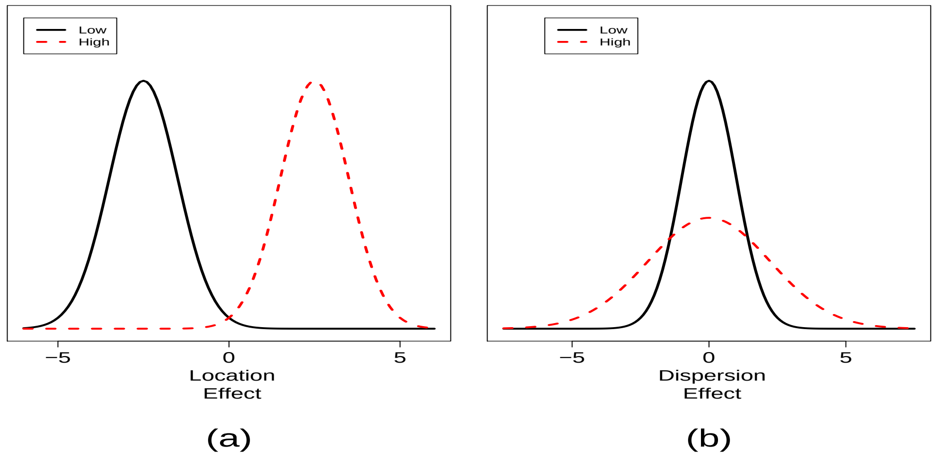

Figure 1.1: Example of (a) location effect and (b) dispersion effect

of a second factor, then the joint effect of the two factors is a two-factor *interaction effect*.

Different experimental designs possess different properties for the estimation of effects. For example, all main effects and two-factor interactions may be estimated independently of each other using one design, while main effect estimates may be independent of two-factor interaction estimates but two-factor interaction estimates correlated with each other using a different design. In evaluating two designs, the first question is whether the two designs are fundamentally the same, or *equivalent*; equivalent designs are also called *isomorphic*.

If two designs are equivalent, then the designs possess the same properties and, subject to practical considerations, either design is an equally good choice. If classes of equivalent designs can be identified, then properties of the class can be described and used to characterize the individual designs in the class. By studying classes of designs instead of individual designs, the amount of work is reduced. Thus, determination of design equivalence is an important problem.

Numerous criteria exist for determining the non-equivalence of two designs. These criteria provide varying degrees of accuracy in determining non-equivalence. In Part I of this work, a new criterion, called the *average squared correlation criterion*, is proposed for the ranking of inequivalent designs. As a consequence of the ranking, the criterion is useful for the determination of non-equivalence. The new criterion can be modified for use with either qualitative or quantitative factors, for which the definition of equivalence differs (see Chapter 2). The proposed criterion will be shown to be more effective for ranking and for determining non-equivalence than certain existing criteria in at least some special cases.

The outline of Part I is as follows. Chapter 2 provides basic information necessary for the definition of the average squared correlation criterion. Description of the special case of evaluation of projection designs is also included in Chapter 2. The $E(s^2)$ criterion for two-level supersaturated designs is discussed briefly in Chapter 3; the average squared correlation criterion is similar to the $E(s^2)$ criterion but for factors with more than two levels. The average squared correlation criterion for evaluating designs with qualitative factors is described and an example is given in Chapter 4. Additionally, in Section 4.3 of Chapter 4 an important theorem is proved showing invariance to the selection of contrasts used in the criterion. The ranking criteria for the average squared correlation is defined and compared to two other criteria in Chapter 5; a discussion of estimation capacity is included in Section 5.3 of Chapter 5. The average squared correlation criterion is extended to designs with quantitative factors in Chapter 6. Chapter 7 provides a brief description of competing methods of rank ordering designs with quantitative factors. A counterexample to the stated interpretation of one of the competing methods is given in Section 7.2 of Chapter 7. The relationship between this same method and the average squared correlation criterion is described in Chapter 8. Chapter 9 examines use of the average squared correlation criterion for ranking designs with quantitative factors. Finally, a summary of findings is provided in Chapter 10.

Methods for the identification of dispersion effects are somewhat less advanced than the methods for location effects. However, with the increased emphasis on quality in manufacturing and other industries, methods for detecting dispersion effects are gaining more attention. In industry, the ability to identify even a small dispersion effect and select a production setting to minimize the variability of the response can result in significant financial benefits for a company.

In studying location effects, replication provides additional power for the detection of significant effects. Replication also provides for an independent estimate of the error variance when fitting a full model, that is, a model in which all independent main effects and interaction effects are included. These same properties of replicated designs should exist for the detection of dispersion effects as well. Therefore, replicated fractional factorial experiments may be more advantageous than unreplicated full factorial experiments when the number of available runs is limited.

When replicated observations are available, these are commonly summarized by a function of the variance, reducing the experiment to a single replicate design, and potentially reducing the power for effect detection. Part II of this work examines different functions of the data that measure dispersion without aggregating the replicate observations into a single measurement. A simulation study is used to compare the effect detection power of the new functions of the data and the summary statistics. This work shows that measures preserving replicate data provide increased power for detecting significant effects as compared to traditional methods. Empirical critical values for the proposed measures are provided. Finally, a recommendation as to a best measure for use in applied settings is made.

The outline of Part II is as follows. Chapter 11 provides an introduction to the problem and a brief review of previous research. In Chapter 12, the model is introduced. The measures examined in Phase I of the current work are described in Chapter 13. The Phase I simulation study which replicates and extends recent research of Mackertich, Benneyan and Kraus (2003) is described in Chapter 14. Chapter 15 provides a discussion of the results and selects the best measures based on the preliminary findings. A revised list of measures to be examined in Phase II of this work is presented in Chapter 16. The simulation for Phase II is described in Chapter 17. Empirical critical values for the selected measures are derived and provided in Chapter 18. Chapter 19 gives a test of the stability of observed Type I error based on these empirical critical values for various location and dispersion models. Based on the test results, the list of measures is further reduced. Chapter 20 examines the power of the measures of interest. A final recommendation is made in Chapter 21.

PART I

AVERAGE SQUARED CORRELATION

CHAPTER 2

INTRODUCTION TO AVERAGE SQUARED CORRELATION

2.1 Orthogonal Arrays

An orthogonal array, $OA(n, p, k, t)$, is an $n \times p$ array with elements taken from a set of k distinct symbols, and strength t, where strength is the property that for any set of t columns of the array each of the k^t possible sets of symbols appear equally often. In this definition, n is equal to the number of design runs and must be a multiple of k^t, p is equal to the number of design factors, and k is equal to the number of levels of each factor.

When an orthogonal array is used as a *design matrix*, $\mathbf{D}$, the columns represent the factors to be studied and the rows represent the treatment combinations observed in the runs of the experiment. The design $\mathbf{D}$ is called a *factorial design* if the $(i, j)^{th}$ element of $\mathbf{D}$ is the level of the j^{th} factor in the i^{th} run. In this work, three-level factors will be considered. The factors levels will be coded (0,1,2); when factors are treated as quantitative, 2 represents the high level, 1 represents the middle level, and 0 represents the low level.

2.2 Screening Experiments and Projection Designs

Orthogonal arrays are useful as screening designs in which a large number of factors are to be studied in a relatively small number of runs. For a fixed number of factors, orthogonal arrays provide designs with fewer runs than other types of designs. For example, a resolution III fractional factorial design for seven factors, each at three levels, requires at least $n = 27$ runs, while an orthogonal array of strength 2 can be found with only $n = 18$ runs. Orthogonal arrays can be found via Sloane's library of orthogonal arrays (Sloane 2005).

Use of orthogonal arrays for screening relies on the idea of factor sparsity (cf. Box and Meyer (1986)). *Factor sparsity* is the belief that, of a large number of factors studied, only a small number of the factors are active. A factor is defined to be *active* if its effect on the response is non-negligible. In screening experiments, it is expected that only a few of the many factors studied will induce a large effect on the response.

In screening experiments two situations can arise. First, it may be the case that only a subset of the columns of the array are needed for the design matrix (Xu and Wu 2001, Ma and Fang 2001, Lin and Draper 1992). For example, it may be known that only four factors are to be studied when an array with seven columns (allowing for seven factors) exists. If only a subset of columns is needed, the design columns can be selected in advance. The question is then which subset of columns to select. Does a design composed of columns (1,2,3,4) possess better properties than a design composed of columns (1,3,5,7)? The second situation arises when a large number of factors is screened and a main effects analysis is used to identify the active factors (Cheng and Wu 2001). Once the active factors are identified, the original design can be projected onto the smaller space of the active factor columns (by deleting the columns of the design corresponding to the inactive factors) in order to clarify the effects on the response. In this situation, for example, seven factors may be screened, but three (corresponding to columns 1, 4, and 5) are found to have a negligible effect on the response. The design array is then projected onto the active effects space (corresponding to columns 2, 3, 6, and 7) by deleting columns 1, 4, and 5. In designing the experiment, it is unknown which subset of columns will be needed. If, in the example, the inactive factors had been assigned to columns 1, 2, and 3, then the projection onto columns 4, 5, 6, and 7 would have been needed instead of the projection onto columns 2, 3, 6, and 7. Therefore, a starting array with a large number of "good" sub-designs is desired. In both of these situations, a subset of columns of the original array is ultimately selected. The resulting sub-design is called a *projection design*.

Clearly, there exists more than one projection design from a given starting array. In fact, there are $\binom{p}{p'}$ possible projection designs, where p is the number of columns of the original array and p' is the selected number of columns. In order either to identify and select the optimal projection or to ensure there are many good projections, all possible projections must be studied. Study of all possible projections quickly becomes expensive in terms of time and computation: for $p = 13$ factors and $p' = 3$ columns, the total number of possible projections is 286.

The number of projections to study can be reduced by examination of only the inequivalent (i.e. non-isomorphic) designs. The reduction is possible since all equivalent designs possess the same properties. Thus, by identifying and ranking all the inequivalent classes, all possible projection designs are effectively being ranked. Precise definitions of equivalence for designs with qualitative factors or with quantitative factors are given in Section 2.5. The use of the proposed average squared correlation pattern criterion for the identification and classification of projection designs is a special case and will be discussed in Section 5.2.2 for designs with qualitative factors and Section 9.2 for designs with quantitative factors.

2.3 Contrasts and Contrast Matrices

A factor with k levels has $k-1$ degrees of freedom for measuring its the main effect. The $k-1$ degrees of freedom allow for comparison of the effects on the response of the different levels of the factor. Then the main effect of a factor can be measured by means of $k-1$ orthogonal contrasts. (See Dean and Voss (1999) page 170.)

Consider the model

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij},$$

where Y_{ij} is the j^{th} replicated observation of the i^{th} treatment combination, μ is the general mean, and τ_i is the effect of treatment combination i on the response. In the case of a single factor model, τ_i denotes the effect of the i^{th} level of the factor; for multi-factor models, τ_i represents the effect of the i^{th} combination of the factors at their respective levels.

A *contrast* is a linear combination of the parameters $\tau_1, \tau_2, \ldots, \tau_v$ of the form $\sum c_i\tau_i$ with $\sum c_i = 0$. (See Dean and Voss (1999) Section 4.2.) A contrast can be represented as a column vector by listing the contrast coefficients $c_1, c_2, \ldots, c_v$; the contrast $c_1\tau_1 + c_2\tau_2 + \ldots c_v\tau_v$ would be represented by $\left[\begin{array}{cccc} c_1 & c_2 & \ldots & c_v \end{array} \right]'$. A set of $k-1$ orthogonal contrasts used to describe the main effect of a k-level factor is called *a complete set of orthogonal contrasts* (Dean and Voss 1999).

For a quantitative factor with $k = 3$ levels, the complete set of $k - 1 = 2$ orthogonal contrasts selected to measure the main effect is generally selected to be the set containing a linear effect contrast and a quadratic effect contrast; such polynomial trend contrasts have no physical interpretation for qualitative factors. The linear and quadratic contrasts provide information about whether the response increases or decreases as the factor levels increase, and whether the rate of increase or decrease remains constant. For example, if the factor is temperature, the response may increase as temperature increases across the levels of temperature studied, leading to a non-zero estimate of the linear contrast. However, the response may increase from the low level of temperature to the middle level of temperature, then decrease from the middle level to the high level of temperature. This second response trend would result in a non-zero estimate of the quadratic contrast. The set of orthogonal polynomial trend contrasts can be expanded to include cubic and higher order effect contrasts for factors with more than three levels, although the presence of higher order trends is rare in practice and experiments involving factors with more than three levels are uncommon.

In this work, because each factor is at three levels, two orthogonal contrasts form the complete set of orthogonal contrasts. The *standard linear* and *standard quadratic* contrast coefficient vectors are given by

$$l = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{2.1}$$

and

$$q = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \tag{2.2}$$

respectively (see, for example, Dean and Voss (1999), Table A.2). Use of these contrasts assumes that the levels of the factor are equally spaced and that the same number of observations will be taken at each level.

Each contrast coefficient vector contains only three elements, one element corresponding to each level of the factor. The contrast vector is constructed by setting the i^{th} element equal to the contrast coefficient corresponding to the factor level setting of the i^{th} element of the design. In constructing the linear contrast vector, the i^{th} element of the contrast vector is set equal to 1.0 if the factor is set at the high level for the i^{th} run of the design, 0.0 if the factor is set the middle level, and -1.0 is the factor is set at the low level. Similarly, the the i^{th} element of the quadratic contrast vector is set equal to 1.0 if the factor is set at either the high level or the low level for the i^{th} run of the design and -2.0 if the factor is set at the middle level. For example, the linear and quadratic contrasts vectors corresponding to the design column $\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 2 & 2 & 0 & 1 \end{bmatrix}'$ are $\begin{bmatrix} -1 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & 0 \end{bmatrix}'$ and $\begin{bmatrix} 1 & -2 & 1 & -2 & 1 & 1 & 1 & 1 & -2 \end{bmatrix}'$ respectively.

The main effect contrasts can then be used to construct interaction contrasts in the following way. The set of interaction contrasts is constructed by element-wise product of the elements of the contrasts of every main effect included in the interaction. The element-wise product of two vectors $\mathbf{c}_i = \begin{bmatrix} c_{1i} & c_{2i} & \ldots & c_{ni} \end{bmatrix}'$ and $\mathbf{d}_j = \begin{bmatrix} d_{1j} & d_{2j} & \ldots & d_{nj} \end{bmatrix}'$denoted $\cdot$, is given by

$$\mathbf{c}_i \cdot \mathbf{d}_j = \begin{bmatrix} c_{1i}d_{1j} \\ c_{2i}d_{2j} \\ \vdots \\ c_{ni}d_{nj} \end{bmatrix}. \tag{2.3}$$

For example, if the interaction effect is between two factors, and the main effect of each factor is described by a linear and quadratic contrast, then the set of contrasts describing the interaction effect would be constructed as the element-wise product of the linear with linear contrasts, the linear with quadratic contrasts, the quadratic with linear contrasts, and the quadratic with quadratic contrasts. For two factors, each with three levels, four orthogonal contrasts are needed to described the interaction effect. In general, if an interaction is composed of i main effects with $k_1, k_2, \ldots, k_i$ levels, then the interaction effect is described by $\prod_{j=1}^{i}(k_j - 1)$ contrasts.

The *contrast matrix*, denoted $\mathbf{C}$, contains two contrasts measuring each main effect and four contrasts measuring each two-factor interaction with the coefficients defined as above.

Each factor represented by a column of the $OA(n, p, k, t)$ is described by a complete set of orthogonal contrasts that is orthogonal to every other set of main effect contrasts in the contrast matrix, $\mathbf{C}$. Therefore, every main effect contrast is estimated independently of every other main effect contrast. The interaction effect contrasts, calculated from the main effect contrasts, are not necessarily orthogonal to the contrasts measuring other effects; the interaction contrasts can be correlated with main effect contrasts or other interaction effect contrasts.

2.4 Factor Aliasing and Correlation

Two factors are said to be completely *aliased* if the two factors cannot be estimated independently. For example, if factors A and B are aliased, then the design is not capable of distinguishing the estimate of A from the estimate of B. If two factors are completely aliased, and the effect is found to be significant, then it is not possible to determine whether it is factor A, factor B, or both factors producing the non-negligible effect. It is desirable, then, to minimize the number of aliased factors; if not all factors can be made independent, then it is desirable also to minimize the degree of aliasing between aliased factors.

The degree of aliasing between contrasts can be described by the correlation between the contrasts. Complete aliasing of two contrasts corresponds to a correlation equal to one, and complete independence of two contrasts corresponds to a correlation equal to zero. The smaller the correlation the less aliased the contrasts and the greater the information that is available about the individual effects.

For regular three-level designs, (i.e. designs constructed through defining relations among factor labels), any two factorial effects are either independent or fully aliased. Orthogonal arrays can be either regular or non-regular. If an orthogonal array is regular, the strength of the array is equal to the resolution of the design minus one. (For a definition of resolution, see for example, Wu and Hamada (2000) page 159.)

In the case of non-regular designs with factors, partial aliasing of contrasts is possible. Partial aliasing of two contrasts exits when the correlation is greater than zero and less than one. For example, if a factor has three levels, then it is possible that a linear orthogonal contrast is independent of all other effect contrasts and only the quadratic orthogonal contrast is aliased (either completely or partially) with one or more interaction effect contrasts. It is also possible that both the linear and quadratic orthogonal contrasts are partially aliased with other contrasts. In either case, the factorial effect is partially aliased with another factorial effect.

Geometrically, the correlation between two contrasts is equal to the cosine of the angle between the two contrasts. For contrasts $\mathbf{c}_i$ and $\mathbf{c}_j$, the correlation between the contrasts is

$$\rho_{i,j} = \frac{\mathbf{c}_i'\mathbf{c}_j}{\sqrt{(\mathbf{c}_i'\mathbf{c}_i)(\mathbf{c}_j'\mathbf{c}_j)}} \tag{2.4}$$

(see Johnson and Wichern (1998) page 122).

In examining the aliasing of effects, only the size (magnitude) of the correlation is important, while the sign of the correlation (i.e. positive or negative) is irrelevant. Therefore, in this work the square of the correlation is used. In addition to removing the sign, squaring the correlation magnifies the differences between correlations as large correlations are reduced less than are small correlations.

2.5 Equivalence

The definition of design equivalence depends upon the type of factors studied. For designs with qualitative factors, two designs are said to be *combinatorially equivalent*, or *combinatorially isomorphic*, if one design matrix can be obtained from the other by

(1) a sequence of row permutations, corresponding to a change in the run order,

(2) a sequence of columns permutations, corresponding to a change in the factor labels, and

(3a) a series of symbol permutations within columns, corresponding to changes in level labels within one or more factors.

For qualitative factors, there is no meaningful order to the levels of the factors, and the assignment of symbols to levels within a factor is completely arbitrary since the symbol assigned to a level has no interpretation. As a result, any permutation can be applied to symbols within a column. For example, for a color factor, the symbols can be assigned so that red = 0, blue = 1, and green = 2. For this factor, any permutation of the symbols 0, 1, and 2, can be applied and is equally appropriate;

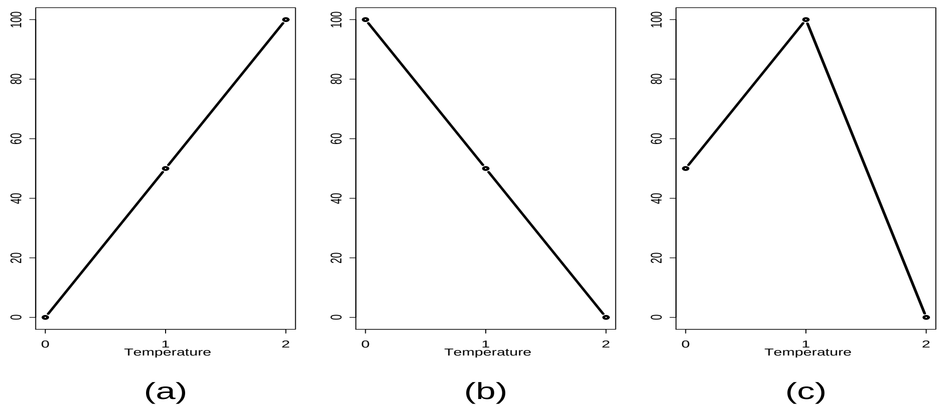

Figure 2.1: Example of change to interpretation of trend due to different factor level labeling for quantitative factors

there is no change in the interpretation of the effects if the symbols are assigned so that red = 1, blue = 0, and green = 2.

In the case of quantitative factors, there exists a true ordering of the levels. For example, there exists an inherent order to temperature levels of 100° C, 50° C, and 0° C. Based on the order, the levels can be labeled such that 100° C is 2 (high), 50° C is "1" (middle), and 0° C is "0" (low). These symbol assignments can be reversed, so that 100° C is 0, 50° C is 1, and 0° C is 2, without changing the intrinsic structure of the order. However, the order structure would change if 1 is assigned to 100° C, 0 is assigned to 50° C, and 2 is assigned to 0° C. In this last case, the pattern of the response over the factor levels can change. The plots in Figure 2.1 present this simple, one-factor example. In each of the plots in Figure 2.1, the data are:

Temperature	Response
100° C	100
50° C	50
0° C	0

The difference between the plots is the labeling of the factor levels:

Left (a)	:	100° C = 2 (high)
	:	50° C = 1 (middle)
	:	0° C = 0 (low)
Center (b)	:	100° C = 0 (low)
	:	50° C = 1 (middle)
	:	0° C = 2 (high)
Right (c)	:	100° C = 1 (middle)
	:	50° C = 0 (low)
	:	0° C = 2 (high)

The plots in Figure 2.1 (a) and Figure 2.1 (b) preserve the natural ordering of the levels; the plot in Figure 2.1 (c) disrupts the level order. Figure 2.1 (a) and (b) suggest a linear pattern when the inherent order is maintained, but Figure 2.1 (c) suggests a quadratic since the original ordering is neglected.

As stated above, the symbol order can be reversed, so long as the inherent order is preserved. This order requirement restricts the allowable symbol exchanges within a column. And this restriction on the exchange of symbols necessarily restricts the possible permutations that result in isomorphic designs with quantitative factors.

For designs with quantitative factors, two designs are isomorphic if one design can be obtained from the other through

(1) a sequence of row permutations, corresponding to a change in the run order,

(2) a sequence of columns permutations, corresponding to a change in the factor labels, and

(3b) a sequence of level order reversals for one or more factors (which require that the order is preserved).

This type of isomorphism is called *geometric isomorphism* (Cheng and Ye 2004). From a geometric viewpoint, with design points viewed as points in $\mathbb{R}^k$, Cheng and Ye (2004) define two designs as geometrically isomorphic if one geometric design object can be obtained from the other by rotating and/or reflecting the design with respect to a super-plane, where rotating corresponds to factor label permutations and reflecting corresponds to level order reversal. This geometric interpretation can be seen in the reflection of Figure 2.1 (a) about level 1 (middle) to obtain Figure 2.1 (b).

Similar to geometric isomorphism, Cheng and Wu (2001) defined *model isomorphism*. Two designs are equivalent in terms of model isomorphism if the model matrix of one design can be obtained from the other by

(1) a sequence of row permutations, corresponding to a change in the run order,

(2) a sequence of columns permutations, corresponding to a change in the factor labels, and

(3c) a sequence of sign changes within columns.

The *model matrix* for a design with n runs and p factors is a coded matrix for the design with columns for each of the factorial contrasts. Notice that model isomorphism is defined with respect to the model matrix. As a result, two designs may be model isomorphic with respect to one model but model non-isomorphic with respect to a different model. Therefore, in determining model isomorphism, the model must be specified. In contrast, combinatorial and geometric equivalence are defined with respect to the design matrix, and invariant to the fitted model.

For two-level designs, combinatorial isomorphism and geometric isomorphism are equivalent since only one symbol permutation exists and necessarily preserves the order of the factor levels. For designs with factors having more than two levels, because of the additional level order restriction, designs which are combinatorially isomorphic may not be geometrically isomorphic. However, all designs which are geometrically isomorphic must also be combinatorially isomorphic.

CHAPTER 3

$E(s^2)$ CRITERION AND SUPERSATURATED DESIGNS

The contrast matrix constructed from an orthogonal array, in which each main effect and each two-factor interaction is represented by a complete set of orthogonal contrasts, can be viewed as a supersaturated design. A two-level *supersaturated design* is usually defined to be a factorial design in which the number of main effect parameters to be estimated, $p + 1$ (equal to the number of factors plus one for the overall mean), is greater than the number of runs, n. (See Wu and Hamada (2000), Section 8.6.) In the case of a contrast matrix involving interactions and/or factors at more than two levels, each contrast represents a factorial effect to be estimated. If the number of contrasts (columns) is greater that the number of runs (rows), then the same estimation problems arise as for two-level supersaturated main effects. For example, for the $OA(18, 7, 3, 2)$, the number of columns in the complete contrast matrix is 98 $(= (7 \times 2) + (\binom{7}{2} \times 4)$, while the number of rows is 18. A projection of $OA(18, 7, 3, 2)$ onto $p' = 3$ columns results in a "saturated" design with eighteen contrast columns and eighteen runs, while a projection onto $p' = 4$ columns is supersaturated.

Orthogonality of design columns and contrast matrix columns is a desirable property. True effects can be hidden and spurious effects exhibited as a result of the correlation structure integral to supersaturated designs (Abraham, Chipman and

Vijayan 1999). Assuming interaction effects are negligible, efficient estimation of main effects is attained in the case when every pair of columns is orthogonal (Plackett and Burman 1946).

The condition that every pair of columns is orthogonal cannot be satisfied if the number of columns is greater than the number of rows minus 1; there are at most n orthogonal columns, including a column of ones, in n-dimensional space. For two-level factors, each factorial effect is described by a single contrast, and the condition can be satisfied if $p > n - 1$. For three-level factors, in which each factorial effect is described by two orthogonal contrasts, the orthogonality condition can be attained for an orthogonal array if interaction contrasts are not included. However, if interaction contrasts are included in the contrast matrix, the number of contrasts is greater than n and the orthogonality condition cannot be satisfied.

In the case when orthogonality of all columns cannot be achieved, it is desired to have all pairs of columns as nearly orthogonal as possible. That is, it is desired to have $c_i'c_j$, $(i \neq j)$, as small as possible for all contrast columns c_i and c_j; $c_i'c_j$ is the numerator of the correlation between two contrasts given in Equation (2.4). Additionally, it is desired to have a minimum value of the maximum $c_i'c_j \neq 0$, and a smaller number of pairs of columns attaining this maximum value (Booth and Cox 1962).

In order to compare two-level supersaturated designs under a main effects model, Booth and Cox (1962) proposed the $E(s^2)$ criterion as a measure of non-orthogonality. The criterion is to minimize $E(s^2)$, where

$$E(s^2) = \sum \frac{s_{i,j}^2}{\binom{p}{2}} \tag{3.1}$$

and

$$s_{i,j} = \mathbf{x}_i'\mathbf{x}_j \qquad (3.2)$$

for two columns, $\mathbf{x}_i$ and $\mathbf{x}_j$, of the design matrix (Lin 1993). Booth and Cox (1962) show that the average variance of an estimated main effect is a function of the variance of $\mathbf{s} = \{s_{ij}\}$, where each s_{ij} is the sum of products $\mathbf{x}_i'\mathbf{x}_j$ of two randomly selected columns of the design matrix (i.e. for a design matrix $\mathbf{X}$, s_{ij} is the $(i,j)^{th}$ element of $\mathbf{X}'\mathbf{X}$). Smaller $var(\mathbf{s})$ produces smaller average variance of the estimate.

A two-level factor has only a single contrast to measure its main effect, so $\mathbf{x}_i \equiv \mathbf{c}_i$ when $\mathbf{x}$ is coded (-1,1) for the low and high levels. Also, for two-level designs, $\mathbf{c}_i'\mathbf{c}_i = n$ for all i, and so $n \times \rho_{i,j} = s_{i,j}$. Similarly, when $\mathbf{c}_i$ and $\mathbf{c}_j$ are normalized, the inner product $\mathbf{c}_i'\mathbf{c}_j$ is equal to the correlation between the two contrasts (see Equation (2.4)), and $\rho_{i,j} = s_{i,j}$. Thus, minimizing $E(s^2)$ is equivalent to minimizing to the average squared correlation, ρ^2, for two-level factor designs.

The current work focuses on three-level factors, which are described by two orthogonal contrasts. Thus, the $E(s^2)$ criterion is not directly applicable. However, the basic idea of $E(s^2)$ can be extended, and provides the foundation for the average squared correlation criterion proposed and developed in Chapter 4 and Chapter 6. The average squared correlation combines information from individual contrasts of equal importance in order to acquire information about the main effect or two-factor interaction. Equal importance of contrasts will be defined differently for qualitative factors and quantitative factors. Minimizing the correlation will be analogous to minimizing $E(s^2)$. Chapter 4 presents the average squared correlation criterion for qualitative factors; average squared correlations for quantitative factors are discussed in Chapter 6.

CHAPTER 4

AVERAGE SQUARED CORRELATION FOR QUALITATIVE FACTORS

As described in Section 2.4, the correlation between two contrasts represents the degree of aliasing of the contrasts. Correlations are the fundamental components of the proposed criterion used to evaluate designs; the average squared correlation criterion is formed from the correlations between main effect and two-factor interaction contrasts and the correlations between pairs of two-factor interaction contrasts given in the contrast matrix, **C**.

In the case of two-level factors, each main effect and each interaction effect is described by a single contrast. The correlation between contrasts measures the degree of aliasing between the factor main effects and interactions. For three-level factors, two orthogonal contrasts describe each main effect and four orthogonal contrasts describe each two-factor interaction. The correlation between a single pair of contrasts does not provide complete information about the aliasing of the factorial effects; correlations from complete sets of orthogonal contrasts must be combined to provide a complete description of the aliasing of effects.

According the *hierarchical ordering principle*, lower order effects are more likely to be important than higher order effects and effects of the same order are equally likely to be important (see, for example, Wu and Hamada (2000), Section 3.5). When the factors are qualitative, the order of the effect is determined only by the number of factors included in the effect and not by any interpretation of the contrasts being measured. For example, a main effect would be of order 1 while a two-factor interaction would be of order 2. Based on this definition of order, the hierarchical ordering principle generates an effect hierarchy for the contrasts labeled c_{i_1} and c_{i_2} as

$$\begin{aligned} & c_{a_1} == c_{a_2} \\ \gg\ & c_{a_1}c_{b_1} == c_{a_1}c_{b_2} == c_{a_2}c_{b_1} == c_{a_2}c_{b_2} \\ \gg\ & c_{a_1}c_{b_1}c_{c_1} == c_{a_1}c_{b_1}c_{c_2} == \ldots == c_{a_1}c_{b_2}c_{c_2} == \ldots == c_{a_2}c_{b_2}c_{c_2} \\ \gg\ & c_{a_1}c_{b_1}c_{c_1}c_{d_1} == c_{a_1}c_{b_1}c_{c_1}c_{d_2} == \ldots == c_{a_2}c_{b_2}c_{c_2}c_{d_2} \gg \ldots \end{aligned} \tag{4.1}$$

where $==$ represents equal importance (e.g. $c_{a_1} == c_{a_2}$ indicates that the c_{a_1} contrast and the c_{a_2} contrast are equally important) and $\gg$ represents greater importance (e.g. $c_{a_2} \gg c_{a_1}c_{b_1}$ indicates that the c_{a_2} contrast is more important than the $c_{a_1} \times c_{b_1}$ interaction contrast). From the effect hierarchy (4.1), the main effects contrasts are more important than the two-factor interaction contrasts; all of the main effect contrasts are equally important and all of the two-factor interaction contrasts are equally important according to (4.1).

Based on a given effect hierarchy, the individual contrast correlations within a given level can be averaged because all contrasts within a given level are equally important. The resulting averages will provide information about the aliasing of main effects with interaction effects and information about aliasing of interaction effects with other interaction effects.

The hierarchical ordering principle and effect hierarchy (4.1) are used as the basis for ranking designs as described in Section 5.2.

4.1 Average Squared Correlations of Order 3 and Order 4

Based on the equation for the correlation between two contrasts given in (2.4), for a given contrast matrix, $\mathbf{C}$, the *correlation matrix*, $\mathbf{R}$, is

$$\mathbf{R} = \mathbf{D}^{-1/2}\mathbf{C}'\mathbf{C}\mathbf{D}^{-1/2} \tag{4.2}$$

where $\mathbf{D}$ is the diagonal matrix composed of the diagonal elements of $\mathbf{C}'\mathbf{C}$ and $\mathbf{D}^{-1/2}$ is the inverse of the square-root matrix of $\mathbf{D}$. The $(i,j)^{th}$ element of $\mathbf{R}$ is the correlation between the i^{th} and j^{th} columns (contrasts) of $\mathbf{C}$. The correlation matrix is used to calculate the average squared correlations of order 3 and order 4.

Correlations between main effects and two-factor interactions are called *correlations of order 3*; the *average square correlation of order 3* for designs with qualitative factors is defined as the arithmetic mean of the squared correlations of any complete set of orthogonal contrasts of a given main effect with any complete set of orthogonal contrasts for a given two-factor interaction. For a given factor, A, with three levels, two orthogonal contrasts that span the space are selected for the main effect and labeled A_1 and A_2. For a two-factor interaction effect, BC, where both B and C are three-level factors, four orthogonal contrasts are needed. The four orthogonal contrasts are taken to be the element-wise products of each combination of the orthogonal main effect contrasts for the two factors B and C: B_1C_1, B_1C_2, B_2C_1, and B_2C_2. Then, given a main effect, A, and a two-factor interaction effect, BC, the average squared correlation of order 3 between these two effects is given by

$$Ave\ \rho_3^2(A, BC) = \frac{1}{8}\Big[\rho^2(A_1, B_1C_1) + \rho^2(A_1, B_1C_2) + \rho^2(A_1, B_2C_1)$$

$$+\rho^2(A_1, B_2C_2) + \rho^2(A_2, B_1C_1) + \rho^2(A_2, B_1C_2)$$
$$+\rho^2(A_2, B_2C_1) + \rho^2(A_2, B_2C_2)\Big] \qquad (4.3)$$

The subscript on $\rho_3^2(A, BC)$ represents the order of the correlations.

Correlations between two-factor main effects and other two-factor interactions are called *correlations of order 4*, with the *average squared correlation of order 4* for designs with qualitative factors defined as the arithmetic mean of the correlations of a complete set of orthogonal contrasts for a two-factor interaction effect with a complete set of orthogonal contrasts for a different two-factor interaction effect. For example, given the AB and CD interaction effects, all combinations of the four orthogonal contrasts that form the AB interaction effect and the four orthogonal contrasts that form the CD interaction effect are averaged to calculate the average squared correlation of order 4:

$$\begin{aligned} Ave\ \rho_4^2(AB, CD) = \frac{1}{16}\Big[&\rho^2(A_1B_1, C_1D_1) + \rho^2(A_1B_1, C_1D_2) + \rho^2(A_1B_1, C_2D_1) \\ &+ \rho^2(A_1B_1, C_2D_2) + \rho^2(A_1B_2, C_1D_1) + \rho^2(A_1B_2, C_1D_2) \\ &+ \rho^2(A_1B_2, C_2D_1) + \rho^2(A_1B_2, C_2D_2) + \rho^2(A_2B_1, C_1D_1) \\ &+ \rho^2(A_2B_1, C_1D_2) + \rho^2(A_2B_1, C_2D_1) + \rho^2(A_2B_1, C_2D_2) \\ &+ \rho^2(A_2B_2, C_1D_1) + \rho^2(A_2B_2, C_1D_2) + \rho^2(A_2B_2, C_2D_1) \\ &+ \rho^2(A_2B_2, C_2D_2)\Big] \qquad (4.4) \end{aligned}$$

Again, the subscript on $\rho_4^2(AB, CD)$ represents the order of the correlations.

4.1.1 Average Squared Correlation Pattern for Designs with Qualitative Factors

The values of the average squared correlations of order 3 and order 4 and the number correlations with each distinct value can be used to construct the *average squared correlation pattern* (ASCP). The ASCP can be used to rank classes of combinatorially equivalent designs. As a consequence of the ranking, the ASCP can also help to identify the classes. Since $Ave\ \rho_3^2(A, BC) = Ave\ \rho_3^2(BC, A)$, only one of these values needs to be included in the pattern. Then, for a three-factor design with factors A, B, and C, the complete set of average squared correlations of order 3 included in the ASCP is

(A,AB)	(B,AB)	(C,AB)
(A,AC)	(B,AC)	(C,AC)
(A,BC)	(B,BC)	(C,BC)

In the case that the design matrix is an orthogonal array of strength two or greater, the ASCP can be simplified. From the definition of an orthogonal array, it follows that the correlation between a main effect and a two-factor interaction effect that includes that main effect is equal to zero. In this example, then, $\rho_3(A, AB) = \rho_3(A, AC) = \rho_3(B, AB) = \rho_3(B, BC) = \rho_3(C, AC) = \rho_3(C, BC) = 0$. Thus, only three average squared correlations of order 3 need to be included in the ASCP:

(A,BC)	(B,AC)	(C,AB)

In general, for a p-factor design, there are $p \times \binom{p}{2}$ total average squared correlations of order 3, with $p \times (p-1)$ average squared correlations of order 3 known to equal zeros. The ASCP consists of $p \times (p-1) \times (p/2-1)$ average squared correlations.

Again, for the same three-factor design, the complete set of average squared correlations of order 4 included in the ASCP is

$$(AB,AC) \qquad (AB,BC) \qquad (AC,BC)$$

For a p-factor design, there is a total of

$$\frac{1}{2}\left[\binom{p}{2} \times \left(\binom{p}{2} - 1\right)\right] = \frac{p(p^2-1)(p-2)}{8}$$

average squared correlations of order 4 included in the ASCP. (There are a total of $\binom{p}{2} \times \left(\binom{p}{2} - 1\right)$ contrast pairs, not including a contrast with itself. Of these, half are redundant, i.e. $\rho(AB, CD)$ and $\rho(CD, AB)$ are equal. Thus, only $\frac{1}{2}\left[\binom{p}{2} \times \left(\binom{p}{2} - 1\right)\right]$ average squared correlations are included in the ASCP.) There are no average squared correlations of order 4 that are known always to be equal to zero in the case of orthogonal arrays of strength two; for orthogonal arrays of strength three or greater, simplification of the ASCP is possible.

The complete sets of distinct, non-zero average squared correlations of order 3 and order 4 are summarized into the ASCP. The ASCP is a two-row array,

$$\left|\begin{array}{cccc|cccc} Ave\ \rho^2_{3(1)} & Ave\ \rho^2_{3(2)} & \dots & Ave\ \rho^2_{3(k)} & Ave\ \rho^2_{4(1)} & Ave\ \rho^2_{4(2)} & \dots & Ave\ \rho^2_{4(m)} \\ r_{3(1)} & r_{3(2)} & \dots & r_{3(k)} & r_{4(1)} & r_{4(2)} & \dots & r_{4(m)} \end{array}\right| \qquad (4.5)$$

where

$$Ave\ \rho^2_{3(1)} < Ave\ \rho^2_{3(2)} < \dots < Ave\ \rho^2_{3(k)},$$

$$Ave\ \rho^2_{4(1)} < Ave\ \rho^2_{4(2)} < \dots < Ave\ \rho^2_{4(m)},$$

$$r_{i(j)} \text{ is the number of } Ave\ \rho^2_i = Ave\ \rho^2_{i(j)},$$

k is the number of distinct values of $Ave\ \rho^2_3$, and m is the number of distinct values of $Ave\ \rho^2_4$.

1	**2**	**3**	**7**
A	B	C	D
1	1	1	1
2	2	2	1
0	0	0	1
1	1	2	1
2	2	0	1
0	0	1	1
1	2	1	2
2	0	2	2
0	1	0	2
1	0	0	2
2	1	1	2
0	2	2	2
1	2	0	0
2	0	1	0
0	1	2	0
1	0	2	0
2	1	0	0
0	2	1	0

Table 4.1: Projection design composed of columns 1, 2, 3, and 7, from $OA(18, 7, 3, 2)$ in Table A.1

4.2 Example: Calculation of Average Squared Correlations of Order 3 and Order 4

Consider the $OA(18, 7, 3, 2)$ presented in Table A.1. Suppose, as an example, we select the projection formed by the first, second, third, and seventh columns of the array and label the columns A, B, C, and D, respectively. Table 4.1 gives the design.

Corresponding to this design, the contrast matrix, $\mathbf{C}$, is constructed. In this example, the contrast matrix utilizes the following pair of orthogonal contrasts for each factor main effect:

$$c_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{4.6}$$

and

$$c_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \tag{4.7}$$

The contrast c_1 is a comparison of the high and low levels of the factor; the middle level of the factor is compared to the average of the high and low levels of the factor by contrast c_2. A set of four contrasts is computed by element-wise multiplication of the main effect contrast columns for each two-factor interaction. The complete contrast matrix, **C**, for this design is given in Table 4.2, where for ease of presentation **C** is divided into the first eight columns, the middle twelve columns and the last twelve columns.

From the contrast matrix, **C**, given in Table 4.2, the correlation matrix is computed using (4.2). The complete correlation matrix for the projection design in this example is given in Table 4.3. The correlation matrix is again divided for ease of presentation.

A_1	A_2	B_1	B_2	C_1	C_2	D_1	D_2
0	-2	0	-2	0	-2	0	-2
1	1	1	1	1	1	0	-2
-1	1	-1	1	-1	1	0	-2
0	-2	0	-2	1	1	0	-2
1	1	1	1	-1	1	0	-2
-1	1	-1	1	0	-2	0	-2
0	-2	1	1	0	-2	1	1
1	1	-1	1	1	1	1	1
-1	1	0	-2	-1	1	1	1
0	-2	-1	1	-1	1	1	1
1	1	0	-2	0	-2	1	1
-1	1	1	1	1	1	1	1
0	-2	1	1	-1	1	-1	1
1	1	-1	1	0	-2	-1	1
-1	1	0	-2	1	1	-1	1
0	-2	-1	1	1	1	-1	1
1	1	0	-2	-1	1	-1	1
-1	1	1	1	0	-2	-1	1

A_1B_1	A_1B_2	A_2B_1	A_2B_2	A_1C_1	A_1C_2	A_2C_1	A_2C_2	A_1D_1	A_1D_2	A_2D_1	A_2D_2
0	0	0	4	0	0	0	4	0	0	0	4
1	1	1	1	1	1	1	1	0	-2	0	-2
1	-1	-1	1	1	-1	-1	1	0	2	0	-2
0	0	0	4	0	0	-2	-2	0	0	0	4
1	1	1	1	-1	1	-1	1	0	-2	0	-2
1	-1	-1	1	0	2	0	-2	0	2	0	-2
0	0	-2	-2	0	0	0	4	0	0	-2	-2
-1	1	-1	1	1	1	1	1	1	1	1	1
0	2	0	-2	1	-1	-1	1	-1	-1	1	1
0	0	2	-2	0	0	2	-2	0	0	-2	-2
0	-2	0	-2	0	-2	0	-2	1	1	1	1
-1	-1	1	1	-1	-1	1	1	-1	-1	1	1
0	0	-2	-2	0	0	2	-2	0	0	2	-2
-1	1	-1	1	0	-2	0	-2	-1	1	-1	1
0	2	0	-2	-1	-1	1	1	1	-1	-1	1
0	0	2	-2	0	0	-2	-2	0	0	2	-2
0	-2	0	-2	-1	1	-1	1	-1	1	-1	1
-1	-1	1	1	0	2	0	-2	1	-1	-1	1

B_1C_1	B_1C_2	B_2C_1	B_2C_2	B_1D_1	B_1D_2	B_2D_1	B_2D_2	C_1D_1	C_1D_2	C_2D_1	C_2D_2
0	0	0	4	0	0	0	4	0	0	0	4
1	1	1	1	0	-2	0	-2	0	-2	0	-2
1	-1	-1	1	0	2	0	-2	0	2	0	-2
0	0	-2	-2	0	0	0	4	0	-2	0	-2
-1	1	-1	1	0	-2	0	-2	0	2	0	-2
0	2	0	-2	0	2	0	-2	0	0	0	4
0	-2	0	-2	1	1	1	1	0	0	-2	-2
-1	-1	1	1	-1	-1	1	1	1	1	1	1
0	0	2	-2	0	0	-2	-2	-1	-1	1	1
1	-1	-1	1	-1	-1	1	1	-1	-1	1	1
0	0	0	4	0	0	-2	-2	0	0	-2	-2
1	1	1	1	1	1	1	1	1	1	1	1
-1	1	-1	1	-1	1	-1	1	1	-1	-1	1
0	2	0	-2	1	-1	-1	1	0	0	2	-2
0	0	-2	-2	0	0	2	-2	-1	1	-1	1
-1	-1	1	1	1	-1	-1	1	-1	1	-1	1
0	0	2	-2	0	0	2	-2	1	-1	-1	1
0	-2	0	-2	-1	1	-1	1	0	0	2	-2

Table 4.2: Complete contrast matrix for example projection design in Table 4.1

	A_1	A_2	B_1	B_2	C_1	C_2	D_1	D_2
A_1	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
A_2	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
B_1	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
B_2	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000
C_1	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
C_2	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000
D_1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000
D_2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
A_1B_1	0.0000	0.0000	0.0000	0.0000	-0.3062	0.1768	0.0000	-0.7071
A_1B_2	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062	0.0000	0.0000
A_2B_1	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062	0.0000	0.0000
A_2B_2	0.0000	0.0000	0.0000	0.0000	0.3062	-0.1768	0.0000	-0.7071
A_1C_1	0.0000	0.0000	-0.3062	0.1768	0.0000	0.0000	0.3062	-0.1768
A_1C_2	0.0000	0.0000	0.1768	0.3062	0.0000	0.0000	-0.1768	-0.3062
A_2C_1	0.0000	0.0000	0.1768	0.3062	0.0000	0.0000	0.1768	0.3062
A_2C_2	0.0000	0.0000	0.3062	-0.1768	0.0000	0.0000	0.3062	-0.1768
A_1D_1	0.0000	0.0000	0.0000	0.0000	0.3062	-0.1768	0.0000	0.0000
A_1D_2	0.0000	0.0000	-0.7071	0.0000	-0.1768	-0.3062	0.0000	0.0000
A_2D_1	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062	0.0000	0.0000
A_2D_2	0.0000	0.0000	0.0000	-0.7071	0.3062	-0.1768	0.0000	0.0000
B_1C_1	-0.3062	0.1768	0.0000	0.0000	0.0000	0.0000	0.3062	-0.1768
B_1C_2	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000	-0.1768	-0.3062
B_2C_1	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062
B_2C_2	0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000	0.3062	-0.1768
B_1D_1	0.0000	0.0000	0.0000	0.0000	0.3062	-0.1768	0.0000	0.0000
B_1D_2	-0.7071	0.0000	0.0000	0.0000	-0.1768	-0.3062	0.0000	0.0000
B_2D_1	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062	0.0000	0.0000
B_2D_2	0.0000	-0.7071	0.0000	0.0000	0.3062	-0.1768	0.0000	0.0000
C_1D_1	0.3062	0.1768	0.3062	0.1768	0.0000	0.0000	0.0000	0.0000
C_1D_2	-0.1768	0.3062	-0.1768	0.3062	0.0000	0.0000	0.0000	0.0000
C_2D_1	-0.1768	0.3062	-0.1768	0.3062	0.0000	0.0000	0.0000	0.0000
C_2D_2	-0.3062	-0.1768	-0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000

Continued

Table 4.3: Complete correlation matrix for example projection design in Table 4.1

Table 4.3 Continued

	A_1B_1	A_1B_2	A_2B_1	A_2B_2	A_1C_1	A_1C_2	A_2C_1	A_2C_2	A_1D_1	A_1D_2	A_2D_1	A_2D_2
A_1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
A_2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
B_1	0.0000	0.0000	0.0000	0.0000	-0.3062	0.1768	0.1768	0.3062	0.0000	-0.7071	0.0000	0.0000
B_2	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062	0.3062	-0.1768	0.0000	0.0000	0.0000	-0.7071
C_1	-0.3062	0.1768	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000	0.3062	-0.1768	0.1768	0.3062
C_2	0.1768	0.3062	0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000	-0.1768	-0.3062	0.3062	-0.1768
D_1	0.0000	0.0000	0.0000	0.0000	0.3062	-0.1768	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000
D_2	-0.7071	0.0000	0.0000	-0.7071	-0.1768	-0.3062	0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000
A_1B_1	1.0000	0.0000	0.0000	0.0000	0.1250	0.2165	-0.2165	0.1250	0.0000	0.0000	0.0000	-0.5000
A_1B_2	0.0000	1.0000	0.0000	0.0000	0.2165	-0.1250	0.1250	0.2165	0.0000	-0.5000	0.0000	0.0000
A_2B_1	0.0000	0.0000	1.0000	0.0000	-0.2165	0.1250	-0.1250	-0.2165	0.0000	-0.5000	0.0000	0.0000
A_2B_2	0.0000	0.0000	0.0000	1.0000	0.1250	0.2165	-0.2165	0.1250	0.0000	0.0000	0.0000	0.5000
A_1C_1	0.1250	0.2165	-0.2165	0.1250	1.0000	0.0000	0.0000	0.0000	0.1250	0.2165	0.2165	-0.1250
A_1C_2	0.2165	-0.1250	0.1250	0.2165	0.0000	1.0000	0.0000	0.0000	0.2165	-0.1250	-0.1250	-0.2165
A_2C_1	-0.2165	0.1250	-0.1250	-0.2165	0.0000	0.0000	1.0000	0.0000	0.2165	-0.1250	-0.1250	-0.2165
A_2C_2	0.1250	0.2165	-0.2165	0.1250	0.0000	0.0000	0.0000	1.0000	-0.1250	-0.2165	-0.2165	0.1250
A_1D_1	0.0000	0.0000	0.0000	0.0000	0.1250	0.2165	0.2165	-0.1250	1.0000	0.0000	0.0000	0.0000
A_1D_2	0.0000	-0.5000	-0.5000	0.0000	0.2165	-0.1250	-0.1250	-0.2165	0.0000	1.0000	0.0000	0.0000
A_2D_1	0.0000	0.0000	0.0000	0.0000	0.2165	-0.1250	-0.1250	-0.2165	0.0000	0.0000	1.0000	0.0000
A_2D_2	-0.5000	0.0000	0.0000	0.5000	-0.1250	-0.2165	-0.2165	0.1250	0.0000	0.0000	0.0000	1.0000
B_1C_1	0.1250	-0.2165	0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250	-0.2500	0.0000	-0.4330	0.0000
B_1C_2	0.2165	0.1250	-0.1250	0.2165	-0.2165	-0.1250	0.1250	-0.2165	-0.4330	0.0000	0.2500	0.0000
B_2C_1	-0.2165	-0.1250	0.1250	-0.2165	0.2165	0.1250	-0.1250	0.2165	-0.4330	0.0000	0.2500	0.0000
B_2C_2	0.1250	-0.2165	0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250	0.2500	0.0000	0.4330	0.0000
B_1D_1	0.0000	0.0000	0.0000	0.0000	-0.2500	-0.4330	-0.4330	0.2500	-0.5000	0.0000	0.0000	0.0000
B_1D_2	0.0000	-0.5000	-0.5000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000	0.0000	0.0000
B_2D_1	0.0000	0.0000	0.0000	0.0000	-0.4330	0.2500	0.2500	0.4330	0.0000	0.0000	-0.5000	0.0000
B_2D_2	-0.5000	0.0000	0.0000	0.5000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000
C_1D_1	-0.2500	-0.4330	-0.4330	0.2500	-0.1250	0.2165	0.2165	0.1250	-0.1250	0.2165	0.2165	0.1250
C_1D_2	0.0000	0.0000	0.0000	0.0000	-0.2165	-0.1250	-0.1250	0.2165	0.2165	0.1250	0.1250	-0.2165
C_2D_1	-0.4330	0.2500	0.2500	0.4330	0.2165	0.1250	0.1250	-0.2165	-0.2165	-0.1250	-0.1250	0.2165
C_2D_2	0.0000	0.0000	0.0000	0.0000	-0.1250	0.2165	0.2165	0.1250	-0.1250	0.2165	0.2165	0.1250

Continued

Table 4.3 Continued

	B_1C_1	B_1C_2	B_2C_1	B_2C_2	B_1D_1	B_1D_2	B_2D_1	B_2D_2	C_1D_1	C_1D_2	C_2D_1	C_2D_2
A_1	-0.3062	0.1768	0.1768	0.3062	0.0000	-0.7071	0.0000	0.0000	0.3062	-0.1768	-0.1768	-0.3062
A_2	0.1768	0.3062	0.3062	-0.1768	0.0000	0.0000	0.0000	-0.7071	0.1768	0.3062	0.3062	-0.1768
B_1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3062	-0.1768	-0.1768	-0.3062
B_2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062	0.3062	-0.1768
C_1	0.0000	0.0000	0.0000	0.0000	0.3062	-0.1768	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000
C_2	0.0000	0.0000	0.0000	0.0000	-0.1768	-0.3062	0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000
D_1	0.3062	-0.1768	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
D_2	-0.1768	-0.3062	0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
A_1B_1	0.1250	0.2165	-0.2165	0.1250	0.0000	0.0000	0.0000	-0.5000	-0.2500	0.0000	-0.4330	0.0000
A_1B_2	-0.2165	0.1250	-0.1250	-0.2165	0.0000	-0.5000	0.0000	0.0000	-0.4330	0.0000	0.2500	0.0000
A_2B_1	0.2165	-0.1250	0.1250	0.2165	0.0000	-0.5000	0.0000	0.0000	-0.4330	0.0000	0.2500	0.0000
A_2B_2	0.1250	0.2165	-0.2165	0.1250	0.0000	0.0000	0.0000	0.5000	0.2500	0.0000	0.4330	0.0000
A_1C_1	0.1250	-0.2165	0.2165	0.1250	-0.2500	0.0000	-0.4330	0.0000	-0.1250	-0.2165	0.2165	-0.1250
A_1C_2	-0.2165	-0.1250	0.1250	-0.2165	-0.4330	0.0000	0.2500	0.0000	0.2165	-0.1250	0.1250	0.2165
A_2C_1	0.2165	0.1250	-0.1250	0.2165	-0.4330	0.0000	0.2500	0.0000	0.2165	-0.1250	0.1250	0.2165
A_2C_2	0.1250	-0.2165	0.2165	0.1250	0.2500	0.0000	0.4330	0.0000	0.1250	0.2165	-0.2165	0.1250
A_1D_1	-0.2500	-0.4330	-0.4330	0.2500	-0.5000	0.0000	0.0000	0.0000	-0.1250	0.2165	-0.2165	-0.1250
A_1D_2	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000	0.0000	0.0000	0.2165	0.1250	-0.1250	0.2165
A_2D_1	-0.4330	0.2500	0.2500	0.4330	0.0000	0.0000	-0.5000	0.0000	0.2165	0.1250	-0.1250	0.2165
A_2D_2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000	0.1250	-0.2165	0.2165	0.1250
B_1C_1	1.0000	0.0000	0.0000	0.0000	0.1250	0.2165	0.2165	-0.1250	-0.1250	-0.2165	0.2165	-0.1250
B_1C_2	0.0000	1.0000	0.0000	0.0000	0.2165	-0.1250	-0.1250	-0.2165	0.2165	-0.1250	0.1250	0.2165
B_2C_1	0.0000	0.0000	1.0000	0.0000	0.2165	-0.1250	-0.1250	-0.2165	0.2165	-0.1250	0.1250	0.2165
B_2C_2	0.0000	0.0000	0.0000	1.0000	-0.1250	-0.2165	-0.2165	0.1250	0.1250	0.2165	-0.2165	0.1250
B_1D_1	0.1250	0.2165	0.2165	-0.1250	1.0000	0.0000	0.0000	0.0000	-0.1250	0.2165	-0.2165	-0.1250
B_1D_2	0.2165	-0.1250	-0.1250	-0.2165	0.0000	1.0000	0.0000	0.0000	0.2165	0.1250	-0.1250	0.2165
B_2D_1	0.2165	-0.1250	-0.1250	-0.2165	0.0000	0.0000	1.0000	0.0000	0.2165	0.1250	-0.1250	0.2165
B_2D_2	-0.1250	-0.2165	-0.2165	0.1250	0.0000	0.0000	0.0000	1.0000	0.1250	-0.2165	0.2165	0.1250
C_1D_1	-0.1250	0.2165	0.2165	0.1250	-0.1250	0.2165	0.2165	0.1250	1.0000	0.0000	0.0000	0.0000
C_1D_2	-0.2165	-0.1250	-0.1250	0.2165	0.2165	0.1250	0.1250	-0.2165	0.0000	1.0000	0.0000	0.0000
C_2D_1	0.2165	0.1250	0.1250	-0.2165	-0.2165	-0.1250	-0.1250	0.2165	0.0000	0.0000	1.0000	0.0000
C_2D_2	-0.1250	0.2165	0.2165	0.1250	0.0000	-0.1250	0.2165	0.2165	0.1250	0.0000	0.0000	1.0000

From the complete correlation matrix, the correlation matrix for the $A \times BC$ interaction is extracted:

	A_1	A_2	B_1C_1	B_1C_2	B_2C_1	B_2C_2
A_1	1.0000	0.0000	−0.3062	0.1768	0.1768	0.3062
A_2	0.0000	1.0000	0.1768	0.3062	0.3062	−0.1768
B_1C_1	−0.3062	0.1768	1.0000	0.0000	0.0000	0.0000
B_1C_2	0.1768	0.3062	0.0000	1.0000	0.0000	0.0000
B_2C_1	0.1768	0.3062	0.0000	0.0000	1.0000	0.0000
B_2C_2	0.3062	−0.1768	0.0000	0.0000	0.0000	1.0000

The average squared correlation of order 3 for the $A \times BC$ interaction is computed as

$$\begin{aligned} Ave\ \rho_3^2(A, BC) &= \frac{1}{8}\Big[\rho^2(A_1, B_1C_1) + \rho^2(A_1, B_1C_2) + \rho^2(A_1, B_2C_1) \\ &\quad + \rho^2(A_1, B_2C_2) + \rho^2(A_2, B_1C_1) + \rho^2(A_2, B_1C_2) \\ &\quad + \rho^2(A_2, B_2C_1) + \rho^2(A_2, B_2C_2)\Big] \\ &= \frac{1}{8}\Big[(-0.3062)^2 + 0.1768^2 + 0.1768^2 + 0.3062^2 \\ &\quad + 0.1768^2 + 0.3062^2 + 0.3062^2 + (-0.1768^2)\Big] \\ &= 0.0625 \end{aligned}$$

By similar calculations, the average squared correlations of order 3 for the other main effect with two-factor interaction pairs are:

$$\begin{aligned} Ave\ \rho_3^2(A, BD) &= 0.1250 \\ Ave\ \rho_3^2(A, CD) &= 0.0625 \\ Ave\ \rho_3^2(B, AC) &= 0.0625 \\ Ave\ \rho_3^2(B, AD) &= 0.1250 \\ Ave\ \rho_3^2(B, CD) &= 0.0625 \\ Ave\ \rho_3^2(C, AB) &= 0.0625 \end{aligned}$$

$$Ave\ \rho_3^2(C, AD) = 0.0625$$

$$Ave\ \rho_3^2(C, BD) = 0.0625$$

$$Ave\ \rho_3^2(D, AB) = 0.1250$$

$$Ave\ \rho_3^2(D, AC) = 0.0625$$

$$Ave\ \rho_3^2(D, BC) = 0.0625$$

Next, the correlation matrix for the $AB \times AC$ interaction is extracted:

	A_1B_1	A_1B_2	A_2B_1	A_2B_2	A_1C_1	A_1C_2	A_2C_1	A_2C_2
A_1B_1	1.0000	0.0000	0.0000	0.0000	0.1250	0.2165	−0.2165	0.1250
A_1B_2	0.0000	1.0000	0.0000	0.0000	0.2165	−0.1250	0.1250	0.2165
A_2B_1	0.0000	0.0000	1.0000	0.0000	−0.2165	0.1250	−0.1250	−0.2165
A_2B_2	0.0000	0.0000	0.0000	1.0000	0.1250	0.2165	−0.2165	0.1250
A_1C_1	0.1250	0.2165	−0.2165	0.1250	1.0000	0.0000	0.0000	0.0000
A_1C_2	0.2165	−0.1250	0.1250	0.2165	0.0000	1.0000	0.0000	0.0000
A_2C_1	−0.2165	0.1250	−0.1250	−0.2165	0.0000	0.0000	1.0000	0.0000
A_2C_2	0.1250	0.2165	−0.2165	0.1250	0.0000	0.0000	0.0000	1.0000

The average squared correlation of order 4 for the $AB \times AC$ interaction is computed as

$$\begin{aligned} Ave\ \rho_4^2(AB, AC) &= \frac{1}{16}\Big[\rho^2(A_1B_1, A_1C_1) + \rho^2(A_1B_1, A_1C_2) + \rho^2(A_1B_1, A_2C_1) \\ &+ \rho^2(A_1B_1, A_2C_2) + \rho^2(A_1B_2, A_1C_1) + \rho^2(A_1B_2, A_1C_2) \\ &+ \rho^2(A_1B_2, A_2C_1) + \rho^2(A_1B_2, A_2C_2) + \rho^2(A_2B_1, A_1C_1) \\ &+ \rho^2(A_2B_1, A_1C_2) + \rho^2(A_2B_1, A_2C_1) + \rho^2(A_2B_1, A_2C_2) \\ &+ \rho^2(A_2B_2, A_1C_1) + \rho^2(A_2B_2, A_1C_2) + \rho^2(A_2B_2, A_2C_1) \\ &+ \rho^2(A_2B_2, A_2C_2)\Big] \\ &= \frac{1}{16}\Big[0.1250^2 + 0.2165^2 + (-0.2165)^2 + 0.1250^2 \\ &+ 0.2165^2 + (-0.1250)^2 + 0.1250^2 + 0.2165^2 \\ &+ (-0.2165)^2 + 0.1250^2 + (-0.1250)^2 + (-0.2165)^2 \end{aligned}$$

$$+ 0.1250^2 0.2165^2 + (-0.2165)^2 + 0.1250^2\Big]$$

$$= 0.0312$$

By similar calculations, the distinct average squared correlations of order 4 for the other two-factor interaction with two-factor interaction pairs are:

$$\begin{aligned}
Ave\ \rho_4^2(AB, AD) &= 0.0625 \\
Ave\ \rho_4^2(AB, BC) &= 0.0312 \\
Ave\ \rho_4^2(AB, BD) &= 0.0625 \\
Ave\ \rho_4^2(AB, CD) &= 0.0625 \\
Ave\ \rho_4^2(AC, AD) &= 0.0312 \\
Ave\ \rho_4^2(AC, BC) &= 0.0312 \\
Ave\ \rho_4^2(AC, BD) &= 0.0625 \\
Ave\ \rho_4^2(AC, CD) &= 0.0312 \\
Ave\ \rho_4^2(AD, BC) &= 0.0625 \\
Ave\ \rho_4^2(AD, BD) &= 0.0625 \\
Ave\ \rho_4^2(AD, CD) &= 0.0312 \\
Ave\ \rho_4^2(BC, BD) &= 0.0312 \\
Ave\ \rho_4^2(BC, CD) &= 0.0312 \\
Ave\ \rho_4^2(BD, CD) &= 0.0312
\end{aligned}$$

The distinct values of $Ave\ (\rho_3^2)$ and $Ave\ (\rho_4^2)$ and the number of each distinct value are indexed to create the ASCP and used to describe a property of the projection of the $OA(18, 7, 3, 2)$ onto these four columns. Following the format of (4.5), for this example, the ASCP is given in Table 4.4.

Projection	Number of $Ave\ (\rho_3^2)$ 0.0625	0.1250	Number of $Ave\ (\rho_4^2)$ 0.0312	0.0625
(1,2,3,7)	9	3	9	6

Table 4.4: ASCP for design in Table 4.1 treating factors as qualitative

The average squared correlations of order 3 and order 4 can be computed to describe every possible projection design of the $OA(18,7,3,2)$ in Table A.1. The ASCP for all combinatorially inequivalent three-, four-, and five-factor projection designs (as given by Evangelaras, Koukouvinos, Dean and Dingus (2005b)) are given in Table 4.5.

Class	Number of $Ave\ \rho_3^2$					Number of $Ave\ \rho_4^2$					
	.0312	.0625	.0938	.1250	.2500	0.0000	.0312	.0625	.0938	.1250	.2500
18.3.1	0	3	0	0	0	0	3	0	0	0	0
18.3.2	0	0	0	3	0	0	0	3	0	0	0
18.3.3	0	0	0	0	3	0	0	0	0	3	0
18.4.1	0	12	0	0	0	0	12	0	3	0	0
18.4.2	0	9	0	3	0	0	9	6	0	0	0
18.4.3	0	3	0	9	0	3	3	9	0	0	0
18.4.4	0	9	0	0	3	3	9	0	0	3	0
18.5.1	0	30	0	0	0	0	30	0	15	0	0
18.5.2	0	21	0	9	0	3	21	18	3	0	0
18.5.3	0	24	0	3	3	6	24	9	3	3	0
18.5.4	0	12	0	18	0	12	12	18	3	0	0

Table 4.5: Average squared correlations of order 3 and order 4 values for the combinatorially inequivalent three-factor , four-factor, and five-factor projection designs from $OA(18,7,3,2)$

4.3 Independence from Choice of Orthogonal Contrast Set

The definitions of average squared correlations of order 3 and order 4 in Section 4.1 and the example calculations in Section 4.2 use the standard linear and quadratic contrast coefficients. For qualitative factors, the linear and quadratic contrasts have no physical interpretation. Theorem 4.3.1 shows that the average squared correlations of order 3 and order 4 do not depend on the choice of orthogonal contrast set; the proof of Theorem 4.3.1 is found on page 44, following Lemma 4.3.3 and proof.

Theorem 4.3.1

(a) The average squared correlation of order 3 for any main effect and any two-factor interaction does not depend on the specific choice of complete sets of orthonormal contrasts.

(b) The average squared correlation of order 4 for any pair of two-factor interactions does not depend on the specific choice of complete sets of orthonormal contrasts.

The average of squared correlations between all pairs of contrasts measuring the effects of factors A and B is denoted $Ave\ \rho^2(\mathbf{C}_A, \mathbf{C}_B)$. If the factorial effect of A is measured by J orthogonal contrasts and the factorial effect of B is measured by K orthogonal contrasts, then

$$Ave\ \rho^2(\mathbf{C}_A, \mathbf{C}_B) = \frac{1}{JK} \sum_{j=1}^{J} \sum_{k=1}^{K} \rho^2(\mathbf{a}_j, \mathbf{b}_k)$$

where $\mathbf{a}_j$ is the j^{th} contrast measuring the effect of A, $\mathbf{b}_k$ is the k^{th} contrast measuring the effect of B, and $\rho^2(\mathbf{a}_j, \mathbf{b}_k)$ is equal to the squared correlation between contrasts $\mathbf{a}_i$ and $\mathbf{b}_j$. For a main effect of factor A and a two-factor interaction $B = C \times D$

effect, the average squared correlation of order 3 is denoted *Ave* $\rho_3^2(\mathbf{C}_A, \mathbf{C}_B)$; the average squared correlation of order 4 for a two-factor interaction $A = E \times F$ effect and a two-factor interaction $B = C \times D$ effect is denoted *Ave* $\rho_4^2(\mathbf{C}_A, \mathbf{C}_B)$. Similarly, the sum of squared correlations between all pairs of contrasts measuring the factorial effects of A and B, denoted *sum* $\rho^2(\mathbf{C}_A, \mathbf{C}_B)$, is

$$sum\ \rho^2(\mathbf{C}_A, \mathbf{C}_B) = \sum_{j=1}^{J}\sum_{k=1}^{K} \rho^2(\mathbf{a}_j, \mathbf{b}_k).$$

Before proving Theorem 4.3.1, the following claim is needed.

Lemma 4.3.2 *Let* $\mathbf{C}_A$ *denote the contrast sub-matrix consisting of* J *orthogonal contrasts* $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_J)$ *that describe the factorial effect of* A*, and* $\mathbf{C}_B$ *denote the contrast sub-matrix consisting of* K *orthogonal contrasts* $(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_K)$ *that describe the factorial effect* B*. Then*

$$\frac{1}{JK} sum\ \rho^2(\mathbf{C}_A, \mathbf{C}_B) = \frac{1}{JK}\sum_{j=1}^{J}\sum_{k=1}^{K} \rho^2(\mathbf{a}_j, \mathbf{b}_k) = \frac{1}{JK} trace(\mathbf{C}_B{}'\mathbf{C}_A\mathbf{C}_A{}'\mathbf{C}_B) \tag{4.8}$$

where $\rho^2(\mathbf{a}_j, \mathbf{b}_k)$ *is equal to the squared correlation between contrasts* $\mathbf{a}_i$ *and* $\mathbf{b}_j$*.*

Proof of Lemma 4.3.2

Let $\mathbf{C}_A$ denote the contrast sub-matrix consisting of J orthogonal contrasts $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_J)$ that describe the factorial effect of A and $\mathbf{C}_B$ denote the contrast sub-matrix consisting of K orthogonal contrasts $(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_K)$ that describe the factorial effect B.

Let the contrasts be normalized so that

$$\mathbf{a}_j^* = \frac{\mathbf{a}_j}{\sqrt{\mathbf{a}_j'\mathbf{a}_j}}$$

and

$$\mathbf{b}_k^* = \frac{\mathbf{b}_k}{\sqrt{\mathbf{b}_k'\mathbf{b}_k}}$$

Then $(\mathbf{a}_1^*, \mathbf{a}_2^*, \ldots, \mathbf{a}_J^*)$ is also a complete set of orthonormal contrasts corresponding to the factorial effect **A** and $(\mathbf{b}_1^*, \mathbf{b}_2^*, \ldots, \mathbf{b}_J^*)$ is also a complete set of orthonormal contrasts corresponding to the factorial effect **B**.

For given contrasts, $\mathbf{a}_j$, $1 \leq j \leq J$, and $\mathbf{b}_k$, $1 \leq k \leq K$, from Equation (2.4) in Section 2.4, the correlation between $\mathbf{a}_j$ and $\mathbf{b}_k$ is

$$\begin{aligned} \rho(\mathbf{a}_j, \mathbf{b}_k) &= \frac{\mathbf{a}_j'\mathbf{b}_k}{\sqrt{(\mathbf{a}_j'\mathbf{a}_j)(\mathbf{b}_k'\mathbf{b}_k)}} \\ &= \mathbf{a}_j^{*\prime}\mathbf{b}_k^* \end{aligned} \tag{4.9}$$

Squaring (4.9) gives

$$\begin{aligned} \rho^2(\mathbf{a}_j^*, \mathbf{b}_k^*) &= (\mathbf{a}_j^{*\prime}\mathbf{b}_k^*)'(\mathbf{a}_j^{*\prime}\mathbf{b}_k^*) \\ &= \mathbf{b}_k^{*\prime}\mathbf{a}_j^*\mathbf{a}_j^{*\prime}\mathbf{b}_k^* \end{aligned} \tag{4.10}$$

The sum of the squared correlations between **A** and **B** is

$$\begin{aligned} \mathit{sum}\ \rho^2(\mathbf{C}_A, \mathbf{C}_B) &= \sum_{j=1}^{J}\sum_{k=1}^{K} \rho^2(\mathbf{a}_j, \mathbf{b}_k) \\ &= \sum_{j=1}^{J}\sum_{k=1}^{K} \mathbf{b}_k^{*\prime}\mathbf{a}_j^*\mathbf{a}_j^{*\prime}\mathbf{b}_k^* \end{aligned} \tag{4.11}$$

Next, calculate $\mathbf{C}_B{}'\mathbf{C}_A\mathbf{C}_A{}'\mathbf{C}_B$:

$$\mathbf{C}_B{}'\mathbf{C}_A\mathbf{C}_A{}'\mathbf{C}_B = \begin{bmatrix} \mathbf{b}_1^{*\prime} \\ \mathbf{b}_2^{*\prime} \\ \vdots \\ \mathbf{b}_K^{*\prime} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1^* & \mathbf{a}_2^* & \cdots & \mathbf{a}_J^* \end{bmatrix} \begin{bmatrix} \mathbf{a}_1^{*\prime} \\ \mathbf{a}_2^{*\prime} \\ \vdots \\ \mathbf{a}_J^{*\prime} \end{bmatrix} \begin{bmatrix} \mathbf{b}_1^* & \mathbf{b}_2^* & \cdots & \mathbf{b}_K^* \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{b}_1^{*\prime}\mathbf{a}_1^* & \mathbf{b}_1^{*\prime}\mathbf{a}_2^* & \cdots & \mathbf{b}_1^{*\prime}\mathbf{a}_J^* \\ \mathbf{b}_2^{*\prime}\mathbf{a}_1^* & \mathbf{b}_2^{*\prime}\mathbf{a}_2^* & \cdots & \mathbf{b}_2^{*\prime}\mathbf{a}_J^* \\ \vdots & & & \\ \mathbf{b}_K^{*\prime}\mathbf{a}_1^* & \mathbf{b}_K^{*\prime}\mathbf{a}_2^* & \cdots & \mathbf{b}_K^{*\prime}\mathbf{a}_J^* \end{bmatrix} \begin{bmatrix} \mathbf{a}_1^{*\prime}\mathbf{b}_1^* & \mathbf{a}_1^{*\prime}\mathbf{b}_2^* & \cdots & \mathbf{a}_1^{*\prime}\mathbf{b}_K^* \\ \mathbf{a}_2^{*\prime}\mathbf{b}_1^* & \mathbf{a}_2^{*\prime}\mathbf{b}_2^* & \cdots & \mathbf{a}_2^{*\prime}\mathbf{b}_K^* \\ \vdots & & & \\ \mathbf{a}_J^{*\prime}\mathbf{b}_1^* & \mathbf{a}_J^{*\prime}\mathbf{b}_2^* & \cdots & \mathbf{a}_J^{*\prime}\mathbf{b}_K^* \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^J \mathbf{b}_1^{*\prime}\mathbf{a}_j^*\mathbf{a}_j^{*\prime}\mathbf{b}_1^* & & & \\ & \sum_{j=1}^J \mathbf{b}_2^{*\prime}\mathbf{a}_j^*\mathbf{a}_j^{*\prime}\mathbf{b}_2^* & & \\ & & \ddots & \\ & & & \sum_{j=1}^J \mathbf{b}_K^{*\prime}\mathbf{a}_j^*\mathbf{a}_j^{*\prime}\mathbf{b}_K^* \end{bmatrix} \quad (4.12)$$

The diagonal elements of the matrix $\mathbf{C}_B{}'\mathbf{C}_A\mathbf{C}_A{}'\mathbf{C}_B$ in (4.12) are the summands of equation (4.11).

Thus,

$$sum\ \rho^2(\mathbf{C}_A, \mathbf{C}_B) = \sum_{j=1}^{J}\sum_{k=1}^{K} \mathbf{b}_k^{*\prime}\mathbf{a}_j^*\mathbf{a}_j^{*\prime}\mathbf{b}_k^* = trace(\mathbf{C}_B{}'\mathbf{C}_A\mathbf{C}_A{}'\mathbf{C}_B).$$

□

Suppose the effect of B is the two-factor interaction $C \times D$. Let $\mathbf{C}_C = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{K_c})$ and $\mathbf{C}_D = (\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_{K_d})$ be the complete sets of orthogonal contrasts for the main effects of factors C and D, respectively. As described in Section 2.3, the complete set of orthogonal contrasts $(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_K)$ is constructed by element-wise product of all pairs of contrasts $(\mathbf{c}_i, \mathbf{d}_j)$, $i = 1, 2, \ldots, K_c$ and $j = 1, 2, \ldots, K_d$. So $\mathbf{C}_B = \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{d}_1 & \mathbf{c}_2 \cdot \mathbf{d}_1 & \ldots & \mathbf{c}_{K_c} \cdot \mathbf{d}_1 & \mathbf{c}_1 \cdot \mathbf{d}_2 & \ldots & \mathbf{c}_{K_c} \cdot \mathbf{d}_{K_d} \end{bmatrix}$. Define

$$\mathbf{C}_C \cdot \mathbf{d}_j = \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{d}_j & \mathbf{c}_2 \cdot \mathbf{d}_j & \ldots & \mathbf{c}_{K_c} \cdot \mathbf{d}_j \end{bmatrix} \quad (4.13)$$

and

$$\mathbf{C}_C \cdot \mathbf{C}_D = \begin{bmatrix} \mathbf{C}_C \cdot \mathbf{d}_1 & \mathbf{C}_C \cdot \mathbf{d}_2 & \ldots & \mathbf{C}_C \cdot \mathbf{d}_{K_d} \end{bmatrix} \quad (4.14)$$

Then $\mathbf{B} = \mathbf{C} \cdot \mathbf{D}$.

The following lemma will also be needed to prove Theorem 4.3.1.

Lemma 4.3.3 *For a matrix,* $\mathbf{Q}$*,*

$$(\mathbf{C}_C\mathbf{Q}) \cdot \mathbf{d}_j = (\mathbf{C}_C \cdot \mathbf{d}_j)\mathbf{Q}.$$

Proof of Lemma 4.3.3

Let $(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{K_c})$ be a complete set of orthogonal contrasts corresponding to a factorial effect $\mathbf{C}$ and let $(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_{K_d})$ be a complete set of orthogonal contrasts corresponding to a factorial effect $\mathbf{D}$. Let $\mathbf{B}$ be the $\mathbf{C} \times \mathbf{D}$ interaction.

Let $\mathbf{Q}$ be a $K_c \times K_c$ matrix.

Then

$$\begin{aligned}
(\mathbf{C}_C\mathbf{Q}) \cdot \mathbf{d}_j &= \left(\begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1K_c} \\ c_{21} & c_{22} & \ldots & c_{2K_c} \\ \vdots & & & \\ c_{n1} & c_{n2} & \ldots & c_{nK_c} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & \ldots & q_{1K_c} \\ q_{21} & q_{22} & \ldots & q_{2K_c} \\ \vdots & & & \\ q_{K_c1} & q_{K_c2} & \ldots & q_{K_cK_c} \end{bmatrix} \right) \cdot \begin{bmatrix} d_{1j} \\ d_{2j} \\ \vdots \\ d_{nj} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{K_c} c_{1i}q_{i1} & \sum_{p=1}^{J} c_{1i}q_{i2} & \ldots & \sum_{i=1}^{K_c} c_{1i}q_{iK_c} \\ \sum_{i=1}^{K_c} c_{2i}q_{i1} & \sum_{i=1}^{K_c} c_{2i}q_{i2} & \ldots & \sum_{i=1}^{K_c} c_{2i}q_{iK_c} \\ \vdots & & & \\ \sum_{i=1}^{K_c} c_{ni}q_{i1} & \sum_{i=1}^{J} c_{ni}q_{i2} & \ldots & \sum_{i=1}^{K_c} c_{ni}q_{iK_c} \end{bmatrix} \cdot \begin{bmatrix} d_{1j} \\ d_{2j} \\ \vdots \\ d_{nj} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{K_c} d_{1j}c_{1i}q_{i1} & \sum_{i=1}^{K_c} d_{1j}c_{1i}q_{i2} & \ldots & \sum_{i=1}^{K_c} d_{1j}c_{1i}q_{iK_c} \\ \sum_{i=1}^{K_c} d_{2j}c_{2i}q_{i1} & \sum_{i=1}^{K_c} d_{2j}c_{2i}q_{i2} & \ldots & \sum_{i=1}^{K_c} d_{2j}c_{2i}q_{iK_c} \\ \vdots & & & \\ \sum_{i=1}^{K_c} d_{nj}c_{ni}q_{i1} & \sum_{i=1}^{K_c} d_{nj}c_{ni}q_{i2} & \ldots & \sum_{i=1}^{K_c} d_{n1}c_{ni}q_{iK_c} \end{bmatrix} \\
&= \begin{bmatrix} c_{11}d_{1j} & c_{12}d_{1j} & \ldots & c_{1K_c}d_{1j} \\ c_{21}d_{2j} & c_{22}d_{2j} & \ldots & c_{2K_c}d_{2j} \\ \vdots & & & \\ c_{n1}d_{nj} & c_{n2}d_{nj} & \ldots & c_{nK_c}d_{nj} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & \ldots & q_{1K_c} \\ q_{21} & q_{22} & \ldots & q_{2K_c} \\ \vdots & & & \\ q_{K_c1} & q_{K_c2} & \ldots & q_{K_cK_c} \end{bmatrix} \\
&= (\mathbf{C}_C \cdot \mathbf{d}_j)\mathbf{Q}
\end{aligned}$$

□

The following corollary follows from Lemma 4.3.3.

Corollary 4.3.4

$$\mathbf{C}_B{}^* = (\mathbf{C}_C\mathbf{Q}) \cdot \mathbf{C}_D = (\mathbf{C}_C \cdot \mathbf{C}_D)\mathbf{Q}.$$

Proof of Corollary 4.3.4

$$\begin{aligned}
(\mathbf{C}_C\mathbf{Q}) \cdot \mathbf{C}_D &= \left[\begin{array}{cccc} (\mathbf{C}_C\mathbf{Q}) \cdot \mathbf{d}_1 & (\mathbf{C}_C\mathbf{Q}) \cdot \mathbf{d}_2 & \ldots & (\mathbf{C}_C\mathbf{Q}) \cdot \mathbf{d}_{K_d} \end{array}\right] \\
&= \left[\begin{array}{cccc} (\mathbf{C}_C \cdot \mathbf{d}_1)\mathbf{Q} & (\mathbf{C}_C \cdot \mathbf{d}_2)\mathbf{Q} & \ldots & (\mathbf{C}_C \cdot \mathbf{d}_{K_d})\mathbf{Q} \end{array}\right] \\
&= (\mathbf{C}_C\mathbf{C}_D)\mathbf{Q}
\end{aligned}$$

□

Now we can prove Theorem 4.3.1.

Proof of Theorem 4.3.1 (a)

Let $\mathbf{C}_A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_J \end{array}\right]$ be a complete set of orthogonal contrasts for a main effect A with $J + 1$ levels.

Let B represent the $C \times D$ two-factor interaction, where C and D have $K_c + 1$ levels and $K_d + 1$ levels, respectively, and $K_c \times K_d = K$. So $\mathbf{C}_B = \mathbf{C}_C \cdot \mathbf{C}_D$.

From Lemma 4.3.2, $sum\ \rho^2(\mathbf{C}_A, \mathbf{C}_B) = trace(\mathbf{C}_B{}'\mathbf{C}_A\mathbf{C}_A{}'\mathbf{C}_B)$, and so

$$Ave\ \rho_3^2(\mathbf{C}_A, \mathbf{C}_B) = \frac{1}{JK} trace(\mathbf{C}_B{}'\mathbf{C}_A\mathbf{C}_A{}'\mathbf{C}_B) \tag{4.15}$$

Next, let $\mathbf{Q}_1$ be a $J \times J$ orthogonal matrix. That is, let $\mathbf{Q}_1$ be a square $J \times J$ matrix such that $\mathbf{Q}_1\mathbf{Q}_1' = \mathbf{Q}_1'\mathbf{Q}_1 = \mathbf{I}_J$.

Then $\mathbf{A}\mathbf{Q}_1$ has the same dimensions as $\mathbf{A}$. And the columns of $\mathbf{A}\mathbf{Q}_1$ are orthogonal, since the columns of both $\mathbf{A}$ and $\mathbf{Q}_1$ are orthogonal. Thus, $\mathbf{A}\mathbf{Q}_1$ is a (possibly) different set of orthonormal contrasts for the main effect $\mathbf{A}$. Every orthonormal basis for the main effect of A is related by an orthogonal transformation to every other orthonormal basis for the main effect of A (see Lemma 8 of Appendix II, Scheffé (1959)).

Then

$$\begin{aligned} Ave\ \rho_3^2(\mathbf{C}_A\mathbf{Q}_1, \mathbf{C}_B) &= \frac{1}{JK} trace(\mathbf{C}_B{}'(\mathbf{C}_A\mathbf{Q}_1)(\mathbf{C}_A\mathbf{Q}_1)'\mathbf{C}_B) \\ &= \frac{1}{JK} trace(\mathbf{C}_B{}'\mathbf{C}_A\mathbf{Q}_1\mathbf{Q}_1'\mathbf{C}_A{}'\mathbf{C}_B) \\ &= \frac{1}{JK} trace(\mathbf{C}_B{}'\mathbf{C}_A\mathbf{I}\mathbf{C}_A{}'\mathbf{C}_B) \\ &= \frac{1}{JK} trace(\mathbf{C}_B{}'\mathbf{C}_A\mathbf{C}_A{}'\mathbf{C}_B) \\ &= Ave\ \rho_3^2(\mathbf{C}_A, \mathbf{C}_B) \end{aligned} \qquad (4.16)$$

and the average squared correlation of order 3 is independent of the set of orthogonal contrasts selected for the main effect of A.

Next, let $\mathbf{Q}_2$ be a $K_c \times K_c$ orthogonal matrix. That is, let $\mathbf{Q}_2$ be a square $K_c \times K_c$ matrix such that $\mathbf{Q}_2\mathbf{Q}_2' = \mathbf{Q}_2'\mathbf{Q}_2 = \mathbf{I}_{K_c}$.

Then $\mathbf{CQ}_2$ has the same dimensions as $\mathbf{C}$. And the columns of $\mathbf{CQ}_2$ are orthogonal, since the columns of both $\mathbf{C}$ and $\mathbf{Q}_2$ are orthogonal. Thus, $\mathbf{CQ}_2$ is a (possibly) different set of orthonormal contrasts for the main effect $\mathbf{C}$. Every orthonormal basis for the main effect of C is related by an orthogonal transformation to every other orthonormal basis for the main effect of C (see Lemma 8 of Appendix II, Scheffé (1959)).

Let $\mathbf{B}^* = (\mathbf{C}_C\mathbf{Q}_2) \cdot \mathbf{C}_D$.

From Corollary 4.3.4,

$$\begin{aligned} \mathbf{C}_{B^*} &= (\mathbf{C}_C\mathbf{Q}_2) \cdot \mathbf{C}_D \\ &= (\mathbf{C}_C \cdot \mathbf{C}_D)\mathbf{Q}_2 \end{aligned} \qquad (4.17)$$

So

$$
\begin{aligned}
\mathbf{C}_{B^*}{}' &= ((\mathbf{C}_C \cdot \mathbf{C}_D)\mathbf{Q}_2)' \\
&= \left[\begin{array}{cccc} (\mathbf{C}_C \cdot \mathbf{d}_1)\mathbf{Q}_2 & (\mathbf{C}_C \cdot \mathbf{d}_2)\mathbf{Q}_2 & \dots & (\mathbf{C}_C \cdot \mathbf{d}_{K_d})\mathbf{Q}_2 \end{array} \right] \\
&= \left[\begin{array}{c} ((\mathbf{C}_C \cdot \mathbf{d}_1)\mathbf{Q}_2)' \\ ((\mathbf{C}_C \cdot \mathbf{d}_2)\mathbf{Q}_2)' \\ \vdots \\ ((\mathbf{C}_C \cdot \mathbf{d}_{K_d})\mathbf{Q}_2)' \end{array} \right] \\
&= \left[\begin{array}{c} \mathbf{Q}_2'(\mathbf{C}_C \cdot \mathbf{d}_1)' \\ \mathbf{Q}_2'(\mathbf{C}_C \cdot \mathbf{d}_2)' \\ \vdots \\ \mathbf{Q}_2'(\mathbf{C}_C \cdot \mathbf{d}_{K_d})' \end{array} \right] \\
&= \left[\begin{array}{c} \mathbf{Q}_2(\mathbf{C}_C \cdot \mathbf{d}_1)' \\ \mathbf{Q}_2(\mathbf{C}_C \cdot \mathbf{d}_2)' \\ \vdots \\ \mathbf{Q}_2(\mathbf{C}_C \cdot \mathbf{d}_{K_d})' \end{array} \right] \\
&= \mathbf{Q}_2(\mathbf{C}_C \cdot \mathbf{C}_D)'
\end{aligned}
$$

Then the average squared correlation of order 3 between $\mathbf{A}$ and the new set of orthogonal interaction contrasts $\mathbf{C}_{B^*}$ is

$$
\begin{aligned}
Ave\ \rho_3^2(\mathbf{C}_A, \mathbf{C}_{B^*}) &= \frac{1}{JK} trace((\mathbf{C}_A)'\mathbf{C}_B\mathbf{C}_{B^*}{}'(\mathbf{C}_A)) \\
&= \frac{1}{JK} trace(\mathbf{C}_A{}'(\mathbf{C}_C \cdot \mathbf{C}_D)\mathbf{Q}_2\mathbf{Q}_2'(\mathbf{C}_C \cdot \mathbf{C}_D)'\mathbf{C}_A \\
&= \frac{1}{JK} trace(\mathbf{C}_A{}'(\mathbf{C}_C \cdot \mathbf{C}_D)\mathbf{I}(\mathbf{C}_C \cdot \mathbf{C}_D)'\mathbf{C}_A
\end{aligned}
\tag{4.18}
$$

$$= \frac{1}{JK} trace(\mathbf{C}_A{}'(\mathbf{C}_C \cdot \mathbf{C}_D)(\mathbf{C}_C \cdot \mathbf{C}_D)'\mathbf{C}_A$$

$$\begin{aligned} &= \frac{1}{JK} trace(\mathbf{C}_A{}'\mathbf{C}_B\mathbf{C}_B{}'\mathbf{C}_A) \\ &= Ave\ \rho_3^2(\mathbf{C}_A, \mathbf{C}_B) \end{aligned} \tag{4.19}$$

and the average squared correlation of order 3 is independent of the set of orthogonal contrasts selected for the main effect component of the two-factor interaction.

Also, the average squared correlation of order 3 between the set of orthogonal contrasts, $\mathbf{AQ}_1$, and the new set of orthogonal interaction contrasts $\mathbf{C}_{B^*}$ is

$$\begin{aligned} Ave\ \rho_3^2(\mathbf{C}_A\mathbf{Q}_1, \mathbf{C}_{B^*}) &= Ave\ \rho_3^2(\mathbf{C}_A, \mathbf{C}_{B^*}) \\ &= Ave\ \rho_3^2(\mathbf{C}_A, \mathbf{C}_B) \end{aligned} \tag{4.20}$$

and the average squared correlation of order 3 is independent of the set of orthogonal contrasts selected for the main effect and the set of orthogonal contrasts selected for the main effect component of the two-factor interaction.

The proof of Theorem 4.3.1 **(b)** follows similarly.

□

Because the average squared correlations are independent of the choice of a complete set of orthogonal contrasts in the case of qualitative factors, the rank ordering of designs based on the ASCP will be identical for all such choices. Even though the standard linear and quadratic orthogonal contrasts do not have a physical interpretation for qualitative factors, these contrast provide the same ranking of designs as all other complete sets of orthogonal contrasts.

4.4 Conditions for Use of the Average Squared Correlation Criterion

The average square correlation criterion is developed to describe properties of orthogonal arrays of strength two and their projections. For orthogonal arrays of strength three, all average squared correlations of order three are equal zero; in this case, the average squared correlations of order four are still useful for rank ordering designs.

As discussed in Section 2.3, a factorial effect with k levels can be measured by means of $k-1$ orthogonal contrasts. Otherwise the columns do not fully describe the effect as shown in the example below, where the two-factor interaction columns do not completely measure the two-factor interaction effects.

As was proven in Section 4.3, the average squared correlations of order 3 and order 4 are independent of the choice of orthogonal contrast set. If the design is an orthogonal array of strength two, for any choice of complete set of orthogonal contrasts the element-wise product of the main effect contrasts will produce columns for the two-factor interaction effect that are contrasts. However, if the design is not an orthogonal array, the columns resulting from element-wise product of the main effect contrasts are not necessarily contrasts. For example, consider the design in Table 4.6 with three factors and twelve runs.

For the design in Table 4.6, the main effects are not pairwise orthogonal. As a result of this non-orthogonality, the interaction effect columns produced by element-wise products of the column pairs are not contrasts (i.e. does not sum to zero). The full matrix, including main effect contrasts and two-factor interaction columns, is given in Table 4.7, where for ease of presentation the matrix is divided into the first

A	B	C
0	0	0
0	0	1
0	1	2
0	1	2
1	2	0
1	2	0
1	2	1
1	2	1
2	0	0
2	0	1
2	1	2
2	1	2

Table 4.6: Example of non-orthogonal design

six columns and the last twelve columns. From Table 4.7 it is clear that each column corresponding to a two-factor interaction is not a contrast and is not orthogonal to the other columns for the same two-factor interaction effect.

While the average squared correlation criterion is developed for orthogonal arrays of strength two, the criterion can be extended to other classes of designs. The extension requires identification of the correct set of orthogonal contrasts to be used. Dean and Draper (1999) provides a method for constructing orthogonal contrasts in a special case. The extension of the average squared correlation criterion to general classes of designs is not undertaken in this work.

A_1	A_2	B_1	B_2	C_1	C_2
-1	1	-1	1	-1	1
-1	1	-1	1	0	-2
-1	1	0	-2	1	1
-1	1	0	-2	-1	1
0	-2	1	1	0	-2
0	-2	1	1	1	1
0	-2	1	1	-1	1
0	-2	1	1	0	-2
1	1	0	-2	1	1
1	1	0	-2	-1	1
1	1	-1	1	0	-2
1	1	-1	1	1	1

A_1B_1	A_1B_2	A_2B_1	A_2B_2	A_1C_1	A_1C_2	A_2C_1	A_2C_2	B_1C_1	B_1C_2	B_2C_1	B_2C_2
1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
1	-1	-1	1	0	2	0	-2	0	2	0	-2
0	2	0	-2	-1	-1	1	1	0	0	-2	-2
0	2	0	-2	1	-1	-1	1	0	0	2	-2
0	0	-2	-2	0	0	0	4	0	-2	0	-2
0	0	-2	-2	0	0	-2	-2	1	1	1	1
0	0	-2	-2	0	0	2	-2	-1	1	-1	1
0	0	-2	-2	0	0	0	4	0	-2	0	-2
0	-2	0	-2	1	1	1	1	0	0	-2	-2
0	-2	0	-2	-1	1	-1	1	0	0	2	-2
-1	1	-1	1	0	-2	0	-2	0	2	0	-2
-1	1	-1	1	1	1	1	1	-1	-1	1	1

Table 4.7: Complete contrast matrix for example projection design in Table 4.1

CHAPTER 5

RANKING AND NON-EQUIVALENCE OF DESIGNS WITH QUALITATIVE FACTORS

5.1 Competing Methods

Numerous methods for ranking and determining non-equivalence of two designs have been proposed. If two designs are determined to be equivalent, the two designs possess the same properties and are assigned the same rank order. Many methods (Ma and Fang 2001, Xu and Wu 2001, Evangelaras et al. 2005b) have been developed primarily to rank designs with qualitative factors. Additionally, if two designs are ranked differentially, then the two designs are not equivalent. This chapter provides a brief review of the methods that will be compared to the average squared correlation criterion in Chapter 5.

Ma and Fang (2001) and Xu and Wu (2001) independently proposed a generalized wordlength pattern (GWP) and a generalized minimum aberration criterion for ranking both regular and non-regular designs. The proposed criterion is independent of the complete set of orthogonal contrasts used to measure the main effects and is, therefore, appropriate for designs with qualitative factors. The GWP of a design D

is defined as

$$W^g(D) = \{A_1^g(D), \ldots, A_j^g(D)\} \tag{5.1}$$

for

$$A_m^g(D) = \frac{1}{n(k-1)} \sum_{i=0}^{j} P_m(i;j) E_i(D) \tag{5.2}$$

where k is the number of levels of the factors,

$$P_m(i;j) = \sum_{r=0}^{m} (-1)^r (s-1)^{m-r} \binom{j}{r} \binom{j-i}{m-r} \tag{5.3}$$

and $E_m(D)$ is the distance distribution of D defined as

$$E_m(D) = \frac{1}{n} \#\{(\mathbf{a}, \mathbf{b}) | \mathbf{a}, \mathbf{b} \in D, d_H(\mathbf{a}, \mathbf{b}) = m\} \tag{5.4}$$

where $d_H(\mathbf{a}, \mathbf{b})$ is the Hamming distance between rows $\mathbf{c}$ and $\mathbf{d}$ of the design matrix $\mathbf{D}$ for the design D. The Hamming distance counts the number of dimensions in which two points do not match; if the Hamming distance is equal to zero, the two points are the same.

The GWP is an extension of the wordlength pattern (WP) for regular designs, $W(D) = \{A_1(D), \ldots, A_j(D)\}$, where $A_m(D)$ is equal to the number of distinct words in the defining relation with length equal to i (see Wu and Hamada (2000) Section 4.2). The GWP reduces to the WP for regular designs (see Ma and Fang (2001)). Like the WP, the GWP describes the aliasing of effects: the greater the value of $A_m^g(D)$, the greater the total aliasing of all j-factor and $(m-j)$-factor effect pairs.

Ma and Fang (2001) stated the theorem that two combinatorially isomorphic designs have the same GWP. Thus, GWP provides a necessary condition for combinatorial equivalence. However, GWP does not provide a sufficient condition for equivalence; two inequivalent designs can have the same GWP.

The GWP can be used to rank both regular and non-regular designs in terms of generalized resolution and generalized aberration (Xu and Wu 2001). Each entry, $A_i(D)$, in the generalized wordlength pattern represents the degree of aliasing between all i-factor interactions and the overall mean. Designs are ranked by sequentially minimizing $A_i(D)$ for $i = 1, 2, \ldots, p$. The *generalized resolution* of a design D is defined as the smallest value of i for which $A_i(D) > 0$. For designs D_1 and D_2, D_1 is said to have less *generalized aberration* than D_2 if $A_t(D_1) < A_t(D_2)$ for the smallest value of t such that $A_t(D_1) \neq A_t(D_2)$; D_1 is said to have *minimum generalized aberration* if no other design (with the same number of levels) has less generalized aberration than D_1 (Xu and Wu 2001).

The α wordlength pattern given by Cheng and Ye (2004) is a redefinition of the GWP of Xu and Wu (2001). The α wordlength pattern is calculated from the coefficients of an indicator function developed to determine non-equivalence and equivalence of designs with quantitative factors. The indicator function will be described in detail in Chapter 7. The relationship between the α wordlength pattern and the current work is examined in Chapter 8.

The α wordlength pattern of Cheng and Ye (2004) (defined in Section 8.2) is a redefinition of the GWP of Xu and Wu (2001). Each entry, $\alpha_i(D)$, in the α wordlength pattern is equal to the corresponding entry, $A_i(D)$, in the GWP. Computationally, the α wordlength pattern is simpler and, therefore, will be used here for comparison.

As discussed in Chapter 4, in this work the average squared correlation pattern criterion is proposed to rank order designs and evaluate non-equivalence. Two equivalent designs must have the same ASCP. This follows from Theorem 4.3.1 since two combinatorially equivalent designs are related by an orthogonal transformation. Also, two designs that have different ASCPs must be non-equivalent. Equality of ASCPs is a necessary but not a sufficient condition for combinatorial equivalence.

5.2 Ranking and Equivalence

The ASCPs of designs can be used to rank order designs. Smaller correlations represent less aliasing between effects; less aliasing indicates greater information for estimation of effects independent from other effects. Smaller average squared correlations of order 3 correspond to better estimation of main effects (independent of two-factor interactions) and smaller average squared correlations of both order 3 and order 4 correspond to better estimation of two-factor interactions (independent of main effects and other two-factor interactions). Thus, average squared correlations of order 3 are considered more important than average squared correlations of order 4, since in this work estimation of main effects independently of two-factor interactions is considered more important than estimation of two-factor interactions. Based on the average squared correlation values, designs can be ranked by sequentially maximizing the $r_{3(1)}$, $r_{3(2)}$, …, $r_{3(k)}$, $r_{4(1)}$, $r_{4(2)}$, …, $r_{4(m)}$ in Array (4.5). By maximizing the $r_{i(j)}$ in this order, the design with the greatest number of the smallest average squared correlations of order 3 is ranked as the best.

5.2.1 Example: Rank Ordering of Combinatorially Inequivalent Design Classes from $OA(18, 7, 3, 2)$

The ASCP and α wordlength patterns for each inequivalent projection design class from the $OA(18, 7, 3, 2)$ in Table A.1 are shown in Table 5.1 in order to examine the optimal design choices based on each ranking method. In general, ranking of designs requires examination of both $Ave\ (\rho_3^2)$ and $Ave\ (\rho_4^2)$. However, for projections of all

Class	Number of $Ave\ \rho_3^2$					Number of $Ave\ \rho_4^2$						α Wordlength
	.03	.06	.09	.12	.25	0.00	.03	.06	.09	.12	.25	
18.3.1	0	3	0	0	0	0	3	0	0	0	0	(0.0,0.0,0.5)
18.3.2	0	0	0	3	0	0	0	3	0	0	0	(0.0,0.0,1.0)
18.3.3	0	0	0	0	3	0	0	0	0	3	0	(0.0,0.0,2.0)
18.4.1	0	12	0	0	0	0	12	0	3	0	0	(0.0,0.0,2.0,1.5)
18.4.2	0	9	0	3	0	0	9	6	0	0	0	(0.0,0.0,2.5,1.0)
18.4.3	0	3	0	9	0	3	3	9	0	0	0	(0.0,0.0,3.5,0.0)
18.4.4	0	9	0	0	3	3	9	0	0	3	0	(0.0,0.0,3.5,0.0)
18.5.1	0	30	0	0	0	0	30	0	15	0	0	(0.0,0.0,5.0,7.5,0.0)
18.5.2	0	21	0	9	0	3	21	18	3	0	0	(0.0,0.0,6.5,4.5,1.5)
18.5.3	0	24	0	3	3	6	24	9	3	3	0	(0.0,0.0,7.0,3.5,2.0)
18.5.4	0	12	0	18	0	12	12	18	3	0	0	(0.0,0.0,8.0,1.5,3.0)

Table 5.1: ASCP and α wordlength patterns for inequivalent projection classes from $OA(18, 7, 3, 2)$

sizes from the $OA(18, 7, 3, 2)$ design ranking is based solely on $Ave\ (\rho_3^2)$. According to the ASCP, the design classes 18.3.1, 18.4.1, and 18.5.1, would be ranked as the best projection designs of three-, four-, and five-factors, respectively. Sequentially minimizing the elements $\alpha_i(D)$ of the α wordlength gives the same ranking.

The ASCP and α wordlength pattern do not provide identical rank orderings of the designs in the case of the four-factor projections. Ranking the designs in decreasing order of quality, the ASCP ranks the designs in the order of 18.4.1 $\gg$ 18.4.2 $\gg$ 18.4.4 $\gg$ 18.4.3, where $\gg$ represents better than. In comparison, the α wordlength pattern of Section 8.2 would rank the designs in order 18.4.1 $\gg$ 18.4.2 $\gg$ 18.4.4 $==$ 18.4.3, where $==$ represents is equally as good as. In this example, the ASCP is able to provide a more detailed rank ordering than the α wordlength pattern.

It is also possible that the ASCP provides a "reversed" ranking compared to the α wordlength pattern. For example, the α wordlength pattern ranks 18.5.3 higher than 18.5.4 while ASCP ranks 18.5.4 higher than 18.5.3. The rank reversal is the result of a few large average squared correlations of order 3 (i.e. the average squared

	p = 3	p = 4	p = 5
Ave ρ^2	3	4	4
α Wordlength/GWP	3	3	4
Actual	3	4	4

Table 5.2: Number of combinatorially inequivalent projection design classes identified for $p = 3, 4, 5$ columns from $OA(18, 7, 3, 2)$ in Table A.1

correlations of order 3 equal to 0.09 and 0.12); these large average squared correlations are included in the α wordlength pattern but do not affect the ASCP ranking since 18.5.3 has more average squared correlations of order 3 equal to the smaller value 0.06 than does 18.5.4.

5.2.2 Combinatorial Non-equivalence of Projection Designs from Orthogonal Arrays

Table 5.1 indicates that the ASCP is able to distinguish all inequivalent classes of three-, four-, and five-factor projection designs from the $OA(18, 7, 3, 2)$ while the α wordlength pattern is not able to distinguish all four-factor projections. Table 5.2, Table 5.3, and Table 5.4 provide the numbers of inequivalent classes identified by the ASCP and the α wordlength pattern for three-, four-, and five-factor projections from the $OA(18, 7, 3, 2)$, $OA(27, 13, 3, 2)$, and $OA(36, 13, 3, 2)$ in Tables A.1, A.2, and A.3, respectively. The actual numbers of inequivalent classes for each design, as presented by Evangelaras et al. (2005b), are also provided for comparison.

Again, from Table 5.2, the ASCP is able to identify all classes for each of the projection sizes for the $OA(18, 7, 3, 2)$; the α wordlength pattern fails to differentiate

	p = 3	p = 4	p = 5
Ave ρ^2	2	3	3
α Wordlength/GWP	2	3	3
Actual	2	3	3

Table 5.3: Number of combinatorially inequivalent projection design classes identified for $p = 3, 4, 5$ columns from $OA(27, 13, 3, 2)$ in Table A.2

two of the inequivalent classes of four-factor projections, identifying three classes and not four.

For the $OA(27, 13, 3, 2)$, both criteria are able to identify correctly each of the combinatorially equivalent design classes (Table 5.3). However, neither algorithm is able to distinguish all the equivalence classes for the $OA(36, 13, 3, 2)$ (Table 5.4). For the $OA(36, 13, 3, 2)$, each criterion can identify all classes of three-factor projections. The ASCP performs considerably better than the wordlength pattern for both four- and five-factor projections: ASCP is 93% and 92% correct for four- and five-factor projections, respectively, while wordlength pattern is only 74% and 42% correct. Thus, the proposed ASCP provides a substantial improvement for determining combinatorially inequivalent design classes of projection designs as compared to the α wordlength pattern/GWP. The superior performance in determining inequivalent classes exemplifies a superior capability to rank order the projection designs.

5.3 Estimation Capacity of Projection Designs

Different designs allow different models to be fit to the data. The size of the model fit and the number of parameters that can be estimated are necessarily restricted by the number of experimental runs. In general, for an experiment with n runs, at most

	p = 3	p = 4	p = 5
Ave ρ^2	6	25	77
α Wordlength/GWP	6	20	35
Actual	6	27	84

Table 5.4: Number of combinatorially inequivalent projection design classes identified for $p = 3, 4, 5$ columns from $OA(36, 13, 3, 2)$ in Table A.3

$n - 1$ factorial contrasts can be estimated in addition to the overall mean. Thus, it is informative to consider which main effect and interaction effect contrasts can be estimated from the projection designs studied in addition to the correlation between contrasts.

As pointed out by Evangelaras et al. (2005b) all projections of the $OA(27, 13, 3, 2)$ in Table A.2 are regular fractional factorial designs. Thus, the estimable effects are known via the defining relations, which are also provided by Evangelaras et al. (2005b).

Given the $OA(18, 7, 3, 2)$ in Table A.1, at most seventeen factorial contrasts can be estimated. Recall that each main effect requires two contrasts and each two-factor interaction effect requires four contrasts. Therefore, if all main effect contrasts are fitted, at most two of the three two-factor interactions can also be fitted for three- and four-factor projections, and at most one of the three two-factor interactions can be fitted for the five-factor projections.

In Appendix B, Tables B.1–B.5 give the degrees of freedom for estimating each main effect and two-factor interaction effect for each of the possible models that can be fitted. In the tables, if a main effect has two degrees of freedom, then that main effect can be estimated independently from all other effects in the model; a two-factor

Class	Number of $Ave\ \rho_3^2$						Number of $Ave\ \rho_4^2$					
	0.016	0.031	0.062	0.078	0.109	0.250	0.008	0.016	0.031	0.039	0.055	0.125
36.3.1	3	0	0	0	0	0	3	0	0	0	0	0
36.3.2	0	0	0	0	3	0	0	0	0	0	3	0
36.3.3	0	0	0	0	0	3	0	0	0	0	0	3
36.3.4	0	0	3	0	0	0	0	0	3	0	0	0
36.3.5	0	3	0	0	0	0	0	3	0	0	0	0
36.3.6	0	0	0	3	0	0	0	0	0	3	0	0

Table 5.5: ASCP for inequivalent three-factor projection classes from $OA(36, 13, 3, 2)$

interaction effect can be estimated independently from all other effects in the model if it has four degrees of freedom. For example, in Table B.1, designs in class 18.3.1 allow independent estimation of all main effects and any pair of two-factor interactions. In contrast, only a single main effect can be estimated using designs in class 18.3.3. For the $OA(18, 7, 3, 2)$, the rank ordering of the design classes based on the ASCP and α wordlength pattern/GWP is the same as the rank ordering based on the estimation capacity.

At most thirty-five factorial contrasts can be estimated using the $OA(36, 13, 3, 2)$ in Table A.3. At most six two-factor interactions can be fit in addition to the main effects for five-factor projections, while all two-factor interactions can be fit for three- and four-factor projections.

Tables B.6–B.14 of Appendix B give the degrees of freedom for estimating each main effect and two-factor interaction effect for each of the possible models that can be fitted. The numbers of models capable of estimating all main effects and i two-factor interactions, for $i = 1, 2, 3, 4, 5, 6$, for each five-factor projection class are given in Tables B.15 – B.20.

For three-factor projections, the three design classes 36.3.1, 36.3.5, and 36.3.6 allow for the independent estimation of all main effects and two-factor interaction effects. In contrast, no main effect or two-factor interaction is estimable using designs in class 36.3.3. The ASCP would similarly rank order the class 36.3.1 as the best design and class 36.3.3 as the least good design; the ASCPs for three-factor projections from the $OA(36, 13, 3, 2)$ are given in Table 5.5. However, the ASCP would rank class 36.3.1 as better than either class 36.3.5 or class 36.3.6. In fact, the ASCP ranks class 36.3.4 as better than class 36.3.6 as well. Thus, it appears that the rank ordering of the poor designs is consistent between the ASCP and the estimation capacity criteria but differs for the optimal designs. This pattern holds for both four- and five-factor projection designs from the $OA(36, 13, 3, 2)$.

Additional discussion of the estimation capacity of the various projection designs is given in (Evangelaras et al. 2005b).

CHAPTER 6

AVERAGE SQUARED CORRELATION FOR QUANTITATIVE FACTORS

In Section 4.1, average squared correlations of order 3 (Equation (4.3)) and order 4 (Equation (4.4)) are defined for qualitative factors. These definitions are based on the effect hierarchy (4.1) in which the order of the effect is defined by the number of factors included in the effect and the two orthogonal contrasts for each main effect are treated as equally important. In the case of quantitative factors, however, the two orthogonal contrasts are specified as linear and quadratic contrasts and are meaningful; when the factor levels possess a natural ordering as with quantitative factors, the existence of a linear or quadratic effect can be important to identify. Also, the linear and quadratic contrasts are not equally important as the linear trend is more likely than the quadratic trend to be non-negligible. Therefore, the contrasts should be grouped differently for quantitative factors than they were for qualitative factors, producing a different effect hierarchy. If the order of the effect is defined as equal to the polynomial degree of the effect contrasts, with a linear contrast having polynomial degree 1 and a quadratic contrast having polynomial degree 2, the effect hierarchy is

$$l$$

$$\gg q == ll$$

$$\gg lq == ql == lll$$

$$\gg qq == llq == lql == qll == llll$$

$$\gg lqq == qlq == qql == lllq == llql == lqll == qlll == lllll \gg \dots \quad (6.1)$$

This effect hierarchy is the basis of the β wordlength pattern of Cheng and Ye (2004), which will be described in Chapter 7 and which will be compared to the average squared correlation criterion in Chapter 9.

The problem with the effect hierarchy (6.1) is that some correlations of order 3 are regarded as equally important as some correlations of order 4. For example, the $A_l \times B_l C_q$, $A_l \times B_q C_l$, and $A_q \times B_l C_l$ correlations of order 3 are included in the same group with the $A_l B_l \times B_l C_l$ and $A_l C_l \times B_l C_l$ correlations of order 4. As a result, it is not possible to differentiate the average squared correlations of order 3 from the average squared correlations of order 4.

If both the number of factors and the polynomial degree of the contrasts are considered to be important, the effect hierarchy (4.1) used for qualitative factors (see Section 4.1) and the effect hierarchy (6.1) can be combined into a single effect hierarchy. The resulting hierarchy ranks first by the number of factors included in the effect and then by the polynomial order of the contrasts of the effect. The resulting effect hierarchy is as follows

$$l \gg q$$

$$\gg ll \gg lq == ql \gg qq$$

$$\gg lll \gg llq == lql == qll \gg lqq == qlq == qql \gg qqq$$

$$\gg llll \gg lllq == \dots == qlll \gg llqq == \dots == qqll$$

$$\gg lqqq == \dots == qqql \gg qqqq \gg \dots \quad (6.2)$$

The effect hierarchy (6.2) was suggested by Cheng and Ye (2004) as an alternative to (6.1), but was not examined by these researchers. The effect hierarchy (6.2) will be the foundation for the ASCP for designs with quantitative factors.

6.1 Average Squared Correlations of Order 3 and Order 4

The average squared correlations of order 3 and order 4 are calculated from the correlation matrix as defined in Equation (4.2) of Section 4.1. Based on the effect hierarchy (6.2), the average squared correlations of order 3 (representing correlations between a main effect and a two-factor interaction) are summarized into four values representing polynomial orders 3, 4, 5, and 6. The set of four average squared correlations for a main effect and a two-factor interactions are called *the complete set of average squared correlations of order 3*. For example, for the $A \times BC$ interaction, the complete set of average squared correlations of order 3 is

$$\begin{aligned}
Ave\ \rho^2_{3,3}(A,BC) &= \rho^2(A_l, B_lC_l) \\
Ave\ \rho^2_{3,4}(A,BC) &= \frac{1}{3}[\rho^2(A_l, B_lC_q) + \rho^2(A_l, B_qC_l) + \rho^2(A_q, B_lC_l)] \\
Ave\ \rho^2_{3,5}(A,BC) &= \frac{1}{3}[\rho^2(A_l, B_qC_q) + \rho^2(A_q, B_lC_q) + \rho^2(A_q, B_qC_l)] \\
Ave\ \rho^2_{3,6}(A,BC) &= \rho^2(A_q, B_qC_q) \qquad (6.3)
\end{aligned}$$

For each average squared correlation, $Ave\ \rho^2_{i,j}$, the subscript i represents the factorial order ($i = 3, 4$) and the subscript j represents the polynomial degree of the correlation ($j = 3, 4, 5, 6, 7, 8$). Similarly, the average squared correlations of order 4 are summarized into five values representing polynomial orders 4, 5, 6, 7, and 8. The set of five average squared correlations for a pair of two-factor interactions are called *the complete set of average squared correlations of order 4*. For example, the complete set of average squared correlations of order 4 for the $AB \times CD$ interaction is

$$\begin{aligned}
Ave\ \rho^2_{4,4}(AB,CD) &= \rho^2(A_lB_l, C_lD_l) \\
Ave\ \rho^2_{4,5}(AB,CD) &= \frac{1}{4}[\rho^2(A_lB_l, C_lD_q) + \rho^2(A_lB_l, C_qD_l) + \rho^2(A_lB_q, C_lD_l) \\
&\quad + \rho^2(A_qB_l, C_lD_l)] \\
Ave\ \rho^2_{4,6}(AB,CD) &= \frac{1}{6}[\rho^2(A_lB_l, C_qD_q) + \rho^2(A_lB_q, C_lD_q) + \rho^2(A_qB_l, C_lD_q) \\
&\quad + \rho^2(A_lB_q, C_qD_l) + \rho^2(A_qB_l, C_qD_l) + \rho^2(A_qB_q, C_lD_l)] \\
Ave\ \rho^2_{4,7}(AB,CD) &= \frac{1}{4}[\rho^2(A_lB_q, C_qD_q) + \rho^2(A_qB_l, C_qD_q) + \rho^2(A_qB_q, C_lD_q) \\
&\quad + \rho^2(A_qB_q, C_qD_l)] \\
Ave\ \rho^2_{4,8}(AB,CD) &= \rho^2(A_qB_q, C_qD_q) \qquad (6.4)
\end{aligned}$$

where, again, the subscripts represent factorial order and polynomial degree. Thus, each design is represented by sets of four average squared correlations of order 3 and five average squared correlations of order 4.

While it is not assumed that the correlation matrix is symmetric, it is true that the complete set of average squared correlations of order 3 for A with BC is the same as the set of average squared correlations of order 3 for BC with A. This equivalence is clear from (6.3), since

$$\rho^2(A_i, B_jC_k) = \rho^2(B_jC_k, A_i) \qquad (6.5)$$

so that, for example,

$$\begin{aligned}
Ave\ \rho^2_4(A,BC) &= \frac{1}{3}[\rho^2(A_l, B_lC_q) + \rho^2(A_l, B_qC_l) + \rho^2(A_q, B_lC_l)] \\
&= \frac{1}{3}[\rho^2(B_lC_q, A_l) + \rho^2(B_qC_l, A_l) + \rho^2(B_lC_l, A_q)] \\
&= Ave\ \rho^2_4(BC, A) \qquad (6.6)
\end{aligned}$$

For the same reason, from (6.4) it follows that the complete set of average squared correlations of order 4 for AB with CD is the same as the complete set of average squared correlations of order 4 for CD with AB. Therefore, only one of each of these pairs needs to be included in the average squared correlation pattern.

6.1.1 Average Squared Correlation Pattern for Designs with Quantitative Factors

The ASCP defined in Section 4.1.1 can be extended for designs with quantitative factors. For designs with quantitative factors, the ASCP is based on the values of the average squared correlations of order 3 and order 4 and the number of each distinct value.

The correlation between any main effect contrast for a factor and any contrast for a two-factor interaction involving that factor is equal to zero and such complete sets of average squared correlations of order 3 do not need to be included in the ASCP. For designs with quantitative factors, the numbers of complete sets of average squared correlations of order 3 and of order 4 are the same as the corresponding numbers of average squared correlations for designs with qualitative factors. (See Section 4.1 for details.)

The distinct, non-zero complete sets of average squared correlations of order 3 and order 4 are summarized into the ASCP. The ASCP is a two-row array,

$$\left| \begin{array}{cccc|cccc} Ave\ \rho^2_{3,3(1)} & Ave\ \rho^2_{3,3(2)} & \cdots & Ave\ \rho^2_{3,3(k_3)} & Ave\ \rho^2_{3,4(1)} & Ave\ \rho^2_{3,4(2)} & \cdots & Ave\ \rho^2_{3,4(k_4)} \\ r_{3,3(1)} & r_{3,3(2)} & \cdots & r_{3,3(k_3)} & r_{3,4(1)} & r_{3,4(2)} & \cdots & r_{3,4(k_4)} \end{array} \right|$$

$$\left| \begin{array}{cccc|cccc} Ave\ \rho^2_{3,5(1)} & Ave\ \rho^2_{3,5(2)} & \cdots & Ave\ \rho^2_{3,5(k_5)} & Ave\ \rho^2_{3,6(1)} & Ave\ \rho^2_{3,6(2)} & \cdots & Ave\ \rho^2_{3,6(k_6)} \\ r_{3,5(1)} & r_{3,5(2)} & \cdots & r_{3,5(k_5)} & r_{3,6(1)} & r_{3,6(2)} & \cdots & r_{3,6(k_6)} \end{array} \right|$$

$$\left| \begin{matrix} Ave\ \rho^2_{4,4(1)} & Ave\ \rho^2_{4,4(2)} & \ldots & Ave\ \rho^2_{4,4(m_4)} \\ r_{4,4(1)} & r_{4,4(2)} & \ldots & r_{4,4(m_4)} \end{matrix} \right| \begin{matrix} Ave\ \rho^2_{4,5(1)} & Ave\ \rho^2_{4,5(2)} & \ldots & Ave\ \rho^2_{4,5(m_5)} \\ r_{4,5(1)} & r_{4,5(2)} & \ldots & r_{4,5(m_5)} \end{matrix} \right|$$

$$\left| \begin{matrix} Ave\ \rho^2_{4,6(1)} & Ave\ \rho^2_{4,6(2)} & \ldots & Ave\ \rho^2_{4,6(m_6)} \\ r_{4,6(1)} & r_{4,6(2)} & \ldots & r_{4,6(m_6)} \end{matrix} \right| \begin{matrix} Ave\ \rho^2_{4,7(1)} & Ave\ \rho^2_{4,7(2)} & \ldots & Ave\ \rho^2_{4,7(m_7)} \\ r_{4,7(1)} & r_{4,7(2)} & \ldots & r_{4,7(m_7)} \end{matrix} \right|$$

$$\left| \begin{matrix} Ave\ \rho^2_{4,8(1)} & Ave\ \rho^2_{4,8(2)} & \ldots & Ave\ \rho^2_{4,8(m_8)} \\ r_{4,8(1)} & r_{4,8(2)} & \ldots & r_{4,8(m_8)} \end{matrix} \right| \tag{6.7}$$

where

$$Ave\ \rho^2_{3,i(1)} < Ave\ \rho^2_{3,i(2)} < \ldots < Ave\ \rho^2_{3,i(k_i)},$$

for $i = 3, 4, 5, 6,$

$$Ave\ \rho^2_{4,j(1)} < Ave\ \rho^2_{4,j(2)} < \ldots < Ave\ \rho^2_{4,j(m_j)},$$

for $j = 4, 5, 6, 7, 8,$

$$r_{i,j(n)} \text{ is the number of } Ave\ \rho^2_{i,j} = Ave\ \rho^2_{i,j(n)},$$

k_i is the number of distinct values of $Ave\ \rho^2_{3,i}$, and m_j is the number of distinct values of $Ave\ \rho^2_{4,j}$.

The conditions for use of the average squared correlation criterion discussed in Section 4.4 for designs with qualitative factors also apply to designs with quantitative factors.

6.2 Example: Calculation of Average Squared Correlations of Order 3 and Order 4

Consider the $OA(18, 7, 3, 2)$ presented in Table A.1. Suppose, as an example, we select the projection formed by the first, second, third, and seventh columns of the array, and label the columns A, B, C, and D, respectively. This is the same example considered in Section 4.2 and will utilize many of the same tables. The design is given in Table 4.1, and corresponding contrasts and correlation matrices in Tables 4.2 and 4.3.

From the complete correlation matrix, the correlation matrix for the $A \times BC$ interaction can be extracted as in the the Example 4.2; that is:

	A_l	A_q	B_lC_l	B_lC_q	B_qC_l	B_qC_q
A_l	1.0000	0.0000	−0.3062	0.1768	0.1768	0.3062
A_q	0.0000	1.0000	0.1768	0.3062	0.3062	−0.1768
B_lC_l	−0.3062	0.1768	1.0000	0.0000	0.0000	0.0000
B_lC_q	0.1768	0.3062	0.0000	1.0000	0.0000	0.0000
B_qC_l	0.1768	0.3062	0.0000	0.0000	1.0000	0.0000
B_qC_q	0.3062	−0.1768	0.0000	0.0000	0.0000	1.0000

Treating the factors as quantitative, the complete set of average squared correlations of order 3 for the $A \times BC$ interaction is computed as

$$\begin{aligned}
Ave\ \rho^2_{3,3}(A, BC) &= \rho^2(A_l, B_lC_l) \\
&= (-0.3062)^2 \\
&= 0.0938 \\
Ave\ \rho^2_{3,4}(A, BC) &= \frac{1}{3}\left[\rho^2(A_l, B_lC_q) + \rho^2(A_l, B_qC_l) + \rho^2(A_q, B_lC_l)\right] \\
&= \frac{1}{3}\left[0.1768^2 + 0.1768^2 + 0.1768^2\right] \\
&= 0.0312 \\
Ave\ \rho^2_{3,5}(A, BC) &= \frac{1}{3}\left[\rho^2(A_l, B_qC_q) + \rho^2(A_q, B_lC_q) + \rho^2(A_q, B_qC_l)\right] \\
&= \frac{1}{3}\left[0.3062^2 + 0.3062^2 + 0.3062^2\right] \\
&= 0.0938 \\
Ave\ \rho^2_{3,6}(A, BC) &= \rho^2(A_q, B_qC_q) \\
&= (-0.1768)^2 \\
&= 0.0312
\end{aligned}$$

By similar calculations, the complete set of average squared correlations of order 3 for A with BD is

$$
\begin{aligned}
Ave\ \rho^2_{3,3}(A, BD) &= 0.000 \\
Ave\ \rho^2_{3,4}(A, BD) &= 0.1667 \\
Ave\ \rho^2_{3,5}(A, BD) &= 0.000 \\
Ave\ \rho^2_{3,6}(A, BD) &= 0.5000
\end{aligned}
$$

The complete set of average squared correlations of order 3 for A with CD is

$$
\begin{aligned}
Ave\ \rho^2_{3,3}(A, CD) &= 0.0938 \\
Ave\ \rho^2_{3,4}(A, CD) &= 0.0312 \\
Ave\ \rho^2_{3,5}(A, CD) &= 0.0938 \\
Ave\ \rho^2_{3,6}(A, CD) &= 0.0312
\end{aligned}
$$

The complete sets of average squared correlations of order 3 for the B, C, and D main effects are calculated similarly.

Next, the correlation matrix for the $AB \times AC$ interaction can be extracted:

	A_lB_l	A_lB_q	A_qB_l	A_qB_q	A_lC_l	A_lC_q	A_qC_l	A_qC_q
A_lB_l	1.0000	0.0000	0.0000	0.0000	0.1250	0.2165	−0.2165	0.1250
A_lB_q	0.0000	1.0000	0.0000	0.0000	0.2165	−0.1250	0.1250	0.2165
A_qB_l	0.0000	0.0000	1.0000	0.0000	−0.2165	0.1250	−0.1250	−0.2165
A_qB_q	0.0000	0.0000	0.0000	1.0000	0.1250	0.2165	−0.2165	0.1250
A_lC_l	0.1250	0.2165	−0.2165	0.1250	1.0000	0.0000	0.0000	0.0000
A_lC_q	0.2165	−0.1250	0.1250	0.2165	0.0000	1.0000	0.0000	0.0000
A_qC_l	−0.2165	0.1250	−0.1250	−0.2165	0.0000	0.0000	1.0000	0.0000
A_qC_q	0.1250	0.2165	−0.2165	0.1250	0.0000	0.0000	0.0000	1.0000

The complete set of average squared correlations of order 4 for the $AB \times AC$ interaction is computed as

$$
Ave\ \rho^2_{4,4}(AB, AC) = \rho^2(A_lB_l, A_lC_l)
$$

$$
\begin{aligned}
&= 0.1250^2 \\
&= 0.0156 \\
Ave\ \rho^2_{4,5}(AB, AC) &= \frac{1}{4}\Big[\rho^2(A_lB_l, A_lC_q) + \rho^2(A_lB_l, A_qC_l) + \rho^2(A_lB_q, A_lC_l) \\
&\quad + \rho^2(A_qB_l, A_lC_l)\Big] \\
&= \frac{1}{4}\Big[0.2165^2 + (-0.2165)^2 + 0.2165^2 + (-0.2165)^2\Big] \\
&= 0.0469 \\
Ave\ \rho^2_{4,6}(AB, AC) &= \frac{1}{6}\Big[\rho^2(A_lB_l, A_qC_q) + \rho^2(A_lB_q, A_lC_q) + \rho^2(A_qB_l, A_lC_q) \\
&\quad + \rho^2(A_lB_q, A_qC_l) + \rho^2(A_qB_l, A_qC_l) + \rho^2(A_qB_q, A_lC_l)\Big] \\
&= \frac{1}{6}\Big[0.1250^2 + (-0.1250)^2 + 0.1250^2 + 0.1250^2 \\
&\quad + (-0.1250)^2 + 0.1250^2\Big] \\
&= 0.0156 \\
Ave\ \rho^2_{4,7}(AB, AC) &= \frac{1}{4}\Big[\rho^2(A_lB_q, A_qC_q) + \rho^2(A_qB_l, A_qC_q) + \rho^2(A_qB_q, A_lC_q) \\
&\quad + \rho^2(A_qB_q, A_qC_l)\Big] \\
&= \frac{1}{4}\Big[0.2165^2 + (-0.2165)^2 + 0.2165^2 + (-0.2165)^2\Big] \\
&= 0.0469 \\
Ave\ \rho^2_{4,8}(AB, AC) &= \rho^2(A_qB_q, A_qC_q) \\
&= 0.1250^2 \\
&= 0.0156
\end{aligned}
$$

By similar calculations, the complete set of average squared correlations of order 4 for AB with AD is

$$
\begin{aligned}
Ave\ \rho^2_{4,4}(AB, AD) &= 0.0000 \\
Ave\ \rho^2_{4,5}(AB, AD) &= 0.0000
\end{aligned}
$$

$$\begin{aligned} Ave\ \rho^2_{4,6}(AB, AD) &= 0.1250 \\ Ave\ \rho^2_{4,7}(AB, AD) &= 0.0000 \\ Ave\ \rho^2_{4,8}(AB, AD) &= 0.2500 \end{aligned}$$

The complete set of average squared correlations of order 4 for AB with BC is

$$\begin{aligned} Ave\ \rho^2_{4,4}(AB, BC) &= 0.0156 \\ Ave\ \rho^2_{4,5}(AB, BC) &= 0.0469 \\ Ave\ \rho^2_{4,6}(AB, BC) &= 0.0156 \\ Ave\ \rho^2_{4,7}(AB, BC) &= 0.0469 \\ Ave\ \rho^2_{4,8}(AB, BC) &= 0.0156 \end{aligned}$$

For AB with BD, the complete set of average squared correlations of order 4 is

$$\begin{aligned} Ave\ \rho^2_{4,4}(AB, BD) &= 0.0000 \\ Ave\ \rho^2_{4,5}(AB, BD) &= 0.0000 \\ Ave\ \rho^2_{4,6}(AB, BD) &= 0.1250 \\ Ave\ \rho^2_{4,7}(AB, BD) &= 0.0000 \\ Ave\ \rho^2_{4,8}(AB, BD) &= 0.2500 \end{aligned}$$

Finally, the complete set of average squared correlations of order 4 for AB with CD is

$$\begin{aligned} Ave\ \rho^2_{4,4}(AB, CD) &= 0.0625 \\ Ave\ \rho^2_{4,5}(AB, CD) &= 0.1406 \\ Ave\ \rho^2_{4,6}(AB, CD) &= 0.0312 \\ Ave\ \rho^2_{4,7}(AB, CD) &= 0.0469 \end{aligned}$$

Projection	Number of Ave ($\rho^2_{3,3}$)		Number of Ave ($\rho^2_{3,4}$)			Number of Ave ($\rho^2_{3,5}$)		Number of Ave ($\rho^2_{3,6}$)		
	0.0000	0.0938	0.0000	0.0312	0.1667	0.0000	0.0938	0.0000	0.0312	0.5000
(1,2,3,7)	15	9	12	9	3	15	9	12	9	3

Projection	Number of Ave ($\rho^2_{4,4}$)				Number of Ave ($\rho^2_{4,5}$)		
	0.0000	0.0156	0.0625	0.2500	0.0000	0.0469	0.1406
(1,2,3,7)	2	9	3	1	3	9	3

Projection	Number of Ave ($\rho^2_{4,6}$)				Number of Ave ($\rho^2_{4,4}$)		Number of Ave ($\rho^2_{4,5}$)		
	0.0156	0.0312	0.0833	0.1250	0.0000	0.0469	0.0000	0.0156	0.2500
(1,2,3,7)	9	3	1	2	3	12	3	9	3

Table 6.1: ASCP for design in Table 4.1 treating factors as quantitative and using the standard linear and quadratic contrasts

$$Ave\ \rho^2_{4,8}(AB, CD) = 0.0000$$

The complete sets of average squared correlations of order 4 for the other pairs of two-factor interactions are calculated in a similar way.

The distinct values of $Ave\ (\rho^2_{3,3})$, $Ave\ (\rho^2_{3,4})$, $Ave\ (\rho^2_{3,5})$, $Ave\ (\rho^2_{3,6})$, $Ave\ (\rho^2_{4,4})$, $Ave\ (\rho^2_{4,5})$, $Ave\ (\rho^2_{4,6})$, $Ave\ (\rho^2_{4,7})$, and $Ave\ (\rho^2_{4,8})$ and the number of each distinct value can be indexed to form the ASCP and used to describe a property of the projection of the $OA(18, 7, 3, 2)$ onto these four columns. For this example, the ASCP is given in Table 6.1, where for ease of presentation the ASCP is divided into the complete sets of average squared correlations of 3 and the complete sets of average squared correlations of order 4.

A complete set of average squared correlations of order 3 and a complete set of average squared correlations of order 4 can be computed to describe every possible projection design of the $OA(18, 7, 3, 2)$ in Table A.1. The ASCP of order 3 for all geometrically inequivalent three-, four-, and five-factor projections are given in Table 6.2; Table 6.3 gives the average squared correlations of order 4. Representatives of

Class	Number of Ave $(\rho^2_{3,3})$ 0.0000	0.0938	Number of Ave $(\rho^2_{3,4})$ 0.0000	0.0312	0.1667	0.1979	0.5000
18.3.1	6	3	6	3	0	0	0
18.3.2	9	0	6	0	3	0	0
18.3.3	9	0	6	0	0	0	3
18.3.4	6	3	6	0	0	3	0
18.4.1	15	9	12	9	3	0	0
18.4.2	15	9	12	9	0	0	3
18.4.3	12	12	12	12	0	0	0
18.4.4	12	12	12	9	0	3	0
18.4.5	15	9	12	3	3	6	0
18.5.1	26	24	20	24	3	0	3
18.5.2	23	27	20	21	3	6	0
18.5.3	23	27	20	24	0	3	3
18.5.4	20	30	20	30	0	0	0
18.5.5	26	24	20	12	6	12	0

Class	Number of Ave $(\rho^2_{3,5})$ 0.0000	0.0938	Number of Ave $(\rho^2_{3,6})$ 0.0000	0.0312	0.5000
18.3.1	6	3	6	3	0
18.3.2	9	0	6	0	3
18.3.3	9	0	6	0	3
18.3.4	6	3	6	3	0
18.4.1	15	9	12	9	3
18.4.2	15	9	12	9	3
18.4.3	12	12	12	12	0
18.4.4	12	12	12	12	0
18.4.5	15	9	12	9	3
18.5.1	26	24	20	24	6
18.5.2	23	27	20	27	3
18.5.3	23	27	20	27	3
18.5.4	20	30	20	30	0
18.5.5	26	24	20	24	6

Table 6.2: Average squared correlations of order 3 values for the geometrically inequivalent three-factor , four-factor, and five-factor projection designs from $OA(18, 7, 3, 2)$

each geometrically inequivalent design class are taken from Evangelaras, Kolaiti and Koukouvinos (2005a). The ASCPs presented are used to rank order the projection designs in Chapter 9.

Class	Number of $Ave\ (\rho^2_{4,4})$ 0.0000	0.0156	0.0625	0.1406	0.2500	Number of $Ave\ (\rho^2_{4,5})$ 0.0000	0.0469	0.1406
18.3.1	0	3	0	0	0	0	3	0
18.3.2	2	0	0	0	1	3	0	0
18.3.3	0	0	0	0	3	3	0	0
18.3.4	0	1	0	2	0	0	3	0
18.4.1	2	9	3	0	1	3	9	3
18.4.2	3	9	0	0	3	6	9	0
18.4.3	0	12	0	3	0	0	15	0
18.4.4	0	13	0	2	0	0	15	0
18.4.5	5	5	0	4	1	6	9	0
18.5.1	8	24	6	3	4	12	27	6
18.5.2	5	29	3	7	1	6	36	3
18.5.3	6	31	0	5	3	9	36	0
18.5.4	0	30	0	15	0	0	45	0
18.5.5	16	16	0	11	2	18	27	0

Class	Number of $Ave\ (\rho^2_{4,6})$ 0.0000	0.0156	0.0312	0.0781	0.0833	0.0990	0.1250	0.1406	0.2500
18.3.1	0	3	0	0	0	0	0	0	0
18.3.2	0	0	0	0	1	0	2	0	0
18.3.3	0	0	0	0	0	0	0	0	3
18.3.4	0	0	0	2	0	1	0	0	0
18.4.1	0	9	3	0	1	0	2	0	0
18.4.2	3	9	0	0	0	0	0	0	3
18.4.3	0	12	0	0	0	0	0	3	0
18.4.4	0	9	0	5	0	1	0	0	0
18.4.5	3	3	0	4	1	2	2	0	0
18.5.1	6	24	6	0	1	0	2	3	3
18.5.2	3	21	3	10	1	2	2	3	0
18.5.3	6	24	0	8	0	1	0	3	3
18.5.4	0	30	0	0	0	0	0	15	0
18.5.5	12	12	0	8	2	4	4	3	0

Class	Number of $Ave\ (\rho^2_{4,7})$ 0.0000	0.0469	Number of $Ave\ (\rho^2_{4,8})$ 0.0000	0.0156	0.1406	0.2500
18.3.1	0	3	0	3	0	0
18.3.2	3	0	0	0	0	3
18.3.3	3	0	0	0	0	3
18.3.4	0	3	0	3	0	0
18.4.1	3	12	3	9	0	3
18.4.2	6	9	3	9	0	3
18.4.3	0	15	0	12	3	0
18.4.4	0	15	0	12	3	0
18.4.5	6	9	3	9	0	3
18.5.1	12	33	12	24	3	6
18.5.2	6	39	6	27	9	3
18.5.3	9	36	6	27	9	3
18.5.4	0	45	0	30	15	0
18.5.5	18	27	12	24	3	6

Table 6.3: Average squared correlations of order 4 values for the geometrically inequivalent three-factor , four-factor, and five-factor projection designs from $OA(18,7,3,2)$

6.3 Non-Independence from Choice of Orthogonal Contrast Set

In Section 4.3, Theorem 4.3.1 shows that, for qualitative factors, the averaged squared correlation of order 3 and average squared correlation of order 4 are independent of the choice of the complete set of orthogonal contrasts. This theorem results from the fact that the *sum* $\rho^2(\mathbf{C}_A, \mathbf{C}_B) = trace(\mathbf{C}_B{}'\mathbf{C}_A\mathbf{C}_A{}'\mathbf{C}_B)$ (Lemma 4.3.2) and that $trace(\mathbf{C}_B{}'\mathbf{C}_A\mathbf{C}_A{}'\mathbf{C}_B)$ is constant for orthogonal rotations of $\mathbf{A}$ and $\mathbf{B}$. In the case of quantitative factors, the average squared correlations are not single values averaged over all contrast pairs for a main effect and two-factor interaction or pair of two-factor interactions. As a result, neither the complete set of average squared correlations of order 3 nor the complete set of average squared correlations of order 4 is equal to $trace(\mathbf{C}_B{}'\mathbf{C}_A\mathbf{C}_A{}'\mathbf{C}_B)$. Thus, Theorem 4.3.1 does not apply to designs with quantitative factors.

Though the average squared correlations of order 3 and order 4, and as a result the ASCPs, are dependent upon the choice of the complete set of orthogonal contrasts, only the standard linear and quadratic contrasts ((2.1) and (2.2), respectively) will be examined in Chapter 9 for ranking. Because the linear and quadratic contrasts possess a physical interpretation for quantitative factors and are commonly used in applications, these contrasts are used in this work.

CHAPTER 7

COMPETING METHODS FOR RANKING AND NON-EQUIVALENCE OF DESIGNS WITH QUANTITATIVE FACTORS

The distinction between combinatorially isomorphic and geometrically isomorphic designs has not always been prominent. While numerous methods (Ma and Fang 2001, Xu and Wu 2001, Evangelaras et al. 2005b) have been proposed for ranking and determining non-equivalence of two designs with respect to qualitative factors, fewer methods have been considered for examination of designs with quantitative factors. This chapter provides a brief review of some of the methods for ranking determining geometric non-equivalence.

Cheng and Wu (2001) examined the special case of evaluating projection designs, though the projection-efficiency criteria they proposed can be used to evaluate general designs as well. The *projection-efficiency criteria* provide two components for evaluating and ranking designs (Cheng and Wu 2001). The broader criterion is called *eligible projection.* A projection design is called *eligible* if it is a second order design; a design is a second order design if all $(n+1)(n+2)/2$ parameters of the second-order model $y = \mu + \sum_{i=1}^{n} \beta_i x_i + \sum_{i=1}^{n} \beta_{ii} x_i^2 + \sum_{1=i<j}^{n} \beta_{ij} x_i x_j + \epsilon$ are estimable in the design. Eligible designs are preferred to ineligible designs, and a design with more eligible

projections is ranked as better than a design with fewer eligible projections. Also, eligible projections onto smaller numbers of factors are more important than eligible projections onto larger numbers of factors based on the factor sparsity principle. Within the set of eligible projection designs, D- and G- efficiencies can be used to rank design classes, with designs having higher estimation efficiency ranked as better. A design is *D-optimal* if the design minimizes volume of the confidence ellipsoid for all possible contrasts (i.e. minimizes the generalized variance of the parameter estimates based on a pre-selected model); a design that minimizes the maximum prediction variance is *G-optimal* (Kiefer 1974, Silvey 1980). Cheng and Wu (2001) utilized a complete computer search to identify the classes of both combinatorially isomorphic and model isomorphic designs (see Section 2.5 for definitions) from several regular and non-regular designs. Every identified model non-isomorphic class of designs is evaluated using the projection-efficiency criteria.

Tsai, Gilmour and Mead (2000) proposed an evaluation criterion, denoted by $Q(\Gamma^{(p)})$, that is an approximation to the average A_s-efficiency criterion; averaging A-efficiency over multiple models was first suggested by Wu (1993). A design is *A-optimal* if the design minimizes the average variance of the parameter estimates based on a pre-selected model (Kiefer 1974). If a design minimizes the average variance of the parameter estimates over a model including only a subset of the parameters of the maximal model, then the design is *A_s-optimal*.

The goal of the criterion $Q(\Gamma^{(p)})$ is to identify designs that have many efficient projections onto multiple lower-dimensional designs. In order to identify designs with projections that are efficient for estimation over a wide range of models and not just for main effects models, the proposed $Q(\Gamma^{(p)})$ criterion averages the A_s-efficiency over all eligible models whose number of parameters is not greater than the number of runs and which include the marginal terms of all terms. (For example, if a model includes x_1x_2, then the model also includes x_1 and x_2.) The overall design criterion $Q(\Gamma^{(p)})$ is the average of the sums of the approximate variances of the parameter estimates in each model (i.e. the A_s efficiencies) over all estimable models from the design. Smaller values of $Q(\Gamma^{(p)})$ indicate better effect estimation over a wide range of possible models. To compare designs with respect to projections, $Q(\Gamma^{(p)})$ can be averaged over all possible k-factor projections for a given number of factors, k. Then smaller values of $Ave\ Q(\Gamma^{(p)})$ indicate a larger number of k-factor projections which are more efficient over a wide range of possible models.

Both the projection-efficiency criteria of Cheng and Wu (2001) and the $Q(\Gamma^{(p)})$ criterion of Tsai et al. (2000) were developed for rank ordering designs, with non-equivalence determined when two designs are ranked individually. In contrast, Cheng and Ye (2004) developed the indicator function to determine equivalence of designs and then provide secondary criteria, the α and β wordlength patterns, for rank ordering.

Viewing the design as a set of points in $\mathbb{R}^p$, the geometric structure of the design is unique and can be uniquely represented in polynomial form by its indicator function. The use of polynomial systems to characterize designs was first proposed by Pistone and Wynn (1996). Using polynomial systems to describe designs, algebraic geometry methods were used to study the properties of the designs. Fontana, Pistone and Rogantin (2000) developed the use of indicator functions for study of the design properties of two-level unreplicated fractional factorial designs; Ye (2003) extended the definition to include replicated designs. Use of the indicator function to determine equivalence of designs was extended to general factorial designs by Cheng and Ye (2004).

As described by Cheng and Ye (2004), for a design D with p factors, the *indicator function* $F_D(\mathbf{x})$ is defined as the number of times the design point $\mathbf{x}$ appears in D. (A design point $\mathbf{x}$ is a specific combination of factor levels to be run.) Alternatively, the indicator function $F_D(\mathbf{x})$ can be expressed as a polynomial expansion of a set of orthogonal contrasts that define each factor. For a factor i with k_i levels, define a complete set of orthogonal contrasts $(C_0^i(x), C_1^i(x), \ldots, C_{k_i-1}^i(x))$ such that, for all $u, v = 0, 1, \ldots, k_i - 1$,

$$\sum_{x \in \{0,1,\ldots,s_i-1\}} C_u^i(x) C_v^i(x) = \begin{cases} 0, & \text{if } u \neq v, \\ k_i, & \text{if } u = v, \end{cases} \tag{7.1}$$

where the superscript i represents which factor the contrasts measure. From the complete set of orthogonal contrasts, define a *statistical orthonormal contrast basis* (SOCB) for the design space of D to be

$$C_{\mathbf{t}}(\mathbf{x}) = \prod_{i=1}^{p} C_{t_i}^i(x_i), \tag{7.2}$$

where $C_0^i(x) = 1$ for all i (i.e. for all factors). When $C_j^i(x)$ is a polynomial of degree j for $j = 0, 1, \ldots, k_i - 1$ and $i = 1, 2, \ldots, p$ then the SOCB is called an *orthogonal polynomial basis* (OPB) (see Draper and Smith (1998), Chapter 22).

The polynomial expansion of the indicator function, $F_D(\mathbf{x})$, is

$$F_D(\mathbf{x}) = \sum_{t \in \mathcal{T}} b_{\mathbf{t}} C_{\mathbf{t}}(\mathbf{x}) \tag{7.3}$$

where $\mathcal{T} = \mathcal{G}_1 \times \mathcal{G}_2 \times \ldots \times \mathcal{G}_p$ for $\mathcal{G}_i = (0, 1, \ldots, k_i - 1) \subset \mathbb{R}$ and the coefficients, $b_{\mathbf{t}}$, of $F_D(\mathbf{x})$ are defined as

$$b_{\mathbf{t}} = \frac{1}{N} \sum_{\mathbf{x} \in D} C_{\mathbf{t}}(\mathbf{x}) \tag{7.4}$$

(Cheng and Ye 2004). For an SOCB, $b_{\mathbf{0}} = n/N$, where n is the number of design points (runs) in D and N is the number of design points (runs) in the full factorial design containing D. In their work, Cheng and Ye (2004) placed no restrictions on the number of design points in D and allowed for replicate runs.

The unique coefficients of the indicator function, $F_D(\mathbf{x})$, provide information about the aliasing of factors. Theorem 7.1 describes the relationship between the indicator function coefficient and the correlation between contrasts.

Theorem 7.1 *(Cheng and Ye (2004), Corollary 2.2) Let* $\{C_{\mathbf{t}}(\mathbf{x}), \mathbf{t} \in \mathcal{T}\}$ *be an SOCB. For disjoint* $C_{\mathbf{u}}$ *and* $C_{\mathbf{v}}$*,*

$$b_{\mathbf{u}+\mathbf{v}} = \frac{1}{N} \sum_{\mathbf{x} \in D} C_{\mathbf{u}}(\mathbf{x}) C_{\mathbf{v}}(\mathbf{x}). \tag{7.5}$$

Furthermore, the correlation of $C_{\mathbf{u}}$ *and* $C_{\mathbf{v}}$ *in* D *is* $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$*.*

Cheng and Ye (2004) claim that this corollary follows directly from (7.2) and (7.4). However, this interpretation of ratios of the indicator function coefficients as correlations is not always correct. A discussion of this issue and an example are presented in Section 7.2. When the interpretation of the ratio of indicator function coefficients as correlations is correct, there exists a direct link between the indicator function of Cheng and Ye (2004) and the average squared correlation criterion developed in this work; the relationship between these two criteria will be described in detail in Chapter 8.

The indicator function was developed specifically for the purpose of determining whether two designs are geometrically equivalent. Cheng and Ye (2004) provide the necessary and sufficient condition for geometric isomorphism using the indicator function.

Cheng and Ye (2004) construct two different wordlength patterns from the indicator function coefficients for the purpose of rank ordering designs. The α wordlength pattern is a redefinition of the GWP of Xu and Wu (2001) defined in Section 5.1. As with the GWP, the α wordlength pattern is appropriate for use with qualitative factors. The β wordlength pattern is based on the effect hierarchy (6.1), and is used with quantitative factors. The α and β wordlength patterns will be described in more detail in Section 8.2. Like the indicator function itself, there is a link between the α and β wordlength patterns and the average squared correlation criterion that will be discussed in Section 8.2.

7.2 Counterexample to the Interpretation of Indicator Function Coefficients as Correlations

Theorem 7.1 (Cheng and Ye (2004), Corollary 2.2) states that, for disjoint $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$, the correlation of contrasts $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$ is equal to $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$:

$$\rho(C_{\mathbf{u}}, C_{\mathbf{v}}) = b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}. \tag{7.6}$$

The contrasts $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$ are disjoint if the elements in the positions of $\mathbf{v}$ corresponding to the positions of non-zero elements of $\mathbf{u}$ are equal to zero. For example, if $\mathbf{u} = 0111$, then $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$ are disjoint if $\mathbf{v} = i000$ for some $i = 1, 2, \ldots, k-1$, where k is the number of levels of the first factor in the design. In particular, Cheng and Ye (2004) interpret $b_{\mathbf{t}}/b_{\mathbf{0}}$ as the correlation between the contrast $C_{\mathbf{t}}$ and the overall mean, represented by $C_{\mathbf{0}}$. In this section, a counterexample will be provided in which the $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$ is not equal to the correlation between contrasts $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$.

Since the indicator function necessarily requires contrasts to be scaled so that they satisfy the constraint (7.1), the linear and quadratic contrasts, (2.1) and (2.2), must be re-scaled. The linear contrasts (7.7) is scaled by a factor of $\sqrt{3/2}$ and the quadratic contrast (7.8) is scaled by a factor of $1\sqrt{2}$; the scaled linear and quadratic contrasts are given by Cheng and Ye (2004)

$$l_s = \begin{bmatrix} -\sqrt{3/2} \\ 0 \\ \sqrt{3/2} \end{bmatrix} \tag{7.7}$$

and

$$q_s = \begin{bmatrix} 1/\sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}. \tag{7.8}$$

In addition to these two contrasts, a constant vector, $C_0(x) = 1$ is required for the indicator function calculations.

Consider the design given in Table 7.1. Table 7.2 gives the contrast correlation matrix based on contrasts (7.7) and (7.8).

A	B	C	D
1	1	1	1
2	2	2	2
0	0	0	0
1	1	2	0
2	2	0	1
0	0	1	2
1	2	1	0
2	0	2	1
0	1	0	2
1	0	0	1
2	1	1	2
0	2	2	0
1	2	0	2
2	0	1	0
0	1	2	1
1	0	2	2
2	1	0	0
0	2	1	1

Table 7.1: Design used to illustrate relationship between indicator function coefficients and average squared correlations

	C_{1000} A_l	C_{2000} A_q	C_{0100} B_l	C_{0200} B_q	C_{0010} C_l	C_{0020} C_q	C_{0001} D_l	C_{0002} D_q
A_l	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
A_q	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
B_l	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
B_q	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000
C_l	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
C_q	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000
D_l	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000
D_q	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
A_lB_l	0.0000	0.0000	0.0000	0.0000	-0.3062	0.1768	0.3062	0.1768
A_lB_q	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062	0.1768	-0.3062
A_qB_l	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062	0.1768	-0.3062
A_qB_q	0.0000	0.0000	0.0000	0.0000	0.3062	-0.1768	-0.3062	-0.1768
A_lC_l	0.0000	0.0000	-0.3062	0.1768	0.0000	0.0000	0.3062	0.1768
A_lC_q	0.0000	0.0000	0.1768	0.3062	0.0000	0.0000	0.1768	-0.3062
A_qC_l	0.0000	0.0000	0.1768	0.3062	0.0000	0.0000	0.1768	-0.3062
A_qC_q	0.0000	0.0000	0.3062	-0.1768	0.0000	0.0000	-0.3062	-0.1768
A_lD_l	0.0000	0.0000	0.3062	0.1768	0.3062	0.1768	0.0000	0.0000
A_lD_q	0.0000	0.0000	0.1768	-0.3062	0.1768	-0.3062	0.0000	0.0000
A_qD_l	0.0000	0.0000	0.1768	-0.3062	0.1768	-0.3062	0.0000	0.0000
A_qD_q	0.0000	0.0000	-0.3062	-0.1768	-0.3062	-0.1768	0.0000	0.0000
B_lC_l	-0.3062	0.1768	0.0000	0.0000	0.0000	0.0000	-0.3062	0.1768
B_lC_q	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062
B_qC_l	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062
B_qC_q	0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000	0.3062	-0.1768
B_lD_l	0.3062	0.1768	0.0000	0.0000	-0.3062	0.1768	0.0000	0.0000
B_lD_q	0.1768	-0.3062	0.0000	0.0000	0.1768	0.3062	0.0000	0.0000
B_qD_l	0.1768	-0.3062	0.0000	0.0000	0.1768	0.3062	0.0000	0.0000
B_qD_q	-0.3062	-0.1768	0.0000	0.0000	0.3062	-0.1768	0.0000	0.0000
C_lD_l	0.3062	0.1768	-0.3062	0.1768	0.0000	0.0000	0.0000	0.0000
C_lD_q	0.1768	-0.3062	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000
C_qD_l	0.1768	-0.3062	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000
C_qD_q	-0.3062	-0.1768	0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000

Continued

Table 7.2: Complete correlation matrix based on contrasts (7.7) and (7.8) for example design in Table 7.1

Table 7.2 Continued

	C_{1100} A_lB_l	C_{1200} A_lB_q	C_{2100} A_qB_l	C_{2200} A_qB_q	C_{1010} A_lC_l	C_{1020} A_lC_q	C_{2010} A_qC_l	C_{2020} A_qC_q	C_{1001} A_lD_l	C_{1002} A_lD_q	C_{2001} A_qD_l	C_{2002} A_qD_q
A_l	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
A_q	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
B_l	0.0000	0.0000	0.0000	0.0000	-0.3062	0.1768	0.1768	0.3062	0.3062	0.1768	0.1768	-0.3062
B_q	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062	0.3062	-0.1768	0.1768	-0.3062	-0.3062	-0.1768
C_l	-0.3062	0.1768	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000	0.3062	0.1768	0.1768	-0.3062
C_q	0.1768	0.3062	0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000	0.1768	-0.3062	-0.3062	-0.1768
D_l	0.3062	0.1768	0.1768	-0.3062	0.3062	0.1768	0.1768	-0.3062	0.0000	0.0000	0.0000	0.0000
D_q	0.1768	-0.3062	-0.3062	-0.1768	0.1768	-0.3062	-0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000
A_lB_l	1.0000	0.0000	0.0000	0.0000	0.1250	0.2165	-0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250
A_lB_q	0.0000	1.0000	0.0000	0.0000	0.2165	-0.1250	0.1250	0.2165	-0.2165	-0.1250	0.1250	-0.2165
A_qB_l	0.0000	0.0000	1.0000	0.0000	-0.2165	0.1250	-0.1250	-0.2165	0.2165	0.1250	-0.1250	0.2165
A_qB_q	0.0000	0.0000	0.0000	1.0000	0.1250	0.2165	-0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250
A_lC_l	0.1250	0.2165	-0.2165	0.1250	1.0000	0.0000	0.0000	0.0000	0.1250	-0.2165	0.2165	0.1250
A_lC_q	0.2165	-0.1250	0.1250	0.2165	0.0000	1.0000	0.0000	0.0000	-0.2165	-0.1250	0.1250	-0.2165
A_qC_l	-0.2165	0.1250	-0.1250	-0.2165	0.0000	0.0000	1.0000	0.0000	0.2165	0.1250	-0.1250	0.2165
A_qC_q	0.1250	0.2165	-0.2165	0.1250	0.0000	0.0000	0.0000	1.0000	0.1250	-0.2165	0.2165	0.1250
A_lD_l	0.1250	-0.2165	0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250	1.0000	0.0000	0.0000	0.0000
A_lD_q	-0.2165	-0.1250	0.1250	-0.2165	-0.2165	-0.1250	0.1250	-0.2165	0.0000	1.0000	0.0000	0.0000
A_qD_l	0.2165	0.1250	-0.1250	0.2165	0.2165	0.1250	-0.1250	0.2165	0.0000	0.0000	1.0000	0.0000
A_qD_q	0.1250	-0.2165	0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250	0.0000	0.0000	0.0000	1.0000
B_lC_l	0.1250	-0.2165	0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250	0.3750	0.2165	0.2165	0.6250
B_lC_q	0.2165	0.1250	-0.1250	0.2165	-0.2165	-0.1250	0.1250	-0.2165	-0.2165	-0.1250	-0.1250	0.2165
B_qC_l	-0.2165	-0.1250	0.1250	-0.2165	0.2165	0.1250	-0.1250	0.2165	-0.2165	-0.1250	-0.1250	0.2165
B_qC_q	0.1250	-0.2165	0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250	0.6250	-0.2165	-0.2165	0.3750
B_lD_l	0.1250	0.2165	-0.2165	0.1250	0.3750	-0.2165	0.2165	-0.1250	0.1250	0.2165	-0.2165	0.1250
B_lD_q	-0.2165	0.1250	-0.1250	-0.2165	0.2165	-0.1250	0.6250	0.2165	0.2165	-0.1250	0.1250	0.2165
B_qD_l	0.2165	-0.1250	0.1250	0.2165	-0.2165	0.6250	-0.1250	-0.2165	-0.2165	0.1250	-0.1250	-0.2165
B_qD_q	0.1250	0.2165	-0.2165	0.1250	-0.1250	-0.2165	0.2165	0.3750	0.1250	0.2165	-0.2165	0.1250
C_lD_l	0.3750	-0.2165	0.2165	-0.1250	0.1250	0.2165	-0.2165	0.1250	0.1250	0.2165	-0.2165	0.1250
C_lD_q	0.2165	-0.1250	0.6250	0.2165	-0.2165	0.1250	-0.1250	-0.2165	0.2165	-0.1250	0.1250	0.2165
C_qD_l	-0.2165	0.6250	-0.1250	-0.2165	0.2165	-0.1250	0.1250	0.2165	-0.2165	0.1250	-0.1250	-0.2165
C_qD_q	-0.1250	-0.2165	0.2165	0.3750	0.1250	0.2165	-0.2165	0.1250	0.1250	0.2165	-0.2165	0.1250

Continued

Table 7.2 Continued

	C_{0110} B_lC_l	C_{0120} B_lC_q	C_{0210} B_qC_l	C_{0220} B_qC_q	C_{0101} B_lD_l	C_{0102} B_lD_q	C_{0201} B_qD_l	C_{0202} B_qD_q	C_{0011} C_lD_l	C_{0012} C_lD_q	C_{0021} C_qD_l	C_{0022} C_qD_q
A_l	-0.3062	0.1768	0.1768	0.3062	0.3062	0.1768	0.1768	-0.3062	0.3062	0.1768	0.1768	-0.3062
A_q	0.1768	0.3062	0.3062	-0.1768	0.1768	-0.3062	-0.3062	-0.1768	0.1768	-0.3062	-0.3062	-0.1768
B_l	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.3062	0.1768	0.1768	0.3062
B_q	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062	0.3062	-0.1768
C_l	0.0000	0.0000	0.0000	0.0000	-0.3062	0.1768	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000
C_q	0.0000	0.0000	0.0000	0.0000	0.1768	0.3062	0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000
D_l	-0.3062	0.1768	0.1768	0.3062	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
D_q	0.1768	0.3062	0.3062	-0.1768	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
A_lB_l	0.1250	0.2165	-0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250	0.3750	0.2165	-0.2165	-0.1250
A_lB_q	-0.2165	0.1250	-0.1250	-0.2165	0.2165	0.1250	-0.1250	0.2165	-0.2165	-0.1250	0.6250	-0.2165
A_qB_l	0.2165	-0.1250	0.1250	0.2165	-0.2165	-0.1250	0.1250	-0.2165	0.2165	0.6250	-0.1250	0.2165
A_qB_q	0.1250	0.2165	-0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250	-0.1250	0.2165	-0.2165	0.3750
A_lC_l	0.1250	-0.2165	0.2165	0.1250	0.3750	0.2165	-0.2165	-0.1250	0.1250	-0.2165	0.2165	0.1250
A_lC_q	-0.2165	-0.1250	0.1250	-0.2165	-0.2165	-0.1250	0.6250	-0.2165	0.2165	0.1250	-0.1250	0.2165
A_qC_l	0.2165	0.1250	-0.1250	0.2165	0.2165	0.6250	-0.1250	0.2165	-0.2165	-0.1250	0.1250	-0.2165
A_qC_q	0.1250	-0.2165	0.2165	0.1250	-0.1250	0.2165	-0.2165	0.3750	0.1250	-0.2165	0.2165	0.1250
A_lD_l	0.3750	-0.2165	-0.2165	0.6250	0.1250	0.2165	-0.2165	0.1250	0.1250	0.2165	-0.2165	0.1250
A_lD_q	0.2165	-0.1250	-0.1250	-0.2165	0.2165	-0.1250	0.1250	0.2165	0.2165	-0.1250	0.1250	0.2165
A_qD_l	0.2165	-0.1250	-0.1250	-0.2165	-0.2165	0.1250	-0.1250	-0.2165	-0.2165	0.1250	-0.1250	-0.2165
A_qD_q	0.6250	0.2165	0.2165	0.3750	0.1250	0.2165	-0.2165	0.1250	0.1250	0.2165	-0.2165	0.1250
B_lC_l	1.0000	0.0000	0.0000	0.0000	0.1250	0.2165	-0.2165	0.1250	0.1250	0.2165	-0.2165	0.1250
B_lC_q	0.0000	1.0000	0.0000	0.0000	0.2165	-0.1250	0.1250	0.2165	-0.2165	0.1250	-0.1250	-0.2165
B_qC_l	0.0000	0.0000	1.0000	0.0000	-0.2165	0.1250	-0.1250	-0.2165	0.2165	-0.1250	0.1250	0.2165
B_qC_q	0.0000	0.0000	0.0000	1.0000	0.1250	0.2165	-0.2165	0.1250	0.1250	0.2165	-0.2165	0.1250
B_lD_l	0.1250	0.2165	-0.2165	0.1250	1.0000	0.0000	0.0000	0.0000	0.1250	-0.2165	0.2165	0.1250
B_lD_q	0.2165	-0.1250	0.1250	0.2165	0.0000	1.0000	0.0000	0.0000	-0.2165	-0.1250	0.1250	-0.2165
B_qD_l	-0.2165	0.1250	-0.1250	-0.2165	0.0000	0.0000	1.0000	0.0000	0.2165	0.1250	-0.1250	0.2165
B_qD_q	0.1250	0.2165	-0.2165	0.1250	0.0000	0.0000	0.0000	1.0000	0.1250	-0.2165	0.2165	0.1250
C_lD_l	0.1250	-0.2165	0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250	1.0000	0.0000	0.0000	0.0000
C_lD_q	0.2165	0.1250	-0.1250	0.2165	-0.2165	-0.1250	0.1250	-0.2165	0.0000	1.0000	0.0000	0.0000
C_qD_l	-0.2165	-0.1250	0.1250	-0.2165	0.2165	0.1250	-0.1250	0.2165	0.0000	0.0000	1.0000	0.0000
C_qD_q	0.1250	-0.2165	0.2165	0.1250	0.1250	-0.2165	0.2165	0.1250	0.0000	0.0000	0.0000	1.0000

b_{0000}	=	0.2222	b_{1000}	=	0.0000	b_{2000}	=	0.0000
b_{0001}	=	0.0000	b_{1001}	=	0.0000	b_{2001}	=	0.0000
b_{0002}	=	0.0000	b_{1002}	=	0.0000	b_{2002}	=	0.0000
b_{0010}	=	0.0000	b_{1010}	=	0.0000	b_{2010}	=	0.0000
b_{0011}	=	0.0000	b_{1011}	=	0.0680	b_{2011}	=	0.0393
b_{0012}	=	0.0000	b_{1012}	=	0.0393	b_{2012}	=	-0.0680
b_{0020}	=	0.0000	b_{1020}	=	0.0000	b_{2020}	=	0.0000
b_{0021}	=	0.0000	b_{1021}	=	0.0393	b_{2021}	=	-0.0680
b_{0022}	=	0.0000	b_{1022}	=	-0.0680	b_{2022}	=	-0.0393
b_{0100}	=	0.0000	b_{1100}	=	0.0000	b_{2100}	=	0.0000
b_{0101}	=	0.0000	b_{1101}	=	0.0680	b_{2101}	=	0.0393
b_{0102}	=	0.0000	b_{1102}	=	0.0393	b_{2102}	=	-0.0680
b_{0110}	=	0.0000	b_{1110}	=	-0.0680	b_{2110}	=	0.0393
b_{0111}	=	-0.0680	b_{1111}	=	0.0833	b_{2111}	=	0.0481
b_{0112}	=	0.0393	b_{1112}	=	0.0481	b_{2112}	=	0.1389
b_{0120}	=	0.0000	b_{1120}	=	0.0393	b_{2120}	=	0.0680
b_{0121}	=	0.0393	b_{1121}	=	-0.0481	b_{2121}	=	-0.0278
b_{0122}	=	0.0680	b_{1122}	=	-0.0278	b_{2122}	=	0.0481
b_{0200}	=	0.0000	b_{1200}	=	0.0000	b_{2200}	=	0.0000
b_{0201}	=	0.0000	b_{1201}	=	0.0393	b_{2201}	=	-0.0680
b_{0202}	=	0.0000	b_{1202}	=	-0.0680	b_{2202}	=	-0.0393
b_{0210}	=	0.0000	b_{1210}	=	0.0393	b_{2210}	=	0.0680
b_{0211}	=	0.0393	b_{1211}	=	-0.0481	b_{2211}	=	-0.0278
b_{0212}	=	0.0680	b_{1212}	=	-0.0278	b_{2212}	=	0.0481
b_{0220}	=	0.0000	b_{1220}	=	0.0680	b_{2220}	=	-0.0393
b_{0221}	=	0.0680	b_{1221}	=	0.1389	b_{2221}	=	-0.0481
b_{0222}	=	-0.0393	b_{1222}	=	-0.0481	b_{2222}	=	0.0833

Table 7.3: Indicator function coefficients for the design in Table 7.1

The indicator function coefficients for this design are given in Table 7.3. Using the definition of the indicator function coefficients (7.4) from Cheng and Ye (2004), the coefficients are calculated from the complete contrast matrix (including all i-factor interactions, $i = 2, 3, 4$).

According to (7.6), $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$ is equal to the correlation between contrasts $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$. The values of $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$ are given in Table 7.4. From the correlation matrix (Table 7.2), it can be seen that the $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$ ratios given in Table 7.5 are equal to the correlations between $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$

The correlation of C_{0000} and C_{1111} should also equal $b_{1111}/b_{0000} = 0.3750$ since $\mathbf{u} = 0000$ and $\mathbf{v} = 1111$ are disjoint and $\mathbf{u} + \mathbf{v} = 1111$. However, given that, by

b_{0000}/b_{0000}	=	1.0000	b_{1000}/b_{0000}	=	0.0000	b_{2000}/b_{0000}	=	0.0000
b_{0001}/b_{0000}	=	0.0000	b_{1001}/b_{0000}	=	0.0000	b_{2001}/b_{0000}	=	0.0000
b_{0002}/b_{0000}	=	0.0000	b_{1002}/b_{0000}	=	0.0000	b_{2002}/b_{0000}	=	0.0000
b_{0010}/b_{0000}	=	0.0000	b_{1010}/b_{0000}	=	0.0000	b_{2010}/b_{0000}	=	0.0000
b_{0011}/b_{0000}	=	0.0000	b_{1011}/b_{0000}	=	0.3062	b_{2011}/b_{0000}	=	0.1768
b_{0012}/b_{0000}	=	0.0000	b_{1012}/b_{0000}	=	0.1768	b_{2012}/b_{0000}	=	-0.3062
b_{0020}/b_{0000}	=	0.0000	b_{1020}/b_{0000}	=	0.0000	b_{2020}/b_{0000}	=	0.0000
b_{0021}/b_{0000}	=	0.0000	b_{1021}/b_{0000}	=	0.1768	b_{2021}/b_{0000}	=	-0.3062
b_{0022}/b_{0000}	=	0.0000	b_{1022}/b_{0000}	=	-0.3062	b_{2022}/b_{0000}	=	-0.1768
b_{0100}/b_{0000}	=	0.0000	b_{1100}/b_{0000}	=	0.0000	b_{2100}/b_{0000}	=	0.0000
b_{0101}/b_{0000}	=	0.0000	b_{1101}/b_{0000}	=	0.3062	b_{2101}/b_{0000}	=	0.1768
b_{0102}/b_{0000}	=	0.0000	b_{1102}/b_{0000}	=	0.1768	b_{2102}/b_{0000}	=	-0.3062
b_{0110}/b_{0000}	=	0.0000	b_{1110}/b_{0000}	=	-0.3062	b_{2110}/b_{0000}	=	0.1768
b_{0111}/b_{0000}	=	-0.3062	b_{1111}/b_{0000}	=	0.3750	b_{2111}/b_{0000}	=	0.2165
b_{0112}/b_{0000}	=	0.1768	b_{1112}/b_{0000}	=	0.2165	b_{2112}/b_{0000}	=	0.6250
b_{0120}/b_{0000}	=	0.0000	b_{1120}/b_{0000}	=	0.1768	b_{2120}/b_{0000}	=	0.3062
b_{0121}/b_{0000}	=	0.1768	b_{1121}/b_{0000}	=	-0.2165	b_{2121}/b_{0000}	=	-0.1250
b_{0122}/b_{0000}	=	0.3062	b_{1122}/b_{0000}	=	-0.1250	b_{2122}/b_{0000}	=	0.2165
b_{0200}/b_{0000}	=	0.0000	b_{1200}/b_{0000}	=	0.0000	b_{2200}/b_{0000}	=	0.0000
b_{0201}/b_{0000}	=	0.0000	b_{1201}/b_{0000}	=	0.1768	b_{2201}/b_{0000}	=	-0.3062
b_{0202}/b_{0000}	=	0.0000	b_{1202}/b_{0000}	=	-0.3062	b_{2202}/b_{0000}	=	-0.1768
b_{0210}/b_{0000}	=	0.0000	b_{1210}/b_{0000}	=	0.1768	b_{2210}/b_{0000}	=	0.3062
b_{0211}/b_{0000}	=	0.1768	b_{1211}/b_{0000}	=	-0.2165	b_{2211}/b_{0000}	=	-0.1250
b_{0212}/b_{0000}	=	0.3062	b_{1212}/b_{0000}	=	-0.1250	b_{2212}/b_{0000}	=	0.2165
b_{0220}/b_{0000}	=	0.0000	b_{1220}/b_{0000}	=	0.3062	b_{2220}/b_{0000}	=	-0.1768
b_{0221}/b_{0000}	=	0.3062	b_{1221}/b_{0000}	=	0.6250	b_{2221}/b_{0000}	=	-0.2165
b_{0222}/b_{0000}	=	-0.1768	b_{1222}/b_{0000}	=	-0.2165	b_{2222}/b_{0000}	=	0.3750

Table 7.4: Values of $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$ using the indicator coefficients $b_{\mathbf{t}}$ from Table 7.3 for the design in Table 7.1

definition,

$$C_{0000} = \left[\begin{array}{cccccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right]'$$

and, from (7.7),

$$C_{1111} = 2.25 \times \left[\begin{array}{cccccccccccccccccc} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]',$$

using equation (2.4), the correlation between C_{0000} and C_{1111} is

$$\begin{aligned} \rho(C_{0000}, C_{1111}) &= \tfrac{6.75}{\sqrt{273.375}} \\ &= 0.4082 \quad \neq \quad 0.3750 \end{aligned}$$

By a similar argument, the correlation of C_{1000} and C_{0111} should equal $b_{1111}/b_{0000} = 0.3750$. Again, from (7.7)

$$C_{1000} = \sqrt{3/2} \times \left[\begin{array}{cccccccccccccccccc} 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \end{array} \right]'$$

b_{0111}/b_{0000} = -0.3062 = $\rho(C_{0100}, C_{0011}) = \rho(C_{0010}, C_{0101}) = \rho(C_{0001}, C_{0110})$
b_{0112}/b_{0000} = 0.1768 = $\rho(C_{0100}, C_{0012}) = \rho(C_{0010}, C_{0102}) = \rho(C_{0002}, C_{0110})$
b_{0121}/b_{0000} = 0.1768 = $\rho(C_{0100}, C_{0021}) = \rho(C_{0020}, C_{0101}) = \rho(C_{0001}, C_{0120})$
b_{0122}/b_{0000} = 0.3062 = $\rho(C_{0100}, C_{0022}) = \rho(C_{0020}, C_{0102}) = \rho(C_{0002}, C_{0120})$
b_{0211}/b_{0000} = 0.1768 = $\rho(C_{0200}, C_{0011}) = \rho(C_{0010}, C_{0201}) = \rho(C_{0001}, C_{0210})$
b_{0212}/b_{0000} = 0.3062 = $\rho(C_{0200}, C_{0012}) = \rho(C_{0010}, C_{0202}) = \rho(C_{0002}, C_{0210})$
b_{0221}/b_{0000} = 0.3062 = $\rho(C_{0200}, C_{0021}) = \rho(C_{0020}, C_{0201}) = \rho(C_{0001}, C_{0220})$
b_{0222}/b_{0000} = -0.1768 = $\rho(C_{0200}, C_{0022}) = \rho(C_{0020}, C_{0202}) = \rho(C_{0002}, C_{0220})$
b_{1011}/b_{0000} = 0.3062 = $\rho(C_{1000}, C_{0011}) = \rho(C_{0010}, C_{1001}) = \rho(C_{0001}, C_{1010})$
b_{1012}/b_{0000} = 0.1768 = $\rho(C_{1000}, C_{0012}) = \rho(C_{0010}, C_{1002}) = \rho(C_{0002}, C_{1010})$
b_{1021}/b_{0000} = 0.1768 = $\rho(C_{1000}, C_{0021}) = \rho(C_{0020}, C_{1001}) = \rho(C_{0001}, C_{1020})$
b_{1022}/b_{0000} = -0.3062 = $\rho(C_{1000}, C_{0022}) = \rho(C_{0020}, C_{1002}) = \rho(C_{0002}, C_{1020})$
b_{1101}/b_{0000} = 0.3062 = $\rho(C_{1000}, C_{0101}) = \rho(C_{0100}, C_{1001}) = \rho(C_{0001}, C_{1100})$
b_{1102}/b_{0000} = 0.1768 = $\rho(C_{1000}, C_{0102}) = \rho(C_{0100}, C_{1002}) = \rho(C_{0002}, C_{1100})$
b_{1110}/b_{0000} = -0.3062 = $\rho(C_{1000}, C_{0110}) = \rho(C_{0100}, C_{1010}) = \rho(C_{0010}, C_{1100})$
b_{1111}/b_{0000} = 0.375 = $\rho(C_{1000}, C_{0111}) = \rho(C_{0100}, C_{1011}) = \rho(C_{0010}, C_{1101}) = \rho(C_{0001}, C_{1110})$
b_{1112}/b_{0000} = 0.2165 = $\rho(C_{1000}, C_{0112}) = \rho(C_{0100}, C_{1012}) = \rho(C_{0010}, C_{1102}) = \rho(C_{0002}, C_{1110})$
b_{1120}/b_{0000} = 0.1768 = $\rho(C_{1000}, C_{0120}) = \rho(C_{0100}, C_{1020}) = \rho(C_{0020}, C_{1100})$
b_{1121}/b_{0000} = -0.2165 = $\rho(C_{1000}, C_{0121}) = \rho(C_{0100}, C_{1021}) = \rho(C_{0020}, C_{1101}) = \rho(C_{0001}, C_{1120})$
b_{1122}/b_{0000} = -0.125 = $\rho(C_{1000}, C_{0122}) = \rho(C_{0100}, C_{1022}) = \rho(C_{0020}, C_{1102}) = \rho(C_{0002}, C_{1120})$
b_{1201}/b_{0000} = 0.1768 = $\rho(C_{1000}, C_{0201}) = \rho(C_{0200}, C_{1001}) = \rho(C_{0001}, C_{1200})$
b_{1202}/b_{0000} = -0.3062 = $\rho(C_{1000}, C_{0202}) = \rho(C_{0200}, C_{1002}) = \rho(C_{0002}, C_{1200})$
b_{1210}/b_{0000} = 0.1768 = $\rho(C_{1000}, C_{0210}) = \rho(C_{0200}, C_{1010}) = \rho(C_{0010}, C_{1200})$
b_{1211}/b_{0000} = -0.2165 = $\rho(C_{1000}, C_{0211}) = \rho(C_{0200}, C_{1011}) = \rho(C_{0010}, C_{1201}) = \rho(C_{0001}, C_{1210})$
b_{1212}/b_{0000} = -0.125 = $\rho(C_{1000}, C_{0212}) = \rho(C_{0200}, C_{1012}) = \rho(C_{0010}, C_{1202}) = \rho(C_{0002}, C_{1210})$
b_{1220}/b_{0000} = 0.3062 = $\rho(C_{1000}, C_{0220}) = \rho(C_{0200}, C_{1020}) = \rho(C_{0020}, C_{1200})$
b_{1221}/b_{0000} = 0.625 = $\rho(C_{1000}, C_{0221}) = \rho(C_{0200}, C_{1021}) = \rho(C_{0020}, C_{1201}) = \rho(C_{0001}, C_{1220})$
b_{1222}/b_{0000} = -0.2165 = $\rho(C_{1000}, C_{0222}) = \rho(C_{0200}, C_{1022}) = \rho(C_{0020}, C_{1202}) = \rho(C_{0002}, C_{1220})$
b_{2011}/b_{0000} = 0.1768 = $\rho(C_{2000}, C_{0011}) = \rho(C_{0010}, C_{2001}) = \rho(C_{0001}, C_{2010})$
b_{2012}/b_{0000} = -0.3062 = $\rho(C_{2000}, C_{0012}) = \rho(C_{0010}, C_{2002}) = \rho(C_{0002}, C_{2010})$
b_{2021}/b_{0000} = -0.3062 = $\rho(C_{2000}, C_{0021}) = \rho(C_{0020}, C_{2001}) = \rho(C_{0001}, C_{2020})$
b_{2022}/b_{0000} = -0.1768 = $\rho(C_{2000}, C_{0022}) = \rho(C_{0020}, C_{2002}) = \rho(C_{0002}, C_{2020})$
b_{2101}/b_{0000} = 0.1768 = $\rho(C_{2000}, C_{0101}) = \rho(C_{0100}, C_{2001}) = \rho(C_{0001}, C_{2100})$
b_{2102}/b_{0000} = -0.3062 = $\rho(C_{2000}, C_{0102}) = \rho(C_{0100}, C_{2002}) = \rho(C_{0002}, C_{2100})$
b_{2110}/b_{0000} = 0.1768 = $\rho(C_{2000}, C_{0110}) = \rho(C_{0100}, C_{2010}) = \rho(C_{0010}, C_{2100})$
b_{2111}/b_{0000} = 0.2165 = $\rho(C_{2000}, C_{0111}) = \rho(C_{0100}, C_{2011}) = \rho(C_{0010}, C_{2101}) = \rho(C_{0001}, C_{2110})$
b_{2112}/b_{0000} = 0.625 = $\rho(C_{2000}, C_{0112}) = \rho(C_{0100}, C_{2012}) = \rho(C_{0010}, C_{2102}) = \rho(C_{0002}, C_{2110})$
b_{2120}/b_{0000} = 0.3062 = $\rho(C_{2000}, C_{0120}) = \rho(C_{0100}, C_{2020}) = \rho(C_{0020}, C_{2100})$
b_{2121}/b_{0000} = -0.125 = $\rho(C_{2000}, C_{0121}) = \rho(C_{0100}, C_{2021}) = \rho(C_{0020}, C_{2101}) = \rho(C_{0001}, C_{2120})$
b_{2122}/b_{0000} = 0.2165 = $\rho(C_{2000}, C_{0122}) = \rho(C_{0100}, C_{2022}) = \rho(C_{0020}, C_{2102}) = \rho(C_{0002}, C_{2120})$
b_{2201}/b_{0000} = -0.3062 = $\rho(C_{2000}, C_{0201}) = \rho(C_{0200}, C_{2001}) = \rho(C_{0001}, C_{2200})$
b_{2202}/b_{0000} = -0.1768 = $\rho(C_{2000}, C_{0202}) = \rho(C_{0200}, C_{2002}) = \rho(C_{0002}, C_{2200})$
b_{2210}/b_{0000} = 0.3062 = $\rho(C_{2000}, C_{0210}) = \rho(C_{0200}, C_{2010}) = \rho(C_{0010}, C_{2200})$
b_{2211}/b_{0000} = -0.125 = $\rho(C_{2000}, C_{0211}) = \rho(C_{0200}, C_{2011}) = \rho(C_{0010}, C_{2201}) = \rho(C_{0001}, C_{2210})$
b_{2212}/b_{0000} = 0.2165 = $\rho(C_{2000}, C_{0212}) = \rho(C_{0200}, C_{2012}) = \rho(C_{0010}, C_{2202}) = \rho(C_{0002}, C_{2210})$
b_{2220}/b_{0000} = -0.1768 = $\rho(C_{2000}, C_{0220}) = \rho(C_{0200}, C_{2020}) = \rho(C_{0020}, C_{2200})$
b_{2221}/b_{0000} = -0.2165 = $\rho(C_{2000}, C_{0221}) = \rho(C_{0200}, C_{2021}) = \rho(C_{0020}, C_{2201}) = \rho(C_{0001}, C_{2220})$
b_{2222}/b_{0000} = 0.375 = $\rho(C_{2000}, C_{0222}) = \rho(C_{0200}, C_{2022}) = \rho(C_{0020}, C_{2202}) = \rho(C_{0002}, C_{2220})$

Table 7.5: Ratios $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$ that are equal to the correlations between $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$

and

$$C_{0111} = \sqrt{27/8} \times \left[\begin{array}{cccccccccccccccc} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \end{array} \right]',$$

using Equation 2.4 gives that the correlation between C_{1000} and C_{0111} is

$$\begin{aligned} \rho(C_{1000}, C_{0111}) &= \tfrac{6.75}{\sqrt{303.75}} \\ &= 0.3873 \quad \neq \quad 0.3750 \end{aligned}$$

Thus, it is clear that,

$$\begin{array}{rcccccc} \rho(C_{0000}, C_{1111}) & = & 0.4082 & \neq & 0.3750 & = & b_{1111}/b_{0000} \\ & & & \text{and} & & & \\ \rho(C_{1000}, C_{0111}) & = & 0.3873 & \neq & 0.3750 & = & b_{1111}/b_{0000} \end{array}$$

and the claim that the correlation of $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$ is equal to $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$ is not true for all disjoint $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$.

Cheng and Ye (2004) do not provide a formal proof of Theorem 7.1, but state that it "follows immediately " from (7.4) and the definition of an orthonormal contrast basis (7.2). From (7.1) and (7.2), it follows that

$$\sum_{\mathbf{x}\in\mathcal{D}} C_{\mathbf{u}}(\mathbf{x})C_{\mathbf{v}}(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{u} \neq \mathbf{v}, \\ N, & \text{if } \mathbf{u} = \mathbf{v}, \end{cases} \tag{7.9}$$

where $\mathcal{D}$ is the full factorial design. Equation (7.9) is true since the sum is taken over the full factorial design $\mathcal{D}$. However, when a fraction of $\mathcal{D}$ is taken, equation (7.9) does not necessarily hold. In the example above, $\sum_{\mathbf{x}\in D} C_{\mathbf{1111}}(\mathbf{x})C_{\mathbf{1111}}(\mathbf{x}) = 15.1875$ and $\sum_{\mathbf{x}\in D} C_{\mathbf{0111}}(\mathbf{x})C_{\mathbf{0111}}(\mathbf{x}) = 16.8750$.

By definition, $b_{\mathbf{u}+\mathbf{v}} = \frac{1}{N}\sum_{\mathbf{x}\in D} C_{\mathbf{u}+\mathbf{v}}(\mathbf{x})$; by (7.2), $\sum_{\mathbf{x}\in\mathcal{D}} C_{\mathbf{u}+\mathbf{v}}(\mathbf{x}) = \frac{1}{N}C'_{\mathbf{u}}C_{\mathbf{v}}$, so $b_{\mathbf{u}+\mathbf{v}}$ is $\frac{1}{N}$ times the numerator of (2.4). Then (7.6) and (2.4) are equivalent if $b_{\mathbf{0}} = \frac{1}{N}\sqrt{C'_{\mathbf{u}}C_{\mathbf{u}}C'_{\mathbf{v}}C_{\mathbf{v}}}$. The denominator of (2.4) can be written as

$$\sqrt{C'_{\mathbf{u}}C_{\mathbf{u}}C'_{\mathbf{v}}C_{\mathbf{v}}} \equiv \sqrt{\left(\sum_{\mathbf{x}\in D} C_{\mathbf{u}}(\mathbf{x})C_{\mathbf{u}}(\mathbf{x})\right)\left(\sum_{\mathbf{x}\in D} C_{\mathbf{v}}(\mathbf{x})C_{\mathbf{v}}(\mathbf{x})\right)}. \tag{7.10}$$

Notice in equation (7.10) that the sum is taken over D and not $\mathcal{D}$. If

$$\sum_{\mathbf{x}\in D} C_{\mathbf{t}}(\mathbf{x})C_{\mathbf{t}}(\mathbf{x}) = n \quad \text{for } t = u, v, \tag{7.11}$$

then

$$\begin{aligned} \frac{1}{N}\sqrt{\sum_{\mathbf{x}\in D} C'_{\mathbf{u}}C_{\mathbf{u}} \sum_{\mathbf{x}\in D} C'_{\mathbf{v}}C_{\mathbf{v}}} &= \frac{1}{N}\sqrt{n * n} \\ &= \frac{n}{N} \\ &= b_{\mathbf{0}}. \end{aligned}$$

In this case,

$$\begin{aligned} \rho(C_{\mathbf{u}}, C_{\mathbf{v}}) &= \frac{C'_{\mathbf{u}}C_{\mathbf{v}}}{\sqrt{C'_{\mathbf{u}}C_{\mathbf{u}}C'_{\mathbf{v}}C_{\mathbf{v}}}} \\ &= \frac{Nb_{\mathbf{u+v}}}{Nb_{\mathbf{0}}} \\ &= \frac{b_{\mathbf{u+v}}}{b_{\mathbf{0}}}, \end{aligned}$$

and Corollary 2.2 is true. As a special case, if $D = \mathcal{D}$, then $\sqrt{C'_{\mathbf{u}}C_{\mathbf{u}}C'_{\mathbf{v}}C_{\mathbf{v}}} = N$ and the corollary is true.

As stated above, Equation (7.11) is not necessarily true when $D \neq \mathcal{D}$. In the case that $\sum_{\mathbf{x}\in D} C_{\mathbf{u}}(\mathbf{x})C_{\mathbf{u}}(\mathbf{x}) = U$ and $\sum_{\mathbf{x}\in D} C_{\mathbf{v}}(\mathbf{x})C_{\mathbf{v}}(\mathbf{x}) = V$ with at least one of U or V not equal to n,

$$\begin{aligned} \rho(C_{\mathbf{u}}, C_{\mathbf{v}}) &= \frac{C'_{\mathbf{u}}C_{\mathbf{v}}}{\sqrt{C'_{\mathbf{u}}C_{\mathbf{u}}C'_{\mathbf{v}}C_{\mathbf{v}}}} \\ &= \frac{Nb_{\mathbf{u+v}}}{\sqrt{U * V}} \\ &= \frac{b_{\mathbf{u+v}}}{\sqrt{U * V}/N} \\ &\neq \frac{b_{\mathbf{u+v}}}{b_{\mathbf{0}}}, \end{aligned}$$

since $\sqrt{U * V}/N \neq n/N = b_{\mathbf{0}}$. Thus, in such a case, the interpretation of $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$ as the correlation of $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$ is not correct.

The incorrect interpretation of $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$ leads to an incorrect interpretation of the elements of the α wordlength pattern. Cheng and Ye (2004) interpret $\alpha_i(D)$, defined in Section 8.2, as a measure of overall aliasing of all i-factor interactions with the overall mean. As shown above, for $\|\mathbf{t}\|_0 = i$, $b_{\mathbf{t}}/b_{\mathbf{0}}$ is not necessarily equal to the correlation between the i-factor interaction and the overall mean, so this interpretation is incorrect. However, a smaller $\alpha_3(D)$ does indicate a lesser degree of aliasing of main effects with two-factor interactions and a smaller $\alpha_4(D)$ does indicate a lesser degree of aliasing of two-factor interactions.

CHAPTER 8

RELATIONSHIP BETWEEN AVERAGE SQUARED CORRELATIONS AND INDICATOR FUNCTION

Despite the fact that the ratio of indicator function coefficients $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}}$ cannot always be interpreted as the correlation between contrasts ($C_{\mathbf{u}}$ and $C_{\mathbf{v}}$), $\rho(C_{\mathbf{u}}, C_{\mathbf{v}})$, there exists a direct relationship between the indicator function coefficients and a subset of the average squared correlations. The relationship exists provided that the design is of strength at least two (see Section 4.4 for design properties required for use of the average squared correlation criterion); for all orthogonal arrays of strength at least two, the condition (7.11) holds for all main effect and two-factor interaction effect contrasts. Because $b_{\mathbf{u}+\mathbf{v}}/b_{\mathbf{0}} = \rho(C_{\mathbf{u}}, C_{\mathbf{v}})$ only for disjoint $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$, only average squared correlations for disjoint effects can be computed from the indicator function coefficients.

8.1 Example: Relationship Between Average Squared Correlations and Indicator Function Coefficients

It is helpful to examine a simple example, using the same set of orthogonal contrasts for the calculation of each criterion. From Section 6.3, the average squared correlation criterion for quantitative factors is dependent upon the choice of orthogonal contrasts.

Consider the design given in Table 7.1 of Section 7.2. The indicator function coefficients, calculated from the complete contrast matrix (including all i-factor interactions, $i = 2, 3, 4$) using the re-scaled linear and quadratic contrasts (7.7) and (7.8), for this design are given in Table 7.3.

Average Squared Correlations of Order 3

In the case of a four-factor projection with qualitative factors, the average squared correlations of order 3 for A with BC, A with BD, and A with CD can be calculated as follows:

$$\begin{aligned} Ave\ \rho_3^2(A, BC) &= \frac{1}{8}\left[\left(\frac{b_{1110}}{b_{0000}}\right)^2 + \left(\frac{b_{1120}}{b_{0000}}\right)^2 + \left(\frac{b_{1210}}{b_{0000}}\right)^2 + \left(\frac{b_{1220}}{b_{0000}}\right)^2 \right. \\ &\quad \left. + \left(\frac{b_{2110}}{b_{0000}}\right)^2 + \left(\frac{b_{2120}}{b_{0000}}\right)^2 + \left(\frac{b_{2210}}{b_{0000}}\right)^2 + \left(\frac{b_{2220}}{b_{0000}}\right)^2\right] \end{aligned} \quad (8.1)$$

$$\begin{aligned} Ave\ \rho_3^2(A, BD) &= \frac{1}{8}\left[\left(\frac{b_{1101}}{b_{0000}}\right)^2 + \left(\frac{b_{1102}}{b_{0000}}\right)^2 + \left(\frac{b_{1201}}{b_{0000}}\right)^2 + \left(\frac{b_{1202}}{b_{0000}}\right)^2 \right. \\ &\quad \left. + \left(\frac{b_{2101}}{b_{0000}}\right)^2 + \left(\frac{b_{2102}}{b_{0000}}\right)^2 + \left(\frac{b_{2201}}{b_{0000}}\right)^2 + \left(\frac{b_{2202}}{b_{0000}}\right)^2\right] \end{aligned} \quad (8.2)$$

$$\begin{aligned} Ave\ \rho_3^2(A, CD) &= \frac{1}{8}\left[\left(\frac{b_{1011}}{b_{0000}}\right)^2 + \left(\frac{b_{1012}}{b_{0000}}\right)^2 + \left(\frac{b_{1021}}{b_{0000}}\right)^2 + \left(\frac{b_{1022}}{b_{0000}}\right)^2 \right. \\ &\quad \left. + \left(\frac{b_{2011}}{b_{0000}}\right)^2 + \left(\frac{b_{2012}}{b_{0000}}\right)^2 + \left(\frac{b_{2021}}{b_{0000}}\right)^2 + \left(\frac{b_{2022}}{b_{0000}}\right)^2\right] \end{aligned} \quad (8.3)$$

If the factors are treated as quantitative, the sets of average squared correlations of order 3 are:

$$\begin{aligned} Ave\ \rho_{3,3}^2(A, BC) &= \left(\frac{b_{1110}}{b_{0000}}\right)^2 \\ Ave\ \rho_{3,4}^2(A, BC) &= \frac{1}{3}\left[\left(\frac{b_{1120}}{b_{0000}}\right)^2 + \left(\frac{b_{1210}}{b_{0000}}\right)^2 + \left(\frac{b_{2110}}{b_{0000}}\right)^2\right] \end{aligned}$$

$$Ave\ \rho^2_{3,5}(A,BC) = \frac{1}{3}\left[\left(\frac{b_{1220}}{b_{0000}}\right)^2 + \left(\frac{b_{2120}}{b_{0000}}\right)^2 + \left(\frac{b_{2210}}{b_{0000}}\right)^2\right]$$

$$Ave\ \rho^2_{3,6}(A,BC) = \left(\frac{b_{2220}}{b_{0000}}\right)^2 \tag{8.4}$$

$$Ave\ \rho^2_{3,3}(A,BD) = \left(\frac{b_{1101}}{b_{0000}}\right)^2$$

$$Ave\ \rho^2_{3,4}(A,BD) = \frac{1}{3}\left[\left(\frac{b_{1102}}{b_{0000}}\right)^2 + \left(\frac{b_{1201}}{b_{0000}}\right)^2 + \left(\frac{b_{2101}}{b_{0000}}\right)^2\right]$$

$$Ave\ \rho^2_{3,5}(A,BD) = \frac{1}{3}\left[\left(\frac{b_{1202}}{b_{0000}}\right)^2 + \left(\frac{b_{2102}}{b_{0000}}\right)^2 + \left(\frac{b_{2201}}{b_{0000}}\right)^2\right]$$

$$Ave\ \rho^2_{3,6}(A,BD) = \left(\frac{b_{2202}}{b_{0000}}\right)^2 \tag{8.5}$$

$$Ave\ \rho^2_{3,3}(A,CD) = \left(\frac{b_{1011}}{b_{0000}}\right)^2$$

$$Ave\ \rho^2_{3,4}(A,CD) = \frac{1}{3}\left[\left(\frac{b_{1012}}{b_{0000}}\right)^2 + \left(\frac{b_{1021}}{b_{0000}}\right)^2 + \left(\frac{b_{2011}}{b_{0000}}\right)^2\right]$$

$$Ave\ \rho^2_{3,5}(A,CD) = \frac{1}{3}\left[\left(\frac{b_{1022}}{b_{0000}}\right)^2 + \left(\frac{b_{2012}}{b_{0000}}\right)^2 + \left(\frac{b_{2021}}{b_{0000}}\right)^2\right]$$

$$Ave\ \rho^2_{3,6}(A,CD) = \left(\frac{b_{2022}}{b_{0000}}\right)^2 \tag{8.6}$$

The average squared correlations of order 3 for the other main effects with disjoint two-factor interactions can be calculated similarly.

The average squared correlations of order 3 for A with AB, A with AC, A with AD, B with AB, B with BC, B with BD, C with AC, C with BC, C with CD, D with AD, D with BD, and D with CD cannot be calculated from the indicator function coefficients since for this interpretation the contrasts $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$ must be disjoint (see Theorem 7.1). For example, $\rho(A_l, A_lB_l) = \rho(C_{1000}, C_{1100}) \neq b_{2100}/b_{0000}$. Also, $\rho(A_q, A_lB_l) = \rho(C_{2000}, C_{1100}) = b_{3100}/b_{0000}$, but b_{3100} is not defined by Cheng and Ye (2004) as part of the indicator function. Thus, average squared correlations of order 3 provide a broader description of the lower order correlations than do the indicator function coefficients.

Using the indicator function coefficients from Table 7.3, for this example, Equations (8.1) – (8.3), give

$$\begin{aligned} Ave\ \rho_3^2(A,BC) &= \frac{1}{8}\left[\left(\frac{-0.0680}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{0.0680}{0.2222}\right)^2 \right. \\ &\quad \left. + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{0.0680}{0.2222}\right)^2 + \left(\frac{0.0680}{0.2222}\right)^2 + \left(\frac{-0.0393}{0.2222}\right)^2\right] \\ &= 0.0625 \end{aligned}$$

$$\begin{aligned} Ave\ \rho_3^2(A,BD) &= \frac{1}{8}\left[\left(\frac{0.0680}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{-0.0680}{0.2222}\right)^2 \right. \\ &\quad \left. + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{-0.0680}{0.2222}\right)^2 + \left(\frac{-0.0680}{0.2222}\right)^2 + \left(\frac{-0.0393}{0.2222}\right)^2\right] \\ &= 0.0625 \end{aligned}$$

$$\begin{aligned} Ave\ \rho_3^2(A,CD) &= \frac{1}{8}\left[\left(\frac{0.0680}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{-0.0680}{0.2222}\right)^2 \right. \\ &\quad \left. + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{-0.0680}{0.2222}\right)^2 + \left(\frac{-0.0680}{0.2222}\right)^2 + \left(\frac{-0.0393}{0.2222}\right)^2\right] \\ &= 0.0625 \end{aligned}$$

Equations (8.4) – (8.6) give

$$\begin{aligned} Ave\ \rho_{3,3}^2(A,BC) &= \left(\frac{-0.0680}{0.2222}\right)^2 \\ &= 0.0938 \end{aligned}$$

$$
\begin{aligned}
Ave\ \rho^2_{3,4}(A,BC) &= \frac{1}{3}\left[\left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2\right] \\
&= 0.0313
\end{aligned}
$$

$$
\begin{aligned}
Ave\ \rho^2_{3,5}(A,BC) &= \frac{1}{3}\left[\left(\frac{0.0680}{0.2222}\right)^2 + \left(\frac{0.0680}{0.2222}\right)^2 + \left(\frac{0.0680}{0.2222}\right)^2\right] \\
&= 0.0938
\end{aligned}
$$

$$
\begin{aligned}
Ave\ \rho^2_{3,6}(A,BC) &= \left(\frac{-0.0393}{0.2222}\right)^2 \\
&= 0.0313
\end{aligned}
$$

$$
\begin{aligned}
Ave\ \rho^2_{3,3}(A,BD) &= \left(\frac{0.0680}{0.2222}\right)^2 \\
&= 0.0938
\end{aligned}
$$

$$
\begin{aligned}
Ave\ \rho^2_{3,4}(A,BD) &= \frac{1}{3}\left[\left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2\right] \\
&= 0.0313
\end{aligned}
$$

$$
\begin{aligned}
Ave\ \rho^2_{3,5}(A,BD) &= \frac{1}{3}\left[\left(\frac{-0.0680}{0.2222}\right)^2 + \left(\frac{-0.0680}{0.2222}\right)^2 + \left(\frac{-0.0680}{0.2222}\right)^2\right] \\
&= 0.0938
\end{aligned}
$$

$$
\begin{aligned}
Ave\ \rho^2_{3,6}(A,BD) &= \left(\frac{-0.0393}{0.2222}\right)^2 \\
&= 0.0313
\end{aligned}
$$

$$Ave\ \rho^2_{3,3}(A, CD) = \left(\frac{0.0680}{0.2222}\right)^2$$
$$= 0.0938$$

$$Ave\ \rho^2_{3,4}(A, CD) = \frac{1}{3}\left[\left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2 + \left(\frac{0.0393}{0.2222}\right)^2\right]$$
$$= 0.0313$$

$$Ave\ \rho^2_{3,5}(A, CD) = \frac{1}{3}\left[\left(\frac{-0.0680}{0.2222}\right)^2 + \left(\frac{-0.0680}{0.2222}\right)^2 + \left(\frac{-0.0680}{0.2222}\right)^2\right]$$
$$= 0.0938$$

$$Ave\ \rho^2_{3,6}(A, CD) = \left(\frac{-0.0393}{0.2222}\right)^2$$
$$= 0.0313$$

These numbers match the average squared correlations of order 3 calculated directly from the correlation matrix (given for design class 18.4.1 in Table 6.2 of Section 6.2). In general, the following lemma holds.

Lemma 8.1.1 *Given an orthogonal array of strength at least two, for a main effect, A, and a disjoint two-factor interaction effect, BC,*

(a) ***Qualitative Factors:*** *The average squared correlation of order 3 calculated from the indicator function coefficients is equal to the average squared correlation of order 3 calculated directly from the correlation matrix.*

(b) ***Quantitative Factors:*** *The the complete set of average squared correlations of order 3 calculated from the indicator function coefficients is equal to the* *complete set of the average squared correlations of order 3 calculated directly from the correlation matrix.*

Average Squared Correlations of Order 4

Because of the constraint that $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$ must be disjoint (Theorem 7.1), it is not possible to calculate any of average squared correlations of order 4 for projections with $p < 4$ factors. Thus, when $p < 4$, average squared correlations of order 4 provide a description of correlations between two-factor interactions and other two-factor interactions when the indicator function coefficients do not.

For $p \geq 4$ factors, only a subset of the set of average squared correlations of order 4 can be calculated from the indicator function coefficients. For a design with $p = 4$ qualitative factors, the indicator function coefficients can be used to calculate the following average squared correlations of order 4:

$$\begin{aligned} Ave\ \rho_4^2(AB, CD) \;=\; & \frac{1}{16}\Bigg[\left(\frac{b_{1111}}{b_{0000}}\right)^2 + \left(\frac{b_{1112}}{b_{0000}}\right)^2 + \left(\frac{b_{1121}}{b_{0000}}\right)^2 + \left(\frac{b_{1122}}{b_{0000}}\right)^2 \\ & + \left(\frac{b_{1211}}{b_{0000}}\right)^2 + \left(\frac{b_{1212}}{b_{0000}}\right)^2 + \left(\frac{b_{1221}}{b_{0000}}\right)^2 + \left(\frac{b_{1222}}{b_{0000}}\right)^2 \\ & + \left(\frac{b_{2111}}{b_{0000}}\right)^2 + \left(\frac{b_{2112}}{b_{0000}}\right)^2 + \left(\frac{b_{2121}}{b_{0000}}\right)^2 + \left(\frac{b_{2122}}{b_{0000}}\right)^2 \\ & + \left(\frac{b_{2211}}{b_{0000}}\right)^2 + \left(\frac{b_{2212}}{b_{0000}}\right)^2 + \left(\frac{b_{2221}}{b_{0000}}\right)^2 + \left(\frac{b_{2222}}{b_{0000}}\right)^2\Bigg] \end{aligned} \qquad (8.7)$$

$$\begin{aligned} Ave\ \rho_4^2(AC, BD) \;=\; & \frac{1}{16}\Bigg[\left(\frac{b_{1111}}{b_{0000}}\right)^2 + \left(\frac{b_{1112}}{b_{0000}}\right)^2 + \left(\frac{b_{1211}}{b_{0000}}\right)^2 + \left(\frac{b_{1212}}{b_{0000}}\right)^2 \\ & + \left(\frac{b_{1121}}{b_{0000}}\right)^2 + \left(\frac{b_{1122}}{b_{0000}}\right)^2 + \left(\frac{b_{1221}}{b_{0000}}\right)^2 + \left(\frac{b_{1222}}{b_{0000}}\right)^2 \\ & + \left(\frac{b_{2111}}{b_{0000}}\right)^2 + \left(\frac{b_{2112}}{b_{0000}}\right)^2 + \left(\frac{b_{2211}}{b_{0000}}\right)^2 + \left(\frac{b_{2212}}{b_{0000}}\right)^2 \end{aligned}$$

$$+ \left(\frac{b_{2121}}{b_{0000}}\right)^2 + \left(\frac{b_{2122}}{b_{0000}}\right)^2 + \left(\frac{b_{2221}}{b_{0000}}\right)^2 + \left(\frac{b_{2222}}{b_{0000}}\right)^2 \Bigg] \quad (8.8)$$

$$\begin{aligned} Ave\ \rho_4^2(AD, BC) \;=\; & \frac{1}{16} \Bigg[\left(\frac{b_{1111}}{b_{0000}}\right)^2 + \left(\frac{b_{1121}}{b_{0000}}\right)^2 + \left(\frac{b_{1211}}{b_{0000}}\right)^2 + \left(\frac{b_{1221}}{b_{0000}}\right)^2 \\ & + \left(\frac{b_{1112}}{b_{0000}}\right)^2 + \left(\frac{b_{1122}}{b_{0000}}\right)^2 + \left(\frac{b_{1212}}{b_{0000}}\right)^2 + \left(\frac{b_{1222}}{b_{0000}}\right)^2 \\ & + \left(\frac{b_{2111}}{b_{0000}}\right)^2 + \left(\frac{b_{2121}}{b_{0000}}\right)^2 + \left(\frac{b_{2211}}{b_{0000}}\right)^2 + \left(\frac{b_{2221}}{b_{0000}}\right)^2 \\ & + \left(\frac{b_{2112}}{b_{0000}}\right)^2 + \left(\frac{b_{2122}}{b_{0000}}\right)^2 + \left(\frac{b_{2212}}{b_{0000}}\right)^2 + \left(\frac{b_{2222}}{b_{0000}}\right)^2 \Bigg] \quad (8.9) \end{aligned}$$

If the factors are treated as quantitative, the complete sets of average squared correlations of order 4 in terms of indicator function coefficients are:

$$\begin{aligned} Ave\ \rho_{4,4}^2(AB, CD) &= \left(\frac{b_{1111}}{b_{0000}}\right)^2 \\ Ave\ \rho_{4,5}^2(AB, CD) &= \frac{1}{4} \Bigg[\left(\frac{b_{1112}}{b_{0000}}\right)^2 + \left(\frac{b_{1121}}{b_{0000}}\right)^2 + \left(\frac{b_{1211}}{b_{0000}}\right)^2 + \left(\frac{b_{2111}}{b_{0000}}\right)^2 \Bigg] \\ Ave\ \rho_{4,6}^2(AB, CD) &= \frac{1}{6} \Bigg[\left(\frac{b_{1122}}{b_{0000}}\right)^2 + \left(\frac{b_{1212}}{b_{0000}}\right)^2 + \left(\frac{b_{1221}}{b_{0000}}\right)^2 + \left(\frac{b_{2112}}{b_{0000}}\right)^2 \\ & \quad + \left(\frac{b_{2121}}{b_{0000}}\right)^2 + \left(\frac{b_{2211}}{b_{0000}}\right)^2 \Bigg] \\ Ave\ \rho_{4,7}^2(AB, CD) &= \frac{1}{4} \Bigg[\left(\frac{b_{1222}}{b_{0000}}\right)^2 + \left(\frac{b_{2122}}{b_{0000}}\right)^2 + \left(\frac{b_{2212}}{b_{0000}}\right)^2 + \left(\frac{b_{2221}}{b_{0000}}\right)^2 \Bigg] \\ Ave\ \rho_{4,8}^2(AB, CD) &= \left(\frac{b_{2222}}{b_{0000}}\right)^2 \quad (8.10) \end{aligned}$$

$$\begin{aligned} Ave\ \rho_{4,4}^2(AC, BD) &= \left(\frac{b_{1111}}{b_{0000}}\right)^2 \\ Ave\ \rho_{4,5}^2(AC, BD) &= \frac{1}{4} \Bigg[\left(\frac{b_{1112}}{b_{0000}}\right)^2 + \left(\frac{b_{1211}}{b_{0000}}\right)^2 + \left(\frac{b_{1121}}{b_{0000}}\right)^2 + \left(\frac{b_{2111}}{b_{0000}}\right)^2 \Bigg] \\ Ave\ \rho_{4,6}^2(AC, BD) &= \frac{1}{6} \Bigg[\left(\frac{b_{1212}}{b_{0000}}\right)^2 + \left(\frac{b_{1122}}{b_{0000}}\right)^2 + \left(\frac{b_{1221}}{b_{0000}}\right)^2 + \left(\frac{b_{2112}}{b_{0000}}\right)^2 \end{aligned}$$

$$+\left(\frac{b_{2211}}{b_{0000}}\right)^2+\left(\frac{b_{2121}}{b_{0000}}\right)^2\Bigg]$$

$$Ave\ \rho^2_{4,7}(AC,BD) = \frac{1}{4}\left[\left(\frac{b_{1222}}{b_{0000}}\right)^2+\left(\frac{b_{2212}}{b_{0000}}\right)^2+\left(\frac{b_{2122}}{b_{0000}}\right)^2+\left(\frac{b_{2221}}{b_{0000}}\right)^2\right]$$

$$Ave\ \rho^2_{4,8}(AC,BD) = \left(\frac{b_{2222}}{b_{0000}}\right)^2 \tag{8.11}$$

$$Ave\ \rho^2_{4,4}(AD,BC) = \left(\frac{b_{1111}}{b_{0000}}\right)^2$$

$$Ave\ \rho^2_{4,5}(AD,BC) = \frac{1}{4}\left[\left(\frac{b_{1121}}{b_{0000}}\right)^2+\left(\frac{b_{1211}}{b_{0000}}\right)^2+\left(\frac{b_{1112}}{b_{0000}}\right)^2+\left(\frac{b_{2111}}{b_{0000}}\right)^2\right]$$

$$Ave\ \rho^2_{4,6}(AD,BC) = \frac{1}{6}\Bigg[\left(\frac{b_{1221}}{b_{0000}}\right)^2+\left(\frac{b_{1122}}{b_{0000}}\right)^2+\left(\frac{b_{1212}}{b_{0000}}\right)^2+\left(\frac{b_{2121}}{b_{0000}}\right)^2$$

$$+\left(\frac{b_{2211}}{b_{0000}}\right)^2+\left(\frac{b_{2112}}{b_{0000}}\right)^2\Bigg]$$

$$Ave\ \rho^2_{4,7}(AD,BC) = \frac{1}{4}\left[\left(\frac{b_{1222}}{b_{0000}}\right)^2+\left(\frac{b_{2221}}{b_{0000}}\right)^2+\left(\frac{b_{2122}}{b_{0000}}\right)^2+\left(\frac{b_{2212}}{b_{0000}}\right)^2\right]$$

$$Ave\ \rho^2_{4,8}(AD,BC) = \left(\frac{b_{2222}}{b_{0000}}\right)^2 \tag{8.12}$$

For both qualitative and quantitative factors, the average squared correlations of order 4 for AB with AC, AB with AD, AB with BC, AB with BD, AC with AD, AC with BC, AC with CD, AD with BC, AD with CD, BC with BD, BC with CD, and BD with CD cannot be calculated from the indicator function coefficients since the two interaction effects are not disjoint. Similar to order 3, $\rho(A_lB_l, A_lC_l) = \rho(C_{1100}, C_{1010}) \neq b_{2110}/b_{0000}$. Also b_{3110}, needed to calculate $\rho(A_qB_l, A_lC_l) = \rho(C_{2100}, C_{1010}) = b_{3110}/b_{0000}$, is not defined by Cheng and Ye (2004) as part of the indicator function. Thus, average squared correlations of order 4 provide a broader description of the lower order correlations than do the indicator function coefficients.

For this example, Equations (8.7) – (8.9), give

$$\begin{aligned}
Ave\ \rho_4^2(AB,CD) &= \frac{1}{16}\left[\left(\frac{0.0833}{0.2222}\right)^2 + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2\right. \\
&\quad + \left(\frac{-0.0481}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{0.1389}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 \\
&\quad + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{0.1389}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{0.0481}{0.2222}\right)^2 \\
&\quad \left. + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 + \left(\frac{0.0833}{0.2222}\right)^2\right] \\
&= 0.0938
\end{aligned}$$

$$\begin{aligned}
Ave\ \rho_4^2(AC,BD) &= \frac{1}{16}\left[\left(\frac{0.0833}{0.2222}\right)^2 + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2\right. \\
&\quad + \left(\frac{-0.0481}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{0.1389}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 \\
&\quad + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{0.1389}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{0.0481}{0.2222}\right)^2 \\
&\quad \left. + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 + \left(\frac{0.0833}{0.2222}\right)^2\right] \\
&= 0.0938
\end{aligned}$$

$$\begin{aligned}
Ave\ \rho_4^2(AD,BC) &= \frac{1}{16}\left[\left(\frac{0.0833}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 + \left(\frac{0.1389}{0.2222}\right)^2\right. \\
&\quad + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 \\
&\quad + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 \\
&\quad \left. + \left(\frac{0.1389}{0.2222}\right)^2 + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{0.0833}{0.2222}\right)^2\right] \\
&= 0.0938
\end{aligned}$$

Equations (8.10) gives

$$\begin{aligned} Ave\ \rho^2_{4,4}(AB,CD) &= \left(\frac{0.0833}{0.2222}\right)^2 \\ &= 0.1406 \end{aligned}$$

$$\begin{aligned} Ave\ \rho^2_{4,5}(AB,CD) &= \frac{1}{4}\left[\left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2 + \left(\frac{0.0481}{0.2222}\right)^2\right] \\ &= 0.0469 \end{aligned}$$

$$\begin{aligned} Ave\ \rho^2_{4,6}(AB,CD) &= \frac{1}{6}\left[\left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{0.1389}{0.2222}\right)^2 + \left(\frac{0.1389}{0.2222}\right)^2\right. \\ &\quad \left. + \left(\frac{-0.0278}{0.2222}\right)^2 + \left(\frac{-0.0278}{0.2222}\right)^2\right] \\ &= 0.1406 \end{aligned}$$

$$\begin{aligned} Ave\ \rho^2_{4,7}(AB,CD) &= \frac{1}{4}\left[\left(\frac{-0.0481}{0.2222}\right)^2 + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{0.0481}{0.2222}\right)^2 + \left(\frac{-0.0481}{0.2222}\right)^2\right] \\ &= 0.0469 \end{aligned}$$

$$\begin{aligned} Ave\ \rho^2_{4,8}(AB,CD) &= \left(\frac{0.0833}{0.2222}\right)^2 \\ &= 0.1406 \end{aligned}$$

The calculation of the average squared correlations of order 4 for the other two-factor interaction pairs follows similarly.

Again, these numbers match the average squared correlations of order 4 calculated directly from the correlation matrix (given for design class 18.4.1 in Table 6.3 of Section 6.2). In general, the following results holds.

Lemma 8.1.2 *Given an orthogonal array of strength at least two with at least four factors, for a two-factor interaction, AB, and a disjoint two-factor interaction effect, CD,*

(a) ***Qualitative Factors:*** *The average squared correlation of order 4 calculated from the indicator function coefficients is equal to the average squared correlation of order 4 calculated directly from the correlation matrix.*

(b) ***Quantitative Factors:*** *The the complete set of average squared correlations of order 4 calculated from the indicator function coefficients is equal to the complete set of the average squared correlation of order 4 calculated directly from the correlation matrix.*

8.2 Relationship Between Average Squared Correlation and Wordlength Patterns

α Wordlength Pattern

The α wordlength pattern of Cheng and Ye (2004) is a redefinition of the GWP of Xu and Wu (2001) based on the indicator function. For $\|\mathbf{t}\|_0$ equal to the number of nonzero elements in $\mathbf{t}$ and a design D, $\alpha_i(D)$ is defined as

$$\alpha_i(D) = \sum_{\|\mathbf{t}\|_0 = i} \left(\frac{b_{\mathbf{t}}}{b_{\mathbf{0}}}\right)^2 \tag{8.13}$$

and the α wordlength pattern is $(\alpha_1(D), \alpha_2(D), \ldots, \alpha_p(D))$.

If D is an orthogonal array of strength two (or greater), for any given $\mathbf{t}$ satisfying $\|\mathbf{t}\|_0 = 3$, $(b_{\mathbf{t}}/b_{\mathbf{0}})^2$ is the squared correlation of a main effect with a disjoint two-factor interaction, and $\alpha_3(D)$ is the sum of these squared correlations. In general, for a three-level factor, $\alpha_3(D)$ is equal to the sum of eight times a subset of average squared correlations of order 3, where the subset is subject to the constraint that each of the $\binom{p}{3}$ factors involved in the main effect and two-factor interaction is included once. For example, for $p = 4$,

$$\alpha_3(D) = 8\times\left(Ave\ \rho_3^2(A, BC) + Ave\ \rho_3^2(A, BD) + Ave\ \rho_3^2(A, CD) + Ave\ \rho_3^2(B, CD)\right). \tag{8.14}$$

Equally,

$$\alpha_3(D) = 8\times\left(Ave\ \rho_3^2(B, AC) + Ave\ \rho_3^2(B, AD) + Ave\ \rho_3^2(C, AD) + Ave\ \rho_3^2(C, BD)\right). \tag{8.15}$$

Various other combinations also exist; the constraint is that each combination of three factors must appear exactly once.

In this example, $Ave\ \rho_3^2(B, AC) = Ave\ \rho_3^2(B, AD) = Ave\ \rho_3^2(C, AD) = Ave\ \rho_3^2(C, BD)$. When all average squared correlations of order 3 are equal, then $Ave\ \rho_3^2 = \alpha_3(D)/8p$. However, this equality for $Ave\ \rho_3^2$ is a special case and is not true in general.

Similarly, if D is an orthogonal array of strength two (or greater), for a given $\mathbf{t}$ satisfying $\|\mathbf{t}\|_0 = 4$, $(b_{\mathbf{t}}/b_{\mathbf{0}})^2$ is the squared correlation of a two-factor interaction with a disjoint two-factor interaction, and $\alpha_4(D)$ is the sum of these squared correlations. In general, for a three-level factor, $\alpha_4(D)$ is equal to sixteen times the sum of a subset of average squared correlations or order 4, where the subset is subject to the constraint that each of the $\binom{p}{4}$ factors involved in the pair of two-factor interactions is included once. For example, for $p = 4$,

$$\alpha_4(D) = 16 \times (Ave\ \rho_4^2(AB, CD)). \tag{8.16}$$

Equally,

$$\alpha_4(D) = 16 \times (Ave\ \rho_4^2(AC, BD)) \tag{8.17}$$

or

$$\alpha_4(D) = 16 \times (Ave\ \rho_4^2(AD, BC)). \tag{8.18}$$

For the example of Section 8.1, using the average squared correlations of order 3, equation (8.14) gives

$$\begin{aligned}
\alpha_3(D) &= 8 \times \Big(Ave\ \rho_3^2(A, BC) + Ave\ \rho_3^2(A, BD) + Ave\ \rho_3^2(A, CD) + Ave\ \rho_3^2(B, CD)\Big) \\
&= 8 \times (0.0625 + 0.0625 + 0.0625 + 0.0625) \\
&= 2.0
\end{aligned}$$

which is equal to the value of $\alpha_3(D)$ calculated directly from the indicator function (given in Table 4.5 of Section 5.2). Using Equation (8.15) gives

$$\begin{aligned}
\alpha_3(D) &= 8 \times \Big(Ave\ \rho_3^2(B, AC) + Ave\ \rho_3^2(B, AD) + Ave\ \rho_3^2(C, AD) + Ave\ \rho_3^2(C, BD)\Big) \\
&= 8 \times (0.0625 + 0.0625 + 0.0625 + 0.0625) \\
&= 2.0
\end{aligned}$$

which is again equal to the value of $\alpha_3(D)$ calculated directly from the indicator function. For $\alpha_4(D)$, equations (8.16), (8.17), and (8.18)

$$\begin{aligned}
\alpha_4(D) &= 16 \times (Ave\ \rho_4^2(AB, CD)) \\
&= 16 \times (Ave\ \rho_4^2(AC, BD)) \\
&= 16 \times (Ave\ \rho_4^2(AD, BC)) \\
&= 16 \times 0.0938 \\
&= 1.5
\end{aligned}$$

and $\alpha_4(D)$ calculated from the average squared correlations of order 4 is equal to $\alpha_4(D)$ calculated directly from the indicator function given in Table 5.1. Thus, there is a direct link between the average squared correlations of order 3 and order 4 and the α wordlength pattern of Cheng and Ye (2004).

Because $\alpha_3(D)$ and $\alpha_4(D)$ sum over individual average squared correlations of order 3 and order 4, respectively, the α wordlength pattern provides less information about the aliasing pattern than the ASCP. Two different sets of averaged squared correlations may produce the same value of either $\alpha_3(D)$ or $\alpha_4(D)$. For example, for both the set $Ave\ \rho^2(A, BC) = 0.0625$, $Ave\ \rho^2(A, BD) = 0.1250$, $Ave\ \rho^2(A, CD) = 0.1250$ and $Ave\ \rho^2(B, CD) = 0.1250$ and the set $Ave\ \rho^2(A, BC) = 0.0625$, $Ave\ \rho^2(A, BD) = 0.0625$, $Ave\ \rho^2(A, CD) = 0.0625$ and $Ave\ \rho^2(B, CD) = 0.2500$, the value of $\alpha_3(D) = 3.5$. Thus, it is expected that the α wordlength pattern will be less able to differentiate combinatorially inequivalent designs.

β Wordlength Pattern

Because all contrast interactions are not equally important when factors are quantitative, the β wordlength pattern based on the polynomial degree of the interaction (i.e. effect hierarchy (6.1)) was proposed by Cheng and Ye (2004). For $\|\mathbf{t}\|_1$ equal to the sum of the polynomial degrees of the contrasts included in $\mathbf{t}$ and a design D,

$$\beta_i(D) = \sum_{\|\mathbf{t}\|_1 = i} \left(\frac{b_{\mathbf{t}}}{b_{\mathbf{0}}} \right)^2 . \tag{8.19}$$

and the β wordlength pattern is $(\beta_1(D), \beta_2(D), \ldots, \beta_P(D))$, $P = \sum_{i=1}^{p} (k_i - 1)$.

Since the summands of $\beta_i(D)$ are again squared correlations, the $\beta_i(D)$ are again sums of average squared correlations of order 3 and order 4. While $\alpha_3(D)$ and $\alpha_4(D)$ maintain separation of the average squared correlations of order 3 and order 4, respectively, the $\beta_i(D)$ combine the average squared correlations of order 3 and order

4. For example, for $p = 4$, the $\beta_i(D)$ can be calculated as

$$\begin{aligned}
\beta_3(D) &= (Ave\ \rho^2_{3,3}(A, BC)) + (Ave\ \rho^2_{3,3}(A, BD)) + (Ave\ \rho^2_{3,3}(A, CD)) \\
&\quad + (Ave\ \rho^2_{3,3}(B, CD)) \\
\beta_4(D) &= 3 \times (Ave\ \rho^2_{3,4}(A, BC)) + 3 \times (Ave\ \rho^2_{3,4}(A, BD)) + 3 \times (Ave\ \rho^2_{3,4}(A, CD)) \\
&\quad + 3 \times (Ave\ \rho^2_{3,4}(B, CD)) + (Ave\ \rho^2_{4,4}(AB, CD)) \\
\beta_5(D) &= 3 \times (Ave\ \rho^2_{3,5}(A, BC)) + 3 \times (Ave\ \rho^2_{3,5}(A, BD)) + 3 \times (Ave\ \rho^2_{3,5}(A, CD)) \\
&\quad + 3 \times (Ave\ \rho^2_{3,5}(B, CD)) + 4 \times (Ave\ \rho^2_{4,5}(AB, CD)) \\
\beta_6(D) &= (Ave\ \rho^2_{3,6}(A, BC)) + (Ave\ \rho^2_{3,6}(A, BD)) + (Ave\ \rho^2_{3,6}(A, CD)) \\
&\quad + (Ave\ \rho^2_{3,6}(B, CD)) + 6 \times (Ave\ \rho^2_{4,6}(AB, CD)) \\
\beta_7(D) &= 4 \times (Ave\ \rho^2_{4,7}(AB, CD)) \\
\beta_8(D) &= (Ave\ \rho^2_{4,8}(AB, CD)) \qquad (8.20)
\end{aligned}$$

As with the $\alpha_i(D)$ for $i = 3, 4$, in general, for a three-level factor, the sums are taken over the set of contrasts subject to the constraint that each of the $\binom{p}{3}$ combinations of factors involved in the main effect and two-factor interaction is included exactly once (for $\beta_3(D)$, $\beta_4(D)$, $\beta_5(D)$, $\beta_6(D)$) and each of the $\binom{p}{4}$ combinations of factors involved in the pair of two-factor interactions is included exactly once (for $\beta_4(D)$, $\beta_5(D)$, $\beta_6(D)$, $\beta_7(D)$, $\beta_8(D)$).

For the example of Section 8.1, using the average squared correlations of order 3 and order 4, Equations (8.20) give

$$\begin{aligned}
\beta_3(D) &= \rho^2_{3,3}(A, BC) + \rho^2_{3,3}(A, BD) + \rho^2_{3,3}(A, CD) + \rho^2_{3,3}(B, CD) \\
&= 0.0938 + 0.0938 + 0.0938 + 0.0938 \\
&= 0.375
\end{aligned}$$

$$
\begin{aligned}
\beta_4(D) &= 3 \times \rho^2_{3,4}(A, BC) + 3 \times \rho^2_{3,4}(A, BD) + 3 \times \rho^2_{3,4}(A, CD) \\
&\quad + 3 \times \rho^2_{3,4}(B, CD) + \rho^2_{4,4}(AB, CD) \\
&= 3 \times 0.0313 + 3 \times 0.0313 + 3 \times 0.0313 + 3 \times 0.0313 + 0.1406 \\
&= 0.5162 \\
\beta_5(D) &= 3 \times \rho^2_{3,5}(A, BC) + 3 \times \rho^2_{3,5}(A, BD) + 3 \times \rho^2_{3,5}(A, CD) \\
&\quad + 3 \times \rho^2_{3,5}(B, CD) + 4 \times \rho^2_{4,5}(AB, CD) \\
&= 3 \times 0.0938 + 3 \times 0.0938 + 3 \times 0.0938 + 3 \times 0.0938 + 4 \times 0.0469 \\
&= 1.3132 \\
\beta_6(D) &= \rho^2_{3,6}(A, BC) + \rho^2_{3,6}(A, BD) + \rho^2_{3,6}(A, CD) + \rho^2_{3,6}(B, CD) \\
&\quad + 6 \times \rho^2_{4,6}(AB, CD) \\
&= 0.0313 + 0.0313 + 0.0313 + 0.0313 + 6 \times 0.1406 \\
&= 0.9688 \\
\beta_7(D) &= 4 \times \rho^2_{4,7}(AB, CD) \\
&= 4 \times 0.0469 \\
&= 0.1876 \\
\beta_8(D) &= \rho^2_{4,8}(AB, CD) \\
&= 0.1406
\end{aligned}
$$

The values of $\beta_3(D)$, $\beta_4(D)$, $\beta_5(D)$, $\beta_6(D)$, $\beta_7(D)$, and $\beta_8(D)$ calculated from the average squared correlation of order 3 and order 4 are equal (up to rounding) to the values calculated directly from the indicator function given in Table 9.1. Thus, there is a direct link between the average squared correlations of order 3 and order 4 and the β wordlength pattern of Cheng and Ye (2004).

Like the α wordlength pattern, because $\beta_i(D)$, $i = 3, 4, 5, 6, 7, 8$, sum over individual average squared correlations of order 3 and order 4, the β wordlength pattern provides less information about the aliasing pattern than the ASCP. Thus, it is expected that the β wordlength pattern will be less able than the ASCP to differentiate geometrically inequivalent designs.

CHAPTER 9

RANKING OF DESIGNS WITH QUANTITATIVE FACTORS

9.1 Ranking and Equivalence

The ASCP can be used to rank order designs with quantitative factors similar to the criterion used for rank ordering designs with qualitative factors described in Section 5.2. Based on the complete sets of average squared correlations of order 3 and order 4, designs can be ranked by sequentially maximizing the $r_{3,3(1)}$, ..., $r_{3,3(k_3)}$, $r_{3,4(1)}$, ..., $r_{3,4(k_4)}$, $r_{3,5(1)}$, ..., $r_{3,5(k_5)}$, $r_{3,6(1)}$, ..., $r_{3,6(k_6)}$, $r_{4,4(1)}$, ..., $r_{4,4(m_4)}$, $r_{4,5(1)}$, ..., $r_{4,5(m_5)}$, $r_{4,6(1)}$, ..., $r_{4,6(m_6)}$, $r_{4,7(1)}$, ..., $r_{4,7(m_7)}$, $r_{4,8(1)}$, ..., $r_{4,8(m_8)}$ in Array (6.7). That is, designs are ranked first by factorial order, then by polynomial degree, selecting designs with a large number of smaller average squared correlations as better designs. For example, using this criterion, a design with more small values of $Ave\ \rho^2_{3,3}$ is better than a design with more large values of $Ave\ \rho^2_{3,3}$. By maximizing the $r_{i(j)}$ in this order, designs are selected for which there is less aliasing of lower degree polynomials of main effects and two-factor interactions.

The *generalized minimum aberration criterion* for designs with quantitative factors sequentially minimizes the $\beta_i(D)$ for $i = 1, 2, \ldots, P$, where $P = \sum_{i=1}^{p}(k_i - 1)$ (Cheng and Ye 2004). The generalized resolution and minimum aberration design are defined as in Section 5.2, replacing $\alpha_i(D)$ with $\beta_i(D)$.

Another possible criterion for rank ordering designs with quantitative factors minimizes the value of $Q(\Gamma^{(p)})$; as described in Chapter 7, lower $Q(\Gamma^{(p)})$ indicates better effect estimation over a wide range of possible models. An additional criterion (not examined here) that can be used to compare designs with respect to projection is to minimize $Q(\Gamma^{(p)})$ averaged over all possible k-factor projections for a given number of factors, k (Tsai et al. 2000).

9.1.1 Example: Rank Ordering of Geometrically Inequivalent Design Classes from $OA(18, 7, 3, 2)$

Patterns of $Ave\ \rho^2_{3,3}$ and $Ave\ \rho^2_{3,4}$, β wordlength patterns, and $Q(\Gamma^{(p)})$ values are shown in Table 9.1 in order to examine optimal design choices for quantitative factors based on each ranking method. Based on $Ave\ (\rho^2_{3,3})$ and $Ave\ (\rho^2_{3,4})$ (which in this case are sufficient to differentiate and rank the projection designs from $OA(18, 7, 3, 2)$) the design classes 18.3.2, 18.4.1, and 18.5.1, would be ranked as the best projection designs of three-, four-, and five-factors, respectively. Sequentially minimizing the elements $\beta_i(D)$ of the β wordlength, classes 18.3.2, 18.4.2, and 18.5.3, would be ranked as optimal. Classes 18.3.1, 18.4.2, and 18.5.1, would be selected as optimal designs based on minimizing the $Q(\Gamma^{(p)})$ values. Thus, these three methods of projection design ranking do not result in exactly the same rankings.

The difference in rankings is the result of the different values of the three criterion. The ASCP assigns values according to the factorial order of the correlation followed by the polynomial degree. In contrast, the β wordlength pattern assigns values according to polynomial degree only. In the case when there are many small correlations with both low polynomial degree correlations and low factorial order, these two criteria induce the same rank ordering; these two criteria induce different rank orderings when the small correlations are of either low polynomial degree but higher factorial order or high polynomial degree but low factorial order.

Both the ASCP and the β wordlength pattern are based on fitting a single model to the data. In contrast, the $Q(\Gamma^{(p)})$ considers a set of possible models to be fit to the data; the set of models used for the $Q(\Gamma^{(p)})$ includes the single models assumed by the ASCP and the β wordlength pattern. If the design is optimal only for the single model and poor for all other possible models, then the $Q(\Gamma^{(p)})$ will select a different optimal design than the ASCP and the β wordlength pattern criteria.

9.2 Geometric Non-equivalence of Projection Designs from Orthogonal Arrays

From Table 9.1, the ASCP is able to distinguish all inequivalent design classes of three-, four-, and five-factor projection designs from the $OA(18, 7, 3, 2)$; Neither the β wordlength pattern nor $Q(\Gamma^{(p)})$ are able to distinguish all four-factor projection design classes. Table 9.2, Table 9.3, and Table 9.4 provide the numbers of geometrically inequivalent classes identified by the ASCP, the β wordlength pattern, and $Q(\Gamma^{(p)})$ for three-, four-, and five-factor projections from the $OA(18, 7, 3, 2)$, $OA(27, 13, 3, 2)$, and $OA(36, 13, 3, 2)$ in Tables A.1, A.2, and A.3, respectively. The actual numbers of inequivalent classes of projection designs are provided for reference; the true numbers of classes are found using the indicator function (Cheng and Ye 2004), the only necessary and sufficient condition for geometric equivalence studied.

Class	Columns	Number of $Ave\ (\rho^2_{3,3})$ 0.0000	0.9380	Number of $Ave\ (\rho^2_{3,4})$ 0.0000	0.0312	0.1667	0.1979	0.5000
18.3.1	(1,2,3)	6	3	6	3	0	0	0
18.3.2	(1,2,7)	9	0	6	0	3	0	0
18.3.3	(1,4,7)	9	0	6	0	0	0	3
18.3.4	(3,5,7)	6	3	6	0	0	3	0
18.4.1	(1,2,3,4)	15	9	12	9	3	0	0
18.4.2	(1,2,3,7)	15	9	12	9	0	0	3
18.4.3	(1,2,4,7)	12	12	12	12	0	0	0
18.4.4	(1,3,4,7)	12	12	12	9	0	3	0
18.4.5	(1,3,5,7)	15	9	12	3	3	6	0
18.5.1	(1,2,3,4,5)	26	24	20	24	3	0	3
18.5.2	(1,2,3,4,7)	23	27	20	21	3	6	0
18.5.3	(1,2,3,5,7)	23	27	20	24	0	3	3
18.5.4	(1,2,4,6,7)	20	30	20	30	0	0	0
18.5.5	(1,3,4,5,7)	26	24	20	12	6	12	0

Class	Columns	β Wordlength	$Q(\Gamma^{(p)})$
18.3.1	(1,2,3)	(0.0000,0.0000,0.0938,0.0937,0.2813,0.0312)	0.5328
18.3.2	(1,2,7)	(0.0000,0.0000,0.0000,0.5000,0.0000,0.5000)	0.5378
18.3.3	(1,4,7)	(0.0000,0.0000,0.0938,0.5938,0.2812,0.0312)	0.5635
18.3.4	(3,5,7)	(0.0000,0.0000,0.0000,1.5000,0.0000,0.5000)	0.5993
18.4.1	(1,2,3,4)	(0.0000,0.0000,0.3750,0.5156,1.3125,0.9688,0.1875,0.1406)	0.9853
18.4.2	(1,2,3,7)	(0.0000,0.0000,0.2813,0.8438,1.4062,0.7812,0.1875,0.0000)	0.9801
18.4.3	(1,2,4,7)	(0.0000,0.0000,0.2813,1.7813,0.8437,0.5937,0.0000,0.0000)	1.0446
18.4.4	(1,3,4,7)	(0.0000,0.0000,0.3750,0.8906,1.3125,0.5938,0.1875,0.1406)	1.0041
18.4.5	(1,3,5,7)	(0.0000,0.0000,0.2813,1.7813,0.8437,0.5937,0.0000,0.0000)	1.0446
18.5.1	(1,2,3,4,5)	(0.0000,0.0000,0.9375,1.6406,3.7500,4.5313,0.9375,0.7031,0.0000,0.0000)	1.7055
18.5.2	(1,2,3,4,7)	(0.0000,0.0000,0.8438,2.5781,3.6797,3.1016,0.9844,1.1250,0.1172,0.0703)	1.7166
18.5.3	(1,2,3,5,7)	(0.0000,0.0000,0.7500,3.0156,3.5625,3.0000,0.9375,0.5781,0.3750,0.2813)	1.7282
18.5.4	(1,2,4,6,7)	(0.0000,0.0000,0.7500,3.8906,2.4375,2.3750,2.0625,0.3281,0.3750,0.2813)	1.7888
18.5.5	(1,3,4,5,7)	(0.0000,0.0000,0.8438,3.0156,3.1172,2.7891,1.5469,1.0000,0.1172,0.0703)	1.7469

Table 9.1: Partial average squared correlations of order 3 pattern, β wordlength patterns, and $Q(\Gamma^{(p)})$ for inequivalent projection classes from $OA(18,7,3,2)$

	p = 3	p = 4	p = 5
Ave ρ^2	4	5	5
β Wordlength	4	4	5
$Q(\Gamma^{(p)})$	4	4	5
Indicator Function	4	5	5

Table 9.2: Number of geometrically inequivalent projection design classes identified for $p = 3, 4, 5$ columns from $OA(18, 7, 3, 2)$ in Table A.1

While Cheng and Ye (2004) published the number of geometrically inequivalent classes of projection designs from the $OA(18, 7, 3, 2)$ for three- and four-factor projections, the numbers provided in Table 9.2 do not match these researchers' numbers. The reason for the difference is due to different sets of permutations considered. The search conducted by Cheng and Ye (2004) included level label permutations within each column that did not preserve the level ordering, in effect searching over a set of non-isomorphic starting arrays; the search undertaken in this work examines a single starting array with a given level labeling and does not include level label permutations that do not preserve the level order.

Again, from Table 9.2, only the ASCP is able to identify all classes for each of the projection sizes for the $OA(18, 7, 3, 2)$ given. Both the β wordlength pattern and the $Q(\Gamma^{(p)})$ differentiate all three- and five-factor projections classes but fail to distinguish between two of the five design classes for four-factor projections.

For the $OA(27, 13, 3, 2)$, all three methods are able to identify correctly each of the geometrically inequivalent design classes (Table 9.3). However, none of the three criteria is able to distinguish all the equivalence classes for the $OA(36, 13, 3, 2)$ (Table 9.4). For the given starting design, $Q(\Gamma^{(p)})$ cannot identify all classes of any

	p = 3	p = 4	p = 5
Ave ρ^2	2	3	3
β Wordlength	2	3	3
$Q(\Gamma^{(p)})$	2	3	3
Indicator Function	2	3	3

Table 9.3: Number of geometrically inequivalent projection design classes identified for $p = 3, 4, 5$ columns from $OA(27, 13, 3, 2)$ in Table A.2

	p = 3	p = 4	p = 5
Ave ρ^2	13	111	439
β Wordlength	13	109	441
$Q(\Gamma^{(p)})$	10	34	75
Indicator Function	13	116	443

Table 9.4: Number of geometrically inequivalent projection design classes identified for $p = 3, 4, 5$ columns from $OA(36, 13, 3, 2)$ in Table A.3

projection size considered, with the proportion of classes identified decreasing as the number of projection columns increases: $Q(\Gamma^{(p)})$ identifies 77% of the three-factor projection classes, 29% of the four-factor projection classes, and only 17% of the five-factor projection classes. Both ASCP and β wordlength pattern identify 100% of the three-factor projection classes from the given $OA(36, 13, 3, 2)$. The ASCP performs slightly better than the β wordlength pattern for $p = 4$ factors, while the β wordlength pattern performs slightly better than the ASCP for $p = 5$ factors; in each case, the difference between the number of classes identified by each method equal to two.

The more effective classification by the β wordlength pattern as compared to the ASCP in the case of five-factor projections of the $OA(36, 13, 3, 2)$ is unexpected. As described in Section 8.2, the elements of the β wordlength pattern, $\beta_i(D)$, are sums of the average squared correlations, $Ave\ \rho^2_{i,j}$. The ASCP preserves more individual information than the β wordlength pattern and, therefore, is expected to be more sensitive to non-equivalence.

CHAPTER 10

SUMMARY OF FINDINGS

The average squared correlation criterion is developed for rank ordering orthogonal arrays of strength two and projection designs of these arrays. Because correlations between contrasts represent the degree of aliasing between the contrasts, the average squared correlation criterion can be used to select an optimal design with respect to effect aliasing. As a consequence of the design ranking, two designs ranked individually can be declared non-equivalent.

In the case of qualitative factors, the average squared correlations are independent of the choice of orthogonal contrast set (Section 4.3). Possible determination of non-equivalence is a result of this theorem.

For designs with qualitative factors, the average squared correlations are grouped by factorial order of the contrast pair. The ASCP for these designs is similar to the GWP (Xu and Wu 2001, Ma and Fang 2001) and the α wordlength pattern (Cheng and Ye 2004) which also group contrasts based on factorial order. However, the ASCP provides a finer grouping than the other criteria. As a result, the ASCP is more sensitive to the detection of non-equivalence and, thereby, the differential ranking of designs. As shown in Chapter 5, the ASCP provides improved ranking as compared to the GWP and α wordlength pattern for projection designs from both the $OA(18, 7, 3, 2)$ and the $OA(36, 13, 3, 2)$.

Contrast grouping by factorial order alone is not adequate for designs with quantitative factors. The average squared correlation criterion, therefore, subdivides groups with a given factorial order by polynomial degree. This methodology is in contrast to the β wordlength pattern (Cheng and Ye 2004) which groups contrast correlations by polynomial degree only. While both criteria are based on squared correlations, the different groupings lead to different behavior under different conditions; in some cases (e.g. four-factor projections of an $OA(36, 13, 3, 2)$) the ASCP provides a more detailed ranking, while in other cases (e.g. five-factor projections of an $OA(36, 13, 3, 2)$) the β wordlength pattern provides a more detailed ranking. Also, in some cases the rankings of the two criteria are the same, while in other cases the ranking differ.

For designs with both qualitative factors and quantitative factors, the average squared correlation pattern provides a meaningful description of the correlations between contrasts. This new criterion offers an alternative to current criteria for rank ordering orthogonal arrays and their projections.

PART II

IDENTIFICATION OF DISPERSION EFFECTS IN REPLICATED EXPERIMENTS

CHAPTER 11

TESTS FOR DISPERSION EFFECTS AND HOMOGENEITY OF VARIANCE

An effect on the variability of response is called a *dispersion effect.* A dispersion effect acts on the variance of the response analogous to a location effect acting on the mean of the response. When the variance of the response is larger at one level of the factor than at the other level(s) of the factor, then a dispersion effect exists (See Figure 1.1 (b)). For example, consider a factorial experiment with a factor A having two levels labeled $High$ and Low. Let the variance of the response, Y, be $\sigma^2_{High}(Y)$ when A is set at the $High$ level, and $\sigma^2_{Low}(Y)$ when A is set at the Low level. If $\sigma^2_{High}(Y) \neq \sigma^2_{Low}(Y)$, then there exists a dispersion effect of factor A; if $\sigma^2_{High}(Y) = \sigma^2_{Low}(Y)$, then no dispersion effect exists for factor A. For example, Figure 1.1 (b) shows the case where $\sigma^2_{High}(Y) > \sigma^2_{Low}(Y)$

A dispersion effect is measured as the difference between the variability of the response when the factor is set at the different levels. In the example above, the dispersion effect of A is measured as $\gamma = \sigma^2_{High}(Y) - \sigma^2_{Low}(Y)$. In the case of a replicated experiment, the dispersion effect can be estimated as the difference between the observed sample variance when factor A is set at the $High$ level and the observed sample variance when factor A is set at the Low level, $\hat{\gamma} = s^2_{High}(Y) - s^2_{Low}(Y)$.

Methods for the identification of dispersion effects have been the focus of recent research in industrial statistics. This shift from a concentration only on the mean response to the study of dispersion is largely the result of emphasis on quality control and quality improvement. The aim of such quality improvement is to select the combination of factor levels that reduces the variability of the product; the reduction in variability produces a significant financial benefit in cost savings as fewer units (or batches) are rejected. Thus, it is necessary first to identify which factors are affecting the variability.

11.1 Analysis of Homogeneity of Variance

The antecedent of the analysis of dispersion effects is the analysis of homogeneity of variance. Analysis of homogeneity of variance questions whether the variances of different samples are equal, not whether a specific factor induces differences in variability. Also, tests of homogeneity of variance have been, and continue to be, used to test the validity of the equal variances assumption required for many statistical procedures (e.g. Analysis of Variance).

An early method for testing homogeneity of variance was the likelihood ratio test proposed by Neyman and Pearson (Neyman and Pearson 1931). The likelihood ratio test requires the assumption that the data are normally distributed; the likelihood ratio test is extremely sensitive to violations of the normality assumption, exhibiting uncontrolled Type I error rates when the assumption is violated. The Neyman-Pearson likelihood ratio test provides the foundation for numerous modifications including Bartlett's Test.

The classical method for testing whether or not two samples have equal variances is the F-test based on the ratio of the two sample variances (for example, see Box, Hunter and Hunter (1978) page 121). This method was extended by Bartlett (1937) to the case of more than two samples. Other early tests of homogeneity of variance include Cochran's Test (Cochran 1941) and Hartley's Test (Hartley 1950), also known as the F-max Test. As discussed in Seber (1977), each of these tests is sensitive to departures from normality. The true level of significance can be very different from the nominal level if the distribution of the sample data is not normal.

Work by Box (1953) emphasizes the extent of the sensitivity of Bartlett's Test to the normality assumptions. Box (1953) suggests a method of reducing this sensitivity by using within-group information in the following way. The Box Test divides the within-treatment replicates into groups of size k, computes $log(s^2)$ for each group, and conducts an analysis of variance with the $log(s^2)$ data as the response variable. However, the results of the Box Test are not unique, but depend on the division of the data points into groups (Box 1953). A second disadvantage of the Box Test is the reduction of the number of data points as a result of grouping. In general, grouping of data leads to a loss of information.

An alternative method for testing equality of variances of two or more samples of equal sizes was proposed by Levene (1960). Like the Box Test, Levene's Test utilizes the analysis of variance methodology. Unlike the Box Test, Levene's Test transforms each individual observation, keeping the original number of data points. Levene's Test is based on the construct $E((X_{ij} - \mu_i)^2) = \sigma_i^2$ for a random sample of observations $X_{i1}, \ldots, X_{iN_i}$ on a random variable X_i from a distribution with mean μ_i and variance σ_i^2. A one-way analysis of variance, therefore, can be performed using $(X_{ij} - \mu_i)^2$ as the response variable. Because in practice the μ_i are not known, Levene's Test applies the analysis of variance to $B_{ij} = (X_{ij} - \bar{X}_i)^2$. Though the B_{ij} are not independent, the correlation of B_{ij} and B_{ik} is of order n^{-2} (except where X_i assumes only two values, each with equal probability). As suggested by Levene (1960), a correlation of this order will not significantly affect the distribution of the F-statistic.

Levene (1960) further generalized his proposed methodology, allowing the use of $W_{ij} = g(|X_{ij} - \bar{X}_i|)$, where $g(x)$ is any monotonically increasing function of x on $(0, \inf)$. In particular, Levene (1960) studied $z_{ij} = |X_{ij} - \bar{X}_i|$, $L_{ij} = log(|X_{ij} - \bar{X}_i|)$, $t_{ij} = |X_{ij} - \bar{X}_i|^{1/2}$, and $s_{ij} = |X_{ij} - \bar{X}_i|^2$. Monte Carlo methods were used to identify the distributions of the F-test statistics. The simulation study found that tests based on t and L provide poor power when the distribution of the data is normal and uncontrolled Type I error when the distribution of the data is not normal. Tests based on z are more powerful than tests based on s but with Type I error greater than the nominal value (Levene 1960).

Because the mean is the best estimate of central tendency only when the underlying distribution is symmetric, $z_{ij} = |X_{ij} - \bar{X}_i|$ is likely to be significantly affected when the underlying distribution is not symmetric. This led Brown and Forsythe (1974) to consider other more robust measures of central tendency. In particular, Brown and Forsythe (1974) examined the use of the within-treatment median, $\widetilde{X}$, and the 10% trimmed mean, $\bar{X}_{-10\%}$. Similar to the work of Levene (1960), Brown and Forsythe (1974) used a Monte Carlo study to compare Levene's Test with the proposed variations. Results of the study show that $W_{10} = |X_{ij} - \bar{X}_{-10\%}|$ is the best choice with respect to level (i.e. Type I error probability) in cases where the data are from a long-tailed, symmetric distribution. When data are from an asymmetric distribution, $W_{50} = |X_{ij} - \widetilde{X}_i|$ is the best choice with respect to level (Brown and Forsythe 1974). The difference in power between each variation and z_{ij} is small relative to the difference in observed Type I error.

Miller (1968) conducted a Monte Carlo study to determine the power of jackknifing in comparison to the classical F-test, Box Test, Levene's Test, Box-Anderson Test which adjusts the degrees of freedom for the classical F-test (Box and Andersen 1955), and Moses Test (Moses 1963) which is the Wilcoxon two-sample rank sum test using the $log(s^2)$ from subgroups of the within-treatment replicates for the comparison of two sample variances. In considering Levene's test, all four original variations (i.e. z_{ij}, L_{ij}, t_{ij}, and s_{ij}) were examined. The results of this study indicate that Levene's Test is robust but less powerful than the jackknife, Box-Andersen test, and Box Test for long-tailed distributions. The Box Test is also shown to be robust, approximately as powerful as the jackknife with groups of size $k = 5$, but less powerful than the Box-Andersen Test and the jackknife with groups of size $k = 1$. This indicates that, in fact, there is a significant loss of power due to dividing the within-treatment samples.

A thorough examination, comparison, and review of tests for homogeneity of variance was conducted by Conover, Johnson and Johnson (1981). In total, fifty-six parametric and nonparametric tests were compared for robustness and power. Different distributions, sample sizes (both equal and unequal), and variance combinations were considered. All four variations of Levene's Test (i.e. z_{ij}, L_{ij}, t_{ij}, and s_{ij}), three variations of Brown and Forsythe's modification of Levene's Test replacing the mean with the median (i.e. $z^*_{ij} = |X_{ij} - \widetilde{X}_i|$, $t^*_{ij} = |X_{ij} - \widetilde{X}_i|^{1/2}$, and $s^*_{ij} = |X_{ij} - \widetilde{X}_i|^2)$, and Box's Test were included in the study. Based on the definition that a test is robust if the maximum Type I error rate is less than 0.10 for an $\alpha = 0.05$ significance level test, Conover et al. (1981) found that only five of the fifty-six tests are robust tests. Brown and Forsythe's variation of Levene's Test using absolute deviation from the median, z^*_{ij}, and the square root of the absolute deviation from the median, t^*_{ij}, were included in the group of five robust tests. A variation of Bartlett's test using the median and two nonparametric tests were also determined to be robust. Additionally, Brown and Forsythe's modification of Levene's Test using absolute deviation from the median, z^*_{ij}, and the two nonparametric tests were found to be more powerful than the other robust tests. The researchers noted that the main benefit of using the median in place of the mean is the reduction of the observed Type I error rates, but only for certain of the tests (Conover et al. 1981). It is interesting but not surprising to note that all of the robust tests use the median rather than the mean.

Although developed for a different purpose, the methods proposed by Levene (1960) and Brown and Forsythe (1974) may provide a foundation for developing a test for the identification of dispersion effects in replicated experiments. It is reasonable to believe that the Levene (1960) and Brown and Forsythe (1974) methods would be powerful for identifying dispersion effects, similar to the results for homogeneity of variance. Extensions of these methods form the foundation for the current research.

11.2 Identification of Dispersion Effects in Unreplicated Experiments

Replicated experiments generally require a large number of runs and, consequently, most research on identification of dispersion effects has focused on unreplicated experiments. In order to reduce further the required number of runs, fractional factorial and other similar designs are most frequently investigated.

Box and Meyer (1986) proposed a method for dispersion effect identification based on the natural logarithm of the ratio of the sums of squared residuals associated with the high and low levels of a factor. Sample variances are computed using the residuals obtained from fitting an identified location model. Use of the residuals is required to correct for the aliasing of location and dispersion effects; in order to eliminate location-dispersion aliasing, "large" location effects, including the overall mean, must be fit to the data. Box and Meyer (1986) conclude that using the residuals to calculate the sample variances is sufficient for the purpose of dispersion effect identification.

Identification of the active dispersion effects using the Box-Meyer method is highly subjective; the logarithm of the ratio of the variances of the sample residuals are plotted and the "large" values are visually identified (Box and Meyer 1986). While normal theory significance values can be marked for guidance, they are not valid critical values since the F-test assumptions are not satisfied. The Box-Meyer test statistic does not possess a well-defined reference distribution since the statistic is based directly on the residuals from fitting a location model identified by the researcher based on the data. This disadvantage is true of all methods based on location model residuals.

Bergman and Hynén (1997) proposed an alternative method for dispersion effect identification that produces a test statistic with a well-defined distribution. This distribution then provides a critical value with which to conduct a formal significance test, removing the subjectivity associated with the Box-Meyer method. To test for a dispersion effect of factor i, the Bergman-Hynén statistic is equal to the natural logarithm of the ratio of the sum of the squared residuals at the high level of the factor to the sum of the squared residuals at the low level of the factor, where the residuals are computed from fitting a location model including the identified active location effects, the effect of the factor i, and all interaction terms between the active location effect factors and factor i. The idea of this expanded location model will be the basis for the residuals studied as a possible measure in the second stage of simulations in the current work (see Chapter 16).

In an unreplicated experiment, misidentification of the location model can have a significant impact on the dispersion effects analysis. While misidentification of nonexistent effects can decrease the efficiency of the the dispersion analysis, it will not affect the validity of the method. The more serious issue arises from the non-identification of small to moderate location effects. The impact of unidentified active location effects on the methods of Box and Meyer (1986) and Bergman and Hynén (1997) was examined by Pan (1999). Using computer simulation, Pan (1999) showed that unidentified location effects can have a serious impact on both the power and error probability of dispersion effect identification. Also, inclusion of borderline location effects is not sufficient to prevent this problem. The impact of unidentified location effects exists for both methods examined. Pan (1999) cites the cause of this issue as the aliasing of location and dispersion effects in unreplicated experiments.

Bursztyn and Steinberg (2005) present a review of a number of methods for dispersion effect screening in unreplicated experiments including, among others, the Box-Meyer and Bergman-Hynén methods as well as the following. Harvey's method (Harvey 1976) and Wang's method (Wang 1989) both rely on fitting a location model using least squares and constructing a test statistic from the model residuals. The nonparametric method of McGrath and Lin (2002) for identifying dispersion effects does not require the assumption that the data are normally distributed; the test instead uses the rank of the location model regression coefficient. McGrath and Lin

(2001) also developed a parametric method to deal with multiple dispersion effects and interactions. Brenneman and Nair (2001) proposed a method combining a modified Wang method and a log-linear dispersion model for joint location and dispersion modeling. Other methods for the identification of dispersion effects have been proposed by researchers including Chowdhury and Fard (2001), Liao (2000), and Holm and Wiklander (1999). Bursztyn and Steinberg (2005) concluded that screening of dispersion effects from small unreplicated experiments should be undertaken with caution. From their review, the Bergman-Hynén method is identified as a good quick screen for large influential dispersion effects; modeling can be used to identify additional dispersion effects.

Ankenman and Dean (2003) also provide a review of various methods for the identification of dispersion effects.

11.3 Identification of Dispersion Effects in Replicated Experiments

The solution to the issue of location-dispersion aliasing is replication; through replication, location and dispersion effects can be completely separated. Extension of certain methods (e.g., Box and Meyer (1986) and Bergman and Hynén (1997)) to replicated experiments was suggested by Pan (1999). The methods of Box and Meyer (1986) and Bergman and Hynén (1997) for unreplicated experiments are extended by Pan (1999) for use in replicated experiments by replacing the sum of squared location model residuals with a measure of with within-treatment replicate variability. The variability is measured by the sum of squared scaled differences between observations for two replicates per treatment; for more than two replicates per treatment, the within-treatment variance can be used. Thus, the identification and fitting of a location model is not required in replicated experiments. While the extension methods proposed by Pan (1999) for replicated experiments eliminate the impact of misidentification of location effects, identification of dispersion effects is still effectively based on a single dispersion measure at each design point.

The extension of the Box-Meyer method to replicated experiments was studied by Nair and Pregibon (1988). The Box-Meyer method was extended by using the sum of the squared residuals for all replicates at the high and low levels of the factor. Nair and Pregibon (1988) found that the natural logarithm of the ratio of the sums of squared residuals is generally biased and can lead to either failing to identify an active dispersion effect or incorrectly "identifying" an inactive dispersion effect. Nair and Pregibon (1988) also found that the method of probability plotting proposed by Daniel (1959) is invalid since the bias and variance of the test statistic depend on the true but unknown model; this dependence also makes it difficult to construct a formal test.

In replicated experiments, a commonly used method for the identification of dispersion effects, based on the work of Bartlett and Kendall (1946), is a least squares analysis of the logarithm of the sample variances or the sample standard deviations of the within-treatment replicates. The natural logarithms of the sample variances have approximate normal distributions with only the mean of the distribution dependent upon the true population variance. Also, the logarithm converts multiplicative relationships into additive relationships. Based on a study of the distribution, Bartlett and Kendall (1946) suggested that the natural logarithm of the variance produces transformed variates which are appropriate for use as the response in a least squares analysis when the samples contain at least ten observations each; when samples contain between five and nine observations, a least squares analysis using the transformed variates of the response should be performed cautiously. Least squares analysis of the natural logarithm of the variance should not be used when the samples contain fewer than five observations each. The natural logarithmic transformation of the within-treatment variances and least squares analysis of the transformed variates yields a test statistic which is unbiased and independent of the error distribution. Nair and Pregibon (1988) found this methodology to be useful for dispersion effect identification but not estimation.

With respect to quality control, Taguchi (1986) proposed the use of a signal-to-noise ratio for jointly studying location and dispersion in screening designs. As discussed by Wu and Hamada (2000), the signal-to-noise ratio is a composite measure combining the sample mean and sample variance into a single measure. The goal of using a signal-to-noise ratio is to identify a factor-level combination that will produce the smallest possible variance and a mean target value. Three main signal-to-noise ratios were proposed by Taguchi (1986) to be used according to the mean response target value: the nominal-the-best signal-to-noise ratio when the target value is specified; the smaller-the-better signal-to-noise ratio when the smallest possible target value is desired; and the larger-the-better signal-to-noise ratio when the largest possible target value is desired. In using the nominal-the-best signal-to-noise ratio, the mean and variance may or may not be confounded. As discussed in Box (1988), it may be possible to divide the factors such that only a subset of the factors will have a dispersion effect while a disjoint subset will have a location effect. In such cases, the location and dispersion are not confounded by use of the signal-to-noise ratio. In contrast, the smaller-the-better and larger-the-better signal-to-noise ratios confound the location and dispersion.

Recently, Mackertich et al. (2003) suggested an alternative method for detecting dispersion effects in replicated experiments. This method is analogous to that of Levene (1960) in that the authors proposed applying a function to each individual observation such that each of the transformed observations provide a measure of the dispersion. Analysis of variance is then performed on the modified data. The advantage of this alternative method is that, for r replicate observations at each of v treatment combinations, the total rv degrees of freedom is conserved since the total rv modified observations are used. The intention is that preservation of the degrees of freedom will increase the probability of detecting a dispersion effect.

One specific function proposed by Mackertich et al. (2003) is the absolute deviation from the mean, which is the same transformation proposed by Levene (1960) for testing homogeneity of variance. A second alternative function examined by Mackertich et al. (2003) is the absolute deviation from the mean raised to the 0.42 power. The power transformation is based on the Kullback-Leibler information and is included as a transformation of the function to achieve approximate normality.

Mackertich et al. (2003) used simulation to rank order the effectiveness of seven traditional methods and their two proposed alternative methods for the detection of dispersion effects. Phase I of the current work replicates and extends this simulation work; see Chapter 13 and Chapter 14. Additional simulations are performed in Phase II to determine empirical critical values and study the Type I error probabilities and the power of selected measures based on the empirical critical values under varying models. The final result is a recommendation of robust and powerful dispersion measures for identifying dispersion effects in replicated experiments.

CHAPTER 12

MODEL

Throughout the current work, the observation for the j^{th} replicate of the i^{th} treatment combination is generated according to the following location-dispersion model:

$$Y_{ij} = \mu_i + \epsilon_{ij} \quad \begin{array}{c} i = 1, 2, \ldots, I \\ j = 1, 2, \ldots, r \end{array} \tag{12.1}$$

where

$$\mu_i = \mathbf{x}'_{\mu,i}\beta \tag{12.2}$$

and $\mathbf{x}'_{\mu,i}$ is the i^{th} row of the location model matrix $\mathbf{X}_\mu$ corresponding to treatment level combination i. The error variables, ϵ_{ij}, are assumed to be independent and to have identical distributions with mean zero and variance σ_i^2, where

$$\sigma_i = g(\mathbf{x}'_{\sigma,i}\gamma). \tag{12.3}$$

and $\mathbf{x}'_{\sigma,i}$ is the i^{th} row of the dispersion model matrix $\mathbf{X}_\sigma$ corresponding to treatment level combination i. Except where indicated, the normal distribution is the assumed distribution for the ϵ_{ij} (i.e., $\epsilon_{ij} \sim N(0, \sigma_i^2)$) . Two different forms of (12.3) are studied in the current work: an additive dispersion model with $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$ and a multiplicative dispersion model with $g(\mathbf{x}'_{\sigma,i}\gamma) = exp(\mathbf{x}'_{\sigma,i}\gamma)$.

12.1 Additive Dispersion Model, $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$

The Phase I model is taken from Mackertich et al. (2003) and assumes an additive dispersion model

$$\sigma_i = \mathbf{x}'_i\gamma. \tag{12.4}$$

In Phase I, values of γ_i are specified and fixed. (See Chapter 15 for the specific location and dispersion models assumed.) The selected values of γ_i produce $\sigma_i > 0$ for all i in these simulations. In general, however, the additive dispersion model would not guarantee positive σ_i for all i.

12.2 Multiplicative Dispersion Model, $g(\mathbf{x}'_{\sigma,i}\gamma) = exp(\mathbf{x}'_{\sigma,i}\gamma)$

For Phase II of the current work, the multiplicative dispersion model is assumed, with

$$\sigma_i = exp(\mathbf{x}'_i\gamma) \tag{12.5}$$

(see for example Wolfinger and Tobias (1998)). One significant advantage of the variance model (12.5) is that the standard deviations σ_{ij} are necessarily positive. The bounded nature of the likelihood function using (12.5) was cited by Harvey (1976) as one of three reasons why this multiplicative variance model is attractive; Harvey (1976) also cites the simpler form of the likelihood ratio test for the multiplicative model and consistency of the dispersion effect estimators. The multiplicative variance model has been supported and used by researchers including Harvey (1976), Cook and Weisberg (1983), Aitkin (1987), Verbyla (1993), and Wolfinger and Tobias (1998).

CHAPTER 13

MEASURES FOR PHASE I

Mackertich et al. (2003) examined a total of nine dispersion measures: seven "traditional" measures and two proposed alternative measures. Let $y_{i1}, \ldots, y_{in_i}$ be a random sample of observations on a random variable Y_i from a distribution with mean μ_i and variance σ_i^2. Except where indicated, the random variable Y_i is assumed to follow a normal distribution, $N(\mu_i, \sigma_i^2)$. The seven traditional dispersion measures studied were:

T1. Within-run sample standard deviation, $s = \sqrt{\frac{1}{n_j-1}\sum_{j=1}^{n_j}(y_{ij} - \bar{y}_{i\cdot})^2}$

T2. Within-run sample variance, $s^2 = \frac{1}{n_j-1}\sum_{j=1}^{n_j}(y_{ij} - \bar{y}_{i\cdot})^2$

T3. Natural logarithm of the within-treatment sample standard deviation (plus 1.0), $ln(s+1)$

T4. Nominal-the-best signal-to-noise ratio, $S/N_{N1} = 10log(\frac{\bar{y}^2}{s^2})$

T5. Alternative nominal-the-best signal-to-noise ratio, $S/N_{N2} = -10log(s^2)$ In this project, S/N_{N2} is modified to $ln(s^2+1)$

T6. Smaller-the better signal-to-noise ratio, $S/N_S = -10log(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{y_i^2})$

T7. Larger-the-better signal-to-noise ratio, $S/N_L = -10log(\frac{1}{n}\sum_{i=1}^{n} y_i^2)$

The two proposed alternative dispersion measures studied were:

M1. Absolute deviation from the within-treatment mean, $|y_{ij} - \bar{y}_i|$

M2. Absolute deviation from the within-treatment mean raised to the 0.42 power, $|y_{ij} - \bar{y}_i|^{0.42}$

In Phase I of the current work, the same seven traditional dispersion measures are examined (see Section 13.1) and, in addition, thirty alternative dispersion measures are examined. The alternative dispersion measures are divided into four groups based on:

1. absolute deviation from the mean, $|y_{ij} - \bar{y}_i|$ (Section 13.2)
2. absolute deviation from the median, $|y_{ij} - \tilde{y}_i|$ (Section 13.3)
3. residuals, $y_i - \mathbf{x}_i'\hat{\beta}$ (Section 13.4)
4. absolute deviation from a trimmed mean, $|y_{ij} - \bar{y}_{i\cdot -k}|$, where k is the number of observations trimmed from each tail, for $k = 1, 2$ (Section 13.5)

13.1 The Traditional Measures

The sample variance, s^2, provides an intuitive, unbiased estimate of the true variance, σ^2. Similarly, the sample standard deviation, s, provides an intuitive estimate of the true standard deviation, σ. Thus, these sample statistics are natural choices for the identification of dispersion effects.

Even when the response variable follows a normal distribution, neither the sample variance nor the sample standard deviation has a normal distribution. The violation of this assumption creates an issue with using the analysis of variance procedure to analyze these two measures. Bartlett and Kendall (1946) showed that, when the response variable follows a normal distribution, the natural logarithm of the sample variance follows approximately a normal distribution. Therefore, analysis of the natural logarithm of the standard deviation is preferred to analysis of the standard deviation when testing for homogeneity of variance. Bartlett and Kendall (1946) stated that "the transformation may be safely used of $n = 10$ and over, more tentatively from $n = 5$ to $n = 9$, and probably not at all below $n = 5$." Use of the natural logarithm of the sample standard deviation is likewise an attempt to attain a transformed observation which follows approximately a normal distribution. Before taking the logarithm of both the sample variance and the sample standard deviation, 1.0 is added to each measure in order to prevent problems due to zero values resulting from rounding.

The three signal-to-noise ratios T4, T6, and T7 were proposed by Taguchi (1986) as a performance measure for quality improvement. Each ratio combines the measurement of location and dispersion. The nominal-the-best signal-to-noise ratio is used for cases when the goal is to achieve a specified target value while minimizing the variability about the target. Generally, this is the goal of most experiments, including the experiments simulated in this project. The smaller-the-better signal-to-noise ratio and the larger-the-better signal-to-noise ratio are designed to achieve the smallest or largest response, respectively, while minimizing variability. As neither of these is the goal of the simulated experiments in this study, it is expected that neither of these ratios will perform well. However, they are included in the simulations in order to compare to the results of the present study with those of Mackertich et al. (2003).

13.2 Measures Based on the Within-Treatment Mean

The absolute deviation from the mean, $|y_{ij} - \bar{y}_i|$, was proposed as a measure of dispersion by Mackertich et al. (2003). The absolute deviation from the mean was the original measure proposed by Levene (1960) for testing for homogeneity of variance.

The mean of the distribution is a measure of the center of the distribution and distance from the mean provides a measure of the spread of the distribution. (By definition, the variance of the distribution is the average of the squared distances from the mean.) The greater the sum of the absolute values of the distances (i.e. the magnitudes of the distances), the greater the variability of the distribution. Thus, each individual deviation from the mean provides information about the variability.

The use of deviation from the mean in place of the variance to identify dispersion effects is analogous to the use of individual observations in place of sample means to identify location effects. In each case, the individual components are used for analysis.

If a full model (i.e. a model with the maximum number of estimable effect parameters included) is fit to the data, the fitted values, $\hat{\beta}' x_j$, are equal to the within-treatment mean, $\bar{y}_{i\cdot}$. The advantage in fitting the full location model is that all location effects, both active and inactive, are fit and, thus, there is no confounding effect of unidentified location effects. In this case, the issue raised by Pan (1999) with respect to unidentified location effects does not apply. However, in this case, small location effects are included in the model, leading to decreased efficiency of the test (Pan 1999).

The disadvantage of using the absolute deviations from the within-treatment means is that these measures are dependent; the r^{th} deviation from the mean can be derived from the other $r - 1$ deviations from the mean. In order to remove the complete dependence, we could use $r-1$ of the r absolute deviations from the mean, where r is the number of replicates per treatment. The disadvantage of this methodology would be the loss of one degree of freedom. However, this loss would not be as great as the loss using traditional measures. The question is whether the removal of complete dependence or the preservation of one degree of freedom is the greater advantage. To find an answer to this question, the set of $r-1$ absolute deviations from the mean was included in the simulations in addition to the complete set of r absolute deviations from the mean.

Starting from the assumption of that the original observations, y_{ij}, come from a normal distribution, it can be shown that the distribution of $|y_{ij}-\bar{y}_i|$ is half-normal with mean $\mu = \sqrt{2(n-1)\sigma_Y^2/n\pi}$ and variance $\sigma^2 = ((n-1)\sigma_Y^2/n)(1-(2/\pi))$, where σ_Y^2 is the variance of the Y_i. As stated above, the $|y_{ij}-\bar{y}_i|$ are not independent. However, Levene (1960) showed that the order of the correlation is n^{-2}, and assumed it to have little effect.

Various power transformations of $|y_{ij}-\bar{y}_i|$, especially $|y_{ij}-\bar{y}_i|^{0.42}$ studied by Mackertich et al. (2003), are included in the current study. It can be shown that the distribution of $Z = |y_{ij}-\bar{y}_i|^a$ is

$$f(z) \quad = \quad \frac{2}{\sqrt{2\pi}\sqrt{\sigma^2}}\frac{1}{a}z^{\frac{1}{a}-1}exp(-\frac{z^{\frac{2}{a}}}{2\sigma^2}). \tag{13.1}$$

When $a=2$,

$$f(z) \quad = \quad \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2}}z^{-\frac{1}{2}}exp(-\frac{z}{2\sigma^2}) \tag{13.2}$$

so that $Z \sim Gamma(1/2, 2\sigma^2)$. The variance of the $Gamma(1/2, 2\sigma^2)$ distribution is $(1/2)\times 2^2 = 2.0$ (see Casella and Berger (1990) Section 3.2) which matches the variance calculated by Levene (1960) since the third moment $\gamma_2 = 0$ for the normal distribution.

13.2.1 Power Transformations

Six power transformations of the absolute deviation from the within-treatment mean are examined in Phase I: 0.35, 0.42, 0.55, 1.5, 2, and 5.

The square of the function, $|y_{ij} - \bar{y}_i|^2$, is proposed for several reasons. Squaring will make large values larger while leaving small values relatively unchanged, thereby magnifying the dispersion information in the transformed data.

The power 0.42 was originally proposed by Mackertich et al. (2003) as the optimal power for $|y_{ij} - \bar{y}_i|$. The exponent value 0.42 is based on the Kullback-Leibler information. The Kullback-Leibler information is a measure of the similarity of two distributions, in this case the distribution of $|y_{ij} - \bar{y}_i|$ and the normal distribution. By raising data from one distribution to a power equal to the Kullback-Leibler information, the transformed data more closely approximate the second distribution (Kullback and Leibler 1951).

The power values of 0.35 and 0.55 were given by Mackertich et al. (2003) as the endpoints of the range of power transformations applied to the absolute deviation from the mean in order to induce random variables that follow approximately a normal distribution. The absolute deviation from the mean raised to the 0.35 and 0.55 powers are included to test the sensitivity of the 0.42 power transformation to the choice of power.

In order to extend the range of powers, the exponent values 1.5 and 5 are included in the analysis.

13.3 Measures Based on the Within-Treatment Median

The absolute deviation from the median, $|y_{ij} - \widetilde{y}_i|$, is similar to the variant of Levene's Test proposed by Brown and Forsythe (1974) (see Section 11.1). Like the mean, the median is a measure of the center of the distribution. If the distribution of the responses, y_{ij}, is symmetric, then the mean and the median are equal. However, the median is less affected than the mean by extreme observations. Thus, for extremely skewed response distributions, the median may be a better estimate of the center of the distribution and may lead to a more robust test for dispersion. The robustness in testing for homogeneity of variance resulting from use of the median in place of the mean is supported by the results of Conover et al. (1981).

The same six power transformations listed in Section 13.2.1 for the absolute deviation from the mean are applied to the absolute deviation from the median.

13.4 Measures Based on the Residuals and Absolute Residuals

For the linear model discussed in Chapter 14 and $\mathbf{x}_i'$ the row of the model matrix $\mathbf{X}$ corresponding to treatment level combination i, the residuals are equal to $y_i - \mathbf{x}_i'\hat{\beta}$. The residuals, then, are a measure of the fit of the model to the data; the residuals measure the difference between the observed data and the model predictions. Large residuals indicate a large variability about the mean response. Differences between the averages of the magnitudes of the residuals at the different level settings of the factor would be an indicator of a possible dispersion effect.

With a replicated experiment, the residuals can be analyzed as individual observations of the dispersion. This methodology can be viewed as an extension of the methods of Box and Meyer (1986) and Bergman and Hynén (1997) to replicated experiments, with the location-dispersion confounding eliminated as a result of the replication.

Different location models can be fit to the data; each different model provides different residuals for analysis. In applications, the true location model is not known. The question is then which location model to fit to the data. It is this question and the fact that it is not absolutely answered that lead to the warning of the effect of unidentified location effects by Pan (1999). As with unreplicated experiments, if the location model is selected based on an imperfect criterion, unidentified location effects may not be included in the model. These unidentified location effects can affect the subsequent dispersion analysis. Conversely, if misidentified nonexistent location effects are included, the efficiency of dispersion detection is decreased (Pan 1999).

Because of the question associated with the selection of a location model and the uncertainty of the impact of fitting the incorrect location model, residuals from more than one model are examined in Phase I of this work. The first location model fit corresponds to perfect knowledge of the true location effects (but not their size). In application, it is possible that the exact location model might be fit as a result of correct identification of the true location effects. The second location model fit to the data is a main effects model, representing the simplest location model that might realistically be fit in practice. The main effects location model may exclude significant location effects and include small location effects. The saturated location model (i.e. the location model including all main effects and two-factor interactions) is also considered in the current work since residuals from fitting the saturated location model are equivalent to absolute deviations from the mean.

If the assumptions of the standard linear model are not violated, the residuals follow a normal distribution with mean $\mu_{R_i} = 0$ and variance $\sigma^2_{R_i} = \sigma^2_Y(1 - p_{ii})$, where σ^2_Y is the variance of the Y_i and p_{ii} is the i^{th} diagonal element of $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. The residuals are not independent since they are based on the fitted values, which are all based on the same fitted model. However, when the sample size is large with respect to the number of parameters in the model, the dependence of the residuals is nonsignificant and can be ignored (Neter, Wasserman and Kutner 1990).

The absolute values of the residuals, $|y_i - \mathbf{x}_i'\hat{\beta}|$, are also examined in this work. By taking absolute value, both large positive residuals and large negative residuals will represent large dispersion. The absolute value residual is equivalent to the absolute deviation from the mean when a saturated location model is fit to the data.

Starting from the known distribution of the residuals, it can be shown that the absolute residuals follow a half-normal distribution with mean $\mu_{AR_i} = \sqrt{\pi/2\sigma^2_{R_i}}$ and variance $\sigma^2_{AR_i} = \sigma^2_{R_i}(1 - 2/\pi)$ and are not independent. The probability density function of $Z = |y_i - \mathbf{x}_i'\hat{\beta}|^a$ is equal to that of $|y_{ij} - \bar{y}_i|$, whose probability density function given in 13.1, with σ^2_i replaced by $\sigma^2_{R_i}$; in the case of $a = 2$, the probability density function is equal to that of $|y_{ij} - \bar{y}_i|^2$, whose probability density function is given in 13.2, with σ^2_i replaced by $\sigma^2_{R_i}$, giving $Z \sim Gamma(1/2, 2\sigma^2_{R_i})$

Similar to the case of the absolute deviations from the mean, the non-independence of the transformed observations using a function of the residuals is a violation of the most important assumption of the ANOVA procedure. Such a violation makes the use of F-tests and F-distribution critical values invalid.

Again, the same six power transformations listed in Section 13.2.1 for the absolute deviation from the mean are applied to the absolute residuals.

13.5 Measures Based on the Within-Treatment Trimmed Mean

Results from pilot simulations for the current work indicated the absolute deviation from the within-treatment median as a leading choice of dispersion measure. In the case of $r = 4$ replicates per treatment, the median is equal to the mean of the two middle observations, which is equal to the the trimmed mean with one observation trimmed from each tail. It is, thus, unclear whether it is the properties of the median or of the trimmed mean that are at work in providing a good dispersion measure.

The trimmed mean, $|y_j - \bar{y}_{-k}|$, is less influenced than the mean by extreme observations since it eliminates the most extreme observations from each tail. Because it is the average of more than one central observation, the trimmed mean utilizes more information than does the median. Therefore, deviation from the within-treatment trimmed mean will be examined. This measure is similar to the variant of Levene's Test proposed by Brown and Forsythe (1974) as discussed in Section 11.1.

Similar to the other measures, the same six power transformations listed in Section 13.2.1 are applied to the absolute deviation from the median.

13.6 Other Dispersion Measures

Two additional alternative dispersion measures are proposed and examined. The within-treatment replicates are divided randomly into two approximately equal-sized groups. The resulting measure is called $[ln(s + 1)]_{HALF}$. The natural logarithm of the standard deviation (plus one) is then calculated for each group and used as the response measure. This measure is similar to the Box Test with $k = 2$ groups (Box 1953). By computing $ln(s + 1)$ for each half, two independent statistics are calculated from the r replicates within each treatment instead of only one. So $[ln(s + 1)]_{HALF}$

captures the good qualities of $ln(s+1)$ while hopefully increasing overall power of the test by increasing the number of total observations. However, $[ln(s+1)]_{HALF}$ reduces the number of observations from the original r to two, thereby still reducing degrees of freedom from the total possible. As with the Box Test, there is a loss of information due to splitting the samples. Also similar to the Box Test, the results from $[ln(s+1)]_{HALF}$ are not unique depending on the division of the replicates into groups.

To retain all original degrees of freedom, $[ln(s+1)]_{ALL}$ is proposed. For $[ln(s+1)]_{ALL}$, all groups of $r-1$ within-treatment replicates are formed and $ln(s+1)$ is calculated for each group. By calculating $ln(s+1)$ for all groups of $r-1$ replicates, a total of r dispersion observations are created. Thus, there is no loss of degrees of freedom. However, since $ln(s+1)$ is computed for all groups of $r-1$, the r dispersion measures are not independent.

CHAPTER 14

PHASE I STUDY SIMULATION

For all simulations in the current study, a 2_V^{5-1} fractional factorial design was used. The single replicate design matrix is shown in Table 14.1. This design supports a full second-order model, allowing for the independent estimation of all main effects and two-factor interaction effects (see, for example, Dean and Voss (1999) Chapter 15).

The procedure for simulating data is as follows. Values of μ_i are calculated according to (12.2) with

$$\beta = \begin{bmatrix} 100 & 10 & -5 & 7 & 0 & 0 & 0 & 0 & 5 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}' \tag{14.1}$$

for each $i = 1, 2, \ldots, 16$ and the treatment combinations in Table 14.1. Similarly, values for σ_i are generated according to (12.3) with $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$,

$$\gamma = \begin{bmatrix} 10 & 1 & 1.5 & -1 & \gamma_4 & 0.75 & 0 & 0 & 0.5 & 0 & -0.75 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}' \tag{14.2}$$

and the value of γ_4 varying from zero to four in increments of 0.05. A vector of rv random variates, ϵ_{ij}, is then generated using a random $N(0,1)$ number data generator intrinsic to the IMSL library in FORTRAN. (See *IMSL Fortran Library User's Guide: STAT/LIBRARY Volume 2 of 2* (1994-2003) Chapter 18 for more information regarding data generators.) The vector of random variates, ϵ_{ij}, is standardized so that

X_1	X_2	X_3	X_4	X_5
-1	-1	-1	-1	1
-1	-1	-1	1	-1
-1	-1	1	-1	-1
-1	-1	1	1	1
-1	1	-1	-1	-1
-1	1	-1	1	1
-1	1	1	-1	1
-1	1	1	1	-1
1	-1	-1	-1	-1
1	-1	-1	1	1
1	-1	1	-1	1
1	-1	1	1	-1
1	1	-1	-1	1
1	1	-1	1	-1
1	1	1	-1	-1
1	1	1	1	1

Table 14.1: Single replicate 2_V^{5-1} design matrix used for simulations

the mean is exactly zero and the variance exactly one; if the ϵ_{ij} are not standardized, the exact mean and variance of the within-treatment samples can vary considerably about the intended values, affecting the simulation results. The simulated data value, y_{ij}, is produced by multiplying σ_i by ϵ_{ij} and adding μ_i. Once the data are simulated, each dispersion measure is calculated.

The calculated dispersion measures described in Chapter 13 are used as the response data for analysis. Following Mackertich et al. (2003), the model fit to the dispersion measure data is the exact dispersion model used to generate the data, (i.e. the model including effects of X_1, X_2, X_3, X_4, X_5, X_2X_3, and X_1X_4), assuming perfect knowledge of the effects in the model. As a result of fitting only the known dispersion effects in the model, the power of the test should be high.

A test of the null hypothesis

$$H_0 : \gamma_4 = 0 \tag{14.3}$$

is completed using the test statistic

$$M = \frac{(\mathbf{A}\hat{\gamma})'(\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{A}')^{-1}(\mathbf{A}\hat{\gamma})}{(f(\mathbf{Y}) - \mathbf{X}\hat{\gamma})'(f(\mathbf{Y}) - \mathbf{X}\hat{\gamma})} \tag{14.4}$$

(Scheffé 1959) where $f(\mathbf{Y})$ is the vector of the transformed observations (e.g. $|y_{ij} - \bar{y}_i|$, $|y_i - \mathbf{x}_i'\hat{\beta}|$, $|y_{ij} - \tilde{y}_i|$, etc.) and $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'$. Each calculated M is compared to a critical value.

Independent sets of of data are generated 100,000 times for each value of γ_4. Based on each of the 100,000 simulations, the total number of times the test rejects is counted. In the case that $\gamma_4 = 0$, the proportion of times the test rejects is a measure of the Type I error rate. In all other cases (i.e. $\gamma_4 \neq 0$), the proportion of times the test rejects is a measure of the power to detect the active dispersion effect, γ_4.

Two different critical values are used to conduct each test. First, each calculated M is compared to a critical value from the F-distribution; the F-distribution with 1 and $v - p = 16 - 8 = 8$ degrees of freedom is used for tests based on the traditional measures while the F-distribution with 1 and $rv - p = (4 \times 16) - 8 = 56$ degrees of freedom is used for tests based on the alternative measures. (For these tests, $v = 16$ is the number of treatment combinations, $r = 4$ is the number of replicates per treatment combination, and $p = 8$ is the number of parameters estimated in the model. For Phase I, only the overall variance and dispersion effects known to be active are included in the model.) Tests against the F-distribution critical values are conducted to compare with the results of Mackertich et al. (2003); the results of these tests are not considered in this work, except with respect to the stability of the simulations (see Section 15.2).

The test against the F-distribution critical value assumes that the calculated dispersion data met the assumptions of the analysis of variance procedure. It is known that some of the measures violate at least one of the analysis of variance assumptions (see Chapter 13), and this test may not be exactly appropriate. In addition to testing against the F-distribution, an empirical critical value is used. An $\alpha = 0.05$ significance level empirical critical value for each dispersion measure is determined from the 100,000 calculated M values as the 95^{th} percentile value of the calculated test statistics with $\gamma_4 = 0$ and $N(0, \sigma_i^2)$ errors. For each Phase I simulation, a unique critical value is determined for each measure.

Simulations are run for different numbers of replicates, r, and different distributions of ϵ_{ij} in Model (12.1). Generating from a normal error distribution, the number of replicates at each treatment combination is varied over the range $r = 2$ to $r = 10$. With $r = 4$, four different error distribution are considered: Normal(0,1), Beta(1/2,1/2), Cauchy(0,1), and Exponential(1). For each error distribution, the random error variates are always standardized to guarantee that the mean is exactly zero and the variance is exactly one to prevent additional, uncontrolled variability in the generated observations.

Finally, with $r = 4$ and a normal error distribution, an active location effect of X_4 is included in the location model for additional simulations. When the active location effect is included in the location model, residuals are calculated based on the known model excluding X_4 in order to determine the effect on the residuals of an unidentified active location effect, as well as the exact known model, main effects model, and the saturated model.

Plots of selected power curves will be presented in Section 15.1, where results will be discussed. In Section 15.1, the Phase I simulation results will be used to rationalize reduction of the list of measures for consideration in Phase II. Tables of results from the Phase I simulations are available upon request.

CHAPTER 15

PHASE I STUDY RESULTS

15.1 Phase I Results

The results of the Phase I simulations are used to identify promising dispersion measures for further study in Phase II under less specific models. In this section, the discussion of the selected results from Phase I focus on dividing the original list of measures (given in Chapter 13) into two groups: one group to be studied in Phase II and one group to be eliminated.

As the goal of this work is to identify dispersion measures that increase the power of effect detection as compared to the current traditional measures, each alternative measure is compared to a selected traditional measure. Results for the traditional measures are, therefore, discussed first (Section 15.1.1). Discussion of the results for the proposed alternative measures are then organized into the four basic measures groups in order of power: residuals (Section 15.1.2), absolute deviation from the mean (Section 15.1.3), absolute deviation from the median (Section 15.1.4), absolute deviation from the trimmed mean (Section 15.1.5), and other measures (Section 15.1.6).

To compare the dispersion measures, the power to detect the dispersion effect of interest, γ_4, across the range of effect sizes using the test against the empirical critical

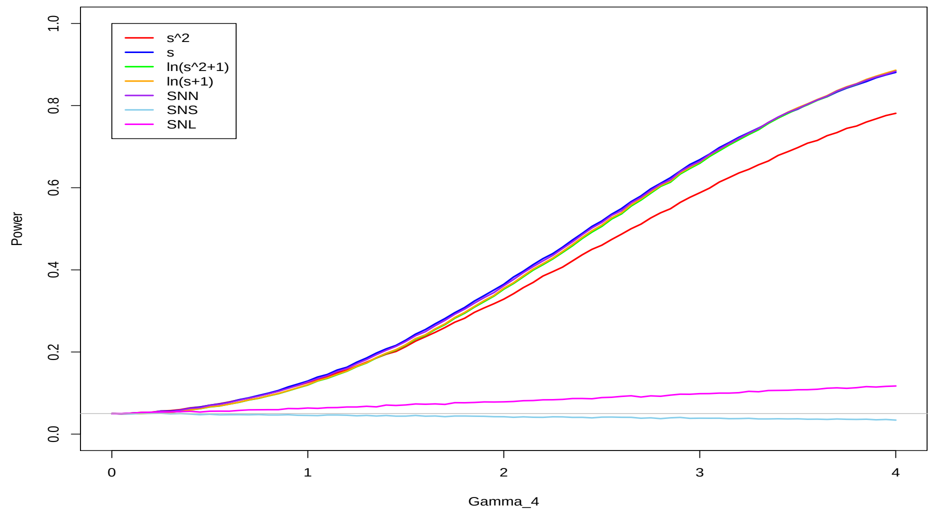

Figure 15.1: Power curves for tests using traditional measures with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 4$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

values to achieve an $\alpha = 0.05$ significance level is used as the standard for judgement. The size of the error rate for tests against the F-distribution critical values is not considered since tables of empirical critical values can be constructed based on Monte Carlo simulation.

15.1.1 Traditional Measures

In this section the performance of the traditional measures is examined. The power curves, based on empirical critical values, for the seven traditional measures are given in Figure 15.1. These power curves are for simulations of a normal error distribution with $r = 4$ replicates per treatment using the model and procedures described in Chapter 14.

As discussed in Chapter 13, it is not expected that the smaller-the-better and the larger-the-better signal-to-noise ratios will perform well in this simulation study. Based on Figure 15.1, it is clear that these two signal-to-noise ratios provide extremely low power for detecting the dispersion effect across all sizes of the effect. The poor performance of these two measures is consistent across all numbers of replicates and across all error distributions. The performance of the smaller-the-better and the larger-the-better signal-to-noise ratios differs when a location effect of X_4 is introduced into the location effects; when $\beta_4 > 0$, both signal-to-noise ratios are powerful to reject the null hypothesis of $\gamma_4 = 0$, even when in truth $\gamma_4 = 0$. These results are, again, not surprising as the signal-to-noise ratio confounds the location and dispersion effect. When a location effect exists, the signal-to-noise ratios are identifying the location effect as the dispersion effect. Therefore, the smaller-the-better signal-to-noise ratio and the larger-the-better signal-to-noise ratio are eliminated from further study.

From Figure 15.1, the within-treatment variance, s^2, has lower power to detect the dispersion effect than the other traditional measures (i.e. s, $ln(s^2+1)$, $ln(s+1)$, S/N_N). The difference in performance between s^2 and the other traditional measures increases as the size of the dispersion effect increases; for the largest effect sizes examined, the difference in performance decreases as the number of replicates per treatment increases. Therefore, the within-treatment variance, s^2 is not studied further.

Traditionally, $ln(s+1)$ is the measure of choice for the type of dispersion effect analysis studied here (based on the work of Bartlett and Kendall (1946)). Based on the current work, $ln(s+1)$ ranks among the best traditional measures for the Phase I simulations. The within-treatment standard deviation, s, the natural logarithm of the

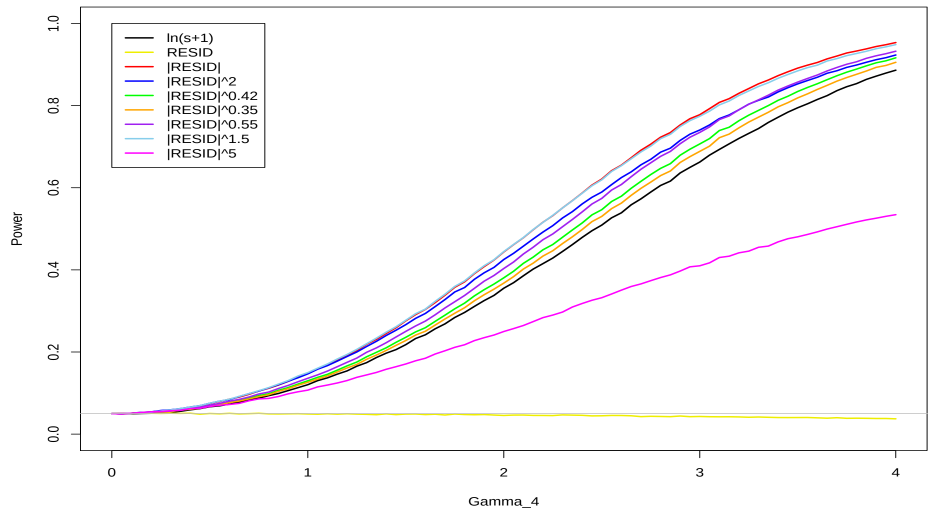

Figure 15.2: Power curves for functions of the residuals fitting known location effects with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 4$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

within-treatment variance, $ln(s^2 + 1)$, the natural logarithm of the within-treatment standard deviation, $ln(s + 1)$, and the nominal-the-better signal-to-noise ratio, S/N_N, all perform similarly according to Figure 15.1. This pattern of similarity exists for all $r \geq 4$ replicates per treatment. For $4 \leq r \leq 8$ replicates, s has the greatest power among the traditional measures when $\gamma_4 = 1$ and $\gamma_4 = 2$; $ln(s + 1)$ has the greatest power when $\gamma_4 = 2$ and $\gamma_4 \geq 3$ for $r > 8$. Because no traditional measure clearly dominates, s, $ln(s + 1)$, $ln(s^2 + 1)$, and S/N_N are all studied in Phase II. For comparison of Phase I results, $ln(s + 1)$ is used as the standard for evaluation of the alternative measures.

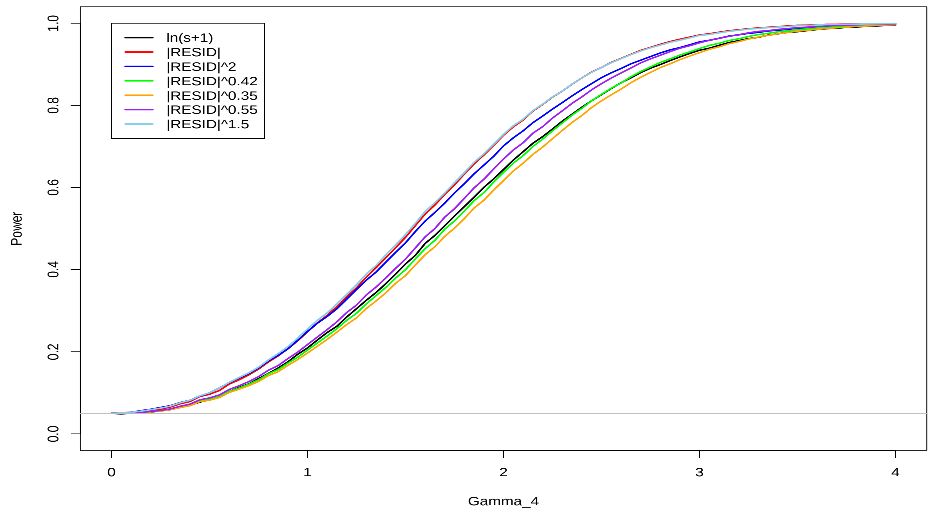

Figure 15.3: Power curves for functions of the residuals fitting known location effects with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 7$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

15.1.2 Absolute Residuals, $|y_i - \mathbf{x}'_i\hat{\beta}|$

Assuming the true location effects are known or can be identified exactly, Figure 15.2 shows the power curves for the residuals, $y_i - \mathbf{x}'_i\hat{\beta}$, the absolute residuals, $|y_i - \mathbf{x}'_i\hat{\beta}|$, and the power transformations of $|y_i - \mathbf{x}'_i\hat{\beta}|$. The absolute residuals and five of the six power transformations show increased power to detect the dispersion effect compared to $ln(s+1)$; only $y_i - \mathbf{x}'_i\hat{\beta}$ and $|y_i - \mathbf{x}'_i\hat{\beta}|^5$ show significantly decreased power compared to $ln(s+1)$ and will be eliminated from further consideration.

The absolute residuals, $|y_i - \mathbf{x}'_i\hat{\beta}|$, and $|y_i - \mathbf{x}'_i\hat{\beta}|^{1.5}$ were found in the Phase I simulations to have the greatest power for detecting the dispersion effect compared to all other measures studied for $r = 4$; $|y_i - \mathbf{x}'_i\hat{\beta}|^{1.5}$ has greater power for smaller

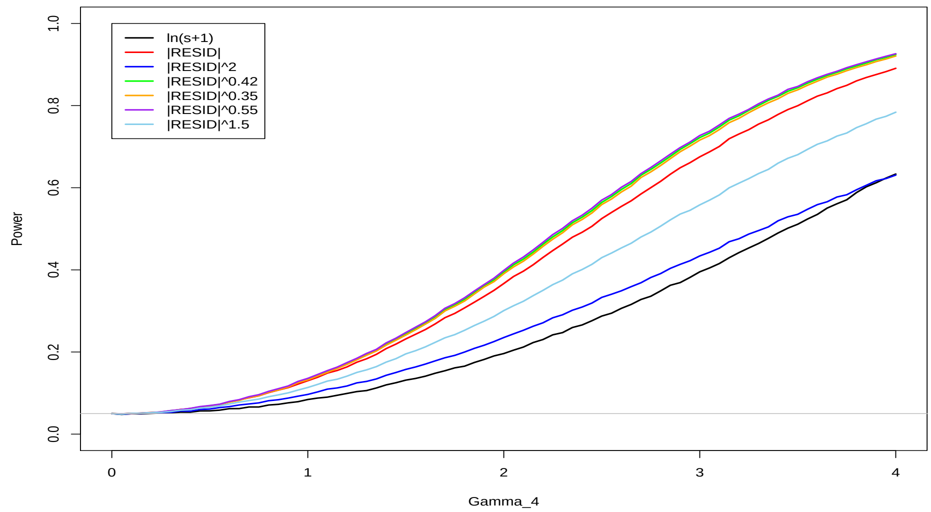

Figure 15.4: Power curves for functions of the residuals fitting known location effects with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 4$ replicates, and exponential error distribution, testing against the empirical critical value to control Type I error equal to 0.05

values of γ_4 while $|y_i - \mathbf{x}'_i\hat{\beta}|$ has greater power for larger values of γ_4. Both $|y_i - \mathbf{x}'_i\hat{\beta}|$ and $|y_i - \mathbf{x}'_i\hat{\beta}|^{1.5}$ will be examined in Phase II.

Figure 15.2 shows that $|y_i - \mathbf{x}'_i\hat{\beta}|^2$, $|y_i - \mathbf{x}'_i\hat{\beta}|^{0.42}$, $|y_i - \mathbf{x}'_i\hat{\beta}|^{0.35}$, and $|y_i - \mathbf{x}'_i\hat{\beta}|^{0.55}$ also perform well, with power greater than $ln(s+1)$. However, already for $r = 7$, $ln(s+1)$ shows greater power to detect the dispersion effect than both $|y_i - \mathbf{x}'_i\hat{\beta}|^{0.42}$ and $|y_i - \mathbf{x}'_i\hat{\beta}|^{0.35}$ (Figure 15.3). The change in the ranking of these measures indicates that $|y_i - \mathbf{x}'_i\hat{\beta}|^{0.42}$ and $|y_i - \mathbf{x}'_i\hat{\beta}|^{0.35}$ are not inducing the same gains in power for each additional replicate added to a treatment as are other measures. Though these two measures provide close to the greatest power to detect the dispersion effect when the error distribution is exponential (Figure 15.4), these two measures will be eliminated from further study.

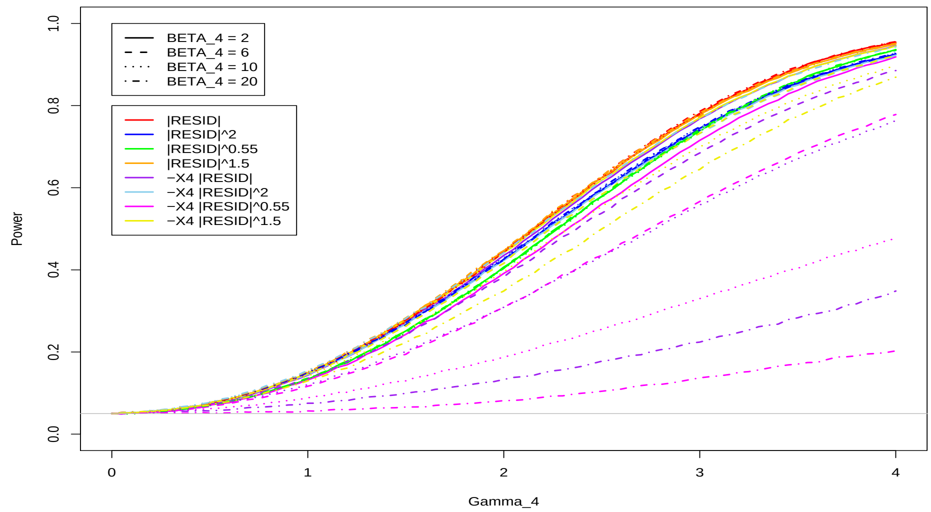

Figure 15.5: Comparison of power curves for functions of the residuals fitting known location effects with data from simulation of model (12.1) with location effects adding $\beta_4 = k$, $k = 2, 6, 10, 20$, to (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 4$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

The $|y_i - \mathbf{x}'_i\hat{\beta}|^{0.55}$ is not studied in Phase II. Though the test using $|y_i - \mathbf{x}'_i\hat{\beta}|^{0.55}$ provides the greatest power when the error distribution is exponential (Figure 15.4), the measure does not perform as well as other measures for other error distributions. Because this measure does not provide any clear benefit compared with $|y_i - \mathbf{x}'_i\hat{\beta}|$, $|y_i - \mathbf{x}'_i\hat{\beta}|^{1.5}$, or $|y_i - \mathbf{x}'_i\hat{\beta}|^2$, $|y_i - \mathbf{x}'_i\hat{\beta}|^{0.55}$ is eliminated from further consideration.

From Figure 15.5, when tests are based on empirical critical values to control Type I error equal to 0.05, the power of the test using $|y_i - \mathbf{x}'_i\hat{\beta}|^2$ is not greatly affected by failing to include the active location effect of X_4 in the location effects; though the test based on $|y_i - \mathbf{x}'_i\hat{\beta}|^2$ has less power to detect the dispersion effect than the test based on $|y_i - \mathbf{x}'_i\hat{\beta}|$ when the exact location effects is fit, the power of the test

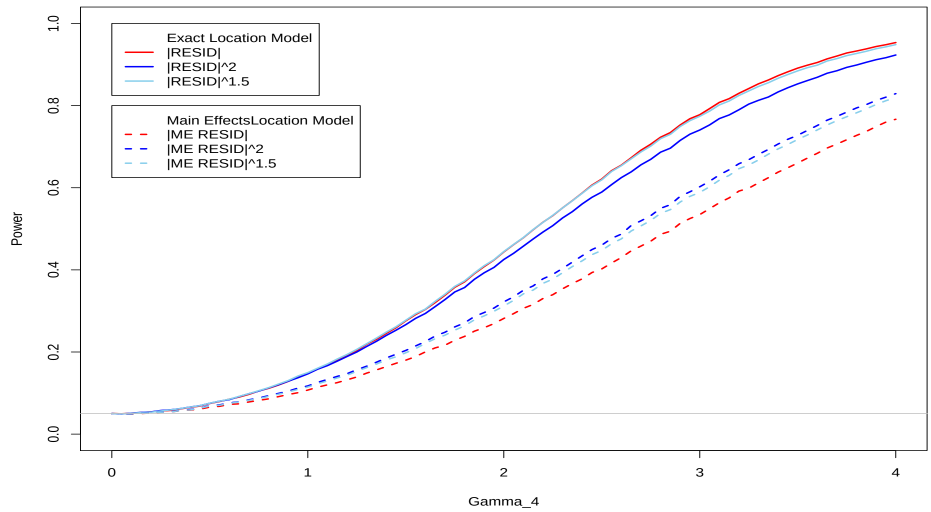

Figure 15.6: Comparison of power curves for functions of the residuals fitting the two different location effects with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 4$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

using $|y_i - \mathbf{x}'_i\hat{\beta}|^2$ is almost constant whether or not the active effect of X_4 is included in the location effects. Due to the robustness of $|y_i - \mathbf{x}'_i\hat{\beta}|^2$ to exclusion of the active location effect, this measure warrants further study in the next stage of simulations.

The preceding discussion is based on the assumption that the exact location effects is known. The power can vary considerably if an incorrect location effects is fit to the data to produce the residuals. Figure 15.6 shows the power curves for the $|y_i - \mathbf{x}'_i\hat{\beta}|$, $|y_i - \mathbf{x}'_i\hat{\beta}|^{1.5}$, and $|y_i - \mathbf{x}'_i\hat{\beta}|^2$ fitting the exact and main effects location models. It is clear from Figure 15.6 that functions of $|y_i - \mathbf{x}'_i\hat{\beta}|$ from fitting the exact model provide greater power than the corresponding functions of $|y_i - \mathbf{x}'_i\hat{\beta}|$ from fitting the main effects model. While this issue of location model selection does not impact the

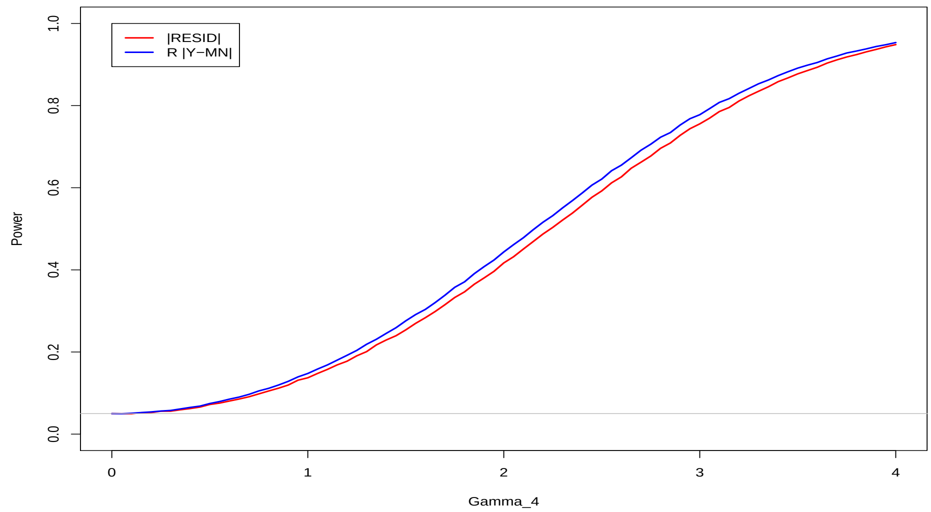

Figure 15.7: Comparison of power curves for residuals fitting known location effects and absolute deviation from the mean with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 4$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

selection of measures for study in Phase II, it is an important consideration in the application of this measure. Identification of the location model for the residuals is examined in Phase II as part of the study of residuals as a practical measure for dispersion effect detection (see Chapter 16).

Based on the Phase I results as presented, $|y_i - \mathbf{x}'_i\hat{\beta}|$, $|y_i - \mathbf{x}'_i\hat{\beta}|^{1.5}$, and $|y_i - \mathbf{x}'_i\hat{\beta}|^2$ are studied in Phase II.

15.1.3 Absolute Deviation from the Within-Treatment Mean, $|y_{ij} - \bar{y}_i|$

Based on the Phase I simulation results for power of the test against the empirical critical values, the absolute deviation from the within-treatment means, $|y_{ij} - \bar{y}_i|$, is

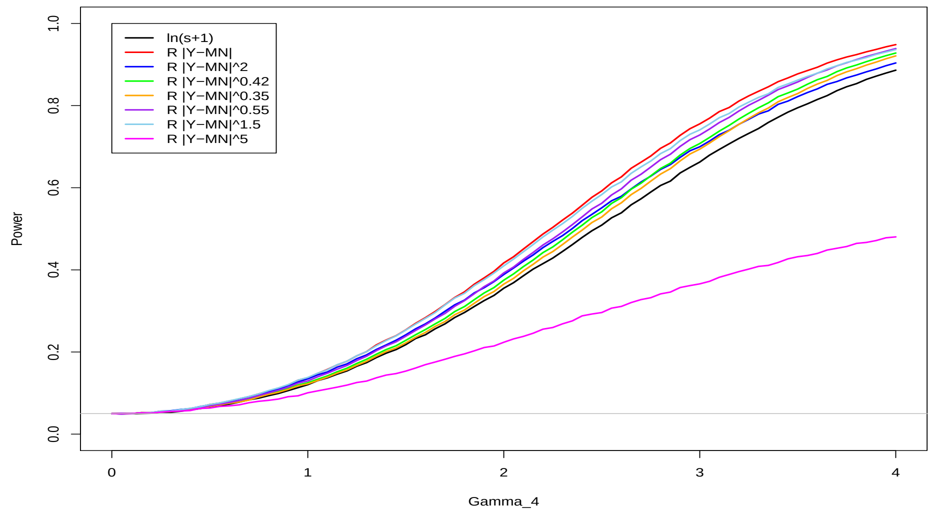

Figure 15.8: Power curves for functions of the absolute deviation from the mean with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 4$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

found to have power close to the power exhibited by the absolute residuals, $|y_i - \mathbf{x}'_i\hat{\beta}|$ (Figure 15.7). From the power curves of the functions of $|y_{ij} - \bar{y}_i|$ shown in Figure 15.8, the test based on $|y_{ij} - \bar{y}_i|$ has greater power to detect the dispersion effect than does $ln(s+1)$. Similar to $|y_i - \mathbf{x}'_i\hat{\beta}|$, five of the six power transformations of $|y_{ij} - \bar{y}_i|$ show increased power of detection compared to $ln(s+1)$, with $|y_{ij} - \bar{y}_i|^{1.5}$ showing the greatest power among these. Only $|y_{ij} - \bar{y}_i|^5$ is dominated by $ln(s+1)$, and thus will not be considered further.

The absolute deviation from the mean is one of the measures proposed by Mackertich et al. (2003). Mackertich et al. (2003) found that the probability of detecting the dispersion effect of interest is consistently higher across all effect sizes for $|y_{ij} - \bar{y}_i|$ than for all of the traditional measures. The increased detection power was observed

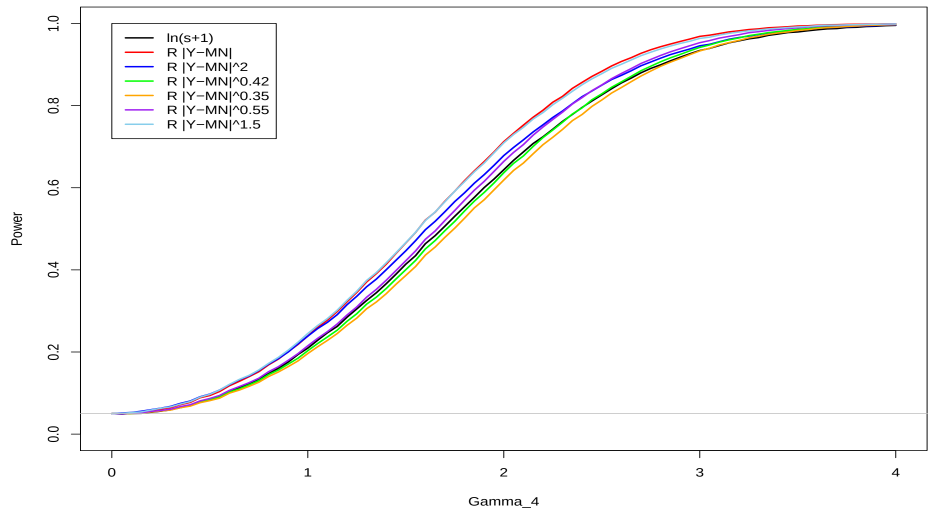

Figure 15.9: Power curves for functions of the absolute deviation from the mean with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 7$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

both for tests against the F-distribution critical value and for tests against the empirical critical values. Consistent with the results of Mackertich et al. (2003), Figure 15.8 shows increased power of $|y_{ij} - \bar{y}_i|$ compared to $ln(s+1)$. The $|y_{ij} - \bar{y}_i|$ measure is among the most powerful measures across all effect sizes for all numbers of replicates studied. For tests against the F-distribution critical value, the (unadjusted) Type I error rate for the test based on $|y_{ij} - \bar{y}_i|$ decreases as the number of replicates per treatment increases, consistent with the results of Mackertich et al. (2003).

Mackertich et al. (2003) also proposed $|y_{ij} - \bar{y}_i|^{0.42}$, suggesting this power transformation would make the distribution of the transformed data more closely approximate a normal distribution. In fact, $|y_{ij} - \bar{y}_i|^{0.42}$ better controls the Type I error rate when testing against the F-distribution critical value than does $|y_{ij} - \bar{y}_i|$, though still above the nominal $\alpha = 0.05$ level. From the power curves shown in Figure 15.8, $|y_{ij} - \bar{y}_i|^{0.42}$ is less powerful than $|y_{ij} - \bar{y}_i|$ for detecting the dispersion effect. As the number of replicates per treatment increases, $|y_{ij} - \bar{y}_i|^{0.42}$ loses the detection power advantage over $ln(s + 1)$ (Figure 15.9). While the results of the simulations indicate that $|y_{ij} - \bar{y}_i|^{0.42}$ is not the optimal measure for identification of dispersion effects, this measure will be carried into Phase II mainly as a continuation of the previous work.

Of the other power transformations of $|y_{ij} - \bar{y}_i|$, the measures $|y_{ij} - \bar{y}_i|^{0.55}$ and $|y_{ij} - \bar{y}_i|^{1.5}$ show power close to $|y_{ij} - \bar{y}_i|$ and greater than $|y_{ij} - \bar{y}_i|^{0.42}$. From Figure 15.9, there is a greater increase in detection power for $|y_{ij} - \bar{y}_i|^{1.5}$ with additional replicates per treatment as compared to $|y_{ij} - \bar{y}_i|^{0.55}$. Therefore, only $|y_{ij} - \bar{y}_i|^{1.5}$ is studied further, while $|y_{ij} - \bar{y}_i|^{0.55}$ is eliminated.

Despite increased power compared to $ln(s + 1)$, $|y_{ij} - \bar{y}_i|^{0.35}$ and $|y_{ij} - \bar{y}_i|^2$ are also discarded due to decreased power relative to $|y_{ij} - \bar{y}_i|$. Only $|y_{ij} - \bar{y}_i|$, $|y_{ij} - \bar{y}_i|^{0.42}$, and $|y_{ij} - \bar{y}_i|^{1.5}$ are examined further, both with respect to Phase I results and Phase II.

The above results are based on use of all r within-treatment values of $|y_{ij} - \bar{y}_i|$. As discussed in Chapter 13, the use of only $r - 1$ of the transformed observations, randomly deleting one value, is examined. Figure 15.10 shows the power curves for the selected functions of $|y_{ij} - \bar{y}_i|$ for both r and $r-1$ of the transformed observations. It is clear that the use of only $r-1$ of the transformed observations provides less power than use of all r transformed observations, and less power than use of $ln(s + 1)$. The power when using only $r - 1$ of the $|y_{ij} - \bar{y}_i|$ is lower across all numbers of replicates

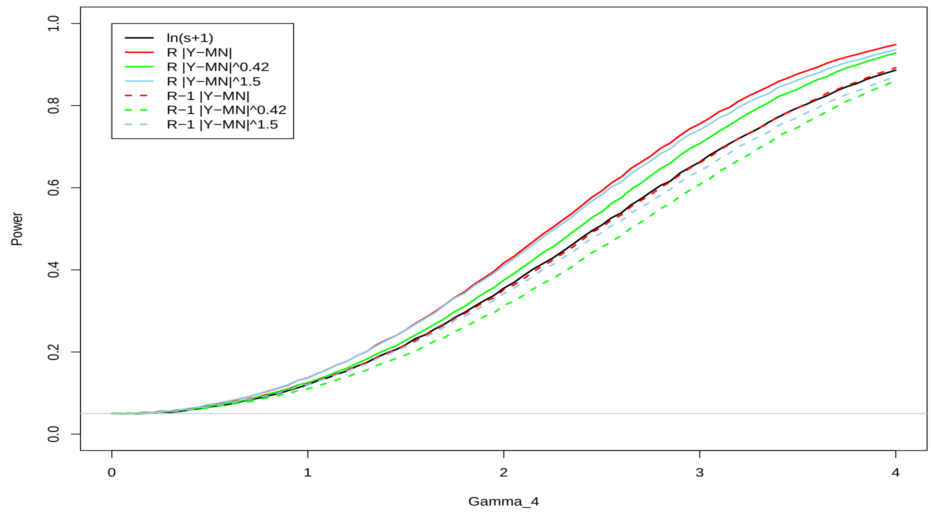

Figure 15.10: Comparison of power curves for r and $r-1$ absolute deviations from the mean with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 4$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

and for all error distributions. Because of the reduction in power, the use of $r-1$ values of the $|y_{ij} - \bar{y}_i|$ is not studied in Phase II of this work.

Based on the Phase I results as presented, all r values of $|y_{ij} - \bar{y}_i|$, $|y_{ij} - \bar{y}_i|^{0.42}$, and $|y_{ij} - \bar{y}_i|^{1.5}$ are studied in Phase II.

15.1.4 Absolute Deviation from the Within-Treatment Median, $|y_{ij} - \tilde{y}_i|$

The absolute deviation from the within-treatment median, $|y_{ij} - \tilde{y}_i|$, is shown in Figure 15.11 to have increased power to detect the dispersion effect over the complete range of effect sizes as compared to $ln(s+1)$; $|y_{ij} - \tilde{y}_i|^{1.5}$ shows increased power compared to $|y_{ij} - \tilde{y}_i|$ over part of the range of effect sizes and increased power compared

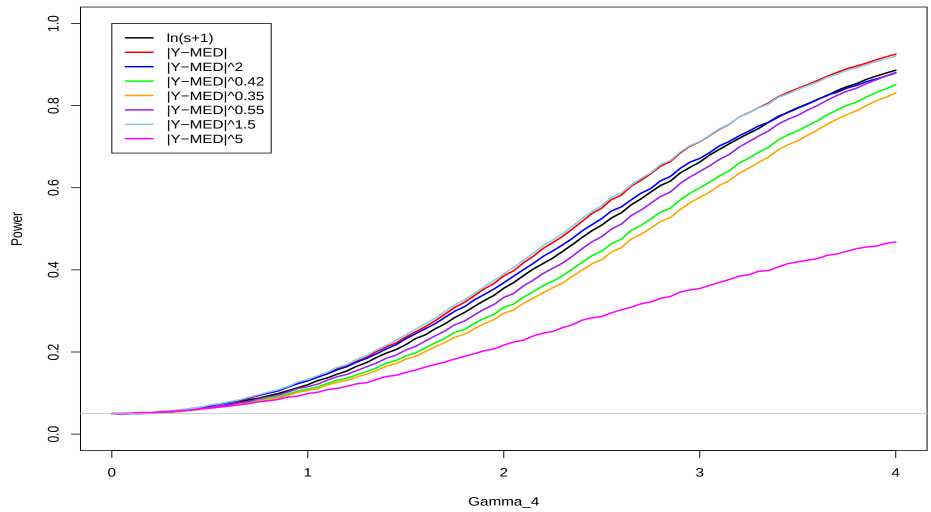

Figure 15.11: Power curves for functions of the absolute deviation from the median with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 4$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

to $ln(s+1)$ over the complete range of effect sizes. From Figure 15.11 it is clear that $|y_{ij} - \tilde{y}_i|^{0.42}$, $|y_{ij} - \tilde{y}_i|^{0.35}$, $|y_{ij} - \tilde{y}_i|^{0.55}$, and $|y_{ij} - \tilde{y}_i|^5$ provide less power to detect the dispersion effect, and these are eliminated from further study. The performance of the $|y_{ij} - \tilde{y}_i|^2$ measure is somewhat less clear, as it shows increased power compared to $ln(s+1)$ for smaller effect sizes ($\gamma_4 \leq 3.5$) but decreased power for larger effect sizes ($\gamma_4 \geq 3.5$). Due to this ambiguity and the relatively decreased power compared to $|y_i - \mathbf{x}'_i\hat{\beta}|$ and $|y_{ij} - \bar{y}_i|$, $|y_{ij} - \tilde{y}_i|^2$ is also eliminated from further study.

Because $|y_{ij} - \tilde{y}_i|^{1.5}$ does not provide a substantial advantage compared to $|y_{ij} - \tilde{y}_i|$, $|y_{ij} - \bar{y}_i|^{1.5}$ is not studied further. Based on the Phase I results as presented, only $|y_{ij} - \tilde{y}_i|$ is studied in Phase II.

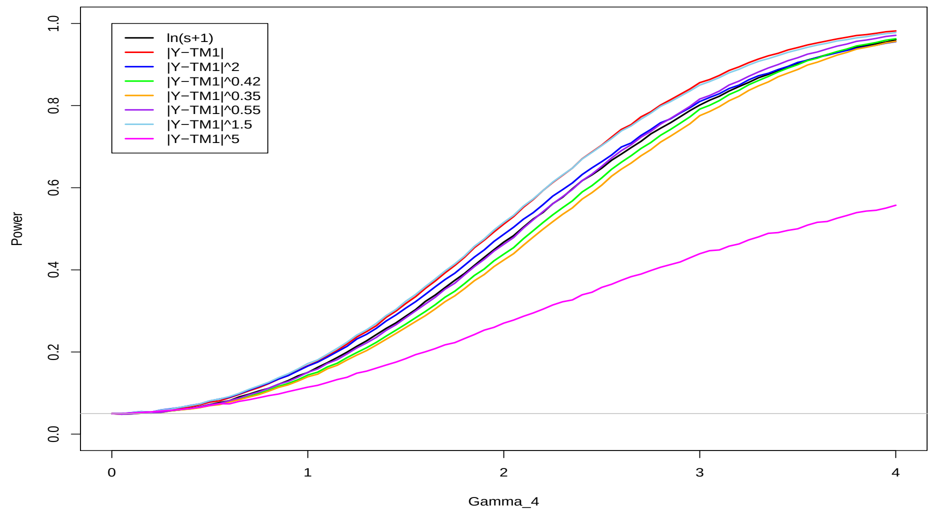

Figure 15.12: Power curves for functions of the absolute deviation from the trimmed mean (-1) with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 5$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

15.1.5 Absolute Deviation from the Within-Treatment Trimmed Mean, $|y_j - \bar{y}_k|$

The performance of $|y_{ij} - \bar{y}_{i(-1)}|$ (Figure 15.12) with $r = 5$ replicates per treatment combination and $|y_{ij} - \bar{y}_{i(-2)}|$ (Figure 15.13) with $r = 7$ replicates per treatment combination shows a similar pattern to the performance of $|y_{ij} - \tilde{y}_i|$. For $k = 1, 2$, the power functions $|y_{ij} - \bar{y}_{i(-k)}|^{0.35}$, $|y_{ij} - \bar{y}_{i(-k)}|^{0.42}$, and $|y_{ij} - \bar{y}_{i(-k)}|^{0.55}$ provide less power than $ln(s + 1)$ to detect the dispersion effect. These measures are, therefore, eliminated from further consideration. Also, for each measure, $|y_{ij} - \bar{y}_{i(-k)}|$ and $|y_{ij} - \bar{y}_{i(-k)}|^{1.5}$ show almost equal power across the range of effect sizes and significantly greater power than $ln(s + 1)$. The difference in power between $|y_{ij} - \bar{y}_{i(-k)}|^2$ and $|y_{ij} - \bar{y}_{i(-k)}|$ is minimal for small effect sizes ($\approx \gamma_4 \leq 1.5$) but increases for larger

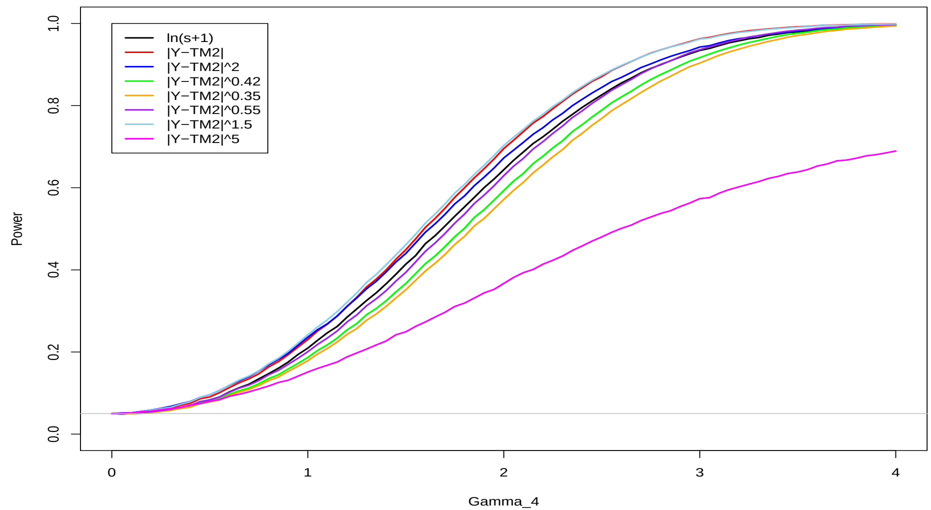

Figure 15.13: Power curves for functions of the absolute deviation from the trimmed mean (-2)with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 7$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

effect sizes; therefore, $|y_{ij} - \bar{y}_{i(-k)}|^2$ is not examined further. Finally, $|y_{ij} - \bar{y}_{i(-k)}|^{1.5}$ is also eliminated from study based on the observed similarity in performance between $|y_{ij} - \bar{y}_{i(-k)}|$ and $|y_{ij} - \tilde{y}_i|$. Only $|y_{ij} - \bar{y}_{i(-1)}|$ and $|y_{ij} - \bar{y}_{i(-2)}|$ are examined further.

The power curves for $|y_{ij} - \tilde{y}_i|$, $|y_{ij} - \bar{y}_{i(-1)}|$, and $|y_{ij} - \bar{y}_{i(-2)}|$ with $r = 10$ replicates per treatment combination are plotted in Figure 15.14 for comparison. Figure 15.14 indicates that the three functions provide similar power for detecting the dispersion effect. At least for $r = 10$ replicates per treatment combination the power of the absolute deviation from the trimmed mean does not depend on the number of observations trimmed from each tail, since $|y_{ij} - \bar{y}_{i(-1)}|$ and $|y_{ij} - \bar{y}_{i(-2)}|$ provide similar power. Since it unlikely that an experiment will have enough replicates per treatment combination to trim two observations per tail (without equivalence to $|y_{ij} - \tilde{y}_i|$),

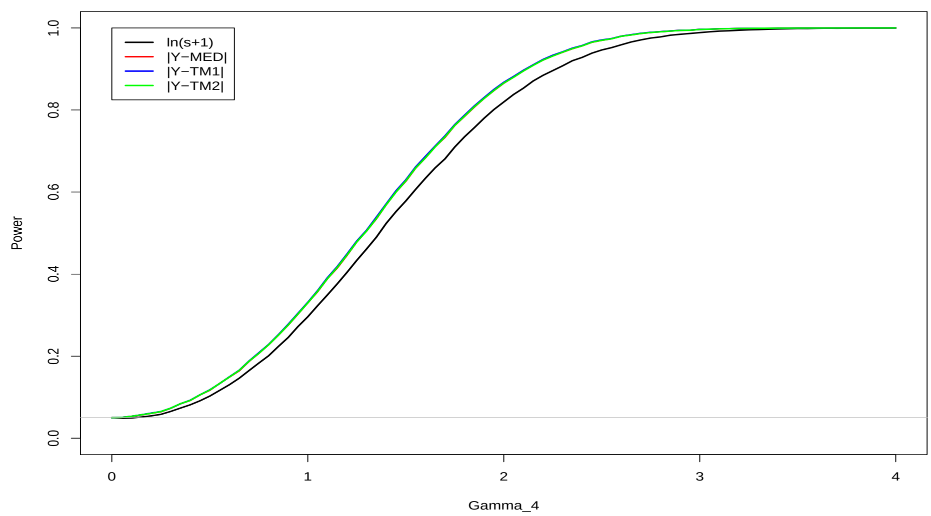

Figure 15.14: Power curves for functions of absolute deviation from the median and trimmed means (-1 and -2) with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 10$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

$|y_{ij} - \bar{y}_{i(-2)}|$ is also eliminated from further consideration. Study of $|y_{ij} - \bar{y}_{i(-1)}|$ follows the work of Brown and Forsythe (1974), who propose use of a 10% trimmed mean; trimming one observation from each tail is equivalent to this proposal for $r = 10$ and the least possible trimming for $r < 10$.

Based on the Phase I results as presented, only $|y_{ij} - \bar{y}_{i(-1)}|$ is studied in Phase II.

15.1.6 Grouped Measures, $[ln(s+1)]_{HALF}$ and $[ln(s+1)]_{ALL}$

The two additional dispersion measures examined, the natural logarithm of the standard deviations of replicates in two approximately equal groups ($[ln(s+1)]_{HALF}$) and the natural logarithm of the standard deviations of replicates in all groups of $r-1$

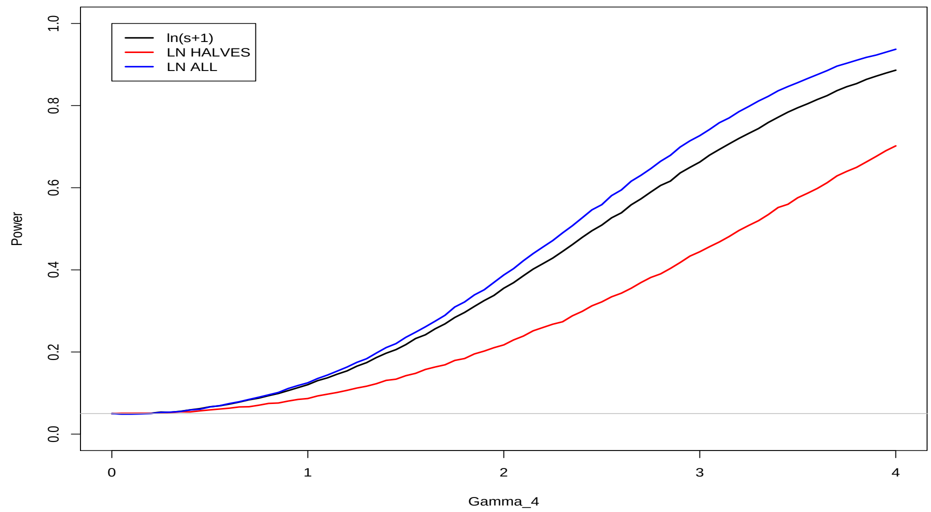

Figure 15.15: Power curves for sub-grouped standard deviations with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 4$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

($[ln(s+1)]_{ALL}$), are both eliminated from further study based on Phase I results. It is clear from the power curves shown in Figure 15.15 that dividing the within-treatment replicates into two groups is considerably less powerful for detecting the dispersion effect of interest than the traditional method of using the natural logarithm of the standard deviation of the whole within-treatment sample. This result is similar to the power decrease of the Box Test observed by Miller (1968). From Figure 15.15, $[ln(s+1)]_{HALF}$ has greatly decreased power to detect true active dispersion effects ($\gamma_4 > 0$).

Figure 15.15 shows that using the standard deviation of all groups of $r-1$ within-treatment replicates is more powerful for detecting the dispersion effect of interest than using the whole within-treatment sample using empirical critical values. As

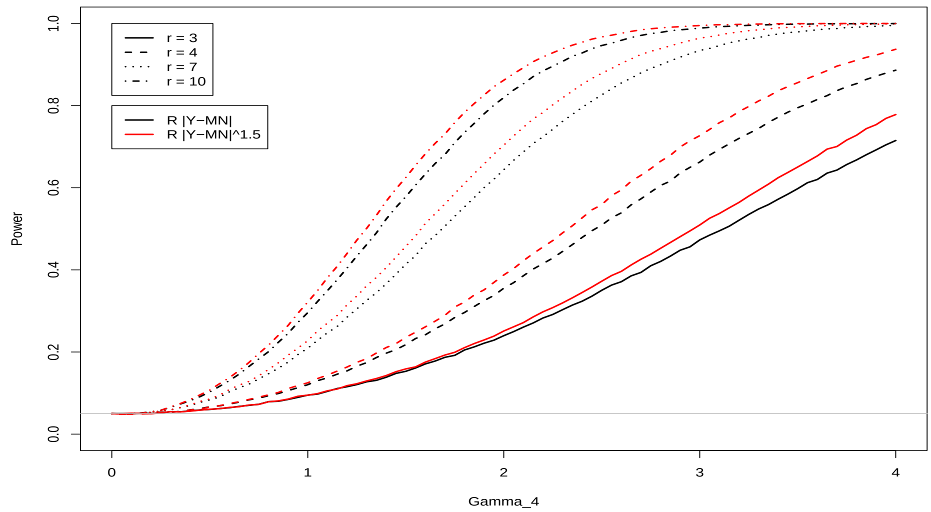

Figure 15.16: Power curves for $ln(s+1)$ and $[ln(s+1)]_{ALL}$with data from simulation of model (12.1) with location effects (14.1), dispersion effects (14.2), and $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$, $r = 3, 4, 7, 10$ replicates, and normal error distribution, testing against the empirical critical value to control Type I error equal to 0.05

the number of replicates per treatment combination increases, the power difference between $[ln(s+1)]_{ALL}$ and $ln(s+1)$ decreases (Figure 15.16). At the same time, as r increases the computational expense for $[ln(s+1)]_{ALL}$ increases. The increase in computation for lesser gains in power makes this measure undesirable.

Based on the Phase I results as presented, neither $[ln(s+1)]_{HALF}$ nor $[ln(s+1)]_{ALL}$ are studied in Phase II.

15.2 Stability of Results

In order to address the issue of the stability of the simulation results, five replicate simulations with $r = 4$ replicates per treatment combination and errors from a normal distribution are conducted. These replicate simulations provide an opportunity to determine whether the observed effect detection probabilities are within an expected range. If the 100,000 tests for a given measure within a single simulation are viewed as 100,000 independent Bernoulli trials with rejection of the null hypothesis considered a success, then an approximate $100(1-\alpha)\%$ confidence interval on each probability can be constructed using the formula

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \tag{15.1}$$

(see Hollander and Wolfe (1999) page 31). In this case, $\hat{p}$ is the proportion of times the test rejects using the F-distribution critical values. Confidence intervals for tests using the empirical critical values cannot be constructed since the empirical critical values are unique for each simulation and, therefore, the test is not identical across simulations. Despite the fact that attention has focused on the tests using the empirical critical values, an examination of the tests using the F-distribution critical value provides relevant information with respect to the stability of the simulation study. The probabilities of detection from the five replicate simulations can be compared with the confidence interval; the calculated $100(1-\alpha)\%$ confidence intervals should include $100(1-\alpha)\%$ of the probabilities of detection from the five replicate simulations.

Confidence intervals are calculated based on the original simulation, labeled Run 0 in the tables. For example, the probability of detecting the dispersion effect using $|y_{ij}-\bar{y}_i|$ when $\gamma_4 = 1$ is $\hat{p} = 0.2106$. Applying (15.1) to this case, the approximate 95% lower confidence limit is

$$\begin{aligned} \hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= 0.2106 - 1.96\sqrt{\frac{0.2106(1-0.2106)}{100000}} \\ &= 0.2106 - 1.96(0.0013) \end{aligned}$$

$$= 0.2081$$

and the approximate 95% upper confidence limit is

$$\begin{aligned} \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= 0.2106 + 1.96\sqrt{\frac{0.2106(1-0.2106)}{100000}} \\ &= 0.2106 + 1.96(0.0013) \\ &= 0.2131 \end{aligned}$$

The approximate 95% confidence intervals and probabilities of detection for the original simulation and each replicate simulation are given in Table C.1, Table C.2, Table C.3, Table C.4, and Table C.5 of Appendix C for $\gamma_4 = 0$, $\gamma_4 = 1$, $\gamma_4 = 2$, $\gamma_4 = 3$, and $\gamma_4 = 4$, respectively.

For each replicate simulation, the probability of detection for each measure is compared to the corresponding confidence interval. The number of times the probability is within the confidence limit is counted and summed in two ways. First, the number of values within the corresponding confidence interval are summed across the measures and within the given effect size. By effect size, the percentage of detection probabilities that are within the stated approximate 95% confidence interval are: 83.33% for $\gamma_4 = 0$; 96.67% for $\gamma_4 = 1$; 100.00% for $\gamma_4 = 2$; 88.33% for $\gamma_4 = 3$; and 95.00% for $\gamma_4 = 4$. Second the number of values within the corresponding confidence interval are summed across effect sizes and within measures. By measure, the percentage of detection probability that are within the stated approximate 95% confidence interval are: 100% for s; 100% for $ln(s^2+1)$; 100% for $ln(s+1)$; 92% for S/N_N; 84% for $|y_i - \mathbf{x}_i'\hat{\beta}|$; 92% for $|y_i - \mathbf{x}_i'\hat{\beta}|^2$; 88% for $|y_i - \mathbf{x}_i'\hat{\beta}|^{1.5}$; 92% for $|y_{ij} - \bar{y}_i|$; 96% for $|y_{ij} - \bar{y}_i|^{.42}$; 76% for $|y_{ij} - \bar{y}_i|^{1.5}$; 96% for $|y_{ij} - \tilde{y}_i|$; and 96% for $|y_{ij} - \bar{y}_{i(-1)}|$. These results indicate that the simulation studies are relatively stable.

CHAPTER 16

MEASURES FOR PHASE II

Based on the results from the Phase I simulations discussed in Chapter 15, the original list of thirty-seven measures is cut to thirteen. The twelve measures selected are:

T1. Within-run sample standard deviation, s

T2. Natural logarithm of the within-treatment sample variance (plus 1.0), $ln(s^2 + 1)$

T3. Natural logarithm of the within-treatment sample standard deviation (plus 1.0), $ln(s + 1)$

T4. Nominal-the-best signal-to-noise ratio, S/N_N

A1. Absolute residuals, $|y_i - \mathbf{x}_i'\hat{\beta}|$

A2. $|y_i - \mathbf{x}_i'\hat{\beta}|^{1.5}$

A3. $|y_i - \mathbf{x}_i'\hat{\beta}|^2$

A4. Absolute deviation from the within-treatment mean, $|y_{ij} - \bar{y}_i|$

A5. $|y_{ij} - \bar{y}_i|^{0.42}$

A6. $|y_{ij} - \bar{y}_i|^{1.5}$

A7. Absolute deviation from the within-treatment median, $|y_{ij} - \widetilde{y}_i|$

A8. Absolute deviation from the within-treatment trimmed mean (-1), $|y_{ij} - \bar{y}_{i(-1)}|$

Because of the variability in the performance of the residuals discussed in Section 15.1.2, it is important to consider which location model to fit and to understand that power to detect the dispersion effect is lost if an incorrect location model is selected. Based on analytic techniques (e.g. F-tests, normal probability plots, etc.) the active location effects can be identified. To test the use of the residuals for dispersion effect identification in the more realistic setting of location model identification, in Phase II a series of F-tests is performed to identify the active location effects. The identified active location effects in addition to the intercept term are then fitted to the data to obtain residuals for use as the response for the detection of the dispersion effects.

Following the work of Bergman and Hynén (1997), a second expanded location model is also fit to the data to calculate residuals in the second phase of study. This second set of residuals is calculated after fitting the location model including all identified active location effects plus the location effect of the factor to be tested for a dispersion effect and all interactions of the identified active location effects with the factor to be tested for a dispersion effect. In Phase II, γ_1 is considered the dispersion effect of interest. Thus, the expanded location model includes all the factors with identified active effects plus the first factor and all two-factor interactions involving the first factor and a factor with an active main effect. If all these factors are already identified as active, the expanded location model is identical to the first location model. Analysis of residuals from the expanded location model should eliminate location-dispersion effect aliasing for the factor of interest, however large or small, and thus should improve the performance of the residuals as a measure for the identification of dispersion effects. The residuals calculated from the expanded location model are denoted $|y_i - \mathbf{x}_i'\hat{\beta}_{exp}|$.

A new measure is added to the list of measures for the second phase of this work. The new measure is:

A9. Absolute deviation from the median, trimming the minimum value, $|y_{ij} - \widetilde{y}_i|_{-1}$

Thus, the set of values of $|y_{ij} - \widetilde{y}_i|_{-1}$ is a subset of the set of values of $|y_{ij} - \widetilde{y}_i|$. It is logical to omit the minimum value of $|y_{ij} - \widetilde{y}_i|$ since if r is even then $min(|y_{ij} - \widetilde{y}_i|)$ is duplicated and if r is odd then $min(|y_{ij} - \widetilde{y}_i|) = 0$.

In addition to adding the new measure, the natural logarithm of each measure is included in the second stage of study. The natural logarithm of each measure is studied due to the use of the multiplicative dispersion model (see Section 12.2) in Phase II. The natural logarithm of the measures may perform better with respect to the new multiplicative model since the natural logarithm converts the multiplicative relationship to an additive relationship.

Note that because $ln(x^a) = a * ln(x)$, the natural logarithm of powers of each measure are proportional to the natural logarithm of each measure and, thus, will lead to equivalent conclusions. Therefore, the natural logarithm of powers of each measure were not included in the second stage of study.

CHAPTER 17

PHASE II SIMULATION

The simulations for Phase II of the current work follow the same format as the Phase I simulations described in Chapter 14 with the following exceptions. The dispersion effect γ_1 is considered the effect of interest throughout Phase II. An additional step is required in Phase II to identify the location model to be fit for calculating residuals. For each data set, a series of F-tests is performed to identify the active location effects; the residuals are then calculated after fitting a model including only the identified location effects or the expanded location model described in Chapter 16. In Phase II, no tests are based on the F-distribution critical values; all tests in Phase II use a set of empirical critical values (described in Chapter 18).

Similar to Phase I, a test of the null hypothesis $H_0 : \gamma_1 = 0$ (14.3) is completed. Because the values of the effect parameters are randomly generated (see Section 17.1), and because the design (Table 14.1) allows for independent estimation of all main effects and two-factor interactions in the model, the results for this test of $H_0 : \gamma_1 = 0$ is typical of all tests of the form $H_0 : \gamma_i = 0$ for $i = 1, 2, \ldots, 16$; the test examined here is representative of tests of a single γ_i, whether γ_i represents a main effect or a two-factor interaction effect.

In performing the test, the model fit to the dispersion measure data is the full model (i.e. the model including all main effects and all two-factor interaction effects). Because of the conservation of the original number of replicate observations, calculation of the M statistic given in (14.4) is possible for the proposed alternative measures, A1-A9. Fitting the full model creates a computational problem for the traditional measures T1-T5. Using these summary measures, the replicated observations are reduced to single measurement. No degrees for freedom for estimating σ^2 are available from the unreplicated design and so Lenth's method (Lenth 1989) is used. Using Lenth's method, the test statistic is computed as

$$t_{PSE,i} = \frac{\hat{\gamma}_i}{PSE} \tag{17.1}$$

for

$$PSE = 1.5 \times median_{\{|\hat{\gamma}_i| < 2.5 s_0\}} |\hat{\gamma}_i| \tag{17.2}$$

with

$$s_0 = 1.5 \times median|\hat{\gamma}_i| \tag{17.3}$$

and where $\hat{\gamma}_i$ is equal to estimate of effect i, the difference between the response average at the high and low levels (See Wu and Hamada (2000) Section 3.13). Lenth's method produces a test statistic similar to the t-test statistic. In this work, $|t_{PSE}|$ is used to simplify testing. Using Lenth's method, all main effects and two-factor interaction effects are estimated, similar to fitting the full model for tests based on the alternative measures.

17.1 Model Generation

For each simulation, data are generated according to (12.1) with $g(x) = exp(x)$ (i.e. a multiplicative dispersion model, see Section 12.2). Except where indicated, the errors, ϵ_{ij}, are generated from a normal distribution. Both first-order and second-order dispersion models are studied with respect to data generation; all location models are second-order. Values of β_i and γ_i are generated randomly according to two scenarios: a single model scenario and a multiple models scenario. For both the single model and multiple model scenarios, the treatment combination means, $\mu_i = \mathbf{x}_i'\beta$, are drawn from a $N(0, 3)$ distribution. From the results of Phase I (Section 15.1), all measures are independent of the location model. (The residuals are not dependent upon the specific location model as long as the correct location model is fit to the data. In Phase II, the location model is identified via a sequence of tests and is assumed to be correct.) The location means, therefore, do not affect the dispersion effect test and can be generated as described. The overall variance parameter, γ_0, is generated according to a $N(0, (1/2)ln2)$ distribution, while the remaining dispersion parameters are generated according to

$$\gamma_i = \begin{cases} Z_i & \text{with probability } 0.4 \\ 0 & \text{with probability } 0.6 \end{cases}$$

where $Z_i \sim N(0, (1/2)ln2)$, for $i = 2, 3, \ldots, 15$. The dispersion effect γ_1 is set to specified values according to each simulation.

17.2 Single Model

For the single model scenario, a single random mean vector and set of dispersion parameters, γ_i, are generated as described above. This single model is used to generate 100,000 data sets. In addition to generating observations according to a randomly generated mean vector and set of dispersion parameters, observations are generated from the null distribution, (i.e. $N(0, 1)$). The single model scenario provides information about the average performance of each measure over multiple experiments for the same underlying model.

17.3 Multiple Models

For the multiple model scenario, a new random mean vector and set of dispersion parameters, γ_i, are generated as described above for each of the 100,000 data sets. In this case, the observed Type I error and power are averaged over a wide range of location and dispersion models. Because a researcher does not know the truth of the model he or she is studying, he or she is concerned with the performance of the analysis method over a set of possible models. It is therefore logical to average the performance over a varying set of models.

17.4 Phase II Simulations

Three sets of simulations are run in Phase II. The first set of simulations (Chapter 18) follows the single model scenario, with data generated according to (12.1) with $\mu_i = 0$ for $i = 1, 2, \ldots, 16$ and $\epsilon_{ij} \sim N(0, 1)$, in order to generate empirical distributions of the test statistics for each dispersion measure and to determine critical values. In this set of simulations, test statistics are calculated but no test against a critical value is performed. The stability of the Type I error rate using these empirical critical values is studied in the second set of simulations (Chapter 19). Both the single model and multiple model scenarios are employed for the stability analysis. If the Type I error is close to the nominal $\alpha = 0.05$ significance level for data generated from either a single or multiple random models, then these critical values are useful for application. In particular, the empirical critical values can be used to study the power of the test to detect true dispersion effects. The size of γ_1 is varied in the third set of simulations (Chapter 20) to study the power of each measure. For the power study, the empirical critical values from the first set of simulations (given in Chapter 18) are used. All power simulations follow the multiple model scenario.

CHAPTER 18

EMPIRICAL CRITICAL VALUES

Even when the original observations come from a normal distribution, the distribution function for each of the dispersion measures is more complicated. As a result, the test statistics (14.4) computed from the measures generally do not follow an F-distribution, and the standard F-distribution critical values are not valid. Therefore, a series of simulations is completed in order to obtain critical values for tests based on each dispersion measure with varying numbers of replicates per treatment combination using the design in Table 14.1. For consistency of methodology, empirical critical values are generated for Lenth's test statistics.

Critical values are obtained for testing the null hypothesis,

$$H_0 : \text{all } \gamma_i = 0, \;\; i = 1, \ldots, p-1,$$

where p is the number of dispersion parameters in the model including the overall variance. Data are generated from the $N(0,1)$ distribution (i.e. data from (12.1) with $g(\mathbf{x}'_{\sigma,i}\gamma) = exp(\mathbf{x}'_{\sigma,i}\gamma)$, $\beta_i = 0$ for all i in (12.2) and $\gamma_i = 0$ for all i in (12.3)) 100,000 times and the calculated test statistics, (14.4) for A1-A9 and (17.1) for T1-T5, saved. Critical values are then extracted from the empirical distribution formed by the simulated test statistics. The critical values are set equal to the $100(1-\alpha)$ percentile of the empirical distribution for significance levels $\alpha = 0.001, 0.005, 0.01, 0.05, 0.1$. For example, the $95,000^{th}$ largest calculated M-statistic value is selected as the $\alpha = 0.05$ level critical value for the alternative dispersion measures. Similarly, the $95,000^{th}$ largest calculated $|t_{PSE}|$ statistic value was selected as the $\alpha = 0.05$ level critical value for the traditional dispersion measures.

Empirical test statistic distributions are obtained for $r = 3, 4, 5, 6, 7, 8, 9, 10, 249, 250$ replicates per treatment combination and for measures T1-T5, A1-A9, and the natural logarithm of A1-A9. Plots of the empirical cumulative distribution functions (CDFs) for the test statistics calculated from each measures are given in Appendix E. For functions of the median, the CDFs for even r are plotted separately from the CDFs for odd r since the definition of the median is different for even numbers and odd numbers of observations.

18.1 Empirical Distribution Critical Values

For each number of replicates, critical values are determined for $\alpha = 0.001$, 0.005, 0.01, 0.05, 0.1. The critical values for each measure, each number of replicates, and each significance level are given in Table 18.1 and Table 18.2; Table 18.1 provides critical values for the $|t_{PSE}|$ statistics (T1-T5) and Table 18.2 provides critical values for the M-test statistics (A1-A9 and natural logarithm of A1-A9). Table 18.2 also provides the critical value from the F-distribution with 1 and $rv - p = 64 - 16 = 48$ degrees of freedom. This critical value corresponds to a design with $r = 4$ replicates at each of the sixteen treatment combinations and estimating sixteen parameters (i.e. the overall mean, all five main effects, and all ten two-factor interaction effects) . For most alternative measures, the empirical critical value is not very different from the F-distribution critical value.

Test statistics and critical values for $ln(|y_{ij} - \widetilde{y}_i|)$ are not available when r is odd due to the fact that $|y_{ij} - \widetilde{y}_i| = 0$, and $ln(|y_{ij} - \widetilde{y}_i|) = \infty$, for some j. Similarly, when $r = 3$, the trimmed mean with one observation trimmed from each tail is equal to the median and $|y_{ij} - \bar{y}_{i(-1)}| = 0$, and $|y_{ij} - \bar{y}_{i(-1)}| = \infty$, for some j. Again, test statistics and critical values are not available for $ln(|y_{ij} - \bar{y}_{i(-1)}|)$ in this case. For $ln(|y_{ij} - \widetilde{y}_i|)$ the problem of missing critical values is avoided by use of $ln(|y_{ij} - \widetilde{y}_i|_{-1})$, the natural logarithm of the absolute deviations from the median excluding the minimum absolute deviation value. Only $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ is studied in the rest of the current work; $ln(|y_{ij} - \widetilde{y}_i|)$ is not considered further. Despite the missing critical value for the single case of $r = 3$, $ln(|y_{ij} - \bar{y}_{i(-1)}|)$ is studied.

The critical values in Table 18.1 and Table 18.2 are used in the remaining simulations to determine both Type I error rates (Chapter 19) and power (Chapter 20).

Measures	r	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.005$	$\alpha = 0.001$
s	3	1.6767	2.1118	3.3858	4.0427	5.8075
s	4	1.6767	2.1148	3.3949	4.0178	5.8473
s	5	1.6790	2.1218	3.4442	4.1298	5.9291
s	6	1.6743	2.1159	3.3801	4.0654	6.0676
s	7	1.6917	2.1272	3.4542	4.1723	6.2417
s	8	1.6792	2.1165	3.4460	4.1302	5.8667
s	9	1.6837	2.1091	3.3247	3.9679	5.8427
s	10	1.6734	2.1126	3.3660	4.0283	5.7315
s	249	1.6787	2.1165	3.4249	4.1152	6.0481
s	250	1.6725	2.1022	3.3774	4.0744	5.9081
$ln(s^2+1)$	3	1.6800	2.1166	3.4228	4.0393	6.0945
$ln(s^2+1)$	4	1.6805	2.1277	3.4096	4.0720	5.9006
$ln(s^2+1)$	5	1.6874	2.1278	3.4692	4.1631	5.9829
$ln(s^2+1)$	6	1.6786	2.1202	3.3886	4.1016	6.2355
$ln(s^2+1)$	7	1.6981	2.1387	3.4671	4.1988	6.2946
$ln(s^2+1)$	8	1.6851	2.1231	3.4506	4.1839	6.0204
$ln(s^2+1)$	9	1.6888	2.1164	3.3442	3.9977	5.8238
$ln(s^2+1)$	10	1.6759	2.1207	3.3922	4.0461	5.7758
$ln(s^2+1)$	249	1.6784	2.1165	3.4252	4.1177	6.0442
$ln(s^2+1)$	250	1.6729	2.1028	3.3749	4.0734	5.9184
$ln(s+1)$	3	1.6955	2.1431	3.4759	4.1263	6.1317
$ln(s+1)$	4	1.6833	2.1257	3.4536	4.0996	5.7824
$ln(s+1)$	5	1.6891	2.1295	3.4814	4.1501	5.9068
$ln(s+1)$	6	1.6784	2.1152	3.3881	4.0660	5.9898
$ln(s+1)$	7	1.6924	2.1439	3.4442	4.1698	5.9821
$ln(s+1)$	8	1.6824	2.1182	3.4796	4.1453	5.9798
$ln(s+1)$	9	1.6874	2.1177	3.3616	3.9642	5.8531
$ln(s+1)$	10	1.6759	2.1182	3.3972	4.0458	5.7861
$ln(s+1)$	249	1.6786	2.1137	3.4091	4.0956	6.0173
$ln(s+1)$	250	1.6733	2.1014	3.3786	4.0641	5.8701
S/N_N	3	1.5871	2.0015	3.2208	3.8595	5.6745
S/N_N	4	1.6225	2.0214	3.1807	3.8123	5.6690
S/N_N	5	1.6181	2.0244	3.2082	3.8294	5.6031
S/N_N	6	1.6212	2.0361	3.1801	3.8299	5.5643
S/N_N	7	1.6175	2.0187	3.1975	3.8226	5.5797
S/N_N	8	1.6204	2.0203	3.1978	3.8217	5.5486
S/N_N	9	1.6184	2.0251	3.1682	3.7909	5.6098
S/N_N	10	1.6190	2.0204	3.2107	3.8109	5.3835
S/N_N	249	1.6150	2.0146	3.1800	3.7718	5.3571
S/N_N	250	1.6132	2.0193	3.2270	3.8620	5.6433

Table 18.1: Critical values for Lenth's test statistic (17.1) obtained from the empirical distributions of test statistics based on traditional dispersion measures (T1-T5) under $H_0^* : \gamma_1 = 0$ (14.3), design of Table 14.1, and $N(0,1)$ observations, for Type I error rates $0.001, 0.005, 0.01, 0.05, 0.1$

Measure	r	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.005$	$\alpha = 0.001$
$\lvert y_{ij} - \widetilde{y}_i \rvert$	3	1.2867	1.8079	3.1045	3.6570	4.9006
$\lvert y_{ij} - \widetilde{y}_i \rvert$	4	2.6987	3.9265	7.0883	8.5621	12.5738
$\lvert y_{ij} - \widetilde{y}_i \rvert$	5	1.7784	2.5103	4.3315	5.1044	6.8635
$\lvert y_{ij} - \widetilde{y}_i \rvert$	6	2.3964	3.4219	5.9610	7.1169	9.8937
$\lvert y_{ij} - \widetilde{y}_i \rvert$	7	2.0088	2.8436	4.9799	5.8658	8.1268
$\lvert y_{ij} - \widetilde{y}_i \rvert$	8	2.3915	3.4227	5.9186	7.0938	9.6414
$\lvert y_{ij} - \widetilde{y}_i \rvert$	9	2.1586	3.0520	5.1946	6.1529	8.4423
$\lvert y_{ij} - \widetilde{y}_i \rvert$	10	2.3996	3.4037	5.9355	7.0917	9.9660
$\lvert y_{ij} - \widetilde{y}_i \rvert$	249	2.6596	3.7789	6.5779	7.8334	10.6543
$\lvert y_{ij} - \widetilde{y}_i \rvert$	250	2.6939	3.7895	6.5618	7.7088	10.5827
$\lvert y_{ij} - \widetilde{y}_i \rvert_{-1}$	3	2.2867	3.3408	6.3681	7.9299	11.8074
$\lvert y_{ij} - \widetilde{y}_i \rvert_{-1}$	4	2.2371	3.2332	5.7728	6.9022	10.1493
$\lvert y_{ij} - \widetilde{y}_i \rvert_{-1}$	5	2.5220	3.6065	6.3914	7.6496	10.5164
$\lvert y_{ij} - \widetilde{y}_i \rvert_{-1}$	6	2.4375	3.4746	6.0167	7.2140	9.8941
$\lvert y_{ij} - \widetilde{y}_i \rvert_{-1}$	7	2.5850	3.6728	6.5175	7.7453	10.7546
$\lvert y_{ij} - \widetilde{y}_i \rvert_{-1}$	8	2.5528	3.6284	6.3235	7.5491	10.3101
$\lvert y_{ij} - \widetilde{y}_i \rvert_{-1}$	9	2.6249	3.7194	6.3824	7.5742	10.5009
$\lvert y_{ij} - \widetilde{y}_i \rvert_{-1}$	10	2.5714	3.6552	6.4078	7.6458	10.5851
$\lvert y_{ij} - \widetilde{y}_i \rvert_{-1}$	249	2.6785	3.8060	6.6246	7.8880	10.7319
$\lvert y_{ij} - \widetilde{y}_i \rvert_{-1}$	250	2.7131	3.8158	6.6042	7.7618	10.6541
$\lvert y_{ij} - \bar{y}_{i(-1)} \rvert$	3	1.2867	1.8079	3.1045	3.6570	4.9006
$\lvert y_{ij} - \bar{y}_{i(-1)} \rvert$	4	2.6987	3.9265	7.0883	8.5621	12.5738
$\lvert y_{ij} - \bar{y}_{i(-1)} \rvert$	5	2.8482	4.0758	7.1780	8.5364	11.5644
$\lvert y_{ij} - \bar{y}_{i(-1)} \rvert$	6	2.8707	4.0824	7.1091	8.5025	11.9141
$\lvert y_{ij} - \bar{y}_{i(-1)} \rvert$	7	2.9212	4.1547	7.3715	8.6616	12.0450
$\lvert y_{ij} - \bar{y}_{i(-1)} \rvert$	8	2.9195	4.1583	7.2555	8.6790	11.7418
$\lvert y_{ij} - \bar{y}_{i(-1)} \rvert$	9	2.9367	4.1357	7.1179	8.4246	11.4113
$\lvert y_{ij} - \bar{y}_{i(-1)} \rvert$	10	2.8840	4.0971	7.1900	8.5807	11.7597
$\lvert y_{ij} - \bar{y}_{i(-1)} \rvert$	249	2.6959	3.8321	6.6619	7.9326	10.7731
$\lvert y_{ij} - \bar{y}_{i(-1)} \rvert$	250	2.7303	3.8372	6.6129	7.8098	10.7605
$\lvert y_{ij} - \bar{y}_i \rvert$	3	5.1431	7.2507	12.3542	14.4767	19.2879
$\lvert y_{ij} - \bar{y}_i \rvert$	4	4.0225	5.7340	10.2916	12.1854	17.3935
$\lvert y_{ij} - \bar{y}_i \rvert$	5	3.6065	5.1783	9.0823	10.8087	14.6000
$\lvert y_{ij} - \bar{y}_i \rvert$	6	3.3494	4.7655	8.2691	9.8497	13.9972
$\lvert y_{ij} - \bar{y}_i \rvert$	7	3.2446	4.6176	8.1007	9.5849	13.2854
$\lvert y_{ij} - \bar{y}_i \rvert$	8	3.1651	4.4978	7.8140	9.3467	12.7258
$\lvert y_{ij} - \bar{y}_i \rvert$	9	3.1128	4.4048	7.5465	8.9887	12.0439
$\lvert y_{ij} - \bar{y}_i \rvert$	10	3.0223	4.2967	7.5245	8.9653	12.3861
$\lvert y_{ij} - \bar{y}_i \rvert$	249	2.6955	3.8339	6.6597	7.9261	10.7692
$\lvert y_{ij} - \bar{y}_i \rvert$	250	2.7307	3.8385	6.6171	7.8177	10.7620
$\lvert y_{ij} - \bar{y}_i \rvert^{0.42}$	3	4.6425	6.7383	12.1823	14.8092	21.1371
$\lvert y_{ij} - \bar{y}_i \rvert^{0.42}$	4	3.7301	5.4241	9.8657	11.8769	17.3913
$\lvert y_{ij} - \bar{y}_i \rvert^{0.42}$	5	3.4225	4.8909	8.7192	10.5377	14.4881
$\lvert y_{ij} - \bar{y}_i \rvert^{0.42}$	6	3.1952	4.5799	8.0579	9.6376	13.8236
$\lvert y_{ij} - \bar{y}_i \rvert^{0.42}$	7	3.1296	4.4544	7.8396	9.3423	13.0290
$\lvert y_{ij} - \bar{y}_i \rvert^{0.42}$	8	3.0486	4.3868	7.6936	9.1618	12.9306
$\lvert y_{ij} - \bar{y}_i \rvert^{0.42}$	9	3.0249	4.3280	7.4463	8.7554	12.0696
$\lvert y_{ij} - \bar{y}_i \rvert^{0.42}$	10	2.9452	4.1943	7.3336	8.8443	12.1842
$\lvert y_{ij} - \bar{y}_i \rvert^{0.42}$	249	2.7074	3.8360	6.6954	8.0141	10.8450
$\lvert y_{ij} - \bar{y}_i \rvert^{0.42}$	250	2.7237	3.8673	6.6532	7.7883	10.8192
$F_{1,48}$		2.8131	4.0427	7.1942	8.6590	12.2855

Continued

Table 18.2: Critical values for the M-test statistic (14.4) obtained from the empirical distributions of test statistics based on alternative dispersion measures (A1-A9 and natural logarithm of A1-A9) under $H_0^* : \gamma_1 = 0$ (14.3), design of Table 14.1, and $N(0,1)$ observations, for Type I error rates $0.001, 0.005, 0.01, 0.05, 0.1$

Table 18.2 Continued

Measure	r	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.005$	$\alpha = 0.001$
$\|y_{ij} - \bar{y}_i\|^{1.5}$	3	5.2477	7.1974	11.4198	13.0379	16.4498
$\|y_{ij} - \bar{y}_i\|^{1.5}$	4	4.1150	5.7535	9.9037	11.7065	16.2960
$\|y_{ij} - \bar{y}_i\|^{1.5}$	5	3.6716	5.2166	8.9178	10.4100	13.9025
$\|y_{ij} - \bar{y}_i\|^{1.5}$	6	3.4000	4.7800	8.0898	9.6735	13.3854
$\|y_{ij} - \bar{y}_i\|^{1.5}$	7	3.3028	4.6509	8.0437	9.5203	13.0998
$\|y_{ij} - \bar{y}_i\|^{1.5}$	8	3.1939	4.5522	7.7646	9.1434	12.5516
$\|y_{ij} - \bar{y}_i\|^{1.5}$	9	3.1396	4.4325	7.5012	8.7755	11.9713
$\|y_{ij} - \bar{y}_i\|^{1.5}$	10	3.0509	4.3226	7.4896	8.8740	12.0759
$\|y_{ij} - \bar{y}_i\|^{1.5}$	249	2.6927	3.8270	6.5950	7.8743	10.8422
$\|y_{ij} - \bar{y}_i\|^{1.5}$	250	2.7183	3.8352	6.5775	7.8114	10.7354
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	3	2.9322	4.2142	7.6037	9.3370	13.4514
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	4	2.8916	4.1031	7.1883	8.7173	12.1966
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	5	2.8543	4.1225	7.1745	8.4868	11.8456
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	6	2.7861	3.9756	6.8976	8.2702	11.5330
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	7	2.7895	3.9739	6.9697	8.2714	11.6540
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	8	2.7692	3.9635	7.0104	8.2987	11.6617
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	9	2.7835	3.9367	6.8317	8.1161	11.2245
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	10	2.7312	3.8928	6.8398	8.1768	11.2795
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	249	2.6824	3.8238	6.6494	7.9179	10.7387
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	250	2.7171	3.8330	6.5920	7.8063	10.7173
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	3	2.9146	4.0716	7.0169	8.4496	12.2134
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	4	2.8682	4.0085	6.6714	7.9564	10.8021
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	5	2.8568	3.9910	6.7277	7.9353	10.8233
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	6	2.7829	3.8794	6.5431	7.7746	10.2177
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	7	2.7917	3.9141	6.6833	7.7992	10.7329
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	8	2.7839	3.9307	6.6553	7.8112	10.5386
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	9	2.7736	3.9034	6.5125	7.6054	10.3973
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	10	2.7624	3.9002	6.5877	7.7608	10.7863
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	249	2.6906	3.8325	6.5497	7.8473	10.7419
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	250	2.6985	3.8128	6.5804	7.6947	10.6788
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	3	2.9347	4.1705	7.3523	8.9218	12.9171
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	4	2.8938	4.0824	7.0064	8.3577	11.6468
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	5	2.8650	4.0710	7.0185	8.2807	11.2803
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	6	2.7808	3.9497	6.7564	8.0573	10.9322
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	7	2.8027	3.9557	6.8322	8.0831	11.1937
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	8	2.7759	3.9657	6.8368	8.0773	11.1679
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	9	2.7872	3.9139	6.6864	7.9161	10.9285
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	10	2.7566	3.8924	6.7566	7.9989	10.9503
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	249	2.6831	3.8234	6.5691	7.8319	10.8643
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	250	2.7093	3.8416	6.5415	7.7769	10.7061
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	3	3.0749	4.4068	7.9230	9.8640	13.7963
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	4	2.9725	4.2389	7.3943	8.9599	12.3604
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	5	2.9209	4.2272	7.3356	8.7285	12.0613
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	6	2.8386	4.0470	7.0215	8.4108	11.8031
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	7	2.8372	4.0464	7.0909	8.3618	11.8212
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	8	2.8070	4.0069	7.1078	8.4284	11.7479
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	9	2.8143	3.9773	6.8993	8.2735	11.1222
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	10	2.7603	3.9436	6.8795	8.2753	11.4634
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	249	2.6827	3.8269	6.6390	7.9045	10.7727
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	250	2.7195	3.8308	6.6059	7.8020	10.6612
$F_{1,48}$		2.8131	4.0427	7.1942	8.6590	12.2855

Continued

Table 18.2 Continued

Measure	r	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.005$	$\alpha = 0.001$
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^2$	3	3.0519	4.2603	7.3199	8.8517	12.3962
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^2$	4	2.9591	4.1287	6.8642	8.1231	11.0632
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^2$	5	2.9225	4.0896	6.9259	8.0926	11.0578
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^2$	6	2.8425	3.9738	6.6911	7.8644	10.4899
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^2$	7	2.8387	3.9780	6.8042	7.9265	10.7515
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^2$	8	2.8261	3.9858	6.7571	7.8883	10.5953
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^2$	9	2.8132	3.9469	6.5808	7.6665	10.4839
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^2$	10	2.7938	3.9365	6.6705	7.8460	10.9224
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^2$	249	2.6916	3.8350	6.5528	7.8521	10.7440
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^2$	250	2.7000	3.8148	6.5796	7.6969	10.6861
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^{1.5}$	3	3.0729	4.3566	7.6596	9.3688	13.2226
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^{1.5}$	4	2.9790	4.2123	7.2459	8.6183	11.8168
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^{1.5}$	5	2.9383	4.1654	7.2077	8.4644	11.7161
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^{1.5}$	6	2.8393	4.0338	6.8940	8.1882	11.0578
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^{1.5}$	7	2.8430	4.0224	6.9800	8.1913	11.3464
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^{1.5}$	8	2.8125	4.0242	6.9736	8.2246	11.3714
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^{1.5}$	9	2.8268	3.9697	6.7974	8.0186	10.9607
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^{1.5}$	10	2.7899	3.9389	6.8394	8.0697	11.0462
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^{1.5}$	249	2.6836	3.8248	6.5708	7.8296	10.8573
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^{1.5}$	250	2.7099	3.8462	6.5440	7.7964	10.6879
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert)$	3	N/A	N/A	N/A	N/A	N/A
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert)$	4	4.7055	6.6028	11.2972	13.3300	19.2526
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert)$	5	N/A	N/A	N/A	N/A	N/A
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert)$	6	3.6561	5.1422	8.7226	10.2558	13.9139
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert)$	7	N/A	N/A	N/A	N/A	N/A
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert)$	8	3.2476	4.5951	7.7938	9.2968	12.3049
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert)$	9	N/A	N/A	N/A	N/A	N/A
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert)$	10	3.0542	4.2993	7.4017	8.8224	11.8960
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert)$	249	N/A	N/A	N/A	N/A	N/A
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert)$	250	2.5888	3.6727	6.3178	7.4403	10.5150
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert_{-1})$	3	2.4599	3.5675	6.7092	8.2898	12.6292
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert_{-1})$	4	2.4319	3.4638	6.1225	7.2755	10.5691
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert_{-1})$	5	2.5813	3.6552	6.3367	7.5782	10.4975
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert_{-1})$	6	2.5297	3.6079	6.2353	7.3515	10.1599
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert_{-1})$	7	2.6089	3.6955	6.4512	7.6237	10.2863
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert_{-1})$	8	2.5710	3.6554	6.3898	7.5826	10.2672
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert_{-1})$	9	2.6386	3.7398	6.5071	7.7018	10.6666
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert_{-1})$	10	2.5936	3.7131	6.3687	7.6885	10.8176
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert_{-1})$	249	2.7042	3.8417	6.6711	7.9406	10.7555
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert_{-1})$	250	2.7153	3.8298	6.6084	7.7923	10.8915
$ln(\lvert y_{ij} - \bar{y}_{i(-1)}\rvert)$	3	N/A	N/A	N/A	N/A	N/A
$ln(\lvert y_{ij} - \bar{y}_{i(-1)}\rvert)$	4	4.7055	6.6028	11.2972	13.3300	19.2526
$ln(\lvert y_{ij} - \bar{y}_{i(-1)}\rvert)$	5	3.2561	4.6487	8.0162	9.5616	13.3910
$ln(\lvert y_{ij} - \bar{y}_{i(-1)}\rvert)$	6	3.1370	4.4614	7.7488	9.3124	12.8196
$ln(\lvert y_{ij} - \bar{y}_{i(-1)}\rvert)$	7	3.0365	4.3380	7.5664	9.0101	12.2842
$ln(\lvert y_{ij} - \bar{y}_{i(-1)}\rvert)$	8	3.0029	4.2610	7.4392	8.7907	12.0292
$ln(\lvert y_{ij} - \bar{y}_{i(-1)}\rvert)$	9	2.9730	4.1951	7.3095	8.5347	11.4839
$ln(\lvert y_{ij} - \bar{y}_{i(-1)}\rvert)$	10	2.8767	4.1247	7.1823	8.5079	11.6863
$ln(\lvert y_{ij} - \bar{y}_{i(-1)}\rvert)$	249	2.7062	3.8532	6.7160	7.8812	10.9219
$ln(\lvert y_{ij} - \bar{y}_{i(-1)}\rvert)$	250	2.7267	3.8514	6.6891	7.9009	10.8813
$F_{1,48}$		2.8131	4.0427	7.1942	8.6590	12.2855

Continued

Table 18.2 Continued

Measure	r	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.005$	$\alpha = 0.001$
$ln(\|y_{ij} - \bar{y}_i\|)$	3	4.0644	5.9312	10.9653	13.6479	20.8504
$ln(\|y_{ij} - \bar{y}_i\|)$	4	3.4294	4.8870	8.7635	10.5498	15.4510
$ln(\|y_{ij} - \bar{y}_i\|)$	5	3.1838	4.5170	7.9433	9.5013	13.2277
$ln(\|y_{ij} - \bar{y}_i\|)$	6	3.0431	4.3045	7.5266	9.0160	12.6400
$ln(\|y_{ij} - \bar{y}_i\|)$	7	2.9991	4.2595	7.4194	8.7313	11.9977
$ln(\|y_{ij} - \bar{y}_i\|)$	8	2.9461	4.1829	7.3094	8.6098	12.1211
$ln(\|y_{ij} - \bar{y}_i\|)$	9	2.9275	4.1808	7.1316	8.4074	11.6415
$ln(\|y_{ij} - \bar{y}_i\|)$	10	2.8628	4.0574	7.0417	8.4351	11.7971
$ln(\|y_{ij} - \bar{y}_i\|)$	249	2.7137	3.8453	6.7218	7.8871	11.1020
$ln(\|y_{ij} - \bar{y}_i\|)$	250	2.7132	3.8533	6.6895	7.8646	11.0238
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	3	2.8898	4.1637	7.3255	8.8981	12.4561
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	4	2.8268	4.0071	6.9016	8.3649	11.6307
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	5	2.8160	4.0089	6.9817	8.2675	11.1966
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	6	2.7678	3.9291	6.7065	8.0612	11.0207
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	7	2.7603	3.9168	6.7860	8.0912	11.1541
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	8	2.7750	3.9513	6.7570	7.9686	10.9720
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	9	2.7546	3.9100	6.7812	8.0148	11.5115
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	10	2.7161	3.8535	6.6279	7.8956	10.8691
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	249	2.6986	3.8205	6.6690	7.9733	10.7694
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	250	2.7154	3.8462	6.6822	7.8107	10.8805
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	3	2.9603	4.2185	7.4918	9.0644	12.9161
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	4	2.8624	4.0672	7.1827	8.5798	12.0492
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	5	2.8693	4.0848	7.0599	8.4200	11.6221
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	6	2.8143	3.9850	6.8521	8.1352	11.3235
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	7	2.8001	3.9435	6.8627	8.1158	11.1792
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	8	2.7769	3.9661	6.8027	8.0237	10.8977
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	9	2.7611	3.9331	6.8330	8.1778	11.2759
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	10	2.7250	3.8818	6.7021	7.9795	10.9376
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	249	2.7012	3.8188	6.6833	7.8974	10.8175
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	250	2.7186	3.8524	6.5955	7.7804	10.8427
$F_{1,48}$		2.8131	4.0427	7.1942	8.6590	12.2855

Five replicate simulations (as described above) are performed for each of $r = 4, 7, 10$ replicates per treatment combination in order to check the precision of the critical values. The critical values for $\alpha = 0.001$, 0.005, 0.01, 0.05, 0.1 from each replicate simulation are given in Tables D.1–tab:cv.r10.001 of Appendix app:rep.cv along with the original critical values from Table 18.1 and Table 18.2. For $\alpha = 0.01$, 0.05, 0.1, all critical values for a given measure are very close; the range of the critical values is increased for $\alpha = 0.001$, 0.005. As all tests in the current work are conducted at the $\alpha = 0.05$ significance level, the empirical critical values given in Table 18.1 and Table 18.2 based on 100,000 values are sufficiently precise and can be used; distributions with more than 100,000 values can be generated to increase the precision of the critical values for $\alpha = 0.001$, 0.005.

18.2 Asymptotic Distribution of Test Statistics

For each alternative measure, the F-distribution with 1 and 48 degrees of freedom appears to approximate the empirical distribution of the test statistic for large r. The distributions of the test statistics from the natural logarithm of each measure (Figures E.22 – E.26), except for $ln(|y_{ij} - \tilde{y}_i|)$ (Figure E.20), closely approximate the F-distribution with 1 and 48 degrees of freedom for all values of r examined; in the case of $ln(|y_{ij} - \bar{y}_{i(-1)}|)$, $r \geq 5$ replicates per treatment combination are needed to achieve a good approximation. The F-distribution with 1 and 48 degrees of freedom is closely approximated by the distribution of M using powers of the residuals, A1-A3 (Figures E.14 – E.19). For the absolute deviation from the median, the F-distribution provides a better approximation to the distribution of tests based on an even number of replicates (Figure E.7) than tests based on an odd number of replicates (Figure E.6); for the absolute deviation from the median trimming the minimum deviation value, no such differentiation between even and odd numbers of replicates exists (Figure E.8).

CHAPTER 19

STABILITY OF CRITICAL VALUES AND TYPE I ERROR

In order to determine whether the Type I error rate for tests using each measure are stable and close to the nominal $\alpha = 0.05$ level, tests are performed on data simulated according to model (12.1) with different means (12.2) and error variances (12.3). For each simulated data set, the value of γ_1 in (12.3) is set equal to zero to represent a nonexistent dispersion effect of factor 1. The study of the observed Type I error rates is conducted for $r = 4$, $r = 7$, and $r = 10$. Both the single model scenario and the multiple model scenario (see Chapter 17) are utilized.

For each data set generated, the appropriate test statistic (either (17.1) or (14.4)) is calculated based for each dispersion measure. Each test statistic is compared to the associated empirical critical value given in Table 18.1 or Table 18.2 and the number of times the test rejects counted.

19.1 Single Model Scenario Results

Following the single model scenario, observations are generated from the $N(0, 1)$ distribution (i.e. null location and dispersion models) in order to validate the critical values. Five additional models are randomly generated as described in Section 17.1.

	Model 4.1	Model 4.2	Model 4.3	Model 4.4	Model 4.5
$\mathbf{x}_1'\beta$	-3.1562	-3.2868	1.4996	3.4623	4.0859
$\mathbf{x}_2'\beta$	-1.8922	2.9029	2.6170	-0.7852	-1.0741
$\mathbf{x}_3'\beta$	-0.7601	-0.9475	-2.6629	-3.5661	-4.1923
$\mathbf{x}_4'\beta$	-1.6783	0.3157	2.9891	0.9612	1.3781
$\mathbf{x}_5'\beta$	3.4927	1.4805	2.9732	0.2246	-1.8942
$\mathbf{x}_6'\beta$	-3.8018	2.4362	1.8551	-1.8688	1.4763
$\mathbf{x}_7'\beta$	5.5456	-6.1198	1.3102	3.4631	-1.0585
$\mathbf{x}_8'\beta$	-4.4924	-1.8044	-0.8524	-0.0586	4.1173
$\mathbf{x}_9'\beta$	1.3371	1.9419	-0.6588	-1.0239	-5.8355
$\mathbf{x}_{10}'\beta$	-0.7760	4.1899	-2.2487	0.7320	-3.6942
$\mathbf{x}_{11}'\beta$	5.1722	5.1685	-0.9034	2.2040	-0.1844
$\mathbf{x}_{12}'\beta$	-1.4262	-2.1259	1.4566	-0.1420	-1.8032
$\mathbf{x}_{13}'\beta$	0.3954	-2.7120	-8.5254	-1.8871	3.6729
$\mathbf{x}_{14}'\beta$	2.8526	-1.5119	3.2986	1.5860	1.4523
$\mathbf{x}_{15}'\beta$	-0.9872	-1.3303	-0.4210	-7.7837	2.4615
$\mathbf{x}_{16}'\beta$	0.1748	1.4032	-1.7269	4.4823	1.0922
γ_0	-0.0201	0.4891	-0.5371	0.1088	-0.4170
γ_1	0.0000	0.0000	0.0000	0.0000	0.0000
γ_2	-0.1779	0.0000	0.0000	-0.3599	-0.1866
γ_3	0.1699	0.0000	-0.2438	0.0000	0.0000
γ_4	0.0101	-0.6218	0.0000	0.0000	0.2464
γ_5	0.0000	0.0000	-0.0094	0.0000	0.4025
γ_6	0.0000	0.0000	0.0000	-0.4626	0.0000
γ_7	0.0000	0.6737	0.0000	-0.1252	0.0000
γ_8	0.0000	0.4172	0.0000	0.0000	0.0000
γ_9	0.0000	-0.2741	0.0000	0.1115	0.1122
γ_{10}	0.0000	0.0000	0.0000	0.1835	0.0000
γ_{11}	0.0625	0.0000	0.0000	0.0499	-0.3016
γ_{12}	0.0000	0.0470	-0.4019	0.0000	0.4152
γ_{13}	0.1608	-0.6101	0.1596	-0.2873	-0.1537
γ_{14}	0.0000	-0.1352	0.0000	-0.0336	-0.0838
γ_{15}	0.0000	0.0000	0.0000	0.0000	0.5470

Table 19.1: Treatment combination means and dispersion parameters for fixed models simulated with $r = 4$ replicates per treatment combination to generate observed Type I errors given in Table 19.4

The means and dispersion parameters for these five models are given in Tables 19.1, 19.2, and 19.3 for $r = 4$, $r = 7$, and $r = 10$ replicates per treatment combination respectively, where the first digit of the model number is equal to r.

The observed Type I errors based on the single model simulations, including the null model simulations, are given in Table 19.4 for $r = 4$ replicates per treatment combination. Observed Type I error rates for data generated from the $N(0,1)$ distribution are approximately equal to the $\alpha = 0.05$ nominal significance level for all

	Model 7.1	Model 7.2	Model 7.3	Model 7.4	Model 7.5
$\mathbf{x}_1'\beta$	-1.7732	-1.7442	-3.1029	1.4396	-5.2996
$\mathbf{x}_2'\beta$	-0.1837	5.9979	2.9354	2.4069	-2.6655
$\mathbf{x}_3'\beta$	-2.6421	-2.0714	-3.6723	-4.9570	5.6097
$\mathbf{x}_4'\beta$	0.2659	0.2204	-2.5308	-0.2028	-1.5852
$\mathbf{x}_5'\beta$	7.1682	-3.5706	4.3505	-3.3541	2.6151
$\mathbf{x}_6'\beta$	1.7979	2.7260	4.7435	-1.3694	-3.8434
$\mathbf{x}_7'\beta$	-0.3733	1.4189	-1.3281	2.4163	4.1002
$\mathbf{x}_8'\beta$	0.0300	2.6335	1.7429	-3.7713	-1.5534
$\mathbf{x}_9'\beta$	-2.2024	4.8184	-1.8158	0.3615	0.8207
$\mathbf{x}_{10}'\beta$	-1.6183	-4.8581	-0.7067	-0.7839	0.1038
$\mathbf{x}_{11}'\beta$	-0.4702	-1.1339	-1.3273	-1.4159	-0.7880
$\mathbf{x}_{12}'\beta$	2.3907	-0.1024	-1.1277	-1.2050	3.1998
$\mathbf{x}_{13}'\beta$	0.7970	-2.8844	2.6023	2.0370	-0.6957
$\mathbf{x}_{14}'\beta$	-0.1299	-0.4967	-4.0309	7.7110	1.6524
$\mathbf{x}_{15}'\beta$	-6.5922	1.1633	4.4130	-0.4586	-2.9377
$\mathbf{x}_{16}'\beta$	3.5356	-2.1167	-1.1453	1.1457	1.2668
γ_0	0.6068	0.2222	-0.7223	-0.2602	0.5224
γ_1	0.0000	0.0000	0.0000	0.0000	0.0000
γ_2	0.0000	0.0000	0.0000	-0.4711	0.0000
γ_3	0.0000	0.0000	-0.0254	0.0000	0.0749
γ_4	0.0000	0.0000	0.2535	-0.3345	0.0000
γ_5	-0.2381	0.0000	0.0000	0.0000	0.0000
γ_6	0.0000	0.0611	-0.1921	0.0000	0.0611
γ_7	0.0000	-0.3410	0.1799	0.2604	0.0000
γ_8	0.1235	-0.3790	0.0000	0.0000	0.0000
γ_9	-0.3244	0.0000	0.0000	0.0000	0.0000
γ_{10}	0.3581	0.0000	0.0000	-0.2744	-0.0450
γ_{11}	0.3527	-0.0456	0.0000	0.0000	0.3631
γ_{12}	0.0000	0.0000	0.2885	0.5773	-0.1935
γ_{13}	0.3000	0.0000	0.0000	0.0000	0.0000
γ_{14}	0.0000	-0.1480	0.0000	0.0000	0.0000
γ_{15}	0.0000	0.0000	0.2786	0.0000	0.0000

Table 19.2: Treatment combination means and dispersion parameters for fixed models simulated with $r = 7$ replicates per treatment combination to generate observed Type I errors given in Table 19.5

measures (see column 2 of Tables 19.4–19.6). These results provide validation of the empirical critical values.

For each location-dispersion model, s, $ln(s^2 + 1)$, and $ln(s + 1)$ control the Type I error rate below the nominal $\alpha = 0.05$ level (see columns 3-7 of Tables 19.4). The signal-to-noise ratio controls the Type I error rate for four of the five location-dispersion models, the exception being Model 4.1. Under Model 4.1, the Type I error rate for the test using S/N_N is 0.1009, twice the nominal level. Because the Type I

	Model 10.1	Model 10.2	Model 10.3	Model 10.4	Model 10.5
$\mathbf{x}_1'\beta$	-1.3653	0.1655	3.6214	-2.0814	1.0647
$\mathbf{x}_2'\beta$	-2.8481	1.7179	-4.5440	-4.5947	-0.3825
$\mathbf{x}_3'\beta$	3.4755	-2.4649	-4.0949	-2.3685	-0.7441
$\mathbf{x}_4'\beta$	-1.3433	0.3949	-2.0369	1.2834	-4.7194
$\mathbf{x}_5'\beta$	1.9524	0.2024	1.9016	0.2506	4.4987
$\mathbf{x}_6'\beta$	3.4318	-5.9878	-2.6037	2.3821	-4.1399
$\mathbf{x}_7'\beta$	0.3354	-1.2208	-0.8734	6.0272	1.0797
$\mathbf{x}_8'\beta$	1.6384	-0.8696	-2.1418	3.8963	1.7428
$\mathbf{x}_9'\beta$	4.0188	1.4523	4.3244	-1.0890	4.8221
$\mathbf{x}_{10}'\beta$	1.6178	0.0707	2.6437	-0.8118	-1.1066
$\mathbf{x}_{11}'\beta$	-1.7915	1.6844	2.7347	0.0494	-1.6381
$\mathbf{x}_{12}'\beta$	-5.3504	-1.6457	3.5349	-5.5191	4.6864
$\mathbf{x}_{13}'\beta$	-4.3380	7.6956	0.6899	0.8110	-1.3056
$\mathbf{x}_{14}'\beta$	-3.0503	-2.8208	-4.0540	2.4511	-4.1988
$\mathbf{x}_{15}'\beta$	-0.1265	3.2096	1.3427	-2.0528	-0.4612
$\mathbf{x}_{16}'\beta$	3.7432	-1.5836	-0.4446	1.3662	0.8019
γ_0	-0.0180	-0.4538	-0.3366	0.1422	0.2849
γ_1	0.0000	0.0000	0.0000	0.0000	0.0000
γ_2	0.0000	0.0000	0.0000	0.1418	0.0000
γ_3	0.0000	0.2350	0.0000	0.0000	0.0000
γ_4	0.4287	-0.3176	0.0000	0.0000	0.0000
γ_5	0.0000	0.0000	0.0000	0.0000	0.0000
γ_6	0.0000	0.0000	0.0423	-0.6052	-0.2481
γ_7	0.0000	0.1772	0.0000	-0.5208	0.0000
γ_8	-0.1695	0.0000	0.0000	-0.5279	0.3916
γ_9	0.0000	-0.4534	0.4171	0.0000	0.0000
γ_{10}	0.0000	0.0566	0.0000	0.0000	0.0000
γ_{11}	0.0000	0.0000	0.0000	0.0000	0.0000
γ_{12}	0.6937	0.7859	0.0000	0.0950	0.0000
γ_{13}	0.0000	0.0000	-0.5894	-0.0287	-0.4263
γ_{14}	0.0000	0.0630	-0.0373	0.2955	0.3703
γ_{15}	0.0000	0.2746	0.0000	0.0000	0.0000

Table 19.3: Treatment combination means and dispersion parameters for fixed models simulated with $r = 10$ replicates per treatment combination to generate observed Type I errors given in Table 19.6

error cannot be guaranteed to be controlled to approximately the nominal level for all models, S/N_N is eliminated from consideration. The power of s^2, s, $ln(s^2+1)$, and $ln(s+1)$ is studied in Chapter 20.

The residuals and expanded residuals produce observed Type I error rates greater than 0.065, and frequently greater than 0.10, for all non-null single models generated (see columns 2-6 of Table 19.4). Similarly, the natural logarithm of the residuals and expanded residuals produce Type I error rates greater than the nominal level

Measure	Null Model	Fixed Model 4.1	Fixed Model 4.2	Fixed Model 4.3	Fixed Model 4.4	Fixed Model 4.5
s	0.0500	0.0222	0.0057	0.0323	0.0162	0.0023
$ln(s^2+1)$	0.0510	0.0220	0.0150	0.0358	0.0229	0.0030
$ln(s+1)$	0.0504	0.0250	0.0127	0.0319	0.0159	0.0028
S/N_N	0.0511	0.1009	0.0075	0.0064	0.0126	0.0000
$\|y_{ij}-\widetilde{y}_i\|$	0.0511	0.0513	0.0953	0.0775	0.1981	0.0734
$\|y_{ij}-\widetilde{y}_i\|_{-1}$	0.0508	0.0507	0.0980	0.0786	0.2088	0.0711
$\|y_{ij}-\bar{y}_{i(-1)}\|$	0.0511	0.0513	0.0953	0.0775	0.1981	0.0734
$\|y_{ij}-\bar{y}_i\|$	0.0514	0.0518	0.0941	0.0759	0.1955	0.0666
$\|y_{ij}-\bar{y}_i\|^{0.42}$	0.0513	0.0502	0.0881	0.0567	0.0870	0.0554
$\|y_{ij}-\bar{y}_i\|^{1.5}$	0.0509	0.0509	0.0801	0.0837	0.2156	0.0655
$\|y_i-\mathbf{x}_i'\hat{\beta}\|$	0.0500	0.0662	0.1202	0.1099	0.3390	0.0906
$\|y_i-\mathbf{x}_i'\hat{\beta}\|^2$	0.0498	0.0692	0.0845	0.1378	0.3336	0.0964
$\|y_i-\mathbf{x}_i'\hat{\beta}\|^{1.5}$	0.0497	0.0687	0.0973	0.1305	0.3579	0.0986
$\|y_i-\mathbf{x}_i'\hat{\beta}_{exp}\|$	0.0496	0.0655	0.1333	0.1041	0.3170	0.0943
$\|y_i-\mathbf{x}_i'\hat{\beta}_{exp}\|^2$	0.0496	0.0681	0.0888	0.1303	0.3303	0.0960
$\|y_i-\mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	0.0495	0.0680	0.1043	0.1241	0.3475	0.1003
$ln(\|y_{ij}-\widetilde{y}_i\|_{-1})$	0.0517	0.0498	0.0516	0.0504	0.0510	0.0519
$ln(\|y_{ij}-\bar{y}_{i(-1)}\|)$	0.0522	0.0500	0.0519	0.0509	0.0516	0.0522
$ln(\|y_{ij}-\bar{y}_i\|)$	0.0507	0.0503	0.0507	0.0510	0.0505	0.0512
$ln(\|y_i-\mathbf{x}_i'\hat{\beta}\|)$	0.0509	0.0561	0.2124	0.0586	0.1580	0.1001
$ln(\|y_i-\mathbf{x}_i'\hat{\beta}_{exp}\|)$	0.0513	0.0558	0.2275	0.0570	0.1247	0.1039

Table 19.4: Observed Type I error for testing $H_0^*:\gamma_1=0$ (14.3) using level $\alpha=0.05$ critical values from null distribution for 100,000 simulations of single randomly generated second-order location and second-order dispersion models with $r=4$ and $\gamma_1=0$

Measure	Null Model	Fixed Model 7.1	Fixed Model 7.2	Fixed Model 7.3	Fixed Model 7.4	Fixed Model 7.5
s	0.0486	0.0037	0.0116	0.0015	0.0003	0.0203
$ln(s^2+1)$	0.0489	0.0147	0.0155	0.0005	0.0003	0.0242
$ln(s+1)$	0.0482	0.0093	0.0218	0.0032	0.0014	0.0254
$ln(\|y_{ij}-\widetilde{y}_i\|_{-1})$	0.0501	0.0505	0.0516	0.0502	0.0500	0.0506
$ln(\|y_{ij}-\bar{y}_{i(-1)}\|)$	0.0501	0.0510	0.0501	0.0501	0.0500	0.0494
$ln(\|y_{ij}-\bar{y}_i\|)$	0.0487	0.0495	0.0495	0.0500	0.0494	0.0490

Table 19.5: Observed Type I error for testing $H_0^*:\gamma_1=0$ (14.3) using level $\alpha=0.05$ critical values from null distribution for 100,000 simulations of single randomly generated second-order location and second-order dispersion models with $r=7$ and $\gamma_1=0$

Measure	Null Model	Fixed Model 10.1	Fixed Model 10.2	Fixed Model 10.3	Fixed Model 10.4	Fixed Model 10.5
s	0.0514	0.0439	0.0000	0.0251	0.0001	0.0007
$ln(s^2+1)$	0.0507	0.0361	0.0000	0.0265	0.0002	0.0033
$ln(s+1)$	0.0507	0.0320	0.0002	0.0283	0.0019	0.0127
$ln(\|y_{ij}-\widetilde{y}_i\|_{-1})$	0.0502	0.0502	0.0508	0.0496	0.0499	0.0489
$ln(\|y_{ij}-\bar{y}_{i(-1)}\|)$	0.0503	0.0499	0.0519	0.0510	0.0512	0.0506
$ln(\|y_{ij}-\bar{y}_i\|)$	0.0506	0.0503	0.0514	0.0509	0.0513	0.0514

Table 19.6: Observed Type I error for testing $H_0^*: \gamma_1 = 0$ (14.3) using level $\alpha = 0.05$ critical values from null distribution for 100,000 simulations of single randomly generated second-order location and second-order dispersion models with $r = 10$ and $\gamma_1 = 0$

in most of the single non-null model simulations with $r = 4$. These results indicate that the empirical critical values for the test statistics from these dispersion measures are unstable and cannot be validated across different models. Lacking valid critical values, the location model residuals and the natural logarithm of the location model residuals are not considered further.

For all models, the natural logarithm of the absolute deviation from the median minus the minimum observation, absolute deviation from the trimmed mean, and absolute deviation from the mean produce Type I error rates controlled to close to the nominal $\alpha = 0.05$. The power of each of these four dispersion measures is studied (Chapter 20).

The original alternative measures (described in Chapter 13) studied in Phase I as well as the new alternative measure, $|y_{ij} - \widetilde{y}_i|_{-1}$ (described in Chapter 16), produce observed Type I error rates greater than 0.065 for most simulations (see columns 2-6 of Table 19.4). These results are not surprising due to the multiplicative nature of the dispersion effects from model 12.1 with $g(x) = exp(x)$. Because these dispersion measures fail to control the observed Type I error rate in general, these measures are eliminated from study.

For s, $ln(s^2+1)$, $ln(s+1)$, S/N_N, $ln(|y_{ij}-\widetilde{y}_i|_{-1})$, $ln(|y_{ij}-\bar{y}_{i(-1)}|)$ and $ln(|y_{ij}-\bar{y}_i|)$, the observed Type I errors based on the single model simulations for $r=7$ and $r=10$ replicates per treatment combination are given in Table 19.5 and Table 19.6, respectively. The results for these simulations are similar to the results for $r=4$. For the single model scenario, the Type I error rate using the empirical critical values is stable and controlled. indicating the critical values can be applied to study the power of the tests.

19.2 Multiple Models Scenario Results

Five sets of multiple models simulations are completed for each of $r=4$, $r=7$, and $r=10$; Table 19.7, Table 19.8, and Table 19.9 give the results for these multiple model simulations for s^2, s, $ln(s^2+1)$, $ln(s+1)$, $ln(|y_{ij}-\widetilde{y}_i|_{-1})$, $ln(|y_{ij}-\bar{y}_{i(-1)}|)$ and $ln(|y_{ij}-\bar{y}_i|)$. The results from the multiple model simulations for these seven dispersion measures are similar to the results from the single model scenario simulations; tests using each of the nine dispersion measures provide control of the observed Type I error close to or below the nominal $\alpha=0.05$.

Measure	Random Models 1	Random Models 2	Random Models 3	Random Models 4	Random Models 5
s	0.0181	0.0182	0.0184	0.0182	0.0175
$ln(s^2+1)$	0.0206	0.0202	0.0207	0.0212	0.0205
$ln(s+1)$	0.0192	0.0192	0.0198	0.0199	0.0188
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	0.0508	0.0508	0.0513	0.0512	0.0498
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	0.0510	0.0509	0.0517	0.0517	0.0495
$ln(\|y_{ij} - \bar{y}_i\|)$	0.0507	0.0507	0.0508	0.0511	0.0502

Table 19.7: Observed Type I error for testing $H_0^* : \gamma_1 = 0$ (14.3) using level $\alpha = 0.05$ critical values from null distribution for 100,000 simulations of multiple randomly generated second-order location and second-order dispersion models with $r = 4$ and $\gamma_1 = 0$

Measure	Random Models 1	Random Models 2	Random Models 3	Random Models 4	Random Models 5
s	0.0198	0.0199	0.0206	0.0211	0.0201
$ln(s^2+1)$	0.0225	0.0226	0.0232	0.0229	0.0237
$ln(s+1)$	0.0191	0.0193	0.0203	0.0202	0.0206
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	0.0490	0.0509	0.0502	0.0503	0.0501
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	0.0498	0.0514	0.0501	0.0498	0.0505
$ln(\|y_{ij} - \bar{y}_i\|)$	0.0498	0.0509	0.0507	0.0501	0.0496

Table 19.8: Observed Type I error for testing $H_0^* : \gamma_1 = 0$ (14.3) using level $\alpha = 0.05$ critical values from null distribution for 100,000 simulations of multiple randomly generated second-order location and second-order dispersion models with $r = 7$ and $\gamma_1 = 0$

Measure	Random Models 1	Random Models 2	Random Models 3	Random Models 4	Random Models 5
s	0.0240	0.0236	0.0242	0.0227	0.0235
$ln(s^2+1)$	0.0266	0.0256	0.0267	0.0252	0.0266
$ln(s+1)$	0.0218	0.0218	0.0226	0.0212	0.0221
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	0.0487	0.0514	0.0497	0.0498	0.0494
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	0.0497	0.0516	0.0520	0.0501	0.0506
$ln(\|y_{ij} - \bar{y}_i\|)$	0.0509	0.0515	0.0525	0.0507	0.0515

Table 19.9: Observed Type I error for testing $H_0^* : \gamma_1 = 0$ (14.3) using level $\alpha = 0.05$ critical values from null distribution for 100,000 simulations of multiple randomly generated second-order location and second-order dispersion models with $r = 10$ and $\gamma_1 = 0$

CHAPTER 20

POWER

In Chapter 19, seven measures are found to have stable observed Type I error using the empirical critical values:

T1. Within-run sample standard deviation, s

T2. Natural logarithm of the within-treatment sample variance (plus 1.0), $ln(s^2 + 1)$

T3. Natural logarithm of the within-treatment sample standard deviation (plus 1.0), $ln(s + 1)$

A1. Natural logarithm of the absolute deviation from the median trimmed the minimum value, $ln(|y_{ij} - \widetilde{y}_i|_{-1})$

A2. Natural logarithm of the absolute deviation from the trimmed mean, $ln(|y_{ij} - \bar{y}_{i(-1)}|)$

A3. Natural logarithm of the absolute deviation from the mean, $ln(|y_{ij} - \bar{y}_i|)$.

These seven measures are next compared with respect to power.

Power to detect a single dispersion effect of interest is studied over a set of randomly generated location-dispersion models. All studies of power follow the multiple models scenario, with a new location-dispersion model used to generate each data set (see Section 17.3). By using the multiple models scenario, the power of the test to detect the dispersion effect is averaged over a range of models.

In the model (12.5), $(1/2)ln(2)$ corresponds to a standard deviation ratio of two for the high and low levels of X_1; the standard deviation of the observations when factor 1 is set at the high level (s_+) and the standard deviation of the observations when factor 1 is set at the low level (s_-), the ratio is

$$\begin{aligned} \frac{s_+}{s_-} &= \frac{exp(\gamma_0 + \gamma_1 + \gamma_2 + \ldots)}{exp(\gamma_0 - \gamma_1 + \gamma_2 + \ldots)} \\ &= exp([\gamma_0 + \gamma_1 + \gamma_2 + \ldots] - [\gamma_0 - \gamma_1 + \gamma_2 + \ldots]) \\ &= exp(2\gamma_1). \end{aligned} \tag{20.1}$$

Then for $\gamma_1 = (1/2)ln(2)$,

$$\begin{aligned} \frac{s_+}{s_-} &= exp(2\gamma_1) \\ &= exp(2\left[\frac{1}{2}ln(2)\right]) \\ &= 2. \end{aligned}$$

The dispersion effect when the standard deviation is twice as large at one factor setting as compared to the other should be detectable. Detection of this ratio of two provides a reasonable performance criterion. Power to detect dispersion effects inducing standard deviation ratios greater than two should be considerably greater, and ideally should approach 1.00. In the current work, the value of γ_1 is incremented from $\gamma_1 = 0$ to $\gamma_1 = (1/2)ln(5)$ (corresponding to a standard deviation ratio equal to five) in fifty steps.

From equation (20.1), it can be seen that the ratio of the standard deviations at the high and low level of the factor of interest are independent of all other dispersion parameters in the model, including the overall variance γ_0. This independence is not seen in the additive model. In the additive dispersion model, for example, for $\gamma_1 = 2$ and assuming all other $\gamma_i = 0$ for $i \neq 0, 1$, if $\gamma_0 = 4$ then $s_+/s_- = 3$ while if $\gamma_0 = 10$ then $s_+/s = 1.5$. In this case, although γ_1 is constant, the power to detect γ_1 is dependent upon the value of γ_0.

The remaining model parameters are generated as described in Section 17.1. Two sets of simulations are conducted: (i.) first order dispersion models (i.e. dispersion models with only main effects) and (ii.) second order dispersion models (i.e. dispersion models with main effects and two-factor interaction effects). For all simulations, data are generated according to a second-order location model. Finally, all errors, ϵ_{ij} are generated from a $N(0, 1)$ distribution.

For each value of γ_1, 100,000 data sets are generated. For each data set, the parameters are generated as described above. The location-dispersion model (12.1) including all main effects and two-factor interaction effects is fit to the data (as described in Chapter 17) for both sets of simulations. Lenth's test statistic, t_{PSE}, is used to conduct tests for s^2, s, $ln(s^2 + 1)$, and $ln(s + 1)$; the M-test statistic is used for $ln(|y_{ij} - \widetilde{y}_i|_{-1})$, $ln(|y_{ij} - \bar{y}_{i(-1)}|)$, and $ln(|y_{ij} - \bar{y}_i|)$. The proportion of times a test rejects the null hypothesis $H_0 : \gamma_1 = 0$ is a measure of the power of the measure to detect a dispersion effect of the given size. Detection powers are found over the range of γ_1 stated above, and power curves are plotted and compared in Sections 20.1–20.6.

20.1 First-Order Dispersion Model with $r = 4$ Replicates

Figure 20.1 shows the power curves for tests based on each of the seven dispersion measures when data are generated from first-order dispersion models with $r = 4$

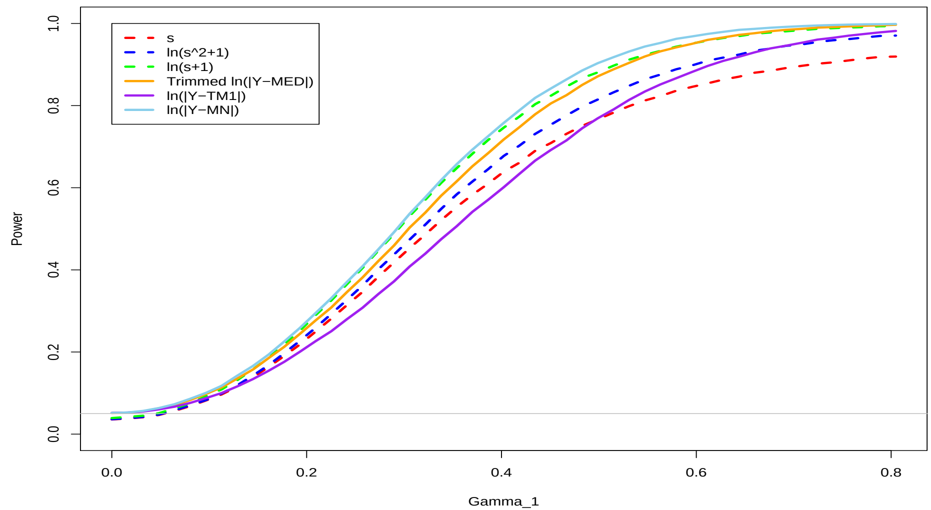

Figure 20.1: Power curves for tests using s, $ln(s^2+1)$, $ln(s+1)$, $ln(|y_{ij} - \tilde{y}_i|_{-1})$, $ln(|y_{ij} - \bar{y}_{i(-1)}|)$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated second-order location models (12.2) and first-order dispersion models (12.3) using empirical critical values, $r = 4$ replicates, and normal error distribution

replicates. Based on these power curves, the test based on the natural logarithm of the absolute deviation from the mean provides greater power to detect the dispersion effect across all effect sizes than all other measures. At the same time, this test (using the empirical critical value given in Table 18.2) controls the Type I error rate close to the nominal $\alpha = 0.05$ significance level. The tests based on the natural logarithm of the standard deviation and the natural logarithm of the absolute deviation from the median trimming the minimum observation provide power close to the test based on $ln(|y_{ij} - \bar{y}_i|)$; all other tests provide power less than the test based on $ln(|y_{ij} - \bar{y}_i|)$. Going forward, only tests based on $ln(|y_{ij} - \bar{y}_i|)$, $ln(|y_{ij} - \tilde{y}_i|_{-1})$, and $ln(s+1)$ will be studied.

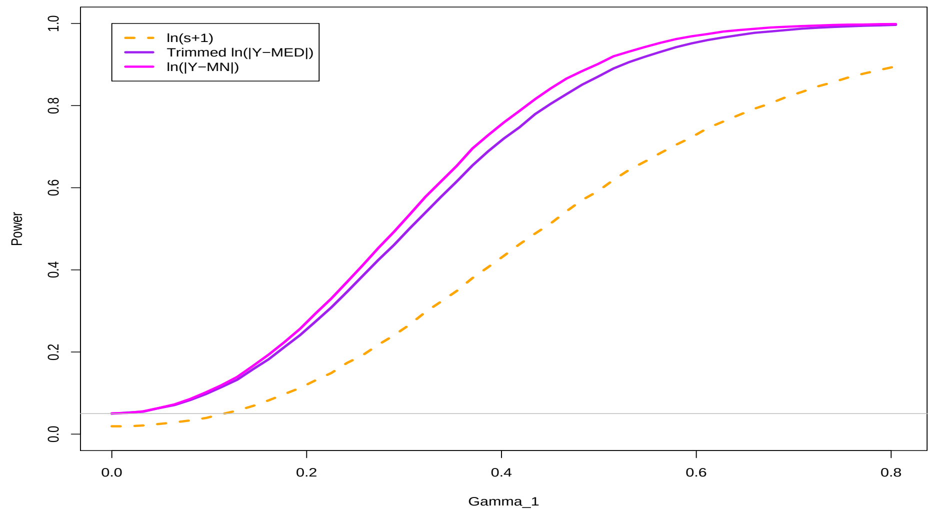

Figure 20.2: Power curves for tests using $ln(s + 1)$, $ln(|y_{ij} - \widetilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated second-order location models (12.2) and second-order dispersion models (12.3) using empirical critical values, $r = 4$ replicates, and normal error distribution

20.2 Second-Order Dispersion Model with $r = 4$ Replicates

The power curves for test based on $ln(|y_{ij} - \bar{y}_i|)$, $ln(|y_{ij} - \widetilde{y}_i|_{-1})$, and $ln(s + 1)$ for data with $r = 4$ replicates per treatment combination from second-order dispersion models are shown in Figure 20.2. Similar to the power for data from first-order dispersion models, the test based on $ln(|y_{ij} - \bar{y}_i|)$ provides the greatest power to detect the dispersion effect; power of the test based on $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ is again close to the power of the test based on $ln(|y_{ij} - \bar{y}_i|)$. Comparing, Figure 20.2 and Figure 20.1, the power of the test based on $ln(s + 1)$ is greatly decreased when data are generated according to a second-order dispersion model. While the power of the test based on $ln(s + 1)$ is close to the power of the test based on $ln(|y_{ij} - \bar{y}_i|)$ when data are generated according to a first-order dispersion model, a large difference between the power of these two tests exists when data are generated according to a second-order dispersion model.

20.3 Power of Tests for Varying Numbers of Replicates Per Treatment Combination

Figures F.1 – F.8 in Appendix F show the power curves for tests based on $ln(|y_{ij} - \bar{y}_i|)$, $ln(|y_{ij} - \widetilde{y}_i|_{-1})$, and $ln(s + 1)$ for both first- and second-order dispersion model data for numbers of replicates from $r = 3$ up to $r = 10$, with power for first-order model data represented by solid lines and power for second-order model data represented by dashed lines. In each case, the power of the test based on $ln(|y_{ij} - \bar{y}_i|)$ and $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ show consistent power whether the data are generated from either first- or second-order models. This consistency is not seen for the test based on $ln(s + 1)$; a significant difference in power for this test based on $ln(s + 1)$ exists between data generated from first-order models as compared to second-order models (Figures F.1 – F.8). For these two measures, while the power may be good in one case (i.e. for first-order data), the power is significantly decreased for the second case (i.e. for second-order data). The reason for this disparity is considered later in Section 20.4. As both types of models will occur in applications, this significant power differential is undesirable.

For $3 \leq r \leq 6$, the test based on $ln(|y_{ij} - \bar{y}_i|)$ provides the greatest power for both first- and second-order data. The test based on $ln(|y_{ij} - \bar{y}_i|)$ provides the greatest power among the tests studied, also, for second-order data for all numbers of replicates, r, examined. The observed Type I error rate for the natural logarithm of the absolute deviation from the mean is close to the nominal $\alpha = 0.05$ significance level for all numbers of replicates, r, considered.

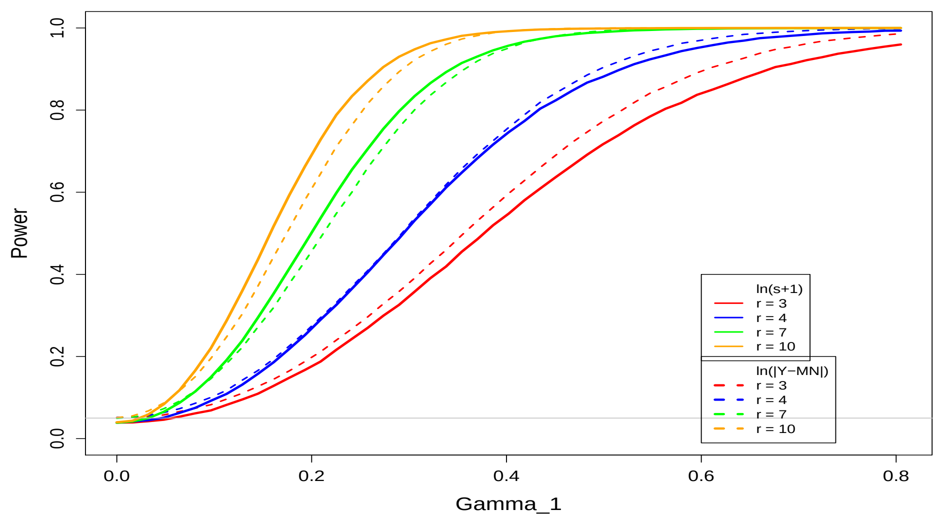

Figure 20.3: Power curves for tests using $ln(s+1)$ and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated second-order location models (12.2) and first-order dispersion models (12.3) using empirical critical values, $r = 3, 4, 7, 10$ replicates, and normal error distribution

In the case of $r > 6$ and data generated from first-order models, the test based on $ln(s+1)$ gives the greatest detection power over part of the range of effect sizes. Figure 20.3 shows the power curves for $ln(s+1)$ and $ln(|y_{ij} - \bar{y}_i|)$ for $r = 3, 4, 7, 10$ replicates per treatment combination. The power of both tests increases as the number of replicates increases, but at different rates. For all r, the test based on $ln(|y_{ij} - \bar{y}_i|)$ shows greater power than the test based on $ln(s+1)$ for smaller effect sizes. For the largest effect sizes, the power of the two tests is similar. The test based on $ln(s+1)$ dominates only over the middle range of effect sizes, and the difference is not great.

In application, the number of replications per treatment combination is usually limited. The use of $r = 3$ or $r = 4$ replicates per treatment combination is more likely to occur in practice than the use of $r \geq 7$ replicates. The superior performance of the test based on $ln(|y_{ij} - \bar{y}_i|)$ is of greater interest and impact; the smaller increase in power of the natural logarithm of the standard deviation for larger numbers of replicates is of less value. Thus, the use of the natural logarithm of the absolute deviation from the mean is still preferred over the natural logarithm of the standard deviation.

The detection power of the test based on $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ is greater than the power of the test based on $ln(s + 1)$ when data are generated from second-order models. This relationship is reversed for data generated from first-order models.

The power curves for $ln(s + 1)$ and $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ for $r = 3, 4, 7, 10$ replicates per treatment combination and first-order data are shown in Figure 20.4. For all numbers of replicates per treatment combination examined, the power of the test based on $ln(s + 1)$ is greater than the power of the test based on $ln(|y_{ij} - \widetilde{y}_i|_{-1})$. The difference in power between the two tests is not great for $r = 3$ and $r = 4$, the numbers of replicates more likely to occur in practice; the difference in power increases as r increases.

As the number of replicates per treatment combination, r, increases, the detection power of the test based on $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ increases. However, up to the number of replicates examined here, the power is never as great as that of the test based on $ln(|y_{ij} - \bar{y}_i|)$.

20.4 Power of Test Based on Natural Logarithm of the Standard Deviation for First-Order vs. Second-Order Dispersion Models

As noted, the power changes significantly for the test based on $ln(s + 1)$ depending on whether data are generated from first- or second-order models, with significantly

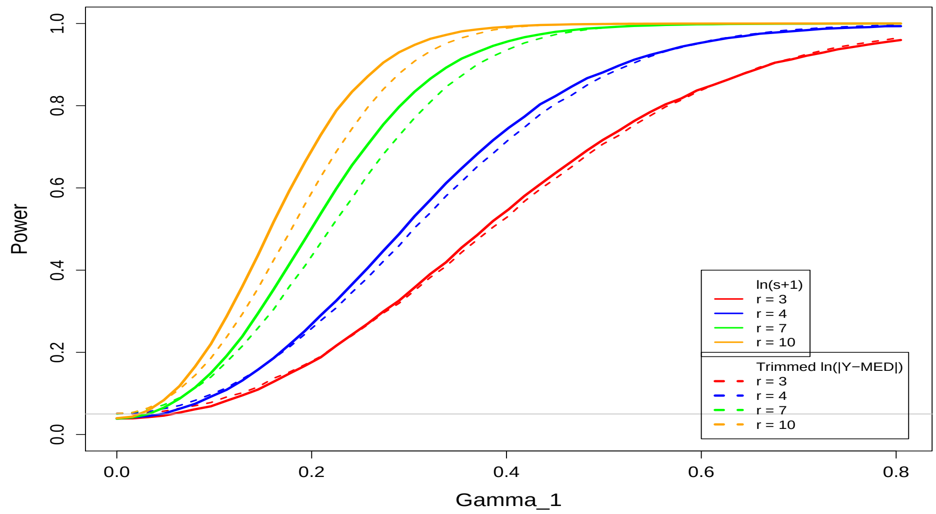

Figure 20.4: Power curves for tests using $ln(s+1)$ and $ln(|y_{ij} - \tilde{y}_i|_{-1})$ with data from randomly generated second-order location models (12.2) and first-order dispersion models (12.3) using empirical critical values, $r = 3, 4, 7, 10$ replicates, and normal error distribution

greater power for data generated from first-order models. This power difference is likely the result of calculation of the pseudo standard error of Lenth's method. From equation (17.2), the pseudo standard error is a scalar times the median of the estimated effects. When data are generated from second-order dispersion models, it is possible that all effects in the fitted model are non-zero. If the median estimated effect is roughly equal to or greater than the effect, γ_1, of interest, then the pseudo standard error will be greater than the effect of interest. In this case, the test statistic, t_{PSE}, given in equation (17.1) will be less than or equal to one. Thus, the power to detect the effect will be decreased.

In order to test the above conjecture that the low power of the test based on $ln(s+1)$ is the result of the possibility of all non-zero effect estimates in the fitted

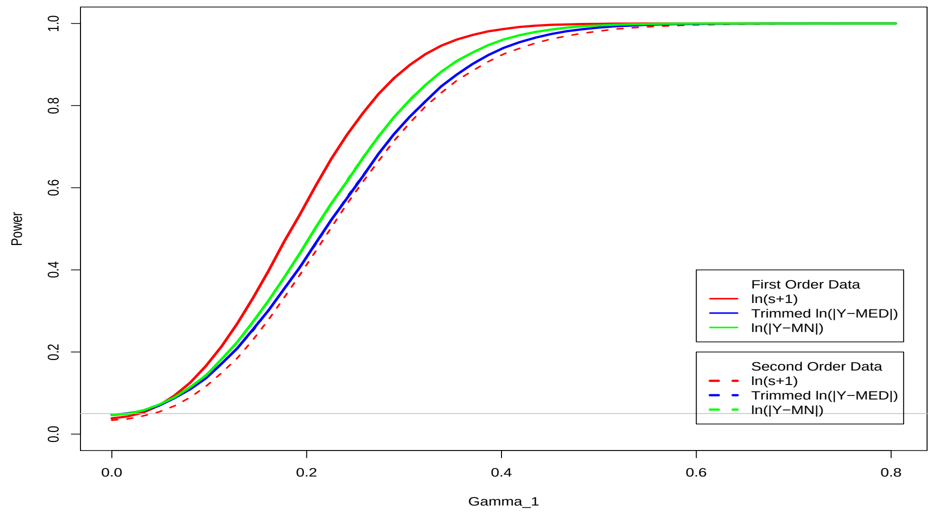

Figure 20.5: Power curves for tests using $ln(s+1)$, $ln(|y_{ij} - \widetilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ from full factorial design with data from randomly generated mean vectors and first- and second-order variance models using empirical critical values from Tables 18.1 and 18.2, $r = 4$ replicates, and normal error distribution

model, the simulation (see Chapter 17) is repeated using a 2^5 full factorial design. Data are again generated from both first- and second-order random dispersion models. Because the full factorial design is used, the full model fit to the data includes three-, four-, and five-factor interaction effects. These higher order effects are known to be zero. As a result of the zero effects, the pseudo standard error should be smaller than previous simulations; the smaller pseudo standard error should provide for increased power to detect the dispersion effect. The power curves from these simulations are shown in Figure 20.5. As can be seen in Figure 20.5, the power of the test based on $ln(s+1)$ is close to the power of the tests based on $ln(|y_{ij} - \bar{y}_i|)$ and $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ for second-order dispersion model data. These results indicate that the addition of inactive effects to the model does increase the power of the test, providing evidence of the given explanation.

The power of the test based on $ln(s+1)$ is again greater for first-order dispersion model data compared to second-order dispersion model data (Figure 20.5). Though both the first- and second-order dispersion models include effects known to be zero, the number of effects known to be zero is different between the two models. The greater the number of effects equal to zero, the smaller the median estimated effect and the smaller the pseudo standard error in the denominator of t_{PSE}. The value of the test statistic, t_{PSE}, is thereby increased, resulting in increased power of the test to reject the null hypothesis.

As before, the power of $ln(|y_{ij} - \bar{y}_i|)$ and $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ is similar for both types of data.

20.5 Non-normal Error Distributions

Based on control of the observed Type I error (Chapter 19) and the observed power (Chapter 20), the tests based on $ln(|y_{ij} - \bar{y}_i|)$ and $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ appear to be the best for detecting a dispersion effect while controlling the Type I error rate. At this point, it is advantageous to examine how these two tests perform under different model assumptions.

In order to study the effect of non-normal error distributions, additional simulations are run assuming either a Cauchy(0,1) or exponential(1) error distribution; simulations are run with $r = 4$ replicates per treatment combination. These non-normal simulations are run following the power simulations described in Chapter 20,

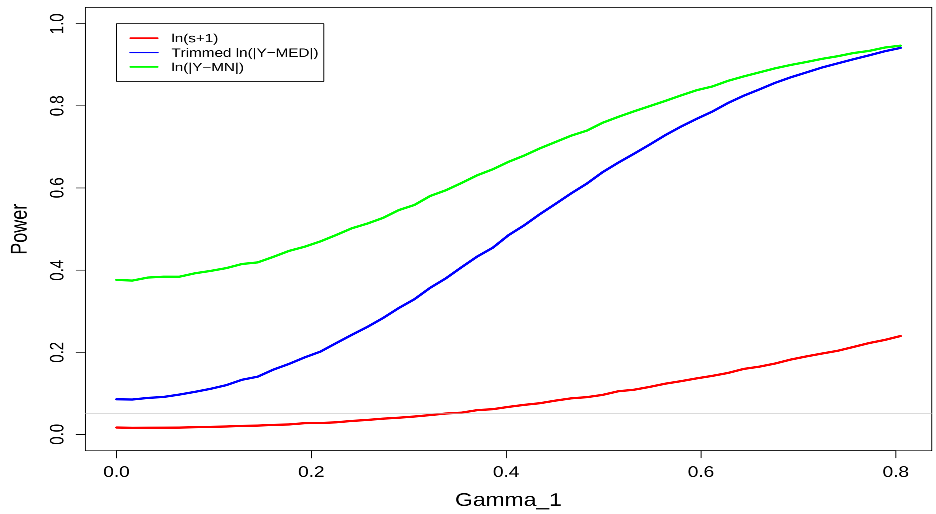

Figure 20.6: Power curves for tests based on $ln(s+1)$, $ln(|y_{ij} - \tilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated second-order location models (12.2) and second-order dispersion models (12.3) using empirical critical values based on normally distributed errors, $r = 4$ replicates, and a Cauchy(0,1) error distribution

generating data from second-order location models and second-order dispersion models. For all tests of the null hypothesis $H_0 : \gamma_1 = 0$, the critical values based on errors following a normal distribution given in Table 18.2 are used. The power curves based on the results from these simulations are shown in Figure 20.6 and Figure 20.7.

From Figure 20.6 and Figure 20.7, the observed Type I error rates for the test based on $ln(|y_{ij} - \bar{y}_i|)$ are 0.3762 and 0.1520 for the Cauchy and exponential error distributions, respectively. The test based on $ln(|y_{ij} - \bar{y}_i|)$ is clearly not a robust test to violations of the normal distribution assumption. If the observed data do not follow approximately a normal distribution, the proposed test should not be used.

The test based on $ln(s + 1)$ controls the Type I error rate to below the nominal $\alpha = 0.05$ significance level for data following a Cauchy or exponential distribution.

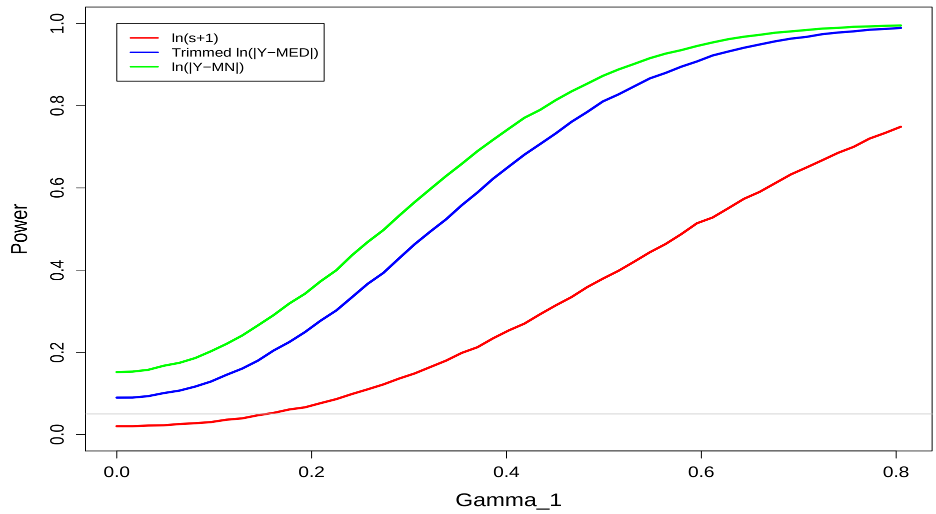

Figure 20.7: Power curves for tests based on $ln(s+1)$, $ln(|y_{ij} - \widetilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated second-order location models (12.2) and second-order dispersion models (12.3) using empirical critical values based on normally distributed errors, $r = 4$ replicates, and an exponential(1) error distribution

The low error rate for this test is at the cost of power; the power of the test based on $ln(s+1)$ is extremely low across the full range of effect sizes for both non-normal error distributions considered.

The observed Type I error rates for the test based on $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ are 0.0855 and 0.0898 when the errors follow a Cauchy or exponential distribution, respectively; these Type I error rates for the test based on $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ are not as great as for the test based on $ln(|y_{ij} - \bar{y}_i|)$. Using the definition from Conover et al. (1981) that a test is robust if the maximum Type I error rate is less than 0.10 for an $\alpha = 0.05$ significance level test, the test based on $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ qualifies as a robust test in these cases. The power of this test is much greater than the power of the test based on $ln(s+1)$ and approaches the power of the test based on $ln(|y_{ij} - \bar{y}_i|)$. For detecting a dispersion effect when the distribution cannot be assumed to normal, a test based on $ln(|y_{ij} - \tilde{y}_i|_{-1})$ is preferred to a test based on either $ln(|y_{ij} - \bar{y}_i|)$ or $ln(s+1)$.

20.6 Additive Dispersion Models

The use of the natural logarithm of either the absolute deviation from the mean of the absolute deviation from the median trimmed the minimum observation is proposed based on the multiplicative model (see Chapter 12). However, in application, the dispersion model may be either multiplicative, as in Phase II, or additive, as in Phase I. The question is then how well these tests perform with respect to the additive model. In order to answer this question, additional simulations are run generating data from random second-order location models and second-order additive dispersion models, i.e. model (12.1) with $g(\mathbf{x}'_{\sigma,i}\gamma) = \mathbf{x}'_{\sigma,i}\gamma$. The mean vector, $\mu_i = \mathbf{x}'_{\mu,i}\beta$, for each data set is generated according to $N(0, 10)$. For these simulations, $\gamma_0 = 10$ and γ_1 is increased from $\gamma_1 = 0$ to $\gamma_1 = 5$ in increments of 0.1. For each model, γ_i, $i \neq 0, 1$, are generated according to

$$\gamma_i = \begin{cases} Z_i & \text{with probability } 0.4 \\ 0 & \text{with probability } 0.6 \end{cases}$$

where $Z_i \sim N(0, 2)$. For each randomly generated vector of γ_i, the standard deviation of each treatment combination is checked to verify that $\sigma_i > 0$ for all i; if $\sigma_i \leq 0$, then the vector of dispersion effects is discarded and a new vector generated. All errors, ϵ_{ij}, are generated from a $N(0, 1)$ distribution. The power curves from these simulations are presented in Figure 20.8.

The observed Type I error for tests based on $ln(|y_{ij} - \bar{y}_i|)$ and $ln(|y_{ij} - \tilde{y}_i|_{-1})$ when the dispersion model is additive is greater than the nominal $\alpha = 0.05$ significance level but less than 0.075. Both tests are considered robust using the definition

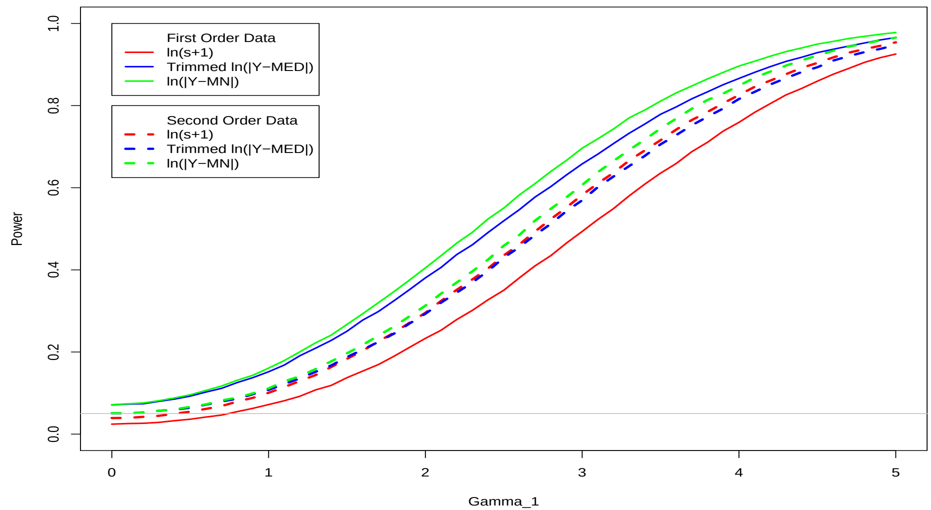

Figure 20.8: Power curves for tests based on $ln(s+1)$, $ln(|y_{ij} - \widetilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated second-order location models (12.2) and second-order additive dispersion models (12.3) using empirical critical values, $r = 4$ replicates, and normal error distribution

from Conover et al. (1981). As with the multiplicative dispersion model, the test based on $ln(|y_{ij} - \bar{y}_i|)$ provides greater power to detect the dispersion effect than the test based on $ln(|y_{ij} - \widetilde{y}_i|_{-1})$; the test based on $ln(s+1)$ provides less power than the test based on either alternative dispersion measure. From these results, $ln(|y_{ij} - \bar{y}_i|)$ and $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ can be used whether the dispersion model is additive or multiplicative.

CHAPTER 21

RECOMMENDATION

Two competing goals must be balanced in recommending a dispersion measure to use for the identification of dispersion effects when experiments are replicated: power to detect a true dispersion effect and control of the Type I error rate. Ideally the same dispersion measure will prove to be both powerful and robust for varying models (e.g. different error distributions, both additive and multiplicative dispersion models). Based on the current work, this is not the case and the recommendation must be qualified by the type of data observed.

From Chapter 19, both the natural logarithm of the absolute deviation from the mean, $ln(|y_{ij} - \bar{y}_i|)$, and the natural logarithm of the absolute deviation from the median trimmed the minimum observation, $ln(|y_{ij} - \widetilde{y}_i|_{-1})$, lead to tests that control the Type I error rate close to the nominal $\alpha = 0.05$ significance level for data following a normal distribution with a multiplicative dispersion model (Tables 19.4–19.9). Of these two dispersion measures, the test based on $ln(|y_{ij} - \bar{y}_i|)$ is shown in Chapter 20 to provide greater power to detect the dispersion effect than the test based on $ln(|y_{ij} - \widetilde{y}_i|_{-1})$, assuming normally distributed data with a multiplicative dispersion model (Figures F.1–F.8); the power of the test based on $ln(|y_{ij} - \bar{y}_i|)$ is similar whether data are generated from a first- or second-order dispersion model. Additionally, the test based on $ln(|y_{ij} - \bar{y}_i|)$ controls the Type I error rate and provides greater power than the test based on $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ when the dispersion model is assumed to be additive (Figure 20.8). Figures 20.6 and 20.7 show, however, that the test based on $ln(|y_{ij} - \bar{y}_i|)$ has Type I error much greater than the nominal $\alpha = 0.05$ significance level when the data follow either a Cauchy(0,1) or exponential(1) distribution. From these results, the recommendation based on the current work is to use the M-test statistic (14.4) based on the natural logarithm of the absolute deviation from the mean, $ln(|y_{ij} - \bar{y}_i|)$, with critical values from Table 18.2 when there is evidence that the data follow a normal distribution; if the distribution of the data is known to be non-normal or is not known, $ln(|y_{ij} - \bar{y}_i|)$ should not be used as the dispersion measure response for the test.

For both the Cauchy and exponential error distributions the test based on the measure $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ is robust according to the definition given by Conover et al. (1981) with Type I error rate less than 2α. The test based on $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ provides high power for detecting the dispersion effect as well as controls the Type I error. While the power of the test based on$ln(|y_{ij} - \widetilde{y}_i|_{-1})$ is exceeded by the power of the test based on $ln(|y_{ij} - \bar{y}_i|)$ when the data follow a normal distribution (Figures F.1–F.8 and Figure 20.8), the difference in power between the two tests is not large. Therefore, when there is little or no evidence that the data follow a normal distribution the recommendation based on the current work is to use the M-test statistic based on the natural logarithm of the absolute deviation from the median trimmed the minimum observation, $ln(|y_{ij} - \widetilde{y}_i|_{-1})$, with critical values from Table 18.2.

Empirical critical values are needed for the implementation of the test methodology using either $ln(|y_{ij} - \bar{y}_i|)$ or $ln(|y_{ij} - \widetilde{y}_i|_{-1})$. A table of empirical critical values for a limited number of error degrees of freedom is given in Table 18.2 of the current work; additional critical values can be generated via Monte Carlo simulation.

Whether $ln(|y_{ij} - \bar{y}_i|)$ is used for data known to follow a normal distribution or $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ is used for data not known to follow a normal distribution, the power of the test based of these alternative measures is greater than the power of the tests based on traditional measures such as $ln(s + 1)$ while controlling the Type I error rate. For experimental designs in which effects can be estimated independently, the dispersion effect test methodology proposed in the current work provides a beneficial alternative to traditional dispersion test methodologies for data from replicated experiments.

APPENDIX A

ORTHOGONAL ARRAYS

1	1	1	1	1	1	1
2	2	2	2	2	2	1
0	0	0	0	0	0	1
1	1	2	0	2	0	1
2	2	0	1	0	1	1
0	0	1	2	1	2	1
1	2	1	0	0	2	2
2	0	2	1	1	0	2
0	1	0	2	2	1	2
1	0	0	1	2	2	2
2	1	1	2	0	0	2
0	2	2	0	1	1	2
1	2	0	2	1	0	0
2	0	1	0	2	1	0
0	1	2	1	0	2	0
1	0	2	2	0	1	0
2	1	0	0	1	2	0
0	2	1	1	2	0	0

Table A.1: $OA(18, 7, 3, 2)$

1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	0	0	1	2	0	2	1	1
0	1	0	0	0	2	2	1	0	2	0	1	1
1	2	1	2	2	2	0	0	1	2	0	2	1
2	2	2	0	0	1	2	0	2	1	1	2	1
0	2	0	1	1	0	1	0	0	0	2	2	1
1	0	1	0	0	0	2	2	1	0	2	0	1
2	0	2	1	1	2	1	2	2	2	0	0	1
0	0	0	2	2	1	0	2	0	1	1	0	1
1	1	2	1	2	2	2	0	0	1	2	0	2
2	1	0	2	0	1	1	0	1	0	0	0	2
0	1	1	0	1	0	0	0	2	2	1	0	2
1	2	2	2	0	0	1	2	0	2	1	1	2
2	2	0	0	1	2	0	2	1	1	2	1	2
0	2	1	1	2	1	2	2	2	0	0	1	2
1	0	2	0	1	1	0	1	0	0	0	2	2
2	0	0	1	2	0	2	1	1	2	1	2	2
0	0	1	2	0	2	1	1	2	1	2	2	2
1	1	0	1	0	0	0	2	2	1	0	2	0
2	1	1	2	1	2	2	2	0	0	1	2	0
0	1	2	0	2	1	1	2	1	2	2	2	0
1	2	0	2	1	1	2	1	2	2	2	0	0
2	2	1	0	2	0	1	1	0	1	0	0	0
0	2	2	1	0	2	0	1	1	0	1	0	0
1	0	0	0	2	2	1	0	2	0	1	1	0
2	0	1	1	0	1	0	0	0	2	2	1	0
0	0	2	2	1	0	2	0	1	1	0	1	0

Table A.2: $OA(27, 13, 3, 2)$

1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	1
0	0	0	0	0	0	0	0	0	0	0	0	1
1	1	1	1	2	2	2	2	0	0	0	0	1
2	2	2	2	0	0	0	0	1	1	1	1	1
0	0	0	0	1	1	1	1	2	2	2	2	1
1	1	2	0	1	2	0	0	1	2	2	0	1
2	2	0	1	2	0	1	1	2	0	0	1	1
0	0	1	2	0	1	2	2	0	1	1	2	1
1	1	0	2	1	0	2	0	2	1	0	2	1
2	2	1	0	2	1	0	1	0	2	1	0	1
0	0	2	1	0	2	1	2	1	0	2	1	1
1	2	0	1	0	2	1	0	0	2	1	2	2
2	0	1	2	1	0	2	1	1	0	2	0	2
0	1	2	0	2	1	0	2	2	1	0	1	2
1	2	0	2	1	1	0	2	0	0	2	1	2
2	0	1	0	2	2	1	0	1	1	0	2	2
0	1	2	1	0	0	2	1	2	2	1	0	2
1	2	1	0	0	0	1	2	2	1	2	0	2
2	0	2	1	1	1	2	0	0	2	0	1	2
0	1	0	2	2	2	0	1	1	0	1	2	2
1	2	2	0	0	1	2	1	1	0	0	2	2
2	0	0	1	1	2	0	2	2	1	1	0	2
0	1	1	2	2	0	1	0	0	2	2	1	2
1	0	2	1	2	0	0	1	0	1	2	2	0
2	1	0	2	0	1	1	2	1	2	0	0	0
0	2	1	0	1	2	2	0	2	0	1	1	0
1	0	2	2	2	1	1	0	2	0	1	0	0
2	1	0	0	0	2	2	1	0	1	2	1	0
0	2	1	1	1	0	0	2	1	2	0	2	0
1	0	0	0	2	0	2	2	1	2	1	1	0
2	1	1	1	0	1	0	0	2	0	2	2	0
0	2	2	2	1	2	1	1	0	1	0	0	0
1	0	1	2	0	2	0	1	2	2	0	1	0
2	1	2	0	1	0	1	2	0	0	1	2	0
0	2	0	1	2	1	2	0	1	1	2	0	0

Table A.3: $OA(36, 13, 3, 2)$

APPENDIX B

ESTIMATION CAPACITY TABLES

Class	A	B	C	AB	AC	BC
18.3.1	2	2	2	4	4	
	2	2	2	4		4
	2	2	2		4	4
18.3.2	2	2	1	1	2	
	2	2	1	1		2
	2	2	2		2	2
18.3.3	2	0	0	0	0	
	0	2	0	0		0
	0	0	2		0	0

Table B.1: Degrees of freedom for estimating all main effects and two two-factor interaction effects for three-factor projections from $OA(18, 7, 3, 2)$

Class	A	B	C	AB	AC	BC
18.3.1	2	2	2	4		
	2	2	2		4	
	2	2	2			4
18.3.2	2	2	1	3		
	2	2	2		4	
	2	2	2			4
18.3.3	2	2	0	2		
	2	0	2		2	
	0	2	2			2

Table B.2: Degrees of freedom for estimating all main effects and one two-factor interaction effect for three-factor projections from $OA(18, 7, 3, 2)$

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
18.4.1	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	1					1
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		1			1	
	2	2	2	2		4				4
	2	2	2	2			1	1		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
18.4.2	2	2	2	1	3	4				
	2	2	2	1	1		2			
	2	2	2	1	3			4		
	2	2	2	1	1				2	
	2	2	2	1	1					2
	2	2	2	2		4	4			
	2	2	2	1		3		3		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			2		2	
	2	0	2	2			2			2
	2	2	2	2				4	4	
	2	2	2	2				4		4
	0	2	2	2					2	2
18.4.3	2	2	2	0	3	3				
	2	2	2	1	1		2			
	2	2	2	0	3			3		
	2	2	2	1	1				2	
	2	2	2	1	3					4
	2	2	2	1		1	2			
	2	2	2	0		3		3		
	2	2	2	1		3			4	
	2	2	2	1		1				2
	2	2	2	1			4	3		
	2	2	2	2			2		2	
	2	2	2	2			2			2
	2	2	2	1				1	2	
	2	2	2	1				1		2
	2	2	2	2					2	2
18.4.4	2	2	2	0	2	2				
	2	2	0	2	2		2			
	2	2	2	0	4			2		
	2	2	0	2	4				2	
	2	0	2	2	4					2
	2	0	2	2		2	2			
	2	2	2	0		4		2		
	2	2	0	2		4			2	
	2	0	2	2		4				2
	2	2	2	0			4	2		
	2	2	0	2			4		2	
	2	0	2	2			4			2
	2	2	0	0				0	0	
	2	0	2	0				0		0
	2	0	0	2					0	0

Table B.3: Degrees of freedom for estimating all main effects and two two-factor interaction effects for four-factor projections from $OA(18, 7, 3, 2)$

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
18.4.1	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
18.4.2	2	2	2	1	3					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
18.4.3	2	2	2	1	3					
	2	2	2	1		3				
	2	2	2	2			4			
	2	2	2	1				3		
	2	2	2	2					4	
	2	2	2	2						4
18.4.4	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	0				2		
	2	2	0	2					2	
	2	0	2	2						2

Table B.4: Degrees of freedom for estimating all main effects and one two-factor interaction effect for four-factor projections from $OA(18, 7, 3, 2)$

Class	A	B	C	D	E	AB	AC	AD	AE	BC	BD	BE	CD	CE	DE
18.5.1	2	2	2	2	2	4									
	2	2	2	2	2		4								
	2	2	2	2	2			4							
	2	2	2	2	2				4						
	2	2	2	2	2					4					
	2	2	2	2	2						4				
	2	2	2	2	2							4			
	2	2	2	2	2								4		
	2	2	2	2	2									4	
	2	2	2	2	2										4
18.5.2	2	2	2	2	1	3									
	2	2	2	2	2		4								
	2	2	2	2	1			3							
	2	2	2	2	2				4						
	2	2	2	2	2					4					
	2	2	2	2	1						3				
	2	2	2	2	2							4			
	2	2	2	2	2								4		
	2	2	2	2	2									4	
	2	2	2	2	2										4
18.5.3	2	2	2	2	1	3									
	2	2	2	2	2		4								
	2	2	2	2	2			4							
	2	2	2	2	2				4						
	2	2	2	2	2					4					
	2	2	2	2	2						4				
	2	2	2	2	2							4			
	2	2	2	2	0								2		
	2	2	2	0	2									2	
	2	2	0	2	2										2
18.5.4	2	2	2	2	1	3									
	2	2	2	2	1		3								
	2	2	2	2	1			3							
	2	2	2	2	2				4						
	2	2	2	2	1					3					
	2	2	2	2	1						3				
	2	2	2	2	2							4			
	2	2	2	2	1								3		
	2	2	2	2	2									4	
	2	2	2	2	2										4

Table B.5: Degrees of freedom for estimating all main effects and one two-factor interaction effect for five-factor projections from $OA(18, 7, 3, 2)$

Class	A	B	C	AB	AC	BC
36.3.1	2	2	2	4	4	4
36.3.2	2	2	2	2	2	2
36.3.3	0	0	0	0	0	0
36.3.4	2	2	2	2	2	2
36.3.5	2	2	2	4	4	4
36.3.6	2	2	2	4	4	4

Table B.6: Degrees of freedom for estimating all main effects and all two-factor interaction effects for three-factor projections from $OA(36, 13, 3, 2)$

Class	A	B	C	AB	AC	BC
36.3.1	2	2	2	4	4	
	2	2	2	4		4
	2	2	2		4	4
36.3.2	2	2	2	4	4	
	2	2	2	4		4
	2	2	2		4	4
36.3.3	2	0	0	0	0	
	0	2	0	0		0
	0	0	2		0	0
36.3.4	2	2	2	4	4	
	2	2	2	4		4
	2	2	2		4	4
36.3.5	2	2	2	4	4	
	2	2	2	4		4
	2	2	2		4	4
36.3.6	2	2	2	4	4	
	2	2	2	4		4
	2	2	2		4	4

Table B.7: Degrees of freedom for estimating all main effects and two two-factor interaction effects for three-factor projections from $OA(36, 13, 3, 2)$

Class	A	B	C	AB	AC	BC
36.3.1	2	2	2	4		
	2	2	2		4	
	2	2	2			4
36.3.2	2	2	2	4		
	2	2	2		4	
	2	2	2			4
36.3.3	2	2	0	2		
	2	0	2		2	
	0	2	2			2
36.3.4	2	2	2	4		
	2	2	2		4	
	2	2	2			4
36.3.5	2	2	2	4		
	2	2	2		4	
	2	2	2			4
36.3.6	2	2	2	4		
	2	2	2		4	
	2	2	2			4

Table B.8: Degrees of freedom for estimating all main effects and one two-factor interaction effect for three-factor projections from $OA(36, 13, 3, 2)$

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.1	2	2	2	2	4	4	4	2	2	2
36.4.2	2	2	2	2	3	4	3	3	4	3
36.4.3	2	2	2	2	4	2	2	4	4	2
36.4.4	2	2	2	2	3	2	3	3	2	3
36.4.5	0	0	2	0	0	4	0	4	0	4
36.4.6	0	0	2	0	0	0	0	0	0	0
36.4.7	2	2	2	2	2	2	2	2	2	2
36.4.8	2	2	2	2	4	2	2	4	4	2
36.4.9	2	2	2	2	4	4	4	4	4	4
36.4.10	0	0	0	0	0	0	0	0	0	0
36.4.11	2	2	2	2	2	2	4	2	4	4
36.4.12	2	2	2	2	4	4	2	2	2	2
36.4.13	2	2	2	2	4	4	4	4	4	4
36.4.14	2	2	2	2	0	0	2	0	2	2
36.4.15	2	2	2	2	4	4	4	4	4	4
36.4.16	2	2	2	2	4	4	4	4	4	4
36.4.17	2	2	2	2	2	2	2	2	2	2
36.4.18	2	2	2	2	2	2	0	2	0	0
36.4.19	2	2	2	2	4	4	4	4	4	4
36.4.20	2	2	2	1	1	1	2	1	2	2
36.4.21	2	2	2	2	2	2	0	2	0	0
36.4.22	2	2	2	2	4	2	2	2	2	2
36.4.23	2	2	2	2	0	0	0	0	0	0
36.4.24	2	2	2	2	0	0	0	0	0	0
36.4.25	2	2	2	2	0	0	2	0	2	2
36.4.26	2	2	2	2	0	0	0	0	0	0
36.4.27	2	2	2	2	0	0	0	0	0	0

Table B.9: Degrees of freedom for estimating all main effects and all two-factor interaction effects for four-factor projections from $OA(36, 13, 3, 2)$

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.1	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	4	4		4	4	2
	2	2	2	2	4		4	2	2	2
	2	2	2	2		4	4	4	4	2
36.4.2	2	2	2	2	4	4	4	4	4	
	2	2	2	2	3	4	3	3		3
	2	2	2	2	4	4	4		4	4
	2	2	2	2	4	4		4	4	4
	2	2	2	2	3		3	3	4	3
	2	2	2	2		4	4	4	4	4
36.4.3	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	2	2	4		2
	2	2	2	2	4	2	2		4	2
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		2	2	4	4	2
36.4.4	2	2	2	2	4	2	3	3	2	
	2	2	2	2	3	3	3	3		3
	2	2	2	2	3	2	4		2	3
	2	2	2	2	3	2		4	2	3
	2	2	2	2	3	3		3	3	3
	2	2	2	2		2	3	3	2	4
36.4.5	0	0	2	0	0	4	0	4	0	
	2	0	2	0	0	4	0	4		4
	0	0	2	0	0	4	0		0	4
	0	2	2	0	0	4		4	0	4
	0	0	2	0	0		0	4	0	4
	0	0	2	2		4	0	4	0	4
36.4.6	0	0	2	0	0	2	0	2	0	
	0	0	2	0	0	0	0	0		0
	0	0	2	0	0	2	0		0	2
	0	0	2	0	0	0		0	0	0
	0	0	2	0	0		0	2	0	2
	0	0	2	0		0	0	0	0	0
36.4.7	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		4	4	4	4	4
36.4.8	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	2	2	4		2
	2	2	2	2	4	2	2		4	2
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		2	2	4	4	2
36.4.9	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		4	4	4	4	4

Continued

Table B.10: Degrees of freedom for estimating all main effects and five two-factor interaction effects for four-factor projections from $OA(36, 13, 3, 2)$

Table B.10 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.10	2	2	0	0	0	0	0	0	0	
	2	0	2	0	0	0	0	0		0
	2	0	0	2	0	0	0		0	0
	0	2	2	0	0	0		0	0	0
	0	2	0	2	0		0	0	0	0
	0	0	2	2		0	0	0	0	0
36.4.11	2	2	2	2	2	2	4	2	4	
	2	2	2	2	2	2	4	2		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	2	2		2	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		4	4	4	4	4
36.4.12	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		2	2	2	2
	2	2	2	2		4	2	2	2	2
36.4.13	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		4	4	4	4	4
36.4.14	2	2	2	2	2	2	4	2	4	
	2	2	2	2	2	2	4	2		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	2	2		2	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		4	4	4	4	4
36.4.15	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		4	4	4	4	4
36.4.16	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		4	4	4	4	4
36.4.17	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		4	4	4	4	4
36.4.18	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	2	2	2		2	2
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		2	2	2	2
	2	2	2	2		4	2	2	2	2

Continued

Table B.10 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.19	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		4	4	4	4	4
36.4.20	2	2	2	1	3	3	4	3	4	
	2	2	2	4	3	3	4	3		4
	2	2	2	2	4	4	4		4	4
	2	2	2	1	3	3		3	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		4	4	4	4	4
36.4.21	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	2	2	2		2	2
	2	2	2	2	4	4		4	4	4
	2	2	2	2	2		2	2	2	2
	2	2	2	2		2	2	2	2	2
36.4.22	2	2	2	2	4	4	4	4	4	
	2	2	2	2	4	4	4	4		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	4	4		4	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		2	2	2	2	2
36.4.23	2	2	2	2	2	2	4	2	4	
	2	2	2	2	2	2	4	2		4
	2	2	2	2	2	2	2		2	2
	2	2	2	2	2	2		2	4	4
	2	2	2	2	2		2	2	2	2
	2	2	2	2		2	2	2	2	2
36.4.24	2	2	2	2	0	0	0	0	0	
	2	2	2	2	0	0	0	0		0
	2	2	2	2	2	2	0		0	0
	2	2	2	2	0	0		0	0	0
	2	2	2	2	2		0	2	0	0
	2	2	2	2		2	0	2	0	0
36.4.25	2	2	2	2	2	2	4	2	4	
	2	2	2	2	2	2	4	2		4
	2	2	2	2	4	4	4		4	4
	2	2	2	2	2	2		2	4	4
	2	2	2	2	4		4	4	4	4
	2	2	2	2		4	4	4	4	4
36.4.26	2	2	2	2	0	0	2	0	2	
	2	2	2	2	0	0	2	0		2
	2	2	2	2	2	2	0		0	0
	2	2	2	2	0	0		0	2	2
	2	2	2	2	2		0	2	0	0
	2	2	2	2		2	0	2	0	0
36.4.27	2	2	2	2	0	0	2	0	2	
	2	2	2	2	0	0	2	0		2
	2	2	2	2	2	2	0		0	0
	2	2	2	2	0	0		0	2	2
	2	2	2	2	2		0	2	0	0
	2	2	2	2		2	0	2	0	0

Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.1	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			2	2	2
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.2	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	3		3	3		3
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.3	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	2	2			2
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		2	2	4		2
	2	2	2	2		2	2		4	2
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4

Continued

Table B.11: Degrees of freedom for estimating all main effects and four two-factor interaction effects for four-factor projections from $OA(36, 13, 3, 2)$

Table B.11 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.4	2	2	2	2	4	4	4	4		
	2	2	2	2	4	3	4		3	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	3		4	3	
	2	2	2	2	4	4		4		4
	2	2	2	2	3	2			2	3
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		2	3	3	2	
	2	2	2	2		4	4	4		4
	2	2	2	2		3	4		3	4
	2	2	2	2		3		4	3	4
	2	2	2	2			4	4	4	4
36.4.5	2	0	2	0	0	4	0	4		
	0	0	2	0	0	4	0		0	
	2	0	2	0	0	4	0			4
	0	2	2	0	0	4		4	0	
	2	2	2	0	2	4		4		4
	0	2	2	0	0	4			0	4
	0	0	2	0	0		0	4	0	
	2	0	2	0	0		0	4		4
	0	0	2	0	0		0		0	4
	0	2	2	0	0			4	0	4
	0	0	2	2		4	0	4	0	
	2	0	2	2		4	2	4		4
	0	0	2	2		4	0		0	4
	0	2	2	2		4		4	2	4
	0	0	2	0			0	4	0	4
36.4.6	2	0	2	0	0	2	0	2		
	0	0	2	0	0	4	0		0	
	2	0	2	0	0	2	0			2
	0	2	2	0	0	2		2	0	
	0	0	2	0	2	0		0		0
	0	0	2	0	0	2			0	2
	0	0	2	0	0		0	4	0	
	0	0	2	0	0		0	2		2
	0	0	2	0	0		0		0	4
	0	2	2	0	0			2	0	2
	0	0	2	0		2	0	2	0	
	0	0	2	0		0	2	0		0
	0	0	2	2		2	0		0	2
	0	0	2	0		0		0	2	0
	0	0	2	2			0	2	0	2
36.4.7	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4

Continued

Table B.11 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.8	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	2	2			2
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		2	2	4		2
	2	2	2	2		2	2		4	2
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.9	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.10	2	2	2	0	0	0	1	0		
	2	2	0	2	0	1	0		0	
	2	0	2	2	1	0	0			0
	2	2	2	0	0	0		0	1	
	2	2	2	0	0	0		0		1
	2	2	2	2	0	0			0	0
	2	2	0	2	0		0	1	0	
	2	2	2	2	0		0	0		0
	2	2	0	2	0		0		0	1
	0	2	2	2	1			0	0	0
	2	2	2	2		0	0	0	0	
	2	0	2	2		0	0	1		0
	2	0	2	2		0	0		1	0
	0	2	2	2		1		0	0	0
	0	2	2	2			1	0	0	0
36.4.11	2	2	2	2	2	2	4	2		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	2	2		2	4	
	2	2	2	2	2	2		2		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4

Continued

Table B.11 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.12	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			2	2	2	2
36.4.13	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.14	2	2	2	2	2	2	4	2		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	2	2		2	4	
	2	2	2	2	2	2		2		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.15	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4

Continued

Table B.11 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.16	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.17	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.18	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			2	2	2	2
36.4.19	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4

Continued

Table B.11 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.20	2	2	2	1	3	3	4	3		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	1	3	3		3	4	
	2	2	2	1	3	3		3		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.21	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.22	2	2	2	2	4	4	4	4		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	4	4		4	4	
	2	2	2	2	4	4		4		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.23	2	2	2	2	2	2	4	2		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	2	2		2	4	
	2	2	2	2	2	2		2		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4

Continued

Table B.11 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.24	2	2	2	2	2	2	4	2		
	2	2	2	2	2	2	2		2	
	2	2	2	2	2	2	2			2
	2	2	2	2	2	2		2	4	
	2	2	2	2	2	2		2		4
	2	2	2	2	2	2			2	2
	2	2	2	2	2		2	2	2	
	2	2	2	2	2		2	2		2
	2	2	2	2	4		0		0	0
	2	2	2	2	2			2	2	2
	2	2	2	2		2	2	2	2	
	2	2	2	2		2	2	2		2
	2	2	2	2		4	0		0	0
	2	2	2	2		2		2	2	2
	2	2	2	2			0	4	0	0
36.4.25	2	2	2	2	2	2	4	2		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	2	2		2	4	
	2	2	2	2	2	2		2		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	4		4		4	4
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		4	4		4	4
	2	2	2	2		4		4	4	4
	2	2	2	2			4	4	4	4
36.4.26	2	2	2	2	2	2	4	2		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	2	2		2	4	
	2	2	2	2	2	2		2		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	2		2		2	2
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		2	2		2	2
	2	2	2	2		4		4	4	4
	2	2	2	2			2	2	2	2
36.4.27	2	2	2	2	2	2	4	2		
	2	2	2	2	4	4	4		4	
	2	2	2	2	4	4	4			4
	2	2	2	2	2	2		2	4	
	2	2	2	2	2	2		2		4
	2	2	2	2	4	4			4	4
	2	2	2	2	4		4	4	4	
	2	2	2	2	4		4	4		4
	2	2	2	2	2		2		2	2
	2	2	2	2	4			4	4	4
	2	2	2	2		4	4	4	4	
	2	2	2	2		4	4	4		4
	2	2	2	2		2	2		2	2
	2	2	2	2		4		4	4	4
	2	2	2	2			2	2	2	2

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.1	2	2	2	2	4	4	4			
	2	2	2	2	4	4		4		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				2	2	2
36.4.2	2	2	2	2	4	4	4			
	2	2	2	2	4	4		4		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4

Continued

Table B.12: Degrees of freedom for estimating all main effects and three two-factor interaction effects for four-factor projections from $OA(36, 13, 3, 2)$

Table B.12 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.3	2	2	2	2	4	4	4			
	2	2	2	2	4	4		4		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		2	2			2
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.4	2	2	2	2	4	4	4			
	2	2	2	2	4	4		4		
	2	2	2	2	4	3			3	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		3	4		3	
	2	2	2	2		4	4			4
	2	2	2	2		3		4	3	
	2	2	2	2		4		4		4
	2	2	2	2		3			3	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.5	2	0	2	0	0	4	0			
	2	2	2	0	2	4		4		
	0	2	2	0	0	4			0	
	2	2	2	0	2	4				4
	2	0	2	0	0		0	4		
	0	0	2	0	0		0		0	
	2	0	2	0	0		0			4
	0	2	2	0	0			4	0	
	2	2	2	0	2			4		4
	0	2	2	0	0				0	4
	2	0	2	2		4	2	4		
	0	0	2	2		4	0		0	
	2	0	2	2		4	2			4
	0	2	2	2		4		4	2	
	2	2	2	2		4		4		4
	0	2	2	2		4			2	4
	0	0	2	2			0	4	0	
	2	0	2	2			2	4		4
	0	0	2	2			0		0	4
	0	2	2	2				4	2	4

Continued

Table B.12 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.6	2	0	2	0	0	4	0			
	2	2	2	0	2	2		2		
	0	2	2	0	0	4			0	
	2	0	2	0	2	2				2
	2	0	2	0	0		0	4		
	0	0	2	0	0		0		0	
	2	0	2	0	0		0			4
	0	2	2	0	0			4	0	
	0	2	2	0	2			2		2
	0	2	2	0	0				0	4
	2	0	2	0		2	2	2		
	0	0	2	2		4	0		0	
	2	0	2	2		2	2			2
	0	2	2	0		2		2	2	
	0	0	2	0		0		0		0
	0	0	2	2		2			2	2
	0	0	2	2			0	4	0	
	0	0	2	2			2	2		2
	0	0	2	2			0		0	4
	0	2	2	2				2	2	2
36.4.7	2	2	2	2	4	4	4			
	2	2	2	2	4	4		4		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.8	2	2	2	2	4	4	4			
	2	2	2	2	4	4		4		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		2	2			2
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4

Continued

Table B.12 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.9	2	2	2	2	4	4	4			
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	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.10	2	2	2	2	1	1	1			
	2	2	2	0	0	0		0		
	2	2	2	2	4	1			1	
	2	2	2	2	1	4				1
	2	2	2	2	4		1	1		
	2	2	0	2	0		0		0	
	2	2	2	2	1		4			1
	2	2	2	2	1			1	1	
	2	2	2	2	1			4		1
	2	2	2	2	1				4	1
	2	2	2	2		4	1	1		
	2	2	2	2		1	4		1	
	2	0	2	2		0	0			0
	2	2	2	2		1		4	1	
	2	2	2	2		1		1		1
	2	2	2	2		1			1	4
	2	2	2	2			1	1	4	
	2	2	2	2			1	1		4
	2	2	2	2			1		1	1
	0	2	2	2				0	0	0
36.4.11	2	2	2	2	4	4	4			
	2	2	2	2	2	2		2		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4

Continued

Table B.12 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.12	2	2	2	2	4	4	4			
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	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.13	2	2	2	2	4	4	4			
	2	2	2	2	4	4		4		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.14	2	2	2	2	4	4	4			
	2	2	2	2	2	2		2		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4

Continued

Table B.12 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.15	2	2	2	2	4	4	4			
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	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.16	2	2	2	2	4	4	4			
	2	2	2	2	4	4		4		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.17	2	2	2	2	4	4	4			
	2	2	2	2	4	4		4		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4

Continued

Table B.12 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.18	2	2	2	2	4	4	4			
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	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.19	2	2	2	2	4	4	4			
	2	2	2	2	4	4		4		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.20	2	2	2	2	4	4	4			
	2	2	2	1	3	3		3		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4

Continued

Table B.12 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.21	2	2	2	2	4	4	4			
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	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.22	2	2	2	2	4	4	4			
	2	2	2	2	4	4		4		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.23	2	2	2	2	4	4	4			
	2	2	2	2	2	2		2		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4

Continued

Table B.12 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.24	2	2	2	2	4	4	4			
	2	2	2	2	2	2		2		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
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	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			0		0	0
	2	2	2	2				4	4	4
36.4.25	2	2	2	2	4	4	4			
	2	2	2	2	2	2		2		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
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	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4
36.4.26	2	2	2	2	4	4	4			
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	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
	2	2	2	2		4			4	4
	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4

Continued

Table B.12 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.27	2	2	2	2	4	4	4			
	2	2	2	2	2	2		2		
	2	2	2	2	4	4			4	
	2	2	2	2	4	4				4
	2	2	2	2	4		4	4		
	2	2	2	2	4		4		4	
	2	2	2	2	4		4			4
	2	2	2	2	4			4	4	
	2	2	2	2	4			4		4
	2	2	2	2	4				4	4
	2	2	2	2		4	4	4		
	2	2	2	2		4	4		4	
	2	2	2	2		4	4			4
	2	2	2	2		4		4	4	
	2	2	2	2		4		4		4
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	2	2	2	2			4	4	4	
	2	2	2	2			4	4		4
	2	2	2	2			4		4	4
	2	2	2	2				4	4	4

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.1	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.2	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.3	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4

Continued

Table B.13: Degrees of freedom for estimating all main effects and two two-factor interaction effects for four-factor projections from $OA(36, 13, 3, 2)$

Table B.13 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.4	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		3			3	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.5	2	2	2	0	2	4				
	2	0	2	0	0		0			
	2	2	2	0	2			4		
	0	2	2	0	0				0	
	2	2	2	0	2					4
	2	0	2	2		4	2			
	2	2	2	2		4		4		
	0	2	2	2		4			2	
	2	2	2	2		4				4
	2	0	2	2			2	4		
	0	0	2	2			0		0	
	2	0	2	2			2			4
	0	2	2	2				4	2	
	2	2	2	2				4		4
	0	2	2	2					2	4
36.4.6	2	2	2	0	2	4				
	2	0	2	0	0		0			
	2	2	2	0	2			4		
	0	2	2	0	0				0	
	2	2	2	0	2					4
	2	0	2	2		4	2			
	2	2	2	0		2		2		
	0	2	2	2		4			2	
	2	0	2	2		2				2
	2	0	2	2			2	4		
	0	0	2	2			0		0	
	2	0	2	2			2			4
	0	2	2	2				4	2	
	0	2	2	2				2		2
	0	2	2	2					2	4
36.4.7	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4

Continued

Table B.13 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.8	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.9	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.10	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	1					1
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		1			1	
	2	2	2	2		4				4
	2	2	2	2			1	1		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.11	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4

Continued

Table B.13 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.12	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.13	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.14	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.15	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4

Continued

Table B.13 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.16	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.17	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.18	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.19	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4

Continued

Table B.13 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.20	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.21	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.22	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.23	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4

Continued

Table B.13 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.24	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.25	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.26	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4
36.4.27	2	2	2	2	4	4				
	2	2	2	2	4		4			
	2	2	2	2	4			4		
	2	2	2	2	4				4	
	2	2	2	2	4					4
	2	2	2	2		4	4			
	2	2	2	2		4		4		
	2	2	2	2		4			4	
	2	2	2	2		4				4
	2	2	2	2			4	4		
	2	2	2	2			4		4	
	2	2	2	2			4			4
	2	2	2	2				4	4	
	2	2	2	2				4		4
	2	2	2	2					4	4

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.1	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.2	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.3	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.4	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.5	2	2	2	0	2					
	2	2	2	2		4				
	2	0	2	2			2			
	2	2	2	2				4		
	0	2	2	2					2	
	2	2	2	2						4
36.4.6	2	2	2	0	2					
	2	2	2	2		4				
	2	0	2	2			2			
	2	2	2	2				4		
	0	2	2	2					2	
	2	2	2	2						4
36.4.7	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.8	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.9	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4

Continued

Table B.14: Degrees of freedom for estimating all main effects and one two-factor interaction effects for four-factor projections from $OA(36, 13, 3, 2)$

Table B.14 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.10	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.11	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.12	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.13	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.14	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.15	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.16	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.17	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.18	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.19	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4

Continued

Table B.14 Continued

Class	A	B	C	D	AB	AC	AD	BC	BD	CD
36.4.20	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.21	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.22	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.23	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.24	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.25	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.26	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4
36.4.27	2	2	2	2	4					
	2	2	2	2		4				
	2	2	2	2			4			
	2	2	2	2				4		
	2	2	2	2					4	
	2	2	2	2						4

Class	N	Class	N	Class	N
36.5.1	0	36.5.31	36	36.5.61	48
36.5.2	0	36.5.32	48	36.5.62	0
36.5.3	0	36.5.33	52	36.5.63	39
36.5.4	0	36.5.34	56	36.5.64	53
36.5.5	0	36.5.35	66	36.5.65	0
36.5.6	0	36.5.36	0	36.5.66	30
36.5.7	0	36.5.37	35	36.5.67	42
36.5.8	0	36.5.38	45	36.5.68	11
36.5.9	0	36.5.39	51	36.5.69	65
36.5.10	0	36.5.40	58	36.5.70	37
36.5.11	0	36.5.41	41	36.5.71	42
36.5.12	0	36.5.42	40	36.5.72	46
36.5.13	0	36.5.43	63	36.5.73	48
36.5.14	4	36.5.44	56	36.5.74	33
36.5.15	47	36.5.45	23	36.5.75	0
36.5.16	59	36.5.46	62	36.5.76	68
36.5.17	60	36.5.47	47	36.5.77	36
36.5.18	62	36.5.48	18	36.5.78	0
36.5.19	44	36.5.49	62	36.5.79	39
36.5.20	50	36.5.50	27	36.5.80	60
36.5.21	63	36.5.51	59	36.5.81	38
36.5.22	58	36.5.52	62	36.5.82	0
36.5.23	56	36.5.53	70	36.5.83	41
36.5.24	41	36.5.54	64	36.5.84	0
36.5.25	48	36.5.55	68		
36.5.26	49	36.5.56	45		
36.5.27	28	36.5.57	68		
36.5.28	44	36.5.58	49		
36.5.29	59	36.5.59	40		
36.5.30	27	36.5.60	62		

Table B.15: Number of models able to estimate all main effects and six two-factor interaction effects for five-factor projections from $OA(36, 13, 3, 2)$

Class	N	Class	N	Class	N
36.5.1	132	36.5.31	144	36.5.61	184
36.5.2	156	36.5.32	193	36.5.62	94
36.5.3	152	36.5.33	201	36.5.63	180
36.5.4	176	36.5.34	194	36.5.64	196
36.5.5	15	36.5.35	218	36.5.65	131
36.5.6	3	36.5.36	36	36.5.66	162
36.5.7	112	36.5.37	184	36.5.67	190
36.5.8	12	36.5.38	189	36.5.68	148
36.5.9	174	36.5.39	176	36.5.69	207
36.5.10	48	36.5.40	200	36.5.70	175
36.5.11	13	36.5.41	181	36.5.71	174
36.5.12	96	36.5.42	170	36.5.72	196
36.5.13	4	36.5.43	223	36.5.73	188
36.5.14	18	36.5.44	208	36.5.74	144
36.5.15	167	36.5.45	184	36.5.75	130
36.5.16	213	36.5.46	211	36.5.76	205
36.5.17	207	36.5.47	190	36.5.77	173
36.5.18	211	36.5.48	171	36.5.78	150
36.5.19	188	36.5.49	199	36.5.79	183
36.5.20	201	36.5.50	180	36.5.80	194
36.5.21	219	36.5.51	219	36.5.81	172
36.5.22	172	36.5.52	208	36.5.82	82
36.5.23	196	36.5.53	221	36.5.83	176
36.5.24	179	36.5.54	208	36.5.84	0
36.5.25	176	36.5.55	207		
36.5.26	190	36.5.56	186		
36.5.27	158	36.5.57	225		
36.5.28	196	36.5.58	189		
36.5.29	205	36.5.59	181		
36.5.30	174	36.5.60	207		

Table B.16: Number of models able to estimate all main effects and five two-factor interaction effects for five-factor projections from $OA(36, 13, 3, 2)$

Class	N	Class	N	Class	N
36.5.1	168	36.5.31	178	36.5.61	180
36.5.2	188	36.5.32	198	36.5.62	166
36.5.3	198	36.5.33	200	36.5.63	196
36.5.4	195	36.5.34	200	36.5.64	202
36.5.5	31	36.5.35	203	36.5.65	191
36.5.6	13	36.5.36	97	36.5.66	186
36.5.7	166	36.5.37	196	36.5.67	200
36.5.8	30	36.5.38	194	36.5.68	171
36.5.9	200	36.5.39	181	36.5.69	200
36.5.10	97	36.5.40	199	36.5.70	187
36.5.11	31	36.5.41	193	36.5.71	191
36.5.12	172	36.5.42	189	36.5.72	199
36.5.13	22	36.5.43	206	36.5.73	198
36.5.14	34	36.5.44	201	36.5.74	178
36.5.15	175	36.5.45	198	36.5.75	191
36.5.16	202	36.5.46	202	36.5.76	203
36.5.17	205	36.5.47	195	36.5.77	179
36.5.18	202	36.5.48	191	36.5.78	179
36.5.19	199	36.5.49	200	36.5.79	194
36.5.20	200	36.5.50	197	36.5.80	198
36.5.21	203	36.5.51	205	36.5.81	193
36.5.22	189	36.5.52	205	36.5.82	150
36.5.23	199	36.5.53	207	36.5.83	193
36.5.24	192	36.5.54	202	36.5.84	0
36.5.25	192	36.5.55	200		
36.5.26	197	36.5.56	192		
36.5.27	174	36.5.57	207		
36.5.28	195	36.5.58	196		
36.5.29	204	36.5.59	194		
36.5.30	185	36.5.60	204		

Table B.17: Number of models able to estimate all main effects and four two-factor interaction effects for five-factor projections from $OA(36, 13, 3, 2)$

Class	N	Class	N	Class	N
36.5.1	110	36.5.31	118	36.5.61	112
36.5.2	119	36.5.32	119	36.5.62	117
36.5.3	119	36.5.33	119	36.5.63	119
36.5.4	118	36.5.34	119	36.5.64	120
36.5.5	34	36.5.35	119	36.5.65	119
36.5.6	22	36.5.36	88	36.5.66	118
36.5.7	110	36.5.37	119	36.5.67	120
36.5.8	34	36.5.38	118	36.5.68	111
36.5.9	119	36.5.39	112	36.5.69	119
36.5.10	88	36.5.40	119	36.5.70	117
36.5.11	34	36.5.41	118	36.5.71	118
36.5.12	112	36.5.42	118	36.5.72	119
36.5.13	30	36.5.43	120	36.5.73	119
36.5.14	35	36.5.44	119	36.5.74	118
36.5.15	111	36.5.45	120	36.5.75	119
36.5.16	119	36.5.46	119	36.5.76	119
36.5.17	120	36.5.47	118	36.5.77	112
36.5.18	119	36.5.48	119	36.5.78	116
36.5.19	119	36.5.49	119	36.5.79	118
36.5.20	119	36.5.50	119	36.5.80	119
36.5.21	119	36.5.51	120	36.5.81	118
36.5.22	118	36.5.52	120	36.5.82	104
36.5.23	119	36.5.53	120	36.5.83	118
36.5.24	118	36.5.54	119	36.5.84	72
36.5.25	118	36.5.55	119		
36.5.26	119	36.5.56	118		
36.5.27	111	36.5.57	120		
36.5.28	118	36.5.58	119		
36.5.29	120	36.5.59	118		
36.5.30	116	36.5.60	120		

Table B.18: Number of models able to estimate all main effects and three two-factor interaction effects for five-factor projections from $OA(36, 13, 3, 2)$

Class	N	Class	N	Class	N
36.5.1	44	36.5.31	45	36.5.61	44
36.5.2	45	36.5.32	45	36.5.62	45
36.5.3	45	36.5.33	45	36.5.63	44
36.5.4	45	36.5.34	45	36.5.64	45
36.5.5	21	36.5.35	45	36.5.65	45
36.5.6	18	36.5.36	42	36.5.66	45
36.5.7	44	36.5.37	45	36.5.67	45
36.5.8	21	36.5.38	45	36.5.68	44
36.5.9	45	36.5.39	44	36.5.69	45
36.5.10	42	36.5.40	45	36.5.70	45
36.5.11	21	36.5.41	45	36.5.71	45
36.5.12	44	36.5.42	45	36.5.72	45
36.5.13	20	36.5.43	45	36.5.73	45
36.5.14	21	36.5.44	45	36.5.74	45
36.5.15	44	36.5.45	45	36.5.75	45
36.5.16	45	36.5.46	45	36.5.76	45
36.5.17	45	36.5.47	45	36.5.77	44
36.5.18	45	36.5.48	45	36.5.78	45
36.5.19	45	36.5.49	45	36.5.79	45
36.5.20	45	36.5.50	45	36.5.80	45
36.5.21	45	36.5.51	45	36.5.81	45
36.5.22	45	36.5.52	45	36.5.82	43
36.5.23	45	36.5.53	45	36.5.83	45
36.5.24	45	36.5.54	45	36.5.84	42
36.5.25	45	36.5.55	45		
36.5.26	45	36.5.56	45		
36.5.27	44	36.5.57	45		
36.5.28	45	36.5.58	45		
36.5.29	45	36.5.59	45		
36.5.30	45	36.5.60	45		

Table B.19: Number of models able to estimate all main effects and two two-factor interaction effects for five-factor projections from $OA(36, 13, 3, 2)$

Class	N	Class	N	Class	N
36.5.1	10	36.5.31	10	36.5.61	10
36.5.2	10	36.5.32	10	36.5.62	10
36.5.3	10	36.5.33	10	36.5.63	10
36.5.4	10	36.5.34	10	36.5.64	10
36.5.5	7	36.5.35	10	36.5.65	10
36.5.6	7	36.5.36	10	36.5.66	10
36.5.7	10	36.5.37	10	36.5.67	10
36.5.8	7	36.5.38	10	36.5.68	10
36.5.9	10	36.5.39	10	36.5.69	10
36.5.10	10	36.5.40	10	36.5.70	10
36.5.11	7	36.5.41	10	36.5.71	10
36.5.12	10	36.5.42	10	36.5.72	10
36.5.13	7	36.5.43	10	36.5.73	10
36.5.14	7	36.5.44	10	36.5.74	10
36.5.15	10	36.5.45	10	36.5.75	10
36.5.16	10	36.5.46	10	36.5.76	10
36.5.17	10	36.5.47	10	36.5.77	10
36.5.18	10	36.5.48	10	36.5.78	10
36.5.19	10	36.5.49	10	36.5.79	10
36.5.20	10	36.5.50	10	36.5.80	10
36.5.21	10	36.5.51	10	36.5.81	10
36.5.22	10	36.5.52	10	36.5.82	10
36.5.23	10	36.5.53	10	36.5.83	10
36.5.24	10	36.5.54	10	36.5.84	10
36.5.25	10	36.5.55	10		
36.5.26	10	36.5.56	10		
36.5.27	10	36.5.57	10		
36.5.28	10	36.5.58	10		
36.5.29	10	36.5.59	10		
36.5.30	10	36.5.60	10		

Table B.20: Number of models able to estimate all main effects and one two-factor interaction effects for five-factor projections from $OA(36, 13, 3, 2)$

APPENDIX C

TABLES OF PROBABILITY OF DISPERSION EFFECT DETECTION AND CONFIDENCE INTERVALS

Dispersion Measure	95% Confidence Interval	Run 0	Run 1	Run 2	Run 3	Run 4	Run 5
s	(0.0463 , 0.0489)	0.0476	0.0465	0.0468	0.0470	0.0475	0.0472
$ln(s^2+1)$	(0.0475 , 0.0501)	0.0488	0.0483	0.0478	0.0478	0.0485	0.0485
$ln(s+1)$	(0.0480 , 0.0506)	0.0493	0.0488	0.0481	0.0480	0.0489	0.0488
S/N_N	(0.0462 , 0.0488)	0.0475	0.0470	0.0465	0.0461	0.0473	0.0470
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	(0.0523 , 0.0551)	0.0537	0.0522	0.0521	0.0531	0.0539	0.0520
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	(0.0473 , 0.0499)	0.0486	0.0470	0.0476	0.0489	0.0485	0.0470
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	(0.0512 , 0.0540)	0.0526	0.0506	0.0509	0.0516	0.0522	0.0504
$\|y_{ij} - \bar{y}_i\|$	(0.0891 , 0.0927)	0.0909	0.0893	0.0900	0.0883	0.0915	0.0883
$\|y_{ij} - \bar{y}_i\|^{.42}$	(0.0802 , 0.0836)	0.0819	0.0809	0.0807	0.0791	0.0809	0.0808
$\|y_{ij} - \bar{y}_i\|^{1.5}$	(0.0903 , 0.0939)	0.0921	0.0897	0.0908	0.0896	0.0922	0.0900
$\|y_{ij} - \widetilde{y}_i\|$	(0.0460 , 0.0486)	0.0473	0.0466	0.0472	0.0469	0.0471	0.0458
$\|y_{ij} - \bar{y}_{i(-1)}\|$	(0.0460 , 0.0486)	0.0473	0.0466	0.0472	0.0469	0.0471	0.0458

Table C.1: Approximate 95% confidence intervals for probability of test rejection using F-distribution critical value and probabilities of detection for the original simulation (Run 0) and each replicate simulation (Run 1 - Run 5) when $\gamma_4 = 0$

Dispersion Measure	95% Confidence Interval	Run 0	Run 1	Run 2	Run 3	Run 4	Run 5
s	(0.1223 , 0.1263)	0.1243	0.1239	0.1243	0.1244	0.1243	0.1239
$ln(s^2+1)$	(0.1154 , 0.1194)	0.1174	0.1182	0.1190	0.1178	0.1189	0.1178
$ln(s+1)$	(0.1172 , 0.1212)	0.1192	0.1200	0.1203	0.1196	0.1208	0.1194
S/N_N	(0.1202 , 0.1242)	0.1222	0.1222	0.1229	0.1226	0.1227	0.1211
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	(0.1532 , 0.1576)	0.1554	0.1551	0.1557	0.1539	0.1563	0.1547
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	(0.1416 , 0.1460)	0.1438	0.1444	0.1449	0.1444	0.1444	0.1434
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	(0.1533 , 0.1577)	0.1555	0.1558	0.1556	0.1550	0.1556	0.1544
$\|y_{ij} - \bar{y}_i\|$	(0.2081 , 0.2131)	0.2106	0.2094	0.2104	0.2101	0.2113	0.2107
$\|y_{ij} - \bar{y}_i\|^{.42}$	(0.1754 , 0.1802)	0.1778	0.1778	0.1790	0.1775	0.1788	0.1792
$\|y_{ij} - \bar{y}_i\|^{1.5}$	(0.2133 , 0.2183)	0.2158	0.2137	0.2149	0.2147	0.2161	0.2148
$\|y_{ij} - \widetilde{y}_i\|$	(0.1227 , 0.1267)	0.1247	0.1253	0.1271	0.1251	0.1248	0.1248
$\|y_{ij} - \bar{y}_{i(-1)}\|$	(0.1227 , 0.1267)	0.1247	0.1253	0.1271	0.1251	0.1248	0.1248

Table C.2: Approximate 95% confidence intervals for probability of test rejection using F-distribution critical value and probabilities of detection for the original simulation (Run 0) and each replicate simulation (Run 1 - Run 5) when $\gamma_4 = 1$

Dispersion Measure	95% Confidence Interval	Run 0	Run 1	Run 2	Run 3	Run 4	Run 5
s	(0.3520 , 0.3580)	0.3550	0.3562	0.3564	0.3562	0.3572	0.3571
$ln(s^2+1)$	(0.3455 , 0.3515)	0.3485	0.3473	0.3486	0.3468	0.3493	0.3486
$ln(s+1)$	(0.3497 , 0.3557)	0.3527	0.3511	0.3530	0.3507	0.3537	0.3526
S/N_N	(0.3491 , 0.3551)	0.3521	0.3525	0.3528	0.3520	0.3530	0.3523
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	(0.4537 , 0.4599)	0.4568	0.4563	0.4564	0.4575	0.4562	0.4581
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	(0.4167 , 0.4229)	0.4198	0.4192	0.4203	0.4205	0.4202	0.4212
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	(0.4510 , 0.4572)	0.4541	0.4540	0.4541	0.4549	0.4548	0.4559
$\|y_{ij} - \bar{y}_i\|$	(0.5274 , 0.5336)	0.5305	0.5313	0.5298	0.5304	0.5300	0.5309
$\|y_{ij} - \bar{y}_i\|^{.42}$	(0.4605 , 0.4667)	0.4636	0.4646	0.4629	0.4645	0.4631	0.4626
$\|y_{ij} - \bar{y}_i\|^{1.5}$	(0.5268 , 0.5330)	0.5299	0.5309	0.5311	0.5305	0.5323	0.5323
$\|y_{ij} - \widetilde{y}_i\|$	(0.3727 , 0.3787)	0.3757	0.3760	0.3746	0.3752	0.3760	0.3742
$\|y_{ij} - \bar{y}_{i(-1)}\|$	(0.3727 , 0.3787)	0.3757	0.3760	0.3746	0.3752	0.3760	0.3742

Table C.3: Approximate 95% confidence intervals for probability of test rejection using F-distribution critical value and probabilities of detection for the original simulation (Run 0) and each replicate simulation (Run 1 - Run 5) when $\gamma_4 = 2$

Dispersion Measure	95% Confidence Interval	Run 0	Run 1	Run 2	Run 3	Run 4	Run 5
s	(0.6544 , 0.6602)	0.6573	0.6584	0.6580	0.6559	0.6566	0.6547
$ln(s^2+1)$	(0.6516 , 0.6574)	0.6545	0.6572	0.6557	0.6543	0.6565	0.6539
$ln(s+1)$	(0.6568 , 0.6626)	0.6597	0.6621	0.6609	0.6595	0.6616	0.6591
S/N_N	(0.6512 , 0.6570)	0.6541	0.6573	0.6561	0.6543	0.6557	0.6546
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	(0.7850 , 0.7900)	0.7875	0.7894	0.7898	0.7890	0.7906	0.7876
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	(0.7335 , 0.7389)	0.7362	0.7352	0.7365	0.7372	0.7362	0.7357
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	(0.7785 , 0.7837)	0.7811	0.7817	0.7825	0.7825	0.7823	0.7814
$\|y_{ij} - \bar{y}_i\|$	(0.8347 , 0.8393)	0.8370	0.8390	0.8393	0.8389	0.8391	0.8385
$\|y_{ij} - \bar{y}_i\|^{.42}$	(0.7785 , 0.7837)	0.7811	0.7822	0.7835	0.7818	0.7812	0.7819
$\|y_{ij} - \bar{y}_i\|^{1.5}$	(0.8262 , 0.8308)	0.8285	0.8310	0.8323	0.8318	0.8317	0.8314
$\|y_{ij} - \widetilde{y}_i\|$	(0.6997 , 0.7053)	0.7025	0.7037	0.7040	0.7045	0.7038	0.7030
$\|y_{ij} - \bar{y}_{i(-1)}\|$	(0.6997 , 0.7053)	0.7025	0.7037	0.7040	0.7045	0.7038	0.7030

Table C.4: Approximate 95% confidence intervals for probability of test rejection using F-distribution critical value and probabilities of detection for the original simulation (Run 0) and each replicate simulation (Run 1 - Run 5) when $\gamma_4 = 3$

Dispersion Measure	95% Confidence Interval	Run 0	Run 1	Run 2	Run 3	Run 4	Run 5
s	(0.8719 , 0.8761)	0.8740	0.8753	0.8746	0.8753	0.8741	0.8765
$ln(s^2+1)$	(0.8797 , 0.8837)	0.8817	0.8828	0.8830	0.8833	0.8822	0.8843
$ln(s+1)$	(0.8826 , 0.8866)	0.8846	0.8858	0.8864	0.8859	0.8855	0.8870
S/N_N	(0.8760 , 0.8800)	0.8780	0.8796	0.8792	0.8794	0.8791	0.8801
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	(0.9556 , 0.9582)	0.9569	0.9573	0.9570	0.9573	0.9551	0.9567
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	(0.9194 , 0.9228)	0.9211	0.9207	0.9217	0.9224	0.9186	0.9215
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	(0.9497 , 0.9523)	0.9510	0.9514	0.9513	0.9524	0.9500	0.9514
$\|y_{ij} - \bar{y}_i\|$	(0.9721 , 0.9741)	0.9731	0.9733	0.9729	0.9734	0.9730	0.9729
$\|y_{ij} - \bar{y}_i\|^{.42}$	(0.9548 , 0.9574)	0.9561	0.9562	0.9555	0.9563	0.9550	0.9549
$\|y_{ij} - \bar{y}_i\|^{1.5}$	(0.9662 , 0.9684)	0.9673	0.9673	0.9674	0.9676	0.9669	0.9677
$\|y_{ij} - \widetilde{y}_i\|$	(0.9198 , 0.9232)	0.9215	0.9214	0.9226	0.9222	0.9216	0.9223
$\|y_{ij} - \bar{y}_{i(-1)}\|$	(0.9198 , 0.9232)	0.9215	0.9214	0.9226	0.9222	0.9216	0.9223

Table C.5: Approximate 95% confidence intervals for probability of test rejection using F-distribution critical value and probabilities of detection for the original simulation (Run 0) and each replicate simulation (Run 1 - Run 5) when $\gamma_4 = 4$

APPENDIX D

TABLES OF SELECTED ORIGINAL CRITICAL VALUES AND REPLICATIONS

D.1 $r = 4$

Significance Level: $\alpha = \mathbf{0.1}$

Measure	r	Original	Replicates				
s^2	4	1.6022	1.6099	1.6043	1.6056	1.6004	1.5932
s	4	1.6767	1.6808	1.6804	1.6755	1.6734	1.6721
$ln(s^2+1)$	4	1.6805	1.6892	1.6857	1.6831	1.6832	1.6784
$ln(s+1)$	4	1.6833	1.6938	1.6909	1.6898	1.6846	1.6802
S/N_N	4	1.6225	1.6124	1.6240	1.6309	1.6212	1.6127
$\lvert y_{ij} - \widetilde{y}_i \rvert$	4	2.6987	2.7186	2.7243	2.7029	2.6999	2.6922
$\lvert y_{ij} - \widetilde{y}_i \rvert_{-1}$	4	2.2371	2.2549	2.2725	2.2552	2.2335	2.2437
$\lvert y_{ij} - \bar{y}_{i(-1)} \rvert$	4	2.6987	2.7186	2.7243	2.7029	2.6999	2.6922
$\lvert y_{ij} - \bar{y}_i \rvert$	4	4.0225	4.0352	4.0438	3.9882	3.9990	4.0018
$\lvert y_{ij} - \bar{y}_i \rvert^{0.42}$	4	3.7301	3.7454	3.7636	3.7286	3.7486	3.7434
$\lvert y_{ij} - \bar{y}_i \rvert^{1.5}$	4	4.1150	4.1209	4.1330	4.0867	4.0764	4.0852
$\lvert y_i - \mathbf{x}_i'\hat{\beta} \rvert$	4	2.8916	2.8661	2.8917	2.8736	2.8814	2.8752
$\lvert y_i - \mathbf{x}_i'\hat{\beta} \rvert^2$	4	2.8682	2.8497	2.8749	2.8507	2.8600	2.8516
$\lvert y_i - \mathbf{x}_i'\hat{\beta} \rvert^{1.5}$	4	2.8938	2.8592	2.8892	2.8694	2.8748	2.8708
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp} \rvert$	4	2.9725	2.9491	2.9718	2.9581	2.9536	2.9442
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp} \rvert^2$	4	2.9591	2.9362	2.9606	2.9401	2.9466	2.9448
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp} \rvert^{1.5}$	4	2.9790	2.9576	2.9782	2.9592	2.9619	2.9714
$ln(\lvert y_{ij} - \widetilde{y}_i \rvert)$	4	4.7055	4.7040	4.7038	4.7133	4.6991	4.7026
$ln(\lvert y_{ij} - \widetilde{y}_i \rvert_{-1})$	4	2.4319	2.4496	2.4468	2.4326	2.4304	2.4376
$ln(\lvert y_{ij} - \bar{y}_{i(-1)} \rvert)$	4	4.7055	4.7040	4.7038	4.7133	4.6991	4.7026
$ln(\lvert y_{ij} - \bar{y}_i \rvert)$	4	3.4294	3.4244	3.4269	3.3820	3.4141	3.3946
$ln(\lvert y_i - \mathbf{x}_i'\hat{\beta} \rvert)$	4	2.8268	2.8468	2.8674	2.8355	2.8297	2.8444
$ln(\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp} \rvert)$	4	2.8624	2.8794	2.9083	2.8834	2.8785	2.8743

Table D.1: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^*: \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.1 for the original simulation and five replicate simulations with $r = 4$

Significance Level: $\alpha = \mathbf{0.05}$

Measure	r	Original	Replicates				
s^2	4	1.9985	2.0006	1.9987	2.0018	2.0031	2.0038
s	4	2.1148	2.1260	2.1079	2.1176	2.1193	2.1100
$ln(s^2+1)$	4	2.1277	2.1327	2.1179	2.1311	2.1310	2.1207
$ln(s+1)$	4	2.1257	2.1443	2.1275	2.1329	2.1277	2.1254
S/N_N	4	2.0214	2.0141	2.0467	2.0407	2.0224	2.0152
$\|y_{ij} - \widetilde{y}_i\|$	4	3.9265	3.9400	3.9513	3.9368	3.8911	3.9123
$\|y_{ij} - \widetilde{y}_i\|_{-1}$	4	3.2332	3.2537	3.2769	3.2427	3.2267	3.2382
$\|y_{ij} - \bar{y}_{i(-1)}\|$	4	3.9265	3.9400	3.9513	3.9368	3.8911	3.9123
$\|y_{ij} - \bar{y}_i\|$	4	5.7340	5.7734	5.7913	5.7399	5.7647	5.7548
$\|y_{ij} - \bar{y}_i\|^{0.42}$	4	5.4241	5.4455	5.4115	5.3727	5.4428	5.3778
$\|y_{ij} - \bar{y}_i\|^{1.5}$	4	5.7535	5.8087	5.8124	5.7455	5.7817	5.7760
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	4	4.1031	4.1254	4.1142	4.0730	4.1043	4.0926
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	4	4.0085	3.9831	4.0102	3.9856	4.0302	3.9811
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	4	4.0824	4.0638	4.0822	4.0373	4.0807	4.0692
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	4	4.2389	4.2569	4.2452	4.2070	4.2578	4.2247
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	4	4.1287	4.1055	4.1540	4.1184	4.1466	4.1127
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	4	4.2123	4.1956	4.2043	4.1746	4.2211	4.1878
$ln(\|y_{ij} - \widetilde{y}_i\|)$	4	6.6028	6.6359	6.6643	6.6231	6.6525	6.6436
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	4	3.4638	3.4653	3.4806	3.4783	3.4787	3.4692
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	4	6.6028	6.6359	6.6643	6.6231	6.6525	6.6436
$ln(\|y_{ij} - \bar{y}_i\|)$	4	4.8870	4.9166	4.9102	4.8813	4.9074	4.9068
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	4	4.0071	4.0569	4.0731	4.0300	4.0408	4.0464
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	4	4.0672	4.1309	4.1640	4.1082	4.1139	4.1181

Table D.2: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^*: \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.05 for the original simulation and five replicate simulations with $r = 4$

Significance Level: $\alpha = \mathbf{0.01}$

Measure	r	Original	Replicates				
s^2	4	3.1765	3.1822	3.2198	3.1630	3.1956	3.1505
s	4	3.3949	3.4030	3.4401	3.4138	3.4282	3.4604
$ln(s^2+1)$	4	3.4096	3.4161	3.4638	3.4255	3.4435	3.4597
$ln(s+1)$	4	3.4536	3.4451	3.4975	3.4525	3.4943	3.4820
S/N_N	4	3.1807	3.2038	3.2178	3.2199	3.2120	3.1733
$\|y_{ij} - \widetilde{y}_i\|$	4	7.0883	7.1019	7.0576	7.1198	7.2329	7.0841
$\|y_{ij} - \widetilde{y}_i\|_{-1}$	4	5.7728	5.7893	5.7304	5.7941	5.8307	5.8194
$\|y_{ij} - \bar{y}_{i(-1)}\|$	4	7.0883	7.1019	7.0576	7.1198	7.2329	7.0841
$\|y_{ij} - \bar{y}_i\|$	4	10.2916	10.1283	10.0414	10.0739	10.2115	10.1285
$\|y_{ij} - \bar{y}_i\|^{0.42}$	4	9.8657	9.7265	9.6506	9.6990	9.8229	9.8456
$\|y_{ij} - \bar{y}_i\|^{1.5}$	4	9.9037	9.8005	9.7593	9.8148	9.9050	9.8659
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	4	7.1883	7.3320	7.2961	7.3129	7.3096	7.2353
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	4	6.6714	6.7973	6.7925	6.7612	6.8058	6.7167
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	4	7.0064	7.1342	7.0633	7.1048	7.1571	7.0314
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	4	7.3943	7.5475	7.5039	7.5180	7.5432	7.4921
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	4	6.8642	7.0115	6.9503	6.9804	6.9829	6.9184
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	4	7.2459	7.3389	7.2782	7.2933	7.3571	7.2679
$ln(\|y_{ij} - \widetilde{y}_i\|)$	4	11.2972	11.3329	11.3115	11.3952	11.3585	11.2403
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	4	6.1225	6.0922	6.1110	6.1158	6.1803	6.1193
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	4	11.2972	11.3329	11.3115	11.3952	11.3585	11.2403
$ln(\|y_{ij} - \bar{y}_i\|)$	4	8.7635	8.6839	8.6610	8.7523	8.8293	8.7826
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	4	6.9016	7.1759	7.1295	7.0478	7.0758	7.0459
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	4	7.1827	7.2844	7.1949	7.1883	7.1766	7.1786

Table D.3: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^*: \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.01 for the original simulation and five replicate simulations with $r = 4$

Significance Level: $\alpha = \mathbf{0.005}$

Measure	r	Original	Replicates				
s^2	4	3.8377	3.7911	3.8375	3.8139	3.8163	3.7984
s	4	4.0178	4.0614	4.0825	4.0376	4.1108	4.1552
$ln(s^2+1)$	4	4.0720	4.0671	4.1258	4.0789	4.1450	4.1540
$ln(s+1)$	4	4.0996	4.1376	4.1648	4.1096	4.1372	4.1440
S/N_N	4	3.8123	3.8668	3.8963	3.8496	3.8733	3.8379
$\|y_{ij} - \widetilde{y}_i\|$	4	8.5621	8.6259	8.5325	8.6034	8.7467	8.5494
$\|y_{ij} - \widetilde{y}_i\|_{-1}$	4	6.9022	6.9925	6.8513	6.9531	7.0545	6.9126
$\|y_{ij} - \bar{y}_{i(-1)}\|$	4	8.5621	8.6259	8.5325	8.6034	8.7467	8.5494
$\|y_{ij} - \bar{y}_i\|$	4	12.1854	12.0845	12.1822	12.1785	12.2897	12.0447
$\|y_{ij} - \bar{y}_i\|^{0.42}$	4	11.8769	11.9127	11.8714	11.8590	12.0076	11.8496
$\|y_{ij} - \bar{y}_i\|^{1.5}$	4	11.7065	11.6346	11.5826	11.6457	11.6860	11.5436
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	4	8.7173	8.8291	8.7393	8.7511	8.8013	8.6574
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	4	7.9564	8.0151	7.9864	7.9970	7.9943	7.9610
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	4	8.3577	8.4841	8.4324	8.4433	8.4016	8.3812
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	4	8.9599	9.0725	9.0037	8.9570	9.0346	8.9576
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	4	8.1231	8.2592	8.2497	8.2795	8.1971	8.1609
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	4	8.6183	8.7275	8.7023	8.6757	8.6673	8.5602
$ln(\|y_{ij} - \widetilde{y}_i\|)$	4	13.3300	13.5154	13.5227	13.4241	13.5298	13.2278
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	4	7.2755	7.3455	7.3686	7.3707	7.3690	7.2543
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	4	13.3300	13.5154	13.5227	13.4241	13.5298	13.2278
$ln(\|y_{ij} - \bar{y}_i\|)$	4	10.5498	10.5127	10.4690	10.5682	10.7048	10.5613
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	4	8.3649	8.4849	8.3449	8.5203	8.4732	8.4073
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	4	8.5798	8.7278	8.5342	8.4718	8.6246	8.5959

Table D.4: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^*: \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.005 for the original simulation and five replicate simulations with $r = 4$

Significance Level: $\alpha = \mathbf{0.001}$

Measure	r	Original	Replicates				
s^2	4	5.5838	5.6095	5.5899	5.6691	5.8013	5.5058
s	4	5.8473	5.8217	5.8586	5.9021	5.9823	6.0057
$ln(s^2+1)$	4	5.9006	5.8672	5.9473	6.1821	6.0432	5.9605
$ln(s+1)$	4	5.7824	5.8588	6.0022	5.8145	6.1387	6.0945
S/N_N	4	5.6690	5.7732	5.7641	5.6715	5.5830	5.4858
$\lvert y_{ij} - \widetilde{y}_i\rvert$	4	12.5738	12.7475	12.6282	12.5286	12.3955	12.1122
$\lvert y_{ij} - \widetilde{y}_i\rvert_{-1}$	4	10.1493	9.8498	10.1771	9.8410	9.9510	9.7070
$\lvert y_{ij} - \bar{y}_{i(-1)}\rvert$	4	12.5738	12.7475	12.6282	12.5286	12.3955	12.1122
$\lvert y_{ij} - \bar{y}_i\rvert$	4	17.3935	17.3590	17.8299	17.2304	17.4818	16.7948
$\lvert y_{ij} - \bar{y}_i\rvert^{0.42}$	4	17.3913	17.0566	17.7045	16.9649	17.2842	16.6382
$\lvert y_{ij} - \bar{y}_i\rvert^{1.5}$	4	16.2960	16.4582	16.5332	16.5459	16.6531	15.8093
$\lvert y_i - \mathbf{x}_i'\hat{\beta}\rvert$	4	12.1966	12.8163	12.5051	12.8013	12.2011	12.3656
$\lvert y_i - \mathbf{x}_i'\hat{\beta}\rvert^2$	4	10.8021	11.3836	10.9327	11.2136	10.8874	10.8792
$\lvert y_i - \mathbf{x}_i'\hat{\beta}\rvert^{1.5}$	4	11.6468	12.0201	11.8865	12.1892	11.5386	11.5556
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert$	4	12.3604	13.2621	12.9971	13.0759	12.4178	12.4123
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^2$	4	11.0632	11.5984	11.2532	11.6054	11.1238	11.1592
$\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert^{1.5}$	4	11.8168	12.3590	12.2761	12.5943	11.6788	11.8111
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert)$	4	19.2526	19.1265	18.6463	18.1021	18.6653	18.6053
$ln(\lvert y_{ij} - \widetilde{y}_i\rvert_{-1})$	4	10.5691	10.8458	10.5646	10.4260	10.6789	10.2215
$ln(\lvert y_{ij} - \bar{y}_{i(-1)}\rvert)$	4	19.2526	19.1265	18.6463	18.1021	18.6653	18.6053
$ln(\lvert y_{ij} - \bar{y}_i\rvert)$	4	15.4510	15.0414	15.6506	15.2165	15.2209	14.9670
$ln(\lvert y_i - \mathbf{x}_i'\hat{\beta}\rvert)$	4	11.6307	11.7659	11.5579	11.8895	11.7746	11.8367
$ln(\lvert y_i - \mathbf{x}_i'\hat{\beta}_{exp}\rvert)$	4	12.0492	12.1129	11.4217	11.8105	12.2102	11.9729

Table D.5: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^* : \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.001 for the original simulation and five replicate simulations with $r = 4$

Significance Level: $\alpha = \mathbf{0.1}$

Measure	r	Original	Replicates				
s^2	7	1.6521	1.6449	1.6518	1.6376	1.6304	1.6368
s	7	1.6917	1.6850	1.6861	1.6740	1.6797	1.6766
$ln(s^2+1)$	7	1.6981	1.6888	1.6900	1.6790	1.6841	1.6805
$ln(s+1)$	7	1.6924	1.6859	1.6915	1.6751	1.6835	1.6798
S/N_N	7	1.6175	1.6265	1.6224	1.6184	1.6178	1.6300
$\|y_{ij} - \widetilde{y}_i\|$	7	2.0088	2.0120	2.0241	1.9936	2.0013	2.0081
$\|y_{ij} - \widetilde{y}_i\|_{-1}$	7	2.5850	2.5843	2.5981	2.5651	2.5647	2.5791
$\|y_{ij} - \bar{y}_{i(-1)}\|$	7	2.9212	2.9286	2.9275	2.9003	2.9129	2.9413
$\|y_{ij} - \bar{y}_i\|$	7	3.2446	3.2605	3.2723	3.2403	3.2312	3.2734
$\|y_{ij} - \bar{y}_i\|^{0.42}$	7	3.1296	3.1239	3.1634	3.1115	3.1011	3.1429
$\|y_{ij} - \bar{y}_i\|^{1.5}$	7	3.3028	3.3139	3.3074	3.2743	3.2837	3.3031
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	7	2.7895	2.8057	2.8217	2.7989	2.7890	2.8089
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	7	2.7917	2.8003	2.8139	2.7948	2.7993	2.8085
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	7	2.8027	2.7981	2.8202	2.7935	2.7967	2.8148
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	7	2.8372	2.8520	2.8621	2.8437	2.8339	2.8483
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	7	2.8387	2.8499	2.8647	2.8411	2.8484	2.8561
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	7	2.8430	2.8443	2.8772	2.8415	2.8473	2.8666
$ln(\|y_{ij} - \widetilde{y}_i\|)$	7	N/A	N/A	N/A	N/A	N/A	N/A
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	7	2.6089	2.6206	2.6084	2.6067	2.5890	2.5995
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	7	3.0365	3.0467	3.0737	3.0588	3.0522	3.0638
$ln(\|y_{ij} - \bar{y}_i\|)$	7	2.9991	2.9855	2.9991	2.9844	2.9634	2.9957
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	7	2.7603	2.7863	2.7904	2.7733	2.7813	2.7874
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	7	2.8001	2.8175	2.8140	2.8117	2.7943	2.8180

Table D.6: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^*: \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.1 for the original simulation and five replicate simulations with $r = 7$

D.2 $r = 7$

Significance Level: $\alpha = \mathbf{0.05}$

Measure	r	Original	Replicates				
s^2	7	2.0692	2.0584	2.0626	2.0655	2.0459	2.0571
s	7	2.1272	2.1142	2.1299	2.1227	2.1284	2.1224
$ln(s^2+1)$	7	2.1387	2.1247	2.1348	2.1362	2.1387	2.1315
$ln(s+1)$	7	2.1439	2.1207	2.1381	2.1264	2.1188	2.1349
S/N_N	7	2.0187	2.0278	2.0242	2.0193	2.0288	2.0387
$\|y_{ij} - \widetilde{y}_i\|$	7	2.8436	2.8330	2.8586	2.8353	2.8538	2.8654
$\|y_{ij} - \widetilde{y}_i\|_{-1}$	7	3.6728	3.6533	3.6931	3.6626	3.6803	3.6948
$\|y_{ij} - \bar{y}_{i(-1)}\|$	7	4.1547	4.1400	4.1891	4.1514	4.1568	4.1869
$\|y_{ij} - \bar{y}_i\|$	7	4.6176	4.6272	4.6594	4.5941	4.6065	4.6310
$\|y_{ij} - \bar{y}_i\|^{0.42}$	7	4.4544	4.4784	4.4824	4.4677	4.4824	4.4796
$\|y_{ij} - \bar{y}_i\|^{1.5}$	7	4.6509	4.6545	4.6744	4.6454	4.6322	4.6950
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	7	3.9739	4.0087	4.0235	4.0050	3.9843	4.0061
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	7	3.9141	3.9533	3.9601	3.9241	3.9212	3.9188
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	7	3.9557	3.9793	4.0015	3.9716	3.9717	3.9711
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	7	4.0464	4.0820	4.0903	4.0508	4.0442	4.0717
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	7	3.9780	4.0292	4.0289	3.9796	3.9943	3.9826
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	7	4.0224	4.0438	4.0709	4.0399	4.0372	4.0350
$ln(\|y_{ij} - \widetilde{y}_i\|)$	7	N/A	N/A	N/A	N/A	N/A	N/A
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	7	3.6955	3.6992	3.7262	3.6930	3.6969	3.6881
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	7	4.3380	4.3394	4.3565	4.3299	4.3354	4.3687
$ln(\|y_{ij} - \bar{y}_i\|)$	7	4.2595	4.2663	4.2696	4.2639	4.2477	4.2711
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	7	3.9168	3.9323	3.9662	3.9490	3.9611	3.9752
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	7	3.9435	3.9865	4.0119	3.9891	3.9856	4.0119

Table D.7: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^* : \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.05 for the original simulation and five replicate simulations with $r = 7$

Significance Level: $\alpha = \mathbf{0.01}$

Measure	r	Original	Replicates				
s^2	7	3.3243	3.2913	3.2901	3.3229	3.2780	3.2756
s	7	3.4542	3.3695	3.4289	3.4775	3.4217	3.4306
$ln(s^2+1)$	7	3.4671	3.3926	3.4619	3.4904	3.4396	3.4610
$ln(s+1)$	7	3.4442	3.3848	3.4488	3.4925	3.4384	3.4334
S/N_N	7	3.1975	3.2173	3.1933	3.2169	3.2383	3.2302
$\|y_{ij} - \widetilde{y}_i\|$	7	4.9799	4.8977	4.9352	4.9626	4.9263	4.9759
$\|y_{ij} - \widetilde{y}_i\|_{-1}$	7	6.5175	6.3832	6.4334	6.4511	6.4406	6.4677
$\|y_{ij} - \bar{y}_{i(-1)}\|$	7	7.3715	7.2346	7.2599	7.2827	7.2672	7.3791
$\|y_{ij} - \bar{y}_i\|$	7	8.1007	8.0516	8.0358	8.1266	8.0751	8.1276
$\|y_{ij} - \bar{y}_i\|^{0.42}$	7	7.8396	7.8229	7.8449	7.9432	7.8770	7.9648
$\|y_{ij} - \bar{y}_i\|^{1.5}$	7	8.0437	7.9193	7.9077	8.0192	7.9132	8.0489
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	7	6.9697	6.8853	7.0465	7.0457	6.9775	7.0528
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	7	6.6833	6.6637	6.6906	6.7293	6.6512	6.6118
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	7	6.8322	6.8256	6.8772	6.9313	6.8410	6.8282
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	7	7.0909	7.0236	7.1778	7.1515	7.1064	7.1770
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	7	6.8042	6.7815	6.7723	6.8354	6.7276	6.7013
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	7	6.9800	6.9465	7.0166	7.0696	6.9237	6.9603
$ln(\|y_{ij} - \widetilde{y}_i\|)$	7	N/A	N/A	N/A	N/A	N/A	N/A
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	7	6.4512	6.3635	6.5054	6.3911	6.4674	6.3591
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	7	7.5664	7.4224	7.5887	7.6223	7.5923	7.6282
$ln(\|y_{ij} - \bar{y}_i\|)$	7	7.4194	7.3250	7.3715	7.5245	7.3051	7.3924
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	7	6.7860	6.7806	6.7979	6.8733	6.8144	6.8987
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	7	6.8627	6.8789	6.8704	6.9481	6.9059	6.9754

Table D.8: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^*: \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.01 for the original simulation and five replicate simulations with $r = 7$

Significance Level: $\alpha = \mathbf{0.005}$

Measure	r	Original	Replicates				
s^2	7	3.9866	3.9297	3.9459	3.9209	3.8996	3.9323
s	7	4.1723	4.0532	4.0871	4.1438	4.0671	4.1338
$ln(s^2+1)$	7	4.1988	4.1002	4.1558	4.2097	4.0989	4.1219
$ln(s+1)$	7	4.1698	4.0197	4.1503	4.1236	4.0588	4.0937
S/N_N	7	3.8226	3.8116	3.8402	3.8400	3.8483	3.8340
$\|y_{ij} - \widetilde{y}_i\|$	7	5.8658	5.7584	5.8588	5.8396	5.8494	5.9036
$\|y_{ij} - \widetilde{y}_i\|_{-1}$	7	7.7453	7.5615	7.6724	7.6766	7.6705	7.7408
$\|y_{ij} - \bar{y}_{i(-1)}\|$	7	8.6616	8.5708	8.6153	8.6916	8.6872	8.8933
$\|y_{ij} - \bar{y}_i\|$	7	9.5849	9.4609	9.5852	9.6048	9.6535	9.8168
$\|y_{ij} - \bar{y}_i\|^{0.42}$	7	9.3423	9.3848	9.3488	9.5754	9.5017	9.5064
$\|y_{ij} - \bar{y}_i\|^{1.5}$	7	9.5203	9.2352	9.3592	9.4732	9.3899	9.6018
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	7	8.2714	8.3000	8.2527	8.4272	8.3534	8.4501
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	7	7.7992	7.8467	7.8145	7.9437	7.7423	7.8529
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	7	8.0831	8.0906	8.1060	8.2540	8.0623	8.2591
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	7	8.3618	8.3848	8.3964	8.6291	8.4443	8.5717
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	7	7.9265	7.9747	7.9346	8.0892	7.9185	7.9698
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	7	8.1913	8.1805	8.1902	8.3831	8.2102	8.3831
$ln(\|y_{ij} - \widetilde{y}_i\|)$	7	N/A	N/A	N/A	N/A	N/A	N/A
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	7	7.6237	7.6019	7.6095	7.5645	7.7632	7.6122
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	7	9.0101	8.8435	9.0276	8.9842	9.0533	9.0646
$ln(\|y_{ij} - \bar{y}_i\|)$	7	8.7313	8.7523	8.6766	8.9020	8.8036	8.8518
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	7	8.0912	8.1126	8.0885	8.2336	8.1636	8.3169
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	7	8.1158	8.1625	8.1379	8.2689	8.2611	8.3819

Table D.9: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^*: \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.005 for the original simulation and five replicate simulations with $r = 7$

Significance Level: $\alpha = \mathbf{0.001}$

Measure	r	Original	Replicates				
s^2	7	5.8185	5.6448	5.9119	5.9131	5.6124	5.6588
s	7	6.2417	5.7960	6.0112	5.8970	5.8588	6.0317
$ln(s^2+1)$	7	6.2946	5.8483	6.1918	5.9786	5.8692	5.9159
$ln(s+1)$	7	5.9821	5.8456	5.8489	5.8567	5.8738	5.8250
S/N_N	7	5.5797	5.4733	5.6679	5.6590	5.6207	5.6675
$\|y_{ij} - \widetilde{y}_i\|$	7	8.1268	7.9324	8.0975	8.1079	8.0377	8.2434
$\|y_{ij} - \widetilde{y}_i\|_{-1}$	7	10.7546	10.5893	10.6786	10.8327	10.7333	10.9089
$\|y_{ij} - \bar{y}_{i(-1)}\|$	7	12.0450	11.8509	12.0505	12.2354	12.3143	12.3327
$\|y_{ij} - \bar{y}_i\|$	7	13.2854	13.0228	13.3873	13.5373	13.5190	13.5813
$\|y_{ij} - \bar{y}_i\|^{0.42}$	7	13.0290	12.7169	13.1347	13.1941	13.4427	13.5282
$\|y_{ij} - \bar{y}_i\|^{1.5}$	7	13.0998	12.8232	13.1945	13.1043	12.8331	13.1833
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	7	11.6540	11.2908	11.2690	11.6401	11.4115	12.0923
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	7	10.7329	10.7840	10.6523	10.5398	10.5985	10.8811
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	7	11.1937	11.1622	11.0283	11.1531	11.2437	11.5428
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	7	11.8212	11.5069	11.5224	11.6376	11.6226	12.1260
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	7	10.7515	10.9290	10.7190	10.6571	10.7448	11.1277
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	7	11.3464	11.3651	11.2476	11.3290	11.3553	11.6827
$ln(\|y_{ij} - \widetilde{y}_i\|)$	7	N/A	N/A	N/A	N/A	N/A	N/A
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	7	10.2863	10.5370	10.7322	10.5772	10.6960	10.4895
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	7	12.2842	12.3250	12.4656	12.3346	12.6137	12.5566
$ln(\|y_{ij} - \bar{y}_i\|)$	7	11.9977	12.0335	12.1039	12.1252	12.3055	12.5219
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	7	11.1541	10.9305	10.8998	11.4243	11.2578	11.4224
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	7	11.1792	11.0880	11.2975	11.1557	11.3157	11.6593

Table D.10: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^* : \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.001 for the original simulation and five replicate simulations with $r = 7$

Significance Level: $\alpha = \mathbf{0.1}$

Measure	r	Original	Replicates				
s^2	10	1.6475	1.6593	1.6526	1.6570	1.6498	1.6484
s	10	1.6734	1.6917	1.6798	1.6855	1.6791	1.6834
$ln(s^2+1)$	10	1.6759	1.6915	1.6827	1.6892	1.6844	1.6837
$ln(s+1)$	10	1.6759	1.6940	1.6816	1.6916	1.6850	1.6863
S/N_N	10	1.6190	1.6186	1.6217	1.6204	1.6221	1.6221
$\|y_{ij} - \widetilde{y}_i\|$	10	2.3996	2.4262	2.4109	2.4081	2.3912	2.4186
$\|y_{ij} - \widetilde{y}_i\|_{-1}$	10	2.5714	2.6105	2.5908	2.5885	2.5758	2.6061
$\|y_{ij} - \bar{y}_{i(-1)}\|$	10	2.8840	2.9287	2.9126	2.9026	2.8878	2.9350
$\|y_{ij} - \bar{y}_i\|$	10	3.0223	3.0688	3.0585	3.0449	3.0227	3.0699
$\|y_{ij} - \bar{y}_i\|^{0.42}$	10	2.9452	3.0073	2.9903	2.9758	2.9575	2.9696
$\|y_{ij} - \bar{y}_i\|^{1.5}$	10	3.0509	3.0974	3.0770	3.0743	3.0620	3.0774
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	10	2.7312	2.7885	2.7615	2.7787	2.7504	2.7757
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	10	2.7624	2.7832	2.7656	2.7757	2.7558	2.7752
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	10	2.7566	2.7844	2.7546	2.7711	2.7545	2.7764
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	10	2.7603	2.8230	2.7865	2.8098	2.7850	2.8035
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	10	2.7938	2.8199	2.8011	2.8073	2.7900	2.8071
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	10	2.7899	2.8179	2.7863	2.8022	2.7834	2.8084
$ln(\|y_{ij} - \widetilde{y}_i\|)$	10	3.0542	3.0856	3.0681	3.0781	3.0700	3.0713
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	10	2.5936	2.6378	2.6118	2.6285	2.6179	2.6197
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	10	2.8767	2.9572	2.9345	2.9227	2.9201	2.9157
$ln(\|y_{ij} - \bar{y}_i\|)$	10	2.8628	2.9071	2.9020	2.9048	2.8825	2.8848
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	10	2.7161	2.7676	2.7542	2.7549	2.7518	2.7473
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	10	2.7250	2.7862	2.7865	2.7678	2.7822	2.7673

Table D.11: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^*: \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.1 for the original simulation and five replicate simulations with $r = 10$

D.3 $r = 10$

Significance Level: $\alpha = \mathbf{0.05}$

Measure	r	Original	Replicates				
s^2	10	2.0769	2.0848	2.0847	2.0800	2.0798	2.0714
s	10	2.1126	2.1302	2.1287	2.1223	2.1222	2.1217
$ln(s^2+1)$	10	2.1207	2.1307	2.1338	2.1294	2.1259	2.1309
$ln(s+1)$	10	2.1182	2.1338	2.1359	2.1301	2.1228	2.1284
S/N_N	10	2.0204	2.0254	2.0348	2.0291	2.0279	2.0270
$\|y_{ij} - \widetilde{y_i}\|$	10	3.4037	3.4479	3.4474	3.4232	3.3935	3.4328
$\|y_{ij} - \widetilde{y_i}\|_{-1}$	10	3.6552	3.7168	3.7009	3.6915	3.6691	3.6976
$\|y_{ij} - \bar{y}_{i(-1)}\|$	10	4.0971	4.1839	4.1424	4.1321	4.1032	4.1509
$\|y_{ij} - \bar{y}_i\|$	10	4.2967	4.3764	4.3539	4.3245	4.3077	4.3529
$\|y_{ij} - \bar{y}_i\|^{0.42}$	10	4.1943	4.2580	4.2657	4.2574	4.2070	4.2501
$\|y_{ij} - \bar{y}_i\|^{1.5}$	10	4.3226	4.3829	4.3554	4.3306	4.3195	4.3671
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	10	3.8928	3.9482	3.9075	3.9479	3.9151	3.9545
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	10	3.9002	3.9176	3.8739	3.8979	3.8668	3.8932
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	10	3.8924	3.9524	3.8911	3.9169	3.8997	3.9203
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	10	3.9436	3.9987	3.9552	3.9887	3.9465	4.0046
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	10	3.9365	3.9657	3.9174	3.9371	3.9078	3.9336
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	10	3.9389	4.0041	3.9478	3.9768	3.9426	3.9752
$ln(\|y_{ij} - \widetilde{y_i}\|)$	10	4.2993	4.3454	4.3328	4.3036	4.3337	4.3440
$ln(\|y_{ij} - \widetilde{y_i}\|_{-1})$	10	3.7131	3.7407	3.7040	3.7260	3.7031	3.7198
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	10	4.1247	4.2152	4.1502	4.1758	4.1171	4.1627
$ln(\|y_{ij} - \bar{y}_i\|)$	10	4.0574	4.1254	4.1242	4.1081	4.0957	4.1022
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	10	3.8535	3.9345	3.8844	3.9152	3.8980	3.8913
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	10	3.8818	3.9507	3.9313	3.9166	3.9324	3.9211

Table D.12: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^*: \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.05 for the original simulation and five replicate simulations with $r = 10$

Significance Level: $\alpha = \mathbf{0.01}$

Measure	r	Original	Replicates				
s^2	10	3.2686	3.3421	3.3543	3.2914	3.2942	3.3102
s	10	3.3660	3.4477	3.4243	3.3733	3.3969	3.4522
$ln(s^2+1)$	10	3.3922	3.4546	3.4396	3.3871	3.4149	3.4792
$ln(s+1)$	10	3.3972	3.4551	3.4449	3.3700	3.4109	3.4707
S/N_N	10	3.2107	3.1977	3.1822	3.2182	3.2089	3.2062
$\|y_{ij} - \widetilde{y}_i\|$	10	5.9355	5.9060	5.8926	5.9022	5.9100	5.9637
$\|y_{ij} - \widetilde{y}_i\|_{-1}$	10	6.4078	6.3596	6.3213	6.4110	6.3580	6.4374
$\|y_{ij} - \bar{y}_{i(-1)}\|$	10	7.1900	7.1594	7.0963	7.1840	7.1429	7.1878
$\|y_{ij} - \bar{y}_i\|$	10	7.5245	7.5212	7.4301	7.4844	7.5105	7.5275
$\|y_{ij} - \bar{y}_i\|^{0.42}$	10	7.3336	7.3810	7.3844	7.3614	7.3988	7.4376
$\|y_{ij} - \bar{y}_i\|^{1.5}$	10	7.4896	7.4251	7.3508	7.4923	7.4077	7.5160
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	10	6.8398	6.8346	6.8252	6.8757	6.8690	6.8585
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	10	6.5877	6.5570	6.5827	6.6164	6.5769	6.6010
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	10	6.7566	6.7521	6.7614	6.7387	6.7743	6.6973
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	10	6.8795	6.9179	6.8913	6.9454	6.9713	6.9247
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	10	6.6705	6.6400	6.6604	6.7127	6.6614	6.6554
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	10	6.8394	6.8225	6.8233	6.8493	6.8482	6.8159
$ln(\|y_{ij} - \widetilde{y}_i\|)$	10	7.4017	7.3134	7.3456	7.3805	7.4956	7.3447
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	10	6.3687	6.4172	6.4082	6.4414	6.4229	6.3877
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	10	7.1823	7.2177	7.1343	7.2385	7.1359	7.1900
$ln(\|y_{ij} - \bar{y}_i\|)$	10	7.0417	7.0620	7.0435	7.1011	7.0426	7.1328
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	10	6.6279	6.6665	6.6642	6.7293	6.7455	6.7714
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	10	6.7021	6.7199	6.7524	6.8160	6.7421	6.7487

Table D.13: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^* : \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.01 for the original simulation and five replicate simulations with $r = 10$

Significance Level: $\alpha = \mathbf{0.005}$

Measure	r	Original	Replicates				
s^2	10	3.9351	3.9361	4.0349	3.8995	3.9511	3.9837
s	10	4.0283	4.0741	4.1149	3.9986	4.0935	4.0813
$ln(s^2+1)$	10	4.0461	4.1184	4.1340	4.0320	4.1198	4.1190
$ln(s+1)$	10	4.0458	4.0780	4.1084	3.9873	4.0523	4.1607
S/N_N	10	3.8109	3.8828	3.8157	3.8400	3.8269	3.8729
$\|y_{ij} - \widetilde{y_i}\|$	10	7.0917	7.0951	7.0650	7.0978	7.0980	7.1682
$\|y_{ij} - \widetilde{y_i}\|_{-1}$	10	7.6458	7.6804	7.5817	7.7020	7.6151	7.7064
$\|y_{ij} - \bar{y}_{i(-1)}\|$	10	8.5807	8.5427	8.4559	8.6405	8.5951	8.6649
$\|y_{ij} - \bar{y}_i\|$	10	8.9653	8.9414	8.7902	9.0128	9.0286	9.0217
$\|y_{ij} - \bar{y}_i\|^{0.42}$	10	8.8443	8.7772	8.7591	8.8265	8.9206	8.8287
$\|y_{ij} - \bar{y}_i\|^{1.5}$	10	8.8740	8.8673	8.7598	8.9016	8.8802	9.0049
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	10	8.1768	8.0634	8.1680	8.1769	8.2584	8.1074
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	10	7.7608	7.7428	7.7462	7.8391	7.7130	7.8394
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	10	7.9989	7.9378	8.0030	8.0415	8.0376	8.0235
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	10	8.2753	8.1627	8.2918	8.3157	8.3119	8.2230
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	10	7.8460	7.8127	7.8596	7.8988	7.8044	7.9022
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	10	8.0697	7.9985	8.1876	8.1482	8.1038	8.1398
$ln(\|y_{ij} - \widetilde{y_i}\|)$	10	8.8224	8.7495	8.6764	8.8398	8.8561	8.6716
$ln(\|y_{ij} - \widetilde{y_i}\|_{-1})$	10	7.6885	7.6659	7.5287	7.7290	7.7263	7.5235
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	10	8.5079	8.5501	8.4877	8.5327	8.5621	8.5658
$ln(\|y_{ij} - \bar{y}_i\|)$	10	8.4351	8.4121	8.4063	8.3585	8.4127	8.5232
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	10	7.8956	7.9372	7.9430	7.9517	8.0799	8.0426
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	10	7.9795	8.0439	7.9439	8.0338	8.0592	8.2243

Table D.14: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^*: \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.005 for the original simulation and five replicate simulations with $r = 10$

Significance Level: $\alpha = \mathbf{0.001}$

Measure	r	Original	Replicates				
s^2	10	5.7141	5.7929	6.0307	5.8142	5.6938	5.8201
s	10	5.7315	5.9010	6.0713	5.8788	5.7810	6.0462
$ln(s^2+1)$	10	5.7758	6.0409	6.0595	5.9623	5.8478	6.0497
$ln(s+1)$	10	5.7861	5.9234	5.9797	5.9231	5.7498	5.9143
S/N_N	10	5.3835	5.7435	5.6844	5.6918	5.4250	5.5460
$\|y_{ij} - \widetilde{y}_i\|$	10	9.9660	9.8331	9.6069	9.8380	10.1370	9.8774
$\|y_{ij} - \widetilde{y}_i\|_{-1}$	10	10.5851	10.5758	10.3927	10.5551	11.0345	10.6658
$\|y_{ij} - \bar{y}_{i(-1)}\|$	10	11.7597	11.6385	11.6891	11.6501	12.3083	12.2096
$\|y_{ij} - \bar{y}_i\|$	10	12.3861	12.2381	12.0870	12.2883	12.9280	12.7921
$\|y_{ij} - \bar{y}_i\|^{0.42}$	10	12.1842	12.1031	12.2005	11.9213	12.3100	12.3370
$\|y_{ij} - \bar{y}_i\|^{1.5}$	10	12.0759	12.0725	11.7393	11.9555	12.4593	12.3945
$\|y_i - \mathbf{x}_i'\hat{\beta}\|$	10	11.2795	11.1449	11.0291	11.2163	11.4236	11.8848
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^2$	10	10.7863	10.5132	10.2206	10.5449	10.7652	11.0480
$\|y_i - \mathbf{x}_i'\hat{\beta}\|^{1.5}$	10	10.9503	10.9424	10.6029	10.9603	11.2008	11.4551
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|$	10	11.4634	11.2325	11.1565	11.3499	11.6261	11.8035
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^2$	10	10.9224	10.6574	10.2861	10.6327	10.8687	11.1591
$\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|^{1.5}$	10	11.0462	11.1535	10.6615	11.1037	11.3522	11.6449
$ln(\|y_{ij} - \widetilde{y}_i\|)$	10	11.8960	12.1374	11.6839	11.9340	12.1245	11.7521
$ln(\|y_{ij} - \widetilde{y}_i\|_{-1})$	10	10.8176	10.5701	10.3253	10.5047	10.6614	10.3036
$ln(\|y_{ij} - \bar{y}_{i(-1)}\|)$	10	11.6863	11.7242	11.6249	11.7987	11.9694	11.8036
$ln(\|y_{ij} - \bar{y}_i\|)$	10	11.7971	11.4052	11.4294	11.3753	11.6869	11.4736
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}\|)$	10	10.8691	10.9908	10.7521	10.9279	10.9469	11.2102
$ln(\|y_i - \mathbf{x}_i'\hat{\beta}_{exp}\|)$	10	10.9376	11.1022	11.1668	11.1416	11.1126	11.0715

Table D.15: Critical values for the t_{PSE} and M-test statistics obtained from the empirical distributions of test statistics under $H_0^*: \gamma_1 = 0$ and $N(0,1)$ observations, for Type I error rates 0.001 for the original simulation and five replicate simulations with $r = 10$

APPENDIX E

NULL DISTRIBUTION EMPIRICAL F CDF PLOTS

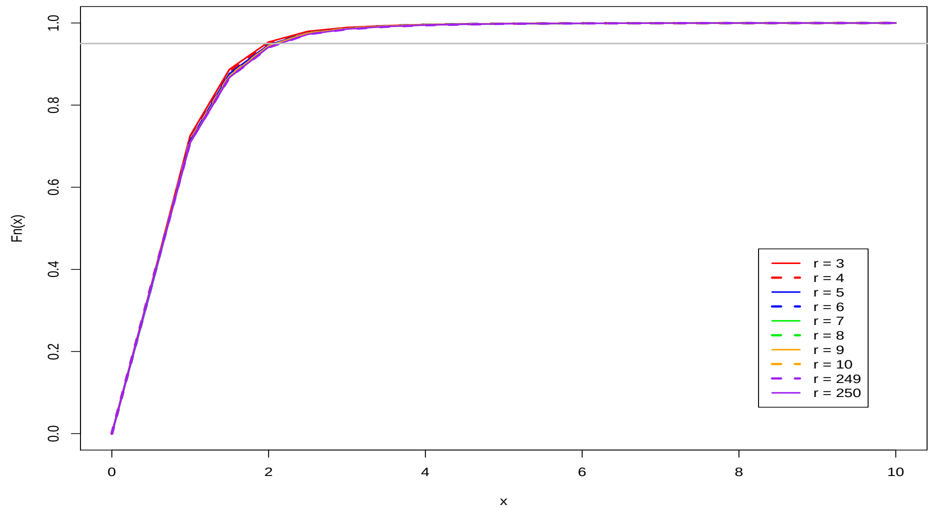

Figure E.1: Empirical CDF for Lenth's $|t_{PSE}|$ test statistic values from s^2 for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

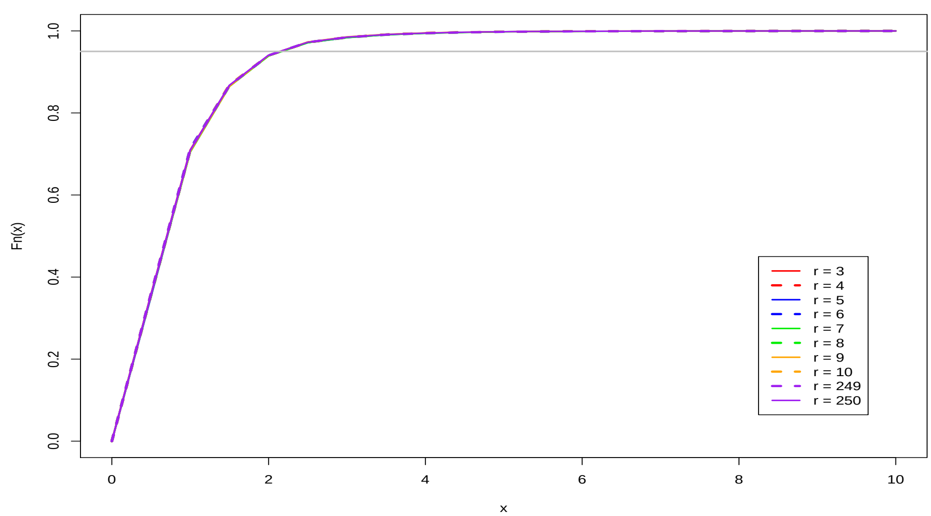

Figure E.2: Empirical CDF for Lenth's $|t_{PSE}|$ test statistic values from s for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

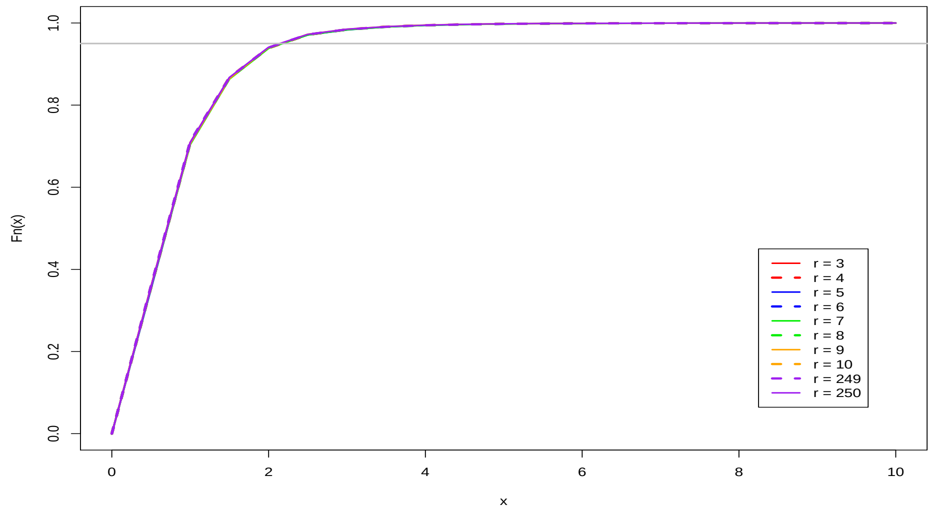

Figure E.3: Empirical CDF for Lenth's $|t_{PSE}|$ test statistic values from $ln(s^2 + 1)$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

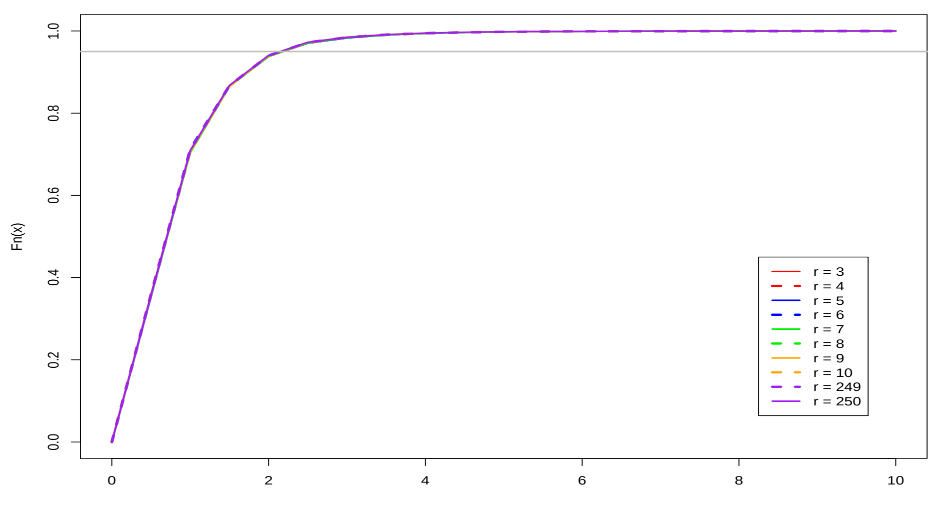

Figure E.4: Empirical CDF for Lenth's $|t_{PSE}|$ test statistic values from $ln(s + 1)$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

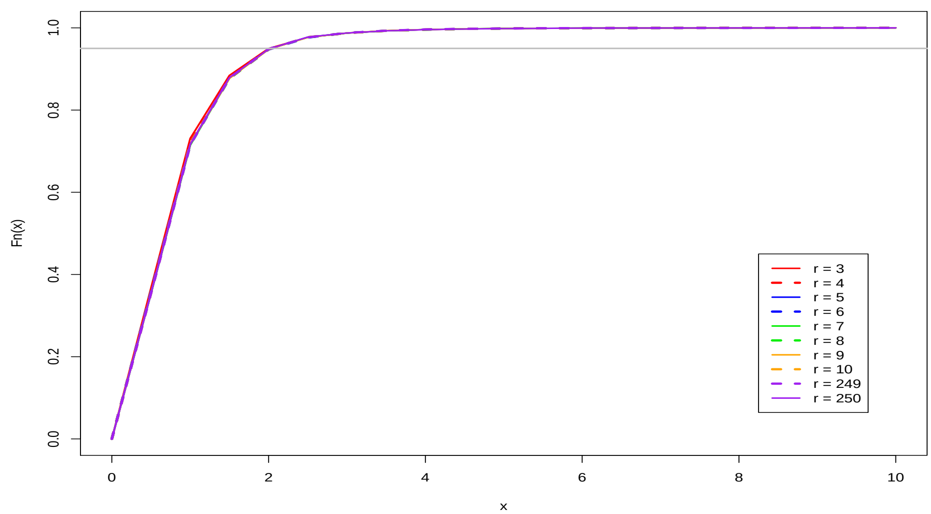

Figure E.5: Empirical CDF for Lenth's $|t_{PSE}|$ test statistic values from S/N_N for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

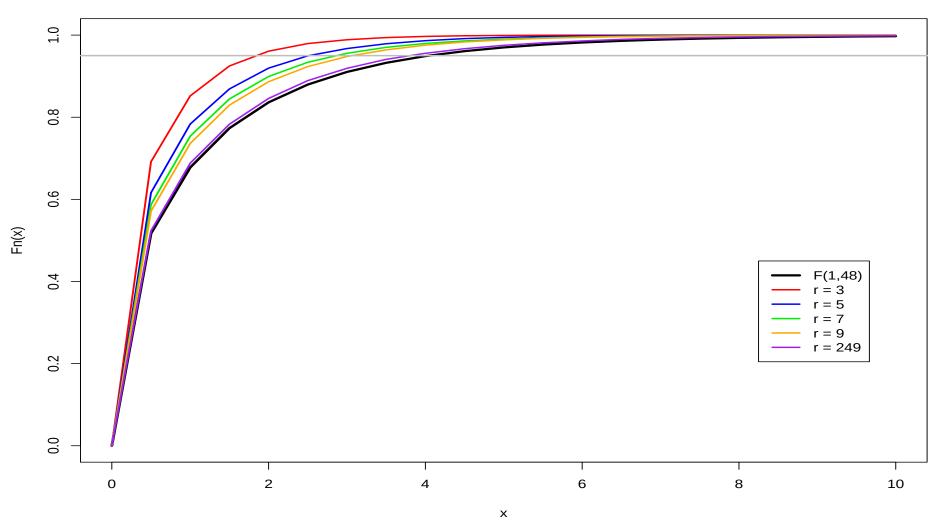

Figure E.6: Empirical CDF for M-test statistic values from $|y_{ij} - \widetilde{y}_i|$ for odd numbers of replicates, $r = 3, 5, 7, 9, 251$

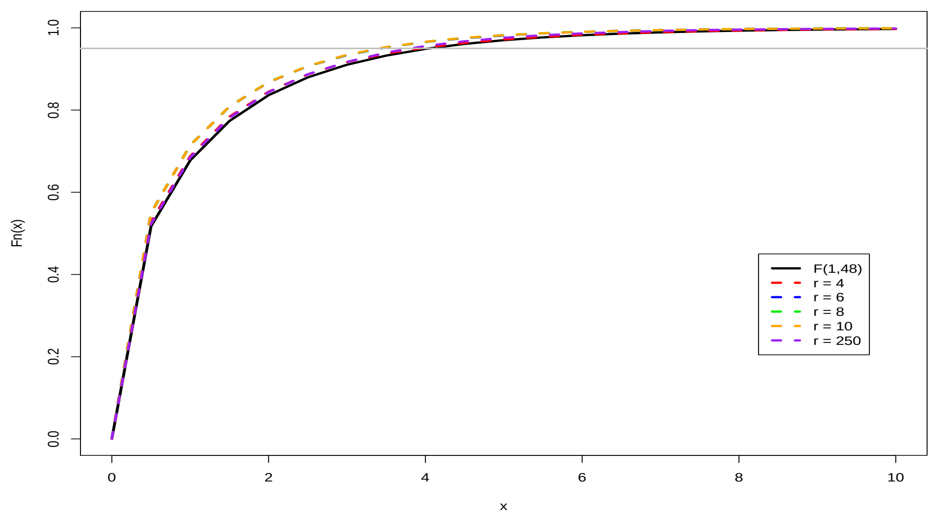

Figure E.7: Empirical CDF for M-test statistic values from $|y_{ij} - \widetilde{y}_i|$ for even numbers of replicates, $r = 4, 6, 8, 10, 250$

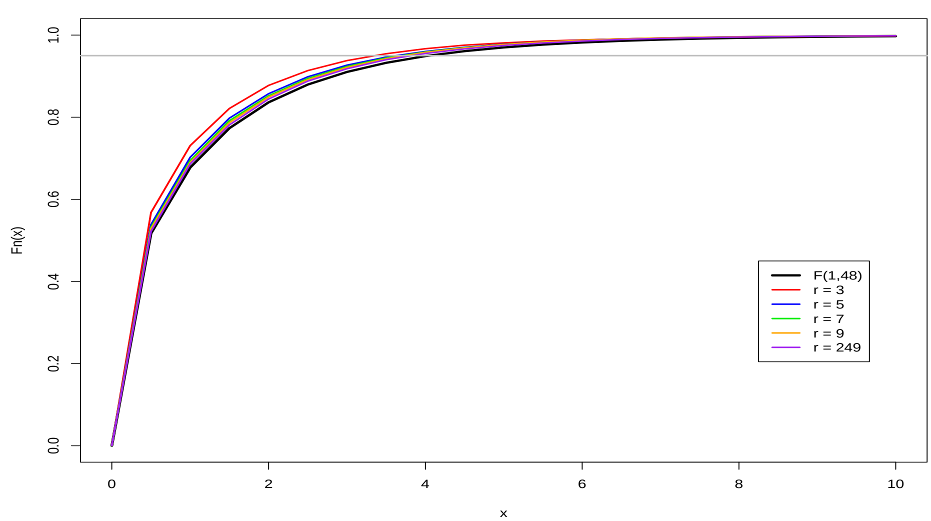

Figure E.8: Empirical CDF for M-test statistic values from $|y_{ij} - \widetilde{y}_i|_{-1}$ for odd numbers of replicates, $r = 3, 5, 7, 9, 251$

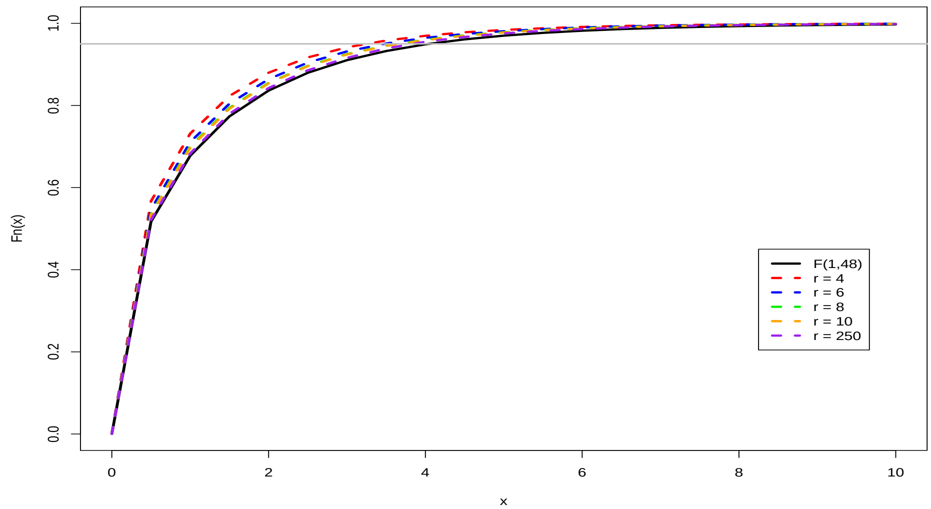

Figure E.9: Empirical CDF for M-test statistic values from $|y_{ij} - \widetilde{y}_i|_{-1}$ for even numbers of replicates, $r = 4, 6, 8, 10, 250$

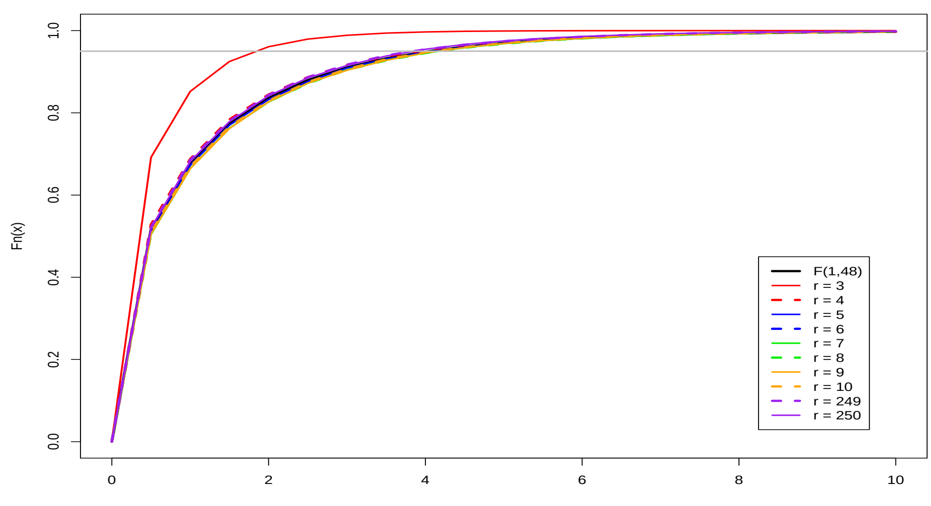

Figure E.10: Empirical CDF for M-test statistic values from $|y_{ij} - \bar{y}_{i(-1)}|$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

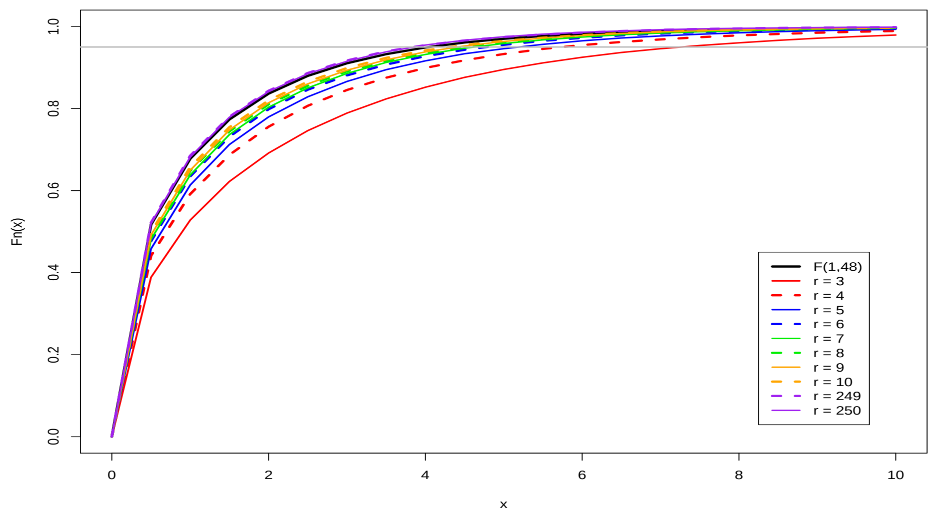

Figure E.11: Empirical CDF for M-test statistic values from $|y_{ij} - \bar{y}_i|$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

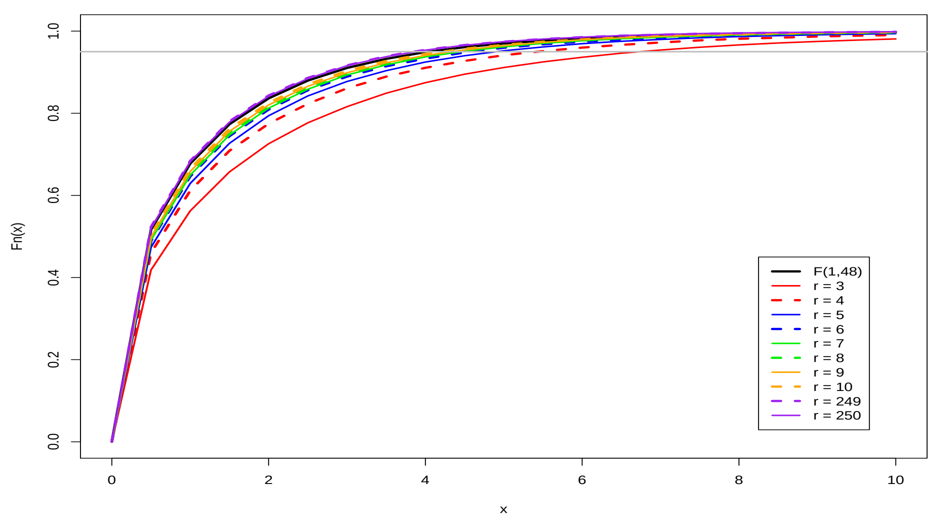

Figure E.12: Empirical CDF for M-test statistic values from $|y_{ij} - \bar{y}_i|^{0.42}$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

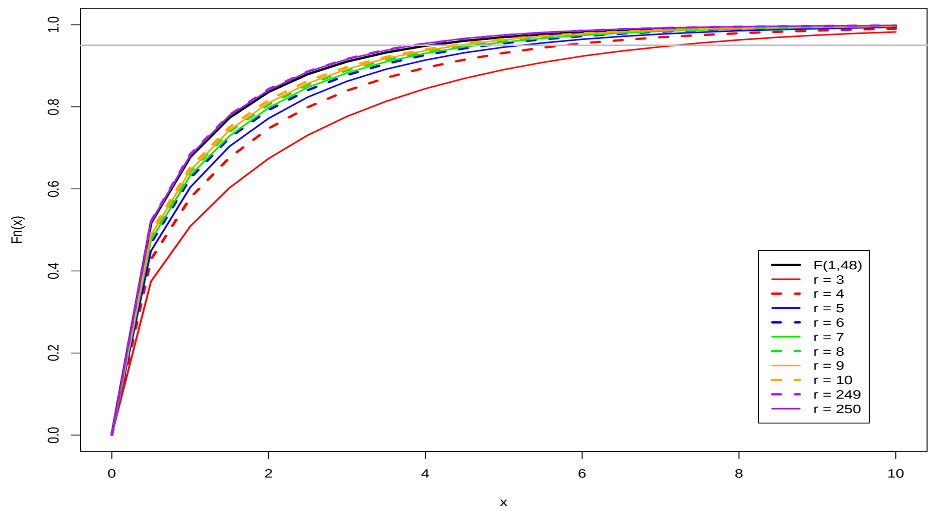

Figure E.13: Empirical CDF for M-test statistic values from $|y_{ij} - \bar{y}_i|^{1.5}$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

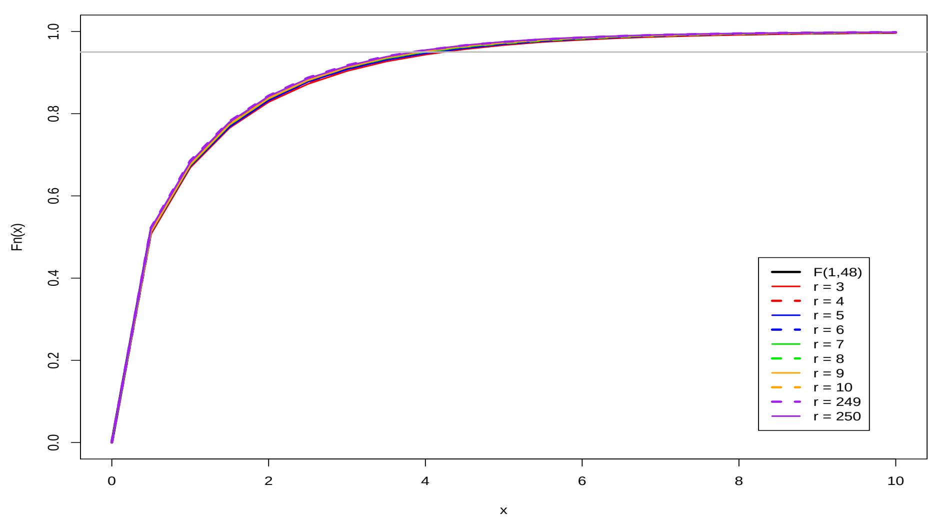

Figure E.14: Empirical CDF for M-test statistic values from $|y_i - \mathbf{x}_i'\hat{\beta}|$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

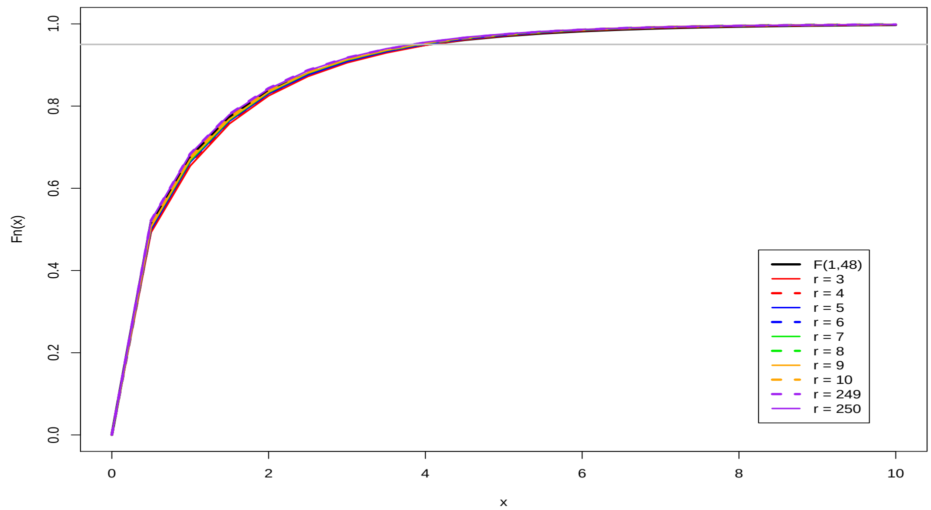

Figure E.15: Empirical CDF for M-test statistic values from $|y_i - \mathbf{x}_i'\hat{\beta}|^2$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

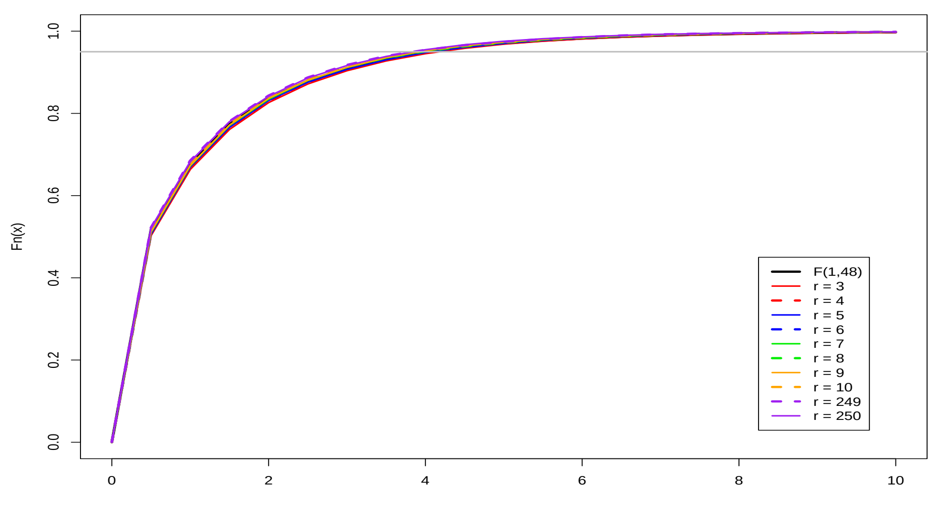

Figure E.16: Empirical CDF for M-test statistic values from $|y_i - \mathbf{x}_i'\hat{\beta}|^{1.5}$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

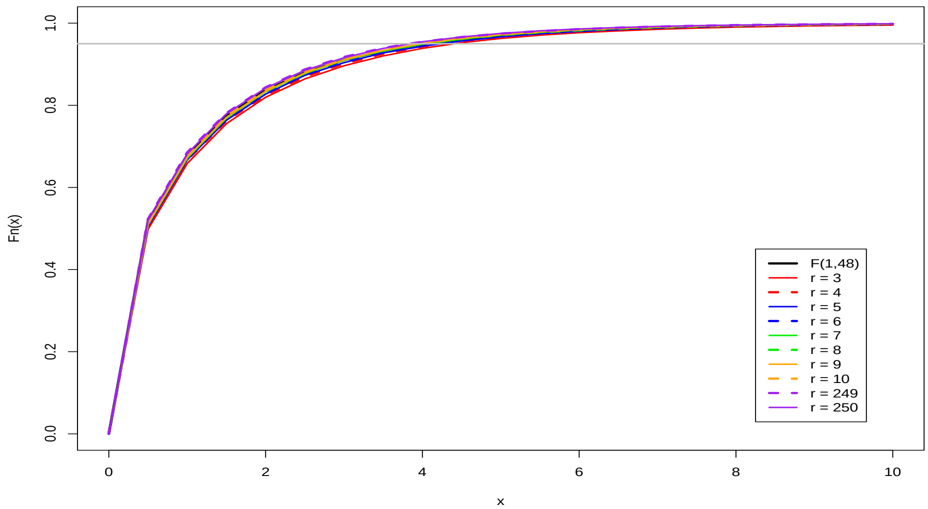

Figure E.17: Empirical CDF for M-test statistic values from $|y_i - \mathbf{x}_i'\hat{\beta}_{exp}|$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

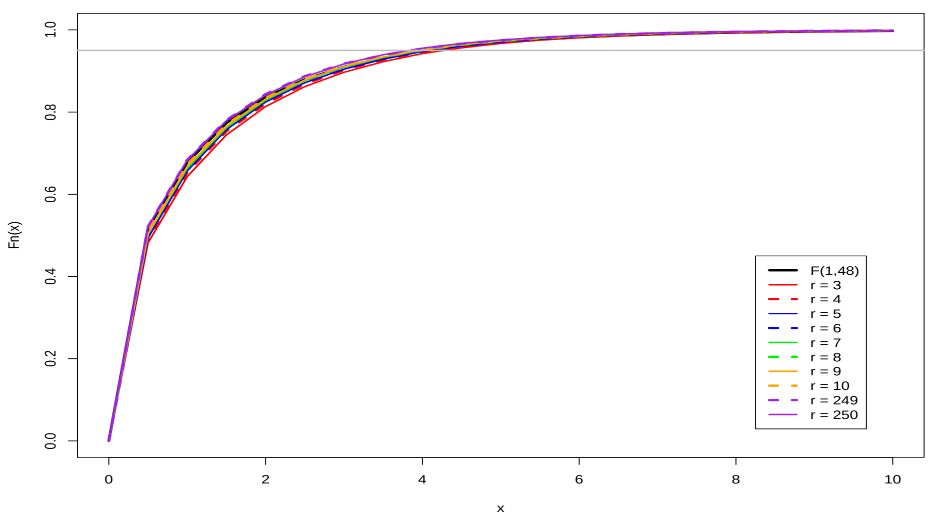

Figure E.18: Empirical CDF for M-test statistic values from $|y_i - \mathbf{x}_i'\hat{\beta}_{exp}|^2$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

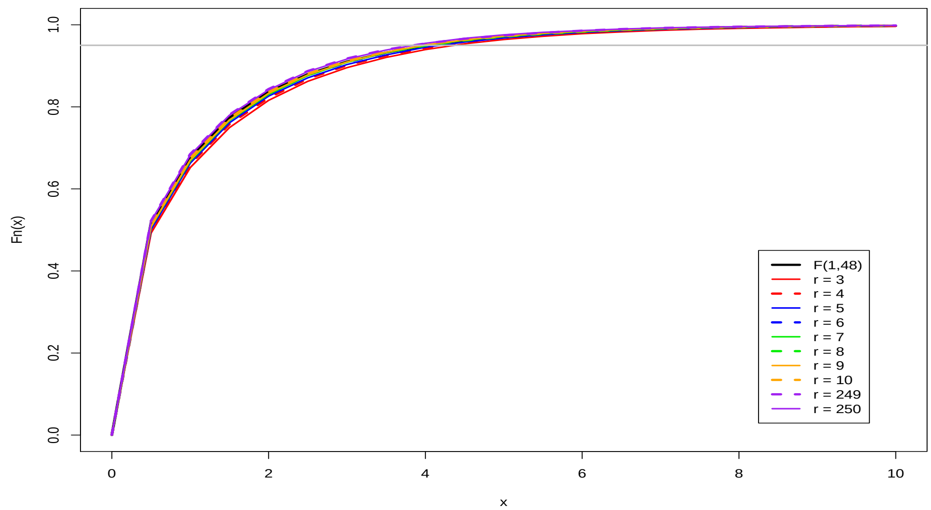

Figure E.19: Empirical CDF for M-test statistic values from $|y_i - \mathbf{x}_i'\hat{\beta}_{exp}|^{1.5}$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

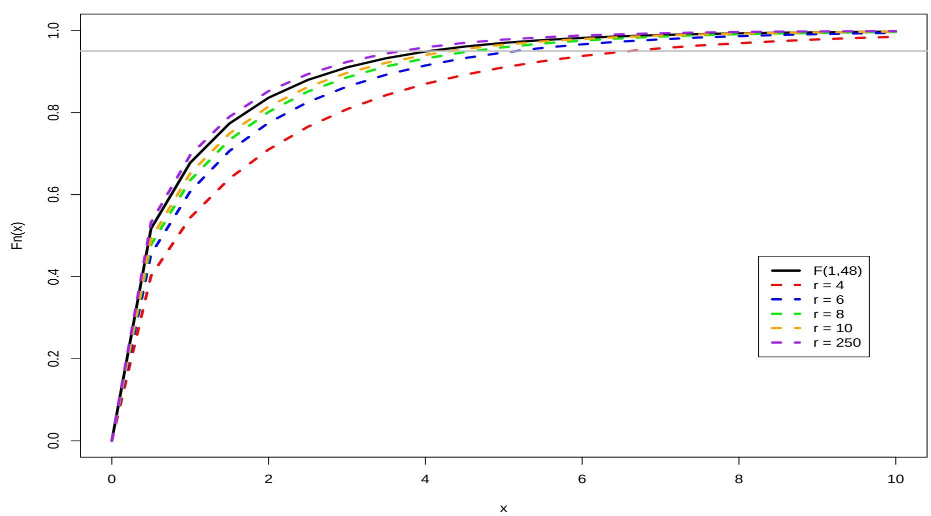

Figure E.20: Empirical CDF for M-test statistic values from $ln(|y_{ij} - \widetilde{y}_i|)$ for even numbers of replicates, $r = 4, 6, 8, 10, 250$

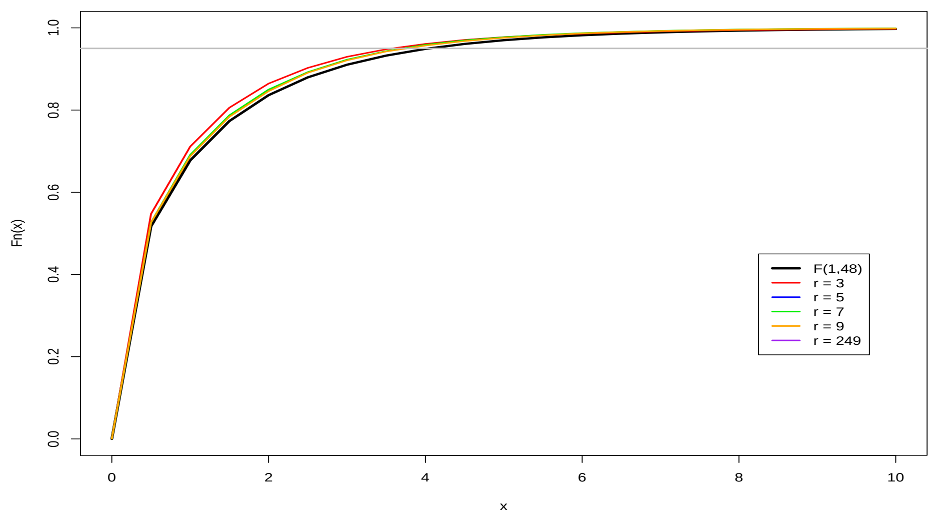

Figure E.21: Empirical CDF for M-test statistic values from $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ for odd numbers of replicates, $r = 3, 5, 7, 9, 251$

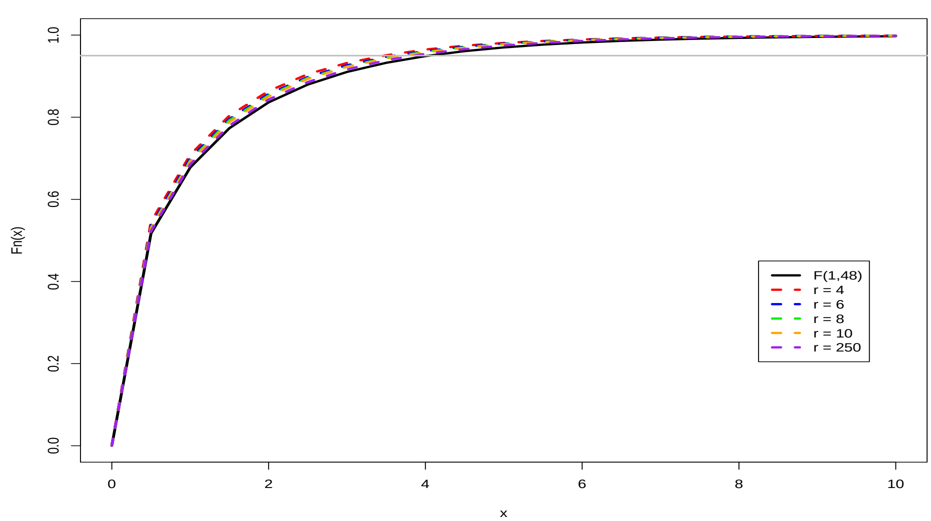

Figure E.22: Empirical CDF for M-test statistic values from $ln(|y_{ij} - \widetilde{y}_i|_{-1})$ for even numbers of replicates, $r = 4, 6, 8, 10, 250$

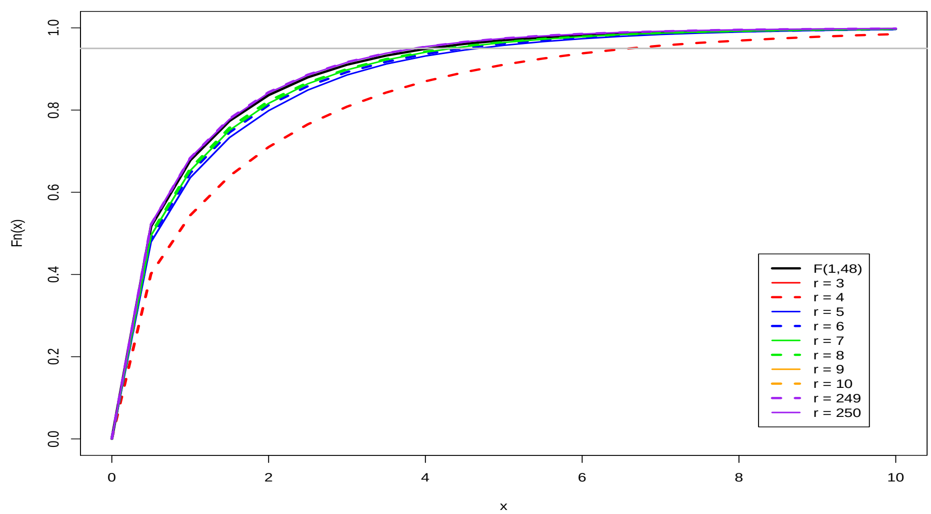

Figure E.23: Empirical CDF for M-test statistic values from $ln(|y_{ij} - \bar{y}_{i(-1)}|)$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

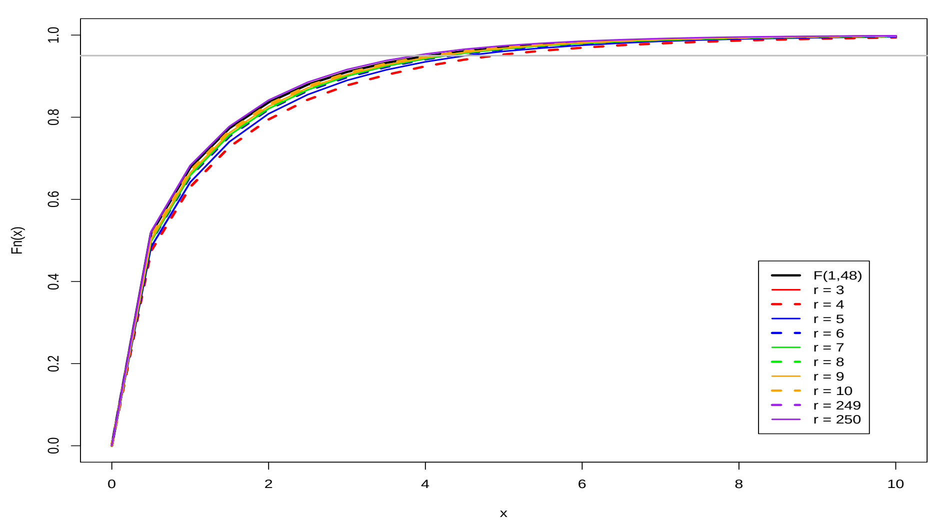

Figure E.24: Empirical CDF for M-test statistic values from $ln(|y_{ij} - \bar{y}_i|)$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

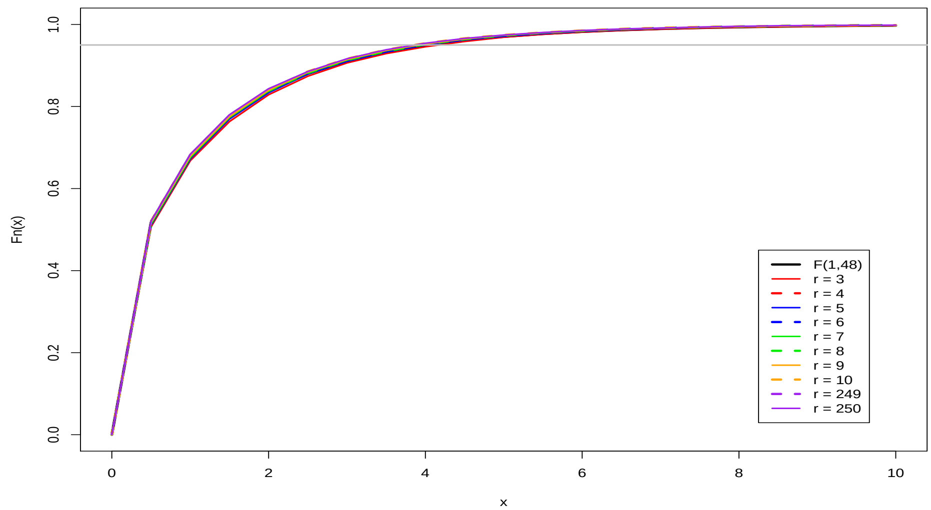

Figure E.25: Empirical CDF for M-test statistic values from $ln(|y_i - \mathbf{x}_i'\hat{\beta}|)$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

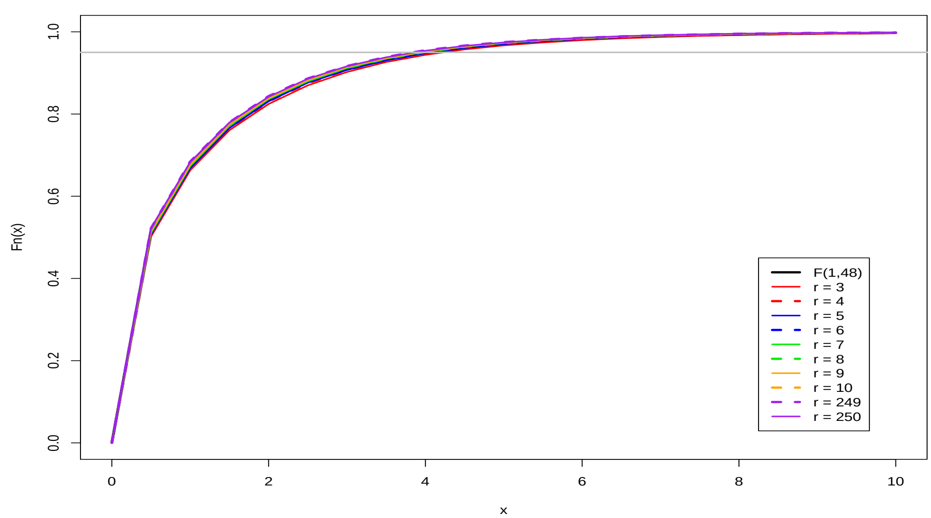

Figure E.26: Empirical CDF for M-test statistic values from $ln(|y_i - \mathbf{x}_i'\hat{\beta}_{exp}|)$ for $r = 3, 4, 5, 6, 7, 8, 9, 10, 250, 251$ replicates per treatment combination

APPENDIX F

POWER CURVES FROM PHASE II POWER STUDY

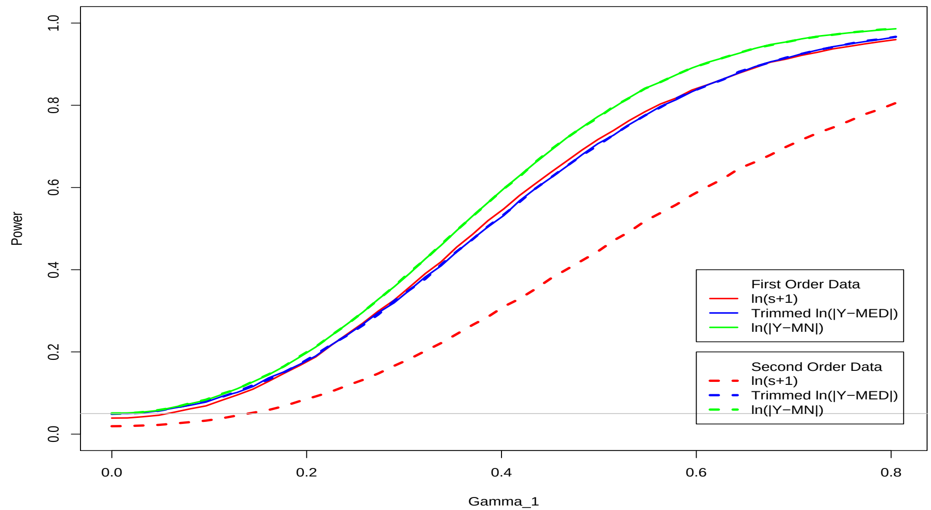

Figure F.1: Power curves for tests using $ln(s+1)$, $ln(|y_{ij} - \tilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated mean vectors and first- and second-order variance models using empirical critical values, $r = 3$ replicates, and normal error distribution

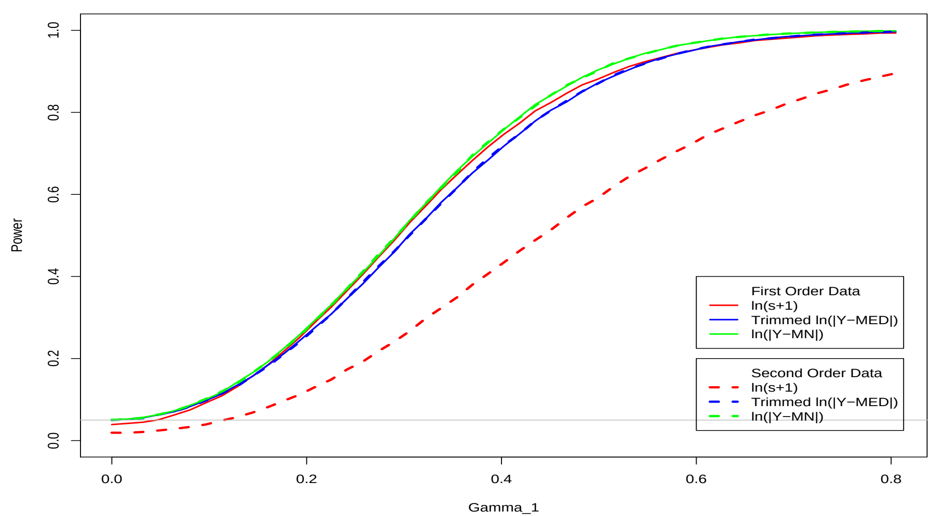

Figure F.2: Power curves for tests using $ln(s+1)$, $ln(|y_{ij} - \tilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated mean vectors and first- and second-order variance models using empirical critical values, $r = 4$ replicates, and normal error distribution

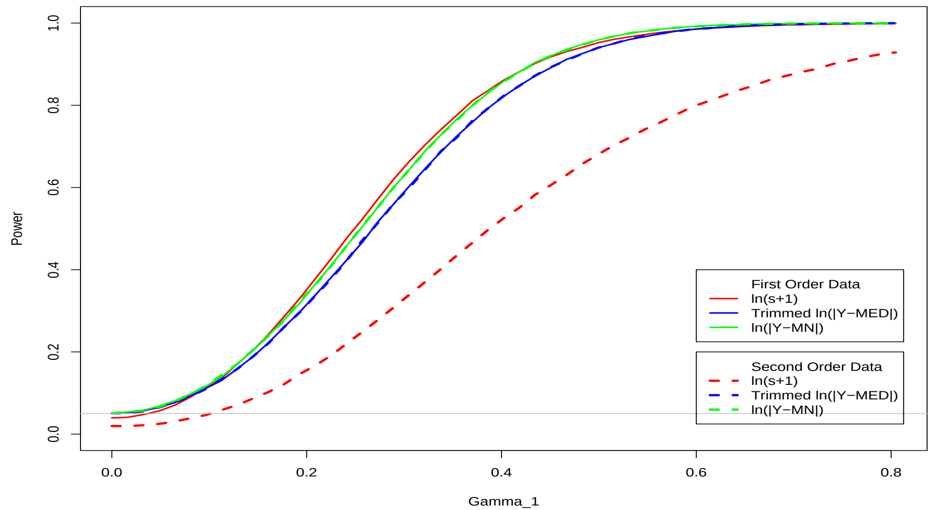

Figure F.3: Power curves for tests using $ln(s+1)$, $ln(|y_{ij} - \tilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated mean vectors and first- and second-order variance models using empirical critical values, $r = 5$ replicates, and normal error distribution

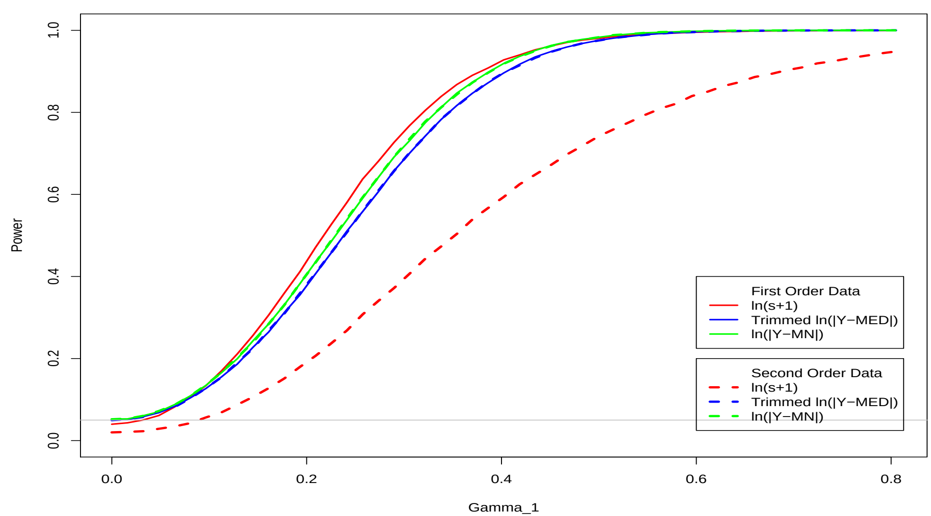

Figure F.4: Power curves for tests using $ln(s+1)$, $ln(|y_{ij} - \tilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated mean vectors and first- and second-order variance models using empirical critical values, $r = 6$ replicates, and normal error distribution

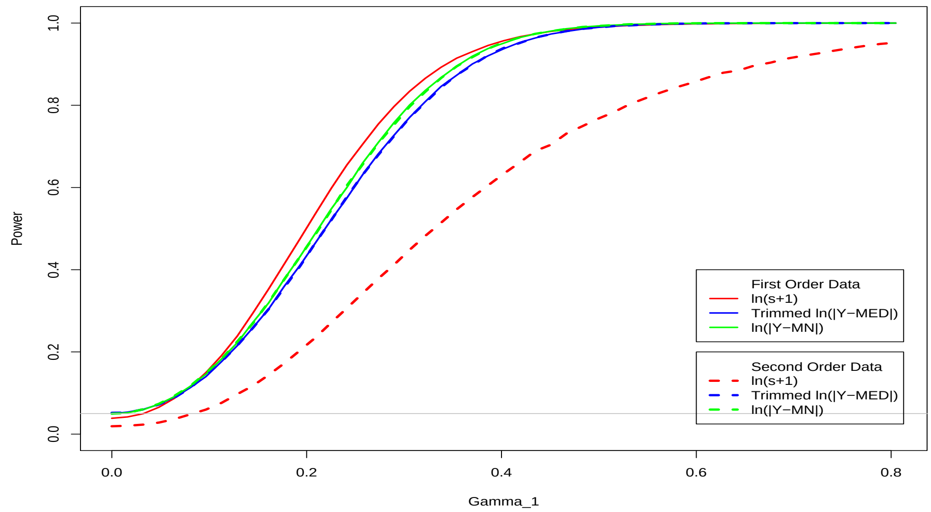

Figure F.5: Power curves for tests using $ln(s+1)$, $ln(|y_{ij} - \tilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated mean vectors and first- and second-order variance models using empirical critical values, $r = 7$ replicates, and normal error distribution

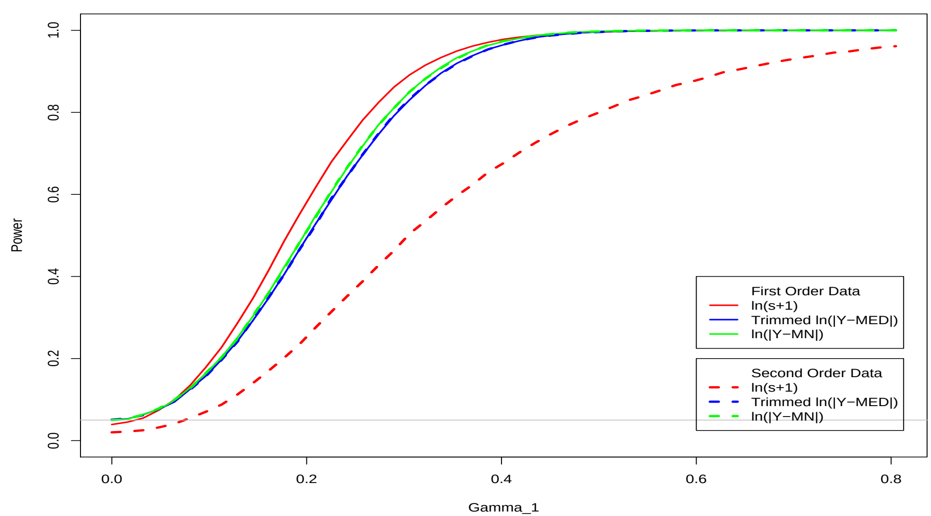

Figure F.6: Power curves for tests using $ln(s+1)$, $ln(|y_{ij} - \tilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated mean vectors and first- and second-order variance models using empirical critical values, $r = 8$ replicates, and normal error distribution

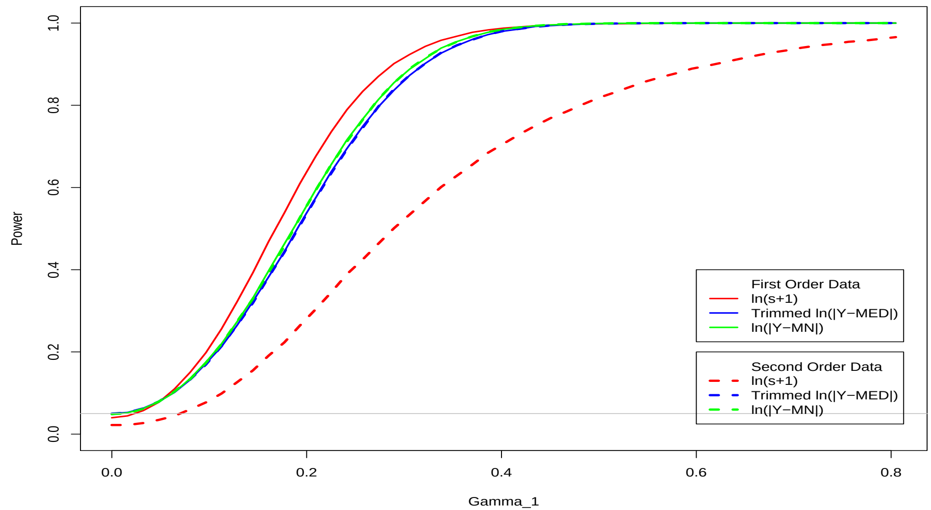

Figure F.7: Power curves for tests using $ln(s+1)$, $ln(|y_{ij} - \tilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated mean vectors and first- and second-order variance models using empirical critical values, $r = 9$ replicates, and normal error distribution

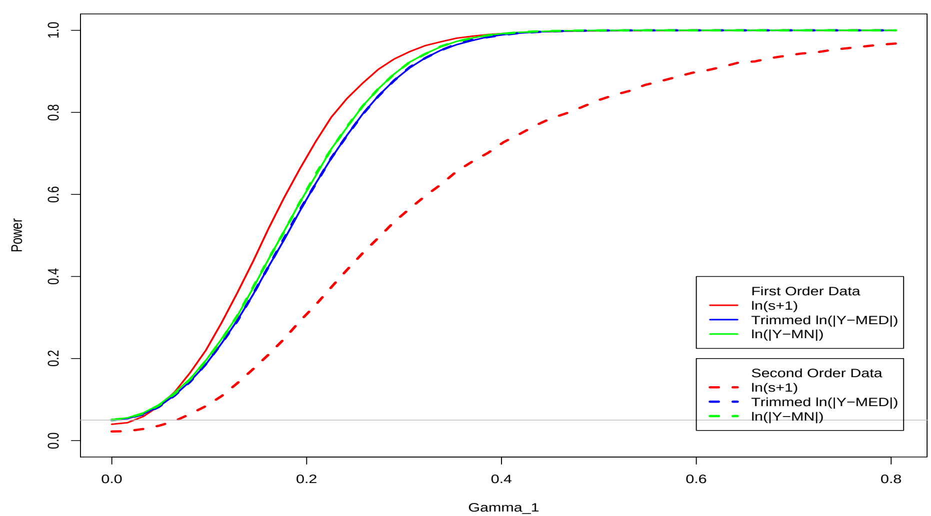

Figure F.8: Power curves for tests using $ln(s+1)$, $ln(|y_{ij} - \tilde{y}_i|_{-1})$, and $ln(|y_{ij} - \bar{y}_i|)$ with data from randomly generated mean vectors and first- and second-order variance models using empirical critical values, $r = 10$ replicates, and normal error distribution

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