CONCEPTUAL AND MATHEMATICAL BARRIERS TO STUDENTS LEARNING QUANTUM MECHANICS

DISSERTATION

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ABSTRACT

Quantum mechanics has revolutionized the way we view the physical world. This theory has required a dramatic revision in the structure of the laws of mechanics governing the behavior of the particles and lead to the discovery of macroscopic quantum effects ranging from lasers and superconductivity to neutron stars and radiation from black holes. Though its validity is well confirmed by the experimental evidence available, quantum mechanics remains somewhat of a mystery.

The purpose of this study is to identify students' conceptual and mathematical difficulties in learning the core concepts of introductory quantum mechanics, with the eventual goal of developing instructional material to help students with these difficulties. We have investigated student understanding of several core topics in the introductory courses, including quantum measurement, probability, Uncertainty Principle, wave functions, energy eigenstates, recognizing symmetry in physical systems, and mathematical formalism. Student specific difficulties with these topics are discussed throughout this dissertation.

In addition, we have studied student difficulties in learning, applying, and making sense out of complex mathematical processes in the physics classroom. We found students' achievement in quantum courses is not independent of their math backgrounds (correlation coefficient 0.547 for P631 and 0.347 for P263). In addition, there is a large

jump in the level of mathematics at which one needs to succeed in physics courses after the sophomore level in The Ohio State University's physics curriculum.

Many students do not have a functional understanding of probability and its related terminologies. For example, many students confuse the "expectation value" with "probability density" in measurement and some students confuse "probability density" with "probability amplitude" or describe the probability amplitude as a "place" or "area."

Our data also suggested that students tend to use classical models when interpreting quantum systems; for example, some students associate a higher energy to a larger amplitude in a wave function. Others, have difficulty differentiating wave functions from energy eigenstates. Furthermore, some students do not use the relationship between the wave function and the wavenumber as a primary resource in for qualitative analysis of wave functions in regions of different potential. Many students have difficulty recognizing mathematical symbols for a given graph and lack the ability to associate the correct functions with their respective graphs. in addition, students do not distinguish an oscillatory function such as e^{-ix} from an exponential decay function such as e^{-x}.

The results reported suggest recommendations for further study of student understanding of quantum mechanics and for the development of materials to aid understanding. These recommendations have potentially important implications for the teaching of introductory quantum mechanics and for the development of teaching aids, texts, and technology resources.

To Brian, Omid, & Parisa

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Lei Bao, Steve Stonebraker, and Homeyra Sadaghiani, "A Flexible Homework System," Submitted to American Journal of Physics (under review).

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FIELDS OF STUDY

Major Field: Physics

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CHAPTER 1

INTRODUCTION

"It is often stated that of all the theories proposed in this century, the silliest is quantum theory. In fact, some say that the only thing that quantum theory has going for it is that it is unquestionably correct." Michio Kaku

1.1 BACKGROUND

In the early 1980s, McDermott [1], Viennot [2], and other physics education researchers [3], [4], found that each student comes into a physics class with a system of commonsense beliefs and intuitions about how the world works. These commonsense beliefs, derived mostly from students' previous personal experiences, are often referred to as misconceptions, preconceptions, or alternative conceptions. Researchers have shown that these commonsense beliefs are very persistent, and traditional instruction does little to change them.

Over the past twenty years, Physics Education Research (PER) has changed our view of student learning in the traditional introductory courses. Regardless of the emphasis on the importance of conceptual understanding of physics, researchers have found that students leave introductory-level physics classes with less understanding of concepts than instructors expect [5]. Researchers have studied student difficulties

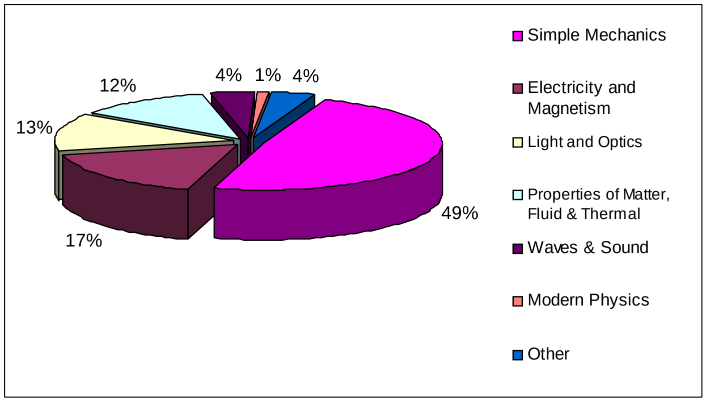

Figure 1.1: Distribution of research on student understanding of different topics in physics

in understanding physics and have focused on developing research-based instructional material to overcome these difficulties.

In a recent review article McDermott and Redish reported a distribution of published studies in physics education [5]. The McDermott and Redish resource letter contains annotated references to empirical studies about students' understanding of science concepts. This article classified the studies as "problem solving," "the effectiveness of laboratory instruction and lecture demonstrations," "the ability to apply mathematics in physics," "students' attitudes and beliefs," and a discussion of research into "reasoning in physics." As shown in Figure 1.1, this resource letter showed that about 49% of research on students' understanding of scientific concepts in physics has focused on mechanics, followed by studies on "electricity and magnetism" at 17%. Research on student understanding of "light and optics" and "properties of matter, fluid mechanics, and thermal physics" received almost the same proportion of researchers' attention, 13% and 12% respectively. Only about 4% of the research was devoted to "waves and sound", and concepts in modern physics at 1% received the most limited attention in the literature.

1.2 TRADITIONAL METHODS OF TEACHING PHYSICS

In most colleges, the structure of a typical introductory physics course has remained essentially static for almost 45 years. The traditional course structure has a number of problems that researchers have documented in recent years. Often, there is so much material to "cover" that students do not have time to develop a solid understanding of any single part. In addition, the issue of coverage does not give instructors time to use classroom techniques that would promote student learning. As a result, students do not get a sense of physics concepts, and after taking one course in physics, many students never take another physics. The traditional course, based on pure lecture, neither helps students to develop good critical thinking skills nor improves their intuition to overcome their misconceptions. A number of studies in physics education research have shown that even students who earn a high grade in typical introductory physics courses often cannot apply basic physical principles to realistic situations, solve a real world problem, or organize the ideas of physics hierarchically [6]. What such students do learn effectively is how to solve standard homework problems using search-for-equation methods that do not require much physical reasoning. A recent study showed that even students who worked more than 100 typical homework problems failed to resolve basic conceptual difficulties with Newtonian mechanics [7].

These problems exist not necessarily because the traditional course was initially poorly designed but at least partly because both the context for the introductory physics course and the type of students taking the course are significantly different now than they were in the early 1960s when the course was designed. Courses now serve a much larger number of students, with a broader range of skills, motivation, and needs. These days, it is important to empower a much more diverse group of students to reason logically about physical problems in a broader set of situations. The different situations and different desired outcomes of instructions call for a different approach to teaching the course.

1.3 TEACHING QUANTUM MECHANICS

Recent developments in nanotechnology, photonics, and superconductivity bring to our everyday life advanced engineering and businesses devices that can be appreciated and explained only through principles of quantum mechanics. The broad development of the applications of quantum technology makes it desirable to introduce some basics of quantum phenomena to a larger population of students early on. However, the abstract nature of quantum mechanics concepts and high level of mathematics involved in this requires different approaches in teaching the course at the introductory level. Traditional teaching of quantum mechanics generally involves one of two distinct approaches to introducing students to the basic concepts of quantum mechanics.

The first approach can be described as a quantitative approach, through which students are introduced to the mathematical algorithms and processes used to solve quantum mechanics problems and in the process they become acquainted with the mathematical tools needed to understand quantum mechanics concepts. Shankar strongly expressed his belief that students must first be introduced to the relevant mathematical skills so that later they "can give quantum theory their fullest attention without having to battle mathematical theorems at the same time [8]." McMurray shared similar concern where she wrote that "students should come to grips with the basic mathematical ideas that are necessary for proper understanding of quantum theory [9]." The idea of teaching students the relevant mathematics first is not unique to quantum mechanics courses. Similar pedagogical approaches have been used in other areas of physics. For example, college freshmen enrolled in a typical calculus-based introductory mechanics course first learned about measurement and vectors.

The second approach to introducing students to the basic concepts of quantum mechanics can be expressed as a historical-conceptual approach. This approach initially emphasizes the history of the development of the experiments, concepts, and theories that have led to the theory of quantum mechanics. The historical-conceptual approach incorporates the "historical development" of quantum mechanics and introduces students to challenges that are similar to those faced by physicists in the early twentieth century.

According to Liboff, a review of the historical development of quantum mechanics and elements of classical mechanics is "important to a firm understanding of quantum mechanics [10]." Supporters of this view argue that since students' misconceptions in science are similar in nature to those that drove the development of various science fields by previous members of the scientific community, omitting certain steps of the historical development would hinder students' conceptual growth and understanding of quantum mechanics [11]. For example, studies on students' conceptual understandings of electricity and magnetism showed that a significant number of students viewed electricity as a "fluid" an idea similar to that held by scientists in the late eighteenth century. Research in the area of introductory mechanics showed that, similarly, many students explained motion in ways similar to Aristotle's explanations. For example, students believed that heavier objects would fall faster than lighter ones [12]. Consequently, the similarity between students' ideas about the physical world and past scientific explanations led researchers to suggest that knowledge about the history of science could help science educators and teachers anticipate the topics about which students would be likely to develop serious misconceptions [13]. Hadzidaki et al. explained that the birth of quantum mechanics introduced a new worldview, or paradigm, in physics. Thus, the authors argued that conceptual understanding of topics in quantum mechanics requires students to develop a new way of thinking. They argued that a qualitative approach could limit the development of serious misconceptions about modern physics topics, and would introduce the core concepts of quantum mechanics to a wider range of students. Additionally, this approach could help students form a more comprehensive understanding that is analogous to contemporary scientific understanding of the macroscopic and microscopic world [14]. Experts have pointed out that it would be possible to introduce the students to important topics and concepts in introductory quantum mechanics without requiring prior advanced mathematical knowledge. It is possible to reduce the technical mathematical skills necessary for the students to learn introductory quantum mechanics by introducing the powerful computer mathematics software programs, which are readily available at most universities [15].

1.4 THE RELATIONSHIP BETWEEN STUDENTS' MATHEMATICAL SKILLS AND THEIR PHYSICS ACHIEVEMENT

Researchers have also studied the relationship between students' mathematics achievement and their physics achievement. Cohen found that students' mathematics scores correlated highly with their physics scores [16]. Other research conducted in this area, however, showed that mathematical skill is only one of several variables prerequisite to understanding the physics concepts presented in a typical introductory mechanics course, and that high scores on mathematics tests are not sufficient indicators of conceptual understanding in physics [17]. Studies of students' understanding in introductory mechanics have also shown that students' understanding is highly fragmented and the scientific concepts that have definite relationships in science are loosely linked, if at all, in students' cognitive structures. Therefore, as some have assumed, the knowledge acquired in mathematics might not transfer as easily to the quantitative teaching approach of topics taught later in physics courses. The results of cognitive research on the "transfer of learning" have indicated similar findings that support the findings in science education. More specifically, cognitive research has shown that when a student was taught in only one subfield of a discipline, transfer of knowledge to related subfields was difficult and cannot be assumed to take place spontaneously [18]. However, up to this point, no research on the relationship between student mathematical and quantum physics achievement has been conducted. Considering the abstract nature of quantum mechanics and the high level of formalism in this theory, there is substantial need for research on the role of mathematics in students' understanding and achievement in quantum mechanics, especially in upper division courses. This study aims to fill this gap and hopes to promote effectiveness in the current practice of teaching quantum mechanics.

1.5 THE PURPOSE OF THIS STUDY

The principal purpose of this research was to study and identify university students' understanding of the core concepts of introductory quantum mechanics, and to determine the mathematical skills that students need to master in order to succeed in a quantum mechanics course. I have studied the most common difficulties students have exhibited in selected topics in quantum mechanics. Since understanding of more conventional mathematics seems as important as a conceptual understanding of these core ideas, I have also studied the relationship between students' mathematical ability and their achievement in quantum courses. A secondary goal was to develop useful classroom materials to aid teaching the problematic concepts identified.

Quantum mechanics understanding as defined for the purpose of this study and the development of insurrectional instruments, consisted of basic knowledge and technical terminology in quantum mechanics, understanding of principles of quantum mechanics theory, and the ability to interpret mathematical and visual representations of quantum mechanical phenomena. Students may be introduced to actual laboratory investigations in quantum mechanics after they have an introductory quantum mechanics course. Therefore, no attempt was made to include knowledge associated with conducting experiments in quantum mechanics. The assessment of the philosophical interpretations of quantum mechanics theory was also not included.

The study of students' conceptual and mathematical difficulties and the development of research-based instructional materials to reduce difficulties are important for several reasons. First, this study extends physics education research on conceptual understanding to a different domain of physics and in a more advanced course. Second, identifying student difficulties with basic concepts can lead to the development of classroom materials to help instructors address these difficulties with more explicit examples and instruction. Finally, the study of students' mathematical difficulties in quantum mechanics can also lead to the development of a better curriculum. While obtaining a conceptual understanding of basic physics concepts has been recognized as important, we cannot deny the role of mathematics and abstract formalism in quantum topics, especially in more advanced courses. However, little research on students' conceptual and mathematical understanding of quantum mechanics, and the development of instructional materials has been conducted. Therefore, there is a substantial need for research on students' understanding in quantum mechanics, and for the development of effective and useful classroom materials that can aid more effective teaching.

Students enrolled in introductory quantum mechanics courses are usually physics majors who have taken advanced physics and mathematics courses in college. Therefore physics educators, and others, tend to assume that students who have passed these courses have mastered the fundamental quantum mechanics concepts presented in these introductory quantum mechanics courses. Based upon research conducted on students' conceptual understanding in other fields of physics, and the findings of this dissertation, there is reason to be very skeptical of such an assumption. But up to this time there has been little or no research on this topic. Thus, the identification of students' conceptual and mathematical difficulties with learning the core concepts in introductory quantum mechanics, with an emphasis upon development of instructional material, is an important step in filling a serious gap in the literature.

As mentioned in section 1.2, experts in physics education have expressed legitimate concerns about current practices and perceptions in the teaching of introductory quantum mechanics. Although these experts have pointed out that it would be possible to introduce students to important topics and concepts in introductory quantum mechanics without requiring prior advanced mathematical knowledge, the findings of this dissertation indicate that students' mathematical skills have a noticeable effect on their achievements in quantum mechanics courses, especially in more advanced courses. It should be emphasized that quantum mechanics is a mathematical theory, and the ability to adopt this theory to our physical world and construct physical meaning from abstract concepts and complex formalism is a logical process that is required for developing of scientific knowledge of quantum phenomena.

Although introducing students to the basic concepts, experiments, and theories that have led to the development of quantum mechanics is important for a firm understanding of the subject, especially for non-major students, we cannot deny the need for a certain level of proficiency in abstract reasoning, mathematical skills, and formalism for physics majors in order to succeed at their studies at the graduate level. In Chapters 4, 5, 6, and 7, I will show how students' mathematical skills in topics such as probability, symmetry, complex numbers, and differential equations affect their success in quantum courses. A general outline of the study and its results follow this section.

1.6 DISSERTATION OVERVIEW AND FINDINGS

An investigation of students' conceptual and mathematical difficulties in learning quantum mechanics, this dissertation consists of 9 chapters:

1. Introduction;
2. Review of previous research;
3. Methodology in physics education research;
4. The relationship between student background knowledge and their learning of quantum mechanics;
5. Identifying student difficulties in understanding quantum measurement;
6. Identifying student difficulties in understanding wave functions;
7. Identifying student difficulties in understanding symmetry;
8. Instructional material;
9. Conclusion and future research.

Chapter **1** is an introduction to the dissertation and provides the reader with background information from PER on student learning in physics and introduces the different approaches in teaching quantum mechanics. Chapter **2** first briefly summarizes the general findings of PER studies of students' understanding of physics, then gives an overview of previous research on student difficulties in modern physics and quantum mechanics topics. The final part of this chapter discusses some of the students' mathematical difficulties in physics.

Chapter **3** contains the history and an overview of research methods in PER. This chapter outlines the methodology for constructing the multiple choice questions that are employed in this study to measure students' understanding of selected topics in introductory quantum mechanics. Based on findings from classroom observation, interviews, and reading students' written homework and exams, I developed 21 on line questionnaires with 10 to 15 questions each. Over three quarters, I administered these questionnaires to students enrolled in sophomore and junior/senior level quantum courses at The Ohio State University. The analysis of students' responses to these questionnaires forms the basis for the data in this study.

In Chapter **4**, the study of the relationship between students' background knowledge in mathematics and physics and their success in quantum courses, shows that:

1. Students' achievement in quantum courses is not independent of their math backgrounds. There is a correlation between students' math scores and their final grades in both $P263$ and $P631$. Although this correlation is small in the case of $P263$ (correlation coefficient 0.347), in $P631$ it is more significant (correlation coefficient 0.547);

2. It seems that there is a large jump in the level of mathematics at which one needs to succeed in physics courses after the sophomore level in The Ohio State University's physics curriculum;

3. Students' background knowledge of classical waves correlate with their final grades in $P263$ (correlation coefficient of 0.487).

Chapter **5** focuses on identifying student difficulties in understanding quantum measurement. I have studied this topic in a variety of different contexts, including mathematics and interpreting formalisms, calculating of expectation value, probability density, the Uncertainty Principle, understanding the probability amplitude, spin measurement in the Stern-Gerlach experiment, and classical probability. The findings on this chapter shows that student difficulties understanding quantum measurement and probability are very complex and are not limited to merely conceptual difficulties. Many students have serious difficulties with both conceptual and mathematical aspects of quantum measurement. In regard to students' conceptual difficulties with topics of quantum measurements I found that:

1. Many students do not have a functional understanding of probability and its related terminologies. For example, many students do not distinguish the "fraction" of particles from the "probability" of measuring certain outcomes in Stern-Gerlach experiments;

2. Some students confuse the "expectation value" of an operator with the "probability density" in measurement;

3. Students' difficulties with the concepts of probability often interfere with their ability to understand and apply the Uncertainty Principle; for example, students tend to mix the "expectation value" with the "amount of uncertainty" in a measurement;

4. Some students interpret the Uncertainty Principle as our inability to make a precise measurement;

5. Some students confuse "probability density" with "probability amplitude" or describe the probability amplitude as a "place" or "area."

In the quantitative aspects involved in learning quantum mechanics the students' main difficulties seem to be dealing with formalisms and abstract materials; many students at the sophomore level who were comfortable using concrete numbers to answer some of the questions correctly seemed to be uncomfortable with the use of abstract symbols to answer questions about classical probability. In summary:

1. Most students show difficulties formalizing their conceptual understandings in terms of mathematical symbols;

2. Students possessing the required knowledge of mathematics often have difficulties translating the formalism in terms of physics and vice versa;

3. Students often perform better with visual questions as compared to similar, but more abstract, questions.

Chapter **6** discusses student difficulties understanding wave functions and their mathematical representations, such as graphs and equations. Findings in this chapter can be summarized as follows:

1. Some students have difficulty with the concept of the wave function as a probability amplitude;

2. Students tend to use classical models when interpreting quantum systems; for example, some students associate a higher energy to a larger amplitude in a wave function;

3. Most students have difficulties calculating a probability density from a given wave function; for example, as a common mistake, students do not square or normalize the wave function before finding the probabilities;

4. Some students have difficulty differentiating wave functions from energy eigenstates;
5. Most students do not write the wave function in terms of its energy eigenfunctions in order to determine the wave function in a later time;
6. Some students do not use the relationship between the wave function and the wave number as a primary resource in qualitative analysis of wave functions in regions of different potential;
7. Many students have difficulty recognizing mathematical symbols for a given graph and lack the ability to associate the correct functions with their respective graphs;
8. Some students do not distinguish an oscillatory function such as e^{-ix} from an exponential decay function such as e^{-x}.

Chapter **7**, discusses the relation between students' mathematical background and their achievements in quantum courses with the example of symmetry. Our findings in this part of the study show that:

1. Some advanced students are not always able to recognize the existence of symmetry in math and physics problems and often fail to use the symmetrical features of a problem to simplify their solutions;
2. Students' mathematical knowledge of symmetry correlates with their ability to solve symmetry-related problems in the context of quantum mechanics;

3. When students lack the appropriate knowledge to solve physics questions, they tend to over-generalize their understanding of some principle related to the problem at hand without careful examination of applicability.

Chapter **8** contains the four research-based instructional worksheets I used with the upper level quantum classes at The Ohio State University, which includes worksheets on the topics of probability, wave functions, and the mathematical solution to the time-independent Schrödinger equation. These worksheets are in the preliminary stages of development. Since many revisions and trials are needed to assess their effectiveness in helping students with learning these topics, claims about the effectiveness of these materials at this stage is not possible. Nevertheless, there is evidence that these materials were popular with the students.

Finally, Chapter **9** contains the conclusion, which summarizes the dissertation results, discusses the implications for instruction and curriculum development, and suggests future research questions.

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CHAPTER 2

REVIEW OF PREVIOUS RESEARCH

"... I believe, very truly at the nature of the quantum rules." Ervin Schrödinger

2.1 INTRODUCTION

Over the past two decades, Physics Education Research (PER) has shown strong evidence that students do not learn much from a traditional lecture course in introductory physics.[1], [2], [3] PER seeks to understand how students learn physics and identify student difficulties in understanding physics to find research-based methods to help students to understand physics. Research in physics education has been successful in pinpointing some of the fundamental problems students have with the understanding classical physics concepts [4]. Furthermore, with the development of instructional material and curricula development, PER has been able to suggest effective ways to overcome some of these difficulties [5].

In general, three different categories can be found in PER. The first one focuses on students' conceptual understanding of physical phenomena and detecting possible misconceived ideas about the topic at hand. The second addresses the design of instructional instruments, such as tutorials and computer software to help students understand complex ideas. The final category involves developing a curriculum to ameliorate or prevent these difficulties.

The first category, understanding students' difficulties, is necessary for developing the research material for the other two. Although PER research has exhausted these three categories in introductory physics courses, very little PER work has been done beyond the first category of research on student understanding of more advanced courses such as quantum mechanical concepts.

In understanding how students learn physics, other areas of educational research, including math and cognitive science, have made great contributions. In particular, because of the complex mathematical formalism required in quantum mechanics, students' ability to apply and interpret math correctly plays a crucial role in their success in these courses. This chapter overviews the previous studies relevant to this study.

This chapter, first gives a brief review of general findings of education research on students' understanding in science and physics (section 2.2), then summarizes the pedagogical research on student understanding of quantum mechanics (section 2.3), and finally, gives a brief overview of research on student difficulties using mathematics in physics (section 2.4).

2.2 RESEARCH ON STUDENT UNDERSTANDING IN SCIENCE AND PHYSICS

The purpose of this section is to review briefly the relatively large body of literature on research on students' understanding in science and in physics. There has been a much international interest on students' understanding of ideas, concepts, and theories taught in science courses. Many studies have revealed that students at different ages with different backgrounds come to science courses with well established ideas that have a powerful influence on what they learn. The students' ideas, in many cases, contradict currently accepted scientific ideas. Although made famous as "misconceptions," researchers have built a strong case for using the term "alternative conceptions" to refer to these type of students' understanding [6]. The term refers to experience-based explanations constructed by students to explain a range of natural plausible phenomena. The term also implies that alternative conceptions are contextually valid and rational from the standpoint of the student. As a result of students' alternative conception research or "Alternative Conception Movement," the following 6 claims are widely accepted [7]:

1. Students come to science instruction with established ideas concerning natural phenomena;
2. Students' alternative conceptions are resistant to change by conventional teaching strategies;
3. Students' alternative conceptions often parallel explanations of natural phenomena offered by earlier generations of scientists;
4. Students' alternative conceptions have their roots in personal experiences, perception, culture, language, school and schooling;
5. Some teachers have alternative conceptions that are similar to those of their students;
6. Students' alternative conceptions conflict with the knowledge presented in formal instruction.

2.3 RESEARCH ON STUDENT UNDERSTANDING OF QUANTUM MECHANICS

The pedagogical research on students' learning of quantum mechanics is rapidly growing. Some researchers have focused on students' understanding of macroscopic phenomena, others have focused on students' understanding of matter waves. Understanding of wave-particle duality of photons and electrons has been another interesting starting point for researchers. However, very few PER researchers have studied students' understanding of advanced quantum mechanics, which would lead to the development of assessment or instructional tools. This section discusses examples of research on students' difficulties, as well as the existing assessments and instructional tools developed in the context of quantum physics. This section ends with some research suggestions for curriculum development.

2.3.1 RESEARCH ON STUDENT UNDERSTANDING OF MODERN PHYSICS

The context for most research on introductory quantum topics is often modern physics courses. Most existing research on modern physics courses focuses on students' ideas of quantization, electron diffraction, the photoelectric effect, light, and models of the atom. For example, Aubrecht studied students ideas about photon prior to taking any quantum courses [8]. Other studies emphasize the development of microscopic models to describe the macroscopic observations. For example, to learn how students develop microscopic models, Beth Thacker interviewed college students to probe their understanding of the charge to mass ratio of an electron, diffraction, and photoelectric effect [9]. In this research, in order to study the "nature of student's preconceived models" and how students develop and alter their models before, during, and after instruction, students were asked about the underlying processes of the macroscopic demonstrations. The results of this study suggest that students' naïve models are often memorized "facts," which do not have strong roots in actual understanding. Therefore, when explaining the observed phenomena, the correct model of the microscopic process was often missing. Thacker further concluded that students need to recognize how macroscopic observations lead to the development of models of the microscopic process in order to gain an understanding of modern physics [10], [11].

There have been other investigations of students' understanding of microscopic and subatomic phenomena. Much of the early work comes from Germany [12], [13] where scientists like Fischler [14], and Lichtfeldt [15] describe the preliminary efforts in understanding students' models of quantum systems. Most of these studies focused on categorizing models that students used when describing microscopic phenomena. However, this early research has not produced any instructional instruments for teaching these topics.

Johnson, Crawford, and Fletcher [16] conducted systematic research on junior undergraduate students at the University of Sydney, Australia to investigate how students learn quantum mechanics. The focus of the study was on students' responses to the following three survey questions:

1. What is a particle?

2. What is a wave?

3. What is the uncertainty?

Attribute (particle)	# of students
Momentum	19
Shape/Size	5
Discrete Energy	4
Photoelectric Effect	3
Reflection	2
Something Irrelevant	1
Not a Particle	1

Attribute (wave)	# of students
Fourier Decomposition	9
Mathematical Model	5
Something Irrelevant	4
Velocity	2
Uncertainty	2
Wave Equation	2
Spectrum	1

Table 2.1: The properties students cited for a particle (table on the left) and wave (on the right)

In this study, the analysis of the results had several stages: categorization analysis, content analysis, and correctness analysis. Johnson et al. categorized students' responses to question one into three levels: (1) a particle is made of stuff, (2) a particle is stuff that travels along a well-defined path, (3) a particle is stuff that travels along a well-defined path and responds to external forces.

In the analysis of the responses to the second question, two different approaches were identified. Seventeen students started with "standard textbook" terminology and described a wave as a localized concept; the remaining 31 students, started with the simple properties of a wave; and 17 out of these 31 had deeper conceptual understanding and were able to see a wave in relation to other physical concepts.

In addition, students associated some properties to a particle and a wave. The most commonly cited properties were "momentum" for a particle and "Fourier decomposition" for a wave. Table 2.1 demonstrates the properties students cited for a particle and a wave, and the frequency of response among students. The responses to the third question concerning "uncertainty" were not analyzed by the authors due to significant fragmentation among the responses.

Another example of research on the topics of modern physics is a pilot study by Euler et al. from Kansas State [17]. They conducted research on the conceptual knowledge of future physics teachers in Germany to investigate their views in the area of modern physics and on ways to change these views. Their main focus was on models of the atom. They split the students into an experimental and a comparison group. All students had taken a course in quantum physics and the students of the experimental group took three special sessions dedicated to models and concepts in quantum mechanics.

The answers to the pretest revealed that most students apply classical models to quantum phenomena. Although the results of the pretest were homogenous between the two groups, the statistical analysis of the results of the post-test showed a significant conceptual change in the experimental group. However, one can claim this difference is due to the fact that the experimental group had received more instruction on the topics.

In the modern physics level, the PER Group at the University of Washington has focused on students' difficulties understanding interference and diffraction patterns [18]. Through interviews of college students on interference and diffraction at the University of Washington, researchers found that most students had difficulty explaining the phenomenon of diffraction or predicting the pattern on the screen when the width of the slit is changed [19]. In addition, researchers in this group have investigated student difficulties in applying a wave model when they study interference and diffraction of light [20]. Vokos et al. found that sophomore and junior students are not

Misconceptions	Examples
The idea of Quantal State	-Energy eigenstates are the only allowed states -A quantal state $\Psi(x)$ is completely specified by its associated probability density $\|\Psi(x)\|^2$ -A wave function describes a single system averaged over some amount of time
Quantum Measurement	- The collapse of the wave packet involves faster-than-light communication
Regarding Identical Particles	- In the two-body expression $\Psi(x_1, x_2)$, the labels 1 and 2 refer to particles

Table 2.2: List of some observed misconceptions by Styer

able to interpret diffraction and interference in terms of basic wave models. Furthermore, these students often treated the de Broglie wavelength as a fixed property of a particle, not as a function of momentum [21].

In summary, most studies on student understanding of microscopic phenomena indicate that the main misconceptions are a result of students' classical worldview, which provoke the overlapping, or mixing-up, of the conceptual frameworks of classical physics and quantum mechanics.

2.3.2 RESEARCH ON STUDENT UNDERSTANDING OF MORE ADVANCED QUANTUM PHYSICS

The amount of pedagogical research in more advanced quantum mechanics is very limited. Some of the findings on students difficulties are the observations of experienced instructors and some are more systematic research that involve interview and

Symbol	Misconception
M1	If the system is initially in an eigenstate of any operator $\hat{Q}$, then the expectation value of another operator Q' will be time independent if $[\hat{Q}, \hat{Q}'] = 0$.
M2	If the system is initially in an eigenstate of an operator $\hat{Q}$, then the expectation value of that operator is time independent.
M3.1	An eigenstate of any operator is a stationary state.
M3.2	If a system is in an eigenstate of any operator $\hat{Q}$, then it remains in the eigenstate of $\hat{Q}$ forever unless an external perturbation is applied.
M3.3	The statement: "The time-dependent exponential factors cancel out in the expectation value" is synonymous with the statement: "The system does not evolve in an eigenstate."
M4	The expectation value of an operator in an energy eigenstate may depend on time.
M5	If the expectation value of an operator $\hat{Q}$ is zero in some initial state, the expectation value cannot have any time dependence.
M6	Individual terms $(H_0, H_{1,} \ldots)$ in a time-dependent Hamiltonian $H = H_0 + H_1 + \ldots$can cause transitions from one eigenstate of H to another.
M7	Time evolution of an arbitrary state cannot change the probability of obtaining a particular outcome when any observable is measured because the time evolution operator is of the form $\exp(-i\hat{H}t/\hbar)$.

Table 2.3: Misconceptions in the domain of quantum measurement and time evolution identified by C. Singh et al.

pretest/post-test techniques. For example, Styer reported 15 common misconceptions[1] in quantum mechanics regarding the topics of quantal state, quantum measurement, and identical particles [22]. His list is based on casual observations and should not be mistaken for well-founded research. Examples of his findings are shown in Table 2.2.

[1] Using the term "misconception" in this article is controversial in the physics education research community, especially with respect to topics of quantum mechanics where students have no prior real world experience with the microscopic phenomena.

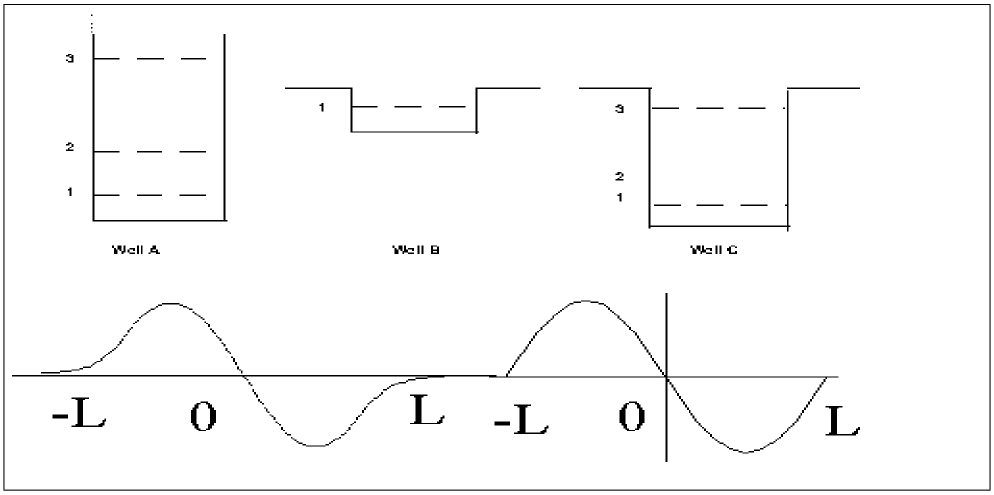

Figure 2.1: Conceptual quiz question on potential well given in Physics 263 at the University of Mary land.

At the University of Pittsburgh another study on similar topics led to the development of a formal survey. This survey was administered at several other universities at the end of a full year in an upper-level physics course. By analyzing responses to this survey, Singh et al. constructed a list of student misconceptions [23]. The results of their categorization are in Table 2.3.

Bao and Redish at the University of Maryland looked at students' understanding of potential wells, probability, and the wave function [24]. The base of this research was a conceptual quiz for junior and senior physics majors. A sample quiz question on the bound-state energy level is illustrated in Figure 2.1 The results of this study showed that students have:

1. A tendency to interpret 1-D potential well as a 2-D gravitational well;

2. Difficulty understanding negative values of total energy;

3. Difficulty using statistical methods to describe a system;

4. Difficulty connecting the energy to the shape of the wave functions.

Further interviews showed that some students think that a particle loses energy as it tunnels through a barrier and that the decaying wave function represents the energy of a particle. Therefore, if the particle does get through the barrier it will have less energy on the other side. In diagrams the students drew, the energy of the particle has connected with the amplitude of the wave function instead of the wavelength. Bao and Redish used these responses to study different models students use in describing a quantum system [25]. For example, they found three categories of student models regarding potential diagrams and wave function:

1. Classical intuitive models;

2. Hybrid models in which students mix the classical and quantum models;

3. Correct models based on a quantum mechanical view.

Bao and Redish have also investigated students' understanding of probabilistic interpretations of physical systems [26]. Based on this study and others, the University of Maryland later developed tutorials to address student difficulties in modern physics courses [27]. This tutorials, called *"New Model Course in Applied Quantum Physics,"* covers introductory level topics such as classical and quantum probability, Fourier analysis, models of tunneling, photoelectric, and potential energy diagrams, among others. One of these tutorials, the photoelectric effect, will be discussed as an example of an instructional tool in the next section. This course provides three different resources for instructors of introductory quantum mechanics or modern physics.

The first resource is a set of physics education research papers on how students learn physics in general, and quantum mechanics in particular. The second resource is an overview of effective classroom teaching strategies. The third one is a set of classroom materials to be used as a supplement.

2.3.3 RESEARCH ON DEVELOPMENT OF INSTRUCTIONAL INSTRUMENTS

There are unavoidable overlaps between curriculum development and the use of instructional instruments in PER. A growing number of researchers interested in curriculum development have focused on computer simulations and other visualizations of quantum phenomena. One example of these research-based instruments is Visual Quantum Mechanics [28], which was developed at Kansas State University by Rebello [29] and Zollman over the last 10 years. Visual quantum mechanics materials are computer simulations aimed at taking quantum mechanics topics into high school. Zollman, in interviewing pre-service science teachers, found evidence of student difficulty with representations and interpretation of quantum theory of atoms and energy bands in solids. While students did not exhibit difficulty with many of the subjects presented during class, they could not transfer their understanding to similar subjects. For example, while students could understand the wave function of electrons, they were unable to relate diverging wave functions in the bound states of a square well potential to the disallowed energy levels in an atom. Findings of Zollman and others at Kansas State University motivated the development of VQM.

Steinberg et al. [30] investigated a computer-based tutorial on the photoelectric effect. The schematic diagram set up of the experiment used in these interviews is shown in Figure 2.2. Students were asked to draw and explain a graph of current

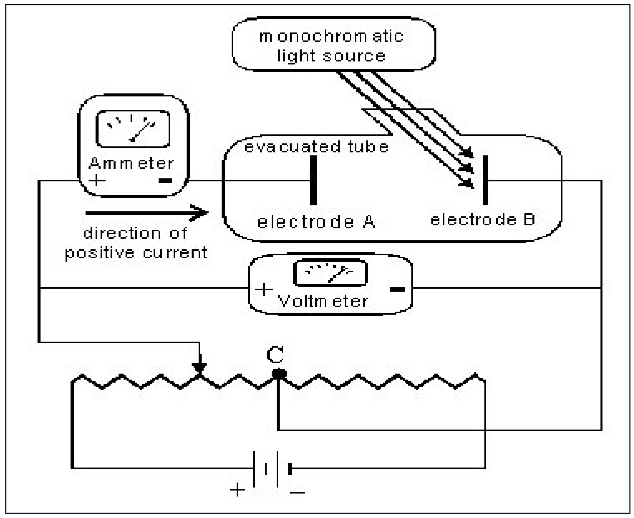

Figure 2.2: A diagrammatic set of an experiment used for interviewing students by Steinberg et al.

versus voltage for a photoelectric experiment and also to describe the effect of change in frequency or intensity of the light on the graph.

Some of student difficulties included:

1. The belief that V = IR applies to the photoelectric experiment;

2. The belief that a photon is a charged object;

3. An inability to make any prediction of an I-V graph for the photoelectric experiment;

4. An inability to give any explanation relating photons to the photoelectric effect;

5. An inability to differentiate between intensity of light (and hence photon flux) and frequency of light (and hence photon energy).

The above-mentioned difficulties persisted even after students received instruction. To address these difficulties and engage students intellectually, Sherwood and Anderson designed a computer tutorial, called Photoelectric Tutor that asked students to draw a qualitative I-V graph [31]. This computer-based tutorial interprets student graphical input and enters into a dialogue with the student. A sample student computer dialogue can be found in Appendix **A.** The improved version of this simulation, as well as the results of other similar research, has become part of the *"A New Model Course in Applied Quantum Physics"* project at the University of Maryland [32].

Robinett, Cataloglu, and Tasar at Pennsylvania State University have developed the first assessment tool to examine the progress of the students' understanding from sophomore level modern physics through junior level quantum courses, and at the first year graduate level [33]. This test is called QMVI (Quantum Mechanics Visualization Instruments) and is focused on students' ability to conceptually understand and visualize some of quantum mechanics' core ideas. Unlike the study by Singh, this test has focused more on visual and less on mathematical aspects of quantum mechanics. There are 25 questions in this test, which cover a variety of topics such as time development of wave packets, the Schrödinger equation, and some semi-classical aspects of quantum courses. For the complete list of the topics and their percentages in the test see Table 2.4**.** This test presents the same topic in several different ways to compare students' approaches to problems in different contexts. In general, students have done well on the topics focused on the wave function, though most students found the problems related to the momentum space and time development of wave packet quite challenging. Figure 2.3 shows question number 20 of this test, which

Main Topic	**Subtopic**	%
A. Historical development & terminology	1. Blackbody radiation 2. Photoelectric effect 3. Bohr Atom 4. Wave Particles duality 5. DeBroglie hypothesis 6. Uncertainty Principle 7. Probability & semi-classical behavior, waves	10
B. Quantum mechanics	8. Postulates of quantum mechanics I-V 9. Observable and operators 10. Eigenfunctions and eigenvalues 11. Dirac Delta function 12. State function and expectation values 13. Hilbert space and its properties	20
C. Schrödinger equation	14. Wave function of a single particle 15. The Schrödinger equation (time dependent & time independent) 16. Scalar products of wave function 17. Normalization and probability density	22
D. Application of Schrödinger equation in one dimension	18. Particle in a box (infinite hard wall) 19. Particle in a time-independent potential 20. Continuity conditions 21. Unbound and bound well 22. Harmonic oscillator 23. Wave packet and scattering (time-independent) 24. Probability density and probability current 25. Scattering by a one dimensional well 26. Tunneling	28
E. Advanced applications	27. Further applications of Schrödinger equation in two and three dimensions 28. Wentzel-Kremes-Brillouin Method 29. Time-independent perturbation theory 30. Angular momentum	20

Table 2.4: Basic content outline of the QMVI

was designed to probe students' ability to connect the energy of a particle and its momentum probability distributions; however, even at the graduate level, students had difficulty answering this question[2]. Robinett suggests this test should be used as a "post-test" in the assessment of effectiveness of any new instructional material.

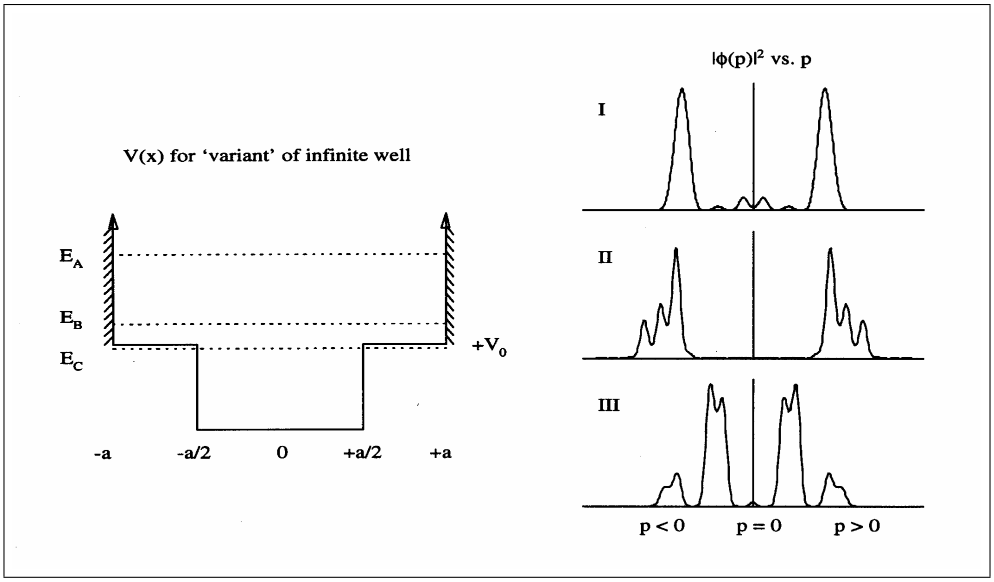

Figure 2.3: Question number twenty on VQMI: "On the left, there is a picture of three different quantized energy levels(E_0, E_B, E_C)in a variant of an infinite square well: it is an infinite well of width 2a with impenetrable walls at $x = +a$, but where $V(x) = 0$ for $-a/2 < x < +a/2$ and $V(x) = +V_0$ in the rest of the well. On the right are three different momentum-space probability distributions, plots of $|\phi(p)|^2$ versus p, corresponding to the three levels given by E_A, E_B,and E_C. Which momentum distribution (I, II, III) goes with which energy level?"

[2]The correct answer for this question is as follows: A particle with E_c bounces back and fourth between the walls at $\pm\frac{a}{2}$ and has $P = \pm\sqrt{2mE_c}$, so the momentum representation will have two peaks like graph I. For $E_B \geq V_0$ we expect larger speed (momentum) for a particle in middle region $p = \pm\sqrt{2m(E_B - V_0)}$ and two smaller peaks at $p = \pm\sqrt{2mE_B}$, case III. Finally, as E increases $E_A >> V_0$ and the difference between two peaks decreases, case II. In brief E_A : II, E_B : III, E_C : I.

As in the case of the model course, which was developed at the University of Maryland, the authors cited pedagogical and PER work as valuable sources in the development of their test [34]. This shows that understanding students' learning processes and their difficulties is the starting point for the development of any instructional material. The authors further stated that the primary goal of this test was to provide input for the development of web-based instruction material.

2.3.4 RESEARCH ON CURRICULUM DEVELOPMENT

Newtonian physics is the starting point for the formation of student understanding of physics, so students often hold strongly to these classical ideas. To overcome this epistemological obstacle, one approach is to introduce the concepts of quantum mechanics before or in conjunction with the classical courses. The teaching of quantum subjects in high school in Germany and the focus on the development of Visual Quantum Mechanics at Kansas State are examples of this approach.

In another curriculum development effort, Fischler [35] designed a new introductory course that omits all analogies and references to classical physics. He suggested avoiding the Bohr model when dealing with the hydrogen atom, introducing electron diffraction before photons when teaching the photoelectric effect, and replacing the dualistic description with statistical interpretation to explain observed phenomena. Fischler's course was tested in several high schools in Berlin. To evaluate the course, a comparison was made between the experimental and control groups. The results showed conceptual changes in the experimental group. For example, 68% of the students in the test group oriented themselves toward the idea of quantized energy, while the students in the control group persisted in the conception of circle and shell to explain models of an atom. Fischler argued that this approach prevents the transfer of macroscopic properties to the particle world and constructs a cognitive network in which quantum is something new without any relation to the classical wave or real particle.

Another approach suggested increased epistemological treatment as a possible way to eliminate the classical point of view in describing quantum topics. For instance, Hadzidaki et al. suggested introducing classical physics and quantum mechanics as independent conceptual systems [36]. Hadzidaki concludes that the initial classical knowledge is insufficient to deal with modern scientific content and the new information cannot be added to the existing knowledge background. Therefore, a revolutionary constructional process should take place. In other words, the instructional process that leads learners to "build" their new models based on their initial knowledge, called radical or weak reconstruction, is inadequate and blocks acquisition of new knowledge. One key instructional tool in this study is computer simulations to symbolize microscopic phenomena and replace the missing sensory experience of quantum mechanics [37]. Using computer simulations, the authors created visualizations of the hydrogen atom's orbital [see Figure 2.4]. They argued that this visual representation accomplishes the following:

1. Breaks the limits of practical knowledge of microscopic systems ($\Delta p \Delta x > \hbar$);
2. Eliminates the classical concepts of fixed orbits and states;
3. Shows that the orbital is a picture formed by the possible positions of the electron;
4. Illustrates that these positions conform to statistical law;

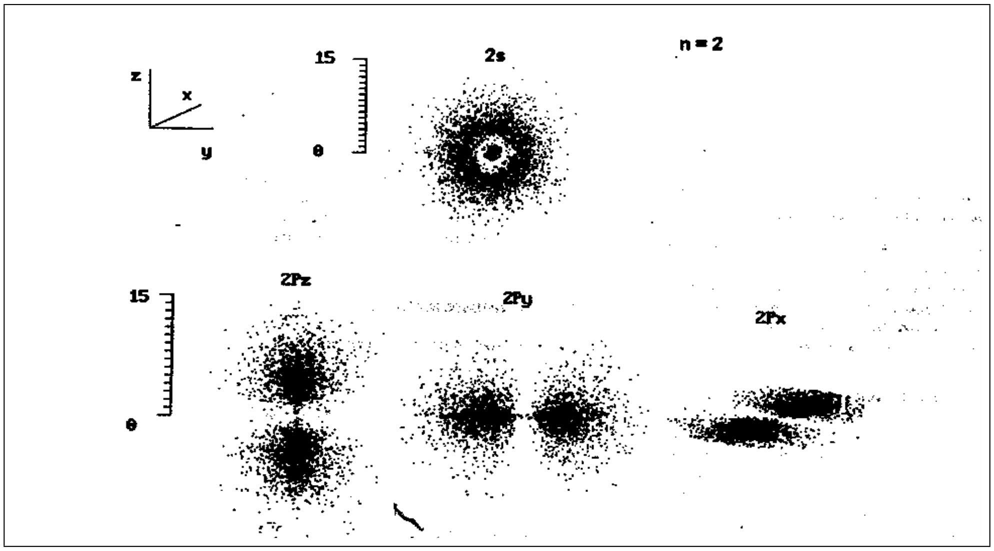

Figure 2.4: An image of the simulation/visualization of hydrogen atom orbits

5. Depicts the density of the point-per-unit volume, which visualizes the probability density of finding the electron inside this volume.

2.3.5 VISUALIZATION

Research has pointed out the importance of visualization in learning and teaching and developing scientific understanding. These suggestions originate in part from the principles of the relatively new sciences of learning, such as in cognitive psychology and neuroscience. By utilizing modern technology neuroscientists have explored and were able to clarify some of the learning mechanisms that took place in human and animal learning. For example, neuroscientists have discovered that visual experiences have a direct effect on the physiology of the central nervous system and an estimate of 50% of the neurons in the brain are associated with vision. McCormick et al. wrote that visualization was a form of communication that transcended application and technological boundaries [38]. The report suggested that visualization was a tool that could promote discovery, scientific understanding, and learning. Learning science was viewed as an active process in which students themselves construct meaningful understanding [39]. In this model of learning, knowledge is selected, discriminated, associated, and elaborated within a previously existing cognitive structure. Research pointed to the advantages of visualization in developing scientific understanding by providing additional opportunities for students to establish connections with preexisting knowledge structures [40]. Thus, scientists made extensive use of visualization tools to translate data into visual images with the hope that consistent or inconsistent patterns will emerge and, utilizing those images and visual models, a new level of scientific understanding will accomplish [41].

Numerous studies in mathematics education and science education have shown the importance of visualization in teaching and learning of mathematics and science concepts. For example, Aiken et al. found that there was a positive relationship between visualization and achievement in mathematics [42]. Researchers also documented a positive correlation between visualization and mathematical aptitude [43]. Interactive computer graphics and graphing calculators are among new technologies that can greatly expand the scope and power of visualization in both mathematics education and science education. A few of these studies discuss recent findings in educational research in the domain of visualization that have the potential to enhance students' conceptual understanding in quantum mechanics courses. For example, Rich found that students who used graphing calculators for the entire year of pre-calculus were far better able to deal with issues of scale on a graph when compared with students in a more traditional instructional setting [44]. Rich also found that students who were taught pre-calculus using graphing calculators better understood the connection between an algebraic representation and its graph. Moreover, the finding revealed that the students viewed graphs more carefully and that they understood the importance of a function's domain, its asymptotic behavior, and its end behavior. Rich also found that pre-calculus students in traditional instruction made almost no use of graphs except in the units dealing explicitly with graphs in the courses. Researchers have found that pre-calculus students who used graphing calculators demonstrated greater ability at higher-order thinking skills than traditional students [45].

2.4 RESEARCH ON STUDENT MATHEMATICAL DIFFICULTIES IN PHYSICS

Mathematics is the symbolic language of physics, which makes it one of the essential components of understanding and communicating physics. A look at the history of physics reveals the profound influence of mathematical invention. One famous example was the invention of analytic geometry and calculus, which was essential to Newton's creation of classical mechanics. Another was the invention of tensor analysis, which was essential to Einstein's creation of the General Theory of Relativity. Understanding physics concepts and using math to solve physics problems are so entangled that it is very hard to separate them. Mathematical language can convey a large amount of data and knowledge in detail very quickly. For example, arithmetic provides familiar structures for organizing multiple parameters.

Schwartz and Moore found that children reasoned proportionality when the associated numbers were within their arithmetic competence, but not when there were difficult numbers or no numbers at all [46]. Unfortunately, students of both introductory and advanced physics have often exhibited difficulty with mathematics. These difficulties appear in both algebra-based and calculus-based physics courses.

2.4.1 NOVICE AND EXPERT

Since students do not necessarily use mathematics in the manner that they are instructed by their teachers, researchers have employed various methods to study student use of mathematics. Some researchers have focused on the differences between experts and novices when using mathematics in physics. For example, Larkins, McDermott, Simon, and Simon stated 4 differences between novices and experts when solving problems [47]. The first is the speed of solution; experts solve problems faster than novices. The second is that, in contrast to the expert, novices tend to attack the problem by determining what the end goal is and working backwards from the end toward the initial conditions in the problem. The third is in their approach to problem solving, the way in which written statements in the problem are translated into algebraic notations. The Final difference is that the novice tends to focus on the syntax of the English statements, whereas the expert tends to translate the English statements semantically, that is, in terms of the physics knowledge relevant to the problem, before constructing algebraic expressions.

A large amount of research has focused on these difficulties and has detected particular difficulties with student use of math in physics. One example indicates students' difficulties with interpreting and using of graphs. Other research has identified students' difficulties with vectors and their algebraic use. A third regarding difficulties with understanding functions, basic algebra, trigonometry, and general mathematical reasoning. Here, we will discuss these difficulties in more detail and our focus is not only on students' difficulties with mathematics in physics, but also their ability in interpreting of math in physics terms and vice versa.

2.4.2 GRAPHS

Students need facility with many mathematical representations when learning physics. For example, students need to be able to interpret physical phenomena based on graphical representations, construct a graph from an experiment, and relate data from a graph to physical quantities. McDermott et al.[48] identified two categories of difficulty understanding graphical representation of physical phenomena: difficulty in connecting graphs to physical concepts and difficulty in connecting graphs to the real world. As an example of the first category, McDermott found that students frequently do not know whether to use the height or the slope of a graph to extract desired information. An example of the second category is student difficulty separating the shape of a graph from the path of the motion.

This latter finding is analogous to research findings in the context of quantum mechanics; similarly, students treat the wave function as the path of particles in quantum mechanics [49]. In addition, Bao found that most students think of a finite or infinite potential well graph as a 2-D physical well that has a particle trapped on it.

2.4.3 VECTORS

Vectors are an essential component of the mathematical language of physics. There has been extensive literature regarding students' preconceptions of the concepts about force and motion at the introductory level. Many students' difficulties with these concepts such as drawing and interpreting free-body diagrams and superposition of forces are associated with students' insufficient vector knowledge. With vectors students need to understand what a vector represents and tie different representations to a well-defined coordinate system.

Steinberg et al. found that introductory students fail to demonstrate a functional understanding of vectors when interpreting vector equations [52]. Rebmann and Viennot studied student performance on problems where coding is required [50]. In their study of over 400 college students they found that many students did not see that determining whether a physical quantity is positive or negative depends on its coding. For example, when a spring is stretched in the positive direction, almost half of students chose "$F = +kx$" to represent the force of the spring. Some could not even set up the sign conventions correctly. Knight investigated the basic knowledge that first year calculus-based introductory physics students had of vectors ($N = 280$). He found that regardless of previous courses on vectors, 50% of beginning physics students have no useful knowledge of vectors at all [51].

2.4.4 FUNCTIONS, EQUATIONS, AND MATHEMATICAL SYMBOLS

Students need to have an understanding of the idea of a function and recognize the relationship between physical situations and the associated equations. A large fraction of students who seem to be comfortable with mathematics required for physics courses, often show obtuseness when relating and interpreting math in their physics courses. For example, many students seem unable to grasp the general idea of functions, equations, and mathematical symbols [52]. Students fail to recognize the relationship

Utterance	Conceptual Schenma	Symbol-Template	Mathematical Expression
"The normal force of a table on a block is balancing the gravitational force of the earth on the block."	Balancing	$\blacksquare = \blacksquare$	$\vec{N}_{\text{T on B}} = \vec{W}_{\text{E on B}}$
"The velocity of block A is the same as the velocity of block B."	Same amount	$\blacksquare = \blacksquare$	$\vec{v}_A = \vec{v}_B$

Table 2.5: Examples of symbol templates and conceptual schema

between the physical situation and the associated expression. Students do not show signs of understanding the idea of a function [53].

Since understanding the relationship between an expression and its physical meaning is an essential part of problem solving in physics, many researchers have tried to gain insight into students' understanding of mathematical equations in physics, and their understanding of the relationship between symbols and physical observables. Sherin developed "symbolic forms" as a framework to study how students understand physics equations [54]. This framework has two parts: "Symbolic templates" and "conceptual schema." The "symbolic template" is a part of knowledge that constructs mathematical expressions and the "conceptual schema" offers the conceptualization of the knowledge contained in the mathematical expressions. Table 2.5 illustrates examples of symbol templates and conceptual schema. As you can see in this table, the same "symbol template" can be used for more than one "conceptual schema."

Clemet et al. observed college science students who were instructed to talk aloud while they were solving simple word problems. They observed students having great difficulties translating the English statements into algebraic equations [55]. One such word problem read:

> Write an equation for the following statement: "*There are six times as many students as professors at this university.*" Use S for number of students and P for the number of professors.

The correct answer to this question is $S = 6P$. Although this question seems very simple, only 37% of the calculus students (N=150), and 57% of the non-science majors (N=47) answered this question correctly. The most common mistake was "word order matching;" that is direct mapping of the English words into algebraic expressions. So students would translate the sentence "There are six times as many students as professors into $6S = P$, merely because of the order in which the words "six," "times," "students," and "professors" appear in this problem. Students read: " Six times students equal a professor."

Researchers found that introductory physics students often fail to see symbols as representations of physical measurements, rather than as numbers. They often plug numbers into equations without parsing equations correctly [56]. They cannot break equations down into their component parts and see the dimensions inherent in an equation.

The reason for these difficulties is not just inadequate mathematics preparation; physics contexts are sufficiently different from math. Many signs and codes mean different things in physics equations than in math. Even different symbols have different meanings . Let us exemplify this. Consider the following equation:

$T(x, y) = (x^2 + y^2)$

What is $T(r, \theta) =$?

While in physics every symbol has a physical meaning and associates with a measurable variable. Here r and θ can represent the polar coordinates and the physics answer to the above question is: $T(r, \theta) = r^2$ [57]. David Hestenes pointed out that multiple mathematical systems contribute to the fragmentation of knowledge. In addition, he listed a number of defects that mathematics has as the language of physics, and suggested a comprehensive language called "Geometrical Algebra," that is a unified mathematical language for the whole of physics that facilitates learning and enhances physical insight [58]. Other studies introduced the low level of formal reasoning as a cause of student difficulties with mathematics in physics, and sought to identify specific factors and procedures to assist with improving formal reasoning [59].

2.4.5 MATH SKILLS AND PHYSICS ACHIEVEMENT

Many studies have shown that mathematics affects students' development of physical understanding. For example, Schwartz et al. showed that conditions that facilitate the application of mathematics lead to improved learning about the balance scale, even when there were no specific instructions or examples [60]. To study the effect of math in physics learning and investigate sources of specific difficulties, it is hard to separate the math from physics; does a lack of vector knowledge affect the understanding of velocity, or do vector errors result from velocity misconceptions?

Several studies have shown that physics achievement correlates strongly with prior mathematics ability and formal reasoning skills [61]. Most of these studies have administered a math diagnostic test as preliminary or have used SAT Math scores to measure students' mathematical ability. For the measurement of students' achievement in physics, their final grades in the course have often been used. This suggests that poor understanding of physics can cause as many math errors as under-prepared math skills can cause in physics. This means that the arrow of causality is double-headed.

2.5 SUMMARY

Findings of empirical research in science education and physics education showed that conventional teaching strategies unfortunately do not highly promote conceptual understanding especially at high-school and first year college level. Consequently, students might leave basics physics courses with alternative concepts that are in contrast with the accepted scientific concepts. These findings suggest that students enrolled in higher level courses such as introductory quantum mechanics courses, might be faced with similar conceptual problems. New teaching approaches are suggested and even tried out. However, most of these approaches failed to report empirically the effectiveness of their new teaching methods.

There have been studies on students' difficulties learning quantum mechanical concepts but a few attempts to provide instructional material to help students' learning. In addition, there have been numerous studies on students' mathematical difficulties in learning physics, which examples of such were discussed.

This dissertation aims to investigate the relationship among students' mathematical skills; their ability to deal with formalism; their conceptual understanding; and their success in quantum courses. Our goal is to show whether or not mathematics influences the development of physical knowledge, and if so, what specific representational properties of mathematics might support the development of specific concepts.

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CHAPTER 3

METHODOLOGY IN PHYSICS EDUCATION RESEARCH

"The fatal pedagogical error is to throw answers, like stones, at the heads of those who have not yet asked the questions." Paul Tillich

Physics education research seeks alternative approaches to pure lectures in order to make physics courses accessible to students from diverse backgrounds while helping them to overcome their misconceptions about the physical world. The goal of PER is to help students achieve a meaningful level of competence in their thinking, problem-solving, and modeling skills. PER aims to motivate students to develop deeper conceptual understandings of physical phenomena and to understand that the principles of physics are logically and hierarchically organized. Facts and formulas evaporate quickly from the student's mind, but developing an ability to think like a physicist in a variety of contexts allows the student to use these skills in various contexts throughout their lives.

Investigation of student difficulties with conceptual understandings of physics is growing in number and variety of topics. Part of the PER community is interested in integrating findings of multi-disciplinary research such as education, cognitive science, and biology, to discover "how people learn;" while another part focuses on practical aspects of this integration in physics classrooms. By gaining deeper insight into student understanding of the physics taught in the classroom and employing findings of cognitive science into education, researchers design curriculum materials that are more effective in student learning. Together, studies of student reasoning and the ways in which they come to an understanding of physics provides a useful tool in the development of an improved curriculum. The study in this dissertation demonstrates students' difficulties in learning quantum mechanics and seeks those ways in which students come to understand physics.

3.1 METHODOLOGY IN PHYSICS EDUCATION RESEARCH

The process of research in PER involves a continuous cycle. This cycle starts from the systematic investigation of students understanding, which often leads to the development of specific instructional strategies to address specific difficulties. This, in turn, leads to testing and revision of the material on the basis of classroom experience. For example, Lillian McDermott and Physics Education Group of the University at Washington has gone through several iterations of this cycle in the development of the material for both tutorials [1] and Physics by Inquiry [2].

Physics education researchers use different techniques to evaluate what students know and what they are learning. A variety of methods have been developed to investigate student ideas, abilities, and conceptual understandings of physical phenomena. Two main categories of research methods are: formal and informal. Formal methods consist of systematic questioning which is designed to target student difficulties and the sources of these difficulties in specific concepts. The formal methods include individual or group interviews, written tests and exams, multiple choice questions, and diagnostics. Informal methods, on the other hand, are observations during lecture, office hours, and help or discussion sessions, in which researchers do not have a full control of the questions being posed to students. In these situations, the primary goal is to help students to come to the correct answer. Multiple methods are often used in a single research endeavor to gain deeper insight into students' reasoning and to enhance the reliability of the results.

3.1.1 INTERVIEW

Interviewing has become an increasingly common technique for studying student understanding of physical principles and their logical reasonings. Interviewing and out-loud thinking tends to provide the most accurate records about the origin and process of students' reasoning. Typically, the verbalizations are recorded and then subjected to theory-based analysis. The interview method is the most effective approach since the researcher is more able to ask unscripted questions on the basis of individual responses. Furthermore, interviewing permits the researcher to observe the functionality of students' knowledge by presenting the student with a variety of contexts. In the think out-loud approach, by listening to students, researchers gain a better understanding of students' thought process and their reasoning patterns. Therefore, interview not only can reveal the difficulties students may have with the topic, but also can point out some of the sources causing these difficulties. Often, interview data forms the main basis for the "model space" that contains all the possible ideas students have shown for a particular physics concepts.

The interviews are videotaped by the researchers, who will transcribe them at a later time. Multiple researchers often read these transcripts and analyze them individually to ensure the reliability of the results. These processes are very time-consuming;

however, since many students share a relatively small number of difficulties, an interview of even a small group usually reveals most of the common difficulties students encounter in a specific topic.

There are difficulties in obtaining a good set of interview data. The main problem with interviewing students is the amount of time involved in this process. First, there is the amount of time the actual interview takes (usually one or more hours per students), and then, there is the time required for watching (and perhaps rewatching) each interview and writing a transcript for each interview. Finally, there is the process of analyzing the transcript individually and in groups. Another common problem with the interview method for researchers is the lack of volunteer students to participate in this part of the research. Due to all these obstacles, other techniques, such as written questions and multiple choice tests, are often employed in conjunction with interviews in order to gather data more easily, in shorter time, and from a larger sample.

3.1.2 MULTIPLE CHOICE QUESTIONS

Multiple choice type tests are one of the most frequently used test types; they have five basic advantages. Multiple-choice tests [6]:

1. are versatile, that is, they can measure multiple levels of specific cognitive skills;
2. allow well distributed content sampling;
3. can be quickly and objectively scored;
4. can provide valuable diagnostic information;
5. are better liked by students when compared to other measures.

Furthermore, multiple-choice questions tend to focus student attention on information in isolation by testing one element at a time. Although this is considered as one of the limitations of this type of questions, for our purpose, identifying student specific difficulties, it servers as an advantage.

The successful data produced in interview processes can be a good source of distracter, the incorrect answers in multiple choice test. In order to examine students' understanding using a multiple-choice test, some researchers focus not on the correct responses students gave on multiple-choice items but focus on their wrong answers. Tamir, for instance, believes that [3] *"when the test constructor writes the alternatives, he follows his own associations, his own ideas, and his own thought patterns. But students' associations and ideas can be quite different. If the major objective of testing is to detect misunderstandings and misconceptions of the student; why not first ask the students for alternative answers and construct the item accordingly?"*

Thus, the construction of distracters for multiple-choice test items should be based on typical students' ideas. Students' responses to interviews, essay questions, and to other open-ended questions could serve as sources for the data. The data collected in this way can provide valuable information about students' misconceptions about a particular scientific area and can be used to construct the distracters for multiple choice questions. In some multiple choice questions, to increases the probability of gaining more information about students' actual understanding, students are provided with extra space to explain their reasoning, their confidence level, or to express reasons different from the reasons provided on the test. For example, through interviewing and written essay questions, Hestenes and Wells explored the range of possible incorrect ideas students have about force and Newtonian mechanics and used these common, yet naive ideas, as alternative responses to design the FCI and the MBLT diagnostic tests [4], [5].

3.2 RESEARCH METHODS IN THIS STUDY

In the study presented by this dissertation, we used a variety of methods in order to enhance our understanding of the nature of student difficulties in quantum mechanics. In addition to formal methods, such as individual interviews, written questions, and multiple choice diagnostics, we have also collected data from informal methods. For example, we have observed many lectures (three full quarters) for undergraduate quantum mechanics courses, studied students' exam and homework assignments, and attended several computer labs where we were able to interact comfortably with students. To gain more insights into students' persisting difficulties after instruction, we have interviewed 18 students, two graders, and held informal conversations with the instructors of the quantum courses at The Ohio State University.

Based on our findings from class observation, interviews, and reading students written homework and exams, we were able initially to identify several common student difficulties with the basic topics of quantum mechanics. To test our speculations we developed on line research questionnaires addressing these difficulties. During the case of this study, we have developed over 21 questionnaires, with 10 to 15 questions each. The questions were often multiple choice questions using students' incorrect ideas from initial findings as the alternative responses. These questionnaires consisted of a series of related questions on a particular topic and were presented weekly to students on line. Since these questionnaires were part of homework assignments, we more able to obtain data from a larger population.

Overall, our investigation had three phases. In phase one, we wanted to identify common difficulties introductory students have in learning quantum mechanics. In phase two, our goal was to determine if any of the students' common difficulties with quantum topics originated from difficulties in required math topics. Finally, in phase three, we developed preliminary instructional materials to guide students step-by-step through these difficulties.

3.2.1 STUDENT POPULATION AND RESEARCH CONTEXT

Most of the data in this research was collected from the following classes at The Ohio State University:

1. The first quarter of upper-level undergraduate quantum mechanics ($P631$) in fall 2003, with 35 students enrolled using "*Introduction to quantum mechanics*" by David Griffiths as the textbook. The class met four times a week for 48 minutes each and was pure lecture format;

2. Introductory quantum mechanics at sophomore level (P263) in spring 2004, with 61 students enrolled and "*Six Ideas that Shaped Physics*" Unit Q by Thomas Moore as the textbook. This class had three lectures and one computer lab session per week. Most computer labs were devoted to completing a pre-written Mathematica notebook on the week's lecture topics;

3. The first quarter of upper-level undergraduate quantum mechanics ($P631$) in fall 2004, with, 60 students enrolled and "*Introductory Quantum Mechanics*," by Liboff as the textbook. This class met for four lectures and one exercise session per week. Some of the material developed in this study was presented

to the students in this class during the exercise sessions. In addition, there was a voluntarily weekly seminar session in the evenings about a paper on quantum mechanics topics, in which students and the instructor had an open discussion.

Of the 61 students who took the introductory quantum course in the spring quarter of 2004, 48 students also enrolled in the advanced course in the fall of 2004. Therefore, we were able to obtain a longitudinal data on these students' performance throughout this time. This allowed us to correlate student background knowledge of certain math topics with their performances in related quantum questions. In the study of students' mathematical background through an online survey in the spring quarter, we learned that 85% of students, prior to taking any quantum courses, had taken at least one math course in differential equations (math 255 or math 415), and one course in linear algebra (math 568 or math 571).

One of the objectives in this study was to understand student difficulty with learning, applying, and making sense out of complex mathematical processes in the physics classroom.. Because of this, we often posed questions from three different perspectives on each research topic: the purely mathematical, the purely physics, and a mix of the two. The research topics we will discuss in this dissertation include, but are not limited to, students' difficulties in:

1. Understanding concepts of probability measurement; (Chapter 5);

2. Application of the Uncertainty Principle (Chapter 5);

3. Understanding wave functions and energy eigenstates (Chapter 6);

4. Recognizing symmetry in physical systems (Chapter 7);

5. Mathematical formalism (Chapters 6, 7).

In addition, in Chapter 4, I report on the general findings of this study and discuss the relationship between students' mathematical skills and their success in quantum courses. The details of student' specific difficulties with selected topics of quantum mechanics are discussed in Chapters 5, 6, and 7.

3.2.2 DATA ANALYSIS

For analyzing the data for this research we used Statistical Packages for Social Science (SPSS). SPSS is an advanced statistical program that performs a variety of statistical analysis, including the computation of correlation and reliability coefficient and is available at The Ohio State University Computation Center. Since we have used both multiple choice and open ended questions in this research, our data for students' scores in these questions were either binary or continuous numbers. Students' responses to the multiple choice questions were coded with 1 for correct and zero for incorrect answers, while for essay and exam questions, after normalization, students score were any number between zero and one. In Chapter 7 we study a relation between categorical data for our multiple choice questions (1 =correct, 0 =incorrect) and continuous data for our exam questions.

There is some difference of opinion among statisticians about whether or not one can mix these two types of data for correlation analysis. Most statisticians believe that, in order to compare the categorical and continuous data, the following two requirements should met. First, as long as a larger categorical number means that the object has more of something, then application of the correlation coefficient is useful. Second, when the nominal categorical scale has only two levels (1 =Male, 2 =Female), the correlation coefficients computed with data of this type may be safely interpreted because the interval property is assumed to be met for these variables [7]. When the data are clearly nominal categorical with more than two levels (1 =Protestant, 2 =Catholic, 3 =Jewish, 4 =Other), application of the correlation coefficient is clearly inappropriate.

Our categorical data meets both requirements: it has only two levels and the large number (one compared to zero) means that the student achieved a higher grade. Nevertheless, in order to obtain more reliable results, we took caution. We made our data compatible by converting the continuous data to categorical data, and could thus perform a chi-square test that is appropriate for categorical data. For example, we divided the range from $0 - 1.0$ into 4 equal intervals and assigned a same number to all the scores in a given interval. For instance, in this method, any scores between $[0.0, 0.25]$ would be replaced by 0, scores between $(0.25, 0.50]$ would be replaced by 1.0, scores in $(0.50, 0.75]$ would be replaced by 2.0 , and scores in $(0.75, 1.0]$ would be replaced by 3.0. The only statistical restriction on this method is to have a minimum of 5 frequency on each interval; based on different distributions in each set of scores we chose 3, 4, or 5 intervals. Then, depending on the type of data, we performed several different statistical tests. Below, we present a brief description of the two statistical tests used in the data analysis in this dissertation.

Chi-square is a non-parametric test of statistical significance for bivariate tabular analysis (also known as crosstabulation and contingency table). Typically, the hypothesis tested with chi-square is whether or not two different samples (of people, texts, scores, whatever) are similar enough in some characteristic or aspect of their behavior that we can generalize from our samples that the populations from which our samples are drawn are also similar in the behavior or characteristic.

A non-parametric test, like chi-square, is a rough estimate of confidence; it accepts weaker, less accurate data as input than parametric tests (like t-tests and analysis of variance, for example) and therefore has less status among statistical tests. Nonetheless, its limitations are also its strengths; because chi-square is more "forgiving" in the data it will accept, it can be used in a wide variety of research contexts.

Chi-square is used most frequently to test the statistical significance of results reported in bivariate tables, and interpreting bivariate tables is integral to interpreting the results of a chi-square test. Bivariate tabular analysis is used when you are trying to summarize the intersections of independent and dependent variables and understand the relationship between those variables. Chi-square analyses are often accompanied by contingency tables (also called cross-tabulation). Contingency tables report the frequencies of variables. Examples of bivariate tables can be found on the data analysis of Chapter 7. Bivariate tabular analysis is good for asking the following kinds of questions: Is there a relationship between any two variables in the data, and how strong is the relationship in the data?

As mentioned before, chi-square is a nonparametric test. It does not require the sample data to be more or less normally distributed, although it relies on the assumption that the variable is normally distributed in the population from which the sample is drawn.

But chi-square, while forgiving, does have some requirements:

1. The sample must be randomly drawn from the population;

2. Data must be reported in raw frequencies (not percentages);

3. Observed frequencies cannot be too small.

Our SPSS tables of chi-square test gave us information about the significance of statistical relationship between any two variable but not the direction of this relation. In order to study the shape and the direction of this relation we used correlation and linear regression analysis.

Linear regression is used to make predictions about a single value. Simple linear regression involves discovering the equation for a line that most nearly fits the given data. That linear equation is then used to predict values for the data. Correlation describes the strength, or degree, of linear relationship. That is, correlation lets us specify to what extent the two variables behave alike or vary together. Correlation analysis is used to assess the simultaneous variability of a collection of variables. The relationships among variables in a correlation analysis are generally not directional.

3.3 DEVELOPMENT OF INSTRUCTIONAL MATERIAL

Recent research in education has shown that instruction in which students are passive learners does not effectively address the conceptual difficulties students may have with the content matter [8]. Many pretest and post-test results have shown that, for many conceptual questions, student performance is essentially the same before and after instruction that is limited to lecture and end-of-chapter problems [9]. In contrast, active engagement of students in the learning process can significantly improve their conceptual understanding [10]. With the aid of previous research in effective instruction (active engagement, group work, etc.) and our insight to student common difficulties in learning quantum mechanics, we have designed a few instructional materials for some of the topics mentioned above. The objective is to provide step-by-step instruction to help students to: (1) develop basic quantum mechanical concepts and use these concepts to interpret different quantum systems, and (2) practice required formalism in this subject and relate and use multiple representations in describing quantum systems. The details of such instructional material will be discussed in Chapter 8 of this dissertation.

3.4 SUMMARY

This chapter outlined research methodology and the construction of multiple choice questions to measure students' understanding of selected topics in introductory quantum mechanics that were employed in this study. Overall, our investigation had three phases. In phase one, we wanted to identify common difficulties introductory students have in learning quantum mechanics. To achieve this goal, we interviewed students, designed research questionnaires, and employed informal methods. Then, in phase two, our goal was to determine if any of the students' common difficulties with quantum topics originated from difficulties in required math topics. To meet this objective, we studied student math background, analyzed their responses to pure math and pure physics questions, and studied whether or not these responses were correlated with their achievements in quantum courses. Finally, in phase three, we have developed preliminary instructional materials to guide students step-by-step through these difficulties.

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CHAPTER 4

THE RELATIONSHIP BETWEEN STUDENT BACKGROUND KNOWLEDGE AND THEIR LEARNING OF QUANTUM MECHANICS

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve." Eugene Wigner

4.1 INTRODUCTION

This chapter focuses on the more general findings of this dissertation and the more specific difficulties students have with the introductory topics of quantum mechanics are discussed in the following three chapters. As is explained in details in Chapter 3, we gave weekly questionnaires to students taking quantum courses. Each questionnaire contained 10-15 questions from different categories such as classical waves, interference and diffraction, probability, mathematical formalism, wave function, and graphical representations [see Appendix **F**]. This chapter first explores the students' average scores in different question categories and then discusses any relationship between the students' average scores on each question categories and their success in quantum courses.

The two main courses in this study are introductory quantum mechanics at sophomore level ($P263$), with 61 students enrolled, and the first quarter of upper-level undergraduate quantum mechanics ($P631$), with 60 students enrolled. Of the 61 students who took the introductory quantum course in the spring quarter of 2004, 48 of them also enrolled in the advanced course in the fall of 2004.[3] Therefore, we were able to obtain a longitudinal data on the students' performances across this time. This allowed us to correlate students' average scores on the questions from different categories with their achievement in these two quantum courses. We used the final course grade, which was calculated using midterms, homework, and final exam grades, as a measure of students' success in the course.

4.2 DIFFERENT QUESTION CATEGORIES

A total of 21 weekly questionnaires on a variety of different topics were administered to students taking the introductory quantum course ($P263$) and the advanced quantum course ($P631$) at the OSU in spring and fall of 2004. Weekly questionnaires covered different topics that seemed to be related to learning quantum physics. For example, some of the main topics of questionnaires included classical waves, interference and diffraction patterns, quantum measurement and probability, mathematical formalism, wave function, physical symmetry, odd and even functions, and graphical representations. In addition to finding students' specific difficulties with these questions, we analyzed the relationship between individual student's average scores in different question categories and her or his final grade in the course. Among all these question categories, the lowest average was 30% on mathematical questions and

[3]We exclude the data for P631 in the fall of 2003 because we are interested in a longitudinal study of the students' performances.

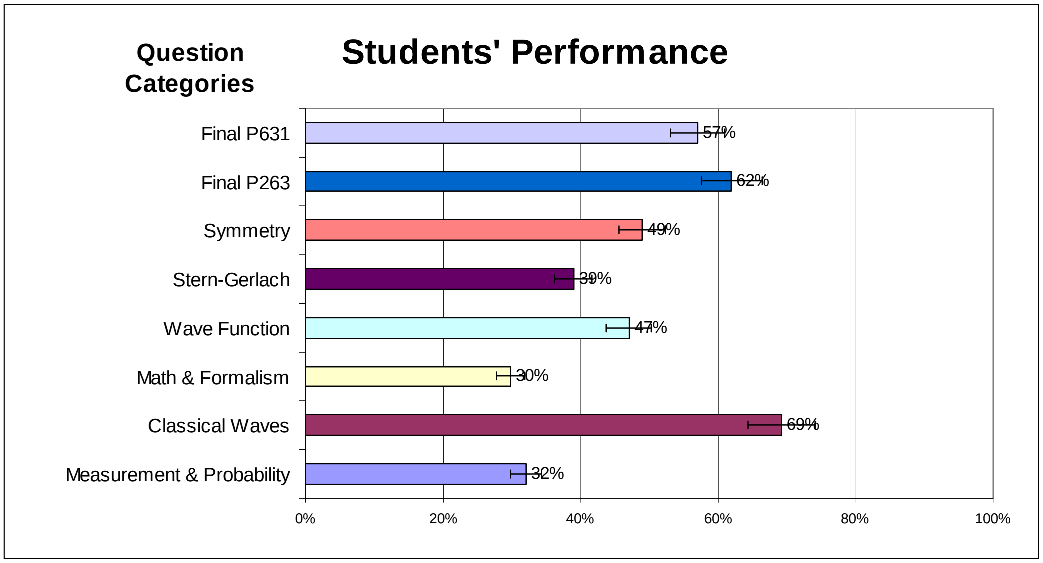

Figure 4.1: Students'final grades and their average scores on different question categories

the second lowest average score was on questions related to understanding quantum probability at 32% [see Figure 4.1].

Using SPSS statistical software we analyzed every correlation between students' scores in these question categories and their final exam grades. Our data shows two significant results: first, there is a positive correlation between students' mathematical background and their final grades; second, students' background knowledge of classical waves correlates with their final grades in the introductory quantum course. In the following sections we discuss these correlations in more details.

4.3 ROLE OF MATHEMATICS IN QUANTUM COURSES

Researchers have studied the relationship between students' mathematics achievement and their physics achievement in introductory courses. Some research found that students' mathematics scores correlated highly with their physics scores [1]. Other research conducted in this area, however, showed that mathematical skills are only one of several variables requisite to the understanding of physics concepts presented in a typical introductory mechanics course, and high scores on mathematics tests are not sufficient indicators of conceptual understanding in physics [2]. In education research literature there are no studies for upper level courses, such as quantum mechanics. To fill this gap in the literature, in a longitudinal research project we studied students' mathematical background and skills and we sought to discover any relationship between students' mathematical knowledge and their achievements in introductory ($P263$) and advanced quantum course ($P631$).

As described in Chapter 3, the questionnaires contained, but were not limited to, pure conceptual and pure mathematical questions. Pure mathematical questions were on the mathematical topics that commonly appear in physics courses, such as Furrier transformation, common differential equations in physics, simple integrations, complex numbers, operators, commutators, multiple representations, and graphical representations of equations. Examples of pure mathematical questions and students' specific difficulties related to the topics of probability measurement, wave function, and symmetry are presented in Appendix **F**. In this chapter, we present students' change in attitude towards the role of mathematics in physics before and after taking quantum courses. In addition, we discuss the relationship between students' math background and their achievement in quantum courses. The data shows that students' mathematical skills in the introductory courses ($P263$) have a small effect on their course final grades. However, there is a moderate positive correlation between students' math scores and their final grades in the advanced courses ($P631$).

4.3.1 STUDENTS' ATTITUDES TOWARDS MATH IN PHYSICS COURSES

It is interesting to note that student attitudes towards math in physics changes from sophomore to junior level courses. The majority of the students enrolled in $P263$ in spring quarter claimed that they encounter mathematical difficulties in only one out of 10 physics problems, while the same students, when they were taking $P631$, stated that their mathematical difficulties are the main drawback in solving quantum physics problems.

For example, students' responses to an on line survey show that for most of the students starting $P263$, mathematics has not been a problem in their physics courses prior to this course:

S1: *"Usually I have no problem with math."*

S2: "*Apart from stupid mistakes I'd say* [I have problems with] *1 or 2 out of 10* [problems]*."*

S3: "*I get stuck only on about 1/10 or 2/10 problems because of mathematical difficulty."*

S4: *"I would estimate that I get stuck less than 1 out of 10 times because of mathematical problems. Most difficulties are of the nature above or complicated wording of the problem."*

S5: *"If I understand what the problem is asking and which formula to use I can usually do the problem. I'd say 2 out of 10 might stump me because of the mathematical difficulties."*

S6: *"Maybe 1 out of 10 times. Usually the homework problems aren't that mathematically complex (i.e. "plug-and-chug"). The bigger problem is the proofs that are done in class."*

S7: *"The math itself I usually have no issues with except for the differential equations sometimes. I find the solutions to the various diff eqs I've run across to be 'fuzzy'."*

S8: *"...usually don't get stuck on the math so far. The math is generally trivial thus far."*

S9: *"Depending on the type of math involved, I might have problems with 1 or 2 out of 10 problems."*

S10: "[I would have problems] *only on the application of the math. I'm not gonna have problems calculating an integral just on getting the actual equation."*

S11: *"Only if* [the problem] *incorporated mathematical concepts that I had not yet learned."*

Perhaps this change in student attitudes towards math in physics is because the mathematics requirements in their previous courses were more of a "plug and chug" that only involved basic algebra, trigonometry, or simple derivatives. In addition, the problems of classical physics are more intuitive and do not require a great deal of interpretation of formalism, as quantum mechanics courses. Though familiar with complex numbers and linear algebra through their math courses, students have not used these topics in the contexts of physics courses; therefore, making sense out of complex and abstract mathematics comes as a challenge for them. When these students start taking a quantum course there is a great boost in their use of mathematical formalism and a need for complex mathematical solutions. After a quarter in a quantum mechanics course most of these students realize that either they do not have a strong enough mathematical background to solve quantum physics problems, or they are not able to make physical interpretations of their mathematical solutions. The following are examples of same students' statements about mathematical difficulties in quantum courses in fall 2004 when they were enrolled in P631:

S1: *"I believe most of the difficulty lies in the mathematical formalization. The concepts are generally easy to pick up with some thought."*

S2: *"50% of the time I get stuck on a QM problem because of math. But the reason I get stuck is because I don't know how to apply the math. I do understand the actual mathematics well enough, just not how to apply them to the problems and how to interpret what the math is telling me."*

S3: *"The most difficult part of QM has been connecting the math in a functional way to the theoretical concepts. An example of this is the idea of Kronecker delta = 1 for n=m, and =0 for n<>m. Mathematically I understand the deviation 100%, and I think I understand the statements in the book in the section. But for some reason, I cannot connect the concept of stationary states with the math."*

S4: *"Mathematical formalism gets me most often. I usually get tripped up on what I consider esoteric trig identities, and long integrals that involve integrating by parts more than once."*

S5: *"Mathematical Formalism, understanding the information in context, the information taught isn't put in context very often. It seems as though many things are assumed as though you already know."*

S6: *" I would say interpreting the formalism, and keeping my vocabulary straight are the hardest things I've encountered so far. Sometimes there will be a question and I will not know how to answer it because I have no idea what it is asking. For example: I don't know what a stationary state is. I think I got that question wrong on the test. I thought a stationary state was just another name for an eigenstate, or maybe it is a state at t=0. But that just shows that I don't really know."*

S7: "*The math is really the hardest part. I was never taught much math in high school just some basic calculus that didn't show many applications at all. I've been reading about the concepts of quantum physics on my own since I was fairly young so the concepts don't seem too hard to understand. I think I remember learning about particle-wave duality in elementary school or junior high. I never taught myself any of the math along with the physics."*

S8: *"I think the hardest physics problems are ones that ask you to come up with your own mathematical model for a physical process."*

S9: "*I often find when i get the answer i have little to no idea how to interpret the answer."*

S10: *"Sometimes it's difficult to take a complex interaction and break it down into simpler parts and model those parts mathematically."*

S11: *"Some times getting stuck in the math can pose great difficulty. Despite understanding the physics find the correct math representation can be difficult."*

There is a change in attitude toward students' level of difficulties with mathematics in physics. Most students in spring 2004 stated that they don't encounter a great deal of difficulty in the application of math and making sense of mathematics in physics and their use of math was mostly limited to: *"finding the right equation and the rest is easy."* Nevertheless, Almost all of these same students stated in fall 2004 that mathematics is one of the biggest problems in their learning in physics courses. Without a doubt there is a greater need for more complex mathematics in higher level physics courses. In addition, in quantum mechanics mathematics is more than just a calculation process. Students need to extract physical meaning from complex and abstract mathematical formalism in a way that they did not have to in classical courses. For example, students need to represent the probability of finding particles in a region in space with the statistical interpretation of wave functions; they need to use mathematical operators to measure a physical observable, or use Dirac notation to compress complex integrations. In other words, instead of terms such as energy, velocity and force, they deal with Hamiltonian, eigenvalues, and wave functions. These latter concepts are counter-intuitive and involve a higher level of mathematical ability that, perhaps, were only mentioned briefly in their math courses.

Another reason for this change in attitude is a sudden increase in the level of mathematics students need in junior level courses as compared to sophomore courses. The physics curriculum relies on students' math skills, which they presumably learned in the math department, yet is often not adequate for their physics classes. A course in the physics department that emphasizes commonly used mathematical methods in physics could help better prepare students for upper level courses.

4.3.2 RELATION BETWEEN STUDENTS' MATH SCORES AND FINAL GRADES IN P263

The course $P263$ is the third quarter of a sophomore level course and is required for all physics majors. The course covers much of the modern physics and introduction to quantum mechanics curriculum, including topics such as photoelectric, Compton scattering, wave superposition, introduction to wave function, orthonormal bases, and the Schrödinger equation. Although the focus in this course is on basic concepts, experiments, and theories that have led to the development of quantum mechanics, students often are also introduced to some of the formalism of this theory.

The study of students' mathematical background, through an online survey in spring quarter, showed that 85% of students, prior (or parallel) to taking any quantum courses, have taken at least one math course in differential equations (math 255 or math 415), and one course in linear algebra (math 568 or math 571). Therefore, almost all students had met the math prerequisites for the course.

In the analysis of the data, we found that there is a relationship between students' total scores in mathematical questions and their final grades in course $P263$. Table 4.1 shows a small positive correlation of 0.347 ($P = 0.05$) between the math scores and the final grade of individual students in $P263$. This correlation shows that the students who answered most of the mathematical questions on the weekly questionnaires correctly, overall scored higher in the course. Although this is not a cause and effect relation, it shows that students' math skills affect their final grades in $P263$ at least slightly.

		P263 Final Grades
Math Scores	Pearson Correlation	.347**
	Sig. (2 tailed)	.033
	N	38

**. Correlation is significant at the 0.05 level (2-tailed).

Table 4.1: The correlation between students' math scores and their final grades in P263

		P631 Final Grades
Math Scores	Pearson Correlation	.547**
	Sig. (2 tailed)	.001
	N	38

**. Correlation is significant at the 0.01 level (2-tailed).

Table 4.2: The correlation between students' math scores and their final grades in P631

4.3.3 RELATION BETWEEN STUDENTS' MATH SCORES AND FINAL GRADS IN P631

$P631$ is the first quarter of a junior/senior level quantum mechanics course that is usually required for all physics majors. This course is a non-relativistic quantum mechanics course where students are typically introduced to basic postulates of quantum mechanics, the Schrödinger wave equation, stationary states, and scattering in one dimension. Overall the course covers the concepts and mathematical formalism of quantum mechanics, both in one dimension. $P263$ and a mathematics course in ordinary and partial differential equations are prerequisites for this junior/senior level course in quantum mechanics.

Further analysis of data shows a positive correlation coefficient of 0.547 between math scores and final grades of the students in $P631$ [see Table 4.2]. Figure 4.3 compares scattered plots of students' total scores in math questions versus their final exam grades in two courses. The more significant correlation between math ability and final grades in $P631$, compared to $P263$, suggests a greater relationship between students' math skills and their achievement in quantum courses. This is perhaps because of the more qualitative nature of course $P263$ and the greater use of complex mathematics in advanced courses such as $P631$.

Descriptive Statistics

	N	Minimum	Maximum	Mean	Std. Deviation
P263 Final Grades	38	35.00	94.00	61.0526	15.41839
P631 Final Grades	45	15.00	88.00	57.0889	15.09338
Math Scores	38	10.00	78.00	27.7632	17.68865
Classical Wave	43	12.00	100.00	51.4884	20.79534
Valid N	38				

Figure 4.2: A discriptive summary of data used in this chapter

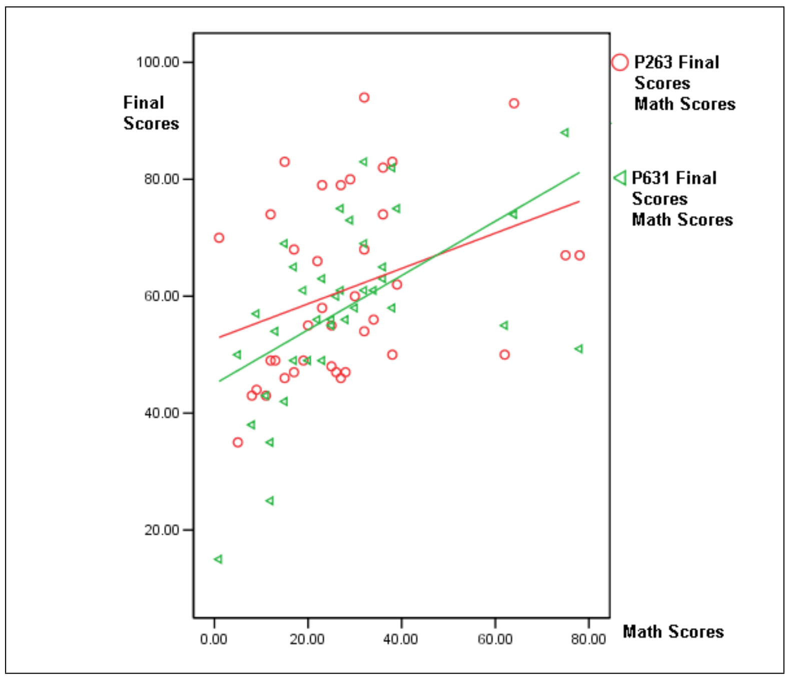

Figure 4.3: Scatter plot of students final grades in P263 and P631 versus their math scores

		P263 Final Grades
Classical Waves	Pearson Correlation	.487**
	Sig. (2 tailed)	.002
	N	38

**. Correlation is significant at the 0.01 level (2-tailed).

Table 4.3: The correlation between students' classical physics background and their final grades in P263

4.4 RELATIONSHIP BETWEEN STUDENTS' BACKGROUND KNOWLEDGE OF CLASSICAL WAVES AND THEIR ACHIEVEMENT IN QUANTUM COURSES

In the study of every possible relationship between the different question categories on our questionnaires and students' achievement in quantum courses, we found that students' average scores on the questions about classical waves relate to students final grades in $P263$. The questions related to classical waves covered topics such as concepts of transverse and longitudinal waves, superposition, traveling pulses, standing waves, diffraction and interference patterns, and phase difference. We found a small positive correlation coefficient of 0.487 between individual student's average scores on the questions related to these topics and their final grades in $P263$ [Table 4.3]. However, we did not observe a significant relationship between the students' scores on classical wave questions and their final grades for $P631$.

Comparing the data on Table 4.3 with the results on Table 4.1 shows that for an introductory course such as $P263$, the effect of students' background knowledge of classical waves on their final grades is greater than the effect of their math scores. This shows that students' learning of basic wave concepts is an important element for their success in learning introductory quantum mechanical courses.

4.5 SUMMARY

As mentioned in Chapter 1, some experts in physics education have pointed out that it would be possible to introduce students to important topics and basic concepts, experiments, and theories that have led to the development of quantum mechanics theory without requiring prior advanced mathematical knowledge. The motivation for this suggestion is to broaden the student audience for introductory quantum mechanics courses and include students from other sciences, mathematics, and engineering. However, others argue that quantum mechanics is a mathematical theory and the ability to adapt this theory to our physical world and construct physical meaning from abstract concepts and complex formalism requires a certain level of logical reasoning and mathematical skill.

Our findings suggest that students' achievement in quantum courses are not independent of their math backgrounds. There is a correlation between students' math scores and their final grades in both $P263$ and $P631$. Although this correlation is small in the case of $P263$ (correlation coefficient 0.347), in $P631$ it is more significant (correlation coefficient 0.547). Perhaps it is possible to reduce the technical mathematical skills necessary to enable students from other sciences to learn core ideas of quantum mechanics at a very introductory level, but in advanced courses we cannot deny the need for a certain level of proficiency in mathematical skills and formalism.

In addition, the students' responses to our survey about their difficulties with math in physics show that their attitudes change from sophomore level to junior level courses. It seems that there is a great jump in the level of mathematics one needs to succeed in physics courses after the sophomore level. Although these students have met all the math prerequisite for their quantum course, they often have difficulties with application and making sense of math in physics. As a result, The Ohio State Physics Department has implemented a recent change in its curriculum for physics majors by replacing the contents of $P263$, from Introductory quantum mechanics, to a course in math methods. In this way, we hope this change will equip students with higher math skills which can help them to succeed in more advanced physics and quantum courses.

Finally, we found that students' background knowledge of classical waves correlates with their final grades in $P263$ (correlation coefficient of 0.487); thus a solid understanding of basic concepts of mechanical wave seems to be one of the prerequisites to learning quantum mechanics. In the next chapter, we discuss students' specific difficulties with quantum measurement and probability.

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CHAPTER 5

IDENTIFYING STUDENT DIFFICULTIES IN UNDERSTANDING QUANTUM MEASUREMENT

"Quantum mechanics is basically a statistical theory. The probability is inherent in a quantum particle which does not have any well defined values." Hsieh Ye

5.1 INTRODUCTION

Since the beginning of history, in a world of uncertainty, man pursued certainty. In fact, the mission of the philosophies of Plato and Aristotle was to discover true and certain knowledge. Later, with the development of Newtonian mechanics, the deterministic worldview remained entrenched. It was only with the advance of thermodynamics that Boltzmann introduced the idea of probability into physical laws. In the same way, for most students a quantum mechanics course is the first time physical concepts are explained in terms of probability [1]. It is often difficult for students to replace the basic convictions of a deterministic worldview with the probabilistic view of quantum mechanics.

Understanding the probabilistic nature of quantum mechanics seems to be a difficult challenge for students, both in introductory and upper level quantum classes [2].

After standard instruction many students have difficulties understanding the concepts of probability and probability density. In addition, students show more difficulties with the formalism of their understanding of physical concepts in terms of abstract mathematical symbols. When it comes to problem solving, the combination of these two difficulties affect their performance in different degrees. Here we address the following questions that are part of the focus of our investigation of students' understanding of quantum mechanics:

Do students in introductory quantum courses have a functional understanding of the concepts of probability and probability density?

Do these students posses the appropriate level of abstract reasoning to formalize their understanding of these concepts in mathematical symbols?

Do students have the ability to integrate their conceptual and mathematical understanding to analyze the probabilistic behavior of a quantum mechanical system?

This chapter discusses student difficulties in understanding and interpreting probability as it relates to quantum measurement and its relevant technical terms, such as expectation value, probability density, and uncertainty. In addition to the conceptual difficulties, some of the students' difficulties associated with the formalism of these concepts are discussed. In brief, this chapter addresses students' conceptual understanding of probability and their ability to formulize and apply their understanding to more analytical and abstract problems. Examples of students' specific problems with classical probability and two-state quantum mechanical spin systems are discussed. In addition, findings in this chapter indicate that students' difficulties with the concepts of probability often interfere with their understanding and applying the Uncertainty Principle.

This chapter first discusses the motivation and initial observations, followed by an overview of research relevant to this part of our study. Then, it gives the details of the research questions covering conceptual and mathematical understanding of probability in quantum measurement. Finally, section 5.5 concludes this chapter with a summary of our findings on students' specific difficulties with quantum measurement.

5.2 MOTIVATION AND INITIAL OBSERVATIONS

During informal observation of quantum classes, in the study of students' written homework, and one-on-one interactions, we observed that students often have trouble with the concepts of probability, probability density, and expectation value. For example, in many cases students tended to confuse the concepts of probability density of an operator with the expectation value of that operator, or described the "expectation value" and "probability density" in very vague terms and even used these two terms interchangeably. Often they had trouble formulating their understanding in order to calculate these quantities mathematically. In addition, interpreting their mathematical findings into physical meanings demonstrated their serious difficulties with understanding basic quantum concepts. For instance, in a part of homework problem students have been asked to show that [3]:

> *".. there is an equal probability for finding any value of momentum for a wave function localized at $x = x_0$."*

The correct answer to this question has two parts. First, in order to calculate the probability density of the momentum ($|\varphi(k)|^2$), one should find the Fourier transform of the localized wave function (delta function) which will result in a constant value of $\frac{1}{2\pi}$. Second, one should conclude that a constant value for the probability density of momentum gives an equal probability for finding any value for momentum. Students had difficulties with both parts of the solution. The complete question, the correct answer, and examples of students' incorrect responses are given in Appendix **B**. Many students were not able to calculate the probability density of the momentum correctly. In addition, their interpretation of their incorrect calculations revealed their conceptual difficulties. Some of the students' reasoning after calculating incorrect values for the probability density of momentum are discussed below:

> *"Since $|\varphi(k)|^2 dk$ is the probability of finding the measurement between $\hbar k$ and $\hbar(k + dk)$, and here, $\varphi(k) = \delta(k)$ with $x = x_0$, this gives us $< \hat{p} >=$ constant."*

This student has difficulty distinguishing that, in this problem, a delta function represents the wave function in the position space and not in the momentum space. But more importantly, she presents a constant value for the "expectation value of momentum" as the reason for "equal probability in measuring" different momentum.

> *"The expectation value of x is the Fourier transform of your delta function, which produces a constant $< p >$, so you have the same probability of any value."*

First, this student confuses the "Fourier transform of the wave function" with the "expectation value of x." Second, he also interprets a constant value for the "expectation value of momentum" as the reason for "equal probability in measuring" different momentum.

Some students calculated $|\varphi(k)|^2 = \infty$, and concluded that:

> *"...this means that any value of momentum is probable."*

In addition to calculating the incorrect value for the probability density, these students interpret the "infinite value for the probability density" as the reason for "equal probability in measuring" different momentum.

Others calculated $< p >= \infty$ incorrectly and stated that:

> "*...therefore, the measurement of momentum can have any value.*"

In this case students present an "infinite value for the expectation value of the momentum" to be the reason for "equal probability in measuring" different momentum.

These initial observations motivated us to investigate students' understanding of probability and its related concepts systematically. One implication of these findings is that students' tend to confuse the terms "expectation value of momentum ($< p >$)" with the "probability density" in measurement ($|\varphi(k)|^2$).

5.3 OVERVIEW OF RESEARCH

Our systematic investigation of students' difficulties with topics of probability includes interviews, written questions, development of multiple choice questions based on students' incorrect ideas (on line questionnaires), and the analysis of students' written homework and exams. The objective here is to understand student specific difficulty and its origin in understanding probability. We are also interested in knowing to what degree, and in which ways, mathematical difficulties affect students' understanding of quantum measurement.

The results presented in this chapter are mostly obtained from the analysis of students' responses to interviews and research questions in three different quarters and classes [4]: two advanced undergraduate quantum mechanics classes ($P631$; fall

2003, $N = 35$ and fall 2004, $N = 60$) and an introductory quantum course ($P263$; spring 2004, $N = 61$).

During the fall of 2003 we observed students' interactions in lectures; we also studied students' written homework and exams in detail to detect any common difficulties students may have in understanding the probabilistic nature of quantum mechanical phenomena. Individual interviews were conducted with 18 students and the grader assigned to the course. All participants in interviews were volunteers and all the students earned grades above the course average.

Three online questionnaires were given to the whole class ($N = 35$). The first questionnaire (week 6 - fall quarter) was designed to investigate students' background knowledge of math and physics. In this questionnaire we also determined the specific math courses that individual students had taken.

The second questionnaire (week 10 - fall quarter) was a series of related interview-like essay questions designed to probe students' understanding of the expectation value of operators, probability density, and the Uncertainty Principle in three different contexts: pure mathematics, pure physics, and mixed. The main objective was to investigate students' understanding of the formalism behind the material and their ability to transfer between physical meanings and mathematical formalisms.

The third questionnaire (week 3 - winter quarter) was a multiple-choice, multiple-response test; in which, students were asked to include their reasoning after each response. We used students' common incorrect understandings obtained from the second questionnaire and interview as alternative answers in the third one. Examples of these multiple choice questions are presented in **Q5.2** and **Q5.3**.

Additional data were obtained from an advanced quantum mechanics class at the Pacific University. Due to the small number of students ($N = 7$), and the consistency between their results and that from OSU, we will focus only on the OSU data.

In spring 2004, we gave $P263$ students 9 weekly questionnaires containing $10 - 15$ questions each. Out of 9 questionnaires, three were designed to probe students' understanding of probability, including classical probability and spin measurement in the Stern-Gerlach Experiment. In fall 2004, we continued the study with $P631$ and tested our findings and our classroom material. Instructional material on probability, along with other classroom worksheets, is discussed later in Chapter 8. In general, this study is carried out in several different contexts, including student understanding of:

1. Mathematics and interpretation of formalisms;

2. Expectation value, probability density, and the Uncertainty Principle;

3. Classical probability;

4. A two state system and examples of spin measurement in Stern-Gerlach experiment.

In addition, student difficulties with probability in context of wave function is discussed in Chapter 6.

5.4 MATHEMATICS AND INTERPRETATION OF FORMALISMS

Our results from the first and the second questionnaires show that a mathematical understanding of the probability concepts seems necessary, but not sufficient, for

Type of questions	**Correct responses**	**Incorrect response**
Pure Math	33%	52%
Pure Physics	43%	47%
Math & Physics	26%	64%

Table 5.1: Results from the second questionnaire show that most students who could answer pure physics and math questions were not able to answer questions that required both.

understanding the physics of it. In other words, there were very few students who could answer pure physics questions without knowing the mathematics behind them. In students' responses to the second questionnaire 33% knew well topics such as operators, commutators, uncertainty, expectation values, probability distributions, and Fourier Transform, in the context of math questions, and 43% of students were able to give a verbal (not mathematical) understanding of these concepts. However, only 26% of students were able to answer the questions that require both a knowledge of math and of physics on these topics [see Table 5.1].

Many students who were able to answer the math and physics questions in isolated contexts were not able to connect and interpret these two in more complex problems, even when they had all the background knowledge required. Most students admitted to this shortcoming during the interview.

For example, students who were able to analyze equation (5.1) in the context of a math question, and were able to recognize $< x >$ or $< p >$ as the average of measurement outcome, were not able to connect equations (5.1) and (5.2), and realize that both carry the same physical meaning.

$$< \ f(x) >= \sum_{-\infty}^{+\infty} f(x)P(x) \tag{5.1}$$

$$< \ \hat{p} >= \int_{-\infty}^{\infty} \Psi^*(x)\hat{p}\Psi(x)dx \tag{5.2}$$

Q5.1(a) For what values of x, is $f(x)$ maximum?

$$f(x) = x(10 - x) \text{ for } 0 < x < 10 \tag{5.3}$$

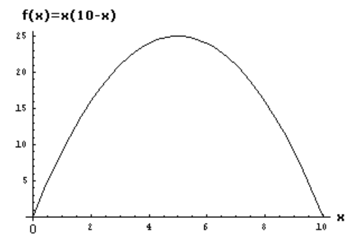
(b) A particle in an infinite square well of length L $(0 < L < a)$ has the initial wave function:

$$\Psi(x) = Ax(a - x). \tag{5.4}$$

If you make a position measurement, where would you more likely find the particle ? Hint: use your results from part (a).

Q5.1 (a) Solution:

$$
\begin{aligned}
f'(x) &= 10 - 2x = 0 && (5.5)\\
\text{at } x &= 5 \;\; f(x) \text{ has maximum or minimum} && (5.6)\\
f''(x) &= -2 < 0 \text{ thus, } f(x) \text{ is maximum at } x = 5 && (5.7)
\end{aligned}
$$

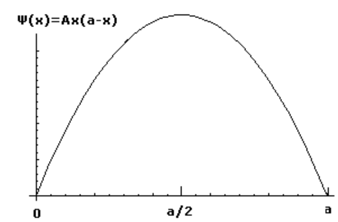


Graph of $f(x) = x(10 - x)$ for $0 < x < 10$

Q5.1(b) Solution 1:

$$
\begin{aligned}
\Psi'(x) &= A(a - 2x) = 0 && (5.8)\\
\text{at } x &= \frac{a}{2}, \Psi(x) \text{ has maximum} && (5.9)
\end{aligned}
$$

therefore, the probability density is maximum at $x = \frac{a}{2}$ and from symmetry $< x >= \frac{a}{2}$



Graph of $\Psi(x) = Ax(a - x)$ for $0 < x < a$

Solution 2:

First normalize the wave function : $\int\limits_0^a \Psi(x)\Psi^*(x)dx = 1$; therefore, $A = \frac{30}{a^5}$.

$$
\begin{aligned}
< \;\; x >= \frac{30}{a^5} \int x^3(a^2 - 2ax + x^2)dx && (5.10)\\
< \;\; x >= \frac{30}{a^5}(\frac{a^6}{4} - \frac{2a^5}{5} + \frac{a^6}{6}) && (5.11)\\
< \;\; x >= \frac{a}{2} && (5.12)
\end{aligned}
$$

Q5.1	**Part (a)**	**Part (b)**		
		correct		incorrect
		Solution1	**Solution2**	
Correct	12	1	5	6
Incorrect	3			3
Total	15	15		

Table 5.2: Students' responses to Q5.1; only one out of six correct responses to the part (b) of this question presented solution 1.

In another example, we presented a pure mathematical problem, **Q5.1** (a), to 15 students, who seemed to show a conceptual understanding of expectation value in the quantum regime, of whom 12 were able to find the maximum value of the given function and graph the function correctly in part (a) [see Table 5.2]. Students were not allowed to use calculators for this task. Of these 12 students, only 6 were able to answer part (b) of **Q5.1** correctly; only one student used the graph of the wave function to answer this question (**Solution 1**), others followed the typical process of normalization and calculation of $< x >=< x|\Psi|x >$ (**Solution 2**). Of the three students who were not able to solve part (a) correctly, none was able to solve part (b) correctly. This is an implication of need for mathematical skills in solving such a quantum mechanical problems.

Students' responses to **Q5.1** have several indications. First, conceptual understanding of expectation value is not sufficient for students to be able to answer this question. Second, most of the students with good math skills are not able to transfer their math knowledge to the quantum mechanics domain. They lack the ability to integrate their conceptual understanding and mathematical skills in solving quantum physics problems. We obtained similar results in a different context of this study.

As an example of student difficulties in transferring knowledge of mathematics to the physical world in the context of the two-state spin system, we have observed that some students think spin up and spin down states for a spin-$\frac{1}{2}$ particle should be perpendicular to each other because $| + z >$ and $| - z >$ states are orthonormal. This example shows the difficulties in understanding orthonormality and the proper use of this concept in math and physics contexts. Our similar findings in the context of classical probability and the Stern-Gerlach Experiment are discussed later in this chapter.

Q5.2. Emma is solving a quantum problem. She is asked to show that, for a particular system any value of the momentum is equally probable. She needs to show that: (pick the best that applies)

1. The probability density of the momentum is infinity.
2. The probability density of the momentum is zero.
3. The probability density of the momentum is a nonzero constant.
4. The expectation value of the momentum is infinity.
5. The expectation value of the momentum is zero.
6. The expectation value of the momentum is same everywhere.

5.5 EXPECTATION VALUE, PROBABILITY DENSITY, AND THE UNCERTAINTY PRINCIPLE

5.5.1 STUDENT TENDENCY TO CONFUSE $<\hat{p}>$ WITH $|\varphi(k)|^2$

Students' responses to the third questionnaire, in which we gave students several pure physics questions in more isolated settings, confirm our preliminary findings regarding students' tendency to confuse the expectation value of an operator ($<\hat{x}>, <\hat{p}>$) with the probability density ($|\Psi(x)|^2, |\varphi(k)|^2$), regardless of the difficulties in mathematics. Questions on probability densities, expectation values, and uncertainties in measurement were given to students in all three classes. Questions **Q5.2** and **Q5.3** are examples of such.

The correct answer for **Q5.2** is choice 3, *"The probability density of the momentum is a nonzero constant."* Choice 6 although generally a correct statement, is not the answer for this question. Although **Q5.2** this question followed instruction on related

Q5.2	P263-04	P631-03	P631-04
Correct	30% ($n = 11$)	31% ($n = 11$)	45% ($n = 25$)
Incorrect	70% ($n = 26$)	69% ($n = 24$)	55% ($n = 29$)
Choice 6	43% ($n = 16$)	34% ($n = 13$)	33% ($n = 18$)
Total	37	35	55

Table 5.3: The summary of students' responses to Q5.2

topics, more than 75% of the students in $P263-04$ and $P631-03$ and more than 50% of the students in $P631-04$ answered **Q5.2** incorrectly [see Table 5.3]. In addition, in all three classes over half of the incorrect choices were choice 6. This suggests some students may have confused "expectation value" with "probability density."

Q5.3. Suppose that at time $t = 0$, a position measurement is made on a particle and $x = x_0$ is found. Assume that measurement was precise enough and the wave function immediately following it is well approximated by a δ function (or a very narrow Gaussian)
Which statements are correct about this system? Choose all that apply and explain your reasoning.

1. $\varphi(k) = \infty$
2. $|\varphi(k)|^2 = \frac{1}{2\pi}$
3. $\varphi(k) = 0$
4. $< p >= 0$
5. $|\varphi(k)|^2 =$ nonzero constant
6. $< p >= \infty$
7. $|\varphi(k)|^2 = 0$
8. $< p >=$ nonzero constant

Q5.3	**P631-03**	**P631-04**
Correct	27% ($n = 6$)	37% ($n = 20$)
Correct Reasoning	9% ($n = 2$)	11% ($n = 6$)
Incorrect	73% ($n = 16$)	64% ($n = 34$)
Total	22	54

Table 5.4: The summary of students' responses to Q5.3

Question **Q5.3** is another example that shows students' conceptual difficulties with the topics of probability. Question **Q5.3** is a multiple-choice, multiple response question concerning related terminologies in probability. Due to the higher-level content of this question, we gave this question only to the students enrolled in the 631 courses [5]. Students were also asked to explain their reasoning.

Of the students who gave at least one correct response to **Q5.3** (27% for $P631-03$ and 37% for $P631-04$) only about 10% gave the correct reasoning [see Table 5.4]. The correct answers to **Q5.3** are choices 2, 4, and 5; nevertheless, none of the students picked the analytical answer, choice 2, and none of the students who picked choice 4: $< p >= 0$, gave correct reasoning for this choice. Of the students who gave at least one correct response, only students who picked choice 5 (˜10%) had also a correct reasoning. An example of incorrect reasoning given by a student who chose 4 as his answer follows:

"Since a position measurement is precise enough to have a function $f(x)$ then to measure $< p >$ with any amount of accuracy is almost null according to the Uncertainty Principle."

This reasoning fails to distinguish among three different concepts of measurement: uncertainty in momentum Δp, expectation value of momentum $< p >$, and the probability density of momentum measurement $|\varphi(k)|^2$. In addition, this reasoning shows the student's lack of ability to apply the Uncertainty Principle correctly.

About 27% of the students picked choice 1 : $\varphi(k) = \infty$ and 7 : $< p >= \infty$; often simultaneously. The first implication of choosing these two choices together is that students treat $< p >$ and $\varphi(k)$ the same. The second implication, considering some of students' reasoning, is that, again, these students fail to apply the Uncertainty Principle correctly. For instance, consider the following reasoning given by a student who chose 1 and 7 as his answers to **Q5.3**:

> *"1 and 7; because if you know the exact position, nothing can be known about the momentum because of the Uncertainty Principle."*

Here, this student interprets the infinite value for expectation value of momentum as maximum uncertainty in momentum measurement.

About one third of the students (33%) picked choice 8 : $< p >=$ nonzero constant. As discussed above, the expectation value of the momentum in this case is zero and this choice is not correct. One possible reason for choosing 8 is that students confuse the expectation value of momentum with the probability density of momentum (choice 3). In addition, some of the students' reasons for choosing this answer show that they interpret a constant expectation value of the momentum as the equal probability for any momentum measurement. This problem is similar to our findings from **Q5.2**, in which students confuse the expectation value of momentum with the probability density of measurement.

5.5.2 QUALITATIVE QUESTIONS

In order to understand the depth of students' understanding of quantum measurement and concepts of probability, we gave written open-ended questions to students taking the introductory quantum course at The Ohio State University (spring 2004, $N = 38$). We used different types of questions that pertain to probability. Examples of these written questions and students' responses are described below.

Q5.4: ***"Why do we use probability to describe a quantum system?"***

> Of 38 student responses to this question, 6 were reasonably correct answers; one example of these responses is:
>
> **S1:** *"If we only looked at one measurement and record the result we probably get some different result the next time we try to reproduce it. This is because the system is going to yield that measurement only a fraction of the time. Probability is the proper model for this. The probabilistic interpretation gives a distribution of results for many identical experiments with identically prepared quanton. In the classical model absolutely identical experiments would give absolutely identical results. This is not the case at the quantum level. Therefore probability is required."*

The remaining 32 students either had mixed ideas or had no clue why we need to use probability in quantum measurement. For instance, 18 students somehow mentioned measurement uncertainty, but often interpreted this uncertainty as our inability to make a precise measurement. These students often concluded that this inability is the reason for using probability in quantum measurement. Some students' quotes are:

> **S2**: *"...man does not have the precision to measure it exactly. We cannot accurately predict the exact location of a quanton at any given time."*
>
> **S3:** *"We use probability because due to the Uncertainty Principle we can never measure the exact position or momentum without disturbing the actual results."*
>
> **S4:** "*Because we can't measure the system with out affecting it so giving a probability is the closest to the truth without affecting the system with measurements. Because if we actually perform a measurement the*

wave function collapses to match that measurement. We lose information pertaining to the system that way."

S5: *"Because there is no way of knowing what is actually going to happen; you can only know what may occur; there is no definite way to predict a single quanton's behavior."*

S6: *" With Heisenberg's Uncertainty Principle we cannot know exactly where an electron is; we want to know its momentum but we can represent its position very well by the Schrödinger Equation."*

S7: *"We cannot know quantities like energy, position, etc. for certain without measuring them; but probability gives us an idea of what we might find. We use probability to describe quantum systems because that is the best we can approximate. We do not know exactly where the electron is but we can still approximate the position with probability."*

The rest of the students had very little idea and even admitted they had no clue:

S8: *"I am not sure; is it because a particle has probability in position?"*

S9: *"...because that's the best we can do... God really does play dice."*

S10: *"...because the actions of qauntons are continuous waves and require statistics as a result."*

S11: *"...because the chance of finding an electron is desired."*

S12: *"...we have no answer that is always true; it's like the particles choose an answer."*

S13: *"The system is too complex to determine anything exactly but we can come up with very reasonable probabilities."*

S14: *"You can't describe a system directly. You have to describe it by probability to say where everything should be at a given time. You can't measure it for certain."*

Q5.5: ***"What is probability amplitude? Describe the relation between probability and probability amplitude."***

Only 4 students were able to clearly describe and distinguish the probability amplitude and the probability density for a given wave function. Fifteen students had incomplete, and often reversed, descriptions for these two concepts, indicating their inability to distinguish the two concepts from each other.

> **S1:** *"I'm not sure; I think the probability amplitude is just the absolute square of the probability density."*
>
> **S2:** *"Probability amplitude is the square of the wave function which is proportional to the probability of finding a particle there."*
>
> **S3:** *"...the amplitude at a point is the probability of finding the quanton at that state."*
>
> **S4:** *"...the probability amplitude is proportional to the probability of getting that measurement."*

The remaining 19 students had different understandings. For example, three students described the probability amplitude as a "place" or "area":

> **S5:** *"It's the place where the most likely particle would be found."*
>
> **S6:** *"...the probability amplitude is equal to the area under the absolute square of the wave function. And this area is also equal to the probability density."*
>
> **S7:** *"...the probability amplitude is the probability of finding an electron in a certain place. Probability is determined at one position and probability amplitude is over a given range."*

Some other ideas include:

S8: *"Probability amplitude shows the number of data points divided by the total number to get the probability."*

S9: *"I really don't know!"*

S10: *"Probability amplitude is the number that determines where a quanton will be."*

Q5.6: *"Consider the 1st excited state of the wave function in an infinite potential well. What does the node of the wave function mean?"*

Only 10 students gave a correct answer to this question; while there was not a clear pattern for students' incorrect responses, we include some examples:

S1: *"The node represents a peak in the energy level."*

S2: *"The number of crests in the wave."*

S3: *"...this is where the quanton is most probable to be found."*

S4: *"...it corresponds to the barrier of the well."*

S5: *"It is the most probable place to find the electron."*

S6: *"Its change in energy is at a maximum."*

S7: *"I'm not sure."*

S8: *"...no clue!"*

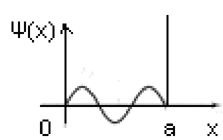
S9: *"...that there is only one place where we are most likely to find the quanton."*

Students' written responses to the above qualitative questions indicate that after a traditional course in quantum mechanics most students have difficulties understandings basic concepts of quantum measurement. Their knowledge is often memorized and fragmented facts that do not equip them to a sufficient ability to apply the principles correctly to analyze a probability wave function.

Q5.7(a) Consider an infinite square well of width L with a single electron in it. If someone performs a measurement of the electron's energy and tells you that they found the electron to have the energy of $n = 2$ eigenstate, at what positions is the electron most likely to be found?

1. $\frac{L}{2}$
2. $\frac{L}{3}, \frac{2L}{3}$
3. $\frac{L}{4}, \frac{3L}{4}$
4. $0, \frac{L}{2}, L$
5. The probability is the same everywhere.

(b) The graph below shows the wave function for a particle trapped in an infinite square well. If you measure the position of this particle, where will you most probably find the particle?



1. $\frac{a}{2}$
2. $\frac{a}{2}, \frac{3a}{2}$
3. $\frac{a}{3}$
4. $\frac{a}{6}, \frac{a}{2}, \frac{5a}{6}$
5. The probability is the same everywhere.

5.5.3 QUANTITATIVE QUESTIONS

On the quantitative questions students showed similar difficulties. The quantitative questions were often in multiple choice format and were given on line. An example of these questions is illustrated in **Q5.7.** Part (a) of this question asks students to find the most probable position for an electron given its energy state ($n = 2$),

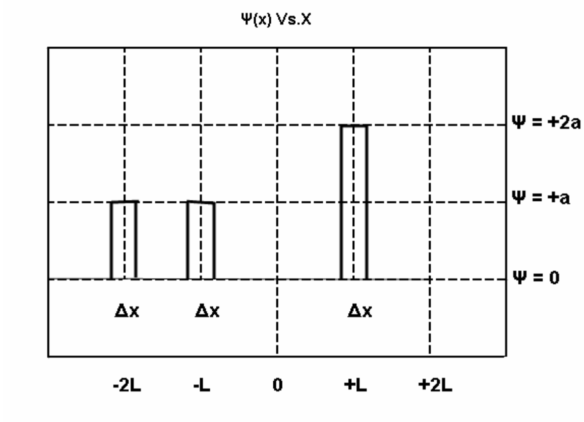
Q5.7	**Part** (a)		**Part** (b)	
	# of students	%	**# of students**	%
Correct	18	47	24	63
Incorrect	20	53	14	37
Total	38		38	

Table 5.5: The summary of students' responses to Q5.7 shows that, although students did not do well in this question, they did much better in part (b), which had a graphical image of wave function.

and in part (b), similarly, students are supposed to find the most probable position for an electron given its wave function graphically.

Of the students in the introductory quantum course $P263$, only 47% picked the correct answer in part (a), choice 3, and 63% in part (b), choice 4. Considering these questions were given to students following traditional lecture instructions, the results are not promising. In addition, students, did much better in the graphical questions, as compared to part (a) [see Table 5.5].

Q5.8. The figure below shows a plot of a (rather artificial) wave function $\Psi(x)$ versus x, over the range of $(-2L, +2L)$. The wave function vanishes for all other values of x. What is the expectation value of x?



1. $< x >= \frac{4L}{6}$
2. $< x >= -L$
3. $< x >= 0$
4. $< x >= \frac{L}{6}$
5. $< x >= \frac{a}{6}$
6. $< x. >= \frac{a}{4}$
7. $< x >= -\frac{L}{4}$

In another quantitative question, **Q5.8,** we asked students to calculate the expectation value of the position for an artificial discrete wave function. This question was also given to the students after instruction in $P631$. Of the 41 students, only 24% picked the correct answer, choice 4,and 76% had incorrect answers [see Table 5.6]. The most common incorrect answers were choice 7 $(37\%, n = 15)$, and choice 2 $(17\%, n = 7)$. Note that choice 7, $-\frac{L}{4}$, is the result of finding expectation value without squaring the wave function in the process:

Q5.8	**# of students**	%
Correct	10	24
Incorrect	31	76
Total	41	

Table 5.6: The summary of students' responses to Q5.8

$$\frac{-2L(a) + (-L)(a) + (L)(2a)}{4a} = \frac{(-3+2)La}{4a} = \frac{-L}{4}, \tag{5.13}$$

and choice 2 results from the above calculation without normalization. In Chapter 8 the results of **Q5.8** are discussed as a pretest question for the evaluation of worksheets developed in this study.

Our findings at this stage show that after taking a course in quantum mechanics students do not have a good understanding of basic quantum concepts. In particular, they have difficulties with quantum measurement and probability. They have difficulties interpreting the wave function as a probability wave and are often unable to distinguish among similar but different concepts such as probability amplitude, probability density, and expectation value [6]. Other difficulties include distinguishing between the expectation value of an operator ($< x >, < p >$) and the probability density ($|\Psi(x)|^2, |\varphi(k)|^2$). Although most students have memorized ideas about the Uncertainty Principle, they are not able to apply this principle when analyzing a quantum system. In addition, students have difficulties with quantitative calculations of probability density, expectation value, etc.

5.5.4 CLASSICAL PROBABILITY

To understand the origin of students' difficulties with quantum measurement and the concepts of probability (classical and quantum mechanical), we interviewed 18 students taking introductory quantum course $P263$ at The Ohio State University. All participants in the interviews were volunteers and all participating students earned grades above the course average. To investigate students' understanding of classical probability we used a modified version of the tutorial developed by the University of Maryland as an interview task [7]. The graphical representation of this tutorial is presented in Figure 5.1 [8]. The figure shows a series of balls moving towards the right at a very small velocity V_0 and falling off from the top to level 1 and then to level 2, ($L >> d$, no friction)

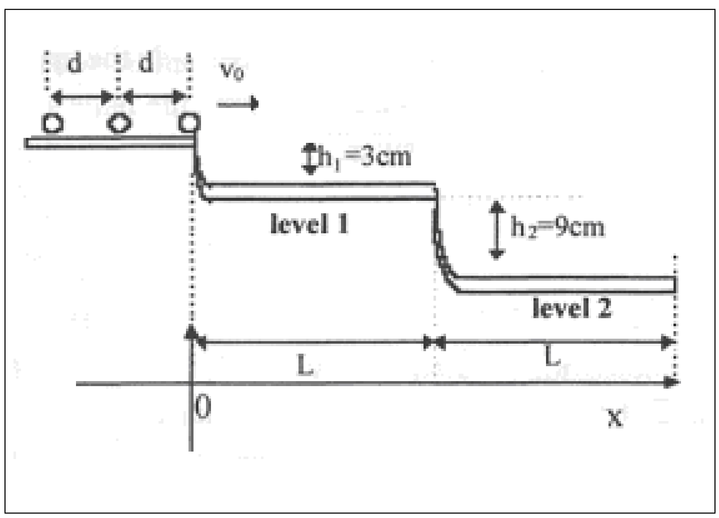


Figure 5.1: A schematic picture of the interview task on classical probability. A series of balls are moving towards the right at a very small velocity V_0 and fall off from the top to level 1 and then to level 2, ($L >> d$, no friction).

In this task, the main question we asked students was to compare the probabilities of finding a given ball on two levels. If students were not able to compare the probabilities, in order to guide them toward the correct answer, we continued with the following three guiding questions until they were able to compare the two probabilities correctly:

1. If you know the speed of a given ball in each level, can you determine the probability of finding that ball on each level? Explain.

2. If you know the time a given ball spends on each level, can you determine the probability of finding that ball on each level? Explain.

3. If ball spends t_1 minutes in level 1 and t_2 minutes in level 2, what is the probability of finding that ball on each level? Explain.

In this task the focus was on students' understanding of probability; therefore, we were more interested in the qualitative reasoning than the specific calculation. Only two students were able to answer the main question correctly without any guidance and further questioning. Seven out of 18 students, who at first seemed clueless about the probabilities of finding a given ball in two levels, were able to answer the main question and compare the probabilities, both qualitatively and quantitatively, after the first, second, or third guiding question. Three students were not able to relate velocity, or the amount of time particles spend in a region, to the probability of finding the particles in that region, and were not able even to give a qualitative answer to the questions.

The remaining six students were able to somehow compare the probabilities qualitatively but not quantitatively even after we walked them through the three guiding questions. These students had difficulties to formulate their qualitative and conceptual understanding of probability into abstract mathematical symbols. For these 6 students we continued the interview with questions similar to the following:

> *"If a ball spends 6 minutes in level 1 and 3 minutes in level 2, what is the probability of finding that ball on each level? Explain."*

All 6 of the students who were not able to use terms such as t_1 and t_2 in question three above to calculate the probability, were able to calculate the correct probabilities using numeric values. When we asked them to explain how they came up with the answer, they often had difficulties explaining. When we asked question three again, they had to work backward from the numbers to derive a formula for their solution. Consider the script below as an example of such (here S stands for the student and I stands for the interviewer):

I: Compare the probabilities of finding a given ball on two levels.

S: *I say it is grater in level one, because they move slower in level one and spends more time on level one.*

I: Can you write the probability in mathematical terms? I mean formalize it.

S: *I have to say we have to find the speed of the ball 2 (level 2), so...*

I: Call them V_1 and V_2 as if you had their values. That way, you don't need to calculate their values.

S: *So* V_0*, I have a* $V_1 = V_0 + mgh$*, do they have masses?*

I: Yes it is m.

S: *Okay, mgh, Oh if* h *equals* .03 *meters then* $V_2 = V_0 + 2mgh$ *, because I am using* $2h$*, No! not* $2mgh$*,* $4mgh$*.Because it is* 3 *and another is* 9*, so that is* 4 *times*3, 12 *times,* 4*,* $V_0 + 4mgh$*, there.*

Oh, I would say the probability is directly proportional to the speed. The actual probability would be, it would be... the time, okay so I have to find the time it spends on level one.

Okay, I have to find the time, it would be VL over V; so the time it spends on one is:

$$T1 = \frac{L}{V_0 + mgh} \tag{5.14}$$

$$T2 = \frac{L}{V_0 + 4mgh} \tag{5.15}$$

Okay so that is the time it spends there. And I am looking at it in one instant of time.... So the probability would depend on how long I would take the picture for , so I had...

I: Can you repeat that?

S: *Yeah, if I have some time T_1 that I know this thing spend on there, the probability of finding that, oh these are spaced space d, okay, I have to worry about that too. Because they are gonna come dropping down, dropping down at a period Tp... the period which is d over V_0 ($Tp = \frac{d}{V_0}$); in every d over V_0, one comes across down. That means,... I can find the average number of the balls on these levels with these two times: would be a period between the drops and time spend on there because ... I can do that by dividing one by the other, yeah:*

$$N_1 = \frac{T_1}{T_p}. \tag{5.16}$$

So call this N_1, this is on average how many balls you find on that level,.. and likewise N_2. Then... of course they get accelerated, but this period is going to be the same. So, that is going to be so... I think this might be upside down. These Ns

are how many balls in average you are going to find in each level. Because they are dropping down and running across.

I: Which one do you think is greater, N_1 or N_2?

S: *I know N_1 is going to be greater, because they are going slower.*

I: You said it goes slower on level one and spends more time there, so the probability of finding the balls on level 1 is greater that two. My question was, can you formalize that?

S: *Oh, let's see. I guess it would be 1 over N : 1 over one of these Ns.*

I: So, what you are saying is that it would be the reverse of N; instead of one over Tp would be Tp over T_1 or T_2. Right?

S: *Yeah, yeah...*

I: Okay, do you want to write down what you just said:

S: *Probability in level 1 is equal to:*

$$\frac{T_p}{T1} \tag{5.17}$$

I might have this upside down....Okay, Probability in level 2 is the same thing:

$$\frac{T_p}{T_2}. \tag{5.18}$$

I: Well does Tp vary for the two cases?

S: *No, I think it does not.*

I: So Tp is the same for these two probabilities. So did you say T_1 is larger or T_2?

S: *T_1 is larger, which makes my second probability larger which does not make sense. I think I have them up side down then.*

I: You mean: Probability in level 1= $\frac{T_1}{TP}$, and probability in level 2= $\frac{T_2}{TP}$?

S: *Yeah,...*

I: Let me ask the question in a different way. If a ball spends 6 minutes in level 1 and 3 minutes in level 2, what is the probability of finding that ball on each level?

S: *That depends on,... it's like if you look at them for one second....so ....Okay, yeah then the probability here would be* 1 *and* 3*, and* 1 *and* 6*. I see one three times here* [level 1]*, and one and a half times there* [level 2]*. The probability of seeing balls in 9 second in level* 1*would be ...* $\frac{2}{3}$*.So* P_1 *is* $\frac{2}{3}$ *and* P_2 *will be* $\frac{1}{3}$.

I: Do you know what you did? Can you go back and use T_1 and T_2 instead of 3 and 6 seconds and formalize the probability for level 1 and 2?

S: *I took the time* T_1 *and I ended up finding the ratio between* T_1 *and* T_2*, wait, one is twice the other, and I think I found the common denominator between two* T_1 *and* T_2*. I think, I am not sure. These two sum up to one, so... this is... I think it was... so, because this is (*P_1*),just* T_1 *over some number* x*,*

P_2 *is* T_2 *over that same number.*

I: What is that number?

S: *Nine, it is* 9 *in the case of numbers, I am trying to think what it means. But if I do this, I get 1 that is* T_1 *+*T_2 *over* x *is equal* 1*; so* x *is equal* $T_1 + T_2$*. So I want, so my probabilities are* T_1 *over sum of two times and* T_2 *over the sum of two times:*

$$P = \frac{T_1}{T_1 + T_2} \text{ and } P_2 = \frac{T_2}{T_1 + T_2} \tag{5.19}$$

I: Does this makes sense to you?

S: *Yeah. It was much easier with numbers... if I know* T_1 *and* T_2 *I can find the probability....*

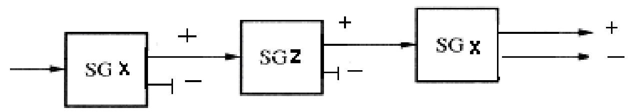
This student is able to answer the conceptual question correctly and understands the probability qualitatively; however, he is not able to formulate his understanding in terms of mathematical symbols to produce a more abstract answer. The above interview is one example of 6 which shows that many students in sophomore level classes have not achieved the required level of abstraction for formalizing their understanding of probability in terms of symbols and mathematical formulas. They often still prefer numbers and understand concrete representations much better than symbols. They have difficulties in interpreting their concrete understandings of probability in terms of mathematical symbols and vice versa.

5.5.5 UNDERSTANDING TWO STATE SPIN MEASUREMENT IN THE STERN-GERLACH EXPERIMENT

Part of the weekly on line questionnaires on probability given to the students taking a junior level quantum mechanics course ($P263$, $N = 44$) were focused on probability measurement in the two-state spin system in the context of the Stern-Gerlach (SG) experiments. Below we provide some examples of these questions, the required concepts to answer the questions correctly, and students' responses and their difficulties.

Question **Q5.9** is an example of the questions related to understanding probability in context of the SG experiment. Part (a) of **Q5.9** includes a visual image in which electrons enter the last device (SGX) in $+Sz$ state; therefore, the probability of exiting in $\pm Sx$ is equal to half, choice 2. About 86% of the students answered this question correctly. However, when we asked a similar question in part (b) (only the order of SGX and SGZ have changed) with no graphical picture, only 32% of the students gave the correct answer, choice 5, [see Table 5.7].

Q5.9.(a) The drawing below shows a sequence of Stern-Gerlach devices. What are the probabilities that an electron entering the last device will come out of the plus and minus channels of this device?



1. $\frac{1}{4}$ and $\frac{3}{4}$, respectively
2. $\frac{1}{2}$ for both channels
3. 0 and 1, respectively
4. $\frac{3}{4}$ and $\frac{1}{4}$, respectively
5. 1 and 0, respectively
6. Some other probabilities

(b) Consider a beam of electrons exiting the Stern-Gerlach device with its inhomogeneous magnetic field parallel to the Z axis (SGZ). We next send the beam with spin $+\frac{1}{2}$ into a SGX device, one with its in-homogeneous magnetic field oriented along the X axis. If we send the beam of the particles exiting the SGX device with $S_x = +\frac{1}{2}$, through another SGZ device, what is the probability of measuring spin $-\frac{1}{2}$ for electrons exiting the second SGZ device?

1. $\frac{1}{8}$
2. $\frac{3}{4}$
3. $\frac{1}{4}$
4. $\frac{1}{3}$
5. $\frac{1}{2}$

This example is another indication of students' difficulties with abstract questions. The concrete representations, such as graphs in **Q5.7** and diagrams in **Q5.9 (a)** seem to reduce the amount of abstraction and promote students' understanding of

Q5.9	**Part** (a)		**Part** (b)	
	# of students	%	**# of students**	%
Correct	38	86	14	32
Incorrect	6	14	30	68
Total	44		44	

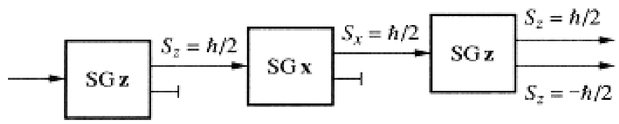
Table 5.7: Students' responses to Q5.9; students did significantly better in part (a) which included a graphical representation of the SGE.

physical phenomena and result a better performance. This finding is in agreement with research findings on other areas of physics [9].

Student Tendency to Confuse the Concept of "Probability" with "fraction"

In addition to difficulties with the abstract nature of probability, many students tend to confuse the concept of "probability" with the "fraction" of particles in the SG experiment. We observed that 25 out of 30 incorrect answers to part (b) of **Q5.9** was $\frac{1}{8}$, which is the "fraction" of electrons exiting the second SGZ device with spin $-\frac{1}{2}$ and not the "probability" of the particles. This, and other similar observations, made us suspect that students confuse the concept of "probability" with "fraction." To investigate this hypothesis we designed several questions targeting this issue.

Q5.10. The drawing below shows a sequence of Stern-Gerlach devices. What fraction of the electrons entering this experiment will exit the second SGZ device with spin $-\frac{1}{2}$?



1. $\frac{1}{4}$
2. $\frac{1}{2}$
3. $\frac{1}{8}$
4. 0
5. 1

Questions **Q5.10** and **Q5.11** are two examples of the questions that test students' ability to distinguish between the concept of probability and the fraction of particles. The correct answer to **Q5.10** is choice 3, $\frac{1}{8}$; however, only half of students gave the correct answer. Of the 22 incorrect answers, 16 students picked choice 2, $\frac{1}{2}$; which is equal to the "probability" of electrons entering this experiment to exit with spin $-\frac{1}{2}$, where the question asks about the "fraction" of electrons.

Q5.11 (a) A beam of electrons pass through a sequence of Stern-Gerlach devices. If after the measurement 25% of the original electrons be in $|+z>$state, this means that probability of electrons exiting the last SG device in $|+z>$ state is $\frac{1}{4}$.

True □ False □

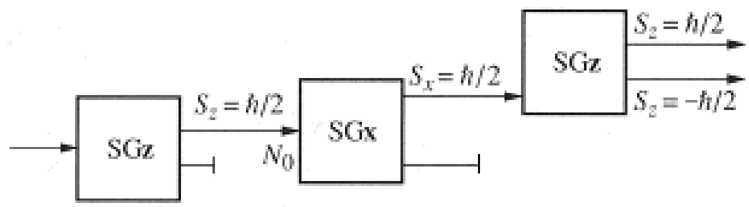
Q5.11 (b) A beam of electrons pass through a sequence of Stern-Gerlach devices. If the probability of measuring $|-z>$ state for the last device be $\frac{1}{2}$, this means that 50% of the original electrons exit this experiment in $|-z>$ state.

True □ False □

Question **Q5.11** is a two-part question that tests students' ability to distinguish between the concept of "probability" and "fraction" of particles. These two statements are both false since the probability and fraction of particles in SG experiments are not necessary the same. However, over 50% of the students (53% in part (a) and 55% for part (b)) decided that these statements are correct and confused the "fraction" of particles with "probability."

In addition to confusing the concept of "probability" with "fraction," we noticed some of the students do not understand a probability measurement should give a unit-less quantity. To elaborate this, consider **Q5.12:** in part (a) we asked about the "fraction" of the particles transmitted through certain parts of the experiment and in part (b) we questioned the "probability" of a certain measurement. Furthermore, in the answer choices for this question, we included choices that involved "No" which stands for the number of particles.

Q5.12 (a) The drawing below shows a sequence of Stern-Gerlach devices. What fraction of the particles transmitted through the first SGZ device will be in $|+z>$ state after the measurements? "No" is the number of particles in the beam exiting the first SGZ device.



1. $\frac{No}{2}$
2. No
3. 1
4. $\frac{1}{4}$
5. $\frac{1}{2}$

(b) In previous question, what is the probability of finding $Sz = -\frac{1}{2}$ for the exiting beam from this experiment?

1. $\frac{No}{2}$
2. No
3. 1
4. $\frac{1}{4}$
5. $\frac{1}{2}$
6. $\frac{No}{4}$

Part (a) in **Q5.12** asks about the "fraction" of the particles, therefore, the correct answer should be $\frac{1}{4}$, choice 4, while part (b) is questioning the "probability," and the correct answer is $\frac{1}{2}$, choice 5. Only 38% of the students answered both part (a) and

Q5.12	Correct	Incorrect	$\frac{1}{2}$ instead of $\frac{1}{4}$	$\frac{No}{2}$ instead of $\frac{1}{2}$
Part (a) fraction	41%	59%	62% of incorrect answers	
Part (b) probability	45%	55%		70% of incorrect answers

Table 5.8: Analysis of students' incorrect responses to Q5.12 shows that students tend to confuse the "friction" of particles with the "probability" of a certain outcome.

(b) correctly. Analysis of students' incorrect choices in Table 5.8 shows that in part (a), 62% of the incorrect answers was $\frac{1}{2}$, that is, the "probability" of the particles exiting in $|+z>$ state instead of the "fraction." In part (b), 70% of the incorrect responses were $\frac{No}{2}$, which includes the "number of the particles" in the answer for probability.

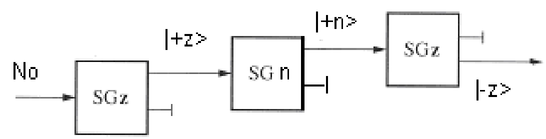
These results have two implications regarding students' specific difficulties with concept of probability. First, students often confuse the concepts of "fraction" with "probability," and, second, some students do not understand that probability is a unit-less quantity. This latter could also result from students' difficulties with understanding normalization. To have a better understanding of the sources of these difficulties, further study and interview of students is required.

Furthermore, we observed that students' performance and the percentage of correct answers drop dramatically when more formalism involved. For example, when we used $<|>$ (ket and bra) notation [see **Q5.13**] the percentage of correct answers for this question was significantly lower than other similar questions in this context [see Table 5.9]; only 17% answered part (a) correctly (choice 4) and 7% answered part (b) correctly (choice 3).

Q5.13 (a). A beam of electrons in state $|+z>$ is sent through a series of three Stern-Gerlach devices. The first, SGZ, transmits only particles in $|+z>$ state. The second device, a SGN device, transmits only particles in $|+n>$ state.

$$|+n>= \cos\frac{\theta}{2}|+z> + \sin\frac{\theta}{2}|-z> \tag{5.20}$$

θ is the angle axis n makes with respect to the z axis in the $x-z$ plane. A last SGZ device transmits only particles in $|-z>$ state. What fraction of the particles transmitted through the first SGZ device will survive the third measurement? "No" is the number of particles in the beam entering the first SG device.



1. $\frac{No}{2}[(\cos\frac{\theta}{2})^2 + (\sin\frac{\theta}{2})^2]$
2. $\frac{No}{2}(\cos\frac{\theta}{2})^2$
3. $\frac{1}{4}(\cos\frac{\theta}{2})^2$
4. $\frac{No}{4}(\sin\frac{\theta}{2})^2$
5. $(\sin\frac{\theta}{2})^2$
6. $\frac{1}{2}(\sin\frac{\theta}{2})^2$

(b) What values of the angle θ of the SGN device maximizes the number of the particles in $|-z>$ state? What is the probability of measuring $|-z>$ for this value of θ?

1. $\theta = 2\pi, Pr(|-z>) = No$, respectively
2. $\theta = 0, Pr(|-z>) = \frac{No}{4}$, respectively
3. $\theta = 2\pi, Pr(|-z>) = 1$, respectively
4. $\theta = 0, Pr(|-z>) = 1$, respectively
5. $\theta = \pi, Pr(|-z>) = \frac{1}{4}$, respectively
6. $\theta = 2\pi, Pr(|-z>) = \frac{No}{4}$, respectively

Q5.13	**Part** (a)		**part** (b)	
	# of Students	%	**# of Students**	%
Correct	5	17	2	7
Incorrect	25	83	28	93
Total	30		30	

Table 5.9: Summary of students' responses to Q5.13

Note that students' responses to **Q5.13**, in addition to confirming their difficulties with greater formalism, are more evidence of their tendency to confuse the "fraction" of particles with "probability" and include "number of particles" in calculation of probability. In part (a), the most common incorrect answer (45%) was choice 5 that is equal to the probability of particles exiting the experiment in $|-z>$ state and not the fraction. In part (b), the most common incorrect answer (43%) was choice 1, and the total of 73% chose answers that involved "No" (choices 1 and 2) for the probability of measuring $|-z>$.

The questions presented in this section are just a few examples of the questionnaires on the SG experiments. These examples indicate that many students entering introductory quantum courses do not have a functional understanding of the concept of probability and a large number of introductory students have difficulties distinguishing between the concepts of "fraction" and "probability."

5.6 SUMMARY

As mentioned at the beginning of this chapter, our goal in this part of study was to seek answers to the following questions:

After standard instruction in math and quantum mechanics:

Do students in introductory quantum courses have a functional understanding of the concepts of probability and probability density?

Do these students posses the appropriate level of abstraction to formalize their understanding of these concepts in mathematical symbols?

Do students have the ability to integrate their conceptual and mathematical understanding to describe the probabilistic behavior of a quantum mechanical system?

Student difficulties with understanding quantum measurement and probability are very complex. They are not limited merely to conceptual difficulties. Even after having a conceptual understanding, most students show difficulties in formalizing their conceptual understandings in terms of mathematical symbols.

Many students have serious difficulties with both qualitative and quantitative aspects of quantum measurement. In the qualitative aspects students do not have a functional understanding of probability and its related terminologies. Many students, even in a classical context, are not able to relate velocity, or the amount of time particles spend in a region, to the probability of finding particles in that region. They often confuse the “fraction” of particles with the “probability” of measuring certain outcomes. Most students have a tendency to confuse the “expectation value” of an operator with the “probability density” in measurement. Furthermore, we found that students’ difficulties with the concepts of probability often interfere with their understanding and applying of the Uncertainty Principle.

In the quantitative aspects the main difficulties seem to be dealing with abstract materials and formalisms. Unfortunately, many of the interview subjects who were comfortable using concrete numbers to answer some of the questions correctly seemed to be uncomfortable with the using abstract symbols to answer questions about probability.

Our findings also show that students often perform better with visual questions as compared to similar but more abstract questions. By reducing the degree of abstraction, for example, a graphical representation often helps students to understand and answer questions better. We have tried to employ these findings in the development of instructional material described in Chapter 8.

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[4] See Chapter 3 of this dissertation for details about the structure of the courses and the student populations.

[5] The topics of delta function and Fourier transform were discussed in detail in the class prior to these questions.

[6] A more detailed discussion of student difficulties with wave functions is presented in the Chapter 5.

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[9] X. Zou, "The Use of Multiple Representations and Visualizations in Student Learning of Introductory Physics: An Example from Work and Energy," Thesis (Ph.D.). The Ohio State University (2000); A. Van Heuvelen and X. Zou, "Multiple Representations of Work–Energy Processes,"American Journal of Physics, **69**, 2, 184 − 194 (2001).

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CHAPTER 6

IDENTIFYING STUDENT DIFFICULTIES IN UNDERSTANDING WAVE FUNCTIONS

"The point of learning is the formation of the new structure, not the accumulation of knowledge. Once the appropriate structure exists, the learner can be said to understand the material and that is often a satisfactory endpoint of the learning process" Rumelhart

6.1 INTRODUCTION

In classical mechanics if we know the mass, m, and the position of a particle at any given time, $x(t)$, we can determine the velocity ($\frac{dx}{dt}$), the momentum ($p = mv$), the acceleration ($a = \frac{dv}{dt}$), and the force by applying Newton's second law ($F = ma$). In quantum mechanics we use the wave function $\Psi(x, t)$ to calculate various quantities of interest. We obtain the wave function by solving the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}. \tag{6.1}$$

Compared to a classical particle, which is localized at a point, the wave function spreads out in space. This and the statistical interpretation of the wave function is disturbing for students. In this chapter, we explore the common difficulties students have with wave function as the probability distribution [1], understanding energy eigenstates, and mathematical difficulties involving the graph of wave functions and the interpretation of the sketch of wave functions in regions with different potentials. We show that after standard instruction in math and physics many students have difficulties calculating the probability distribution, recognizing energy eigenfunctions from the wave functions, associating a given graph with the correct functions, and understanding the time evolution and the stationary states.

This chapter has two main parts. The first part, sections 6.2 – 6.3, discusses students' conceptual difficulties with understanding of quantum wave functions and energy eigenstates. The second part, section 6.4, explores students' mathematical difficulties with the representations of wave functions.

To test students' qualitative and analytical abilities to understand the wave function, we gave several questions to the students taking junior and senior level quantum mechanics courses at The Ohio State University [2]. These questions were given through weekly on line questionnaires that were part of students' homework assignments. The questions that pertained to wave functions can be categorized as follows:

- Understanding the wave function as a probability distribution;
- Recognizing wave functions from energy eigenstates;
- Mathematical difficulties and multiple representations of wave functions.

Below we discuss examples of the questions from each category, the correct answers, students' responses, and the most common difficulties.

6.2 UNDERSTANDING THE WAVE FUNCTION AS A PROBABILITY DISTRIBUTION

Quantum mechanics theory uses Born's Statistical interpretation of the wave function to describe the state of a particle. According to this interpretation, the square of the wave function $|\Psi(x,t)|^2$ gives the probability of finding the particle between x and $x+dx$, at any time t. In other words, $|\Psi(x,t)|^2$ is the probability density for finding the particle at point x at time t. To complete the statistical interpretation, the integral of $|\Psi(x,t)|^2$ all over the space must be 1, giving a definite probability for finding the particle somewhere. This means that the wave function must be normalizable:

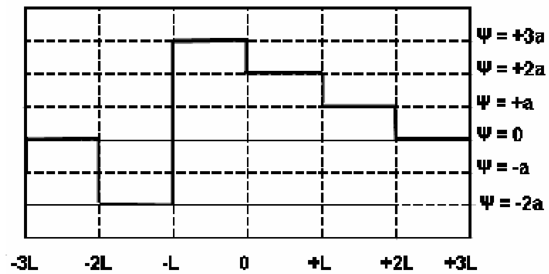
$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1. \tag{6.2}$$

Therefore, physical states of the particle correspond to the "square-integralable" solution of Schrödinger equation.

One simple, but common, problem students have with the calculation of probability density is that they don't square the wave function. This difficulty, perhaps, is due to their classical understanding of probability in which they only need to find the fraction of certain outcomes to the total number of possibilities. Persisting classical views of the world are often big obstacles in student learning of quantum theory [3].

For instance, consider **Q6.1**, where students were asked to calculate the probability density for a discrete wave function. This question was given to both introductory ($P263$) and advanced students ($P631$) after instruction. In addition, students had many homework exercises on calculation of probability density for continuous wave functions.

Q6.1. The figure below shows a plot of a (rather artificial) wave function $\Psi(x)$ versus x, over the range of $(-3L, +3L)$. The wave function vanishes for all other values of x.



What is the probability of finding the particle in the range from $x = 0$ to $x = +2L$?

1. $\frac{3a}{8}$
2. $\frac{3}{8}$
3. $\frac{5}{18}$
4. $\frac{5a}{8}$
5. $\frac{13a}{18}$
6. $\frac{13}{18}$

The analytical solution for **Q6.1**, presented below, involves normalization of the wave function and calculation of the probability wave distribution between 0 and $+2L$:

$$1 = \int_{-\infty}^{+\infty} |\Psi(x)|^2 dx \tag{6.3}$$

$$= \int_{-3L}^{+3L} |\Psi(x)|^2 dx \tag{6.4}$$

$$= [(-2a)^2 + (+3a)^2 + (+2a)^2 + (a)^2]L \tag{6.5}$$

$$= (4 + 9 + 4 + 1)a^2 L \tag{6.6}$$

$$= 18a^2 L, \tag{6.7}$$

therefore,

$$a^2 = \frac{1}{18L}, \tag{6.8}$$

and

$$P[\ 0 \ < \ x < +2L\ ] = \int_{0}^{+2L} |\Psi(x)|^2 dx \tag{6.9}$$

$$= (2a)^2(L) + (a)^2(L) \tag{6.10}$$

$$= 5a^2 L, \tag{6.11}$$

therefore;

$$P[\ 0 < x < +2L\ ] = \frac{5}{18}. \tag{6.12}$$

A faster qualitative solution can be obtained by dividing the probability between 0 and $2L$, the sum of the squares of the wave function in this interval, by the total probability, the sum of the squares of the wave function from $-3L$ to $3L$. This will result in $\frac{5}{18}$ (choice 3) with no need to find a since it cancels out:

$$\begin{aligned}
\text{probability between 0 and } 2L &= (2a)^2 + a^2 && (6.13)\\
\text{total probability} &= (-2a)^2 + (3a)^2 + (2a)^2 + (a)^2 && (6.14)\\
P[\,0 &< x < +2L\,] = \frac{5a^2}{18a^2}, && (6.15)\\
&= \frac{5}{18}. && (6.16)
\end{aligned}$$

More than 50% of the students in both classes missed the correct answer for this question [See Table 6.1]. The most common incorrect choice in both classes was choice 2, $\frac{3}{8}$. Of 54 students in $P631$, 33% picked choice 2 as their answers, and from 49 students in $P263$, 37% also picked this choice. Choice 2, $\frac{3}{8}$, is the result of dividing the units between 0 and $2L$ (3) by the total squares between $-3L$ and $3L$ (8) (not squaring the wave function for each interval to obtain the probability density). This shows that these students did not consider squaring the wave function for each interval to obtain the probability density. Many students (over 20%) in both classes chose answers that involved a (choices $1, 4$, and 6), showing their difficulty in understanding the concepts of probability and the normalization of the wave function. This problem is very similar to students' difficulties in SG experiments discussed in Chapter 5, where students tend to confuse the number of particles with probability and did not understand that probability is unit-less.

Another possible reason for students not squaring the wave function could be their poor understanding of the terminologies involved in probability measurement.

Q6.1	**# of Students** $P631$	%	**# of Students** $P263$	%
Correct	22	41	19	39
Incorrect	32	59	30	61
Total	54		49	

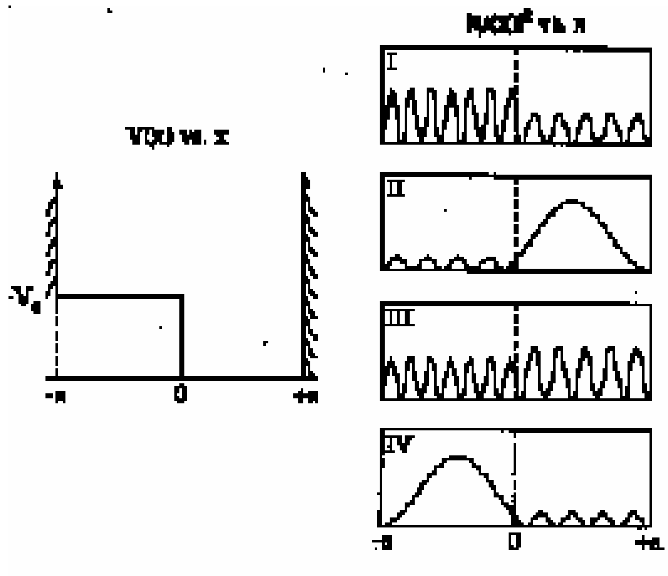
Table 6.1: Summary of students' responses to Q.6.1

For example, as it is discussed in Chapter 5, students have difficulties in distinguishing "probability amplitude" from "probability density" and some students think of probability density as "place" or"area."

In another example, **Q6.2,** we asked students to determine the correct wave function for a given potential well. Question **Q6.2** is a modified version of a question from the Visual Quantum Mechanics Instruments (VQMI). In this test the authors suggest a semi-classical approach for solving this problem and argue that the correct answer for this question is the plot on IV, choice 5 with the reasoning as follows [4]:

> *"An allowed energy state with $E < V_0$ would have most of the probability confined to the right side, with the possibility of tunneling wave functions in the left hand side. (II) is like this except that its behavior for $-a < x, a$ is oscillatory and not damped, so it's not appropriate. Solutions corresponding to $E > V_0$ will have smaller kinetic energies on the left side, and hence have (i) larger amplitudes (more likely to find it there since it's moving slowly) and (ii) fewer 'wiggles' on the left. Choices I and III have more 'wiggles' on the left side and are consistent with the classical dynamics of the system."*

Q6.2. The plot below shows a potential energy function, $V(x)$ versus x, corresponding to an "asymmetric" infinite well. The infinite well is of width $2a$, with impenetrable walls at $x = +a$, but where $V(x) = +V_0$ for x between $(-a, 0)$ and $V(x) = 0$ for x between $(0, +a)$.



Of the figures above which is/are most likely to be physically acceptable energy eigenstate solutions for the time-independent Schrödinger equation for this well? Explain your reasoning.

1. I and IV only
2. II only
3. I only
4. II and III only
5. IV only

Although qualitative arguments similar to the one above were discussed in the class, more than $\frac{2}{3}$ of the students in $P631$ and $\frac{1}{3}$ in $P263$ missed the correct answer [see Table 6.2]. The most popular incorrect answer in both classes was plot II. In a follow-up interview some of the students argued that the larger kinetic energy on

Q6.2	**# of Students** $P631$	%	**# of Students** $P263$	%
Correct	13	33	19	38
Incorrect	26	67	18	62
Total	39		37	

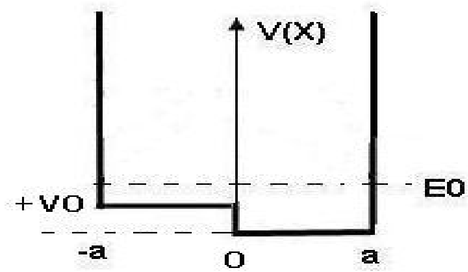
Table 6.2: Summary of students' responses to Q.6.2

the right side gives a larger amplitude to the wave function. This reasoning clearly has roots in students' knowledge of the classical waves. This is another example of students using their classical knowledge to explain quantum systems. It is very common for introductory students to use classical models to describe the quantum domain, which is not familiar to them. This observation is in agreement with previous research on students' understanding of quantum mechanics [5].

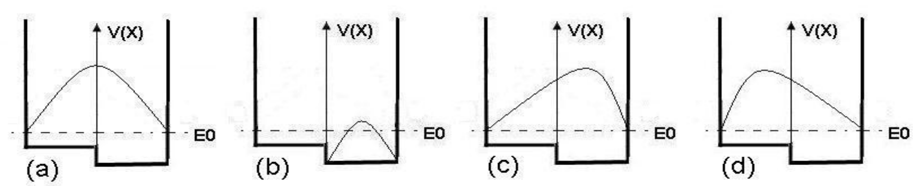
It is interesting to note that although the students in $P631$ have already seen and discussed this problem when they were taking $P263$, still did not answer the question correctly. This has two implications: first, it shows the degree of difficulty students have associating the wave function with the kinetic energy of a particle, and second, it suggests that the qualitative reasoning based on time and velocity can be forgotten and perhaps does not constitute an in-depth conceptual understanding.

Another interesting observation is that the $P631$ students scored slightly better on **Q6.2** when they were enrolled in $P263$ the previous spring [see Table 6.2]. This means some of these students who had a correct answer in the spring quarter for **Q6.2**, missed the correct answer in the fall quarter. This is probably because the introductory course $P263$ had more emphasis on the qualitative aspects of quantum mechanics than mathematical processes and students in this course had encountered similar questions to **Q6.2** more than when they were taking $P631$. This is more evidence that students' qualitative reasoning does not always have strong roots in their understanding and can be memorized and be forgotten after a short time.[4]

Q6.3. The Plot below shows a potential energy function, $V(x)$ versus x, corresponding to an "asymmetric" infinite well. The infinite well is of width $2a$, with impenetrable walls at $x = \pm a$, but where $V(x) = +V_0$ for x between $(-a, 0)$ and $V(x) = 0$ for x between $(0, +a)$.



Of the figures below, which is/are most likely to be physically acceptable energy eigenstate solutions for the time-independent Schrödinger equation for this well? Please explain your reasoning.



Some students used a semi-classical argument in answering **Q6.2**. According to this argument the particle has smaller kinetic energy and velocity on the left side of the well; therefore it spends more time there and the amplitude must be larger in this region. Although this semi-classical approach, based on the time particles spend in each region, is often used in qualitative answers to many quantum questions involving

[4] Similar results have been observed on repeated conceptual questions on probability (see Chapter 4) where students did better in the spring quarter, P263, compared to the same students in the fall quarter, P631.

wave functions in introductory level courses to make answering somewhat simple, this approach is not always the best way to obtain the correct answer especially if one does not have a deep understanding of the relation between the shape of a wave function and the wave number. For example, **Q6.3** shows how using a semi-classical approach can lead to an incorrect answer. Arguing that a larger kinetic energy domain would have a higher amplitude for the wave function can lead to selecting choice (d) in **Q6.3**, which is an incorrect answer.

The larger kinetic energy on the right side results in a larger wave number and smaller wavelength for the wave function. Therefore, as long as the width is equal on both sides of the step, the wave function has to curve down before the middle boundary on the right side to give a larger wave number for the function on the right side as compared to the one on the left side. Only (c) is in agreement with this principle and satisfies the boundary conditions. Choice (a) is incorrect since it results an equal wave number and thus equal kinetic energy in both sides. Choice (b) is incorrect: in general for E_0 larger than the potential on the left side the probability of finding particles on the left side should not be zero; however, to meet the boundary conditions here, the ground state should have energy larger than V_0. Finally, choice (d) reflects a higher kinetic energy and wave number on the left side.

We gave **Q6.3** to the students in the advance quantum course, $P631$, only. The percentage of correct answers was approximately half of what we had for **Q6.2.** Results in Tables 6.2 and 6.3 show that of the students answering **Q6.2** correctly, about half did not answer **Q6.3** correctly and chose answer (d) instead. On a follow-up interview we learned that these students used semi-classical reasoning based on the time particles spend in each region and interpreted the bump on the left side in choice

Q6.3	**# of Students** $P631$	**%**
Correct	7	17
Incorrect	34	83
Total	41	

Table 6.3: Summary of students' responses to Q.6.3

(d) as higher probability. These students did not show any understanding of the relation between the wave number and the wave function. Questions such as **Q6.2,** which can be easily answered by simple semi-classical reasoning based on the time a particle spends on each region, can overstate students' real understanding.

6.3 RECOGNIZING WAVE FUNCTIONS FROM ENERGY EIGENSTATES

One of the challenging concepts for students learning quantum mechanics is understand the difference between wave functions and energy eigenstates. Wave functions and energy eigenstates are abstract and unfamiliar concepts for most students starting quantum courses. Many students do not have the ability to distinguish these two concepts and often fail to answer related questions correctly. These students usually confuse wave functions with energy eigenstates. The following three research questions demonstrate the details of student difficulties with these two concepts.

Q6.4 (a). The wave function for a particle in a box is given at $t = 0$ by:

$$\Psi(x, t = 0) = c_1\psi_1(x) + c_2\psi_2(x). \tag{6.17}$$

Where ψ_i are energy eigenstates for the particle in a box: $H\psi_i = E_i\psi_i$. Is $\Psi(x, t = 0)$ an eigenstate of H? Why?

Question **Q6.4** is an example of the type of questions we gave to students taking physics $P631$ in fall 2004 at OSU. To answer this question correctly one needs to: (i) distinguish the wave function Ψ, and energy eigenstates ψ_i and (ii) understand that for Ψ to be an eigenstate for Hamiltonian H, the following relationship must stand:

$$H\Psi = E\Psi. \tag{6.18}$$

However, here $\Psi(x, t = 0)$ does not satisfy (6.18) since we have:

$$H\Psi(x, t = 0) = Hc_1\psi_1(x) + Hc_2\psi_2(x), \tag{6.19}$$

$$H\Psi(x, t = 0) = E_1c_1\psi_1(x) + E_2c_2\psi_2(x), \tag{6.20}$$

and therefore, in general $\psi(x, t = 0)$ is not an eigenstates of H :

$$H\Psi(x, t = 0) \neq E(c_1\psi_1(x) + c_2\psi_2(x)). \tag{6.21}$$

Here, only if ψ_1 and ψ_2 were degenerate states with the same energy, $E_1 = E_2 = E$ in (6.20), the wave function $\Psi(x, t = 0)$ would be an eigenstate of Hamiltonian H.

Of the 35 students answering this question, 12 students (34%) said, correctly, that $\Psi(x)$ is not an eigenstate of H since it does not satisfy $H\Psi = E\Psi$. Eight students

(23%) agreed that $\Psi(x)$ is not an eigenstate of H but had vague reasons for their arguments. Some examples of these students' reasons are as follows:

S1: *"No, because $\Psi(x, t = 0)$ is a superposition of two waves and you cannot get a real value from measurement for function that is in a superposition."*

S2: *"No, this will be an eigenfunction."*

S3: *"Not necessarily, because the second half of the wave function may not be an eigenstate and thus add a part to the function that is not $E_i\psi_2$."*

The remaining 15 students, for variety of different reasons, argued incorrectly that $\Psi(x, t = 0)$ in equation 6.17 is an eigenstate of Hamiltonian H with reasoning such as:

S4: *"Yes, because it yields an eigenvalue with the original eigenstate."*

S5: *"Since $H\Psi = E\Psi$, I'd say that Ψ is an eigenstate of H because you get the state back after the operation."*

S6: *"Yes, because after applying the operator H to Ψ a scalar multiple of the same state is the result."*

S7: *"Yes, because this is the definition of energy eigenstates."*

S8: *"Yes, because it is a valid variation of the wave function."*

S9: *"Yes, because it meets all of the specifications for an eigenstate. It has two constants and meets the Schrödinger eq."*

S10: *"Yes, because they are the eigenvalues."*

S11: *"Yes, because the Hamiltonian will produce eigenvalues."*

S12: "$\Psi(x, t = 0)$ *would be an eigenstate of* H *if* H *is an allowed eigenstate in this region."*

S13: *"Yes, any linear combination of eigenstates is also an eigenstate."*

S14: *"Yes, because they are normalized."*

Some of these students distinguished the wave function Ψ and energy eigenstates ψ_i, but they were not able to investigate the equation 6.18 thoroughly. Examples of these students are **S4**, **S5**, and **S6**. On the other hand, other students' reasoning reflected a larger confusion with these concepts (**S7** to **S14)**. For example, the argument in **S13** confuses the wave function with the eigenstate, and **S14** argues that since Ψ is normalized it is an eigenstate of the Hamiltonian.

Q6.5. What is the first thing you should do when asked for $\Psi(x, t)$ given $\Psi(x, t = 0)$?

In another research question we observed that most students do not recognize that in order to find the wave function in a later time, first, they need to write the wave function $\Psi(x, t = 0)$ in terms of its energy eigenfunctions. Students' responses to questions **Q6.5** and **Q6.6** illustrate this further. The first question, **Q6.5,** is a free response question we gave to the students in $P631$, of which only 6 students (17%) wrote the correct answer. The remaining 29 students (83%) had basically no idea how to answer this question. Some of students' incorrect responses are:

Q6.6. The wave function for a particle in a box is given at $t = 0$ by:

$$\Psi(x, t=0) = c_1\psi_1(x) + c_5\psi_5(x) + c_7\psi_7(x), \tag{6.22}$$

where ψ_i are energy eigenstates for the particle in a box, $H\psi_i = E_i\psi_i$. What is $\Psi(x, t > 0)$?

1. $\Psi(x,t) = E_1\psi_1(x)e^{-iEt/\hbar} + E_5\psi_5(x)e^{-iEt/\hbar} + E_7\psi_7(x)e^{-iEt/\hbar}$
2. $\Psi(x,t) = c_1\psi_1(x)e^{-iE_1t/\hbar} + c_5\psi_5(x)e^{-iE_5t/\hbar} + c_7\psi_7(x)e^{-iE_7t/\hbar}$
3. $\Psi(x,t) = E_1\psi_1(x)e^{iE_1t/\hbar} + E_5\psi_5(x)e^{iE_5t/\hbar} + E_7\psi_7(x)e^{iE_7t/\hbar}$
4. $\Psi(x,t) = E\psi(x)e^{-iEt/\hbar}$
5. $\Psi(x,t) = E\psi(x)e^{iEt/\hbar}$

S1: *"Check to see if the Hamiltonian is cyclic in time. If it is, then* $\Psi(x,t)$ *is trivially found."*

S2: *"...find the time dependent relationship between them."*

S3: *"In 263 I'm pretty sure we had an equation for the time-evolution of an eigenstate but all I remember is that it had a bunch of exp(i times theta) and stuff like that."*

S4: *"...plug* $t = 0$ *in and find the starting point of the wave function."*

S5: *"...put it in Dirac notation."*

S6: *"...you should square it to find the probability density."*

S7: *"Panic!!! You think I'm kidding but honestly I'm not sure."*

The second question, **Q6.6**, is a similar question to **Q6.5** in a multiple choice format. Of the 48 students answering this question [6], 18 of them (36%) picked the correct answer, choice 2, and of the 30 (64%) incorrect answers, the majority of them (23) were choices 4 and 5. One possible reason for choosing answers 4 or 5 is that students treat the wave function Ψ as an eigenstate of the energy and do not recognize they need to write out the wave function in terms of its energy eigenfunctions before finding its time evolution. Throughout this study we have observed similar mistakes on students' homework and exam papers as well as on their responses to other research questions.

Q6.7. The wave function for a particle at time $t = 0$ happens to be identical to the harmonic oscillator ground state energy eigenfunction:

$$\Psi(x, t = 0) = \varphi_0(x) = Ce^{-\frac{ax^2}{2}} \tag{6.23}$$

(a) What happens to $|\Psi(x,t)|^2$ at later times if the particle is under the influence of a harmonic oscillator potential? $H = H$ (1D harmonic oscillator)

1. $|\Psi(x,t)|^2$ becomes $\varphi_0(x)e^{-iE_0t/\hbar}$.
2. $|\Psi(x,t)|^2$ becomes a broader Gaussian.
3. $|\Psi(x,t)|^2$ is time independent.
4. $|\Psi(x,t)|^2$ oscillates with time.

(b) What happens to $|\Psi(x,t)|^2$ at later times if the particle is free?

1. $|\Psi(x,t)|^2$ becomes $\varphi_0(x)e^{-iE_0t/\hbar}$.
2. $|\Psi(x,t)|^2$ becomes a broader Gaussian.
3. $|\Psi(x,t)|^2$ is time independent.
4. $|\Psi(x,t)|^2$ oscillates with time.

To test students' ability to recognize that it is crucial for a wave function to be written in terms of its energy eigenfunctions before one can determine the wave function in a later time, we asked students in $P631$ question **Q6.7**. To answer part (a) of this question, one needs to recognize that since the wave function is an eigenstate of Hamiltonian H (1D harmonic oscillator), this state is a stationary state. Thus, the probability density of position is time independent:

$$\Psi(x) = \varphi_0(x) = Ce^{-\frac{ax^2}{2}}, \tag{6.24}$$

$$\Psi(x,t) = (Ce^{-\frac{ax^2}{2}})e^{-iE_0t/\hbar}, \tag{6.25}$$

$$|\Psi(x,t)|^2 = |\Psi(x)|^2 = |C|^2e^{-ax^2}. \tag{6.26}$$

However, if the wave function is not the eigenfunction of the Hamiltonian (free particle), which is the case in part (b), the initial Gaussian wave function of the free particle, which is a superposition of the energy eigenfunctions, flattens with time [7]. This is due to the cross terms in the calculation of $|\Psi(x,t)|^2$:

$$\Psi(x) = Ce^{-\frac{ax^2}{2}}, \tag{6.27}$$

$$= \sum C_n\varphi_n(x) \tag{6.28}$$

$$\Psi(x,t) = \sum C_n\varphi_n(x)e^{-iE_nt/\hbar} \tag{6.29}$$

Therefore, the correct answer for part (a) of **Q6.7** is choice 3 and for part (b) it is choice 2.

Most of the students (61%) answered part (a) correctly, however, in part (b) 64% of the students did not show understanding of the difference between wave functions and energy eigenfunctions. Therefore, they failed to recognize that the wave function in equation (6.23) is not an eigenfunction of a free particle Hamiltonian; therefore, the

Q6.7	Part (a)		Part (b)	
	# of Students	%	# of Students	%
Correct	25	61	15	36
Incorrect	16	39	26	64
Total	41		41	

Table 6.4: Summary of students' responses to Q.6.7

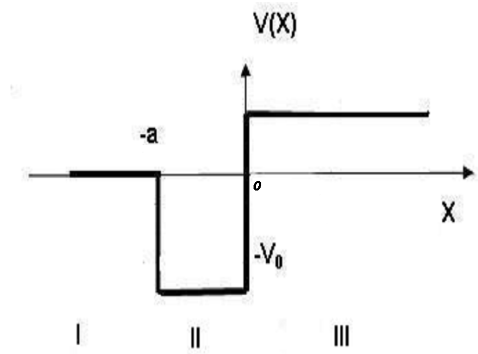
probability density for this case is time dependent [see Table 6.4]. Of the 26 incorrect responses to part (b) of **Q6.7**, more than half were choice (3). This incorrect answer implies that these students perhaps calculated the time evolution of the wave function for the free particle to be $\varphi_0(x)e^{-iE_0t/\hbar}$, which would result a constant probability density $|\Psi(x,t)|^2$.

6.4 MATHEMATICAL DIFFICULTIES AND MULTIPLE REPRESENTATIONS OF WAVE FUNCTION

In the following questions we probed students' mathematical knowledge of graphing exponential and imaginary functions in the context of quantum potential wells. The goal was not to measure students' ability to graph the correct qualitative wave function, but rather to investigate their ability in associating a given graph with the correct function and vice versa.

One motivation for this study was our observation of students' difficulties with some mathematical topics such as complex numbers and graphing real and complex functions in the context of their quantum mechanics courses. For example, during interview tasks many students graphed a function such as $e^{-i\theta}$ as if it was an exponential decay function and failed to recognize its oscillatory nature. Examples of students' difficulties with graphing complex functions in interview tasks are presented in Appendix **D**.

Q6.8. The figure below shows a plot of a potential $V(x)$ versus x where V_0 is a positive constant. Which one of the equations below represents an acceptable physical wave function for a particle with $E < 0$ in region I ($x < -a$)? ξ is a positive real number.



1. $\pm e^{\xi x}$
2. $\pm e^{-\xi x}$
3. $\pm e^{-i\xi x}$
4. $e^{\xi x}$
5. $e^{\pm\xi x}$
6. $e^{i\xi x}$
7. $e^{\pm i\xi x}$

The question demonstrated on **Q6.8** is an example of the mathematical questions we used to test our initial observations of students' difficulties. To answer **Q6.8** one needs to know that (i) if Ψ is a correct wave function for the region I, then $-\Psi$ is a correct answer as well, and (ii) in region I of **Q6.8** we need to have an increasing exponential function that goes to zero on the far left side and can match up to the function in region II on the boundary condition on the right side. Of 41 students,

Q6.8 Choices	**# of Students** $P631$	%
1	12	29
2	8	20
3	2	5
4	7	17
5	5	12
6	2	5
7	5	12
Total	41	

Table 6.5: Details of students' responses to Q.6.8; choice 1 is the correct answer

only 12 gave the correct answer to **Q6.8**, choice 1, and 7 students picked a partial correct answer, choice 4. The latter ones suggested $e^{\xi x}$ as the correct answer but did not agree that $-e^{\xi x}$ was a correct answer as well.

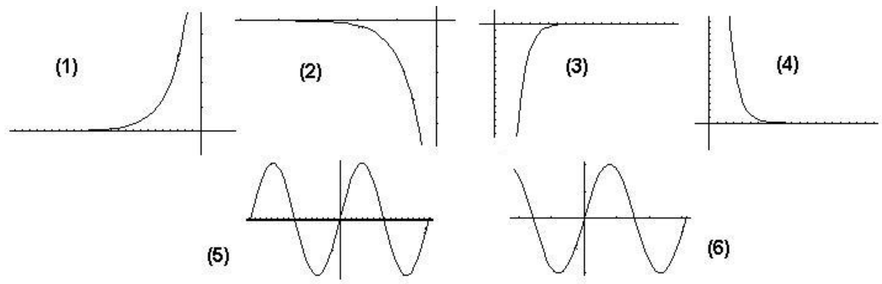
The other 23 students (54%) chose completely incorrect answers. Approximately 20% of the students did not recognize that the correct equation to represent the wave function in region I is not an exponential decay and picked choice 2. A more interesting problem was that 22% of students suggested a complex exponential (choices 3, 6, and 7) for the wave function in region I. The remaining 12% suggested that we can have both exponential decay and increase in region I by choosing answer 5 [see Table 6.5].

Since the students in $P631$ who answered **Q6.8** had worked, on many occasions, on the qualitative sketch of the wave function for similar potential wells in $P263$, we believe that the sketch of the wave function was not the problem.[5] Furthermore, to make sure that the students knew what the wave function in this problem should look like we asked question **Q6.9** right after **Q6.8.** The results convinced us that

[5]These students have also worked on a special worksheet (presented in Chapter 8) focused on sketching wave functions while taking $P263$.

recognizing the shape of the wave function was not the reason for most of incorrect answers to **Q6.8.**

Q6.9. Which one of the plots below is/are an acceptable physical wave function for a particle with $E < 0$ in region III $(0 < x)$ in question **Q6.8**?



1. (1) and (3)
2. (3) and (4)
3. (5) and (6)
4. only (2)
5. only (4)

In question **Q6.9,** we asked students to pick the correct sketch for the wave function in region III of **Q.6.8**. Comparing the results in Tables 6.5 and 6.6 shows the percentage of the correct answers in **Q6.9** (83%, Table 6.6) increased dramatically compared to **Q6.8** (29%, Table 6.5)**.** This boost in correct answers confirms that most students knew the graphical shape of the correct wave function for the potential well in these questions, but were not able to associate the correct equation with the shape they knew**.** This examples show students' difficulties with the multiple representation of wave functions: remembering a graphical representations of wave functions does not prove students' ability to recognize mathematical symbols for the same functions. In addition, this is another evidence that students score better in visual questions.

Q6.9	**# of Students** $P631$	%
Correct	34	83
Incorrect	7	17
Total	41	

Table 6.6: Summary of students' responses to Q.6.9

Choice 5 was the most commonly selected incorrect answer for **Q6.9**. Students who picked choice 5 suggested correctly that the decay function in graph number 4 was the correct answer but they failed to recognize that graph number 3 is also a correct answer, which is produced by reflection of graph number 4 with respect to x axis. In another words, the function presenting graph number 3 is the same function in graph number 4 with a minus sign in front.

To further illustrate the depth of students' difficulties in this topic, consider question **Q6.10** as an example of students' mathematical ability in distinguishing between exponential and oscillatory functions in solution to a differential equation that appears frequently in the solution of the Schrödinger equation. In this question students were asked to suggest the general solution for the equation (6.33). Since here k is a positive and real number, the second derivative of the function must be equal to the positive multiple of the function itself; therefore, regardless of boundary conditions, the answer is an exponential, choice 2:

$$\frac{d\Psi(x)}{dx} = -kA\exp(-kx) + kB\exp(kx), \tag{6.30}$$

$$\frac{d^2\Psi(x)}{dx^2} = k^2(Aexp(-kx) + Bexp(kx)), \tag{6.31}$$

$$\frac{d^2\Psi(x)}{dx^2} = k^2\Psi(x). \tag{6.32}$$

Q6.10. What is/are the general solutions for differential equation below?

$$\frac{d^2\Psi(x)}{dx^2} = k^2\Psi(x) \tag{6.33}$$

k is real and positive.

1. $\Psi(x) = Aexp(-ikx) + Bexp(ikx)$
2. $\Psi(x) = Aexp(-kx) + Bexp(kx)$
3. $\Psi(x) = Asin(kx) + Bcos(kx)$
4. 1 and 2
5. 2 and 3
6. 1 and 3
7. It depends to the boundary conditions

Although we asked **Q6.10** after students had a lecture on the solutions of the Schrödinger equation for a free particle and a particle in a box, more than half of the students gave an incorrect answer to this question. The analysis of students' responses to this question, in Table 6.7, not only shows students' difficulties in solving this differential equation (overall 51%), but also indicates students' difficulties in distinguishing between an exponential and oscillatory function. For example, 22% of the students failed to demonstrate that they can make this distinction by choosing answers 4 or 5. In addition, 17% of the students did not recognize that answers 1 and 3 demonstrate the same function and chose only one of them.

Q6.10 Choices	**# of Students** $P631$	%
1	3	7
2	20	49
3	4	10
4	5	12
5	4	10
6	0	0
7	5	12
Total	41	

Table 6.7: Details of students' responses to Q.6.10, choice 2 is the correct answer

6.5 CONCLUSION

Our findings show that some students have difficulty with the concept of the wave function as a probability distribution. For example, students tend to bring their knowledge of classical waves to the quantum regime by associating a higher energy to a large amplitude in a wave function. This is not very surprising since wave mechanics is a familiar environment for students, and to some extent, has the largest overlap with what students learn in quantum physics.

There exist difficulties with the concept of probability when it comes to calculating a probability density from a given wave function (also see Chapter 4). In addition, students do not recognize that the probability density is the square of the wave function and, as a common mistake, they do not consider squaring the wave function before finding the probabilities.

Our findings also suggest that students have difficulty differentiating wave functions from energy eigenstates. Students' lack of distinction in this area often leads to difficulties in understanding time evolution of the wave function; thus, most students do not understand the need for writing the wave function in terms of its energy eigenfunctions before determining the wave function in a later time.

Some students' do not use the relationship between wave functions and wave numbers as the primary principle for qualitatively analyzing the shape of wave functions in regions of different potential. In the sketch of wave functions a semi-classical reasoning based on the time particles spend in one region alone does not seem to represent students' conceptual understanding of the relationship between wave functions and the wave numbers. Qualitative reasoning that is not strongly rooted in understanding can be memorized and quickly forgotten.

In the mathematical aspects of students' difficulties with the topics of wave function we found that many students have difficulty recognizing mathematical symbol for a given graph. For instance, they do not recognize an oscillatory function such as e^{-ix} from an exponential decay function such as e^{-x}. They lack the ability to associate the correct functions with their graphs.

BIBLIOGRAPHY

[1] Students' specific difficulties regarding understanding probability and quantum measurement are discussed in Chapter 5 of this dissertation.

[2] See Chapter 3 for details about the structure of the course and the student population.

[3] L. Bao and E. Redish, "Understanding Probabilistic Interpretation of Physical Systems: A Prerequisite to Learning Quantum Physics," American Journal of Physics, **70** (3) (2003).

[4] This question is chosen from VQMI by E. Cataloglu and R. Robinett, "Testing the Development of Student Conceptual and Visualization Understanding in Quantum Mechanics Through the Undergraduate Career," Gordon Conference on Research and Science Education: Quantum Mechanics (2002).

[5] L. Bao, "Dynamics of Student Modeling: A Theory, Algorithms, and Application to Quantum Mechanics," Ph.D. dissertation, University of Maryland (1999).

[6] The different number of the students reported for **Q6.5** and **Q6.6** is due to the fact that these two questions were asked at two different times (questionnaire 3 and questionnaire 4 respectively, $P631$ fall 2004). Therefore, there were a different number of students who responded to these questions.

[7] The wave packet will reform exactly into the initial state after a time T such that:

$$\frac{n^2\pi^2\hbar}{2mL^2}T = \text{integer multiple of } 2\pi, \tag{6.34}$$

$$T = \frac{4mL^2}{\hbar\pi n^2}. \tag{6.35}$$

The time-dependency of the wave packet is given by $e^{-iE\frac{t}{\hbar}}$; at a time T the exponential factors will all return to unity ($e^{-iE\frac{t}{\hbar}} = e^{-i2\pi} = 1$) and the initial state will be recovered exactly.

CHAPTER 7

IDENTIFYING STUDENT DIFFICULTIES IN UNDERSTANDING SYMMETRY

"... it is impossible to explain honestly the beauties of the laws of nature in a way that people can feel, without their having some deep understanding in mathematics. I am sorry but this seems to be the case."
Richard Feynman

7.1 INTRODUCTION

In this chapter, we discuss student understanding of the geometrical symmetry and its advantages in solving some potential well problems in quantum mechanics. In addition, we explore student understanding of the relation of odd or even functions and symmetry. Symmetry is an important property of nature, and a good understanding of its qualities not only makes problem solving a simpler and shorter task, but it also introduces students to the idea of parity.

Students are first introduced to the concepts of symmetry and its properties early in their study of math courses, and are later taught to use properties of symmetry in their physics courses. In early college courses, students see the advantages of using symmetry in solving problems in mechanics, geometrical optics, and electromagnetism. For example, students use features of symmetry to simplify the calculation of an electric field near a symmetric charged object. In advanced courses, students are expected to recognize the symmetric aspects of a given system and use these features to simplify the solution.

In this chapter, we show that after standard instruction in math and physics, many students have difficulties recognizing the existence of symmetry in a given system, and fail to apply the properties of symmetry to simplify the solution to the problem at hand. We suspected that lack of appropriate mathematical understanding of symmetry and inability to transfer math knowledge to the quantum mechanics regime are important factors in creating these difficulties. Therefore, we studied students' mathematical background on this topic and then we tested the same students' performances on the symmetry-related questions given on the final examination. At the end of this chapter, we will analyze students' individual responses to math and quantum questions to see if there is any relationship between their math background and achievements in quantum courses. Here, we address the following questions that are part of the focus in our investigation of students' understanding of quantum mechanics: After standard instruction in math and quantum mechanics:

Are students able to recognize the presence of symmetry in a given system?

Are students able to take advantages of symmetry in solving quantum mechanics problems?

Do students possess the appropriate math knowledge of symmetry and are they able to use this knowledge in quantum physics?

7.2 OVERVIEW OF RESEARCH

We gave several symmetry-related questions to the students taking junior and senior level quantum mechanics courses ($P631$) at The Ohio State University. (See Chapter 3 for details about the structure of the course and the student population). These questions were designed to test both students' qualitative and quantitative reasoning ability in the context of symmetric wave functions and potential well in quantum physics and related mathematical topics. Most of these questions were posed through on line weekly questionnaires, some on course examination, and others on homework assignments. The results in this chapter are obtained from the analysis of students' responses to our research questions.

7.2.1 RESEARCH TASKS

We gave different types of questions to students in order to identify their difficulties with the concepts of symmetry. Below we provide description of these questions and the required concepts to answer them correctly.

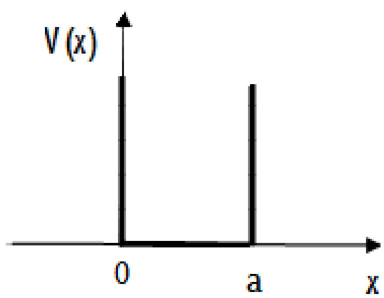
Please note that some of the questions in this chapter can be analyzed for students' difficulties on other quantum mechanical concepts such as difficulties associated with the sketch of wave functions (see Chapter 5). However, here we do not discuss those aspects of the questions and mainly focus on the students' difficulties with use of symmetry [1].

I. Questions about symmetry in an infinite potential well

Consider the following two potential well questions. The first one, **Q7.1,** deals with solving the time-independent Schrödinger equation for a particle in a square potential well with walls at $x = 0$ and $x = a$. The second question, **Q7.2,** asks the solution for the time-independent Schrödinger equation for an infinite square well centered at the origin.

Q7.1. An electron is confined to a one-dimensional, infinitely deep potential energy well of width a depicted below.

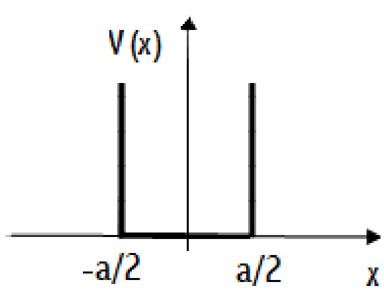
$$V(x) = \begin{cases} 0 & \text{for} \quad 0 < x < a \\ +\infty & \text{for} \quad x < 0, x > a \end{cases}$$



Solve the time-independent Schrödinger equation with appropriate boundary conditions for this square well.

Q7.2. How does your answer change for the infinite square potential well centered at the origin:

$$V(x) = \begin{cases} 0 & \text{for} \quad -\frac{a}{2} < x < \frac{a}{2} \\ +\infty & \text{for} \quad |x| > \frac{a}{2} \end{cases}$$



The first question is the "standard" infinite well, discussed both in the textbook [2] and the lecture, and the second one was part of homework assignment. The standard analytic solution to these kinds of problems, in brief, is as follows:

1. Solving the time-independent Schrödinger equation in position representation;
2. Applying the appropriate boundary conditions;
3. Normalizing the wave function to ensure the probability interpretation is valid.

Taking advantages of symmetry in the second question makes solving the problem much easier. Solving this problem requires students to know that in situations where $V(x) = V(-x)$, it is often convenient to solve separately for even and odd solutions: $\Psi_{even}(x) = \Psi_{even}(-x)$ and $\Psi_{odd}(-x) = -\Psi_{odd}(x)$. Boundary conditions need then only be imposed at one end, e.g., $x = \frac{a}{2}$. In addition, students are expected to choose the appropriate periodic odd or even wave function (sin or cos) based on the boundary conditions. For example, in the first question where we expect the wave function to vanish at $x = 0$ and $x = a$, from the general solution $A\sin(kx) + B\cos(kx)$, ($k = \frac{2n\pi}{a}$), one should choose the sin function ($B = 0$) since it vanishes at the $x = 0$ and $x = a$ for any values of $n = 0, 1, 2, 3, \ldots$. However, the case of the potential well centered at the origin is not as simple. To solve this problem independently, for the even solutions, $\Psi_{even} = Bcos(k_n x)$; ($k_n = \frac{(2n-1)\pi}{a}$), which vanishes at $\pm\frac{a}{2}$ and for the odd solutions, $\Psi_{odd} = A\sin(k_n x)$, ($k_n = \frac{2n\pi}{a}$), which also vanishes at $\pm\frac{a}{2}$. In addition, the two potentials and their solutions can be obtained from each other by shifting the center to origin (substituting $x \rightarrow x - \frac{a}{2}$). The solutions for the potential well centered at the origin have additional properties of evenness and oddness due to the symmetric feature of the well. The complete solutions to **Q7.1** and **Q7.2** are presented in Appendix **E.**

In the study of students' homework and one-on-one interactions during office hours, we observed that many students after instruction on the first problem, through lecture and the textbook, were not able to solve the second problem. Some common problems were:

1. Some students have realized the new boundary condition and chose to have only a cosine function as their solutions and they disregarded the odd solutions;

2. Others seemed lost in the middle of unnecessary complicated mathematics;

3. Most students presented the same solution of the first problem to the second one with a reasoning similar to the following statement:

   "The shift of the coordinate system does not change the physical system, therefore it does not change the answer to the Schrödinger equation."

These observations, along with other difficulties students have shown on the related topics, motivated us to develop targeted questions to study whether or not these difficulties are related to student mathematical difficulties.

II. Questions about energy levels and the wave function in a symmetric infinite potential well

Question **Q7.3** is an example that probes student understanding of a symmetric potential well, the type of wave functions (odd- even), and corresponding energy levels it can have. Students are expected to recognize the presence of odd and even wave functions in the symmetric potential well shown in graph (a), and realize that in case (b), only antisymmetric wave functions survive. Cutting the potential well in half, eliminates the symmetric wave functions in potential well (b) and the possibility of corresponding even energy levels $(E0, E2, E4, E6, ...)$.

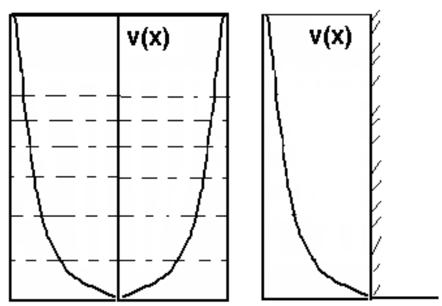
Q7.3. Figure (a) below shows a one-dimensional potential energy function given by $V(x) = kx^2$.The 6 lowest allowed energy eigenstates are shown by dashed lines. Figure (b) shows the "half well" version of this potential, where

$$V(x) = \begin{cases} \infty & \text{for} \quad x > 0 \\ kx^2 & \text{for} \quad x < 0 \end{cases}$$

The energy eigenvalues for potential (a) are given:

$$E = (E_0, E_1, E_2, E_3, E_4, E_5, E_6, ...)$$

What is the most likely pattern of energy eigenvalues in the "half-well" potential on the "half well" (b) on the right ?



(a) (b)

1. The same energy spectrum.
2. $E = (E_1, E_3, E_5, ...)$
3. $E = (E_0, E_2, E_4, E_6, ...)$
4. $E = \frac{1}{2}(E_0, E_1, E_2, E_3, E_4, E_5, E_6, ...)$
5. $E = 2(E_0, E_1, E_2, E_3, E_4, E_5, E_6, ...)$

We gave this question to students who were enrolled in the upper-level undergraduate quantum course, $P631$, after instructions on variety of potential wells including infinite potential well and harmonic oscillator. Only 27% of students picked the correct answer, choice 2. Approximately one-third of students (30%) believed that the half-well potential (b) would have all the energy levels of the full well (a) with twice the values, choice 5. In a follow-up interview with these students we learned the main source of this mistake. The common reasoning was as follows:

> *"Since the well (b) has half width of the well (a), so the wavelength will be half and the wave number will be doubled. That means the energy levels will be doubled."*

It is true that if we reduce the width of an infinite square potential well and keep the same boundary conditions, the energy levels will increase due to decrease of the wavelength for the corresponding energy levels:

$$E = \frac{n^2\pi^2\hbar^2}{2ma^2}, \text{ if } a \rightarrow \frac{a}{2} \Rightarrow E \rightarrow 4E. \tag{7.1}$$

This reasoning is incorrect in the case of **Q7.3**. First, one has only considered the width change of the well and ignored the boundary conditions. Second, this reasoning results in quadrupled energy levels for case (b). Finally, the infinite potential well in this problem is not a square one; although in the case of a harmonic oscillator, similarly, the relation between energy levels ($E = (n + \frac{1}{2})\hbar\omega$, where $\omega = \sqrt{\frac{k}{m}}$) and the width of the potential, $V(x) = \frac{1}{2}m\omega^2 x^2$, is not a linear relationship.

These students have not considered that only antisymmetric (odd) solutions of the wave function in the full potential well in (a) satisfy the boundary conditions of the

Q7.3 Choices	**# of Students**	%
1	5	15
2	9	27
3	3	9
4	6	18
5	10	30

Table 7.1: Details of students' responses to Q7.3

Q7.3	**# of Students**	%
Correct	9	27
Incorrect	24	73
Total	33	

Table 7.2: Summary of students' responses to Q7.3

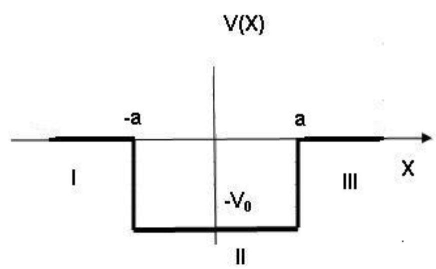
"half-well" in (b), and therefore the symmetric wave functions and their corresponding even energy levels in (a) cannot exist in potential well in (b). This example shows that students often over-generalize their understanding of physics concepts without recognizing the differences in systems and context. The details of students' responses to this question is presented in Table 7.1**,** and a summary is given in Table 7.2.

III. Wave function in a finite potential well

In context of symmetry in a finite potential well, we gave two multiple choice questions, **Q7.4** and **Q7.5**, to the students in $P631$. Solving the first question, **Q7.4**, requires students to understand that the wave function for the bound ground state in an even finite potential well is always a symmetric function, and for other energy levels, alternates from symmetric to antisymmetric. Therefore, the wave function for the even bound states (e.g., ground, second, fourth, sixth, ...) are always symmetric and for the odd bound states (e.g., first, third, fifth, ...) are always antisymmetric. Since we have total of 7 bound states in **Q7.4,** the highest energy level will be the sixth excited state and therefore will have a symmetric wave function. Only half of the students answered this question correctly, choice 2 [see Table 7.3 and 7.4].

Q7.4. An electron is confined to a one-dimensional, finitely deep potential energy well of width $2a$ depicted below. V_0 is a positive constant.

$$V(x) = \begin{cases} -V_0 & \text{for} \quad -a < x < a \\ 0 & \text{for} \quad |x| > a \end{cases}$$



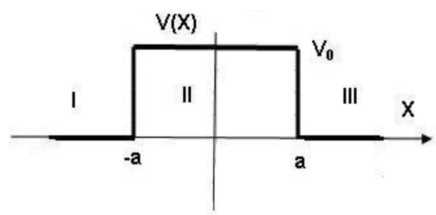
After solving the time-independent Schrödinger equation, you find that there are exactly 7 bound states for this system. Which one of the following statements is correct:

1. The highest energy bound state can have a symmetric or antisymmetric wave function.
2. The highest energy bound state has a symmetric wave function.
3. The highest energy bound state has an antisymmetric wave function.
4. The lowest energy bound state can have a symmetric or antisymmetric wave function.
5. The lowest energy bound state has an antisymmetric wave function.

Q7.5. A beam of electrons coming from the left scatter off the one dimensional rectangular barrier depicted below (The incident beam comes from $x = -\infty$):

$$V(x) = \begin{cases} 0 & x < -a \quad \text{Region I} \\ V_0 & -a < x < +a \quad \text{Region II} \\ 0 & x > +a \quad \text{Region III} \end{cases}$$

Which one of the following statements is correct:



1. Since the potential is an even function, we can assume with no loss of generality that the solutions of Schrödinger equation for this potential are either even or odd.
2. Since the potential is an odd function, we can assume with no loss of generality that the solutions of Schrödinger equation for this potential are either even or odd.
3. Since the potential is an odd function, we can assume with no loss of generality that $\Psi(-x) = \pm\Psi(x)$.
4. Since the potential is an even function, we can assume with no loss of generality that $\Psi(-x) = \pm\Psi(x)$.
5. We cannot make any assumption that solutions are either even or odd.

Q7.4	**# of Students**	%
Correct	22	50
Incorrect	22	50
Total	44	

Table 7.3: Summary of students' responses to Q7.4

Q7.4 Choices	**# of Students**	%
1	5	11
2	22	50
3	9	20
4	2	5
5	6	14

Table 7.4: Details of students' responses to Q7.4

In **Q7.5**, although the potential is an even function, we cannot solve the problem for half well, since in the scattered states symmetry is broken due to the incoming beam from the left. However, almost half of the students (44%) missed the correct answer, choice 5; approximately, 30% of students realized that the potential function is an even function (choice 1 and 4), but did not have a physical understanding of a symmetric system, and 13% of the students were not able to distinguish even functions from odd (choice 2 and 3) [see Table 7.5 and 7.6].

Q7.5 Choices	**# of Students**	**%**
1	5	11
2	4	9
3	2	4
4	9	20
5	25	56

Table 7.5: Details of students' responses to Q7.5

Q7.5	**# of Students**	**%**
Correct	25	56
Incorrect	20	44
Total	45	

Table 7.6: Summary of students' responses to Q7.5

7.3 MATHEMATICAL DIFFICULTIES WITH UNDERSTANDING SYMMETRY: INTEGRALS,GRAPHS, AND FUNCTIONS

Our data in the previous section shows that many students have difficulties recognizing the existence of symmetry in a variety of contexts. Our speculation is that part of students' difficulties with symmetry is due to their poor mathematical background. This section presents students' responses to math questions on recognizing odd and even functions, and whether or not they possess the appropriate math knowledge of symmetry. We probe students' mathematical background in understanding symmetry with examples of odd and even functions in multiple presentations, including graphical and algebraic expressions [see **Q7.6**].

Q7.6.(a) Consider the three integrals below, where $m \neq n$. Which of the following is the correct answer for these integrals?

$$1) \int_{-\pi}^{\pi} \sin[mx]\cos[nx]dx$$
$$2) \int_{-\pi}^{\pi} \cos[mx]\cos[nx]dx$$
$$3) \int_{-\pi}^{\pi} \sin[mx]\sin[nx]dx$$

(b) Consider the three integrals below, where $m \neq 0$. Which of the following is the correct answer for these integrals?

$$1) \int_{-\pi}^{\pi} \sin[mx]\cos[mx]dx$$
$$2) \int_{-\pi}^{\pi} \cos[mx]\cos[mx]dx$$
$$3) \int_{-\pi}^{\pi} \sin[mx]\sin[mx]dx$$

1. All are equal to 1.
2. (1) is equal to zero, (2) and (3) are equal to 1.
3. (3) is equal to 1, (1) and (2) are equal to π.
4. (2) is equal to π, (1) and (3) are equal to 1.
5. All are equal to 2π.
6. All are equal to zero.
7. (1) is equal to zero, (2) and (3) are equal to π.
8. I don't know.
9. It requires a great deal of calculation to answer this question.

The integration of all three integrals in part (a) of **Q7.6** would result in zero. One way to see this without calculation is considering the graphical representations of these integrals [see Figure 7.1]. In part (a), the first integral is an antisymmetric (odd) function whose integration over its period will result in zero. The second and third integrals in part (a) are even functions that are symmetric with respect to the y axis. However, the total area under the graphs is zero. Therefore, all three integrals in part (a) of **Q7.6** are zero, choice 6.

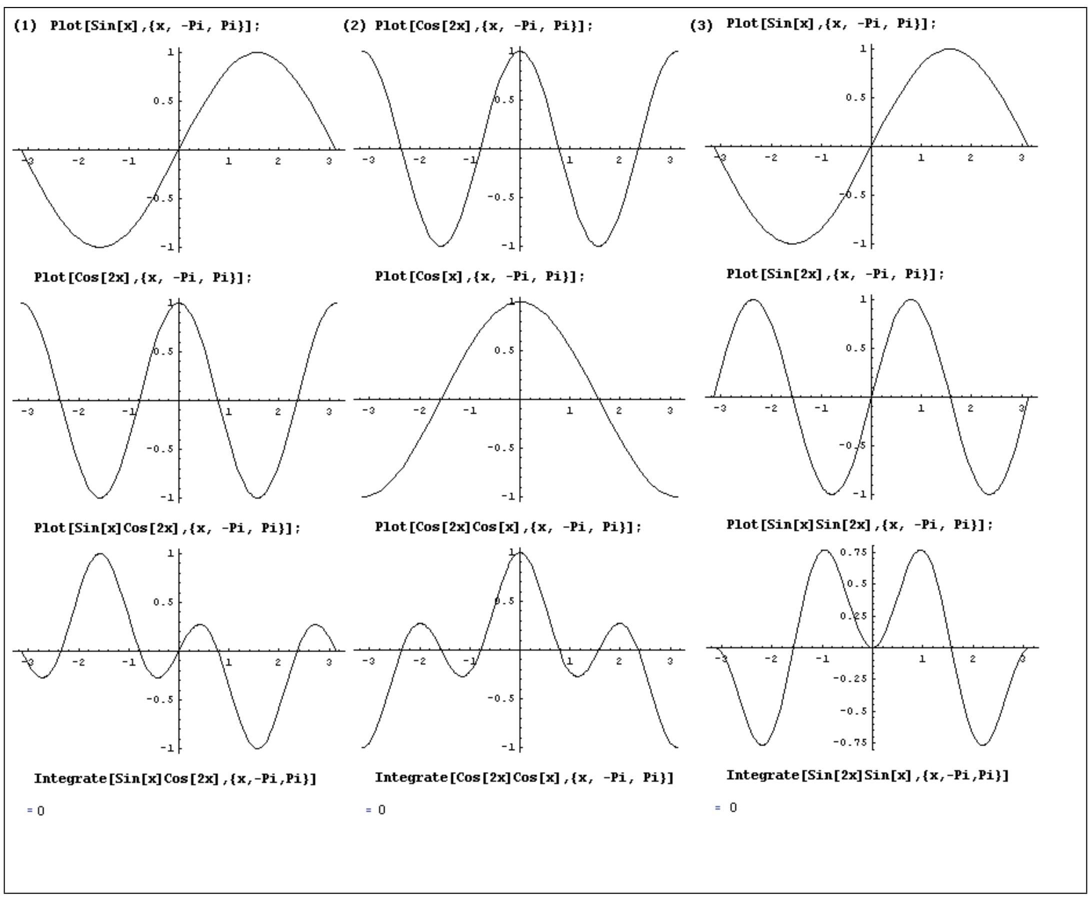


Figure 7.1: Graphical representations of integrals in part (a) of **Q7.6** for $m = 1$ and $n = 2$

In part (b), the first integral is an antisymmetric (odd) function that will result in zero for the integral, while the second and third integrals are symmetric functions with the total area under the graph being positive and equal to π.[see Figure 7.2]. Therefore, the correct answer for the part (b) of **Q7.6** is choice 7.

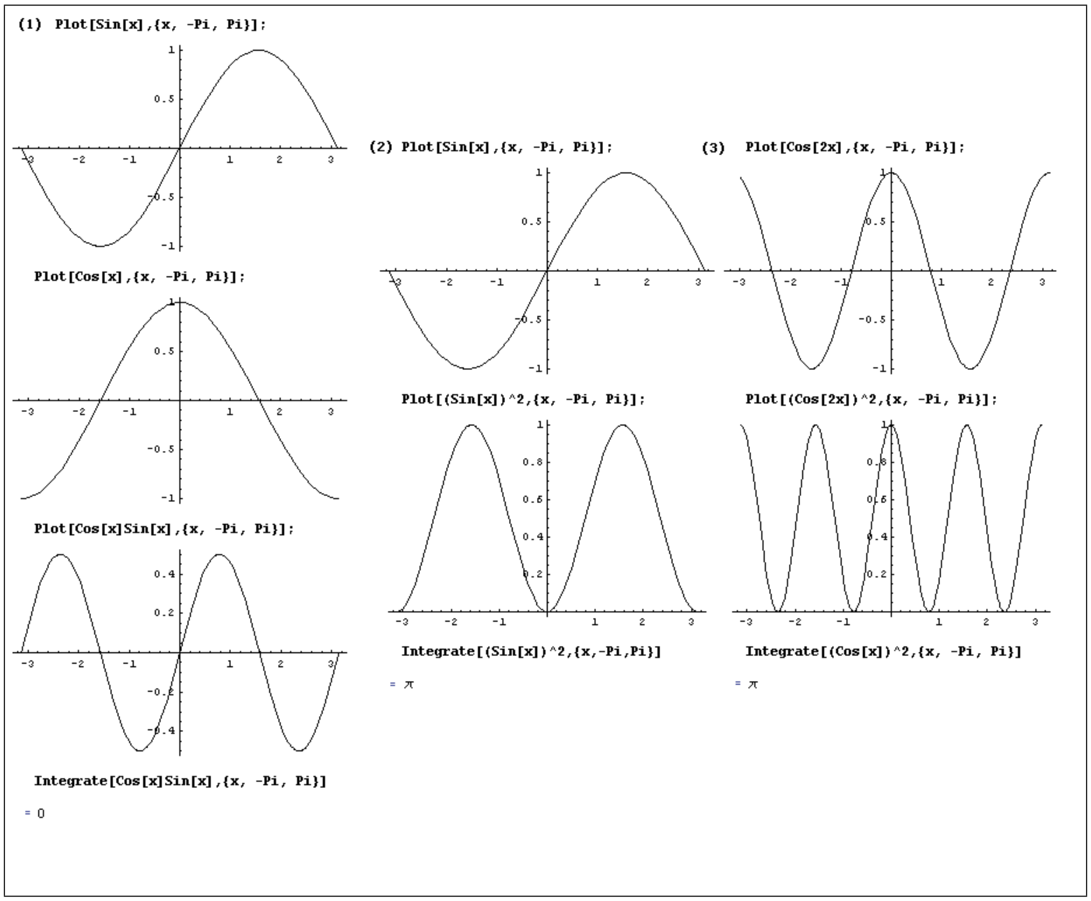


Figure 7.2: Graphical representations of integrals in part (b) of **Q.7.6** for $m = 1$ and $n = 2$

Q7.6	Part (a)		Part (b)	
	Number of Students	%	Number of Students	%
Correct	21	40	16	30
Incorrect	21	40	24	45
Don't know	7	13	8	15
Great deal of calculation	4	7	5	9
Total	53		53	

Table 7.7: Summary of students' responses to Q6.6.

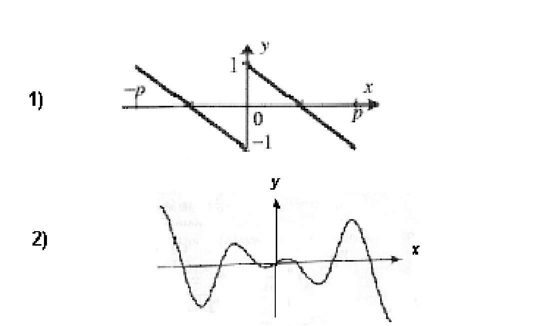
Only 40% of students answered part (a) correctly, and 30% answered part (b) correctly [see Table 7.7]. There was not a clear pattern among incorrect answers for the part (a). However, the most common incorrect answer for part (b) was choice 2 (23%), which has only the correct answer for the first integral in this part (zero). Perhaps, students knew the correct answer for the first integral in part (b), from the orthonormality of sine and cosine functions and picked choice 2 just by guessing. Many students (21%) over-generalized their answer to the first integral in part (b) and picked choice 6 (all are equal to zero). Since the above integrals widely appear in physics courses; it is important for students to be familiar with these integrals and have a good understanding of their graphical representations. Nevertheless, over 20% of the students explicitly stated that either they "don't know" the answer or they estimated a "great deal of calculation" for these integrals.

To learn more about students' aptitude in distinguishing odd and even functions in both symbolic and graphical representations, we gave **Q7.7** to the students enrolled in an introductory quantum course at the sophomore level, $P263$. Although students did better in the graphical representation of odd and even functions (68%), more than half of the students (58%) chose the incorrect answer for the symbolic representations [see Table 7.8]. The correct answer for part (a) is choice 2 since both functions are even,[6] and in part (b) both graphs represent odd functions that has an antisymmetric graphical representation (choice 1).

Q7.7 (a). For the following periodic functions, state whether the function is even, odd, or neither.

$$
\begin{aligned}
1)\quad f(x) &= a\left(1-\left(\frac{x}{p}\right)^2\right), \quad -p \le x \prec p,\ (a \neq 0) \\
2)\quad f(x) &= e^{-c|x|}, \qquad\qquad (c \neq 0), \quad \text{for } |x| \le p
\end{aligned}
$$

(b). For the following graphs, state whether the graph represents an even or odd function.



1. Both are odd.
2. Both are even.
3. (1) is even and (2) is odd.
4. (1) is odd and (2) is even.
5. Neither of them is odd nor even.

[6] One may argue that graph 1 in part (b) of **Q7.7** is not an even nor odd function due to the discontinuity at $x = 0$. However, only one student picked choice 5 in this question.

Q7.7	**Part** (a)		**Part** (b)	
	# of Students	%	**# of Students**	%
correct	22	42	36	68
Incorrect	31	58	17	32
Total	53		53	

Table 7.8: Summary of students' responses to Q7.7

In a follow-up essay question, we asked students to describe the properties of graphical representations of odd and even functions. Out of 53 students in this class, 43% gave a complete correct answer to this question; the remaining 57% were not able to describe the properties of odd or even functions completely and clearly. An example of correct answer for this question is:

> *"The graph of an even function is symmetric with respect to the y-axis. The graph of an odd function is symmetric with respect to the origin [3]."*

Some examples of students' incorrect or incomplete answerers include:

> *"Odd functions peak an odd number of times and even functions peak an even number of times."*

> "*Odd functions are mirrored along the* $y = -x$ *line even functions are mirrored along the* $y = 0$ *line.*"

> "*Not very sure... I think sine is odd and cosine is even.*"

> "*It is an even function if the graph passes through* $x = 0, y = 0$."

> "*Odd functions are reflected about* $y = x$ *and even functions are reflected about* $y = 0$."

> "*Odd- reflection across the line* $x = y$ *through the origin even- reflection at the y-axis.*"

> "*Even goes through the origin and odd doesn't.*"

> "*Odd functions are symmetrical about the line* $y = x$*, even functions are symmetrical about the y-axis.*"

Students' poor performances in these math questions show that most students do not distinguish odd and even functions in different representations and they are not well familiar with the properties of these functions. In addition, we found a correlation between students' responses to some of the above math questions and their scores on certain questions on the quantum mechanics final examination. The details of these relationship will be discussed later in this chapter.

7.4 EXAM QUESTIONS

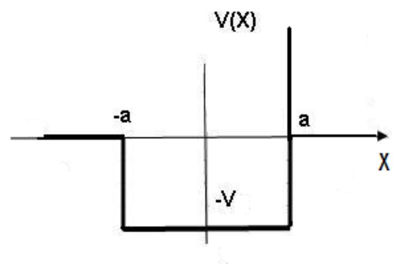
To study any relation between students' mathematical background and their ability to solve related quantum mechanical problems, in the final examination of the advanced quantum class, $P631$ in fall 2004, we asked students two questions that required an understanding of symmetry. The first question, **Q7.8,** requires the sketch of antisymmetric wave functions and the second one, **Q7.9,** requires the integration of an odd function over its period.

7.4.1 SKETCH OF A WAVE FUNCTION IN AN ASYMMETRIC WELL

Question **Q7.8** probes students' ability to recognize that an asymmetric well is not an even function; therefore, it does not have the properties of a symmetric potential well such as symmetric and antisymmetric wave functions.

Q7.8.(a). There are three bound states in the potential well shown below. Sketch these three bound states.

$$V(x) = \begin{cases} 0 & x < -a \\ -V & -a < x < +a \\ \infty & x > +a \end{cases}$$



(b). There are two bound states in the potential well shown below. Sketch these two bound states.

$$V(x) = \begin{cases} 0 & x < -a \\ -V & -a < x < 0 \\ \infty & x > 0 \end{cases}$$

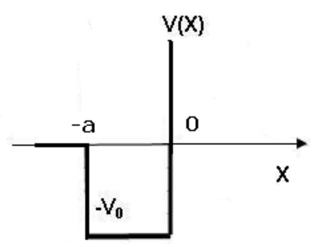


Figure 7.3. shows the correct answer to **Q7.8**. Neither of the potentials in this problem are symmetric functions; therefore, the wave functions would not be symmetric or antisymmetric. However, many students (53% in part (a), and 31% in part (b)) treated these potentials as if they were symmetric finite potential wells. Some of

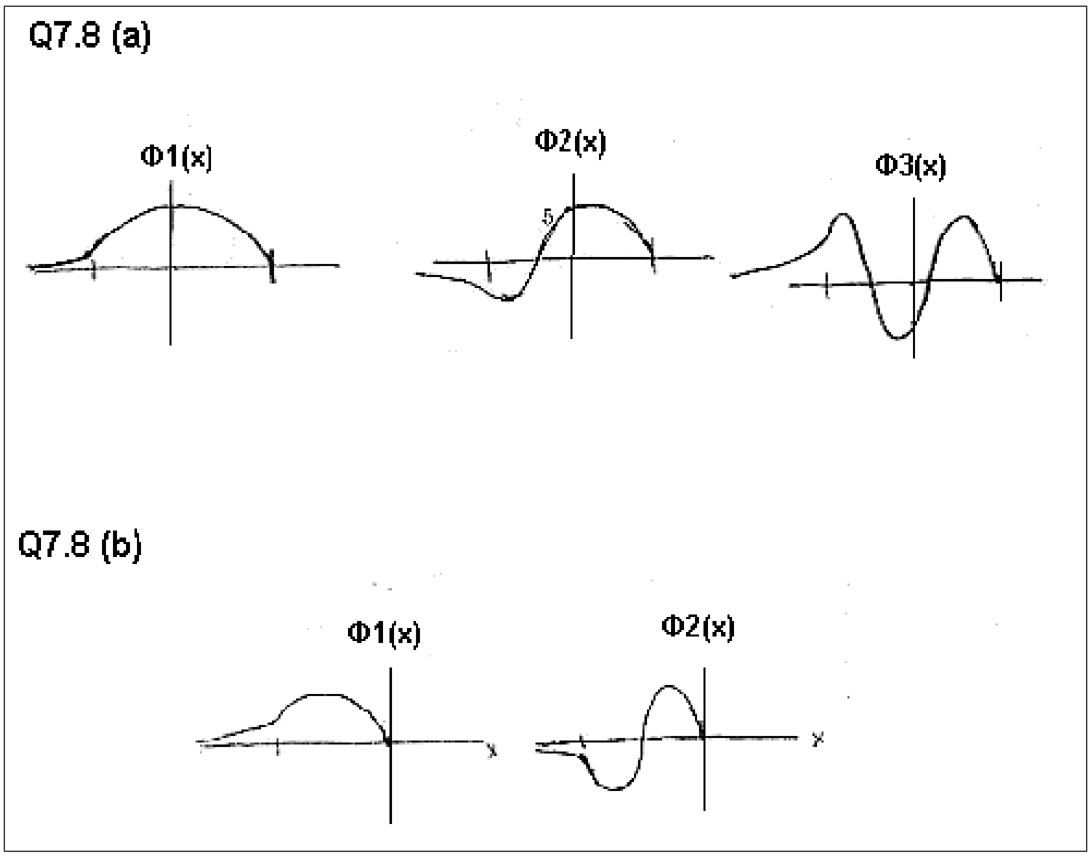


Figure 7.3: The correct sketch of the bound states for **Q.7.8.** The wave functions are not "symmetric" or "antisymmetric"

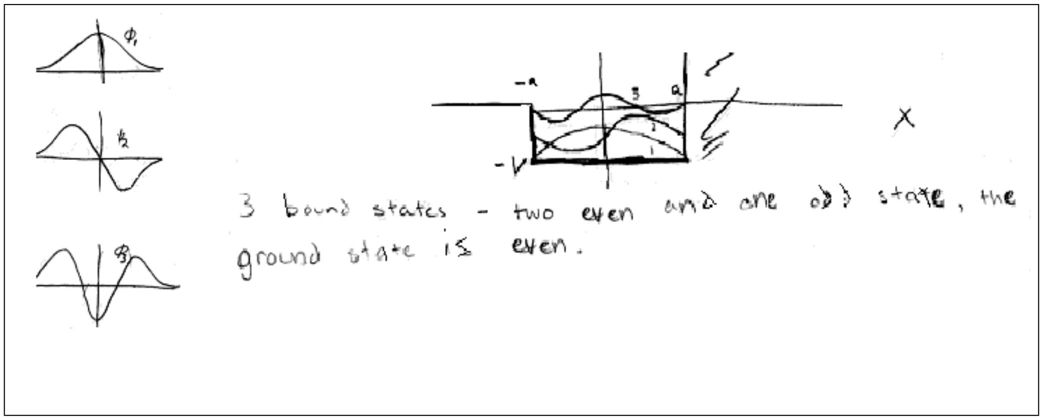


Figure 7.4: An example of students' incorrect sketch for **Q7.8** part (a). The students explicitly stated that there are *" 3 bound states- two even and one odd state, the ground state is even"*

these students explicitly stated that of three bound states in **Q7.8 (a)**, two are even (symmetric) and one is odd (antisymmetric) [see Figure 7.4 and 7.5]. Others sketched symmetric and antisymmetric wave functions the same as a finite potential well. The higher percentage of incorrect answers for part (a) is perhaps due to the position of the well in the coordinate system, which evokes the familiar finite potential well problem that, in fact, does have symmetric and antisymmetric bound states.

Here, students overlooked the different boundary conditions, and that different potential wells are presented. Students then failed to recognize that these potentials are not symmetric. Thus, they do not possess the characteristics of a symmetric potential well including the odd and even functions for the bound states. This is another example of students' over-generalization of what seems to be correct in a different context. Perhaps the fact that the graph of this problem is very similar to the one for finite potential well causes this over-generalization.

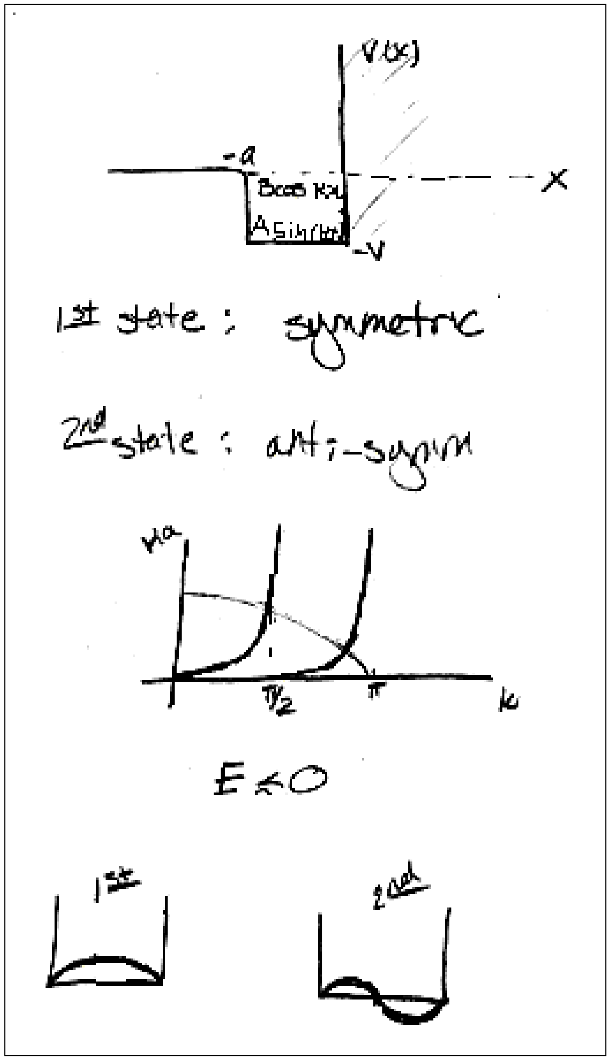


Figure 7.5: An example of students' incorrect sketch for **Q7.8** part (b). The student explicitly stated that the wave function for the first state is "symmetric" and for the second state is "antisymmetric"

The issues related to the understanding of the boundary condition and sketch of wave functions are not discussed here [1]. Therefore, the reason for incorrect answers in (a) is not limited to difficulties in understanding the boundary conditions. In addition, in another part of our study we tested students' ability to sketch wave functions in different boundary conditions in more isolated problems, and our results do not show any significant difficulties in recognizing finite from infinite potential boundaries [see section 6.4].

7.4.2 INTEGRATION OF AN ODD FUNCTION OVER ITS PERIOD

To probe students' ability to recognize the symbolic representation of an odd function and their mathematical skills in calculating integrals of odd function, consider **Q7.9** that was given on the final examination of $P631$ in fall 2004.

> **Q7.9.** A free particle wave function is given by:
>
> $$\Psi(x) = A\exp[\frac{-x^2}{L^2}] \tag{7.2}$$
>
> What is $< \widehat{P} >$?

The correct answer to this question requires students to (1) know how to calculate the expectation value of the momentum from a given wave function, (2) be able to take the partial derivative of the wave function with respect to x, and (3) solve the integral. The derivative of the wave function produces x and the total integral becomes an odd function [4]:

$$< \widehat{P} >= \int_{-\infty}^{\infty} dx \Psi^*(x)(-i\hbar\frac{\partial}{\partial x})\Psi(x) \tag{7.3}$$

$$= A^2 \int_{-\infty}^{\infty} dx(\frac{2i\hbar x}{L^2})e^{\frac{-2x^2}{L^2}}$$

$$= 0. \quad \text{The integral is odd.} \tag{7.4}$$

Approximately 55% of the students were able to set up the integral:

$$< \Psi(x)|i\hbar\frac{\partial}{\partial x}|\Psi(x) > \tag{7.5}$$

and take the partial derivative of the wave function correctly, but were not able to recognize the odd features of the resulting function to be integrated. Figures 7.6 and 7.7 demonstrate examples of students' solutions. These students had no problem setting up the integral and taking the derivative of the wave function, but they left the integration unsolved.

If students would recognize the odd nature of the function under the integral and the fact that the integration of this odd function between $-\infty$ and $+\infty$ results in zero, they could easily conclude that $< \widehat{P} >= 0$. Alternatively, if they would consider the graphical representation of this wave function in momentum space, and would recognize that the new Gaussian function (Fourier transform of a Gaussian is a Gaussian) also peaks at zero, they could conclude that the most probable value for momentum is zero.

Only 8 out of 60 students (13%) recognized the odd function and gave the correct answer. However, a great number of students ($n = 33, 55\%$) were able to set up the

(2b) What are $< \hat{P} > (t)$ and $< \hat{H} > (t)$? You should be able to completely evaluate any integrals in this part.

S1 $\Psi(x,t) = A\exp\left[\frac{(2it-mL^2)x^2}{mL^4}\right]$

$$\langle\hat{P}\rangle = A^2\int_{-\infty}^{\infty}\exp\left[\frac{(-2it-mL^2)x^2}{mL^4}\right]\left(\hat{P}\, e^{\frac{(2it-mL^2)x^2}{mL^4}}\right)dx$$

$$= -A^2 i\int\exp\left[\frac{(-2it-mL^2)x^2}{mL^4}\right]\exp\left[\frac{(2it-mL^2)x^2}{mL^4}\right]\frac{2x(2it-mL^2)}{mL^4}$$

$$= -A^2 i\,\frac{2(2it-mL^2)}{mL^4}\int x\exp\left[\frac{-2\cancel{m}L^2x^2}{\cancel{m}L^{4}}\right]dx$$

$$= ?$$

S2

$$\langle\hat{p}\rangle = \langle\psi|-i\frac{d}{dx}|\psi\rangle$$

$$= -i\langle\psi|\frac{d\psi}{dx}\rangle = -i\frac{2}{\sqrt{\pi}}\left(\frac{L^2}{2}\right)^{3/2}\langle x e^{-\frac{x^2}{L^2}}e^{i\omega t}\,|\,e^{-\frac{x^2}{L^2}} - \frac{2}{L^2}xe^{-\frac{x^2}{L^2}}\rangle$$

$$= -\frac{2i}{\sqrt{\pi}}\left(\frac{L^2}{2}\right)^{3/2}\int_{-\infty}^{\infty}\left(xe^{-\frac{2x^2}{L^2}} - \frac{3x^3}{L^2}e^{-\frac{3x^2}{L^2}}\right)dx = ?$$

S3

$$\langle\hat{P}\rangle_t = \int dp\, P|\Psi(p)|^2 \to \int dx\,\Psi^*(x)\left(-i\frac{\partial}{\partial x}\Psi(x)\right)$$

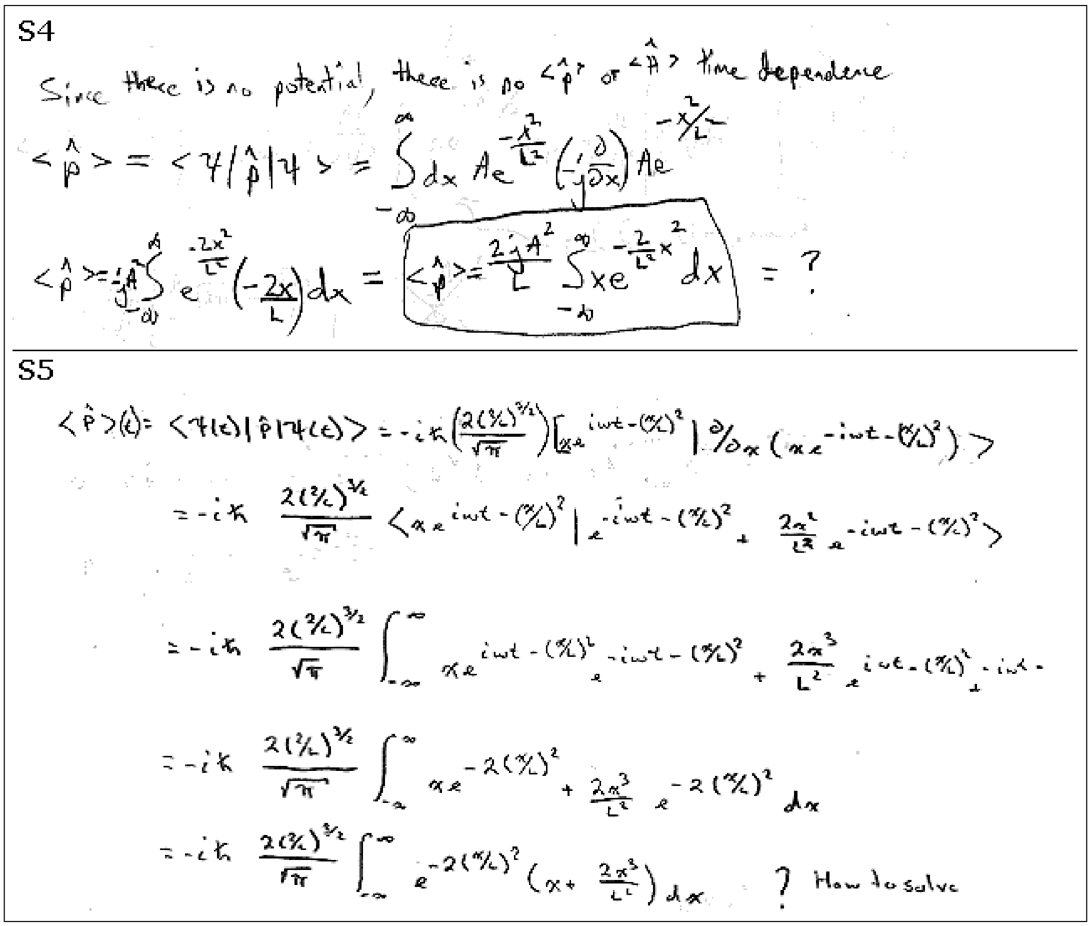
$$= \int dx\, A\exp\left[\frac{-x^2}{L^2}\right]\left(i\,\frac{-2xA}{L^2}\exp\left[\frac{-x^2}{L^2}\right]\right)$$

$$= \frac{-2A^2}{L^2}\int dx\,\exp\left[\frac{-x^2}{L^2}\right] i\,x\,\exp\left[\frac{-x^2}{L^2}\right]$$

$$= \frac{-2A^2}{L^2}\int dx\; ix\,\exp\left[\frac{2x^2}{L^2}\right]$$

$$= ?$$

Figure 7.6: Examples of students' inability to recognize the integration of an odd function in their solution to **Q.7.9**



S4

Since there is no potential, there is no $\langle\hat p\rangle$ or $\langle\hat A\rangle$ time dependence

$$\langle \hat p\rangle = \langle \psi|\hat p|\psi\rangle = \int_{-\infty}^{\infty} dx\, A e^{-\frac{x^2}{L^2}}\left(-i\frac{\partial}{\partial x}\right) A e^{-x^2/L^2}$$

$$\langle \hat p\rangle = -iA^2\int_{-\infty}^{\infty} e^{-\frac{2x^2}{L^2}}\left(-\frac{2x}{L}\right)dx = \boxed{\langle\hat p\rangle = \frac{2iA^2}{L}\int_{-\infty}^{\infty} x e^{-\frac{2}{L^2}x^2}\,dx} = ?$$

S5

$$\langle \hat p\rangle(t) = \langle\psi(t)|\hat p|\psi(t)\rangle = -i\hbar\left(\frac{2(1/L)^{3/2}}{\sqrt\pi}\right)\left[x e^{i\omega t-(x/L)^2}\right| \partial/\partial x\left(x e^{-i\omega t-(x/L)^2}\right)\rangle$$

$$= -i\hbar\, \frac{2(1/L)^{3/2}}{\sqrt\pi}\,\langle x e^{i\omega t-(x/L)^2}\,|\, e^{-i\omega t-(x/L)^2} + \frac{2x^2}{L^2}e^{-i\omega t-(x/L)^2}\rangle$$

$$= -i\hbar\, \frac{2(1/L)^{3/2}}{\sqrt\pi}\int_{-\infty}^{\infty} x e^{i\omega t-(x/L)^2} e^{-i\omega t-(x/L)^2} + \frac{2x^3}{L^2}e^{i\omega t-(x/L)^2}e^{-i\omega t-}$$

$$= -i\hbar\, \frac{2(1/L)^{3/2}}{\sqrt\pi}\int_{-\infty}^{\infty} x e^{-2(x/L)^2} + \frac{2x^3}{L^2}e^{-2(x/L)^2}\,dx$$

$$= -i\hbar\, \frac{2(1/L)^{3/2}}{\sqrt\pi}\int_{-\infty}^{\infty} e^{-2(x/L)^2}\left(x+\frac{2x^3}{L^2}\right)dx$$

? How to solve

Figure 7.7: More examples of students' inability to recognize the integration of an odd function in their solution to **Q7.9**

integral correctly, but did not recognize that (1) the function under integral is an odd function, and (2) the integration of this odd function between $\pm\infty$ will produce a zero value for the $< \widehat{P} >$. Most of these students have left the integral unsolved and admitted that they do not know how to solve the integral. The remaining 32% either were not able to set up the integral to calculate the expectation value of the momentum correctly or had other mathematical difficulties.

7.5 ANALYSIS OF STUDENTS' RESPONSES AND THEIR SPECIFIC DIFFICULTIES

Most of our data in this part of study are from two consecutive quarters: the first introductory quantum course in spring 2004 ($P263$) and the first quarter of advanced undergraduate quantum course in fall 2004 ($P631$). Sixty one students took P263 in spring quarter, of whom 48 students also took the advanced course P631 in fall. Therefore, we have a longitudinal study on these students that includes their individual responses to the questions in this chapter and their exams and final grades in both courses. Since responses to some of the research questions presented here were voluntary and we did not have a 100% participation, we do not have a complete data set for all the 48 students.

I. Student ability to sketch a correct wave function in an asymmetric potential well correlates with their ability to recognize the graph of an odd or even function.

In the study of students' responses to different questions related to understanding of symmetry, we found that students responses to **Q7.7(b)** and **Q7.8** are highly correlated. Question **Q7.7(b)** is a multiple choice question that tests students' ability to recognize the graphical representation of an odd or even function. **Q7.8** is part of

	Q7.8			Total
	.00	.50	1.00	
Q7.7(b) .00	3	6	0	9
1.00	2	4	13	19
Total	5	10	13	28

Table 7.9: Q7.8 - Q7.7b Crosstabulation

	Value	**df**	**Sig. (2-sided)**
Pearson Chi-Square	11.495	2	.003
Likelihood Ratio	14.974	2	.001
Linear-by- linear Association	8.733	2	.003
N of Valid Cases	28		

Table 7.10: Chi-Square Tests for Q7.8 and Q7.7b

a problem on the final examination that asks students to graph the wave function for an asymmetric potential well.

Since question **Q7.7(b)** is a multiple choice question, students' responses to this question produced binary data (1 for correct answers and 0 for incorrect), while students' scores on **Q7.8** were any number between 0 to 5. In order to make these two sets of data statistically comparable, we first normalized the scores for **Q7.8** and then converted these continuous data to a categorical data. That is, the scores between $0.0 - 0.33$ were converted to 0.0, the scores between $3.3 - 6.6$ were converted to 0.5, and the scores between $6.6 - 1.0$ were converted to 1.0. Then we used Chi-Square Test to find any statistical relationship between these two scores [5]. Tables 7.9 and 7.10 show the summary of these analysis.

Table 7.9 shows that there were 28 students who answered these two questions, from which 13 students answered both **Q7.(b)** and **Q7.8** correctly; 3 students answered both questions incorrectly, and 4 out of 10 students who received a partial credit for **Q7.8** had chosen the correct answer for **Q7.7(b)**. Table 7.10 demonstrates a statistical significant ($P = 0.003$) for Chi-Square Test between the scores for these two questions; indicating that students' responses to these two questions are statistically related. Further analysis of the data using linear regression shows that this relation is a positive correlation ($R = 0.569$). This means that students with low scores on **Q7.7(b)**, who did not recognize the graph of odd or even functions ($P263$, spring 2004) did poorly on graphing the wave function for asymmetric wells in **Q7.8** ($P631$, fall 2004).

One interpretation of this result suggests that students' understanding of a graphical representation of odd or even functions plays an important role in their ability to sketch a correct graphical representation of wave functions in a quantum mechanical system. In order to draw a correct wave function, one needs to recognize the potential graph. It seems that students' mathematical background in use of graphical representation in general, and their ability to recognize the symmetric characteristics of odd and even functions are specially, important prerequisites in understanding the symmetric characteristics of different physical systems such as potential well, wave functions, etc.

II. Students' ability to recognize an even or odd function correlates with their achievements in quantum courses.

We found a significant correlation between students' responses to **Q7.6** (both part (a) and (b)) and **Q7.9** on the final examination. Question **Q7.6** is a multiple choice

		Q7.9			Total
		.00	.50	1.00	
Q7.6	.00	15	3	2	20
	.50	6	7	4	17
	1.00	0	0	2	2
Total		21	10	8	39

Table 7.11: Q7.6 - Q7.9 Crosstabulation

question (scores 0 and 1 for each part (a) and (b)). We calculated students' average scores for part (a) and (b) by assigning 0 for those with incorrect responses for both part (a) and (b), 0.5 for those who answered one of the two parts correctly, and 1.0 for students who gave the correct answers for both part (a) and (b). We categorized students' scores in **Q7.9** in the same way as for **Q7.7** described above.

Table 7.11 shows that 15 students who were not able to answer question **Q7.7** correctly, also did not solve the **Q7.9**; only 2 students who did not choose the correct answer for **Q7.6** were able to solve the **Q7.9** completely. In addition, the two students who chose the correct answer for **Q7.6** are the only students who received a complete grade in **Q7.9.** Table 7.12 demonstrates a statistical significant ($P = 0.007$) for Chi-Square test for these two questions. Further analysis of these results, using linear regression, confirms that this correlation is positive ($R = 0.490$), meaning that most of the students who did not answer **Q7.6** correctly ($P263$, spring 2004, $n = 20$) did not do well in **Q7.9** either ($P631$, fall 2004, $n = 18$).

The interpretation of these results is twofold:

1. Most students in advanced physics courses still have difficulties with basic mathematical topics (e.g., difficulties recognizing odd or even functions, recognizing

	Value	df	Sig. (2-sided)
Pearson Chi-Square	14.135	4	.007
Likelihood Ratio	12.847	4	.012
Linear-by- linear Association	9.113	1	.003
N of Valid Cases	39		

Table 7.12: Chi-Square Tests for Q7.6 and Q7.9

symmetric and antisymmetric graphs, integration of odd functions, etc.). Since this difficulty exists in the context of both math and physics, it is different from a typical transfer issue.

2. A poor mathematical background affects students' achievements in quantum mechanics. This is perhaps due to the more abstract nature of quantum mechanics and higher level of formalism required in this course.

We found some other correlations between students' scores on pure math questions and their grades on final examination of $P631$, but they were not as significant as the two we discussed above. For example, students' responses to question **Q7.7(a)** correlates with responses to **Q7.9** ($R = 0.546$) and **Q7.7(a)** ($R = 0.354$). This means that students who were not able to distinguish the symbolic representation of odd and even functions in the context of a pure math question, **Q7.7(a),** were not able to do so in the context of a quantum mechanics problem, **Q7.9**.

7.6 SUMMARY

As it mentioned at the beginning of this chapter, our goal in this part of study was to seek answers to the following questions: After standard instruction in math and quantum mechanics:

Are students able to recognize the presence of symmetry in a given problem?

Are students able to take advantages of symmetry in solving quantum mechanics problems?

Do students possess the appropriate math knowledge of symmetry and are they able to use this knowledge in quantum physics?

The data presented in this chapter shows that students are not always able to easily recognize the existence of symmetry in math and physics problems and often fail to use the symmetrical features of a problem to simplify problem solutions. There is a correlation between students' mathematical knowledge of symmetry and their ability to solve symmetry related problems in the context of quantum mechanics. This may be partly due to students' lack of experience and partly due to their weak mathematical background. On the other hand, in some other topics investigated in this dissertation, we found cases in which students have the required mathematical background, but they fail to apply it in the context of physics questions. However this does not seem to be the case for students difficulties with symmetry. In addition, when students lack the appropriate knowledge to solve physics questions, they tend to over-generalize their understanding of some principle related to the problem at hand, without careful inspection of applicability.

The data in this chapter suggest that students' understanding of a graphical representation of odd or even functions plays an important role in their ability to sketch correct graphical representations of wave functions in a quantum mechanical potential. We found a positive correlation between students' scores on mathematical questions pertaining to symmetry and their grade on symmetry related quantum exam problems.

BIBLIOGRAPHY

[1] The details of students' difficulties with wave functions and the sketch of wave functions are discussed in Chapter 5 of this dissertation.

[2] This problem is discussed on the both textbooks; "*Introduction to Quantum Mechanics,*" by Griffiths and "*Introductory Quantum Mechanics,*" by Liboff.

[3] This answer is chosen from students' correct responses.

[4] One can calculate 'A' from normalization of the wave function to be equal to: $A = \sqrt{\sqrt{\frac{2}{\pi}}\frac{1}{L}}$.

[5] For more details about the data and the statistical analysis used here, please see Chapter 3 of this dissertation.

CHAPTER 8

INSTRUCTIONAL MATERIAL

"How many of those who undertake to educate the young appreciate the necessity of first teaching them how to acquire knowledge?" John Comenius

8.1 INTRODUCTION

In the history of physics education research the identification of student specific difficulties often has led to two important results. First is development of instructional materials for classroom use, such as tutorials [1], and the second is development of assessment tools such as FCI and MBLT [2], [3].

This study has examined students' understanding of introductory quantum mechanics with regard to three main topics of measurement, wave functions, and symmetry, which were discussed in detail in Chapters 5, 6, and 7. After learning what students' specific and common difficulties are, our ultimate goal has been to develop instructional materials to explicitly address these difficulties.

The objective for the design of such material is to provide step-by-step instruction to help students:

1. Develop basic quantum mechanical concepts;

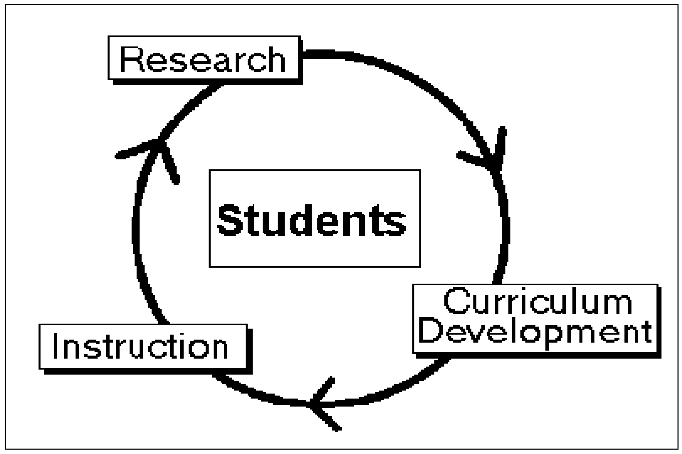


Figure 8.1: The McDermott Wheel: three components of dynamically improving teaching methods

2. Use these concepts to interpret different quantum mechanical systems;

3. Develop scientific reasoning skills;

4. Practice the required formalism in this subject;

5. Relate and use multiple representations in describing quantum systems.

Since the ultimate goal is helping students to learn, physics education researchers [4] suggest that instruction has to be based on the results of research. In turn, the feedback from instruction has to be used to develop research instruments, which leads to further improvement in instruction. Therefore, Research, Curriculum development, and Instruction are three components of dynamically improving teaching methods [see Figure 8.1].

To date our efforts in this area have led to preliminary worksheets on the topics of probability, wave functions, and mathematical solution to the time-independent

Schrödinger equation. The worksheets contain materials on both basic quantum mechanical concepts and mathematical skills related to the formalism of these concepts. They also include multiple choice exercises, supplementary problems, and graphical representations.

Since the design of the worksheets in this study is in a very preliminary stage, it has not been widely tested. To be able to assess the usefulness of any such material, they need to go through several stages of testing and alterations, but we have only been able to administer these worksheets once to the students in $P263$ and $P631$. In addition, for the resulting data from these students, in most cases we did not have a reliable baseline to study the effect of the worksheets on students' performance. A copy of the worksheets developed in this study is presented at the end of this chapter.

8.2 WORKSHEETS RELATED TO QUANTUM PROBABILITY

There are two worksheets on the topics of probability. The first one covers concepts of probability density and its relation to the wave function in a variety of different potentials and boundary conditions (e.g., free particle, infinite and finite potential). On this worksheet there are questions that clue students to relate energy, wavelength, and wave number to the shape of wave functions and the graphs of probability density. There are some quantitative questions on this worksheet, but the focus is more on the qualitative concepts of probability and wave functions. These questions are designed to address students' specific difficulties on probability as addressed in Chapter 5.

We gave this worksheets to the students enrolled in $P631$ in Fall 2004. Students were given one lecture session, 48 minutes, to work on this worksheet. The instructor and two TAs were also in the class to answer any questions students might have. The material focused on helping students to develop basic quantum mechanical concepts, to use these concepts to interpret different quantum mechanical system, and to relate and use multiple representations in describing quantum systems.

Unfortunately, for several reasons, we were not able to assess this worksheet statistically. For example, we did not have a proper pretest and post-test administered. Additionally, students had received other lectures and course materials on the same topics. Nevertheless, more than 85% of the students said they found this worksheet useful. Some of students' quotes, from one of our surveys, show students' attitudes towards these material:

S1: "[After the worksheet on probability]*... Most of the stuff makes sense now."*

S2: "*A good worksheet handled all of the cases for a finite square potential well. (Decaying or propagating waves). However I did not have enough time to go all over it in class.*"

S3: *"I think those worksheets are helpful I wish we had one every week. It was similar to one we did before and I think if we did one of those weekly I would understand what's going on a lot better."*

S4: *"It would have been better if we could have actually gone through it and talked more about it."*

S5 *"It was a good worksheet although it would be nice to have the answers posted (if they are not already up) so I can make sure I got the correct answers and am understanding the material correctly."*

S6: *"It was nice to work on- but to learn a lot from it I would have liked a longer class time...."*

S7: *"It was very helpful. It would have helped to have gone over it more ...."*

S8: *"It would be a nice study tool to put the worksheet online with the answers."*

S9: *"It would help to see a copy of the answers posted on the website."*

S10: *"It would really be nice to talk about these worksheets that we do from time to time if only briefly to make sure I'm not making any conceptual errors in filling them out."*

While most of the students found this worksheets helpful, they were dissatisfied with the time given to this exercise and the fact that, like any other tutorial material, we did not post the solutions.

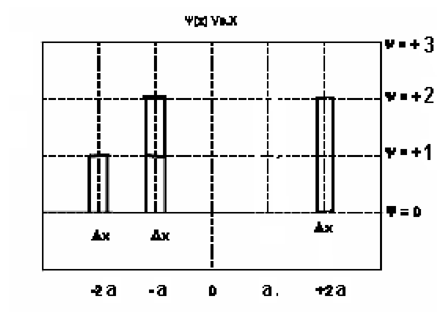
The second worksheet, on probability, discusses a quantitative problem of finding the probability density and expectation values for a discrete (artificial) wave function. This worksheet breaks down a complex problem into a few simple ones and guides students through to find the uncertainty in position and momentum. This worksheet was developed after we found out that students have difficulties in distinguishing related but different concepts of probability, as well as calculating different terms. For example, as discussed in Chapter 5, students have a tendency to confuse the probability density with expectation value and uncertainty in measurement. The goal of this worksheet has mostly to help students with understanding of these basic quantum mechanical concepts and terminologies, practicing required formalism, and relating and using multiple representations in describing theses concepts.

Question	**Q6.1**	%	**Q5.8**	%	**Q8.1**	%	**Q8.2 Part (a)**	%	**Q8.2 Part (b)**	%
Correct	22	41%	10	24%	35	66%	30	73%	22	54%
Incorrect	32	59%	31	76%	18	34%	11	27%	19	46%
Total	54		41		53		41		41	

Table 8.1: Summary of students' responses to pre-test and post-test questions for quantitative worksheet on probability [see section 8.5]. Q6.1 and Q5.8 were pretest and Q8.1 and Q8.2 were post-test questions

We used **Q6.1** from Chapter 6 and **Q5.8** from Chapter 5 as pretest questions. Since this worksheet was given to the students in an optional class time, only 33 out of 61 students worked on it. The post-test questions are shown in **Q8.1** [5] and **Q8.2.**

Q8.1. Post-Test. The wave function for a particle is sketched below. Assume $\Delta x \ll a$.



(a) What is $< \hat{X} >$?

(b) List possible values of position that can be measured and their probability.

Students' performance on pretest and post-test questions are summarized in table 8.1. Students did better overall on post-test questions, however, the performance of the 33 students who worked on the worksheet has not significantly been better than the rest of the class.

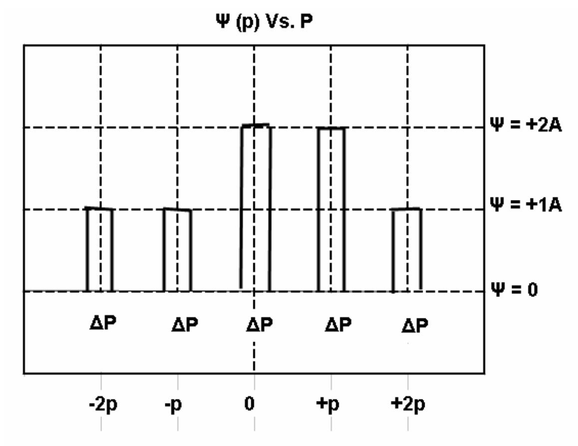
After seeing the pretest questions and working on the related worksheet, many students still were not able to solve these problems and gave incorrect answers, especially on part (b) of **Q8.2** where almost half of the students did not give the correct answer. Questions **Q5.8**, **Q8.1**, and **Q8.2** are very similar. However, **Q8.2** shows the wave function in momentum space and in part (b) asks for the expectation value of the momentum $< p >$ instead of $< x >$. This seems to make the problem difficult for many students. This is, perhaps, because of the students' difficulties with momentum representations of wave functions [6]. In addition, in many incorrect responses, students did not square the wave function in order to find the probability of certain measurements [7]. Only 54% of the students solved part (b) of **Q8.2** correctly. The correct answer for part (a) of **Q8.2** is $\frac{1}{11}$ and the solution for part (b) is as follows:

$$< P >= (-2p)P[-2p] + (-p)P[-p] + 0p[0] + (+p)P[+p] + 2pP[2p] \tag{8.1}$$

$$= -2p(\frac{1}{11}) - p(\frac{1}{11}) + 0(\frac{4}{11}) + p(\frac{4}{11}) + 2p(\frac{1}{11}) \tag{8.2}$$

$$< P >= \frac{3p}{11} \tag{8.3}$$

Q8.2. Post-Test. The figure below shows a plot of a (rather artificial) wave function $\Psi(p)$ versus p, given by the 5 'spikes' each of width Δp over the range of $(-2p, +2p)$. The wave function vanishes for all other values of p. If you measure the momentum of this particle a hundred times, where are you most likely to find the particle?



a) How often will you expect to find the momentum of the particle to be $P = +2p$?

b) In the previous question, what is the expectation value of the momentum, namely $< P >$?

8.3 WORKSHEETS RELATED TO QUANTUM WAVE FUNCTIONS

There were two worksheets on topics related to quantum wave functions: one on the sketch of wave functions and the other on the mathematical solution to the Schrödinger equation. The worksheet on the sketch of wave functions was developed for students in $P263$ and given to the students at the end of one computer lab session. This worksheet had two purposes: first, it was an exercise in sketching wave functions on the regions of different potentials, and second, it aimed to measure the effectiveness of a prior computer lab on students' learning of the sketch of wave functions. Therefore, in addition to being an instructional tool, it was also an assessment tool on students' ability to draw the correct wave functions for different potentials. The assessment of the computer software on the sketch of quantum wave functions is not discussed in this study. Although no formal evaluation of this worksheet was made, students' informal feedback indicate that they found it a good exercise on relating and using multiple representations (mostly graphical) to describe quantum wave functions.

The worksheet on the mathematical solution to the Schrödinger equation was developed for students in $P631$ based on our findings of student difficulties in solving differential equations and their inability to distinguish a decay function from an oscillatory function, as discussed in Chapter 6. The students worked on this worksheet in groups during one lecture session of 48 minutes. Although this material was covered in lecture, redoing the calculations and recognizing when the solution to the differential equations is a sine and cosine function and when it is an exponential decay was a challenge for most students. This worksheet helped students to associate their mathematical solution with their conceptual and qualitative understanding of the shape of wave functions, a specific difficulty that is discussed in Chapter 5 under mathematical difficulties with the associating graph and equation representation of wave functions. Here the main objective is on the practice of required formalism and relating and using multiple representations to describe quantum wave functions. Below are some of the students' comments about this worksheet:

S1: *"I think it helped me to do the little packet* [worksheets] *a few weeks ago that took you step-by-step through the particle in a box problem. It broke it down enough so I could understand it."*

S2: *"Sometimes in the text the math will be applied only in a very specific way it was helpful to see examples with a more general use of the math in the worksheet to help better grasp the full power of the formalism."*

S3: "[For the worksheet on the solution to Schrödinger equation] ... *I did not know how to go about doing quite a few of the calculations but I think I have a good understanding of the more qualitative questions."*

S4: "*I like working in groups in class and seeing if I understand something or just think I do as the professor is writing it on the board."*

S5: "[The worksheet]*...states the initial state then the final state then shows the math to get from state to state so its easier to follow where we are going with it."*

S6: *"It seemed like a pretty straight-forward application of our knowledge of how a wave function should behave given information about its energy and the shape of the potential. Not that difficult and certainly much easier than the homework."*

S7: *"Mostly I was lost I didn't know the difference between bound free and scattering states (but that got cleared up nicely)..."*

According to the students' responses on the evaluation of this worksheet, the material was helpful and popular by them. However, since we used the same set of questions as pretest and post-test, and the answers for the pretest questions were somehow discussed in class, we cannot claim any statistical improvement in students' performance after the worksheet.

8.4 LIMITATIONS OF THESE WORKSHEETS

There were several limitations in these worksheets that make the assessment of their usefulness very difficult. The first, and the most severe was the limitation on the amount of class time we could spend on these worksheets during the quarters. Second, we did not have a control group for this study; these courses are offered only once per academic year with only one class in any given quarter at OSU. In addition, these worksheets were not the only instruction students had received on these topics. Therefore, any improvement in students' performance could have been influenced from other materials used in these courses. Finally, at this point, lack of reliable pretest and post-test data for some of these worksheets is another limitation in doing a better assessment at this point.

We believe that before we can claim any usefulness of these materials we should administer them to different populations of students many times and revise and improve the instructions based on the feedback from each trial. Therefore, at this time these materials are in a very preliminary stage and we hope to test and improve them in the near future.

8.5 WORKSHEETS

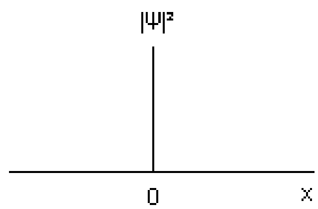
8.5.1 FINDING THE PROBABILITY DENSITY

1. A free moving particle in space on which no net force acts, is described by :

$$\Psi(x) = \Psi_0(x)e^{ikx} \tag{1}$$

where Ψ_0 is a constant. Find the probability density $|\Psi|^2$ for this particle.

2. Plot the probability density $|\Psi|^2$ versus x for this free particle. Interpret your graph. What can you say about the probability of finding a free particle in space?



Uncertainty Principle

1. Recall Heisenberg's Uncertainty Principle:

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}. \tag{2}$$

If you are to define Δx and Δp_x for this free particle what would be the value of Δx and Δp_x? Please explain your answer.

Infinite Potential Well

1. An electron is confined to a one-dimensional, infinitely deep potential energy well of width $L = 100$ pm.

$$V(x) = 0, \qquad \text{if } 0 \leq x \leq L$$

$$V(x) = \infty, \qquad \text{otherwise}$$

Solve the time independent Schrödinger equation for this electron and find the normalization constant.

2. What are the possible values for the electron's energy?

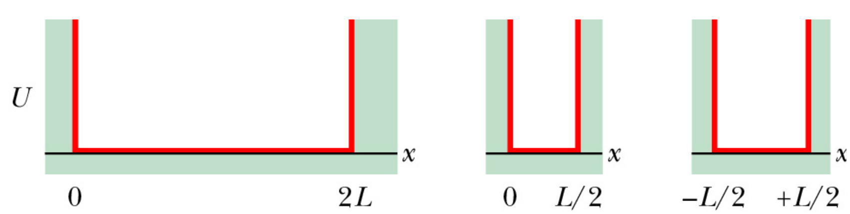
3. Find the probability density $|\Psi|^2$ for this particle for the lowest three energies.

4. Make a plot of the probability density $|\Psi|^2$ versus x (qualitatively) for each wave function.

a) Interpret your graph. What can you say about the probability of finding an electron trapped in a one-dimensional infinite well for each of the plots. State where electron is most likely to be found in each plot.

b) What is the probability of finding the electron outside the well. How does this probability change as energy increases?

5. Consider three infinite potential wells of width $2L$, $L/2$, and L; each contain an electron in the state for which $n = 7$. Rank the wells according to (a) the number of maxima for the probability density of the electron, (b) the value of Δp_x, the uncertainty on measuring the momentum . Explain your answer.



6. Consider an electron confined to a one-dimensional, infinitely deep potential energy well of width L and ground state energy E_0. What happens to the ground state energy of this particle as we decrease the size of the well to $\frac{L}{4}$?

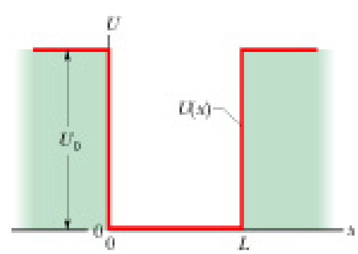
Finite Potential Well

1. An electron is confined to a one-dimensional, finitely deep potential energy well of width $L = 100$ pm.

$$V(x) = -V_0, \qquad \text{for } 0 < x < L$$
$$V(x) = 0, \qquad \text{for } |x| > L$$

where V_0 is a positive constant.

Solve the time independent Schrödinger equation for the bound states with $E < 0$. You may not be able to solve this problem all the way, but try to go as far as you can using the principles you have learned for the infinite potential well.



The finite square well.

If you are not able to solve the Schrödinger equation, use properties of the wave function to predict the probability density $|\Psi|^2$ for the lowest three energies and plot the probability density $|\Psi|^2$ versus x (qualitatively) for each, assuming all three states are bounded.

a) Interpret your graph. What can you say about the probability of finding an electron trapped in a one-dimensional finite well for each of the plots? State where electron is most likely to be found in each plot.

b) What is the probability of finding the electron outside the well? How does this probability change as energy increases?

c) Compare your plots for a finite well with the ones for an infinite well. What are the differences and how do you explain them?

d) Compare the wavelength λ for any given quantum state for an electron trapped in a finite well to the one that is trapped in an infinite well of the same width.

e) Compare the energy E for any given quantum state for an electron trapped in a finite well to the one trapped in an infinite well of the same width.

2. For a potential well with $V_0 = 450$ eV and $L = 100$ pm, calculate the energy of the electron for $n = 3$ and $n = 4$ using:

$$E_n = (\tfrac{h^2}{8mL^2})n^2, \qquad n = 1, 2, 3, ...$$

Using your answer, predict the plot for the probability density of this electron at the $n \gg 3$ state.

Interpret your answer with a physical meaning.

3. The lowest energy state of a particle bound in a finite square well has longer wavelength than of the one in the infinite square well.

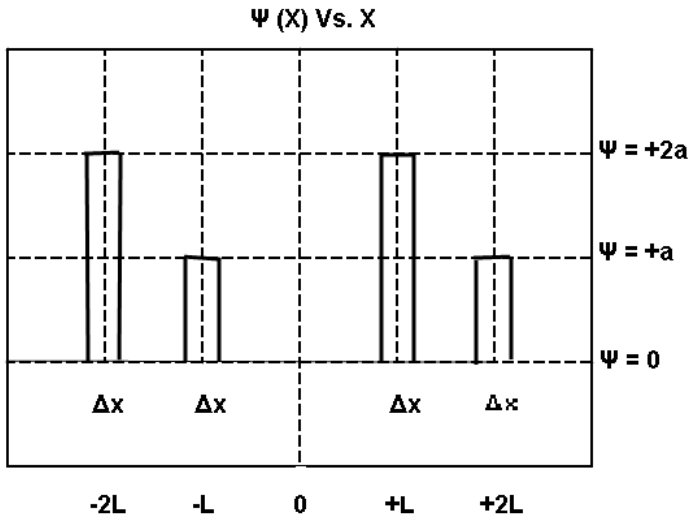
A) True B) False C) It depends

4. The energies for the finite well are, respectively, slightly lower than the ones for the infinite well.

A) True B) False C) It depends

8.5.2 QUANTITATIVE QUESTIONS ON PROBABILITY

1. The figure below shows a plot of a (rather artificial) wave function $\Psi(x)$ versus x , given by the 4 'spikes' each of width Δx over the range of (-2L, +2L). The wave function vanishes for all other values of x. What is your physics interpretation of the spikes of the wave function on this plot?



2. If you measure the position of this particle hundred times where do you expect to find the particle more likely? How often will you expect to find the particle at $x = +2L$? How often do you expect to find the particle at $x = -2L$?

3. What is the probability of finding the particle at $x = -2L$?

4. What is the probability of finding the particle at $x = -L$?

5. What is the probability of finding the particle at $x = 0$?

6. What is the probability of finding the particle at $x = +L$?

7. What is the probability of finding the particle at $x = +2L$?

8. What is the expectation value of x, namely $< x >$?

9. Calculate $< x^2 >$.

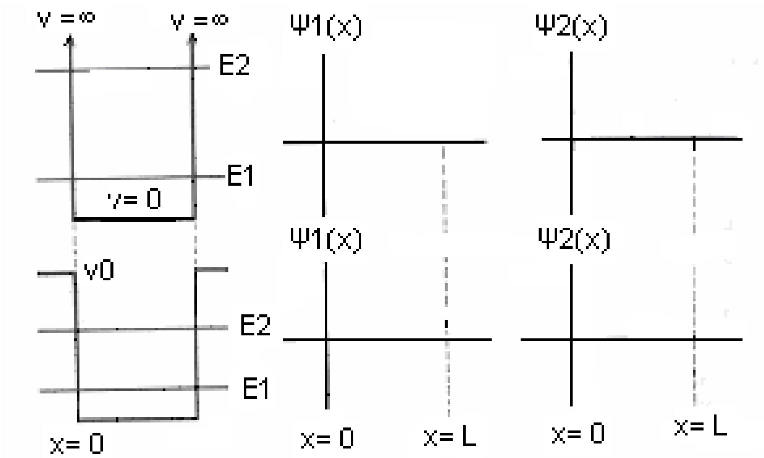
10. What is the value of uncertainty in the position measurement, namely Δx?

11. What is the value of uncertainty in the position measurement, namely Δp_x?

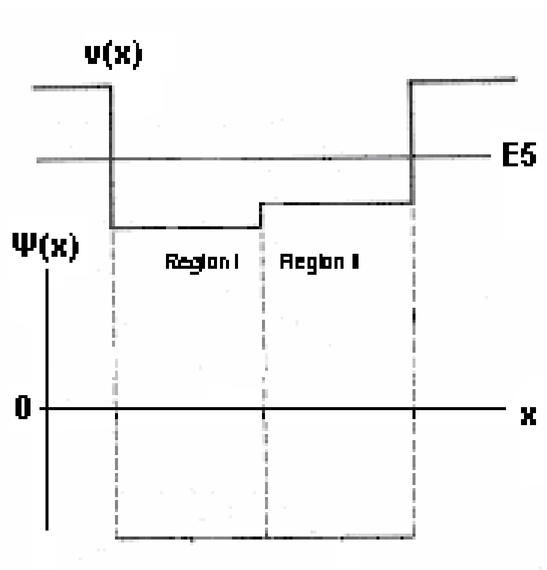
12. What are the physical meaning of $< x >$ and Δx ?

8.5.3 SKETCHING QUANTUM WAVE FUNCTIONS

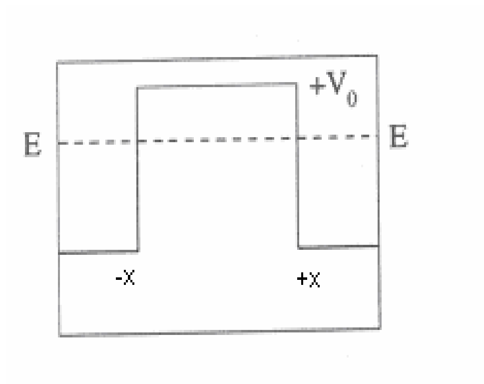
1. An infinite and a finite square well are sketched below. To the right of each well, sketch the wave functions for the two lowest states, respectively, and describe the differences between your answers for the two wells.



2. Sketch qualitatively the wave function for the fifth energy level in this step well. Pay special attention to the maximum values of the wave function in different regions. What physical meaning do these amplitudes have?



3. The figure below shows a particle of energy $E < V_0$ near the boundary of a step up, then step down potential. Solve the time-independent Schrödinger equation in the region (-x, x) using this energy and potential,

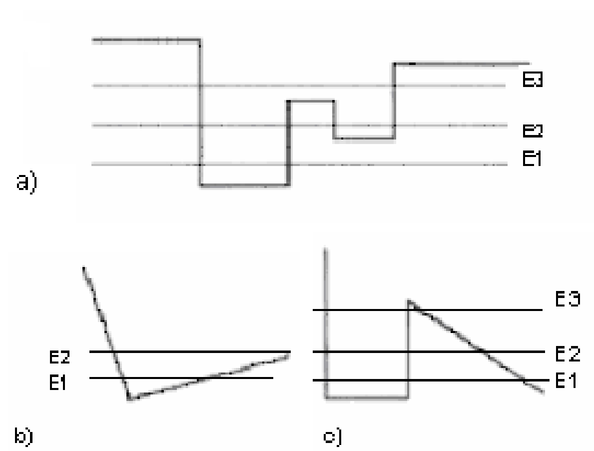


a) What kind of wave functions do you expect to see in the region (-x, x)?

b) Is E a bound state energy?

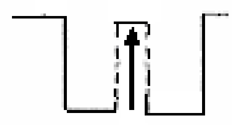
c) Sketch a qualitative possible wave function for this energy.

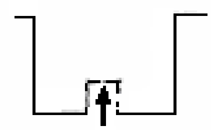
4. Make qualitative plots for wave functions for the indicated energies in each of the potentials below.



5. In the double potential well below, show what happens to the lowest energy wave function as the central barrier grows from zero to infinite height; sketch the wave function for the two heights.

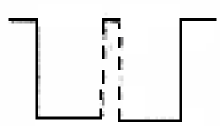
For which well do you expect a greater lowest energy? Why?

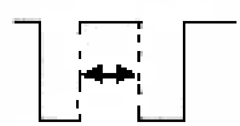




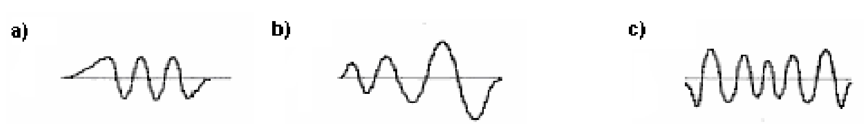
6. In the double potential well below, show what happens to the lowest energy wave function as central barrier grows from zero to full width, sketch the wave function for the two widths.

For which well do you expect to have a greater lowest energy? Why?





7. Sketch the one-dimensional potentials that would give rise to each of the wave functions shown below. Include qualitative features where possible. Draw a horizontal line on each of your diagrams to indicate an appropriate energy for the state in question. Considering the lowest energy level as the first level,what is the number of each energy level shown below?



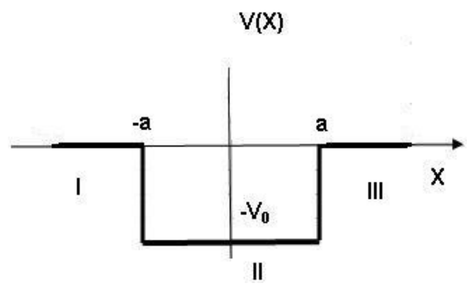
8.5.4 MATHEMATICAL SOLUTIONS TO TIME INDEPENDENT SCHRÖDINGER EQUATION

Finite Potential Well

An electron is confined to a one-dimensional, finitely deep potential energy well of width $2a$; V_0 is a positive constant.

$$V(x) = -V_0, \qquad \text{for } -a < x < a$$

$$V(x) = 0, \qquad \text{for } |x| > a$$



1. Do you expect to have bound, scattering, or free states in this potential well? Explain your reasoning for different values of energy.

2. For each of the three regions **I** $(x < -a)$, **II** $(-a < x < a)$, and **III** $(x > a)$ write down the corresponding time independent Schrödinger equation.

region **I** $(x < -a)$:

region **II** $(-a < x < a)$:

region **III** $(x > a)$:

3. For an electron with $E < 0$, calculate the $\frac{d^2\phi}{dx^2}$ in each region:

region **I** $\quad \frac{d^2\phi_I}{dx^2} =$

region **II** $\quad \frac{d^2\phi_{II}}{dx^2} =$

region **III** $\quad \frac{d^2\phi_{III}}{dx^2} =$

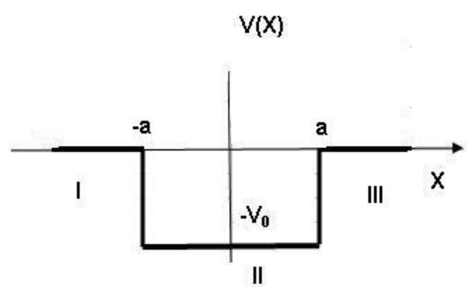
4. Using your answers in part **(3),** write down the general solution for each of the three regions **I, II,** and **III.** ($\phi_{I,}$ $\phi_{II,}$ and ϕ_{III}.)

region **I** $\quad \phi_I \ (x < -a) =$

region **II** $\quad \phi_{II} \ (-a < x < a) =$

region **III** $\quad \phi_{III} \ (x > a) =$

5. Plot qualitative graphs for your answers to part **(4)** on the graph below:



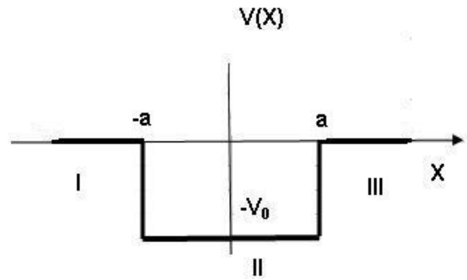
5. A particle under influence of a finite potential well can have unlimited number of bound state energies.

A) True B) False C) It depends on how deep the well is.

6. An electron is confined to a one-dimensional, finitely deep potential energy well of width $2a$; V_0 is a positive constant.

$$V(x) = -V_0, \qquad \text{for } -a < x < a$$

$$V(x) = 0, \qquad \text{for } |x| > a$$



For scattering states where $E > 0$, are the energy states also eigenfunctions of momentum?

1. Yes, because the wave function becomes the same as the wave function for the free particle where the momentum and energy eigenstates are the same.

2. Yes, because there is no potential acting on the particle.

3. No, because the superposition of transfer and reflection waves do not satisfy the $\hat{P}|\Psi(x)>= p|\Psi(x)>$ for all the three regions.

4. No, although we have a free particle wave function in region **I** and **III**, the wave function in region **II** is not the same as free particle wave function.

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[1] See for example, L. McDermott and P. S. Shaffer, "Tutorials in Introductory Physics," Prentice Hall, Upper Saddle River, NJ (1997); R. Steinberg, G. Oberem and L. McDermott, "Development of a Computer Based Tutorial on the Photoelectric Effect," American Journal of Physics **64**, 1370-1379.

[2] D. Hestenes, M. Wells and G. Swackhamer, "Force Concept Inventory," The Physics Teacher **30**, 141-158 (1992).

[3] D. Hestenes and M. Wells, "A Mechanics Baseline Test," The Physics Teacher **30**, 159-166 (1992).

[4] See for example, L. McDermott, "Millikan Lecture 1990: What We teach and What is Learned- Closing the Gap," American Journal Physics, **59,** 301 (1991); L. McDermott and P. Shaffer," Research as a Guide for Curriculum Development: An Example from Introductory Electricity, Part I: Investigation of Student Understanding," American Journal of Physics, **60**, 994 (1992); P. Shaffer and L. McDermott," Research as a Guide for Curriculum Development: An Example from Introductory Electricity, Part II: Design of Instructional Strategies," American Journal of Physics, **60**, 1034 (1992).

[5] Question **Q8.1** was given to the students enrolled in $P631$ on the midterm examination.

[6] Throughout this study we have observed indications of students' difficulties with changing representations to momentum space; however, the specific difficulties with this topic are not discussed.

[7] See Chapter 6 for the details of difficulties students have with the calculation of probability from wave functions.

CHAPTER 9

CONCLUSION AND FUTURE RESEARCH

Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them. Joseph Fourier

Quantum physics is an abstract topic that not only deals with the inaccessible venues and concepts of the microscopic world, but also requires a certain degree proficiency in mathematical skill. This is because the principles of quantum theory are formulated in the language of mathematics. Most student difficulties in quantum mechanics are due to (1) the counter-intuitive nature of the theory and (2) the complicated mathematical procedures involved [1]. The framework of quantum mechanics introduces ideas of probabilities, wave-particle duality, and non-locality into the foundation of physics. These new ideas are abstract and usually challenging for most college students. In addition, although most college students have taken the math courses required for quantum classes, they often still do not have a sufficient mathematical background for understanding the basic postulates of quantum mechanics such as probabilistic interpretation. Thus, one of the difficulties with quantum mechanics relates to formalisms and the translation of physics into mathematical symbols. The other difficulty relates to conceptual understanding of quantum mechanics.

By systematically investigating student learning of quantum mechanics, this study aimed to determine the most common mathematical and conceptual difficulties students encounter in undergraduate quantum mechanics courses [2]. The major objective of this investigation is to build a research base for designing a curriculum to help students develop a functional understanding of quantum phenomena.

This study used different techniques to investigate students' common conceptual and mathematical difficulties in the basic introductory topics in quantum mechanic courses. A variety of methods, formal and informal, were employed to investigate students' ideas and their conceptual understandings of physical phenomena. Following the common process of research on PER, we started with a systematic investigation of students' understanding of basics topics in quantum mechanics by interviewing 21 students enrolled in quantum courses at The Ohio State University. In addition, we observed lectures and read students' written homework and exam papers. This led to the development of more specific weekly questionnaires, which included multiple choice and open ended questions, that were administered to a larger population of students. In multiple choice questions we used students' incorrect ideas as distracters. Twenty-one different weekly questionnaires, with $10-15$ questions each, were given to sum of 156 students enrolled in quantum courses over three different quarters. These questionnaires contained pure physics, pure mathematics and mixed (involved both math and physics) questions. After analyzing students' specific difficulties in selected topics, our ongoing efforts have been toward the developing instructional strategies to address specific difficulties. Of course, these materials needed much testing and revision on the basis of classroom experience before we could actually assess their usefulness.

The students who took part in this study were enrolled in 2 different courses in three distinct quarters: The first quarter of upper-level undergraduate quantum mechanics ($P631$) in fall 2003 and fall 2004, and introductory quantum mechanics at the sophomore level ($P263$) in spring 2004. Of the 61 students who took the introductory quantum course in the spring quarter of 2004, 48 students also enrolled in the advanced course in fall 2004. This allowed us to correlate student background knowledge of certain math topics with their performance on related quantum questions. Section 3.3.1 of Chapter 3 gave specific details about the student population and research context.

9.1 STUDENTS' SPECIFIC CONCEPTUAL DIFFICULTIES

To achieve its primary goal, that is to identify students' conceptual and mathematical difficulties in learning introductory quantum mechanics, this dissertation studied students' understanding of a variety of quantum topics and looked for any relationship between math skills and quantum achievement. A brief review of our findings of student conceptual difficulties is discussed in this section and a review of student difficulties with mathematics is presented in the next section (9.2).

In this study we have focused on 3 main topics: quantum measurement, wave function, and symmetry. Among these topics, quantum measurement and understanding of the probabilistic nature of quantum mechanics, which incorporate the most abstract concepts, are especially difficult challenge for students in both introductory and upper level quantum classes [3]. Findings that support this conclusion are:

a) in Chapter 4 we showed that students' mean scores on this topic were the lowest among the three main topics (excluding the mean scores for math and formalism);

b) student difficulties with probability appear in a variety of different contexts, such as calculating expectation value, probability density, the Uncertainty Principle, probability wave function, spin measurement in the Stern-Gerlach experiment, and classical probability;

c) The analyses of students' understanding, presented in Chapter 5, revealed that student difficulties with understanding quantum measurement and probability are very complex and are not limited to merely conceptual difficulties;

d) from this research it is evident that students often fail to differentiate between the concepts of expectation value and the probability density in a measurement;

e) many students do not have a functional understanding of probability and its related terminologies;

f) students' difficulties with the concept of probability often interfere with their understanding and application of the Uncertainty Principle;

g) some students interpreted uncertainty Principle as human's inability to make a precise measurement and concluded that this inability is the reason for using probability in quantum measurement;

h) students tend to confuse the terms "expectation value" with the "amount of uncertainty" in measurement;

i) some students confuse "probability density" with "probability amplitude" or describe the probability amplitude as a "place" or "area."

One plausible explanation for the difficulties with quantum measurement is that the abstract concepts such as expectation value, probability density, and measurement in momentum space are unfamiliar ideas to students and calculating these terms often requires more sophisticated mathematics than what they are used to it. In addition, the concepts of probability appear to play an important, perhaps overarching role, in the conceptual understanding of quantum mechanics concepts.

Our findings of students' difficulties in understanding topics related to wave functions can be summarized as follows:

a) students had difficulty with the concept of the wave function presenting a probability distribution and calculating its related terms, such as probability density, expectation value, and uncertainty;

b) as a common and persistent difficulty, many students did not square wave functions before finding the probabilities;

c) students had difficulty in understanding the relation between a wave functions and energy eigenstates;

d) students had difficulty with the time evolution of wave functions, for example most students did not write the wave functions in terms of their energy eigenfunctions in order to determine the wave functions in a later time;

e) students appeared to apply semi-classical reasoning to the analysis of wave function at a satisfactory level; however, they over-generalized this approach and they did not use the relationship between the wave function and the wave number as the primary principle for qualitatively analyzing of wave functions in regions of different potential;

f) students tended to use their classical models to interpret quantum systems, for example some students associated a higher energy to a larger amplitude in a wave function.

On the topic of symmetry, mathematical difficulties play an important role in students' ability to solve quantum problems. Our findings suggest that: a) some advanced students are not always able to easily recognize the existence of symmetry in math and physics problems and often fail to use the symmetrical features of a problem to simplify their solutions; b) students' mathematical knowledge of symmetry correlates with their ability to solve symmetry related problems in the context of quantum mechanics; and c) when students lack the appropriate knowledge to solve physics questions, they tend to over-generalize their understanding of some principles related to the problem at hand without carefully inspecting of the applicability.

To help students with these difficulties we developed four instructional worksheets to guide them through step-by-step processes. Chapter 8 provided some examples of the instructional worksheets that we used with the upper level quantum classes at The Ohio State University. They included worksheets on the topics of probability, wave functions, and the mathematical solution to the time-independent Schrödinger equation. However, these materials are in preliminary stage of development. Many revisions and trials will be needed to assess their effectiveness in helping students with their learning of these topics before any claims can be made. The current data collected in this study do not enable the researcher to assess the usefulness of the material developed in this study; hence more research is needed to develop effective materials. Nevertheless, there is anecdotal evidence that these materials were popular with the students.

9.2 ROLE OF MATHEMATICS

This study found three important implications about the role of mathematics and formalism in quantum courses. The first finding is about the relationship between students' mathematics and physics achievements. Our findings on the relationship between students' scores on pure math questions and their final course grade in quantum mechanics show a positive correlation. Although this correlation is small in the case of $P263$ (correlation coefficient 0.347), in $P631$ it is more significant (correlation coefficient 0.547). In Chapter 1, we presented the results of previous research and its various findings. Some researchers found that students' mathematics scores correlated highly with their physics scores; others, however, showed that mathematical skills are only one of several variables prerequisite to the understanding of the physics concepts presented in a typical introductory mechanics course, and that high scores on math tests is not a sufficient indicator of conceptual understanding in physics.

The second finding has ramifications for the two teaching approaches to quantum mechanics. Section 1.2 of Chapter 1 discussed the two distinct approaches to teaching quantum mechanics: quantitative and historical-conceptual. In the quantitative approach, students are introduced to the mathematical algorithms and the processes used to solve quantum mechanics problems. In this process they become acquainted with the mathematical tools needed to understand quantum mechanics concepts. Shankar and McMurray believe in this approach. In the qualitative approach, however, a historical-conceptual mode incorporates the "historical development" of quantum mechanics and introduces students to challenges similar to those faced by physicists in the early 20th century. Our study suggests that students' mathematical abilities seem to be necessary, though not sufficient, for their success in solving related quantum mechanics problems, especially in advanced courses. In Chapter 5, section 5.3.1, we showed that only students who had the required math skills (those who answered pure math questions correctly) were able to answer certain quantum physics questions (mixed math and physics) correctly. However, not all the students with the required knowledge of math could do well in solving quantum problems. In other words, students with poor math backgrounds scored lower in related quantum problems and some of the students possessing the required knowledge of mathematics still had difficulties relating the mathematical formalism.

One plausible reason for the latter is that, as researchers in science and cognitive education have shown, students' understandings of physics are highly fragmented and compartmentalized and are loosely linked in students' cognitive structure, if at all. Therefore, the acquired knowledge in mathematics might not transfer as easily as one might assume. The difficulties with transferring knowledge seems to be greater in quantum courses due to the abstract nature and high level of formalism in this theory.

This brings us to our third result that relates to students' ability to deal with abstract material and move beyond concrete reasoning by using mathematical symbols and formalism. There are many examples in this study that show students' difficulty with abstract reasoning and mathematical formalism: a) in Chapter 5 we show how students fail to reason abstractly about classical probability and lack the ability to use abstract symbols to formalize their conceptual understanding; b) in Chapter 6 the data supported that student have difficulty recognizing mathematical symbols for a given graph and showed their lack of ability to associate functions with their graphs; c) some students can not differentiate an oscillatory function, such as e^{-ix}, from an exponential decay function, such as e^{-x}, (Chapter 6); and d) students often perform better on questions that include visual aspects of the problem, as compared to similar but more abstract questions[7]. The examples of student difficulties with mathematical questions on the topic of symmetry was discussed in Chapter 7.

In college courses students are expected to go beyond the concrete stage of reasoning and attune to connections and meaning instead of numbers and physical objects. Our findings, however, suggest that many undergraduate physics students in junior/senior level courses have not gone beyond the concrete stage of reasoning and they focus on immediate reality, real numbers, and exact directions. These students are not able to use abstract mathematical symbols instead of numbers and often fail to look at the big picture to get an overall impression of what is happening. In the following section, we will discuss possible reasons for this problem and the implications for instructors.

9.3 IMPLICATIONS FOR INSTRUCTORS AND CURRICULUM DEVELOPMENT

The study of students' conceptual and mathematical difficulties can lead to development of research-based instructional materials that can help students in learning difficult concepts. The information gathered from such research can help instructors to develop a more effective curriculum; for example, recognizing student specific difficulties and explicitly addressing the problematic topics can lead to better instruction.

In addition, this dissertation has three important findings. First, most students in sophomore and even junior/senior level courses have still difficulties with abstract concepts and basic mathematical formalisms. Secondly, students in introductory quantum courses, even the good ones, tend to extend their use of classical physics

[7]The effect of visualization is not a concern here. The point is that visual cues provide more concrete features and often reduce the level of abstraction in most questions.

models to analyzing quantum mechanical systems. Finally, use of visual cues can help students with understanding of abstract materials in quantum mechanics. Below we discuss these three results with the implications to instructors in more details.

9.3.1 ABSTRACT VERSUS CONCRETE

One important result of this study is the discovery that many college students have a poor ability for abstract reasoning. According to McKinnon et al. the majority of college freshman do not enter college with adequate skills to argue logically and abstractly about the importance of given principle. They have difficulty applying a known principle when the context in which they have used it is slightly altered [4].

The first evaluation method of the ability to think logically has been developed and verified by a Swiss psychologist, Jean Piaget. During many years of research with children, Piaget found that children progress through various stages of mental manipulation and that these steps cannot be circumvented. Prior to thinking about abstract ideas, students must undergo a period in which they physically manipulate objects using the basic principles upon which the abstraction to be developed depends [5]. This stage Piaget identifies as the concrete stage of thought. Students may handle concrete ideas quite adequately, but until they have had many experiences of manipulating the objects they cannot recognize those concepts in the context of a broader generalization, of which the manipulative experiences and the concepts are simply a subset. Most students should become formally operational, i.e., capable of abstract logical thought from $11-15$ years of age, well before they start college courses [6].

Although the value of concrete reasoning and real-world examples in learning physics is undeniable, it seems that in recent education programs the importance of concrete stages of learning is overemphasized. For example, sometimes educators in college level courses omit the abstract stages of instruction in order to make it more accessible to students. Nevertheless, this teaching approach hinders students' development beyond the concrete reasoning stage. According to education psychology, abstract reasoning is often an important stage of learning models. For example, David Kobal's proposed model for learning, which relies heavily on the ideas of Dewey and Piaget, includes "the formation of Abstract Conceptualizations" as one of the 4 stages of learning [7]. In general, learning originates in concrete experience but experience is not the whole thing. In fact, it is just a beginning. Learning depends on experience, but it also requires reflection, developing abstraction, and active testing of our abstractions.

In the meantime, state departments of education insist that local school districts maintain minimum requirements for graduation. To meet these standards educators developed applied courses of which some do not contain the same scope and sequence of principles and concepts as the basic high school courses taken by the higher-ability students to prepare them for college. Subsequently, some applied academic courses evolved into low-expectation courses. Therefore, colleges and universities cannot assume students' high school education provides sufficient background for students' success in higher education.

Omitting or bypassing the abstract stage of learning can impede students' growth and ability to use abstract symbols and formal notations in science courses. In section

4.3.1, we discussed students' attitude change towards mathematics from spring quarter, when they were enrolled in $P263$, to fall when they were taking $P631$. This change in attitude probably occurred because most students were not prepared enough for the level of mathematics and abstraction they faced in $P631$ and they found their mathematical ability insufficient for understanding the problems in upper level courses.

Students enrolled in introductory quantum mechanics courses are usually physics majors who have taken required physics and mathematics courses. Therefore, physics educators and others tend to assume that students who have passed these courses have mastered the required skills and concepts presented in them. Based on this research, and research conducted on students' concepts in other fields of physics, this assumption often is not valid. Many students have trouble moving from the relatively low level of mathematical and intellectual rigor of the introductory courses in the sophomore year to the much more demanding courses that begin in the junior year. Even some better students are not entering the course with the command of the subject matter that instructors expect them to achieve.

One practical suggestion to help with this problem is to teach mathematical courses in physics departments. This can result in two advantages. Firstly, a course in the physics department could focus on the topics that are widely used in physics courses, and secondly, could provide a physics context for most of the subjects that students have been exposed to in previously math courses. In this way, the physics curriculum could ensure that students have received the mathematical instructions on the topics that often appear in physics courses [9]. Additionally, the physics context of these courses can help with the problem of transferring math into physics. Although these types of courses are very common at the graduate level, just recently many physics departments have recognized the crucial role of such a course in preparing students' for the level of mathematics and abstract reasoning necessary in more advanced undergraduate courses [10].

9.3.2 CLASSICAL PHYSICS

Another important finding of this study relates to students' classical physics background. First, we found that students' background knowledge of classical waves affect their achievement in introductory quantum courses (correlation coefficient of 0.487). Therefore, a solid understanding of mechanical waves and related topics, such as superposition, standing waves, Fourier decomposition, interference and diffraction, seem to be a prerequisite to understanding introductory topics in quantum courses.

Second, students with a good background knowledge of classical physics tend to bring these classical models to quantum domain. For instance, some students used classical models when calculating probability density, and others tended to associate a high amplitude wave function with higher energy for particles. These findings are in agreement with Bao's findings about students' tendency to interpret a 1-D quantum potential well as a 2-D classical gravitational well [11].

Since students' physical intuition is so firmly grounded in classical mechanics, they have little choice but to advance these intuitions as far as they can into quantum mechanics. However, sometimes they are not careful in recognizing the limitation of classical mechanics and the fundamental differences of these two domains. They have to recognize when quantum systems can be laid out with the help of classical mechanics and when classical physics is unable to explain a quantum context.

Perhaps, by implementing a lesson that explicitly compares classical and quantum mechanical models description of different systems and different contexts, instructors can help students recognize the limitations of classical models in the quantum regime. For example, explicitly stating the similarities such as wavelength and wavenumber, while stressing the different interpretation of amplitude in mechanical and quantum waves, conceivably could prevent students' incorrect use of classical models in quantum mechanics.

9.3.3 VISUALIZATION

Over a variety of contexts and in over 250 of the questions we gave to students in this study, overall, it seemed students understood and scored better on the questions that had a visual image. For example, in the topic of probability and SG experiments, students' average scores were higher on the questions that contained a graphical picture of the experiment as compared to similar questions in more formal and abstract settings. This should not be a surprise since the visual aspects of these questions reduced the level of abstraction and gave students a more concrete understanding of what the problem was about. Since quantum mechanics theory is a very abstract topic by nature, the visual aspects of quantum mechanics play an important role in student conceptual understanding of quantum phenomena. Physics instructors should take advantage of this and develop and use available resources to teach the visual aspects of quantum mechanics with the hope of extending their students' experiences in learning and their holistic understanding of these concepts.

Finally, instructors can benefit from number of methods that PER has proven to improve students' learning. Though these methods are developed for introductory physics courses they are robust and applicable to other areas of education. For example, active participation (as opposed to passive note-taking), exploring examples before discussing the general theory, group activities and peer instruction, and a clear focus on content objectives could be implemented in more advanced courses such as quantum mechanics. We believe that the need for these teaching strategies should not vanish between the sophomore and junior years, so we suggest this proven pedagogical approach.

9.4 LIMITATIONS OF THIS STUDY

There were several limitations to this study that restrict the generalization of its results. The main results of the present study were generated by analyzing students' responses to our questionnaires during three quarters (fall 2003, spring 2004, fall 2004) at the Ohio State University. The sample consisted of undergraduate students enrolled in sophomore and junior/senior level quantum courses ($P263$ and $P631$). The sample size was 61 in $P263$, and 95 in $P631$ (sum of the two quarters). Forty-eight students were the same in both these groups. Ideally, more students should be involved in such a study to validate its findings. Due to the nature of the concepts, and the unique population, it would have been extremely difficult to administer several iterations to hundreds of students as recommended by some experts. While the students who participated in this study were most likely representative of groups at similar universities, due to the small sample size, the limited number of universities involved, and the predominantly male population caution is warranted in interpreting and applying these findings to other populations.

Furthermore, the choice of topics in this study initially was based on researchers' class observation and informal interviews with the instructors of these courses. Students' difficulties in these topics were confirmed later by interviewing individual students. Hence, there exist other topics of quantum mechanics that are as fundamental as the ones discussed in this study. In addition, this dissertation might not have been able to cover all the possible difficulties students have with the three main topics of this study.

As noted in Chapter 3 section 3.3.1, the main population for this study was undergraduate physics students and the questionnaires used in this research were developed based principally on contemporary physics curricula in quantum mechanics. Hence, the core content and topics of our research questionnaires is limited to the physics curriculum. Information should be gathered about the content of the topics taught in introductory quantum mechanics courses in related fields, such as chemistry, for further study of student difficulties in this course.

Finally, part of the data gathered in this study was gathered principally from volunteers whose final course grades were not based upon their performance on the questionnaires or interviews. If these data were gathered in more naturalistic environments such as in classrooms in which students' scores was a specified proportion of the class grade, the outcome might have been different. In addition, the voluntary basis of these questionnaires in some cases caused us to lose data. That is, some students did not participate on all the weekly questionnaires; as a result we could not include them in the longitudinal analysis of the data in Chapters 4 and 7.

9.5 RECOMMENDATIONS FOR FUTURE RESEARCH

Further research is needed to support the findings of this dissertation. However, the results of this study can provide information about the relationships between students' specific difficulties in quantum topics and their mathematical abilities. Two specific sets of recommendations emanate from this research. First are recommendations pertaining to the enhancement of research reliability. These recommendations are based upon the methodology of research and interpretation of the results presented in Chapters 5, 6, and 7. Second are the recommendations pertaining to the future use of this study in furthering understanding in physics education.

9.5.1 RECOMMENDATIONS PERTAINING TO THE ENHANCEMENT OF RESEARCH RELIABILITY

As discussed in the limitations section, the research methodology followed in this study and the data collection procedures limit the ability to generalize and the nature of the findings. Expanding the student population and examining data for larger numbers of more diverse students will play an important role in extending the validity and the ability to generalize the findings. Hence, the following recommendations pertaining to the improvement of research are suggested:

1. Alternative forms of information and feedback should be gathered from participating students to ascertain their rationales for specific responses to the questions in the study. For example, in multiple choice questions one could ask for more explanations and ask for students to state their confidence levels in their answers. This could give more information when analyzing data;

2. The results of this study provided interesting data on students' understanding of introductory quantum mechanics. Their understanding of these concepts is important and should be investigated further. Hence, the following recommendations associated with examining students' understanding in introductory quantum mechanics also emanate from this study:

   (a) The construction of the research questions could have received further feedback from experts and the instructors of these courses;

   (b) The nature of the relationship between visual understanding of specific quantum mechanics concepts and the verbal and mathematical understanding of these concepts is very important and should be investigated more specifically;

   (c) Identifying principal mathematical skills and subtopics important to understanding quantum mechanics concepts could help improve the teaching and learning of quantum mechanics concepts. Such studies should inform the development of more effective textbooks and technology related teaching materials;

   (d) The questions in this study should be administered at different universities in different settings to further assess students' understanding of introductory quantum mechanics. The results obtained from earlier studies of this kind can be compared with the baseline data provided by this study;

   (e) The research questions in this study should be administered to more diverse groups (e.g., gender, ethnicity, institutional, etc.) to examine the effects of culture and context;

(f) Similar studies should be done on students in quantum mechanics on related fields beyond physics, such as in electrical engineering, materials science, chemistry, etc., and the results obtained from such studies could be used to inform the development of courses and teaching resources.

9.5.2 RECOMMENDATIONS PERTAINING TO THE FUTURE USE OF THIS STUDY

Recommendations on possible use of this research material in future studies can be summarized as follows:

1. The research questions in this study could be used to examine the effects of specific teaching strategies on students' understanding of introductory quantum mechanics;

2. Furthermore, these questionnaires could be used to study the relationship between students' understanding of introductory quantum mechanics and other ancillary concepts such as understanding of classical waves, understanding of statistical physics, mathematical skills, etc.;

3. The results of such studies have potential to improve knowledge and understanding of the teaching and learning of quantum mechanics and to inform the development of improved teaching materials that can enhance students' understanding of quantum mechanics concepts. For instance, the identification of student difficulties in learning quantum mechanics can help with the content of more effective computer simulations, visualization aids, and tutorials for teaching these courses.

9.6 FINAL REMARKS

A great deal of empirical research has been conducted regarding teaching and learning physics. Over the years teaching practices, textbooks, and instructional resources have been developed, which continue to influence methods of teaching and learning physics. This study resulted in several research-based instructional materials that seem popular with the students. However, they are in a preliminary stages of development and many revisions and trials are needed to assess their effectiveness in helping students learn these topics before we can make any definitive claims.

Our hypothesis was that students have difficulties in a number of quantum topics. We investigated these difficulties in three areas of learning: (1) understanding physics, (2) procedural knowledge of mathematics, and (3) interpreting the language of mathematics into physics and vice versa. This study resulted in the identification of several specific student difficulties in a variety of quantum mechanical topis. In addition, this study found a correlation between students' mathematical ability and their success in quantum courses. Finally, the limited instructional worksheets in this study may be starting point for the development of more effective classroom material for these courses.

The results of this study have implications not only for instructors, but also for reform efforts towards improving the physics curriculum. There is a need for explicit instruction to address the conceptual and mathematical difficulties that students have with topics such as quantum measurement, expectation values, probability densities, the Uncertainty Principle, wave functions, time evolution, and symmetry. Research has shown that unless there is explicit instruction the difficulties with fundamental materials are likely to persist and prevent advanced learning [12].

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[1] D. Griffiths, *" Introduction to Quantum Mechanics,"* Prentice-Hall, NJ (1995).

[2] H. Sadaghiani and L. Bao, AAPT Ann. 33 (2) 113, 172 (2004).

[3] These results are consistent with findings of previous research in this are; for example, students responses to VQMT displayed a more limited understanding of probability and quantitative concepts.

[4] J. McKinnon and John W. Renner, "Are Colleges concerned with Intellectual Development?" American Journal of Physics **39** 1047 (September 1971).

[5] J. Piaget, J. Res. Sci. Teach., 2, 176 (1964).

[6] J. Piaget and Inhelder, *"Growth of Logical Thinking"*, Basic Books, New York, 337 (1958).

[7] D. Kolb, *"Experiential Learning: Experience as the Source of Learning and Development,"* New Jersey: Prentice Hall (1984).

[8] Although it is the most commonly cited, whether Kolb's 4-stage Experiential learning Cycle best represents learning in all situations is debatable. Here it is only an example of many learning models that include abstract conceptualizations.

[9] For example, the data from one of our surveys show that many students in $P263$ had not taken a collage course on statistics and had no formal instructions on probability beyond high school.

[10] For example, The Ohio State University recently modified it physics curriculum, by replacing $P263$, introduction to quantum mechanics, with a course on math methods.

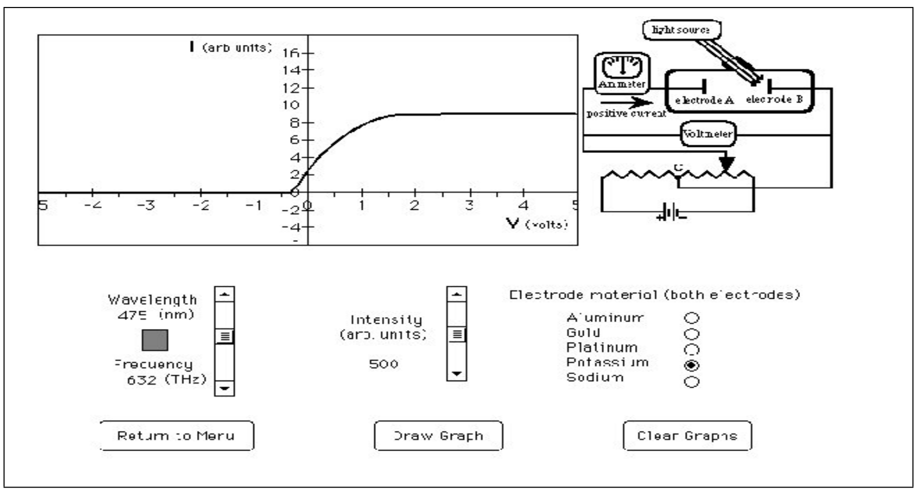
[11] L. Bao, "Dynamics of Student Modeling: A Theory, Algorithms, and Application to Quantum Mechanics," Ph.D. dissertation, University of Maryland (1999).

[12] L. McDermott, P. Shaffer, and M. Somers, "Research as a Guide for Teaching Introductory Mechanics: An Illustration in the Context of the Atwood's Machine," American Journal of Physics, **62**, 46- 55 (1994).

APPENDIX A. PHOTOELECTRIC TUTORIAL

. .

A sample dialogue between computer (**C**) and student (**S**) that occurs after the student draws an I-V graph which includes a negative value for current.



The interactive computer program in the Photoelectric Tutorial interprets students' graphical input and enters into a dialogue with the student.

C: Okay. Let's discuss the graph you have drawn. When there is a current flowing in the circuit, what are the charge carriers?

S: electrons

C: Okay Electrons are the charge carriers. When there is a reading on the ammeter, is there a flow of electrons in the space between the electrodes?

S: Yes

C: Okay! There must be a flow of charge everywhere in the circuit. If there is a flow of electrons in the vacuum tube, they must be ejected from an electrode. What makes these electrons be ejected from the electrode?

S: the voltage between the electrodes

C: Do you think 5 volts between two pieces of metal separated by several inches would be enough to remove electrons from either piece of metal?

S: no

C: I agree. You would need a much larger voltage. So, what makes these electrons be ejected?

S: the light incident on the electrode

C: Good! Light (photons) striking an electrode can cause electrons to be ejected. Would you expect electrons to be ejected into the vacuum from: a) only electrode A, b) only electrode B, or c) both electrodes? (Type a, b, or c.)

S: c

C: We agreed that light is needed for electrons to be ejected into the vacuum. No light is incident on electrode A. So, from where are the electrons being ejected

S: b

C: Right! ...

APPENDIX B. STUDENT TENDENCY TO MIX <P> WITH PROBABILITY DENSITY

. .

Student Tendency to Mix $< \hat{p} >$ with $|\varphi(k)|^2$

Examples of students' homework that shows their difficulties with the topics expectation value and probability are presented in this Appendix. The question and the correct solution are included.

> What is the Fourier Transform of $\delta(x)$? Using Plancherel's theorem, show that [a]:
>
> $$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \tag{9.1}$$
>
> **b)** What is the Fourier Integral Representation of $\delta(x - x_0)$?
> **c)** Suppose that at time $t = 0$, a position measurement is made on a particle and $x = x_0$ is found. Assume that measurement was precise enough and the wave function immediately following it is well approximated by a δ- function (or a very narrow Gaussian). Argue, based on your answer to (b) that there is equal probability to find any value of the momentum immediately afterwards. This is, of course, in agreement with the Uncertainty Principle [**?**].
>
> [a] The first part of this question is a problem from the textbook: D. Griffiths, *"Introduction to Quantum mechanics"* problem 2.25.

Solution to Part (a):

$$
\begin{aligned}
\mathcal{F}[f(x)] &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-ikx} f(x) dx && (9.2)\\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-ikx} \delta(x) dx && (9.3)\\
&= \frac{1}{\sqrt{2\pi}} e^{-ik0} = \frac{1}{\sqrt{2\pi}} && (9.4)
\end{aligned}
$$

The Plancherel's theorem defines the inverse Fourier transform as

$$
\begin{aligned}
\mathcal{F}^{-1}[f(k)] &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{ikx} f(k) dk && (9.5)\\
\mathcal{F}[\frac{1}{\sqrt{2\pi}}] &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{ikx} \frac{1}{\sqrt{2\pi}} dx && (9.6)\\
\delta(x) &= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{ikx} dx && (9.7)
\end{aligned}
$$

Solution to part (b):

The Fourier transform of $\delta(x - x_0)$ using (2) is:

$$
\begin{aligned}
\mathcal{F}[\delta(x - x_0)] &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-ikx} \delta(x - x_0) dx && (9.8)\\
&= \frac{1}{\sqrt{2\pi}} e^{-ikx_0} && (9.9)
\end{aligned}
$$

and the inverse Fourier Transform of this according to (5) is:

$$F^{-1}[\frac{1}{\sqrt{2\pi}}e^{-ikx_0}] = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{ikx}(\frac{1}{\sqrt{2\pi}}e^{-ikx_0})dk \tag{9.10}$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-ik(x-x_0)}dk \tag{9.11}$$

therefore:

$$\delta(x-x_0) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-ik(x-x_0)}dk \tag{9.12}$$

Solution to part **(c)**:

if

$$\Psi(x, t=0) = \delta(x-x_0) \tag{9.13}$$

then

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-ikx}\delta(x-x_0)dx \tag{9.14}$$

$$\frac{1}{\sqrt{2\pi}}e^{-ikx_0} \tag{9.15}$$

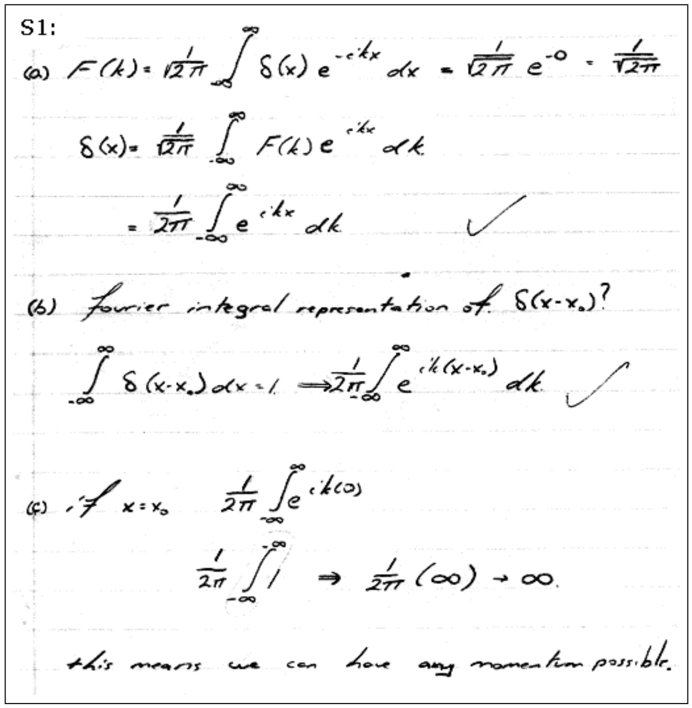
Since the probability of finding a particle momentum, from $\hbar k$ to $\hbar(k+dk)$ is:

$$\phi^*(k)\phi(k)dk = \frac{1}{\sqrt{2\pi}}e^{ikx_0}(\frac{1}{\sqrt{2\pi}}e^{-ikx_0})dk \tag{9.16}$$

$$= \frac{1}{2\pi}dk \tag{9.17}$$

This value is independent of k, so all momentum values must be equally probable, since by varying k, the probability of finding that momentum does not change.

Examples of Student Incorrect Responses on Part (c)



S1:

(a) $F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x)\, e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} e^{-0} = \frac{1}{\sqrt{2\pi}}$

$\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk$

(b) fourier integral representation of $\delta(x-x_0)$?

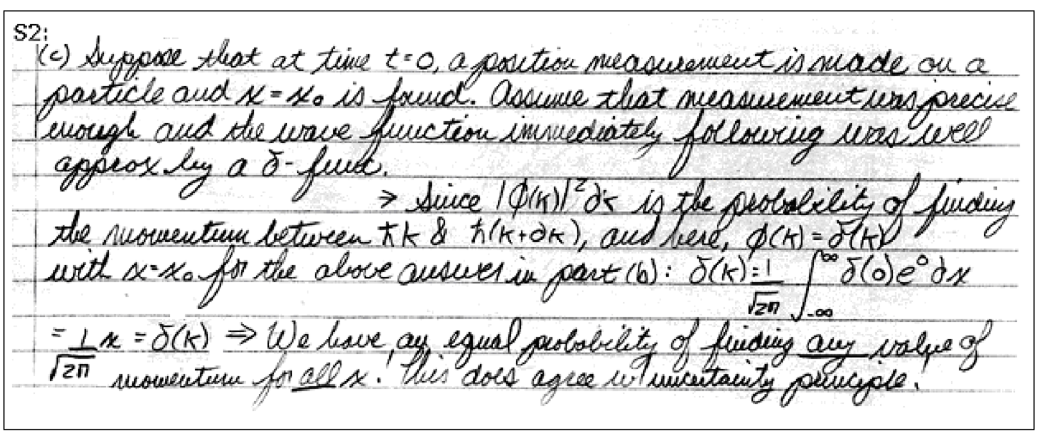
$\int_{-\infty}^{\infty} \delta(x-x_0)\, dx = 1 \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)}\, dk$

(c) if $x = x_0$ $\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(0)}$

$\frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \Rightarrow \frac{1}{2\pi}(\infty) \to \infty.$

this means we can have any momentum possible.

An example of students incorrect work: **S1** incorrectly calculated $\phi(k)$ to be ∞, and concluded: *"This means we can have any momentum possible."*



S2:

(c) Suppose that at time $t=0$, a position measurement is made on a particle and $x = x_0$ is found. Assume that measurement was precise enough and the wave function immediately following was well approx by a δ-func.

→ Since $|\phi(k)|^2 dk$ is the probability of finding the momentum between $\hbar k$ & $\hbar(k+dk)$, and here, $\phi(k) = \delta(k)$ with $x = x_0$ for the above answer in part (b): $\delta(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(0) e^{0}\, dx$ $= \frac{1}{\sqrt{2\pi}} x = \delta(k)$ ⇒ We have an equal probability of finding any value of momentum for all x! This does agree w/ uncertainty principle.

An example of students incorrect work: **S2:** "*Since* $|\phi(k)|^2 dk$ *is the porobability of finding the measurement between* $\hbar k$ *&* $\hbar(k+dk)$*, and here,* $\phi(k) = \delta(k)$ *with* $x = x_0$ *for the above answer in part (b):* $\delta(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(0) e^0 dx = \frac{1}{\sqrt{2\pi}} x = \delta(x) =>$ *We* *have equal probability of finding any value of momentum for all x! This does agree with Uncertainty principle."*

S3:

c) $\langle p \rangle = \int_{-\infty}^{\infty} \Psi^* i\hbar \frac{\partial}{\partial x} \Psi \, dx = \int_{-\infty}^{\infty} \delta^*(x)\, i\hbar \frac{\partial}{\partial x} \delta(x)\, dx$

$\langle p \rangle = \int_{-\infty}^{\infty} \delta(x)\, i\hbar \cdot \underbrace{\frac{1}{2\pi} e^{ikx}} dx = \frac{i\hbar}{2\pi}$ ← constant, not x-dependent

An example of students incorrect work: **S3:** "$<p>= i\hbar r/2\pi$ *this is constant, not x dependent."*

S4:

c) THE FOURIER TRANSFORM OF DELTA FUNC IN K-SPACE GIVES spike @ THAT pt. EXPECTATION VALUE OF MOMENTUM IN K-SPACE IS A CONSTANT.

An example of students incorrect work: **S4:** *"The Fourier Transform of a δ−function in k-space gives spike at that point. Expectation value of momentum in k-space is a constant."*

S5:

~~c) $t=0$, $x=x_0$, so: $\delta \to \frac{1}{2\pi} e^{i(kx_0)} = \Psi(x,t)$~~

~~so, $\langle \hat{p} \rangle = \int \Psi^* \left(i\hbar \frac{\partial}{\partial x}\right) \Psi \, dx = \int \frac{ik}{2\pi} e^{-ikx_0} e^{ikx_0} dx = \frac{ik}{2\pi}$ → constant so equal l'hood~~

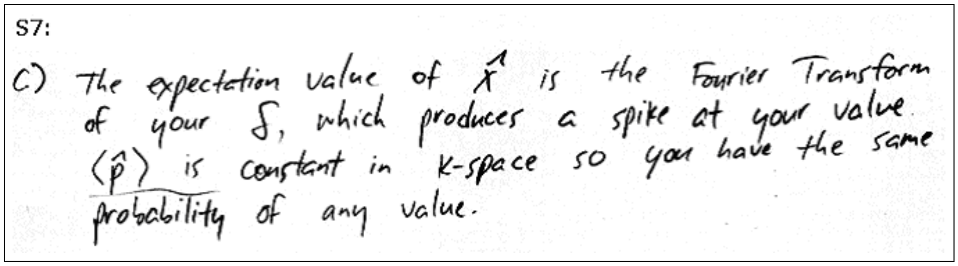
An example of students incorrect work: **S5:** "$t = 0, x = x_0$, *so:* $\frac{1}{2\pi}e^{i(kx_0)} = \Psi(x,t)$ *so,* $<p>= \int \Psi^* p \Psi dx = ... = \frac{ik}{2\pi}$ *constant so equal likely."*

S6:

c) if $x = x_0$ $\quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(0)} = \frac{1}{2\pi} \int e^{(0)} = \frac{1}{2\pi} \infty$

def. position relates to any possible momentum the finer you

An example of students incorrect work: **S6:** *"if* $x = x_0, \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{ik(0)} = \frac{1}{2\pi}\int e^0 = \frac{1}{2\pi}\infty$ *definite position relates to any possible momentum that is fine with you."*


S7:

c) The expectation value of $\hat{x}$ is the Fourier Transform of your δ, which produces a spike at your value. $\langle \hat{p} \rangle$ is constant in k-space so you have the same probability of any value.

An example of students incorrect work: **S7:** "*The expectation value of* $\hat{x}$ *is the Fourier transform of your* δ*, which produces a spike at your value* $< \hat{p} >$ *is constant in* k*-space so you have the same probability of any value."*

S8:

"$\phi(k) = \int \delta(x - x_0)dx = e^{(ikx_0)}$

$|\phi(k)|^2 = 1 =>$ *probability of finding momentum=* dk *between* k $\hbar k$ *and* $\hbar(k + dk)$ *the momentum probability density is a constant independent of* x *and* t*;the probability of measurement any momentum is the same."*

S9:

"$t = 0, x = x_0$

$\delta(x_0) = \frac{1}{2\pi}\int e^{(ik_0)}dk = \infty,$ *The momentum can have any value."*

S10:

"The Fourier transfer of the delta function in k*-space forms a spike at this point. The expectation value of momentum is a constant in* k*-space; this means that momentum probability is equal."*

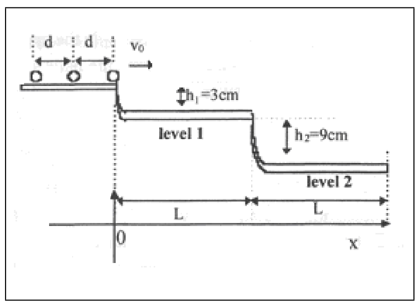
S11:

"$< x >$ is the Fourier transform of a delta function in k-space returns the spike at that point, and the expectation value of the momentum is a constant tin k-space, thus making the probability of the momentum equal."

. .

Classical Probability of a Position Measurement (Version 1)

Consider the experiment shown below. A series of balls is set moving towards the right at a very small velocity V_0 and L >>d (Ignore friction.)



1. Is the probability of finding a given ball on level 1 greater than, less than, or equal to that of finding on level 2? Explain your reasoning.

2. If you know the speed of a given balls in each level, can you determine the probability of finding that ball on each level? Explain.

3. If you know the time a given ball spends on each level, can you determine the probability of finding that ball on each level. Explain.

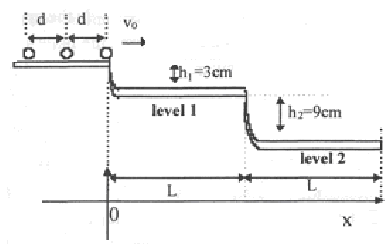
4. If ball spends t_1 minutes in level 1 and t_2 minutes in level 2, what is the probability of finding that ball on each level? Explain.

5. If ball spends 6 minutes in level 1 and 3 minutes in level 2, what is the probability of finding that ball on each level? Explain.

6. How did you come up with this answer?

Classical Probability of a Position Measurement (Version 2)

Consider the experiment shown below. A series of balls is set moving towards the right at a very small velocity V0. (Ignore friction)



(Main question) Is the probability of finding a given ball on level 1 greater than, less than, or equal to that of finding on level 2? Explain your reasoning.

1. If you are taking pictures of this experiment at random times, will the pictures show more balls on level one or level 2? Explain your reasoning.

2. If you know the time given ball spends on each level, can you determine the probability of finding that ball on each level. Explain.

3. For a given ball, compare the speed of ball on level 1 and 2: (V_1 and V_2)

   (a) Qualitatively

   (b) Quantitatively

4. For a given ball, compare the amount of time that ball spends on each level: (t_1 and t_2)

   (a) Qualitatively

   (b) Quantitatively

Probability Density:

1. Do you know what the probability density of a measurement means?

2. Can you describe the meaning of it for this problem? What is the unit of probability density in this case?

3. Draw a graph of probability density versus position, from $x = 0$ to $x = 2L$. Be sure to label the relevant values of the vertical axis.

4. What is the probability of finding a ball between 0 and $2L$? How could you represent this condition in terms of $P(x)$?

5. Imagine splitting level 1 into two unequal segments: segment "1A" from $x = 0$ to $x = 1/3L$, and segment "1B" from $x = 1/3L$ to $x = L$. How many times more likely is to find a given ball in segment 1A than in segment 1B? Explain.

6. Find the probability of finding a given ball along that segment divided by the length of the segment.

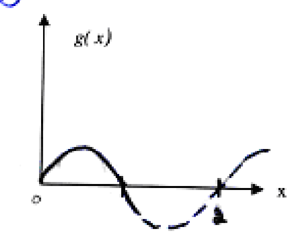
APPENDIX D. EXAMPLES OF STUDENTS' DIFFICULTIES WITH THE GRAPHING COMPLEX FUNCTIONS

. .

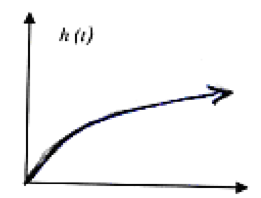
Consider the function below:

$$\Psi(x,t) = \frac{\sqrt{2}\,\mathrm{Sin}[\pi\, x]}{\sqrt{L}}\; e^{-iEt/\hbar} = g(x)\, h(t)$$

a. In the space below predict what the plot of ~~Ψ~~g *(X)* would look like?

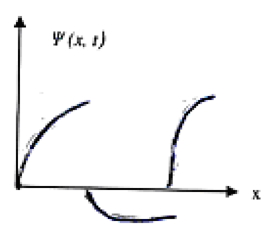


b. In the space below predict the plot of the ~~Ψ~~h *(t)* would look like?



c. Using your answer for part (a) and (b), in the space below predict the plot of the *Ψ (x,t)* would look like at:

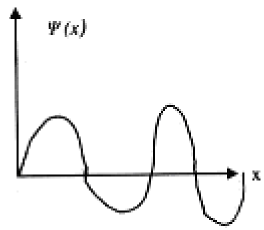
1) t = 0?
2) t = π/2
3) t = π
4) t =3/2π
5) t =2π



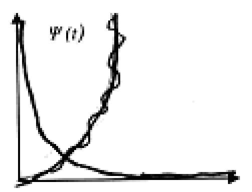
If the wavefunction for a particle in an infinite square well of width L, given as:

$$\Psi(X,t) = \frac{\sqrt{2}\,\mathrm{Sin}[\pi\, x]}{\sqrt{L}}\, e^{-iEt/\hbar} = \Psi(X)\,\Psi(t) \qquad -i\,E\left(\frac{t}{\hbar}\right)$$

a. In the space below predict what the plot of $\Psi(X)$ would look like?

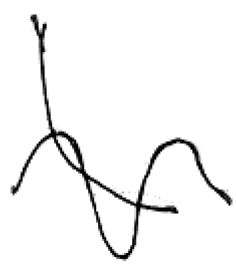


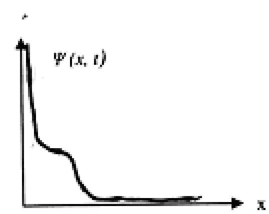
b. In the space below predict the plot of the $\Psi(t)$ would look like?



c. Using your answer for part (a) and (b), in the space below predict the plot of the $\Psi(x,t)$ would look like at:

1) $t = 0$?
2) $t = \pi/2$
3) $t = \pi$
4) $t = 3/2\pi$
5) $t = 2\pi$

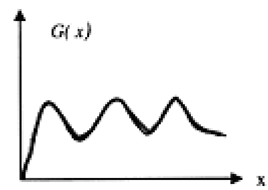




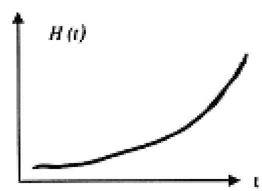
Consider the function below:

$$\Psi(x,t) = \frac{\sqrt{2}\ \mathrm{Sin}[\pi x]}{\sqrt{L}}\ e^{-iEt/\hbar} = G(x)\, H(t)$$

a. In the space below predict what the plot of $G(X)$ would look like?



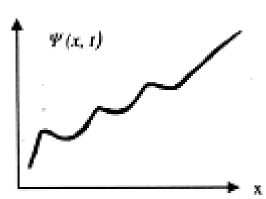
b. In the space below predict the plot of the $H(t)$ would look like?



c. Using your answer for part (a) and (b), in the space below predict the plot of the $\Psi(x,t)$ would look like at:

1) $t = \pi/2$

2) $t = \pi$



APPENDIX E. SOLUTIONS TO Q7.1 AND Q7.2

. .

Q7.1. An electron is confined to a one-dimensional, infinitely deep potential energy well of width a depicted below.

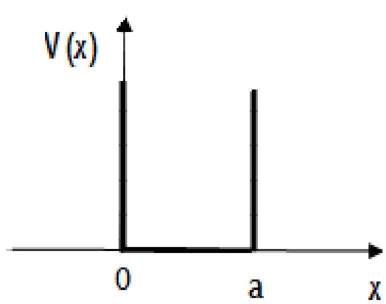
$$V(x) = \begin{cases} 0, & \text{for} \quad 0<x<a \\ +\infty, & \text{for} \quad x \prec 0,\ x>a \end{cases}$$

Solve the time-independent Schrödinger equation with appropriate boundary conditions for this square well.

The time-independent Schrödinger equation becomes:

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} = E\Psi(x)$$

$$\frac{d^2\Psi(x)}{dx^2} = -\frac{2mE}{\hbar^2}\Psi(x) = -k^2\Psi(x) \tag{9.18}$$



where

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

The general solution for equation(1) is:

$$\Psi(x) = A\sin(kx) + B\cos(kx)$$

Since the wave function should be continuous and is zero outside the box, we should have:

$$\Psi(0) = \Psi(a) = 0$$

$$A\sin(0) + B\cos(0) = A\sin(ka) + B\cos(ka) = 0$$

This gives us: $B = 0$ and $A\sin(kx) = 0$. Since for $A = 0$, $\Psi(x)$ vanishes everywhere, $\sin(ka) = 0$ or $k_n a = n\pi$;n=1,2,3,...

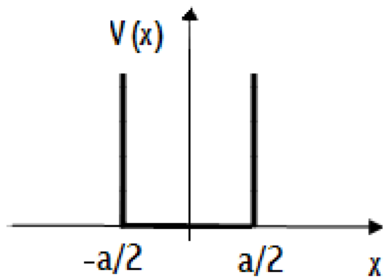
$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

After normalization:

$$\begin{aligned} \Phi_n(x) &= A\sin(\frac{n\pi x}{a}) \\ \int_{-\infty}^{+\infty} \Phi_n^2(x)dx &= A^2 \int_{-\infty}^{+\infty} \sin^2(\frac{n\pi x}{a})dx = 1 \end{aligned}$$

the wave functions are:

$$\begin{aligned} \Phi_n(x) &= \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a}) \\ n &= 1, 2, 3, ... \end{aligned}$$



Q7.2. How does your answer change for the infinite square potential well centered at the origin:

$$V(x) = \begin{cases} 0, & \text{for} \quad -\frac{a}{2}<x<\frac{a}{2} \\ \infty, & \text{for} \quad |x|>\frac{a}{2} \end{cases}$$

Using the general solution for equation(1) is:

$$\Psi(x) = A\sin(kx) + B\cos(kx)$$

The wave function should be continuous and is zero outside the box, we should have:

$$\Psi(-\frac{a}{2}) = \Psi(\frac{a}{2}) = 0$$

$$\begin{aligned} -A\sin(\frac{ka}{2}) + B\cos(\frac{ka}{2}) &= A\sin(\frac{ka}{2}) + B\cos(\frac{ka}{2}) = 0 \\ A\sin(\frac{ka}{2}) &= 0 \quad \text{and} \quad B\cos(\frac{ka}{2}) = 0 \end{aligned}$$

Since for $A = B = 0$, $\Psi(x)$ vanishes everywhere, A and B cannot be zero at the same time.

Furthermore, since $V(x) = V(-x)$, one can solve separately for even and odd solutions.

For even answers we have:

$$\begin{aligned}
\Psi(\frac{a}{2}) &= B\cos(\frac{ka}{2}) = 0 \Rightarrow \frac{ka}{2} = \frac{(2n-1)\pi}{2} \\
k^2 &= \frac{2mE}{\hbar^2} \Rightarrow E = E_{2n-1} = \frac{\hbar^2\pi^2(2n-1)^2}{2ma^2} \\
\Psi_{2n-1} &= \sqrt{\frac{2}{a}}\cos[\frac{(2n-1)\pi}{a}x]; n = 1, 2, 3, ...
\end{aligned}$$

And for odd answers we have:

$$\begin{aligned}
\Psi(\frac{a}{2}) &= B\sin(\frac{ka}{2}) = 0 \Rightarrow \frac{ka}{2} = n\pi \\
k^2 &= \frac{2mE}{\hbar^2} \Rightarrow E = E_n = \frac{\hbar^2\pi^2 n^2}{ma^2} \\
\Psi_{2n-1} &= \sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x]; n = 1, 2, 3, ...
\end{aligned}$$

APPENDIX F. EXAMPLES OF RESEARCH QUESTIONS USED IN THIS STUDY

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This Appendix provides examples of the questions used in this research to study student understanding of different topics. These questions are categorized in the following order:

1. Classical waves;
2. Wave and particle survey;
3. Math and formalism;
4. Math survey;
5. Symmetry;
6. Quantum probability;
7. Stern-Gerlach experiments;
8. Wave functions;
9. Miscellaneous.

CLASSICAL WAVES

- The following three questions refer to a Young's two-slit interference apparatus.

1. What happens to the phase difference between waves arriving at the $m = 3$ dark fringe when the apparatus is immersed in water (initially it was in air):

   (a) The phase difference increases.

   (b) The phase difference decreases.

   (c) The phase difference does not change.

2. What happens to the phase difference between waves arriving at the $m = 3$ dark fringe when the spacing between the slits is increased:

   (a) The phase difference increases.

   (b) he phase difference decreases.

   (c)The phase difference does not change.

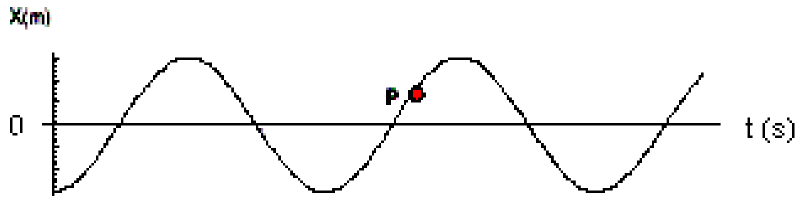
3. What happens to the phase difference between waves arriving at the $m = 3$ dark fringe when the screen is moved further away:

   (a) The phase difference increases.

   (b) The phase difference decreases.

   (c) The phase difference does not change.

4. A mass attached to the end of a spring oscillates back and forth as indicated in the position versus time plot below. At point P, the mass has:



(a) Positive velocity and positive acceleration.

(b) Positive velocity and negative acceleration.

(c) Negative velocity and positive acceleration.

(d) Negative velocity and negative acceleration.

(e) Zero velocity but is accelerating.

5. If the region between the slits and the screen in a Young's two-slit apparatus is filled with a liquid, the distance between the 5th dark fringe and the central bright fringe will:

(a) Decrease.

(b) Increase.

(c) Stay the same.

6. In a Young's two-slit interference apparatus, what is the phase difference between waves from the two slits that arrive at the second minimum?

(a) π

(b) $\frac{3\pi}{2}$

(c) $\frac{5\pi}{2}$

(d) 3π

7. The $m = 3$ bright fringe from a two-slit interference apparatus is observed before and after the apparatus is modified. Which of the following will change the phase difference between waves arriving at the $m = 3$ bright fringe?

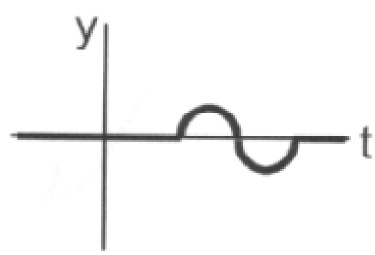
(a) Increasing the spacing between the slits

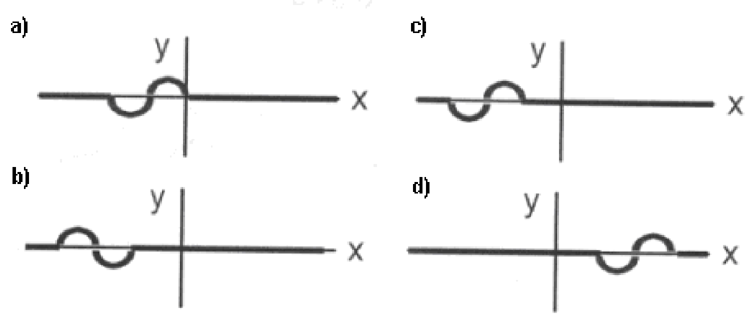
(b) Immersing the apparatus in water

(c) Both of the above

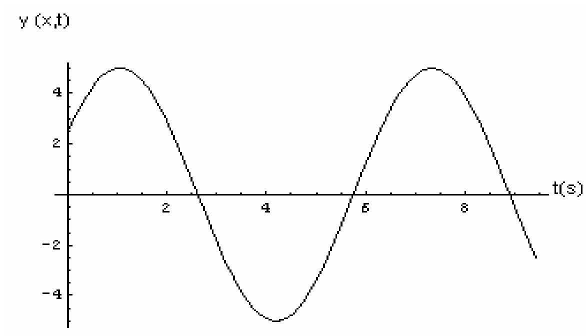
(d) Neither of the above

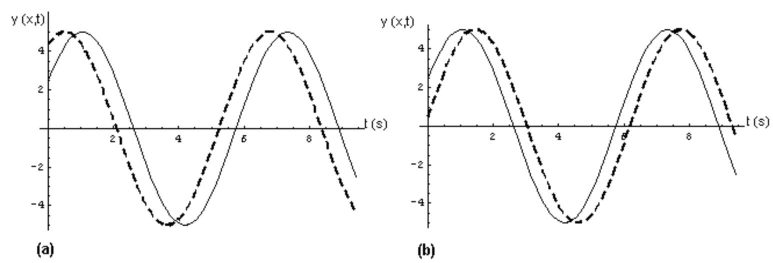
8. The graph below shows the time variation of the displacement of a string element at $X = 0$. The displacement was due to a string wave pulse propagating in the $-X$ direction. Select the letter to graph that best shows the string's displacement as a function of position after some time has passed.





9. The motion of a simple harmonic oscillator is given by $x = x_m \cos(\omega t + \varphi_0)$ and sketched below. How does the graph change if φ_0 is decreased a little?





10. According to superposition principle, waves pass through each other without interacting.

(a) True.

(b) False.

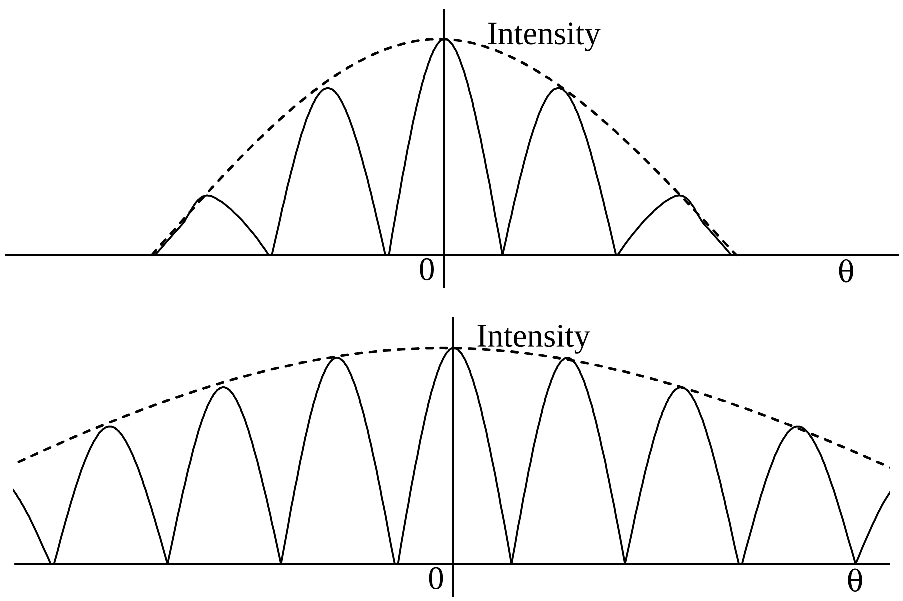
11. Any general oscillation of a medium can be described as a simple superposition of normal mode oscillations.

(a) True.

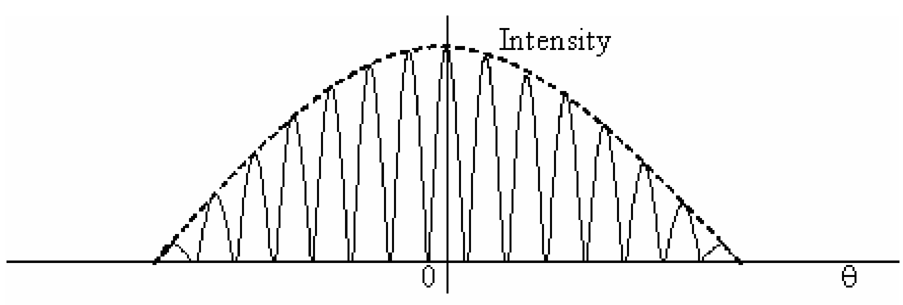
(b) False.

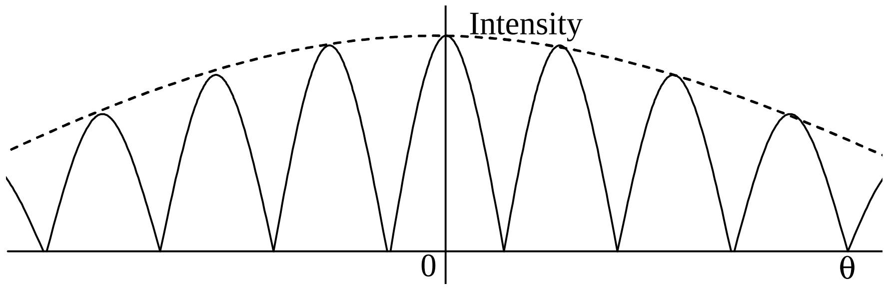
12. In a set of physical optics experiments using an identical laser, different patterns are observed on a distant screen (see the diagrams below). Here, 'a' is the width of a single slit, 'd' is the distance between two neighboring slits, and 'n' is the total number of slits that you shine the laser on. Compare and contrast the three parameters of the slits used in the different experiments.(E.g., you may write $a_1 = a_2$ because ..., or $n_1 > n_2$ because...)

Experiment 1:

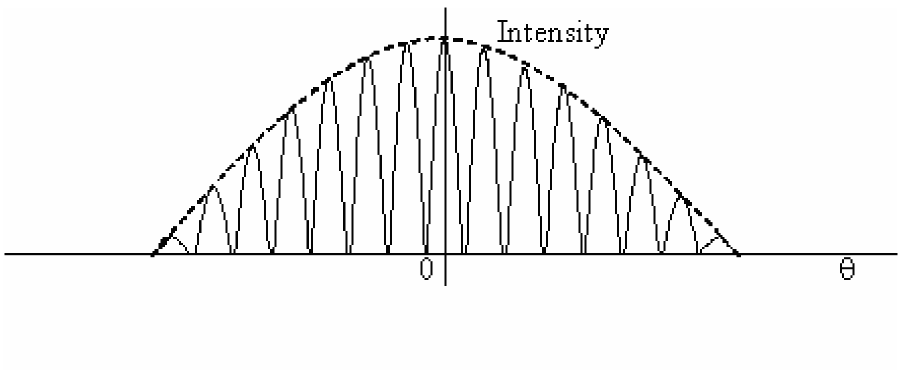


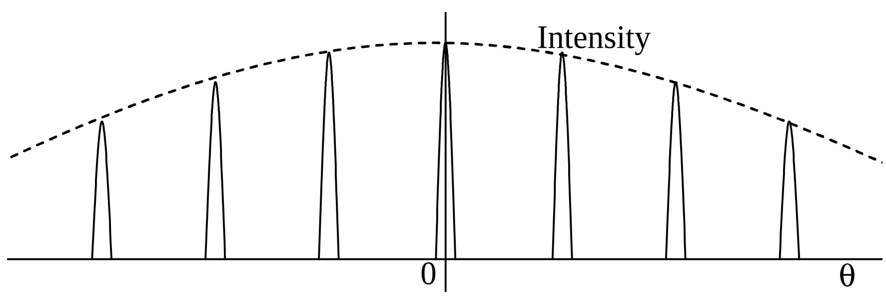
Experiment 2:



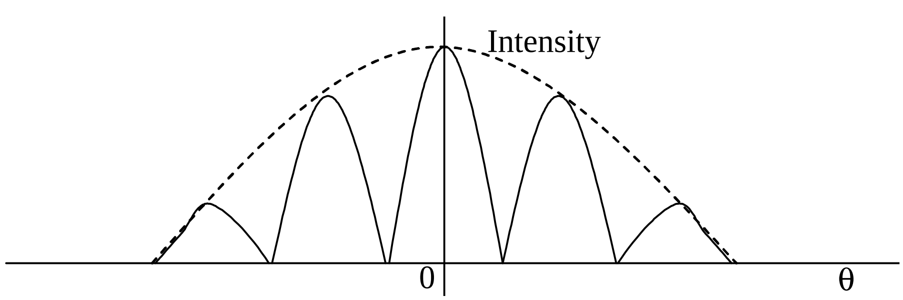


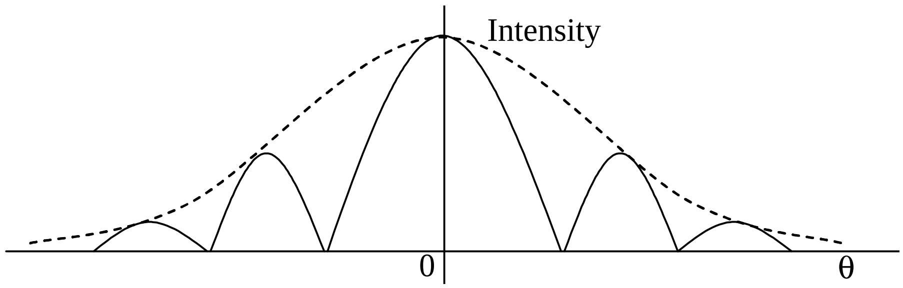
Experiment 3:



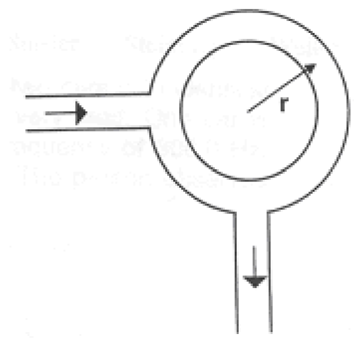


Experiment 4:





13. A sound wave at frequency f propagating at speed V enters a circular pipe of radius r and exits it as shown below. For what will frequencies the sound intensity at the exit be minimum?



(a) $f = \frac{2mv}{r\pi}$

(b) $f = \frac{mv}{r\pi}$

(c) $f = \frac{(m+\frac{1}{2})v}{r\pi}$

(d) $f = \frac{(2m+1)v}{r\pi}$

14. Which one of following is correct about sound waves? Choose all that apply:

(a) At places that the displacement is maximum, pressure is maximum.

(b) At places that the displacement is maximum, pressure is minimum.

(c) At places that the displacement is zero, pressure is maximum.

(d) At places that the displacement is zero, pressure is minimum.

WAVE AND PARTICLE SURVEY

1. What is the most essential property of a particle that makes it different from a wave?

    (a) Discrete energy and well defined path.

    (b) Well defined path and size and shape.

    (c) Superposition, momentum.

    (d) Size, shape, and velocity.

    (e) Momentum and position.

    (f) Responds to force.

    (g) Well defined path.

    (h) Momentum.

    (i) Position.

2. What is the most essential property of a wave that makes it different from a particle?

    (a) Discrete energy, interference and diffraction.

    (b) Velocity, wavelength, and frequency.

    (c) Well defined path and wavelength.

    (d) Superposition and wavelength.

    (e) Momentum and superposition.

    (f) Size, shape, and energy.

    (g) Position and shape.

    (h) Interference.

    (i) All the above.

3. What is the first image that comes to mind when you hear the word photon?

(a) Quanta of electromagnetic radiation.

(b) A small round charged object.

(c) A packet of energy.

(d) A unit of light.

(e) A particle.

(f) A wave.

4. Single photons are directed, one by one, toward a double slit. After some time, the distribution pattern of impacts that makes it through to detector behind the slits is:

(a) Some pattern but different from the light interference.

(b) Identical to an interference pattern.

(c) Individual dots with no pattern.

(d) Two individual bright dots.

5. What is light?

(a) Depends on the situation physicists want to describe; they build different models to describe their observation of the reality

(b) It is a particle with special probability distribution.

(c) Sometimes is wave, sometimes particle.

(d) I don't know, I am confused.

(e) A particle.

(f) A wave.

(g) Neither.

6. Do you think neutrons, with no electric charge, still would have a similar diffraction pattern as electrons?

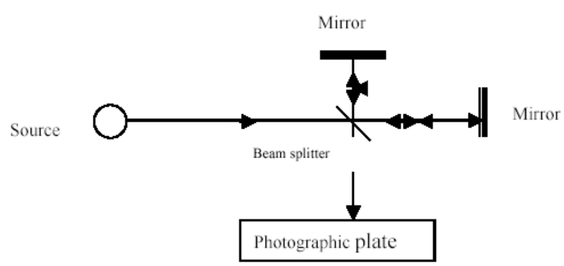
(a) Yes, because the diffraction pattern has nothing to do with the charge of the particle.

(b) No, the pattern for electron is a result of the repulsive force between electrons.

(c) No, electron is different and acts sometimes like a wave.

(d) Yes, because everything is a wave.

7. In a Michelson interferometer, a beam of light is split into two parts of equal intensity, and the two parts are subsequently recombined to interfere with one another. When a single photon is sent through the interferometer, the photographic plate shows:



(a) A single dot somewhere on the plate because the photon chooses one of the two paths through the splitter and then returns and strikes the plate.

(b) A single dot, which is more likely to lie in some regions than others, because of the interference between the two paths.

(c) An interference pattern because the interferometer splits the photon into two waves that subsequently interfere at the plate.

8. A beam of light is directed toward a double slit and creates an interference pattern. Suppose we repeat this experiment with the beam of electrons. What do you expect to see on the screen? (Explain your answer briefly)

9. What does the word "quantization" mean to you?

MATH AND FORMALISM

1. (a) Consider the three integrals below, where $m \neq n$. Which of the following is the correct answer for these integrals?

$$1) \int_{-\pi}^{\pi} \sin[mx] \cos[nx] dx$$
$$2) \int_{-\pi}^{\pi} \cos[mx] \cos[nx] dx$$
$$3) \int_{-\pi}^{\pi} \sin[mx] \sin[nx] dx$$

(b) Consider the three integrals below, where $m \neq 0$. Which of the following is the correct answer for these integrals?

$$1) \int_{-\pi}^{\pi} \sin[mx] \cos[mx] dx$$
$$2) \int_{-\pi}^{\pi} \cos[mx] \cos[mx] dx$$
$$3) \int_{-\pi}^{\pi} \sin[mx] \sin[mx] dx$$

(a) (1) is equal to zero, (2) and (3) are equal to π.

(b) (1) is equal to zero, (2) and (3) are equal to 1.

(c) (2) is equal to π, (1) and (3) are equal to 1.

(d) (3) is equal to 1, (1) and (2) are equal to π.

(e) All are equal to zero.

(f) All are equal to 2π.

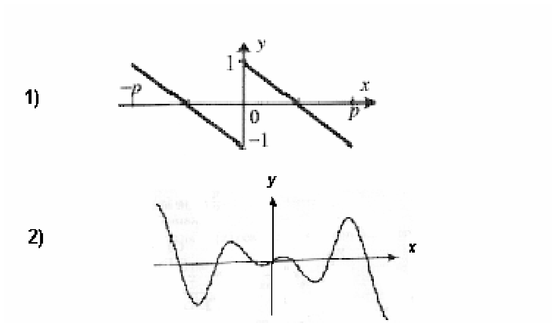
(g) All are equal to 1.

(h) I don't know.

2. (a) For the following periodic functions, state whether the function is even, odd, or neither.

$$
\begin{aligned}
1) \quad f(x) &= a\left(1-\left(\frac{x}{p}\right)^2\right), \quad -p \le x < p, \ (a \neq 0) \\
2) \quad f(x) &= e^{-c|x|}, \qquad (c \neq 0), \quad \text{for } |x| \le p
\end{aligned}
$$

(b) For the following graphs, state whether the graph represents an even or odd function.



(a) Neither of them is odd or even.

(b) (1) is even and (2) is odd.

(c) (1) is odd and (2) is even.

(d) Both are odd.

(e) Both are even.

3. (a) What is the physical meaning of an expectation value for an operator? Using your answer how would you justify the given mathematical definition for expectation value below?

$$< x >= \int_{-\infty}^{\infty} \Psi^*(x) x \Psi(x) dx$$

(b) What does the equation below mean to you?

$$f(j) = \sum f(j) p(j)$$

(c) Is there any relation between this equation and the following equation for expectation value of an operator?

$$< x >= \int_{-\infty}^{\infty} \Psi^*(x) x \Psi(x) dx$$

4. What is the mathematical meaning of an operator? If two operators do not commute, what does this mean mathematically?

5. What is an operator in a quantum system? If two operators do not commute, what does this mean physically?

6. Given two mathematical operators $\hat{A}$ and $\hat{B}$, what happens mathematically when we change the order of two operators acting on a wave function in the following situations?

(a) When the two operators commute

(b) When the two operators do not commute

7. What does it mean physically to change the order of two operators acting on a wave function? (From the point of view of measurement.) Explain.

8. What happens physically when we change the order of two operators acting on a wave function in the following situations?

(a) When the two operators commute

(b) When the two operators do not commute

9. Any arbitrary function can be decomposed into the sum of an even and an odd function.

(a) True.

(b) False.

10. Describe the properties of graphs for odd and even functions.

11. The Fourier theorem says that we can represent any periodic and reasonably continuous wave function as a sum over an infinite number of sinusoidal wave functions: $(n = 1, 2, ...)$

$$f(x) = \int_{-\pi}^{\pi} C_n \sin(nx) dx$$

Can you suggest a way to determine C_n ? Explain your answer. (The C_ns are the constant coefficients of the expansion.)

12. What is/are the general solution for differential equation below? Assume k is real and positive.

$$\frac{d^2\Psi(x)}{dx^2} = k^2\Psi(x)$$

(a) $\Psi(x) = Aexp(-ikx) + Bexp(ikx)$.

(b) $\Psi(x) = Aexp(-kx) + Bexp(kx)$.

(c) $\Psi(x) = Asin(kx) + Bcos(kx)$.

(d) a and b.

(e) b and c.

(f) a and c.

(g) It depends to the boundary conditions.

13. What is the integral below equal to?

$$\int_{-\infty}^{\infty} \delta(x - x_0)f(x)dx$$

(a) $f(x - x_0)$

(b) $f(x_0)$

(c) $f(0)$

(d) 0

(e) 1

MATH SURVEY

1. How well can you apply your knowledge of mathematics to answer quantum problems?

(a) Just okay.

(b) Pretty well.

(c) Not at all.

(d)Very well.

2. If you cannot apply your knowledge of math in quantum course "very well," where is the problem?

(a) Difficulty in formalism (putting the problem into mathematical symbols).

(b) Difficulty in making physical sense out of the mathematical formalism.

(c) Difficulty in understanding the problem and its terminology.

(d) Poor math background.

3. In your experience, which, if any, of the following do you think makes learning quantum physics difficult?

(a) I understand the problems, but I cannot formalize my understanding in terms of math symbols and/or interpret the mathematical formalism into physical meanings.

(b) I have difficulty with the probabilistic nature of quantum mechanics.

(c) The mathematical calculations of the problems are hard.

(d) The understanding of questions being posed is hard.

(e) The topics are hard to visualize.

(f) Others.

4. Have you had a college course in math that discussed complex numbers?

    (a) Yes, in the space below please type what course(s) you took.

    (b) No.

5. How well do you think you understood the math course on complex numbers?

    (a) I understood 90% or more of the material.

    (b) I understood between 80% and 90% of the material.

    (c) I understood between 70% and 80% of the material.

    (d) I understood between 60% and 70% of the material.

    (e) I understood less than 60% of the material.

6. Have you had a college course in math that discusses differential equations?

    (a) Yes, please in the space below type what course(s) you took.

    (b) No.

7. How well do you think you understood the course on differential equations?

    (a) I understood 90% or more of the material.

    (b) I understood between 80% and 90% of the material.

    (c) I understood between 70% and 80% of the material.

    (d) I understood between 60% and 70% of the material.

    (e) I understood less than 60% of the material.

8. Have you had a college course in statistics that discussed probability?

    (a) Yes, in the space below please type what course(s) you took.

    (b) No.

9. How well do you think you understood the math course on probability?

(a) I understood 90% or more of the material.

(b) I understood between 80% and 90% of the material.

(c) I understood between 70% and 80% of the material.

(d) I understood between 60% and 70% of the material.

(e) I understood less than 60% of the material.

10. If you have 10 quantum mechanics problems to solve, how often do you get stuck just because of mathematical difficulties? (E. g. 10 out of 10 problems, 4 out of 10, etc.)

11. When you do get stuck in physics problems, how do you handle the situation?

(a) Just try to get to the final results somehow, even if I know part of my work is incorrect.

(b) Go to my math book and other notes to review the related material.

(c) Ask for help from a friend/classmate/TA.

(d) Use computer to do the math part.

(e) Stop trying.

12. Suppose your intuition about a physics problem comes in conflict with what you have calculated mathematically. (Assume that you can not find any mistake in your calculation.) Which of these do you trust more? In the space below give your choice and explain your reasoning.

(a) math

(b) physics

(c) neither

(d) other

13. If your intuition about a physics problem comes in conflict with what your book or your instructor calculates mathematically, how do you resolve this conflict?In the space below please explain you reason.

14. How do you learn physics? List all the strategies that work for you in learning physics. (E. g. reading, lecture, problem solving, discussion, etc.)

SYMMETRY

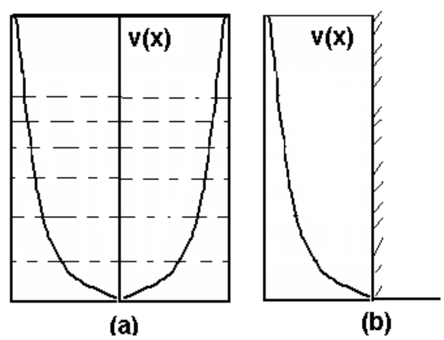
1. Figure (a) below shows a one-dimensional potential energy function given by $V(x) = kx^2$. The six lowest allowed energy eigenstates are shown by dashed lines. Figure (b) shows the "half well" version of this potential, where:

$$V(x) = \begin{cases} \infty & \text{for} \quad x > 0 \\ kx^2 & \text{for} \quad x < 0 \end{cases}$$

The energy eigenvalues for potential (a) are given:

$$E = (E_0, E_1, E_2, E_3, E_4, E_5, E_6, ...)$$

What is the most likely pattern of energy eigenvalues in the "half-well" potential on the "half well" (b) on the right ?



(a) $E = 2\ (E_0, E_1, E_2, E_3, E_4, E_5, E_6, ...)$.

(b) $E = \frac{1}{2}(E_0, E_1, E_2, E_3, E_4, E_5, E_6, ...)$.

(c) $E = (E_0, E_2, E_4, E_6, ...)$.

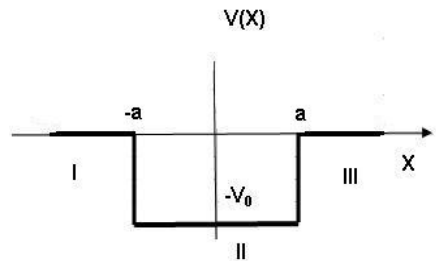
(d) $E = (E_1, E_3, E_5, ...)$.

(e) $E = (E_0, E_1, E_2, E_3)$.

(f) The same energy spectrum.

2. An electron is confined to a one-dimensional, finitely deep potential energy well of width $2a$ depicted below. V_0 is a positive constant.

$$V(x) = \begin{cases} -V_0 & \text{for} \quad -a < x < a \\ 0 & \text{for} \quad |x| > a \end{cases}$$



After solving the time-independent Schrödinger equation, you find that there are exactly seven bound states for this system. Which one of the following statements is correct:

(a) The lowest energy bound state can have a symmetric or antisymmetric wave function

(b) The highest energy bound state can have a symmetric or antisymmetric wave function

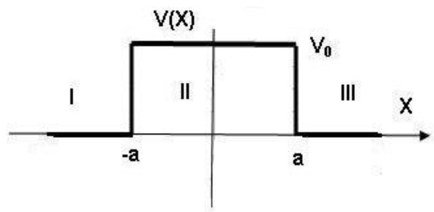
(c) The lowest energy bound state has an antisymmetric wave function.

(d) The highest energy bound state has an antisymmetric wave function.

(e) The highest energy bound state has a symmetric wave function.

3. A beam of electrons coming from the left scatter off the one-dimensional rectangular barrier depicted below. (The incident beam comes from $x = -\infty$:)

$$V(x) = \begin{cases} 0 & \text{for} \quad x < -a \ ; \quad \text{Region I} \\ V_0 & \text{for} \quad -a > x > a \ ; \quad \text{Region II} \\ 0 & \text{for} \quad x > +a \ ; \quad \text{Region III} \end{cases}$$



Which one of the following statements is correct:

(a) Since the potential is an even function, we can assume with no loss of generality that the solutions are either even or odd.

(b) Since the potential is an odd function, we can assume with no loss of generality that the solutions are either even or odd.

(c) Since the potential is an even function, we can assume with no loss of generality that $\Psi(-x) = \Psi(x)$.

(d) Since the potential is an odd function, we can assume with no loss of generality that $\Psi(-x) = -\Psi(x)$.

(e)We cannot make any assumption that solutions are either even or odd.

QUANTUM PROBABILITY

1. An electron is trapped in a one-dimensional infinite potential well that is $L = 100$ pm wide; the electron is in its ground state. What is the probability that you can detect the electron in an interval $\Delta x = 5.0$ pm wide centered at $x = 50$ pm? The wave function of an electron in an infinite potential well is given by:

$$\Psi(x) = \sqrt{\frac{2}{L}} Sin(\frac{n\pi x}{L}),$$

where L is the width of the well. The interval Δx is so narrow that you can take the probability density to be constant within it.

(a) 10%

(b) 25%

(c) 50%

(d) 75%

(e) 100%

2. Regions of smaller momentum have larger maximum values of the amplitude than adjacent regions of larger momentum.

(a) True.

(b) False.

3. What is the physical meaning of $< x >$?

4. The Uncertainty Principle implies that:

(a) If we measure, for example, the position of a particle with certainty, the result of the measurement for that system if repeated has some distribution and results are close to each other but not exactly the same.

(b) If we measure, for example, the position of a particle with certainty, we have changed its momentum so the result of measurement for momentum is uncertain.

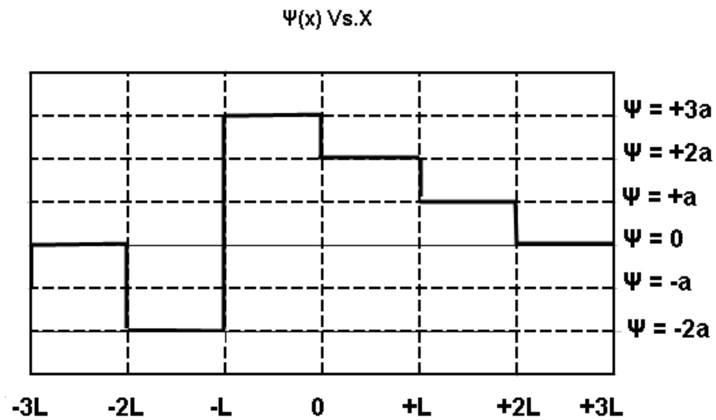
(c) We cannot measure either momentum or position of a microscopic particle, because it acts like a wave and wave is not localized.

(d) In a microscopic system, we don't have good tools to measure momentum and position at a single instant of time.

(e) The electron has a position but no trajectory and an unknown momentum.

(f) The electron in an atom has a definite but unknown position.

5. The figure below shows a plot of a (rather artificial) wave function $\Psi(x)$ versus x , over the range of $(-3L, +3L)$. The wave function vanishes for all other values of x. What is the probability of finding the particle in the range from $x = 0$ to $x = +2L$?



(a) $\frac{3a}{8}$

(b) $\frac{3}{8}$

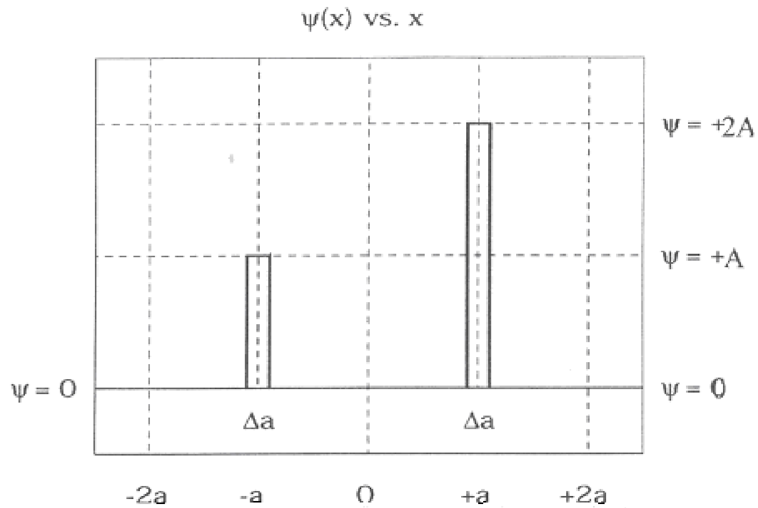
(c) $\frac{5}{18}$

(d) $\frac{5a}{8}$

(e) $\frac{13a}{18}$

(f) $\frac{13}{18}$

6. The figure below shows a plot of a wave-function, $\Psi(x)$ versus x, given by the two spikes shown, each of width Δa. The wave function vanishes for all other values of x not shown. Which one of the expressions below is closest to the value of the uncertainty in the position variable, namely Δx?



(a) $\Delta x = 2a$

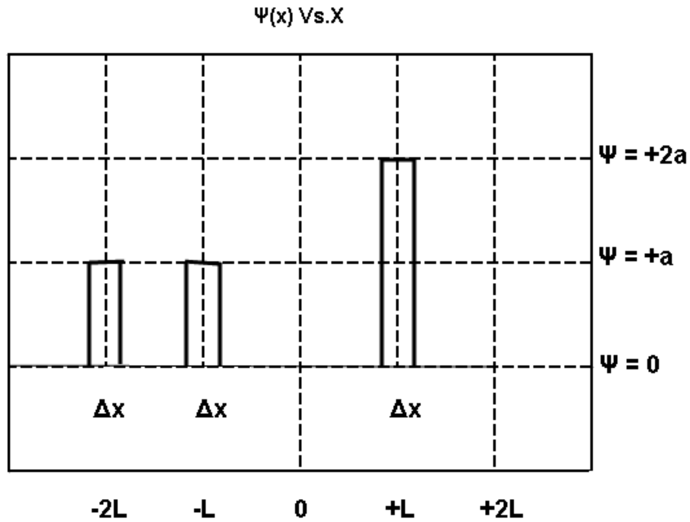
(b) $\Delta x = 2\Delta a$

(c) $\Delta x = \Delta a$

(d) $\Delta x = \frac{3a}{5}$

(e) $\Delta x = \frac{4a}{5}$

7. The figure below shows a plot of a (rather artificial) wave function $\Psi(x)$ versus x , over the range of $(-2L, +2L)$. The wave function vanishes for all other values of x. What is the expectation value of x?



(a) $\frac{4L}{6}$

(b) $\frac{a}{6}$

(c) a

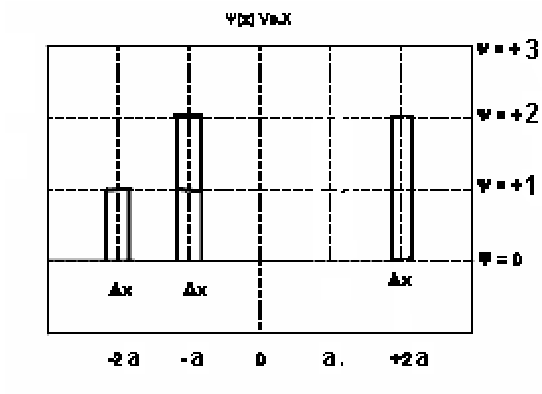
(d) $\frac{L}{6}$

(e) $\frac{4a}{6}$

(f) 0

(g) $-L$

8. The wave function for a particle is sketched below. (Assume $\Delta x \ll a$.)



(a) Calculate the expectation value of position, namely, $< x >$.

(b) List possible values of position that can be measured and their probability.

9. Emma is solving a quantum problem. She is asked to show that, for a particular system, any value of the momentum is equally probable. She needs to show that: (pick the best that applies.)

(a) The expectation value of the momentum is the same everywhere.

(b) The probability density of the momentum is a nonzero constant.

(c) The probability density of the momentum is infinity.

(d) The probability density of the momentum is zero.

(e) The expectation value of the momentum is zero.

(f) The expectation value of the momentum is infinity.

10. Suppose that at time $t = 0$, a position measurement is made on a particle and $x = x_0$ is found. Assume that measurement was precise enough and the wave function immediately following it is well approximated by a $\delta-$function (or a very narrow Gaussian). If you next measure the system's momentum, what possibilities are there?

(a) The value measured for the momentum will be the same everywhere.

(b) There is an equal probability to find any value of the momentum.

(c) There is only one possible value for momentum.

(d) Only certain values for momentum are possible.

11. A free moving particle in space, on which no net force acts, is described by:

$$\Psi(x) = \Psi_0(x)e^{ikx},$$

where Ψ_0 is a constant. Recall Heisenberg's Uncertainty Principle:

$$\Delta x \Delta p_x \geq (\frac{\hbar}{2})$$

If you are to define Δx and Δp_x for this free particle, what would be their values? Explain your answer.

12. (a) For what values of x is $f(x)$ maximum?

$$f(x) = x(10 - x) \text{ for } 0 < x < 10$$

(b) A particle in an infinite square well of length L $(0 < L < a)$ has the initial wave function:

$$\Psi(x) = Ax(a - x).$$

If you make a position measurement, where would you most likely find the particle? Hint: use your results from part (a).

13. A particle in an infinite square well has the initial wave function:

$$\Psi(x) = Ax(a - x).$$

If you make a position measurement, where would you most likely find the particle?

(a) $\frac{a}{2}$

(b) a

(c) $\frac{1}{A}$

(d) $\frac{a}{4}$

(e) 0

(f) The probability of finding the particle is equal everywhere in space.

14. In the previous question, what is the expectation value of the momentum?

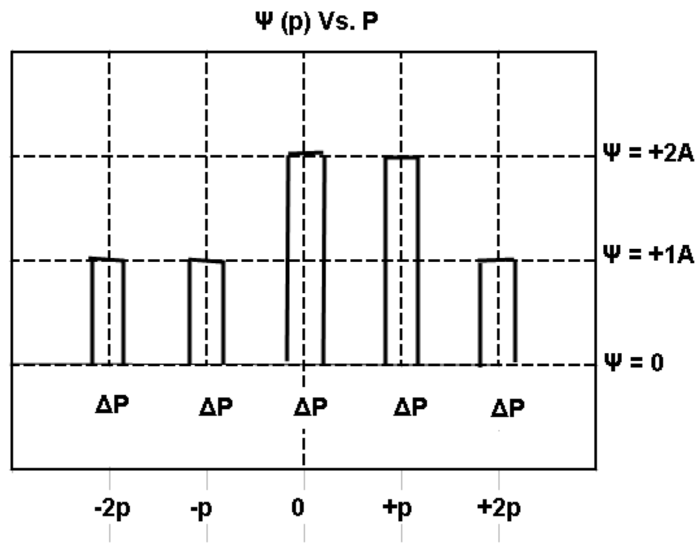
(a) $i\hbar(\partial/\partial t) < x >$

(b) a

(c) $\frac{\hbar}{a}$

(d) 0

15. The figure below shows a plot of a (rather artificial) wave function $\Psi(p)$ versus p, given by the five 'spikes' each of width Δp over the range of $(-2p, +2p)$. The wave function vanishes for all other values of p. (a) What is the probability of measuring $P = +2p$ for momentum?



(a) p

(b) $\frac{p}{7}$

(c) $\frac{1}{11}$

(d) $\frac{1}{19}$

(e) $\frac{p}{11}$

(f) $\frac{1}{7}$

(g) 0

(h) $\frac{p}{19}$

(b) What is the expectation value of the momentum, namely, $< P >$?

(a) $\frac{3}{11}$

(b) $\frac{3p}{7}$

(c) $\frac{3p}{19}$

(d) $\frac{3}{7}$

(e) $\frac{3}{19}$

(f) $\frac{3p}{11}$

(g) We cannot answer this question from this graph.

STERN-GERLACH EXPERIMENTS

1. What are the possible values of a measurement for spin-1 particles (L_z)? The answers are given in units of ħ.

   (a) $1\ ,\ \frac{1}{2},\ 0,\ -\frac{1}{2},\ -1$

   (b) $1, \frac{1}{2}, -\frac{1}{2}, 1$

   (c) $1,\ 0,\ -1$

   (d) $\frac{1}{2}, -\frac{1}{2}$

   (e) $0,\ 1$

   (f) $1,\ -1$

   (g) 1

2. What are the possible values of a measurement for the spin of electrons (L_z)? The answers are given in units of ħ.

   (a) $1, \frac{1}{2}, 0, -\frac{1}{2}, -1$

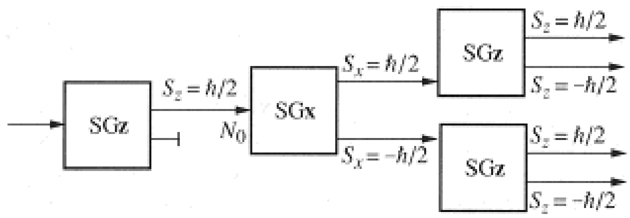
   (b) $1, \frac{1}{2}, -\frac{1}{2}, -1$

   (c) $\frac{1}{2},\ 0,\ -\frac{1}{2}$

   (d) $\frac{1}{2},\ -\frac{1}{2}$

   (e) $1,\ -1$

3. (a) The drawing below shows a sequence of Stern-Gerlach devices. What fraction of the particles transmitted through the first SGZ device will be in the $|+z>$ state after the measurements? 'No' is the number of particles in the beam exiting the first SGZ device.



(a) $\frac{No}{2}$

(b) No

(c) 1

(d) $\frac{1}{8}$

(e) $\frac{No}{4}$

(b) For the preceding question, what is the probability of finding $Sz = -\frac{1}{2}$ for the exiting beam from this experiment?

(a) $\frac{No}{2}$

(b) No

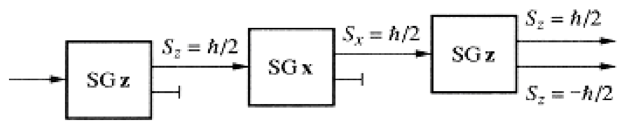
(c) $\frac{No}{4}$

(d) $\frac{1}{4}$

(e) $\frac{1}{2}$

(f) 1

4. The drawing below shows a sequence of Stern-Gerlach devices. What do you think are the probabilities that an electron entering the last device will come out of the plus and minus channels of this device?



(a) $\frac{1}{2}$ for both channels.

(b) 1 and 0, respectively.

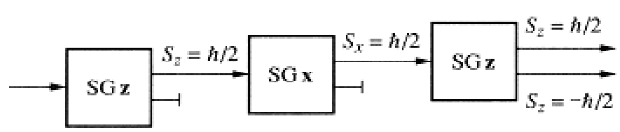
(c) 0 and 1, respectively.

(d) $\frac{1}{4}$ and $\frac{3}{4}$, respectively.

(e) $\frac{3}{4}$ and $\frac{1}{4}$, respectively.

(f) Some other probabilities.

5. The drawing below shows a sequence of Stern-Gerlach devices. What fraction of the electrons entering this experiment will exit the second SGZ device with spin $-\frac{1}{2}$?



(a) $\frac{1}{4}$

(b) $\frac{1}{2}$

(c) $\frac{1}{8}$

(d) 0

(e) 1

6. For the preceding question, what fraction of the electrons entering this experiment will exit the second SGX device with spin $+\frac{1}{2}$?

(a) $\frac{1}{4}$

(b) $\frac{1}{2}$

(c) $\frac{1}{8}$

(d) $\frac{3}{4}$

(e) $\frac{1}{3}$

7. Consider a beam of electrons exiting the Stern-Gerlach (SGZ) device with its inhomogeneous magnetic field parallel to the Z-axis. We next send the beam with spin $+\frac{1}{2}$ into a SGX device, one with its inhomogeneous magnetic field oriented along the $X-$axis. If we send the beam of the particles exiting the SGX device with $Sx = +\frac{1}{2}$, through another SGZ device, what fraction of the electrons entering this experiment will exit the second SGZ device with spin $-\frac{1}{2}$?

(a) $\frac{1}{4}$

(b) $\frac{1}{2}$

(c) $\frac{1}{8}$

(d) 1

(e) $\frac{1}{3}$

(f) 0

8. For the preceding question, what is the probability of finding $Sz = -\frac{1}{2}$ for the beam exiting from the second SGZ?

(a) $\frac{1}{4}$

(b) $\frac{1}{2}$

(c) $\frac{1}{8}$

(d) 1

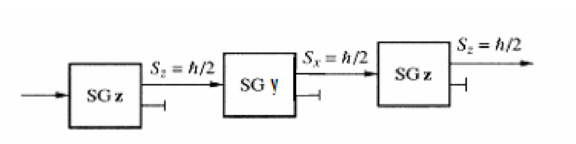
(e) $\frac{1}{3}$

(f) 0

(g) $\frac{2}{3}$

(h) $\frac{1}{16}$

9. A beam of spin-$\frac{1}{2}$ particles is sent through a series of three Stern-Gerlach devices, as illustrated below. What fraction of the particles transmitted through the first SGZ device will survive the third measurement?



(a) $\frac{1}{2}$

(b) $\frac{3}{4}$

(c) $\frac{2}{3}$

(d) 1

(e) 0

(f) $\frac{1}{3}$

(g) $\frac{1}{4}$

(h) $\frac{1}{8}$

10. In previous question, what is the probability of finding $Sz = -\frac{1}{2}$ for exiting beam from this experiment?

(a) $\frac{No}{4}$

(b) $\frac{No}{6}$

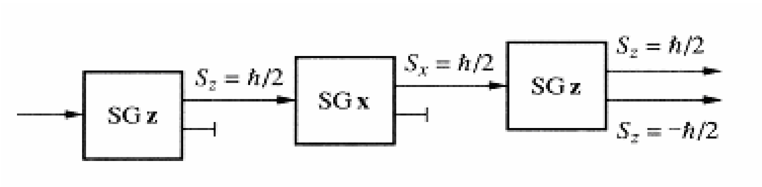
(c) $\frac{1}{6}$

(d) $\frac{1}{2}$

(e) $\frac{No}{2}$

(f) $\frac{1}{4}$

11. The drawing below shows a sequence of Stern-Gerlach devices. What fraction of the electrons entering this experiment will exit the second SGZ device with spin $-\frac{1}{2}$?



(a) $\frac{1}{4}$

(b) $\frac{1}{2}$

(c) $\frac{1}{8}$

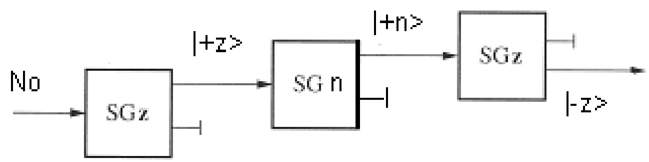
(d) 1

(e) 0

12. (a) A beam of electrons in state $|+z>$ is sent through a series of three Stern-Gerlach devices. The first SGZ, transmits only particles in the $|+z>$ state. The second device, a SGN device, transmits only particles in the $|+n>$ state.

$$|+n>= \cos\frac{\theta}{2}|+z> + \sin\frac{\theta}{2}|-z>$$

Tata (θ) is the angle axis n makes with respect to z axis in the xz plane. A last SGZ device transmits only particles in the $|-z>$ state. What fraction of the particles transmitted through the first SGZ device will survive the third measurement? 'No' is the number of particles in the beam entering the first SG device.



(a) $\frac{No}{2}[\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}]$

(b) $\frac{No}{2}(\cos^2\frac{\theta}{2})$

(c) $\frac{No}{4}(\cos^2\frac{\theta}{2})$

(d) $\frac{No}{4}(\sin^2\frac{\theta}{2})$

(b) What values of the angle θ of the SGN device maximizes the number of the particles in the $|-z>$ state? What is the probability of measuring spin $|-z>$, $Pr(|-z>)$, for this value of θ?

(a) $\theta = 2\pi, Pr(|-z>) = No$, respectively.

(b) $\theta = 0, Pr(|-z>) = \frac{No}{4}$, respectively.

(c) $\theta = 2\pi, Pr(|-z>) = 1$, respectively.

(d) $\theta = 0, Pr(|-z>) = 1$, respectively.

(e) $\theta = \pi, Pr(|-z>) = \frac{1}{4}$, respectively.

(f) $\theta = 2\pi, Pr(|-z>) = \frac{No}{4}$, respectively.

13. For question (12), what is the spin state vector of the transmitted particles?[8]

(a) $|+z> or|-z>$

(b) $[1 \quad 0]$

(c) $[0 \quad 1]$

(d) $[0 \quad 0]$

(e) I don't know.

14. For question (12), what values of the angle θ of the SGN device maximizes the number of the particles in the $|-z>$ state? What is the probability of measuring $|-z>$ for this value of θ?

(a) $\theta = \pi$, and $Pr(|-z>) = No$, respectively.

(b) $\theta = 0$, and $Pr(|-z>) = 1$, respectively.

(c) $\theta = 2\pi$, and $Pr(|-z>) = \frac{No}{4}$, respectively.

(d) $\theta = 2\pi$, and $Pr(|-z>) = 1$, respectively.

(e) $\theta = \pi$, and $Pr(|-z>) = \frac{1}{4}$, respectively.

(f) $\theta = 0$, and $Pr(|-z>) = No$, respectively.

[8] This question could be categorized as math and formalism.

15. For question (12), what fraction of the particles survives the last measurement if the SGN device is simply removed from the experiment?

(a) $\frac{1}{2}$

(b) $\frac{No}{2}$

(c) $\frac{No}{4}$

(d) $\frac{1}{4}$

(e) 1

(f) 0

(g) No

16. A beam of electrons passes through a sequence of Stern-Gerlach devices. If after the measurement 25% of the original electrons are in the $|+z>$ state, this means that the probability of electrons exiting the last SG device in the $|+z>$ state is $\frac{1}{4}$.

(a) True.

(b) False.

17. A beam of electrons passes through a sequence of Stern-Gerlach devices. If the probability of measuring the $|-z>$ state for the last device is $\frac{1}{2}$, this means that 50% of the original electrons exit this experiment in the $|-z>$ state.

(a) True.

(b) False.

WAVE FUNCTIONS

1. The lowest energy state of a particle bound in a finite square well has a longer wavelength than the lowest energy state of one in an infinite square well.

(a) True.

(b) False.

2. The energies for the finite well are, respectively, slightly lower than the ones for the infinite well.

(a) True.

(b) False.

3. In general, if the wave function $\Psi(x)$ represents a possible set of quantum amplitudes, then $-\Psi(x)$ is equally acceptable as an equivalent and physically indistinguishable wave function.

(a) True.

(b) False.

4. For all bound states for the finite well, the wavelength of each wave function inside the well is slightly longer than that of the wave function of corresponding state for the infinitely deep square well.

(a) True.

(b) False.

5. Regions of smaller momentum have larger maximum values of the amplitude than adjacent regions of larger momentum.

(a) True.

(b) False.

6. An electron in a one-dimensional box with walls at $x = (0, a)$ is in the quantum state $\Psi(x)$. What is the lowest energy of the electron that will be measured in this state? ψ_i are energy eigenstates for the electron in a box.

$$\Psi(x) = 2\psi_3 + 4\psi_5 - \psi_6$$

(a) $\frac{9\hbar^2\pi^2}{2ma^2}$

(b) $\frac{3\hbar^2\pi^2}{2ma^2}$

(c) $\frac{3\hbar^2\pi^2}{ma^2}$

(d) $\frac{\hbar^2\pi^2}{2ma^2}$

(e) 0

7. The wave function for a particle in a box is given at $t = 0$ by

$$\Psi(x, t = 0) = c_1\psi_1(x) + c_5\psi_5(x) + c_7\psi_7(x), \tag{1}$$

where Ψ_i are energy eigenstates for the particle in a box, $H\psi_i = E_i\psi_i$. What is $\Psi(x, t > 0)$?

(a) $\Psi(x, t) = E_1\psi_1(x)e^{-iEt/\hbar} + E_5\psi_5(x)e^{-iEt/\hbar} + E_7\psi_7(x)e^{-iEt/\hbar}$

(b) $\Psi(x, t) = c_1\psi_1(x)e^{-iE_1t/\hbar} + c_5\psi_5(x)e^{-iE_5t/\hbar} + c_7\psi_7(x)e^{-iE_7t/\hbar}$

(c) $\Psi(x, t) = E_1\psi_1(x)e^{iE_1t/\hbar} + E_5\psi_5(x)e^{iE_5t/\hbar} + E_7\psi_7(x)e^{iE_7t/\hbar}$

(d) $\Psi(x, t) = E\psi(x)e^{-iEt/\hbar}$

(e) $\Psi(x, t) = E\psi(x)e^{iEt/\hbar}$

8. For equation (1) of question (7), how would you answer if somebody asked you: *"What energy does a particle in a state described by the wave function have?"*

9. The wave function for a particle in a box is given at $t = 0$ by

$$\Psi(x, t = 0) = C_1\psi_{_1}(x) + C_2\psi_2(x),$$

where Ψ_i are energy eigenstates for the particle in a box, $H\psi_i = E_i\psi_i$.

(a) What is $\Psi(x, t > 0)$?

(b) Is $\Psi(x, t = 0)$ an eigenstates of $\hat{H}$? Why?

10. Consider a system with Hamiltonian $\hat{H}$. What are the possible outcomes of an experimental measurement of the energy in general?

11. Consider a system with Hamiltonian $\hat{H}$. What are the possible outcomes of an experimental measurement of the energy for the case when

$$\Psi(x, t) = C_1\psi_1(x)e^{-iE_1t/\hbar} + C_5\psi_5(x)e^{-iE_5t/\hbar} + C_{300}\psi_{300}(x)e^{-iE_{300}t/\hbar},$$

where ψ_i are energy eigenstates for the particle in a box: $H\psi_i = Ei\psi_i$.

(a) Is $\Psi(x, t = 0)$ an eigenstates of $\hat{H}$? Why?

(b) What happens after an actual energy measurement has been carried out?

(c) What is the first thing you should do when asked for $\Psi(x, t)$ given the wave function $\Psi(x, t = 0)$?

12. The solution of Schrödinger equation for a particle in a box on the interval $x = [0, a]$ is

$$\Psi(x) = \sqrt{\frac{2}{a}} \sin(\frac{n\pi x}{a}).$$

Why do we get a sin function and not a cos function? What determines the shape and the type of the wave function in the box?

13. A particle is in a one-dimensional box with walls at $x = 0$ and $x = a$ with ground state energy E_1. One of the walls is moved to the position $x = 2a$. Compare the ground state energy in this new box to the ground state energy of the initial box.

(a) The ground state energy for the new box is one-fourth of the ground state energy for the initial box.

(b) The ground state energy for the new box is four times the ground state energy for the initial box.

(c) The ground state energy for the new box is larger than the ground state energy for the initial box, since we put energy into the system by moving the walls.

(d) The ground state energy for the new box is smaller than the ground state energy for the initial box, since it has a shorter wavelength for the wave function.

(e) The ground state energy for the new box is the same as for the initial box, because the transition conserves energy.

(f) The ground state energy for the new box is twice the ground state energy for the initial box.

14. An electron in a one-dimensional box with walls at $x = (0, a)$ is in the quantum state

$$\Psi(x) = \sqrt{\frac{1}{14}}(2\psi_1 + 3\psi_5 - \psi_6).$$

Measurement of the energy finds the value $E = E_1$ for this particle. What is the probability of finding this particle at $x = (\frac{a}{4})$? (ψ_i are energy eigenstates for the electron in a box.)

(a) $P(x = \frac{a}{4}) = |\sqrt{\frac{1}{14}}[2\phi_1(\frac{a}{4})]|^2$

(b) $P(x = \frac{a}{4}) = |\Psi(\frac{a}{4})|^2$

(c) $P(x = \frac{a}{4}) = \frac{1}{a}$

(d) $P(x = \frac{a}{4}) = \frac{a}{16}$

(e) It is zero for states with odd n, and it is one for states with even n.

(f) This probability is time-dependent and we don't know when the measurement of the position is carried out.

15. The wave function for a particle at time $t = 0$ happens to be identical to the harmonic oscillator ground state energy eigenfunction

$$\Psi(x, t = 0) = \phi_0(x) = Ce^{-\frac{ax2}{2}}.$$

What happens to $|\Psi(x, t)|^2$ at later times if the particle is under the influence of a harmonic oscillator potential? ($\hat{H} = 1D$ harmonic oscillator Hamiltonian)

(a) $|\Psi(x, t)|^2$ oscillates with time.

(b) $|\Psi(x, t)|^2$ becomes a broader Gaussian.

(c) $|\Psi(x, t)|^2$ is time independent.

(d) $|\Psi(x, t)|^2$ becomes $\phi_0(x)e^{-iE_0t/\hbar}$.

16. For the preceding question, what happens to $|\Psi(x,t)|^2$ at later times if the particle is free, $\hat{H} = \hat{H}_{free}$?

(a) $|\Psi(x,t)|^2$ becomes $\phi_0(x)e^{-iE_0t/\hbar}$.

(b) $|\Psi(x,t)|^2$ becomes a broader Gaussian.

(c) $|\Psi(x,t)|^2$ oscillates with time.

(d) $|\Psi(x,t)|^2$ is time independent.

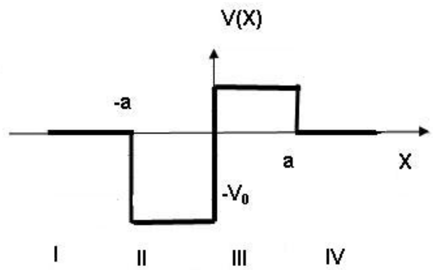
17. A particle under influence of a finite potential well can have unlimited number of bound state energies.

(a) False

(b) True.

(c) It depends on how deep the well is.

18. The figure below shows a plot of a potential $V(x)$ versus x between $-a < x < a$, where V_0 is a positive constant. Which of the functions below presents an acceptable physical wave function for a particle with $E < 0$ in region III $(0 < x < a)$? ξ is a positive real number.



(a) $e^{\xi x}$

(b) $e^{-\xi x}$

(c) $e^{-i\xi x}$

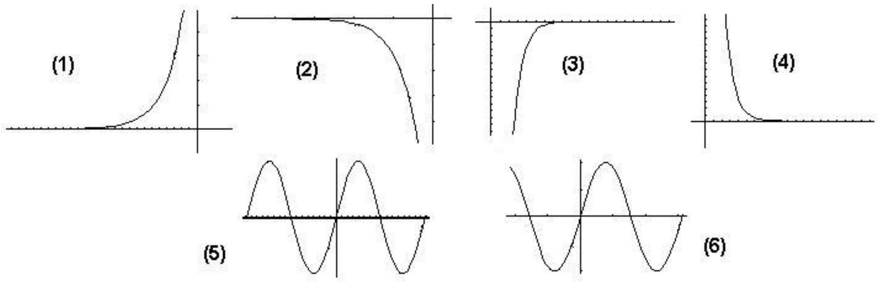
(d) $e^{\xi x}$

(e) $e^{\pm \xi x}$

(f) $e^{i\xi x}$

(g) $e^{\pm i\xi x}$

19. Which of the plots below is of an acceptable physical wave function for a particle with $E < 0$ in region III $(0 < x < a)$ in the preceding question ?



(a) (4) and (5).

(b) (1), (2), and (3).

(c) (1), (2), (3), and (4).

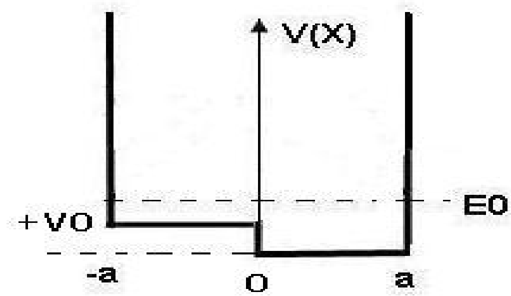
(d) (5) and (6).

(e) (2), (3), and (4).

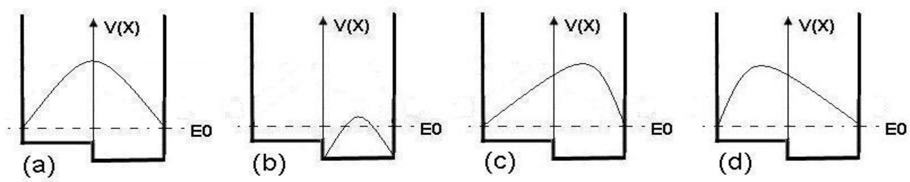
(f) Only (4).

(g) Only (3).

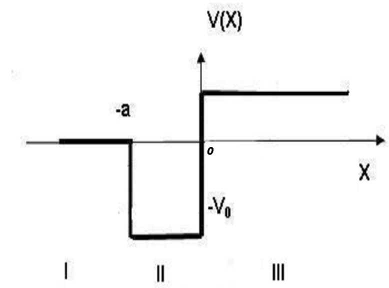
20. The plot below shows a potential energy function, $V(x)$ versus x, corresponding to an "asymmetric" infinite well. The infinite well is of width $2a$, with impenetrable walls at $x = \pm a$, but where $V(x) = +V_0$ for x between $(-a, 0)$ and $V(x) = 0$ for x between $(0, +a)$.



Of the figures below, which is/are most likely to be physically acceptable energy eigenstate solutions for the time-independent Schrödinger equation for this well? Explain your reasoning.



21. The figure below shows a plot of a potential $V(x)$ versus x where V_0 is a positive constant.



(a) Which of the expressions below represents an acceptable physical wave function for a particle with $E < 0$ in region I ($x < -a$)? ξ is a positive real number.

(a) $\pm e^{\xi x}$

(b) $\pm e^{-\xi x}$

(c) $\pm e^{-i\xi x}$

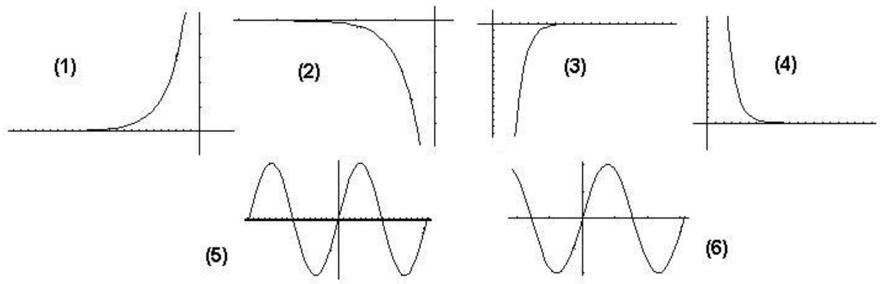
(d) $e^{\xi x}$

(e) $e^{\pm \xi x}$

(f) $e^{i\xi x}$

(g) $e^{\pm i\xi x}$

(b) Which of the plots below is/are an acceptable physical wave function for a particle with $E < 0$ in region III $(0 < x)$?



(a) (1) and (3).

(b) (3) and (4).

(c) (5) and (6).

(d) Only (2).

(e) Only (4).

MISCELLANEOUS

1. The group velocity of a wave packet for a free particle of mass m and wave number k ($\Psi_k(x,t) = Ae^{i(kx-\omega t)}$) is the same as the classical velocity of a particle of mass m and momentum $p = \hbar k$.

   (a) True.

   (b) False.

2. A harmonic oscillator is in the superposition state

$$\Psi(x, t = 0) = C_n\phi_n + C_m\phi_m,$$

where ϕ_n and ϕ_m are harmonic oscillator normalized energy eigenfunctions. For what values of m relative to n can Ψ be nonzero?

   (a) $m > n$

   (b) $m = n$

   (c) $m < n$

   (d) $m = n - 1$

   (e) $m = n + 1$

   (f) $m = n \pm 1$

   (g) The answer does not depend on the relative value of m and n.

3. As time progresses, a free particle wave packet spreads and the real and imaginary parts of the wave packet appear more "wiggly" in the leading edge of the wave compared to the trailing edge. This is because:

(a) The real and imaginary parts get increasingly out of phase with each other.

(b) The different momentum components of the wave packet travel at different speeds.

(c) The real and imaginary parts of the wave packet have different phase velocity.

4. What is the propagation speed of Ψ?

(a) It is half of the classical velocity of a particle of mass m and momentum $p = \hbar k$.

(b) It is twice of the classical velocity of a particle of mass m and momentum $p = \hbar k$.

(c) It has no relationship with the classical velocity of a particle of mass m and momentum $p = \hbar k$.

(d) It is same as the classical velocity of a particle of mass m and momentum $p = \hbar k$.

5. In a one-dimensional infinite square potential well of width a, consider the time evolution of a wave packet.

$$\Psi(x,t) = \sum C_n \phi_n(x) e^{-iE_n t/\hbar}$$

The wave packet travels between the two walls and bounces back and forth. Which one of the statements below best describes the behavior of the wave packet after some time?

(a) The wave packet vanishes after a time that is determined by a, m, and $\hbar$.

(b) The wave packet reforms exactly into the initial state after a time that is determined by a, m, and $\hbar$; it repeats the same cycle over and over.

(c) The wave packet reforms exactly into the initial state after some time and then stays unchanged.

(d) The wave packet reforms exactly into the initial state after a time that is determined by initial energy of the wave packet.

(e) The wave packet spreads as it bounces back and forth and flattens to a constant probability over the entire well for all later times.

(f) The wave packet vanishes after a time that is determined by initial energy of the wave packet.

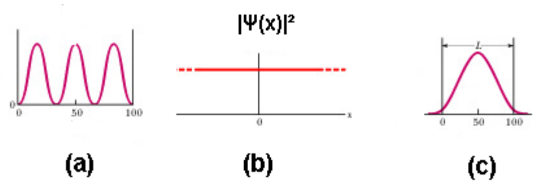
6. Only imaginary wave functions propagate and have current.

(a) True.

(b) False.

7. In what stage of solving $H\psi_i = Ei\psi_i$ do discrete energy levels appear?

8. The picture below shows the probability density of an electron in three different systems. For each graph, give an example of a physical system that can have this probability density for an electron.



9. The solution to the time-independent Schrödinger equation for a free particle gives the following wave function:

$$\Psi(x) = Ae^{ikx};\ \text{where } k \equiv \sqrt{2mE/\hbar}.$$

Can we have a free particle with a definite energy? Why or why not?

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