RIGOROUS EXPONENTIAL ASYMPTOTICS FOR A NONLINEAR THIRD ORDER DIFFERENCE EQUATION

DISSERTATION

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By

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ABSTRACT

In the present thesis, we study a particular 3-D map with a parameter $\varepsilon > 0$, which has two fixed points. One fixed point has a 1-D unstable manifold, while the other has a 1-D stable manifold. The main result is that we prove the smallest distance between the two manifolds is exponentially small in ε for small ε. We first prove in the limit of $\varepsilon \to 0^+$, bounded away from $+\infty$ or $-\infty$, both the stable and unstable manifolds asymptotes to a heteroclinic orbit for a differential equation. Then we show there exists a parameterization of the manifolds so that they differ exponentially in ε. By examining the inner region around the nearest complex singularity of the limiting solution, and using Borel analysis, we relate the constant multiplying the exponentially small term to the Stokes constant of the leading order inner equation.

To my husband

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CHAPTER 1

INTRODUCTION

In the present thesis we prove that the splitting of stable manifold at $w_j = 0$ and unstable manifold at $w_j = 1$ is exponentially small in terms of the parameter ε for the discrete map

$$w_{j+3} = w_j + \varepsilon g(w_{j+2}) \tag{1.1}$$

where $g(w) = w - w^{k+1}$ and $k \in \mathbb{N},\ k \geq 2$.

Discrete maps arise in many applications. Discrete maps arise naturally in numerical calculation of differential equations in a finite difference scheme. The result here is interesting in that it shows that a discretized system may behave significantly differently from the continuous system when $j \to \infty$ or $j \to -\infty$. As we shall see, the heteroclinic connection in the continuous system may break up after discretization and the corresponding manifold in the discrete map does not stay close to the limiting flow uniformly in time. Discrete maps also arises in the study of continuous dynamical system through Poincare maps.

The interest in this particular map is due to a model for ABC [2] flow in fluid dynamics, in which particle trajectories are observed after a discrete interval in time. The incompressibility of fluid is reflected in the volume preserving nature of the map written in the vector form

$$\begin{pmatrix} w_j \\ w_{j+1} \\ w_{j+2} \end{pmatrix} \mapsto \begin{pmatrix} w_{j+1} \\ w_{j+2} \\ w_{j+3} \end{pmatrix} = \begin{pmatrix} w_{j+1} \\ w_{j+2} \\ w_j + \varepsilon g(w_{j+2}) \end{pmatrix} \tag{1.2}$$

(1.2) has two fixed points $\mathbf{0} = (0,0,0)$ and $\mathbf{1} = (1,1,1)$. Linear analysis shows that $\mathbf{0}$ and $\mathbf{1}$ have a one dimensional unstable and stable manifolds respectively. In the formal limit $\varepsilon \to 0^+$, these two manifolds are the same. The objective here is to calculate the splitting of the manifolds for $\varepsilon > 0$ but small.

Calculation of splitting of stable and unstable manifolds of fixed points of dynamical system is usually done through a method due to Melnikov [13]. However, this method does not apply if the splitting is exponentially small, as is the case here.

There are presently a few studies in literature for exponentially small splitting for particular discrete maps. These include the standard map which is an area-preserving diffeomorphism of the two dimensional torus $\mathbb{T}^2 = \mathbb{R}/(2\pi\mathbb{Z})^2$, and is defined by

$$SM : (x, y) \mapsto (x + y + \varepsilon \sin x, y + \varepsilon \sin x). \tag{1.3}$$

SM has been studied by Lazutkin [12] , Hakim and Malllick [14] through formal asymptotic methods and later their result has been rigorously proved by Gelfreich [11], following a method originally proposed by Lazutkin. In 2-D, stable and unstable manifolds of discrete area preserving maps generically intersect at a set of homoclinic points. Homoclinic invariance introduced by Lazutkin provide a coordinate free description of the splitting which is also independent of the particular choice of homoclinic point in one homoclinic trajectory. To be precise, let $(x^{\pm}(t), y^{\pm}(t))$ be parameterization of the stable and unstable separatrices respectively, and let P be a homoclinic point. The homoclinic invariant is defined by

$$\omega = \det \begin{pmatrix} \dot{x}^-(P) & \dot{x}^+(P) \\ \dot{y}^-(P) & \dot{y}^+(P) \end{pmatrix} \tag{1.4}$$

However, for generic three dimensional maps, there are no obvious generalization of the concept of homoclinic invariant. While in two dimensional area preserving maps we have a theorem stating that the stable and unstable manifolds intersect infinitely many times, in three dimensional volume preserving maps by contrast, two one dimensional manifolds generically do not intersect. As a consequence no homoclinic point exists and homoclinic invariant does not make sense. Instead, we estimate the nearest distance between the two curves.

The problem in the present thesis was previously studied by V. Rom-Kedar, *et al* [3]. In their paper, they formally showed that as $\varepsilon \to 0^+$, there exists a heteroclinic limiting flow, and then argued that the splitting is exponentially small in terms of ε by comparing the size of the terms near the complex singularity of the limiting flow. The exponentially small term together with algebraic prefactor is determined by linearization about the limiting flow. While no rigorous proofs were given, they support their conclusion by numerical calculations.

Recent rigorous development in the general area of exponential asymptotics include a general theory for generic behavior of solution to nonlinear ordinary differential equations for large values of the independent variable. Costin [1] considered

$$\mathbf{y}' = \mathbf{f}_0(x) - \Lambda\mathbf{y} - \frac{1}{x}B\mathbf{y} + \mathbf{g}(x, \mathbf{y}) \tag{1.5}$$

where $\mathbf{f}_0(\mathbf{x}) = \mathcal{O}(x^{-2})$, for large x and $\mathbf{g}(x, \mathbf{y}) = \mathcal{O}(|\mathbf{y}|^2, x^{-2}\mathbf{y})$. Λ and B are matrices with constant coefficients satisfying some nonresonance condition. Later Braaksma [8] extended the theory to generic difference equation

$$\mathbf{y}(x+1) = \Lambda(x)\mathbf{y}(x) + \mathbf{g}(x, \mathbf{y}) \tag{1.6}$$

where

$$\Lambda(x) := diag(e^{-\mu_1}(1+x^{-1})^{a_1}, \cdots, e^{-\mu_n}(1+x^{-1})^{a_n}) \tag{1.7}$$

and $\mathbf{g}(x, \mathbf{y})$ is analytic $\mathbb{C}^n$ valued function of $(\frac{1}{x}, \mathbf{y})$ in a neighborhood of $(0, 0)$. $\mu_m, a_m \in \mathbb{C}$ are constants with $\mu_m \neq \mathbf{k} \cdot \mu \mod 2\pi i$ for $m \in \{1, \cdots, n\}$, $\mathbf{k} \in \mathbb{N}^n$, except for $\mathbf{k} = \mathbf{e}_m$.

$$\mathbf{g}(x, \mathbf{y}) = \sum_{\mathbf{l} \in \mathbb{N}^n} \mathbf{g}_{\mathbf{l}}(x) y^{\mathbf{l}} \text{ with } \mathbf{g}_{\mathbf{l}}(x) = \mathcal{O}(x^{-2}) \text{ if } |\mathbf{l}| \leq 1 \text{ as } x \to \infty \tag{1.8}$$

These theories involve behavior of solutions for large independent variable . The study in the thesis involves asymptotics for small parameter ε. While the two seem unrelated, they are actually connected in the study of inner region near the singularity of the limiting flow.

Our contribution has several aspects. In the present thesis, the leading order equation in the inner variable is degenerate, so Braaksma's analysis does not apply immediately. Nevertheless, the methods can be adapted to the inner problem for large value of the independent variable .

In this thesis, we provide the only rigorous proof of exponentially small splitting of separatrices for three dimensional volume preserving map that we know of. As in many other problems of nonlinear analysis, many of the proofs rely on contraction arguments. The choice of the proper Banach space is usually the key.

There is another property of the discrete map which adds to the difficulty of the problem. In the case of autonomous differential equations, the nearest distance between two one dimensional manifolds described by solutions $\mathbf{u}_1(x)$ and $\mathbf{u}_2(x)$ can be characterized by

$$\inf_{x,c} |\mathbf{u}_1(x+c) - \mathbf{u}_2(x)| \tag{1.9}$$

for all possible constants c. However, the expression of manifold separation for solution to difference equation is much more complicated. In our case we had to look for

$$\inf_{x,a(x)} |\mathbf{u}_1(x+a(x)) - \mathbf{u}_2(x)| \tag{1.10}$$

where $x + a(x)$ is a general parametrization of one of the manifolds. $a(x)$ need not be a constant and is, generally, a periodic function of x. This is part of the difficulty. We used the fact that $\mathbf{u}_1(x+a(x)) = \tilde{\mathbf{u}}(x)$ also satisfies the same nonlinear difference equation and asymptotically approaches one of the fixed points to find the general representation of the manifolds.

The main result of the thesis is given at the end of chapter 4 (Theorem 4.43) . The outline of the thesis is the following: The original results are all contained in chapter 3 and chapter 4. However, to familiarize the reader, we describe eyponential asymptotics and its applications as well as how to solve linear inhomogeneous difference equation in Chapter 2. We also describe Borel summation and illustrate through a simple study of difference equation and how its asymptotics can be recovered rigorously through this technique.

In chapter 3, we first prove the limiting flow is close to both the stable and unstable manifold in certain regions, then establish that there exists a reparameterization of the stable manifold that differs from the unstable manifold by exponentially small term and we argue that this is the closest distance between the two manifolds.

In chapter 4, we prove the Borel transform of the leading order inner equation has unique solution in certain region and this solution is Laplace transformable. We also give the ramified analytic structure of this solution at zero. By studying the singularity structure of the nearest singularity of the solution, we show that the leading order inner equation has two solutions that differs only eyponentially. Then we show that the full inner equation has solutions that is asymptotic to the solutions of the leading order equation. By matching the inner and outer solution, we relate the constant multiplying the exponentially small term in the difference of stable and unstable manifolds to the Stoke constant of the leading order inner equation. This leads to the main theorem 4.43.

CHAPTER 2

EXPONENTIAL ASYMPTOTICS AND DIFFERENCE EQUATION

2.1 Exponential Asymptotics

This section is adapted from [5].

Classical asymptotics as Poincare considered is formally defined as the follows.

Definition 2.1. *Given a function $\phi(\varepsilon)$, the series $\sum_{n=0}^{\infty} \phi_n \varepsilon^n$ is said to be an asymptotic expansion of $\phi(\varepsilon)$ as $\varepsilon \to 0^+$ if for every nonnegative integer N,*

$$\lim_{\varepsilon \to 0^+} \frac{\phi(\varepsilon) - \sum_{n=0}^{N} \phi_n \varepsilon^n}{\varepsilon^N} = 0 \tag{2.1}$$

and we write

$$\phi(\varepsilon) \sim \sum_{n=0}^{\infty} \phi_n \varepsilon^n. \tag{2.2}$$

Note ϕ might also depends on other parameter, say in the form of $\phi(x, \varepsilon)$.

Definition 2.2. *We say $\phi(x, \varepsilon)$ asymptotic to $\sum_{n=0}^{\infty} \phi_n(x) \varepsilon^n$ on a closed interval D, as $\varepsilon \to 0^+$ if for any $x \in D$, and every nonnegative integer N, we have*

$$\lim_{\varepsilon \to 0^+} \frac{\phi(x; \varepsilon) - \sum_{n=0}^{N} \phi_n(x) \varepsilon^n}{\varepsilon^N} = 0 \tag{2.3}$$

and we write

$$\phi(x;\varepsilon) \sim \sum_{n=0}^{\infty} \phi_n(x)\varepsilon^n. \tag{2.4}$$

Remark 2.1. .

- *The endpoints of closed interval D can be made to depend on ε.*
- *Instead of simple power series in ε, more generally, we can approximate ϕ by aid of an asymptotic series $\{\zeta_j(\varepsilon)\}_{j=1}^{\infty}$ where $\zeta_j(\varepsilon) \gg \zeta_{j+1}(\varepsilon)$ as $\varepsilon \to 0$.*
- *Asymptotic sequence is not unique.*

An asymptotic series need not converge for $\varepsilon \neq 0$. However, one can accurately approximate the value of a function by using first a few terms from its divergent asymptotic series. For example, Copson mentions a series due to Euler:

$$\sum_{n=1}^{M} \frac{1}{n} \sim \ln M + \gamma + \frac{1}{2M} + \sum_{n=1}^{\infty} \frac{B_{2k}}{(2k)M^{2k}}, \tag{2.5}$$

where $\gamma = 0.5772\cdots$ is the Euler's constant, $\{B_{2k}\}$ are Bernoulli numbers, and the small parameter (ε) is $\frac{1}{M}$. For every finite M, the infinite series in (2.5) diverges. But Euler used the series with $M = 10$ to calculate γ correctly to 15 decimal places.

It is to be noted that asymptotic series like (2.2) only capture terms algebraic in ε, and exponentially small terms like $e^{-1/\varepsilon}$ are smaller than any of the terms in the series as $\varepsilon \to 0^+$; hence they will not be represented. If (2.2) is true, then $\phi(\varepsilon) + e^{-1/\varepsilon}$ will also have the same asymptotic expansion. Such transcendentally small terms are said to lie beyond all orders of the asymptotic expansion.

In most of the applications, exponentially small terms are insignificant and can be ignored. However, exceptional problems in which these small terms have great practical interest are known in many branches of science, including dendritic crystal growth, viscous fingering [6], quantum tunnelling, KAM Theory, etc (see [5]).

We mention two ways in which the exponentially small term can have practical interest. First, transcendentally small terms can be important since they can change a qualitative feature of the function. Second, when we use an asymptotic series to approximate the function value for fixed (small) ε , the transcendentally small terms can be numerically important in practice.

2.2 Difference equation

In this section we will discuss the general methods of solving difference equations, which are similar to solving differential equations.

2.2.1 First Order Linear Difference Equation

Given $h > 0$, define operator Δ_h as

$$\Delta_h[f](x) := f(x+h) - f(x) \tag{2.6}$$

First consider the simplest difference equation

$$\Delta_h[f](x) = 0 \tag{2.7}$$

Obviously any solution of (2.7) is of the type $f(x) = c(x)$ where $c(x)$ is periodic function of period h.

Now consider methods for solving the inhomogeneous difference equation

$$\Delta_h f = g. \tag{2.8}$$

Direct substitution shows that both

$$f(x) = -\sum_{n=0}^{\infty} g(x+nh), \quad \text{and} \quad f(x) = \sum_{n=1}^{\infty} g(x-nh) \tag{2.9}$$

satisfies (2.8), provided the infinite sum converges. This motivates us to define two inversions $\Delta_{h,-}^{-1}$ in a left infinity region and $\Delta_{h,+}^{-1}$ in a right infinity region defined below.

Definition 2.3. *A region $\mathcal{D}_-$ is left infinity type, if it is open and if $s \in \mathcal{D}_- \Rightarrow s-t \in \mathcal{D}_-$ for $t \in \mathbb{R}^+$.*

In a left infinity type region $\mathcal{D}_-$, for $\mu > 0$, define norm

$$\|g\|_\mu := \sup_{x\in\mathcal{D}_-} |e^{-\mu x} g(x)|. \tag{2.10}$$

Let $\mathcal{S}_\mu(\mathcal{D}_-)$ be the function space of complex valued analytic function defined in $\mathcal{D}_-$ continuous in $\overline{\mathcal{D}_-}$ with $\|\cdot\|_\mu < \infty$. It is easy to see that $\mathcal{S}_\mu$ equipped with $\|\cdot\|_\mu$ is a Banach space. Let

$$\Delta_{h,-}^{-1} : \mathcal{S}_\mu(\mathcal{D}_-) \mapsto \mathcal{S}_\mu(\mathcal{D}_-) \text{ defined by } \Delta_{h,-}^{-1} g(x) = \sum_{n=1}^{\infty} g(x-nh) \tag{2.11}$$

Remark 2.2. *Later, we extend definition of $\Delta_{h,-}^{-1}$ on space of algebraically decaying continuous functions $g(x) = \mathcal{O}(x^{-\nu})$ as $x \to -\infty$ in $\mathcal{D}_-$ with $\nu > 1$.*

Proposition 2.4. *The induced operator norm $\|\Delta_{h,-}^{-1}\| = \sup_{g\in\mathcal{S}_\mu(\mathcal{D}_-)} \frac{\|\Delta_{h,-}^{-1}[g]\|_\mu}{\|g\|_\mu}$ has the bound*

$$\|\Delta_{h,-}^{-1}\| \leq \frac{1}{1-e^{-\mu h}} \tag{2.12}$$

Proof.

$$\begin{aligned} |e^{-\mu x}\sum_{n=1}^{\infty} g(x-nh)| &\leq \sum_{n=1}^{\infty} |e^{-\mu(x-nh)} g(x-nh) e^{-\mu n h}| \\ &\leq \|g\|_\mu \sum_{n=1}^{\infty} e^{-\mu n h} \leq \|g\|_\mu \frac{1}{1-e^{-\mu h}} \end{aligned} \tag{2.13}$$

Therefore we get (2.12). ☐

Remark 2.3. *$\Delta_{h,-}^{-1}[g]$ is a particular solution of (2.8). The following lemma gives the structure of general solution to (2.8) as sum of a particular plus a general solution to homogenous equation, analogous to differential equations.*

Lemma 2.5. *For $g \in \mathcal{S}_\mu(\mathcal{D}_-)$ with $\mu > 0$, any solution of (2.8) in $\mathcal{D}_-$ can be written as*

$$f(x) = c(x) + \Delta_{h,-}^{-1}[g](x) \tag{2.14}$$

where $c(x)$ is periodic with period h in $\mathcal{D}_-$.

Proof. Let $c(x) = f(x) - \Delta_{h,-}^{-1}[g](x)$, then $c(x)$ satisfies $\Delta_h[c](x) = 0$. Therefore $c(x)$ is a periodic function with period h in $\mathcal{D}_-$. ☐

Similarly, we define a right infinity region:

Definition 2.6. *$\mathcal{D}_+$ is right infinity type, if it is open and if $s \in \mathcal{D}_+ \Rightarrow s+t \in \mathcal{D}_+$ for $t \in \mathbb{R}^+$.*

Define $\mathcal{S}_\mu(\mathcal{D}_+)$ similar with $\mathcal{S}_\mu(\mathcal{D}_-)$ with $\mathcal{D}_-$ replaced by $\mathcal{D}_+$. $\mathcal{S}_\mu(\mathcal{D}_+)$ is equipped with norm $\|\cdot\|_{\mu,+}$ for $\mu > 0$.

$$\|g\|_{\mu,+} := \sup_{x\in\mathcal{D}_+} |e^{\mu x} g(x)|. \tag{2.15}$$

is a Banach space. Let

$$\Delta_{h,+}^{-1} : \mathcal{S}_\mu(\mathcal{D}_+) \mapsto \mathcal{S}_\mu(\mathcal{D}_+) \quad \Delta_{h,+}^{-1} g(x) = -\sum_{n=0}^{\infty} g(x+nh) \tag{2.16}$$

We also have a bound on the norm of $\Delta_{h,+}^{-1}$ proved in a manner similar to Proposition 2.4.

Proposition 2.7. *The induced norm* $\|\Delta_{h,+}^{-1}\| = \sup_{g\in\mathcal{S}_\mu(\mathcal{D}_+)} \frac{\|\Delta_{h,+}^{-1}[g]\|_{\mu,+}}{\|g\|_{\mu,+}}$ *has the bound*

$$\|\Delta_{h,+}^{-1}\| \leq \frac{1}{1-e^{-\mu h}} \tag{2.17}$$

In the present thesis, we sometimes need to solve (2.8) where g is only defined in a finite region $\mathcal{D}$, which is neither left infinity type nor right infinity type. Here we use a method adapted from Gelfreich [11]. The idea is that if $\mathcal{D}$ is intersection of a left infinity region and a right infinity region, then we project g into two functions that have analytic continuation in a left infinity and a right infinity region with appropriate decay as $x \to \mp\infty$ respectively. Then we can apply the methods developed above.

Consider a finite rectangular open region $\mathcal{D} = (-a, a) \times (-b, b)i$, where $a > 1$, $0 < h \leq 1$ and $b > 0$. Then $\mathcal{D}$ can be represented as $\mathcal{D} = \mathcal{D}_- \cap \mathcal{D}_+$ where $\mathcal{D}_- = (-\infty, a) \times (-b, b)i$, $\mathcal{D}_+ = (-a, \infty) \times (-b, b)i$.

Assume that $a > 1$. Take a $\mathcal{C}^\infty$ monotonically increasing real function χ which has the property $\chi(t) = 0$ for $t \leq -1$ and $\chi(t) = 1$ for $t \geq 1$. We extend χ onto complex domain $\mathbb{C}$ by $\chi(t + si) = \chi(t)$, where $t, s \in \mathbb{R}$.

Remark 2.4. *The extended* $\chi(x)$ *is Lipschitz continuous in* $\partial\mathcal{D}$ *. It is not difficult to see that* $|\chi(x) - \chi(y)| \leq K_0|x - y|$ *,* $x, y \in \mathcal{D}$*, for some constant* K_0 *independent of* x, y*.*

Remark 2.5. *If $a < 1$ depends on a small parameter δ_0, for instance $a = K\delta_0$, where K is a fixed constant and δ_0 is small, we will use $\tilde{\chi}(t) := \chi(\frac{t}{4K\delta_0})$ instead of $\chi(t)$.*

Lemma 2.8. *For any function g analytic in $\mathcal{D}$ and continuous in $\overline{\mathcal{D}}$, let*

$$h_-(x) := \frac{1}{2\pi i}\int_{\partial\mathcal{D}} \frac{\chi(\xi)g(\xi)}{\xi - x}d\xi \tag{2.18}$$

$$h_+(x) := \frac{1}{2\pi i}\int_{\partial\mathcal{D}} \frac{(1-\chi(\xi))g(\xi)}{\xi - x}d\xi \tag{2.19}$$

Then $h_-(x)$, $h_+(x)$ are analytic functions in $\mathcal{D}$ which have analytic continuation onto $\mathcal{D}_-$ and $\mathcal{D}_+$ that are continuous in $\overline{\mathcal{D}}_-$ and $\overline{\mathcal{D}}_+$ respectively.

Proof. Let h^-_{int}, h^-_{ext} denote the two distinct analytic function defined by formulae $\frac{1}{2\pi i}\int_{\partial\mathcal{D}} \frac{\chi(\xi)g(\xi)}{\xi-x}d\xi$ when z is inside $\mathcal{D}$ and outside $\mathcal{D}$ respectively. By Plemelj formulae [20], for any z_0 on the left side edge of rectangle $\partial\mathcal{D}$, we have

$$h^-_{int}(z_0) - h^-_{ext}(z_0) = \chi(z_0)g(z_0) = 0 \tag{2.20}$$

which implies h^-_{int} and h^-_{ext} are analytic continuation of each other in $\mathcal{D}_-$, we denote it by $h_-(x)$.

Similarly, since $1 - \chi(z_0) = 0$ on right edge of $\partial\mathcal{D}$, we get $\frac{1}{2\pi i}\int_{\partial\mathcal{D}} \frac{(1-\chi(\xi))g(\xi)}{\xi-x}d\xi$ defined inside $\mathcal{D}$ can be analytic continued to $\mathcal{D}^+$. □

Let

$$\mathcal{S}_0(\mathcal{D}) := \{g \ : \ g \text{ analytic in } \mathcal{D}, \text{ continuous in } \overline{\mathcal{D}}\} \tag{2.21}$$

It is obvious that once equipped with $\|\cdot\|_\infty$ norm in $\mathcal{D}$, $\mathcal{S}_0(\mathcal{D})$ forms a Banach space.

Define

$$\mathcal{P}_- : \mathcal{S}_0(\mathcal{D}) \mapsto \mathcal{S}_0(\mathcal{D}_-) \quad \mathcal{P}_-[g] = h_-; \tag{2.22}$$

$$\mathcal{P}_+ : \mathcal{S}_0(\mathcal{D}) \mapsto \mathcal{S}_0(\mathcal{D}_+) \quad \mathcal{P}_+[g] = h_+ \tag{2.23}$$

where $h^{\pm}$ is defined in (2.18) and (2.19) respectively.

Lemma 2.9. *For any $g \in \mathcal{S}_0(\mathcal{D})$, and $x \in \mathcal{D}$ we have*

$$g(x) = \mathcal{P}_+[g](x) + \mathcal{P}_-[g](x) \tag{2.24}$$

Proof.

$$\mathcal{P}_+[g](x) + \mathcal{P}_-[g](x) = h_-(x) + h_+(x) = \frac{1}{2\pi i}\oint_{\partial\mathcal{D}} \frac{g(\xi)}{\xi - x} d\xi = g(x) \tag{2.25}$$

□

Lemma 2.10. *If $\mathcal{D}$ is subset of a square with size $R > 1$, $\|\mathcal{P}_\pm\| \le C \ln R$ where C is a constant depending on χ alone and independent of R.*

Proof. Let $g \in \mathcal{S}_0(\mathcal{D})$, to estimate the norm of corresponding h_-, noticing that analytic function can only has maximal value on the boundary, and since $\partial\mathcal{D}$ is finite, we have as $x \to \infty$, $h_-(x) \to 0$, we only need to estimate h_- on $\partial\mathcal{D}^-$. For x that has distance greater than 1 away from $\partial\mathcal{D}$, we have for $R > 1$,

$$|h_-(x)| \le \|g\| \int_{\partial\mathcal{D}} \frac{|d\xi|}{|\xi - x|} \le C\|g\| \ln R \tag{2.26}$$

For $x \in \partial\mathcal{D}$, or $x \in \partial\mathcal{D}^- \setminus \partial\mathcal{D}$ and distance between x and $\partial\mathcal{D}$ is less than or equal to 1,

$$h_-(x) = \chi(x)g(x) + \frac{1}{2\pi i}\int_{\partial\mathcal{D}} \frac{\chi(\xi) - \chi(x)}{\xi - x} g(\xi) d\xi \tag{2.27}$$

Break up the second integral into two, one on $l := \{\xi \in \partial\mathcal{D} \; : \; |\xi - x| < 1\}$ and the other on $\partial\mathcal{D} \setminus l$. The second integral can be bounded by $C\|g\| \ln R$. By Remark 2.4, there exists $\alpha > 0$ and constant $K > 0$ such that $|\chi(\xi) - \chi(x)| \leq K|\xi - x|^\alpha$ for $x, y \in \mathbb{C}$, so we have that

$$\begin{aligned} \left|\frac{1}{2\pi i}\int_l \frac{\chi(\xi) - \chi(x)}{\xi - x} g(\xi) d\xi\right| &\leq \|g\| \int_l K \frac{|\xi - x|^\alpha}{|\xi - x|} |d\xi| \\ &\leq C\|g\| \end{aligned} \tag{2.28}$$

Therefore the second term in (2.27) is bounded by $C \ln R \; \|g\|$ for R goes to infinity. The first term is clearly bounded by $\|g\|$. Hence the lemma follows. □

Under the assumption that the region $\mathcal{D}$ has a smallest distance of $d_1 > 0$ from $\pm\frac{\pi}{\mu}i$, define Δ_h^{-1} as

$$\Delta_h^{-1} : \mathcal{S}_0(\mathcal{D}) \mapsto \mathcal{S}_0(\mathcal{D}) \qquad \Delta_h^{-1} = \Delta_{h,+}^{-1}\mathcal{I}_+^{-1}\mathcal{P}_+\mathcal{I}_0 + \Delta_{h,-}^{-1}\mathcal{I}_-^{-1}\mathcal{P}_-\mathcal{I}_0 \tag{2.29}$$

where

$$\mathcal{I}_0 : \mathcal{S}_0(\mathcal{D}) \mapsto \mathcal{S}_0(\mathcal{D}), \quad \mathcal{I}_0[g](x) = (e^{\mu x} + e^{-\mu x})g(x); \tag{2.30}$$

$$\mathcal{I}_\pm : \mathcal{S}_\mu(\mathcal{D}_\pm) \mapsto \mathcal{S}_0(\mathcal{D}_\pm), \quad \mathcal{I}_\pm[g](x) = (e^{\mu x} + e^{-\mu x})g(x) \tag{2.31}$$

Remark 2.6. *$(e^{\mu x} + e^{-\mu x})$ is chosen to insure the decaying rate that $\Delta_{h,\pm}^{-1}$ requires. We can avoid zeros of this factor by choose $\mu < \frac{\pi}{b}$.*

Lemma 2.11. *Given $g \in \mathcal{S}_0(\mathcal{D})$ then $\Delta_h^{-1}g$ is a solution of (2.8) for $x \in \mathcal{D}$.*

Proof. Proposition 2.9 shows that for $x \in \mathcal{D}$, we have $(\mathcal{P}_+\mathcal{I}_0 + \mathcal{P}_-\mathcal{I}_0)[g](x) = (e^{\mu x} + e^{-\mu x})g(x)$ Apply Δ_h to $\Delta_h^{-1}[g]$ we get

$$\begin{aligned}\Delta_h\Delta_h^{-1}[g](x) &= \Delta_h(\Delta_{h,+}^{-1}\mathcal{I}_+^{-1}\mathcal{P}_+\mathcal{I}_0 + \Delta_{h,-}^{-1}\mathcal{I}_-^{-1}\mathcal{P}_-\mathcal{I}_0)[g](x) \\ &= \frac{1}{e^{\mu x} + e^{-\mu x}}(\mathcal{P}_+\mathcal{I}_0 + \mathcal{P}_-\mathcal{I}_0)[g](x) = g(x) \qquad (2.32)\end{aligned}$$

□

Lemma 2.12. *If $\mathcal{D}$ is subset of a square with size $R > 1$, then $\Delta_h^{-1} : \mathcal{S}_0(\mathcal{D}) \mapsto \mathcal{S}_0(\mathcal{D})$ in Lemma 2.11 has the bound on its norm*

$$\|\Delta_h^{-1}\| \le \frac{C}{1 - e^{-\mu h}} \ln R. \qquad (2.33)$$

where C is a constant .

Proof.

$$\|\Delta_h^{-1}\| \le \|\Delta_{h,+}^{-1}\| \; \|\mathcal{I}_+^{-1}\| \; \|\mathcal{P}_+\| \; \|\mathcal{I}_0\| + \|\Delta_{h,-}^{-1}\| \; \|\mathcal{I}_-^{-1}\| \; \|\mathcal{P}_-\| \; \|\mathcal{I}_0\| \qquad (2.34)$$

It is easy to see that $\|\mathcal{I}\| \le C$, and $\|\mathcal{I}_\pm^{-1}\| \le C_1$. Using these bounds and Proposition 2.4, Proposition 2.7 and Lemma 2.10, we get (2.33). □

Now consider equation

$$\Delta f = g \quad \text{where } \Delta[f](s) := f(s+1) - f(s) \qquad (2.35)$$

where g is algebraically decreasing. $g \in \Upsilon_\gamma$, where

$$\begin{aligned}\Upsilon_\gamma(\mathcal{D}(B, \pm)) := \{g \;\; : \;\; & g \text{ analytic in } \mathcal{D}(B, \pm) \text{ continuous in } \overline{\mathcal{D}}(B, \pm), \\ & \text{with } \|g\|_\gamma < \infty\} \qquad (2.36)\end{aligned}$$

$$\mathcal{D}(B,\pm) := \{s \ : \ \Im s < -B, \pm\Re s < B\} \tag{2.37}$$

$$\|g\|_\gamma = \sup_{s\in\mathcal{D}(B,\pm)} |s^\gamma g(s)| \tag{2.38}$$

For $\gamma > 0$, $\lambda \geq 0$, define

$$\Delta_-^{-1} : \Upsilon_{\gamma+\lambda+1}(\mathcal{D}(B,-)) \mapsto \Upsilon_\lambda(\mathcal{D}(B,-)) \quad \Delta_-^{-1}g(s) = \sum_{n=1}^{\infty} g(s-n) \tag{2.39}$$

$$\Delta_+^{-1} : \Upsilon_{\gamma+\lambda+1}(\mathcal{D}(B,+)) \mapsto \Upsilon_\lambda(\mathcal{D}(B,+)) \quad \Delta_+^{-1}g(s) = -\sum_{n=0}^{\infty} g(s+n) \tag{2.40}$$

Lemma 2.13. *For $g \in \Upsilon_{\gamma+\lambda+1}(D(B,\pm)))$, $\Delta_\pm^{-1}[g] \in \Upsilon_\lambda(D(B,\pm))$ are solutions to (2.35).*

$$\|\Delta_\pm^{-1}\| \leq \frac{K}{\gamma}B^{-\gamma} \tag{2.41}$$

where $\lambda \geq 0$, $\gamma > 0$ and K is a constant independent of B, γ .

Proof. Substituting $\Delta_\pm^{-1}[g]$ into (2.35), we can easily see that they are solutions.

$$\begin{aligned} s^\lambda|\Delta_-^{-1}[g](s)| &\leq \|g\|_\gamma \sum_{n=1}^{\infty} \frac{1}{|s-n|^{\gamma+1}} \frac{s^\lambda}{|s-n|^\lambda} \\ &\leq C_1\|g\|_\gamma |s|^{-\gamma} \int_0^\infty \frac{1}{|\frac{s}{|s|}-t|^{\gamma+1}}dt \leq C\frac{B^{-\gamma}}{\gamma}\|g\|_\gamma \end{aligned} \tag{2.42}$$

Therefore $\|\Delta_-^{-1}\| \leq \frac{K}{\gamma}B^{-\gamma}$ Similarly, $\|\Delta_+^{-1}\| \leq \frac{K}{\gamma}B^{-\gamma}$. ☐

For $\mathcal{D}(B) = \mathcal{D}(B,-) \cap \mathcal{D}(B,+)$, let

$$\Delta^{-1} : \Upsilon_0(\mathcal{D}(B)) \mapsto \Upsilon_0(\mathcal{D}(B)) \qquad \Delta^{-1} = \Delta_+^{-1}\mathcal{J}_+^{-1}\mathcal{P}_+\mathcal{J} + \Delta_-^{-1}\mathcal{J}_-^{-1}\mathcal{P}_-\mathcal{J} \tag{2.43}$$

where

$$\mathcal{J} : \Upsilon_0(\mathcal{D}) \mapsto \Upsilon_0(\mathcal{D}) \quad \mathcal{J}[g](s) = s^{1+\gamma}g(s); \tag{2.44}$$

$$\mathcal{J}_\pm : \Upsilon_0(\mathcal{D}(B,\pm)) \mapsto \Upsilon_{1+\gamma}(\mathcal{D}(B,\pm)) \quad \mathcal{J}_\pm[g](s) = s^{1+\gamma}g(s) \tag{2.45}$$

Lemma 2.14. *If $\mathcal{D}(B)$ is subset of a square with size $R > 1$, then $\Delta^{-1} : \Upsilon_0(\mathcal{D}(B)) \mapsto \Upsilon_0(\mathcal{D}(B))$ has the bound on its norm*

$$\|\Delta^{-1}\| \leq \; CB^{-\gamma}(B+R)^{1+\gamma} \ln R. \tag{2.46}$$

where C is a constant depending on γ.

Proof.

$$\|\Delta^{-1}\| \leq \|\Delta_+^{-1}\| \; \|\mathcal{J}_+^{-1}\| \; \|\mathcal{P}_+\| \; \|\mathcal{J}_0\| + \|\Delta_-^{-1}\| \; \|\mathcal{J}_-^{-1}\| \; \|\mathcal{P}_-\| \; \|\mathcal{J}_0\| \tag{2.47}$$

It is easy to see that $\|\mathcal{J}_0\| \leq C(B+R)^{1+\gamma}$, and $\|\mathcal{J}_\pm^{-1}\| \leq C$. Using these bounds and Lemma 2.13, and Lemma 2.10, we get (2.33). □

2.2.2 Higher Order linear Inhomogeneous Difference Equation

In this subsection, we will consider solving higher order linear inhomogeneous difference equation

$$\mathcal{L}[v] = g \tag{2.48}$$

where g is a given function and $\mathcal{L}$ is a linear difference operator . Here, for illustrative purpose, we will take $\mathcal{L}$ to an linear operator of third order.

Analogous to differential equations, the general solution to inhomogeneous difference equation can be represented as a general solution to the homogeneous equation added to a particular solution.

General solutions of the homogeneous equation where all the coefficients in the linear operator $\mathcal{L}$ are constant in some simple non-resonance case can be found by plugging $f = a^x$ into $\mathcal{L}[f] = 0$, where a is an undetermined constant, and then solving for a. However, in the general case, finding the general solutions of the homogeneous equation explicitly is difficult.

Suppose we get 3 solutions $\{V_\alpha\}_{\alpha=1,2,3}$ to the third order difference equation $\mathcal{L}[v] = 0$, we say they form a fundamental set of solutions if the discrete Wronskian $W(x)$ is nonzero , where $W(x) = \det(D(x))$, and

$$D(x) = \begin{bmatrix} V_1(x+h) & V_2(x+h) & V_3(x+h) \\ V_1(x+2h) & V_2(x+2h) & V_3(x+2h) \\ V_1(x+3h) & V_2(x+3h) & V_3(x+3h) \end{bmatrix} \tag{2.49}$$

Notice

$$W(x) = \begin{vmatrix} V_1(x+h) & V_2(x+h) & V_3(x+h) \\ \Delta_h V_1(x+h) & \Delta_h V_2(x+h) & \Delta_h V_3(x+h) \\ \Delta_h V_1(x+2h) & \Delta_h V_2(x+2h) & \Delta_h V_3(x+2h) \end{vmatrix} \tag{2.50}$$

Lemma 2.15. *If $\{V_\alpha\}_{\alpha=1,2,3}$ is a fundamental set of solutions to $\mathcal{L}[v] = 0$ and coefficient of $v(x+3h)$ is 1 , then for any solution $v(x)$ of the equation , there exists $c_\alpha(x)$ for $\alpha = 1, 2, 3$ that are periodic with period h such that*

$$v(x) = \sum_{\alpha=1}^{3} c_\alpha(x) V_\alpha(x) \tag{2.51}$$

The proof is standard; so it is omitted here.

Now we will concentrate on the problem of given the fundamental set of solutions $\{V_\alpha\}_{\alpha=1,2,3}$ of the homogeneous equation $\mathcal{L}[v] = 0$, how to find solutions to (2.48).

We will assume $V_\alpha(x)$ to be analytic in $\mathcal{D}^\pm$. The method used here is variation of parameter, which reduces the problem to solving first order difference equations whose solutions are explained in section 2.2.1.

Suppose

$$v(x) = \sum_{\alpha=1,2,3} C_\alpha(x) V_\alpha(x) \tag{2.52}$$

where $C_\alpha(x)$'s are undetermined as yet. In the expression

$$v(x+h) = \sum_{\alpha=-1,0,1} C_\alpha(x) V_\alpha(x+h) + \sum_{\alpha=1,2,3} \Delta_h C_\alpha(x) V_\alpha(x+h), \tag{2.53}$$

we choose $C_\alpha(x)$ in such a way that the second term vanishes. So,

$$v(x+2h) = \sum_{\alpha=1,2,3} C_\alpha(x) V_\alpha(x+2h) + \sum_{\alpha=-1,0,1} \Delta_h C_\alpha(x) V_\alpha(x+2h) \tag{2.54}$$

Again, choose $C_\alpha(x)$ so that the second term vanishes. These equations implies

$$\mathcal{L}[v](x) = \sum_{\alpha=-1,0,1} \Delta_h C_\alpha(x) V_\alpha(x+3h)$$

$\Delta_h C_\alpha(x)$ satisfies the following equation:

$$D(x) \begin{bmatrix} \Delta_h C_1(x) \\ \Delta_h C_2(x) \\ \Delta_h C_3(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathcal{L}_h[v](x) \end{bmatrix}$$

Then $\Delta_h C_\alpha(x)$ can be solved in terms of C_α.

$$\Delta_h C_\alpha(x) = \frac{M_\alpha(x)}{W(x)} g(x) \tag{2.55}$$

where M_α is the cofactor of the last element in the α column of $D(x)$. So we reduced the problem to one already solved.

Lemma 2.16. *For g, such that $\frac{M_\alpha}{W} g \in \mathcal{S}_\mu(\mathcal{D}_\pm)$, $\alpha = 1, 2, 3$, let*

$$\mathcal{L}_\pm^{-1}[g](x) := \sum_{\alpha=-1,0,1} V_\alpha(x) \Delta_{h,\pm}^{-1}[\frac{M_\alpha}{W} g](x) \tag{2.56}$$

Then the most general solution of (2.48) is given by

$$v(x) = \sum_{\alpha=-1,0,1} d_\alpha(x) V_\alpha(x) + \mathcal{L}_\pm^{-1}[g] \tag{2.57}$$

where $d_\alpha(x)$ is periodic with period h. $v(x)$ is analytic in $\mathcal{D}_\pm$ if and only if $d_\alpha(x)$ is analytic in $\mathcal{D}_\pm$.

Proof. Let $C_\alpha(x) := \Delta_{h,\pm}^{-1} \frac{M_\alpha(x)}{W(x)} g(x)$. Then $C_\alpha(x)$ satisfies (2.55). But in the above, we already showed that C_α satisfies (2.55) is equivalent to $v_0(x) := \sum_{\alpha=1}^3 C_\alpha(x) V_\alpha(x)$ satisfying (2.48). The lemma follows from Lemma 2.15, noting that analyticity of $\mathcal{L}_\pm^{-1}[g]$ implies that $\sum_\alpha V_\alpha$ is analytic if and only if v is analytic. From analyticity of V_α and $W(x) \neq 0$, it follows $d(x)$ is analytic iff v is. ☐

Lemma 2.17. *For $g \in \mathcal{S}_0(\mathcal{D})$,let*

$$\mathcal{L}^{-1}[g](x) := \sum_{\alpha=-1,0,1} V_\alpha(x) \Delta_h^{-1}[\frac{M_\alpha}{W} g](x) \tag{2.58}$$

Then the most general solution of (2.48) is given by

$$v(x) = \sum_{\alpha=-1,0,1} d_\alpha(x) V_\alpha(x) + \mathcal{L}^{-1}[g] \tag{2.59}$$

where $d_\alpha(x)$ is periodic with period h. $v(x)$ is analytic in $\mathcal{D}$ if and only if $d_\alpha(x)$ is analytic in $\mathcal{D}$.

Proof. The proof is similar to the proof of Lemma 2.16. ☐

For nonlinear equation, we have the following lemmas.

Lemma 2.18. *Let v be an analytic solution to*

$$\mathcal{L}[v] = \mathcal{N}[v] \tag{2.60}$$

in $\mathcal{D}_\pm$ (or $\mathcal{D}$) , where $\mathcal{N}$ is an operator $\mathcal{N} : \mathcal{S}_\mu(\mathcal{D}_\pm) \mapsto \mathcal{S}_\mu(\mathcal{D}_\pm)$ (or $\mathcal{N} : \mathcal{S}_0(\mathcal{D}) \mapsto \mathcal{S}_0(\mathcal{D})$). If $\mathcal{L}_\pm^{-1}[\mathcal{N}[v]]$ (or $\mathcal{L}^{-1}[\mathcal{N}[v]]$) exists, then there exists d_α, $\alpha = 1, 2, 3$ analytic in $\mathcal{D}_\pm$ ($\mathcal{D}$), periodic with period h, such that $d_\alpha(x)$ satisfying

$$v(x) = \sum_{\alpha=-1,0,1} d_\alpha(x) V_\alpha(x) + \mathcal{L}^{-1}[\mathcal{N}[v]] \tag{2.61}$$

Proof. Using v satisfies (2.60), we get

$$\mathcal{L}[v - \mathcal{L}^{-1}[\mathcal{N}[v]]] = \mathcal{L}[v] - \mathcal{L}\mathcal{L}^{-1}[\mathcal{N}[v]]] = \mathcal{L}[v] - \mathcal{N}[v] = 0 \tag{2.62}$$

By Lemma 2.15, there exists d_α, $\alpha = 1, 2, 3$ analytic in $\mathcal{D}_\pm$ ($\mathcal{D}$), periodic with period h, such that $d_\alpha(x)$ satisfies (2.61). □

Remark 2.7. *Lemma 2.16 also hold for $h = 1$ and $\frac{M_\alpha}{W} g \in \Upsilon_\gamma(\mathcal{D}(B, \pm))$, $\alpha = 1, 2, 3$. Lemma 2.17 also hold for $g \in \Upsilon_0(\mathcal{D}(B))$. Lemma 2.18 holds for $\mathcal{N} : \Upsilon_\gamma(\mathcal{D}(B, \pm)) \mapsto \Upsilon_\gamma(\mathcal{D}(B, \pm))$ and $x \in \mathcal{D}(B, \pm)$; or $\mathcal{N} : \Upsilon_0(\mathcal{D}(B)) \mapsto \Upsilon_0(\mathcal{D}(B))$, $x \in \mathcal{D}(B)$*

Remark 2.8. *Sometimes the exponential growth along imaginary direction $V_\alpha(x)$ introduces difficulty in finding suitable bounds for the operator $\mathcal{L}^{-1}$. In such cases we can first divide $\mathcal{D}$ into horizontal strips with given width h (except that the lowest strip may have width less than h): $\mathcal{D} = \bigcup_{n=0}^{N(h)} \mathcal{D}^{(n)}$ where $N(h) = \lceil \frac{2b}{h} \rceil$, $b_0 := b$, $b_n := b_{n-1} - h$,$n \in \mathbb{N}$, and*

$$\mathcal{D}^{(n)} = \mathcal{D} \cap \{x : b_{n+1} < \Im x < b_n\} \tag{2.63}$$

In each strip $\mathcal{D}^{(n)}$, define $\mathcal{L}^{-1}_{(n)}$ based on $\Delta_h^{-1} = \Delta^{-1}_{h,(n)}$, where $\Delta^{-1}_{h,(n)}$ as in (2.29) with region $\mathcal{D}$ replaced by $\mathcal{D}^{(n)}$. Clearly, for $g \in \mathcal{S}_0(\mathcal{D})$, $\mathcal{L}^{-1}_{(n)}[g](x)$ thus defined is analytic inside $\mathcal{D}^{(n)}$ and continuous in $\overline{\mathcal{D}}^{(n)}$.

Remark 2.9. *Lemma 2.18 also holds for $\mathcal{L}^{-1}_{(n)}$ in corresponding $\mathcal{D}^{(n)}$, where the periodic function δ_α are different for different n, denoted by $\delta_\alpha^{(n)}$. However, they are related.*

Let $\mathcal{L}^{-1}_{(T)}$ be the inversion of $\mathcal{L}$ based on $\Delta^{-1}_{h,-}$ where the integration path is $\partial\mathcal{D}$. Let v be an analytic solution to equation (2.60) such that $\mathcal{N}[v] \in \mathcal{S}_0(\mathcal{D})$. By Lemma 2.18 and Remark 2.9, there exists

$$v_0^{(T)} = \sum_{\alpha=-1,0,1} d_\alpha^{(T)}(x)V_\alpha(x), \qquad v_0^{(n)} = \sum_{\alpha=-1,0,1} d_\alpha^{(n)}(x)V_\alpha(x), \quad n = 1, \cdots, N(h), \tag{2.64}$$

where $d_\alpha^{(T)}(x)$, $d_\alpha^{(n)}(x)$ are analytic and periodic with period h, such that

$$v = v_0^{(T)} + \mathcal{L}^{-1}_{(T)}[\mathcal{N}[v]] \tag{2.65}$$

$$v = v_0^{(n)} + \mathcal{L}^{-1}_{(n)}[\mathcal{N}[v]] \tag{2.66}$$

2.3 Borel Transform and Laplace Transform

This section is an introduction to Borel and Laplace Transform and is adapted from [7].

Definition 2.19. *Define Borel transform of $x^{-\beta}$ for any $\beta > 0$ in terms of a dual variable p:*

$$\mathbf{B}[x^{-\beta}](p) = \frac{p^{\beta-1}}{\Gamma(\beta)} \tag{2.67}$$

where $\Gamma(\beta)$ is the Gamma function.

Applying Laplace transform $\mathbf{L}$ to the right hand side of (2.67) we get

$$\mathbf{L}\left[\frac{p^{\beta-1}}{\Gamma(\beta)}\right](x) = \frac{1}{\Gamma(\beta)}\int_0^{\infty} s^{\beta-1}e^{-s}ds = x^{-\beta} \tag{2.68}$$

More generally, Borel transform transforms one series

$$\tilde{f}(x) = x^{-\beta}\sum_{k=0}^{\infty} a_k x^{-\alpha x} \text{ with } \alpha, \beta > 0 \tag{2.69}$$

into another series:

$$F(p) = [\mathbf{B}\tilde{f}](p) = \sum_{k=0}^{\infty}\frac{a_k p^{\alpha k+\beta-1}}{\Gamma(k\alpha+\beta)} \tag{2.70}$$

Because of division by $\Gamma(k\alpha+\beta)$, it is clear that if $\tilde{f}$ is divergent for large k, with a_k growing at a factorial rate comparable or less than $\Gamma(k\alpha+\beta)$, then $F(p)$ will be convergent. Generally, an asymptotic series $\tilde{f}$ is rarely convergent. In the case when $\tilde{f}(x)$ is convergent, from term by term application of inverse Laplace transform, we get for $\Re p > 0$,

$$\mathbf{L}^{-1}\tilde{f} = \mathbf{B}\tilde{f} \tag{2.71}$$

However, the Borel and inverse Laplace transform are not the same because for $\Re p < 0$, $\mathbf{L}^{-1}\tilde{f} = 0$, while $\mathbf{B}\tilde{f}$ is defined by the analytic continuation of the series representation (2.70), which is nonzero except for the trivial case of $\tilde{f} = 0$.

Definition 2.20. *If the Borel transform $\mathbf{B}\tilde{f}$ satisfies the following two conditions*

- *1 The series for $F(p) = [\mathbf{B}\tilde{f}](p)$ is convergent in a neighborhood of $p = 0$.*
- *2 Analytical continuation of $F(p)$ along a ray $\arg p = \theta$ results in $e^{-c|p|}F(p) \in \mathrm{L}^1(0, \infty e^{i\theta})$ for some $c > 0$.*

then we say $\tilde{f}$ is Borel summable.

Suppose $\tilde{f}$ is Borel summable, then

$$f(x) = [\mathbf{L}_\theta F](x) = [\mathbf{L}_\theta \mathbf{B}\tilde{f}](x) \tag{2.72}$$

exists for $\arg x \in (-\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta)$ for large enough $|x|$ and by Watson's Lemma, we have

$$f(x) \sim \sum_{k=0}^{\infty} \int_0^{\infty} e^{i\theta} \frac{a_k p^{k\alpha+\beta-1}}{\Gamma(\beta + k\alpha)} = \sum_{k=0}^{\infty} a_k x^{-k\alpha-\beta} = \tilde{f}(x) \tag{2.73}$$

The association of $\tilde{f}$ with an actual function f given by (2.72) is not unique because different values of θ may result in different f. However, we can make the association unique by choosing $\theta = -\arg x := -\phi$.

Definition 2.21. *If $\tilde{f}$ is Borel transformable, $\mathbf{B}\tilde{f}(p)$ is analytic along $\arg p = -\arg x = -\phi$, and exponentially bounded, then Borel-sum $\sum^{\mathbf{B}} \tilde{f}$ is defined as*

$$\sum^{\mathbf{B}} \tilde{f}(x) = [\mathbf{L}_{-\phi}\mathbf{B}\tilde{f}](x) \tag{2.74}$$

In general, $\mathbf{B}\tilde{f}$ will have singularities. However, it is possible [1] to modify the definition and use a process called "balanced averaging" involving weighting over different possible paths of integration avoiding singularities that yields good algebraic properties of the association between $\tilde{f}$ and its Borel sum f.

The ray in the complex-x-domain characterized by $\arg x = \phi$ for which $\arg p = -\phi$ is a singular direction of $[\mathbf{B}\tilde{f}]$ in the p-plane will be referred to as Stokes line. If ϕ_s is a Stokes line, its associated anti-Stokes lines is defined as $\arg x = \phi_s \pm \frac{\pi}{2}$. Stokes and anti-Stokes lines play a crucial role in asymptotics , as will be seen later.

We define the Borel Transform of the product of series $\tilde{f}$ and $\tilde{g}$ through the relation

$$\mathbf{B}[\tilde{f}\tilde{g}] = [\mathbf{B}\tilde{f}] * [\mathbf{B}\tilde{g}] \tag{2.75}$$

where the convolution operation $*$ is defined as

$$[F * G](p) = \int_0^p F(p-s)G(s)ds \tag{2.76}$$

It is not difficult to prove that if $\tilde{f}$ and $\tilde{g}$ are Borel summable, so is the product $\tilde{f}\tilde{g}$.

2.3.1 Borel Analysis for a Difference Equation

One way to use Borel Transform to solve some differential or difference equation is to find a formal power series solution $\tilde{f}(x) = \sum_{n=0}^{\infty} a_k x^{-k}$, prove it is Borel summable by considering the rate of growth of a_k, then we may find $f(x)$ through (2.72). However, this way is very difficult. Instead, we can apply the Borel Transform on the equation itself and study the analytic properties of the transform, assuming *a priori* that it exists. We will illustrate this analysis through an simple nonlinear difference equation.

$$y(x+1) + y(x) = \frac{1}{x^2} + y^2 \tag{2.77}$$

We seek for a solution that behaves as $\frac{1}{2x^2}$ in half plane $-\frac{\pi}{2} - \theta < \arg x < \frac{\pi}{2} - \theta$ for some $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Suppose that y has a formal series representation $\tilde{y}(x) = \sum_{n=2}^{\infty} \frac{a_n}{x^n}$ that is Borel transformable. Let $\mathbf{B}[\tilde{y}](p) = Y(p)$. Borel transforming both sides of (2.77), we get $Y(p)$ satisfies

$$e^{-p}Y + Y = p + Y * Y \tag{2.78}$$

therefore

$$Y = \mathcal{N}[Y](p) := \frac{p}{e^{-p}+1} + \frac{Y * Y}{e^{-p}+1} \tag{2.79}$$

If we can show $Y(p)$ is locally integrable and exponentially bounded at infinity along some direction $\arg p = \theta$, then we know that $Y(p)$ is Laplace transformable and $y(x) = \mathbf{L}_\theta[Y](x)$ solves (2.77). If further we can show $Y(p)$ is analytic at the origin, then by Watson's lemma from asymptotic behavior of $Y(p)$ for small p, we can get the asymptotic behavior of $\mathbf{L}_\theta[Y](x)$ for large x, and thereby prove $y(x)$ has the desired asymptotic behavior.

To accomplish the above two tasks, we prove that $\mathcal{N}$ is a contraction in some ball of $\mathcal{A}$, the space of analytic functions in region $\mathcal{D}$ and continuous in $\overline{\mathcal{D}}$, with finite $\|\cdot\|_b$ norm , where

$$\mathcal{D} := \left\{ p \ : \ |p| < \pi - \delta \text{ or } \arg p \in (-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta) \text{ or } \arg p \in (\frac{\pi}{2}+\delta, \frac{3\pi}{2}-\delta) \right\} \tag{2.80}$$

$$\|Y\|_b = M_0 \sup_{p\in D} (1+|p|^2) e^{-b|p|} |Y(p)|, \tag{2.81}$$

and

$$M_0 = \sup_{s\geq 0} \left\{ \frac{2(1+s^2)\{\ln(1+s^2) + s\arctan s\}}{s(s^2+4)} \right\} = 3.76\ldots. \tag{2.82}$$

$b > 4$ will be chosen sufficiently large later. Then $\mathcal{A}$ equipped with norm $\|\cdot\|_b$ is a Banach space. It is to be noted that restricted on $p < 1$, $\|\cdot\|_b$ is equivalent to the uniform norm, and for any Y that $\|Y\|_b$ is finite, we have Y is exponentially bounded at ∞ for $\arg p \in (-2\pi + \delta, -\delta)$.

We will first prove a few lemmas to establish property of the norm.

Proposition 2.22. *For any $Y \in \mathcal{A}$ with the property $\sup_{p\in\mathcal{D}} |p^{-r}e^{-\rho|p|}Y(p)| \le K$ for some $\rho > 0$, some integer $r > 0$ and constant K, then we have $\|Y\|_b \le C\frac{2^r}{b^r}\Gamma(r+1)$, for b large enough, where C depends on Y, but not on b, ρ and r.*

Proof. For $|p| < \delta < 1$, we have

$$\begin{aligned}\sup_{|p|<\delta}(1+|p|^2)e^{-b|p|}|Y(p)| &\le 2K(b-\rho)^{-r}\sup_{|p|<\delta} e^{-(b-\rho)|p|}|p|^r(b-\rho)^r \\ &\le \frac{2K\Gamma(r+1)}{(b-\rho)^r} \le C\frac{2^r}{b^r}\Gamma(r+1) \end{aligned} \tag{2.83}$$

where the last step follows from the maximum value of function $e^{-x}x^r$ occurs at $x = r$. For $|p| \ge \delta$, $(1+|p|^2)e^{-b|p|}|Y(p)|$ is exponentially decaying for b large. □

Corollary 2.23. *There exists a constant K independent of b, such that*

$$\left\|\frac{p}{e^{-p}+1}\right\|_b \le \frac{K}{b} \tag{2.84}$$

Proposition 2.24. *For any $Y_1, Y_2 \in \mathcal{A}$, we have $Y_1 * Y_2 \in \mathcal{A}$, and*

$$\|Y_1 * Y_2\|_b \le \|Y_1\|_b\|Y_2\|_b \tag{2.85}$$

Proof.

$$[Y_1 * Y_2](p) = \int_0^p Y_1(s)Y_2(p-s)ds = p\int_0^1 Y_1(ps)Y_2(p(1-s))ds \tag{2.86}$$

In the latter integral the integrand is analytic in p and L^1 in s, therefore $Y_1 * Y_2$ is analytic in p. Since Y_1, Y_2 are continuous in $\bar{\mathcal{D}}$, we have so is $Y_1 * Y_2$.

$$\begin{aligned}|[Y_1 * Y_2](p)| &\le \left|\int_0^p Y_1(p-t)Y_2(t)dt\right| \\ &\le e^{b|p|}\|Y_1\|_b\|Y_2\|_b M_0^{-2}\int_0^p \frac{d\tilde{t}}{\left(1+(|p|-\tilde{t})^2\right)\left(1+\tilde{t}^2\right)} \\ &\le \frac{e^{b|p|}}{M_0(1+|p|^2)}\|Y_1\|_b\|Y_2\|_b \end{aligned} \tag{2.87}$$

□

Proposition 2.25. *For $Y_1, Y_2 \in \mathcal{A}$,*

$$\|Y_1^{*2} - Y_2^{*2}\|_b \leq (\|Y_1\|_b + \|Y\|_b)\|Y_1 - Y_2\|_b \tag{2.88}$$

Proof.

$$Y_1^{*2} - Y_2^{*2} = Y_1 * (Y_1 - Y_2) + Y_2 * (Y_1 - Y_2) \tag{2.89}$$

By Proposition 2.24, take $\|\cdot\|_b$ of both sides we get (2.88). □

The following lemma shows the existence and uniqueness of the solution to equation (2.79).

Lemma 2.26. *For b large enough, $\mathcal{N}$ is a contraction in the ball $\mathcal{B}_b$ where $\mathcal{B}_b$ is ball of size $2K/b$ in $\mathcal{A}$ centered at zero. Thus (2.79) has a unique solution in $\mathcal{B}_b$.*

Proof. For $p \in \mathcal{D}$, $|\frac{1}{1+e^{-p}}| \leq C$ for some constant $C > 0$. Using Corollary 2.23 and Proposition 2.24 for large enough b, we have

$$\|\mathcal{N}[Y]\|_b \leq \left\|\frac{p}{1+e^p}\right\|_b + \|C(|Y| * |Y|)\|_b \leq \frac{K}{b} + \frac{4CK^2}{b^2} \leq \frac{2K}{b} \tag{2.90}$$

Therefore $\mathcal{N}$ maps $\mathcal{B}_b$ back to itself for large enough b.

For $Y_1, Y_2 \in \mathcal{B}_b$, using Proposition 2.25, we get

$$\|\mathcal{N}[Y_1] - \mathcal{N}[Y_2]\|_b \leq C\|Y_1^{*2} - Y_2^{*2}\|_b \leq \frac{2CK}{b}\|Y_1 - Y_2\|_b \tag{2.91}$$

Therefore $\mathcal{N}$ is a contraction in $\mathcal{B}_b$, and the lemma follows. □

Remark 2.10. *Since $\delta > 0$ can be arbitrarily small, $Y(p)$ is analytic in $|p| < \pi$ or $\arg p \in (-\frac{\pi}{2}, \frac{\pi}{2})$ or $\arg p \in (\frac{\pi}{2}, \frac{3\pi}{2})$.*

Lemma 2.27. *For fixed* $\arg(x) \in (-\pi, \pi)$ *(* $\arg \in (-2\pi, 0)$*),*

$$y_0(x) := \mathbf{L}_\theta Y(x) = \int_0^{\infty e^{i\theta}} e^{-px} Y(p) dp \tag{2.92}$$

is a solution to (2.77), where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *(* $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$*) is chosen so that* $\theta + \arg x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *. Furthermore, for large* $|x|$*,*

$$y_0(x) \sim \frac{1}{2x^2} + \sum_{n=3}^{\infty} \frac{a_n}{x^n} \tag{2.93}$$

Proof. By Lemma 2.26, $Y(p)$ is analytic in the origin, hence so is $[Y * Y](p)$. Let

$$Y(p) = \sum_{n=0}^{\infty} A_n p^n \tag{2.94}$$

be the Taylor expansion of Y. Then

$$[Y * Y](p) = p \sum_{n=0}^{\infty} B_n p^n \tag{2.95}$$

where b_n are related to $A_0, \cdots, A_n$. Expanding $\frac{p}{1+e^{-p}}$ into Taylor series, plugging (2.94) and (2.95) into (2.79), comparing coefficient, we get $A_0 = 0$, $B_0 = A_0^2 = 0$, and $A_1 = \frac{1}{2}$ and $Y(p) = \sum_{n=1}^{\infty} A_n p^n$.

Y is exponentially bounded and the choice of θ implies the integral in (2.92) is convergent for sufficiently large $|x|$. By Watson's Lemma

$$y_0(x) \sim \sum_{n=1}^{\infty} \frac{A_n \Gamma(n+1)}{x^{n+1}} = \frac{1}{2x^2} + \sum_{n=3}^{\infty} \frac{a_n}{x^n} \tag{2.96}$$

The proof that $y_0(x)$ satisfies the difference equation (2.77) follows from that $\mathbf{L}_\theta[e^{-p}Y] = y(x+1)$ and $\mathbf{L}_\theta[Y * Y] = y_0^2$ as follows from using Fubini's theorem in the convolution, knowing a priori that the functions are integrable. □

2.3.2 Singularity Analysis of $Y(p)$ at $p = \pi i$ and Stokes Phenomena

Let $\arg x \in (-\pi, \pi)$, and let $y_0(x) = \mathbf{L}_\theta Y(x)$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is chosen so that $\arg(px) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Now if instead we choose $\hat{\theta}$ suitably in $(\frac{\pi}{2}, \frac{3\pi}{2})$, $\hat{y}_0(x) = \mathbf{L}_{\hat{\theta}} Y(x)$ is also a solution to (2.77), and $\hat{y}_0(x) \sim \tilde{y}(x)$ for $\arg x \in (-2\pi, 0)$. Though $y_0(x)$ and $\hat{y}_0(x)$ are both Laplace transforms of the same function $Y(p)$, they are not analytic continuation of each other due to the singularities of $Y(p)$ at $p = \pi i$, $p = 3\pi i$, etc. For $\arg x \in (-\pi, 0)$,

$$y_0(x) - \hat{y}_0(x) = \int_C Y(p) e^{-px} dp \tag{2.97}$$

where C is the deformed contour in Figure 2.1.

From the Watson's Lemma, the leading order asymptotic contribution from $\int_C$ for large $|x|$ in this sector is of the type $S x^\gamma e^{-\pi i x}$ for some constant γ and S (called Stokes constant). γ depends on the nature of leading order singularity of $Y(p)$ at $p = \pi i$. If $Y(p)$ has a simple pole at $p = \pi i$, then $\gamma = 0$.

Relation (2.97) provides the analytic continuation of y_0 for $\arg x \in (-2\pi, -\frac{\pi}{2})$. Therefore as $\arg x$ decreases to $-\pi$, the anti-Stokes line is approached, and $y_0(x) \sim \tilde{y}(x)$ is no longer true because $e^{-\pi i x}$ term is no longer small compared to $\tilde{y}(x)$. The phenomenon that a single analytic function has different asymptotics in different sectors of the complex plane is called Stokes phenomenon.

Next, we will find the ramified structure of $Y(p)$ at the singularity $p = \pi i$. We recall the Borel plane equation for $Y(p)$ is

$$(1 + e^{-p}) Y = p + Y * Y \tag{2.98}$$

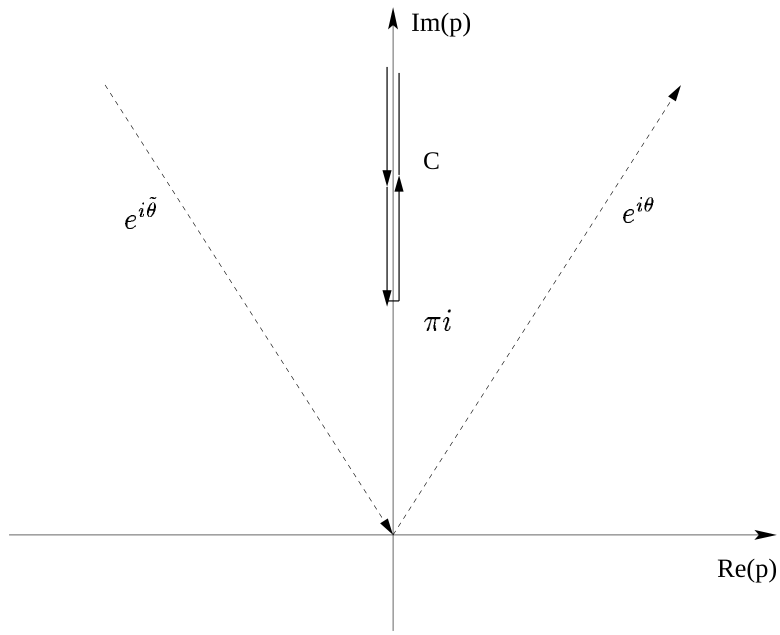

Figure 2.1: Integration path C

The dashed line is the original integration path. The thick line is the deformed one.

It is convenient to define

$$H(p) := \begin{cases} Y(p) & \text{if } |p| < \pi - \nu \\ 0 & \text{otherwise.} \end{cases} \tag{2.99}$$

$h(p) := Y(p) - H(p)$. Consider

$$\mathcal{D}_\nu := \{p : |p - \pi i| < \nu,\ \arg(\pi i - p) \in (-\frac{\pi}{2}, \frac{3\pi}{2})\} \tag{2.100}$$

For $p \in \mathcal{D}_\nu$, equation (2.98) becomes:

$$(1 + e^{-p})h(p) = p + H * H + 2H * h + h * h \tag{2.101}$$

Proposition 2.28. *For $p \in i(\pi - \nu, \pi)$, and $\nu < \frac{\pi}{4}$, we have $h * h(p) = 0$ and it analytically extends to the zero analytic function in $\mathcal{D}_\nu$.*

Proof.

$$h * h(p) = \int_{\pi i - \nu i}^{p} h(s)h(p-s)ds = \int_{0}^{p-\pi i+\nu i} h(s)h(p-s)ds = 0 \tag{2.102}$$

The last step is due to $\nu - (\pi - \nu) < \nu < \pi - \nu$ hence $h(s) = 0$. □

Proposition 2.29. *For $p \in (\pi - \nu, \pi)i$ with $\nu < \pi/4$*

$$H * H(p) = \int_{\nu i - (\pi i - p)}^{2\nu i} Y(s)Y(p-s)ds + \int_{2\nu i}^{\pi i - \nu i} Y(s)Y(p-s)ds \tag{2.103}$$

and the above expression extends to an analytic function for any $p \in \mathcal{D}_\nu$.

Proof. For $p \in (\pi i - \nu i, \pi i)$,

$$\begin{aligned}
[H * H](p) &= \int_{0}^{\pi i - \nu i} H(s)H(p-s)ds = \int_{\nu i - (\pi i - p)}^{\pi i - \nu i} H(s)H(p-s)ds \\
&= \int_{\nu i - (\pi i - p)}^{\pi i - \nu i} Y(s)Y(p-s)ds \\
&= \int_{\nu i - (\pi i - p)}^{2\nu i} Y(s)Y(p-s)ds + \int_{2\nu i}^{\pi i - \nu i} Y(s)Y(p-s)ds
\end{aligned} \tag{2.104}$$

Since $Y(p-s)$ is analytic for $p \in \mathcal{D}_\nu$, and s in the range of integration, the right hand side of (2.103) extends to an analytic function for any $p \in \mathcal{D}_\nu$. □

Proposition 2.30. *For any $p \in (\pi - \nu, \pi)i$, and $\nu < \pi/4$,*

$$[H * h](p) = \int_{\pi i - \nu i}^{p} H(p-s)h(s)ds = \int_{\pi i - \nu i}^{p} Y(p-s)h(s)ds \tag{2.105}$$

The right side of (2.105) extends to an analytic function for any $p \in \mathcal{D}_\nu$.

Proof. First consider $p \in (\pi i - \nu i, \pi i)$, we have

$$\begin{aligned} [H * h](p) &= \int_0^p H(s)h(p-s)ds = \int_0^{\pi i - \nu i} H(s)h(p-s)ds \\ &= \int_{p-\pi i+\nu i}^p H(p-s)h(s)ds = \int_{\pi i-\nu i}^p H(p-s)h(s)ds \\ &= \int_{\pi i - \nu i}^p Y(p-s)h(s)ds \end{aligned} \tag{2.106}$$

Since $Y(p-s)$ is analytic for $p \in \mathcal{D}_\nu$, and s in the range of integration and $h(s)$ is known to be integrable on any ray that avoids $s = \pi i$, it follows that the above provides analytic continuation for $[H * h](p)$ for any $p \in \mathcal{D}_\nu$. □

Lemma 2.31. *For $p \in \mathcal{D}_\nu$, $Y(p)$ has the ramified representation*

$$Y(p) = -\frac{A_1(1-p)}{\pi i - p} - \ln(\pi i - p)A_1'(\pi i - p) - A_2'(\pi i - p) \tag{2.107}$$

where $A_1(z)$ and $A_2(z)$ are analytic for $|z| < \nu$.

Proof. For $p \in \mathcal{D}_\nu$, it is convenient to define $Q(p) := \int_{\pi i - \nu i}^p h(s)ds$. Using integration by parts and $Y(0) = 0$, we get

$$[H * h](p) = \int_{\pi i - \nu i}^p H'(p-s)Q(s)ds \tag{2.108}$$

Hence for $p \in \mathcal{D}_\nu$, (2.98) can be written as

$$\begin{aligned} (1+e^{-p})Q'(p) &= p + [H*H](p) + 2\int_{\pi i - \nu i}^{\pi i} H'(p-s)Q(s)ds \\ &+ 2\int_{\pi i}^p H'(p-s)Q(s)ds \end{aligned} \tag{2.109}$$

We define $z := \pi i - p$ and $Q(p) = Z(\pi i - p)$. Then replacing $s = z(1-t)$ in the above integration, we obtain

$$-zZ'(z) = A(z) + zA_3(z)Z'(z) + 2z\int_0^1 H'(z(t-1))Z(zt)dt \tag{2.110}$$

where $A(z) = (\pi i - z) + [H * H](\pi i - z) + 2\int_{\pi i - \nu i}^{\pi i} H'(p-s)Q(s)ds$ and $A_3(z) := \frac{e^z - 1 - z}{z}$. Hence $z^{-1}A_3(z)$ is analytic at $z = 0$. Dividing both sides of (2.110) by $-z$ and integrating from $z = \nu i$ to z, noticing $Z(\nu i) = 0$, we get

$$
\begin{aligned}
Z(z) = \mathcal{J}[Z](z) &:= -A(0)(\ln z - \ln(\nu i)) - \int_0^z \frac{A(z') - A(0)}{z'} dz' \\
&- \int_{\nu i}^0 \frac{A(z') - A(0)}{z'} dz' - A_3(z)Z(z) + \left(\int_{\nu i}^0 + \int_0^z \right) A_3'(z')Z(z')dz' \\
&- 2\left(\int_{\nu i}^0 + \int_0^z \right) \left[\int_0^1 H'(z'(t-1))Z(z't)dt \right] dz' \qquad (2.111)
\end{aligned}
$$

We now claim that $Z(z)$ is a ramified analytic function for $|z| < \nu$, with the unique decomposition,

$$
Z(z) = A_1(z) \ln z + A_2(z) \qquad (2.112)
$$

where $A_1(z), A_2(z)$ are analytic in $|z| < \nu$.

To show this first note that

$$
\begin{aligned}
&\int_0^z \left[\int_0^1 Y'(z'(t-1))[A_1(z't)\ln(z't) + A_2(z't)]dt \right] \\
&= z\int_0^1 d\tau \left[\int_0^1 Y'(z\tau(t-1))[A_1(z\tau t)[\ln(z\tau t) + A_2(z\tau t)]dt \right] \\
&= z\tilde{A}(z)\ln z + z\tilde{A}_2(z) \qquad (2.113)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^z A_3'(z')Z(z')dz' &= z\int_0^1 A_3'(z\tau)[A_1(z\tau)\ln(z\tau) + A_2(z\tau)]d\tau \\
&= z\hat{A}_1(z)\ln(z) + z\hat{A}_2(z) \qquad (2.114)
\end{aligned}
$$

for some analytic functions $\tilde{A}_1$,$\tilde{A}_2$, $\hat{A}_1$, and $\hat{A}_2$ related to A_1 and A_2. Thus, the linear operator $\mathcal{J}$ preserves the ramified analytic structure of $Z(z)$. If we introduce the norm

$$
\|Z\|_R := \|A_1\|_\infty + \|A_2\|_\infty \qquad (2.115)
$$

Then since $|z| < \nu$, we find

$$\|\mathcal{J}[Z]\|_R \le K\varepsilon \|Z\|_R \tag{2.116}$$

Therefore, $Z(z)$ has a unique solution in this ramified analytic space. Since the solution to (2.98) is unique, we get

$$\int_{\nu i}^{p} Y(p')dp' = A_1(\pi i - p)\ln(\pi i - p) + A_2(\pi i - p) \tag{2.117}$$

for p near πi. Thus, the ramified analytic structure of $Y(p)$ at $p = \pi i$ is given by (2.107). □

Remark 2.11. *Lemma 2.31 shows that the leading order singularity of $Y(p)$ is a simple pole with residue $S = A_1(0)$, however the presence of logarithmic term implies $p = \pi i$ is also a branch point. Therefore in the choice of contour $\int_C$ in Figure 2.1, we have a branch cut along the imaginary axis from $p = \pi i$ to $p = \infty i$.*

Lemma 2.32.

$$y_0(x) - \hat{y}_0(x) \sim Se^{-\pi i x} \text{ as } x \to \infty, -\pi < \arg x < 0 \tag{2.118}$$

Proof. From (2.97), (2.107) and by Watson's Lemma

$$\begin{aligned} y_0(x) - \hat{y}_0(x) &\sim \int_C \left(\frac{A_1(\pi i - p)}{\pi i - p} + \ln(\pi i - p) A_1'(\pi i - p) \right) e^{-px} dp \\ &\sim A_1(0) e^{-\pi i x} \end{aligned} \tag{2.119}$$

□

CHAPTER 3

OUTER EXPANSION

3.1 Formal Separatrix

Consider a discrete map

$$w_{j+3} = w_j + \varepsilon g(w_{j+2}) \tag{3.1}$$

where $g(w) = w - w^{k+1}$ and $k \geq 1$. In terms of $\mathbf{W}_j := (w_j, w_{j+1}, w_{j+2})$, equation (3.1) reads as

$$\mathbf{W}_{j+1} = G(\mathbf{W_j}) \tag{3.2}$$

where $G(w_j, w_{j+1}, w_{j+2}) = (w_{j+1}, w_{j+2}, w_j + \varepsilon g(w_{j+2}))$. In this form, the map is clearly volume preserving and is one to one. The inverse of map G is

$$G^{-1}(w_j, w_{j+1}, w_{j+2}) = (w_{j+2} - \varepsilon g(w_{j+1}), w_j, w_{j+1}) \tag{3.3}$$

This equation has two fixed points $\mathbf{0} \equiv (0,0,0)$, and $\mathbf{1} \equiv (1,1,1)$. Near $\mathbf{0}$, linearizing equation(3.2) for small $\mathbf{W}$ gives

$$V_{j+1} = \mathbf{D}G(\mathbf{0})V_j = (v_{j+1}, v_{j+2}, v_j + \varepsilon v_{j+2}). \tag{3.4}$$

Similarly, near $\mathbf{1}$, decomposing $W_j = \mathbf{1} + \hat{V}_j$ and linearizing the inverse of map G for small $\hat{V}_j$, we get

$$\hat{V}_{j+1} = \mathbf{D}G^{-1}(\mathbf{1})\hat{V}_j = (\hat{v}_{j+2} - \varepsilon g'(1)\hat{v}_{j+1}, \hat{v}_j, \hat{v}_{j+1}). \tag{3.5}$$

The characteristic equation of $\mathbf{D}G(\mathbf{0})$ is

$$\Lambda_\alpha^3 = 1 + \varepsilon\Lambda_\alpha^2, \tag{3.6}$$

and for $\mathbf{D}G^{-1}(\mathbf{1})$ it is

$$\Gamma_\alpha^3 = 1 + k\varepsilon\Gamma_\alpha, \tag{3.7}$$

where and $\Lambda_\alpha, \Gamma_\alpha$ are the three roots of the equation (3.6) and (3.7) respectively indexed by $\alpha = -1, 0, 1$. For $0 < \varepsilon \ll 1$, the three distinct roots Λ_α and Γ_α are asymptotically given by

$$\Lambda_\alpha = \Omega^\alpha(1 + \frac{1}{3}\varepsilon\Omega^{2\alpha}) + \mathcal{O}(\varepsilon^2), \tag{3.8}$$

and

$$\Gamma_\alpha = \Omega^{-\alpha}(1 + \frac{1}{3}k\varepsilon\Omega^{-\alpha}) + \mathcal{O}(\varepsilon^2) \tag{3.9}$$

where $\Omega = e^{2\pi i/3}$ is a cubic root of unity.

It is convenient to define $\Lambda := \Lambda_0$ and $\Gamma := \Gamma_0$. Clearly for $\varepsilon > 0$, $\Lambda > 1$ and $|\Lambda_{\pm 1}| < 1$. By the unstable manifold theorem [15], there is an analytic one dimensional unstable manifold of map G corresponding to eigenvalue Λ, denoted by $u^-(t)$, and the equation of the unstable manifold at 0 in scalar form is

$$u^-(\Lambda^3 t) = u^-(t) + \varepsilon g(u^-(\Lambda^2 t)) \tag{3.10}$$

with initial condition

$$u^-(0) = 0, \qquad \lim_{t\to 0}\frac{u^-(t)}{t} = 1. \tag{3.11}$$

The solution $u^-(t)$ to (3.10) satisfying $u^-(0) = 0$ does not uniquely determine the parameterization, since the equations are invariant under the change of independent variable $t \to a(t)t$, where $a(t)$ is an arbitrary nonzero smooth function that satisfies $a(\Lambda t) = a(t)$ for any t. The second relation in (3.11) is used to determine the solution and therefore the parameterization uniquely. Following arguments given by Costin[1] in a more general context, it is easily shown using contraction argument, $u^-(t)$ is analytic in a neighborhood at $t = 0$.

Similar analysis shows $\Gamma > 1$ and $|\Gamma_{\pm 1}| < 1$. There exists a smooth one dimensional unstable manifold of map G^{-1} at $v = 1$ corresponding to eigenvalue Γ, denoted by $v^+(t)$, and the equation of the unstable manifold in scalar form is

$$v^-(\Gamma^3 r) = v^-(r) - \varepsilon g(v^-(\Gamma r)) \tag{3.12}$$

with initial condition

$$v^-(0) = 1, \qquad \lim_{r \to 0} \frac{v^-(r) - v^-(0)}{r} = -1. \tag{3.13}$$

The latter condition in (3.13) fixes the parameterization of the unstable manifold. To get a parameterization $u^-(t)$ of stable manifold of map G at $\mathbf{1}$, we reparameterize by $r = r(t) = \frac{1}{k} t^{-\frac{\ln \Gamma}{h}}$, which satisfies $r(\Lambda t) = \frac{1}{\Gamma} r(t)$. Define $u^+(t) := v^-(r(t))$. Then $u^+(t)$ satisfies the same equation (3.10) with superscript " $-$ " replaced by "$+$", but with different initial condition

$$\lim_{t \to \infty} u^+(t) = 1, \qquad \lim_{t \to \infty} \frac{u^+(t) - 1}{r(t)} = -1 \tag{3.14}$$

It is convenient to rewrite equation in terms of $x = \ln t$. Let $\breve{z}^{\pm}(x) := u^{\pm}(t(x))$ respectively. Using (3.6) and (3.8) we obtain for some set $\{\alpha_m\}_{m=2}^{\infty}$

$$h := ln\Lambda = \frac{\varepsilon}{3} + \sum_{m=2}^{\infty} \alpha_m \varepsilon^m. \tag{3.15}$$

We will use $z(x)$ as a generic symbol to represent either $\breve{z}_-(x)$ or $\breve{z}_+(x)$. Then $z(x)$ satisfies

$$z(x+3h) = z(x) + \varepsilon g(z(x+2h)). \tag{3.16}$$

To find the limiting flow as $\varepsilon \to 0$ we write equation (3.16) as

$$\frac{z(x+3h) - z(x)}{\varepsilon} = g(z(x+2h)) \tag{3.17}$$

From (3.11), we obtain from initial conditions:

$$\lim_{x\to-\infty} \breve{z}_-(x) = 0 \qquad \lim_{x\to-\infty} \frac{\breve{z}_-(x)}{e^x} = 1 \tag{3.18}$$

We obtain from (3.13) the conditions for $\breve{z}_+(x)$,

$$\lim_{x\to\infty} \breve{z}_+(x) = 1 \qquad \lim_{x\to\infty} \frac{\breve{z}_+(x) - 1}{\frac{1}{k}\exp\{-x\frac{\ln\Gamma}{h}\}} = -1. \tag{3.19}$$

Formally by letting $\varepsilon \to 0$ in equation (3.17), we find

$$z_0'(x) = g(z_0(x)) \tag{3.20}$$

Initial condition (3.18) gives rise to the unique solution to (3.20)

$$z_0(x) = \frac{e^x}{(1+e^{kx})^{1/k}}, \tag{3.21}$$

Noticing that as $\varepsilon \to 0$, the second relation in (3.19) becomes

$$\lim_{x\to\infty} \frac{\breve{z}_+(x) - 1}{\frac{1}{k}e^{-kx}} = -1 \tag{3.22}$$

we get that $z_0(x)$ also satisfies (3.22).So the two manifolds are the same in the formal limit $\varepsilon \to 0^+$. In fact the splitting of separatrices is exponentially small in ε in the real domain, as will be proved later.

To the leading order as $\varepsilon \to 0^+$, the closest complex singularity (from the real domain) of the leading order outer asymptotic expansion $z_0(x)$ is at π/ki, but in fact the singularity of higher order terms depends on ε, which will become clear when we match the outer solution with inner solution. It is convenient to consider a shift of the independent variable of $\breve{z}^{\pm}$ such that the singularities are fixed with respect to ε. Define

$$z_{\pm}(x) = \breve{z}_{\pm}(x + \varrho\varepsilon \ln h) \tag{3.23}$$

where $\varrho := \frac{k+1}{6k}$. Then $z_-(x)$ satisfies initial condition

$$\lim_{x\to-\infty} z_-(x) = 0 \qquad \lim_{x\to-\infty} \frac{z_-(x)}{e^x} = e^{-\varrho\varepsilon \ln h} = 1 - \varrho\varepsilon \ln h + \mathcal{O}(\varepsilon \ln h)^2 \tag{3.24}$$

Plugging

$$z_-(x) = p(x) + v(x), \text{ where } p(x) := z_0(x) + \varepsilon \ln h \; z_1(x) + \varepsilon z_2(x) \tag{3.25}$$

into (3.16) and expanding in terms of ε and $\varepsilon \ln h$, then setting coefficient of ε, $\varepsilon \ln h$, ε^2 to match both in (3.16) and in (3.24), we obtain $z_0(x)$ is as in (3.21), and

$$z_1(x) = \varrho \frac{e^x}{(1+e^{kx})^{1+1/k}} \qquad z_2(x) = -\varrho \frac{e^x}{(1+e^{kx})^{1+1/k}} \ln(1+e^{kx}) \tag{3.26}$$

From (3.23) and (3.19) we get

$$\lim_{x\to\infty} z_+(x) = 1, \qquad \lim_{x\to\infty} \frac{z_+(x) - 1}{\frac{1}{k}\exp\{-x\frac{\ln\Gamma}{h} - \varrho\varepsilon \ln h\}} = -1. \tag{3.27}$$

Similarly, plugging $z_+(x) = z_0^+(x) + (\varepsilon \ln h)\, z_1^+(x) + \varepsilon z_2^+(x) + v^+(x)$ into (3.16), we get $z_i^+ = z_i(x)$ for $i = 0, 1, 2$.

We expect $z_-(x) \sim p(x)$, as $\varepsilon \to 0$ in some complex region adjoining real axis that is bounded away from the nearest complex singularities of $p(x)$, $x = \pm\frac{\pi i}{k}$. We expect to show $v(x) = O(\varepsilon^2)$ in a certain region mentioned earlier. To get an equation for $v(x)$, we substitute (3.25) into (3.16) to obtain

$v(x)$ satisfies

$$\mathcal{L}_0[v] = \mathcal{L}_1[v] + \mathcal{N}_1[v] - f \tag{3.28}$$

where

$$\mathcal{L}_0[v](x) = v(x+3h) - v(x) - \varepsilon g'[z_0(x+2h)]v(x+2h) \tag{3.29}$$

$$\mathcal{L}_1[v](x) := \varepsilon(p^k(x+2h) - z_0^k(x+2h))v(x+2h) \tag{3.30}$$

$$\mathcal{N}_1[v](x) = \varepsilon \sum_{l=2}^{k+1} \binom{k+1}{l} v^l(x+2h)(p(x))^{k+1-l} \tag{3.31}$$

$$f(x) = p(x+3h) - p(x) - \varepsilon g(p(x+2h)) \tag{3.32}$$

where $p(x)$ is defined in (3.25).

As $x \to -\infty$, $z_0(x) \to 0$ and $g'(z_0(x+2h)) \to 1$, $\mathcal{L}_0[v](x)$ formally simplifies to

$$\mathcal{L}_{-\infty}[v](x) = v(x+3h) - v(x) - \varepsilon v(x+2h). \tag{3.33}$$

Equation $\mathcal{L}_{-\infty}[v] = 0$ has three independent solutions e^x, $\exp\{x\frac{\ln\Lambda_{-1}}{\ln\Lambda}\}$, $\exp\{x\frac{\ln\Lambda_1}{\ln\Lambda}\}$. We define three independent solution to $\mathcal{L}_0[v](x)$ by $\tau_{0,\alpha}(x)$, for $\alpha = -1, 0, 1$ so that as $x \to -\infty$, $\tau_\alpha(x) \sim e^{x\frac{\ln\Lambda_\alpha}{h}}$. For arbitrary x, we will prove in Lemma 3.11 that $\tau_{0,\alpha}(x) \sim \tau_{e,\alpha}(x)$ where

$$\tau_{e,\alpha}(x) = \tilde{V}_\alpha(x)\exp\{x\frac{\ln\Lambda_\alpha}{\ln\Lambda}\}, \quad \tilde{V}_\alpha(x) = \big(g(z_0(x))e^{-x}\big)^{\Omega^{2\alpha}} = (1+e^{kx})^{\beta_\alpha}. \tag{3.34}$$

where $\beta_\alpha := -(\frac{k+1}{k})\Omega^{2\alpha}$, $\alpha = -1, 0, 1$. The choice of $\tilde{V}_\alpha$ is made so that $\tau_{e,\alpha}$ comes close to satisfying $\mathcal{L}_0[\tau] = 0$. To see this, we follow argument in Rom-Kedar *et al*

[3] and let $\tau_{0,\alpha}(x) = W_\alpha(x)\exp\{x\frac{\ln\Lambda_\alpha}{\ln\Lambda}\}$ be the exact solution of the homogeneous equation $\mathcal{L}_0[\tau](x) = 0$, then W_α satisfies

$$\Lambda_\alpha^3 W_\alpha(x+3h) = W_\alpha(x) + \varepsilon g'[z_0](x+2h)\Lambda_\alpha^2 W_\alpha(x+2h) \tag{3.35}$$

On taking the formal limit $\varepsilon \to 0^+$ and using (3.8) , it is expected that $W_\alpha \sim \tilde{V}_\alpha$ where $\tilde{V}_\alpha$ satisfies

$$\Omega^{3\alpha}\tilde{V}_\alpha(x) + \frac{d}{dx}\tilde{V}_\alpha(x) = g'[z_0](x)\Omega^{2\alpha}\tilde{V}_\alpha(x). \tag{3.36}$$

With normalization condition $\lim_{x\to-\infty}\tilde{V}_\alpha(x) = 1$, we obtain (3.34). We will prove rigorously later on that indeed, as expected from formal argument here, $\tau_{0,\alpha} \sim \tau_{e,\alpha}$.

Define $\mathcal{D}^-(\delta_0)$ to be the region (See Figure 3.1)

$$\begin{aligned}\mathcal{D}^-(\delta_0) = \{x \in \mathbf{C} \quad : \quad & |\Im x| < \frac{\pi}{k},\ \Re x < A, \\ & -\pi < \arg\{x - \frac{\pi}{k}i + \frac{\delta_0}{\sin\delta_a}\} < -\delta_a, \\ & \delta_a < \arg\{x + \frac{\pi}{k}i + \frac{\delta_0}{\sin\delta_a}\} < \pi\}\end{aligned} \tag{3.37}$$

for some fixed positive A and δ_a.

Let $\mathcal{D}^+(\delta_0)$ be the reflection of $\mathcal{D}^-(\delta_0)$ about the imaginary axis. Later, we will show the δ_0 dependence in $\mathcal{D}^\pm$ only when needed. We define

$$\mathcal{D}_0 := \{x \in \mathcal{D}^- \cap \mathcal{D}^+ : |\Im x| < |\frac{\pi}{k} - \tilde{\delta}_0|\} \tag{3.38}$$

where $\tilde{\delta}_0 := \delta_0(\sec\delta_a + n_0\tan\delta_a)$, and $n_0 > 4$ is a fixed constant.

Remark 3.1. *We chose $\mathcal{D}^-$ so that $z_-(x)$ is expected to be close to $z_0(x)$ in $\mathcal{D}^-$. So $\Re(x) \to \infty$ is avoided as are singularities of z_0.*

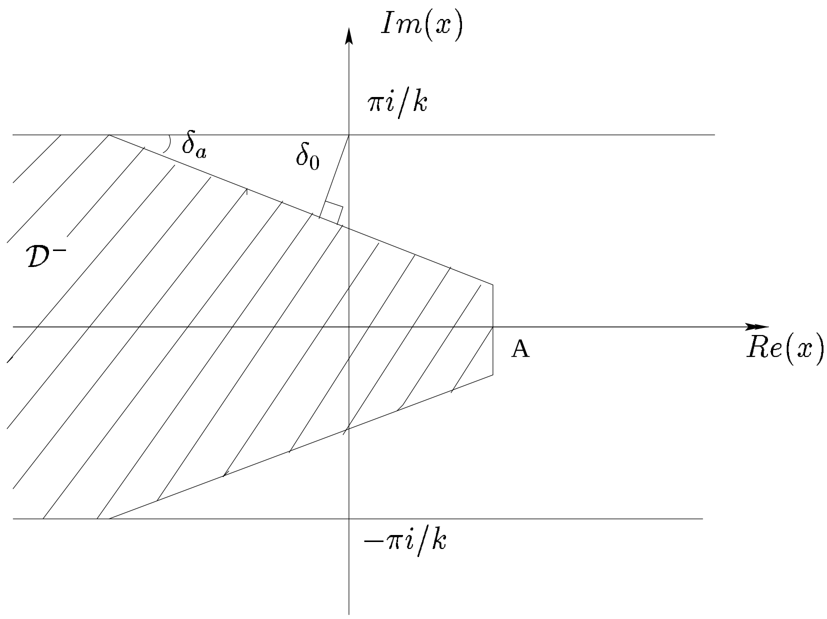

Figure 3.1: Region $\mathcal{D}^-$

Next, we are going to establish that the formal approximation $p(x)$ is close to $z_-(x)$ in $\mathcal{D}^-$ and is close to $z_+(x)$ in $\mathcal{D}^+$. The main result of this section is the following Lemma.

Lemma 3.1. *There exists unique $z_-(x)$ satisfying (3.16) and (3.24).*

$$z_-(x) = z_0(x) + (\varepsilon \ln h) z_1(x) + \varepsilon z_2(x) + v_-(x) \tag{3.39}$$

and we have $|e^{-x} v_-(x)| \leq K\varepsilon^2 \delta_0^{-2-1/k} \ln^2(\delta_0/h)$ for $x \in \mathcal{D}^-$ and $\delta_0 \gg \varepsilon$ where K is a constant independent of δ_0, ε.

Similarly, there exists unique $z_+(x)$ satisfying (3.16) and (3.27),

$$z_+(x) = z_0(x) + (\varepsilon \ln h) z_1(x) + \varepsilon z_2(x) + v_+(x) \tag{3.40}$$

and $|e^{-kx} v_+(x)| \leq K \varepsilon^2 \delta_0^{-2-1/k} \ln^2(\delta_0/h)$ *for* $x \in \mathcal{D}^+$ *and* $\delta_0 \gg \varepsilon$.

3.1.1 Preliminary Lemmas

In this section, we prove some of the preliminary lemmas needed later for proof of Lemma 3.1.

$$\mathcal{S}_\mu(\mathcal{D}^-) := \{\nu(x) : \nu(x) \text{ analytic in } \mathcal{D}^-, \text{continuous in} \overline{\mathcal{D}^-}\} \tag{3.41}$$

Equipped with norm

$$\|\nu\|_\mu = \sup_{x \in \mathcal{D}^-} |e^{-\mu x} \nu(x)| \tag{3.42}$$

where $\mu \geq 2 + \Re(\ln \Lambda_1)/h > 1$. $\mathcal{S}_\mu$ forms a Banach space.

If $\mathcal{L}$ is a linear operator of third order and $\mathcal{L}[w] = 0$ has a fundamental set of solution

$$\tau_\alpha = \tau_{e,\alpha}(1 + \mathcal{O}(\varepsilon \kappa(\delta_0))), \alpha = -1, 0, 1 \tag{3.43}$$

for $x \in \mathcal{D}^-$, where $\kappa(\delta_0)$ is some function of δ_0, $\tau_{e,\alpha}$ is defined in (3.34). Define the inverse of $\mathcal{L}$ as:

$$\mathcal{L}_-^{-1} : \mathcal{S}_\mu(\mathcal{D}^-) \to \mathcal{S}_\mu(\mathcal{D}^-) \tag{3.44}$$

$$\mathcal{L}_-^{-1}[g](x) := \sum_{\alpha=-1,0,1} \mathcal{L}_{-,\alpha}^{-1}[g](x) \qquad \mathcal{L}_{-,\alpha}^{-1}[g](x) := \tau_\alpha \Delta_{h,-}^{-1} \left[\frac{M_\alpha}{W} g \right](x), \tag{3.45}$$

where $W(x)$ is the difference Wronskian of $\{\tau_\alpha\}_{\alpha=-1,0,1}$ and $M_\alpha(x)$ is the cofactor of the last element in the α column of $D(x)$.

In the class of continuous functions g, it is convenient to define operator

$$\tilde{\mathcal{L}}_{-}^{-1}[g] := \sum_{\alpha=-1,0,1} |\tau_\alpha| \Delta_{h,-}^{-1} \left| \frac{M_\alpha}{W} g \right| (x) \tag{3.46}$$

The following Proposition is helpful in proving $\mathcal{L}_{-}^{-1}$ is well defined and bounded. Let $X_s^{\pm} = \pm \frac{\pi}{k} i$. So $X_s^{\pm}$ are the two singularities of $p(x)$ that is closest to the real axis. For $d_0 > \delta_0$, define

$$\mathcal{D}_1^{-}(d_0, \delta_0, \pm) := \mathcal{D}^{\pm} \cap \{x \in \mathbb{C} : |x - X_s^{\pm}| \leq d_0\} \tag{3.47}$$

$$\mathcal{D}_2^{\pm}(d_0, \delta_0) = \mathcal{D}^{\pm} \setminus (\mathcal{D}_1^{-}(d_0, \delta_0, +) \cup \mathcal{D}_1^{-}(d_0, \delta_0, -)) \tag{3.48}$$

Proposition 3.2. *Suppose $|g(x)| \leq K|x - X_s|^{-\gamma}$ for $x \in \mathcal{D}_1^{-}(d_0, \delta_0, \pm)$, if $\gamma > 1$ then we have*

$$|\sum_{n=1}^{N(x)} g(x - nh)| \leq \frac{KK_1(\delta_a)}{h} |x - X_s|^{-\gamma+1} \tag{3.49}$$

where $N(x)$ is the largest integer satisfying $|x - X_s - N(x)h| \leq d_0$, $K_1(\delta_a)$ is a constant depending on γ and δ_a. (δ_a is defined in (3.37)).

If $\gamma = 1$, then we have

$$|\sum_{n=1}^{N(x)} g(x - nh)| \leq \frac{KK_1(\delta_a)}{h} \ln \left(\frac{d_0}{|x - X_s|} \right) \tag{3.50}$$

Proof. We will only show for the "−" case since the other case is similar. For $\gamma > 1$, $x \in \mathcal{D}_1^-(d_0, \delta_0, -)$

$$\begin{aligned} |\sum_{n=1}^{N(x)} g(x-nh)| &\leq \sum_{n=1}^{N(x)} K|x - X_s - nh|^{-\gamma} \\ &\leq \frac{C_1 K}{h} \int_0^{2d_0} |x - X_s - t|^{-\gamma} dt \\ &\leq C_1 \frac{K}{h} |x - X_s|^{-\gamma+1} \int_0^{\frac{2d_0}{|x-X_s|}} |e^{i\phi} - \tilde{t}|^{-\gamma} d\tilde{t} \end{aligned} \tag{3.51}$$

where $\phi := \arg(x - X_s)$. For $\gamma > 1$, the integral in (3.51) can be bounded by $\int_0^\infty |e^{i\phi} - \tilde{t}|^{-\gamma} d\tilde{t}$. Since $x \in \mathcal{D}^-(d_0, \delta_0, -)$, by (3.37), we have $-\pi < \phi < -\delta_a$, therefore $|e^{i\phi} - t| \geq \sin(\delta_a)$ for t in the integration path of (3.51). Hence,

$$\left| \sum_{n=1}^{N(x)} g(x - nh) \right| \leq K_1(\delta_a) \frac{K}{h} |x - X_s|^{-\gamma+1} \tag{3.52}$$

, where $K_1(\delta_a)$ is a constant depending on δ_a and γ.

For $\gamma = 1$, the integral in (3.51) into two: $(0, 1)$ and $(1, \frac{2d_0}{|x-X_s|^\gamma})$ we have the integral is bounded by $C \ln\left(\frac{2d_0}{|x-X_s|}\right)$. □

Lemma 3.3. *For small enough ε, the induced norm of operator $\mathcal{L}_-^{-1}$ has the following bound*

$$\|\mathcal{L}_{-,\alpha}^{-1}\| \leq C \delta_0^{-1-1/k} \varepsilon^{-1}, \quad \alpha = -1, 0, 1 \tag{3.53}$$

where C is a constant independent of ε.

If $g \in \mathcal{S}_\mu(\mathcal{D}^-)$ satisfies the following two conditions

$$|g(x)| \leq K\|g\|_\mu \left(\frac{\delta_0}{|x - X_s|} \right)^\gamma, \quad x \in \mathcal{D}_1(d_0, \delta_0, \pm) \tag{3.54}$$

$$|g(x)| \leq K \left(\frac{\delta_0}{d_0}\right)^{\gamma} \|g\|_\mu, \quad x \in \mathcal{D}_2^-(d_0, \delta_0) \tag{3.55}$$

where $\gamma \geq 0$, K *is a constant, then*

$$\|\mathcal{L}_{-,\alpha}^{-1}[g]\|_\mu \leq CK\|g\|_\mu \frac{\delta_0}{\varepsilon}, \qquad \textit{if } \gamma > 2 + 1/k \tag{3.56}$$

$$\|\mathcal{L}_{-,\alpha}^{-1}[g]\|_\mu \leq CK\|g\|_\mu \frac{\delta_0 \ln \delta_0}{\varepsilon}, \qquad \textit{if } \gamma = 2 + 1/k. \tag{3.57}$$

$$\|\mathcal{L}_{-,\alpha}^{-1}[g]\|_\mu \leq CK\|g\|_\mu \frac{\delta_0^{\gamma-1-1/k} \ln \delta_0}{\varepsilon}, \qquad \textit{if } 0 \leq \gamma < 2 + 1/k. \tag{3.58}$$

Proof. (3.43) implies that for $x \in \mathcal{D}_2^-(d_0, \delta_0)$(defined in (3.48)),

$$\begin{aligned} \frac{M_\alpha(x)}{W(x)} &= e^{-x\frac{\ln \Lambda_\alpha}{h}}(1+e^{kx})^{-(1+1/k)\Omega^{2\alpha}}(K_\alpha + \mathcal{O}(\varepsilon d_0^{-\frac{1+k}{k}}, \varepsilon\kappa(d_0))) \\ &= \tau_{e,\alpha}^{-1}(K_\alpha + \mathcal{O}(\varepsilon d_0^{-\frac{1+k}{k}}, \varepsilon\kappa(d_0))) \end{aligned} \tag{3.59}$$

where K_α is a constant. For $n \in \mathbb{N}$ and $\kappa(\delta_0)\varepsilon << 1$, we have using

$$\left|\tau_\alpha(x)\frac{M_\alpha(x-nh)}{W(x-nh)}\right| \leq C|\Lambda_\alpha|^{-n}\left|\frac{\tilde{V}_\alpha(x)}{\tilde{V}_\alpha(x-nh)}\right| \leq C|\Lambda_\alpha|^{-n} d_0^{-1-1/k}, \tag{3.60}$$

where $n \in \mathbb{N}$ and C only depends on α. By above inequality,

$$\begin{aligned} |e^{-\mu x}\mathcal{L}_{-,\alpha}^{-1}[g](x)| &= \left|e^{-\mu x}\tau_\alpha(x)\sum_{n=1}^{\infty}\left[\frac{M_\alpha(x-nh)}{W(x-nh)}e^{-\mu(x-nh)}g(x-nh)e^{-\mu nh}\right]\right| \\ &\leq C_1 K d_0^{-\frac{k+1}{k}}\|g\|_\mu \sum_{n=1}^{\infty}|\Lambda_\alpha|^{-n}|e^{-\mu nh}| \\ &\leq CKd_0^{-\frac{k+1}{k}}\varepsilon^{-1}\|g\|_\mu \end{aligned} \tag{3.61}$$

where C_1, C are constants. In the last step we used that for $\mu \geq 2 + \Re(\ln \Lambda_1)/h$,$\alpha = -1, 0, 1$,

$$|\Lambda_\alpha e^{-\mu h}| \leq |\Lambda_\alpha \Lambda_1^{-1}\Lambda^{-2}| \leq |\Lambda\Lambda_1|^{-1} = |\Lambda_{-1}| \leq 1 - K_1\varepsilon \tag{3.62}$$

where $K_1 > 0$ is a constant.

By replacing d_0 by δ_0 in above proof, we get (3.53). If $g(x)$ satisfies (3.55), then $\|g\|_\mu$ in (3.61) can be replaced by $C(\delta_0/d_0)^\gamma \|g\|_\mu$.

For $x \in \mathcal{D}_1^-(d_0, \delta_0, \pm)$, we have

$$\left|\frac{M_\alpha(x)}{W(x)}\right| \leq C_1 |e^{-x\frac{\ln \Lambda_\alpha}{h}}| \, |x - X_s|^{\Re(\beta_\alpha)} \tag{3.63}$$

$$|\tau_\alpha(x)| \leq C_2 |e^{x\frac{\ln \Lambda_\alpha}{h}}| \, |x - X_s|^{\Re(\beta_\alpha)} \tag{3.64}$$

If $g(x)$ satisfies $|g(x)| \leq K\delta_0^\gamma \|g\|_\mu |x - X_s|^{-\gamma}$, for $x \in \mathcal{D}_1^-(d_0, \delta_0, -)$, let $N(x)$ be the largest integer such that $\tilde{x} := x - N(x)h \in \mathcal{D}_1^-(d_0, \delta_0, -)$. Breaking up the summation in Δ_-^{-1}, using Proposition 3.2, for $\gamma > 2 + 1/k$, noticing (3.63), we find for $x \in \mathcal{D}_1^-(d_0, \delta_0, -)$,

$$\begin{aligned} |\mathcal{L}_{-,\alpha}^{-1}[g](x)| &\leq \left|\frac{\tau_\alpha(x)}{\tau_\alpha(\tilde{x})}\right| \mathcal{L}_{-,\alpha}^{-1}[g](\tilde{x}) + \left|\sum_{n=1}^{N(x)} \left[\tau_\alpha(x) \frac{M_\alpha(x-nh)}{W(x-nh)} g(x-nh)\right]\right| && (3.65) \\ &\leq C_1 \frac{d_0^{\frac{k+1}{2k}}}{|x - X_s|^{1+1/k}} \varepsilon^{-1} \left(\frac{\delta_0}{d_0}\right)^\gamma \|g\|_\mu + \frac{C_2 K}{h} \frac{\delta_0^\gamma \|g\|_\mu}{|x - X_s|^{\gamma-1}} && (3.66) \\ &\leq K\frac{C}{\varepsilon} \delta_0 \|g\|_\mu && (3.67) \end{aligned}$$

Therefore we have (3.53).

For $\gamma = 2 + 1/k$, using (3.50) of Proposition 3.2 in (3.65) we get (3.57).

If $\gamma < 2 + 1/k$, noticing (3.54) implies $|g(x)| \leq K\|g\|_\mu \frac{\delta_0^\gamma}{|x - X_s|^{2+1/k}}$, similar with

(3.65), we get

$$\begin{aligned} & |e^{-\mu x}\mathcal{L}_{-,\alpha}^{-1}[g](x)| \\ \leq\ & C_1 \frac{d_0^{\frac{k+1}{2k}}}{|x-X_s|^{1+1/k}}\varepsilon^{-1}\left(\frac{\delta_0}{d_0}\right)^r \|g\|_\mu + \frac{C_2 K}{h}\frac{\delta_0^\gamma \|g\|_\mu \ln(\frac{2d_0}{\delta_0})}{|x-X_s|^{1+1/k}} && (3.68) \\ \leq\ & \frac{CK}{\varepsilon}\delta_0^{\gamma-1-1/k}\ln\left(\frac{1}{\delta_0}\right)\|g\|_\mu && (3.69) \end{aligned}$$

□

Remark 3.2. *In the proof of Lemma 3.3, (3.66), (3.68) show that stronger results: If g satisfies (3.54) and (3.55), and $x \in \mathcal{D}_1(d_0, \delta_0, \pm)$, we have*

$$|e^{-\mu x}\mathcal{L}_{-,\alpha}^{-1}[g]| \leq C\frac{K}{\varepsilon}\|g\|_\mu \frac{\delta_0^\gamma \ln\delta_0 \ln^n(1/\delta_0)}{|x-X_s|^{\tilde{\gamma}}} \tag{3.70}$$

where if $\gamma \leq 2+1/k$, then $\tilde{\gamma} = 1+1/k$ and $n=1$; if $\gamma > 2+1/k$ then $\tilde{\gamma} = \gamma - 1$ and $n = 0$.

Remark 3.3. *Clearly Lemma 3.3 and Remark 3.2 hold for $\mathcal{L}_-^{-1}$ replaced by $\tilde{\mathcal{L}}_-^{-1}$ as well.*

Consider w analytic in $\mathcal{D}^-$ satisfying equation

$$\mathcal{L}[w] = \mathcal{Q}[w], \tag{3.71}$$

by Lemma 2.18, w satisfies

$$w = w_0 + \mathcal{L}_-^{-1}\left[\mathcal{Q}[w]\right] \tag{3.72}$$

with $w_0(x) := \sum_{\alpha=-1,0,1} c_\alpha(x)\tau_\alpha$, where $c_\alpha(x)$ are periodic with period h and analytic in $\mathcal{D}^-$.

If $\mathcal{Q}$ is small, we expect that w to be close to w_0. To be precise, suppose $w_0(x) \neq 0$ for $x \in \mathcal{D}^-$, let $\eta(x) = \frac{w(x)-w_0(x)}{w_0(x)}$, then $\eta(x)$ satisfies

$$\eta(x) = \mathcal{J}_-[\eta](x) := \frac{1}{w_0(x)} \mathcal{L}_-^{-1} \left[\mathcal{Q}[w_0(1+\eta)] \right](x) \tag{3.73}$$

Let $\mathcal{B}$ be a ball of size δ centered at zero in $\mathcal{S}_0(\mathcal{D}^-)$.

Lemma 3.4. *Suppose operator $\mathcal{Q} : \mathcal{S}_0(\mathcal{D}^-) \mapsto \mathcal{S}_0(\mathcal{D}^-)$ satisfies the following two conditions: for $\eta, \zeta \in \mathcal{B}$ and $x \in \mathcal{D}_1^-(d_0, \delta_0, \pm)$,*

$$\left| \frac{\mathcal{Q}[w_0(1+\eta)](x-y)}{w_0(x)} \right| \leq K(x,y)\varepsilon^{r_0} |e^{-\mu y}| (1 + \|\eta\|_\infty); \tag{3.74}$$

$$\begin{aligned} &\left| \frac{\mathcal{Q}[w_0(1+\eta)](x-y)}{w_0(x)} - \frac{\mathcal{Q}[w_0(1+\zeta)](x-y)}{w_0(x)} \right| \\ \leq\ & K(x,y)\varepsilon^{r_0} |e^{-\mu y}| \|\eta - \zeta\|_\infty \end{aligned} \tag{3.75}$$

where $K(x,y) = K_0 |x - X_s|^{-r_1} |x - y - X_s|^{-r_2}$ for $x \in \mathcal{D}_1^-(d_0, \delta_0, \pm)$ and $K(x,y) = K_0$ for $x \in \mathcal{D}_2(d_0, \delta_0)$; $r_0 > 1$, $r_1 \geq -1 - 1/k$, and $r_2 \geq 0$, $\mu \geq 2 + \frac{\Re(\ln \Lambda_1)}{h}$ are constants.

Then (3.73) has an unique solution $\eta(x)$ of (3.73) in $\mathcal{B}$ for small enough ε, and $\eta(x) = \mathcal{O}(K_0 \varepsilon^{r_0 - 1} \delta_0^{-r'} \ln^n(\delta_0/h))$, $x \in \mathcal{D}^-$ where $r' = r_1 + r_2 - 1$ and $n = 0$ if $r_2 > 2 + 1/k$; $r' = r_1 + 1 + 1/k$ and $n = 1$ if $0 \leq r_2 \leq 2 + 1/k$.

Proof. We will show that $\mathcal{J}$ is a contraction in $\mathcal{B}$ for small enough ε with $\delta = C\varepsilon^{r_0 - 1} \delta_0^{-r'} \ln^n(\delta_0/h)$ for some constant C.

Suppose $K(x,y)$ satisfies the assumption in the lemma. Let $V(x) := |x - X_s|^{-r_2} e^{\mu x}$, then clearly $|V(x)| \le C\delta_0^{r_2}\|V\|_\mu |x - X_s|^{-r_2}$ for $x \in \mathcal{D}_1^-(d_0, \delta_0, \pm)$ By Lemma 3.3 and Remark 3.2, 3.3, using (3.74)for $x \in \mathcal{D}^-$ we have

$$\begin{aligned}
& |\mathcal{J}[\eta](x)| \\
= & \left| \frac{1}{w_0(x)} \sum_{\alpha=-1,0,1} \tau_\alpha(x) \sum_{n=1}^{\infty} \left[\frac{M_\alpha(x-nh)}{W(x-nh)} \mathcal{Q}[w_0(1+\eta)](x-nh) \right] \right| \\
\le & \; C_1 \; (1 + \|\eta\|_\infty)\varepsilon^{r_0} |x - X_s|^{-r_1} \{ |e^{-\mu x}| |\tilde{\mathcal{L}}_-^{-1}[V](x)\} | \\
\le & \; K\varepsilon^{r_0 - 1}\delta_0^{-r'} \ln^n(\delta_0/h)(1 + \|\eta\|_\infty)
\end{aligned} \tag{3.76}$$

where K is a constant. Choose

$$\delta = 2K\varepsilon^{r_0-1}\delta_0^{-r'} \ln^n(\delta_0/h), \tag{3.77}$$

then for ε small enough, we have $|\mathcal{J}[\eta](x)| \le \delta$. Similarly, using (3.75),

$$|\mathcal{J}[\eta](x) - \mathcal{J}[\zeta](x)| \le C \; \varepsilon^{r_0-1}\delta_0^{-r'} \ln^n(\delta_0/h)\|\eta - \zeta\|_\infty \tag{3.78}$$

Hence $\mathcal{J}$ is a contraction in $\mathcal{B}$. Therefore there exists a unique solution of (3.73) in $\mathcal{B}$. The estimate $\eta(x) = \mathcal{O}(K\varepsilon^{r_0-1}\delta_0^{-r'} \ln^n(\delta_0/h))$ follows from (3.77). □

Now consider solving (3.71) in $\mathcal{D}_0$. This is necessary when $\mathcal{Q}[w]$ defined only in this region.

Define the inverse of $\mathcal{L}$ as $\mathcal{L}^{-1} : \mathcal{S}_0(\mathcal{D}_0) \to \mathcal{S}_0(\mathcal{D}_0)$,for $x \in \mathcal{D}_0$

$$\mathcal{L}^{-1}[g](x) := \sum_{\alpha=-1,0,1} \mathcal{L}_\alpha^{-1}[g], \text{ where } \mathcal{L}_\alpha^{-1}[g] := \tau_\alpha \Delta_{h,\alpha}^{-1} \left[\frac{M_\alpha}{W} g \right](x). \tag{3.79}$$

where $\Delta_{h,\alpha}^{-1}[g](x)$ is defined on region $\mathcal{D}_0$ as (2.29) with $\mathcal{I}_0$, $\mathcal{I}_\pm$ replaced by $\mathcal{I}_{0,\alpha}$, and $\mathcal{I}_{\pm,\alpha}$ defined as following:

$$\mathcal{I}_{0,\alpha} : \mathcal{S}_0(\mathcal{D}_0) \mapsto \mathcal{S}_0(\mathcal{D}_0), \quad \mathcal{I}_0[g](x) = (e^{\mu x} + e^{-\mu x})\tau_\alpha(x)\omega(x)g(x); \tag{3.80}$$

$$\mathcal{I}_{\pm,\alpha} : \mathcal{S}_\mu(\mathcal{D}_\pm,) \mapsto \mathcal{S}_0(\mathcal{D}_\pm), \quad \mathcal{I}_\pm[g](x) = (e^{\mu x} + e^{-\mu x})\tau_\alpha(x)\omega(x)g(x) \tag{3.81}$$

$$\omega(x) := \left([\tau_{-1} + \tau_1]^2(t) + [\tau_{-1} + \tau_1]^2(t+h) + [\tau_{-1} + \tau_1]^2(t+2h)\right)^{-1/2}, \tag{3.82}$$

By Remark 2.8 on horizontal strips of width h covering

$$\mathcal{D}_0 = \bigcup_{n=0}^{N(h)} \mathcal{D}^{(n)} \tag{3.83}$$

$$\mathcal{D}^{(n)} := \mathcal{D}_0 \cap (-\infty, \infty) \times (b_n, b_{n+1})i \tag{3.84}$$

where $b_{n+1} = b_n + h$. Define the inverse of $\mathcal{L}$ as $\mathcal{L}^{-1}_{(n)} : \mathcal{S}_0(\mathcal{D}^{(n)}) \to \mathcal{S}_0(\mathcal{D}^{(n)})$,for $x \in \mathcal{D}^{(n)}$

Let δ_n be the smallest distance between $\mathcal{D}^{(n)}$ and $\pm X_s$.

Lemma 3.5. *For ε small enough, we have*

$$\|\mathcal{L}^{-1}_{\alpha,(n)}\| \leq C\delta_n^{-1-1/k}\varepsilon^{-1} \qquad \alpha = -1, 0, 1 \tag{3.85}$$

$$\|\mathcal{L}^{-1}_{(n)}\| \leq C\delta_n^{-1-1/k}\varepsilon^{-1} \tag{3.86}$$

where C is a constant. In particular, we have

$$\|\mathcal{L}^{-1}_{(t)}\| \leq C\delta_0\varepsilon^{-1} \tag{3.87}$$

Proof. By (3.59), we have that for $x \in \mathcal{D}^{(n)}$,

$$\left|\frac{M_\alpha(x)}{W(x)}\tau_\alpha(x)\right| \leq C \tag{3.88}$$

Similar with Lemma 2.12, we have

$$\|\tau_\alpha \Delta^{-1}_{h,\alpha}\left[\frac{M_\alpha}{W}g\right]\| \leq \frac{C\|w\|_n\|w^{-1}\|_n\|g\|}{1 - e^{-\mu h}} \tag{3.89}$$

By (3.79) and Lemma 2.12, we have

$$|\mathcal{L}_{\alpha,(n)}^{-1}[g]| \leq\leq \frac{C\|w\|_n\|w^{-1}\|_n\|g\|}{1-e^{-\mu h}} \tag{3.90}$$

Noticing (3.43),we get

$$\|\tau_\alpha\|_n = \|\tau_{e,\alpha}\|_n(1+\mathcal{O}(\varepsilon\kappa(\delta_n))) \qquad \text{and } \|\tau_{e,\alpha}^{-1}\| = \|\tau_{e,\alpha}^{-1}\|_n(1+\mathcal{O}(\varepsilon\kappa(\delta_n))) \tag{3.91}$$

But for $x \in \mathcal{D}^{(n)}$ and small enough ε,

$$\|\tau_{e,\alpha}\|_n\|\tau_{e,\alpha}^{-1}\|_n \leq C_1\delta_n^{-1-1/k}e^{\ln\Lambda_\alpha} \leq C\delta_n^{-1-1/k}, \tag{3.92}$$

where C_1, C are constants independent of ε , and n. (3.86) follows from (3.85). □

For $\mathcal{L}_{(t)}^{-1}$ defined in $\mathcal{D}^{(t)}$, that is the strip closest to π/ki, we have special estimate in Lemma 3.8. The following proposition is a preparation of Lemma 3.8.

Proposition 3.6. *If $q \in \mathcal{S}_0(\mathcal{D}^{(t)})$ satisfying*

$$|q(x)| \leq K\frac{\delta_0^\gamma}{|x-X_s|^\gamma}\|q\|, \quad x \in \mathcal{D}^{(t)}; \tag{3.93}$$

for some constant $\gamma > 1$, then

$$|\mathcal{P}_{(t)}^{\pm}q(x)| \leq C\delta_0^\gamma|x-X_s|^{-\gamma}\|g\| \tag{3.94}$$

for some constant C independent of δ_0.

Proof. By the definition of $\mathcal{P}^-$ (2.22), similar with proof of Lemma 2.8, using (2.27)

$$|\chi(x)q(x)| \leq \frac{K\delta_0^\gamma}{|x-X_s|^\gamma}\|q\| \tag{3.95}$$

$$\left|\int_{\partial\mathcal{D}^{(t)}}\frac{\chi(\xi)-\chi(x_0)}{\xi-x_0}q(\xi)d\xi\right| \leq \delta_0^\gamma\|q\|\int_{\partial\mathcal{D}^{(t)}}K_L|\xi-X_s|^{-\gamma}|d\xi| \leq C\delta_0\|q\| \tag{3.96}$$

(2.27), (3.95) and (3.96) imply (3.94). □

Corollary 3.7. *If $q(x)$ satisfies conditions (3.93) in Proposition 3.6 with $\gamma > 1$, then*

$$\|\Delta_{h,(t)}^{-1}[q]\| \leq C\delta_0\varepsilon^{-1}\|q\| \tag{3.97}$$

where C is a constant independent of ε, q and δ_0.

Proof. This a consequence of Proposition 3.2, 3.6 and 2.29. □

Lemma 3.8. *If $q \in \mathcal{S}_0(D_0)$ satisfying the two conditions (3.93) with some constant $r > 2 + 1/k$, then*

$$\|\mathcal{L}_{(t)}^{-1}q\|_1 \leq C\delta_0\varepsilon^{-1}\|q\|_1 \tag{3.98}$$

where $\|\cdot\|_1$ is the sup norm on $\mathcal{D}^{(t)}$.

Proof. If q satisfies (3.93) , then $\tau_\alpha q$ satisfies condition of Proposition 3.6 with $r > 1$, hence by Corollary 3.7, $\|\Delta_h^{-1}\tau_\alpha^{-1}q\|_1 \leq C_1\delta_0\varepsilon^{-1}\|\tau_\alpha q\|_1$. By 3.88

$$\begin{aligned} |\mathcal{L}_\alpha^{-1}[q]| &\leq \|\tau_\alpha\|_1\, \|\Delta_h^{-1}\tau_\alpha^{-1}q\|_1 \leq C_2\|\tau_\alpha\|_n\delta_0\varepsilon^{-1}\|\tau_\alpha^{-1}q\|_1 \\ &\leq C\delta_0\varepsilon^{-1}\|q\|_1 \end{aligned} \tag{3.99}$$

In the last step, we used $\|\tau_\alpha\|_1\|\tau_\alpha^{-1}\|_1 \leq C$. □

3.1.2 Proof of Lemma 3.1

Proposition 3.9. *There exist a third order difference operator $\mathcal{L}_e$ such that*

$$\tau_{e,\alpha} := \exp\{x\frac{\ln\Lambda_\alpha}{h}\}\tilde{V}_\alpha, \qquad \alpha = -1, 0, 1 \tag{3.100}$$

are exact solutions of

$$\mathcal{L}_e[\tau](x) = 0 \tag{3.101}$$

where $\mathcal{L}_e$ is given in (3.114).

Proof. It is convenient to define $m_{2,\alpha} = \frac{\Delta_h \tau_{e,\alpha}}{\tau_{e,\alpha}}$, $m_{3,\alpha} = \frac{\Delta_h^2 \tau_{e,\alpha}}{\tau_{e,\alpha}}$ and

$$R_\alpha := \frac{\mathcal{L}_0[\tau_{e,\alpha}]}{\tau_{e,\alpha}}. \tag{3.102}$$

Note that $m_{2,0}, m_{3,0} = \mathcal{O}(\varepsilon)$ and $m_{2,\pm 1}, m_{3,\pm 1} = \mathcal{O}(t)$ as $\varepsilon \to 0^+$. Estimating residual $\mathcal{L}_0[\tau_{e,\alpha}](x)$ for small ε, $x \in \mathcal{D}^-$, it is seen for Taylor expansion for small ε that

$$|\mathcal{R}_\alpha| = \left| \frac{\mathcal{L}_0[\tau_{e,\alpha}](x)}{\tau_{e,\alpha}} \right| \leq C\varepsilon^3 |e^{kx}| \delta_0^{-3-1/k} \ln^2(\delta_0/h) \tag{3.103}$$

for $x \in \mathcal{D}^-$. Define

$$M := \begin{bmatrix} \tau_{e,-1} & \tau_{e,0} & \tau_{e,1} \\ \Delta_h \tau_{e,-1} & \Delta_h \tau_{e,0} & \Delta_h \tau_{e,1} \\ \Delta_h^2 \tau_{e,-1} & \Delta_h^2 \tau_{e,0} & \Delta_h^2 \tau_{e,1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ m_{2,-1} & m_{2,0} & m_{2,1} \\ m_{3,-1} & m_{3,0} & m_{3,1} \end{bmatrix} \begin{bmatrix} \tau_{e--1} & 0 & 0 \\ 0 & \tau_{e,0} & 0 \\ 0 & 0 & \tau_{e,1} \end{bmatrix} \tag{3.104}$$

$$Q_1 := (\Delta_h M - Q_0 M) M^{-1} \tag{3.105}$$

where

$$Q_0 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q_1 & q_2 & q_3 \end{bmatrix}; \tag{3.106}$$

$q_3 := -3 + \varepsilon g'[z_0(x+2h)]$, $q_2 := 2q_3 + 3$, $q_1 := -1 + q_2 - q_3$. Hence the elements of last row of $\Delta_h M - Q_0 M$ are $\mathcal{L}_0[\tau_\alpha]$, $\alpha = -1, 0, 1$.

Using (3.102) , (3.105) and (3.106), we get that

$$Q_1 M = \Delta_h M - Q_1 M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R_{-1} & R_0 & R_1 \end{bmatrix} \begin{bmatrix} \tau_{e,-1} & 0 & 0 \\ 0 & \tau_{e,0} & 0 \\ 0 & 0 & \tau_{e,1} \end{bmatrix} \tag{3.107}$$

Thus,

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_1 & b_2 & b_3 \end{bmatrix} \tag{3.108}$$

where

$$b_2 := [(R_{-1} - R_0)(m_{3,0} - m_{3,1}) - (R_0 - R_1)(m_{3,-1} - m_{3,0})]/T, \tag{3.109}$$

$$b_3 := -[(R_{-1} - R_0)(m_{2,0} - m_{2,1}) - (R_0 - R_1)(m_{2,-1} - m_{2,0})]/T \tag{3.110}$$

$$b_1 := R_1 - b_2 m_{2,1} - b_3 m_{3,1} \tag{3.111}$$

$$T = (m_{2,-1} - m_{2,0})(m_{3,0} - m_{3,1}) - (m_{2,0} - m_{2,1})(m_{3,-1} - m_{3,1}) \tag{3.112}$$

Notice that $T = \mathcal{O}(1)$ as $\varepsilon \to 0$ or $x \to -\infty$. It follows from (3.105) that the matrix M satisfies the difference equation

$$\Delta_h M - (Q_0 + Q_1)M = 0 \tag{3.113}$$

In particular, the third row of the equation reads that τ_α satisfies the same third order homogeneous difference equation:

$$\mathcal{L}_e \tau_\alpha := \Delta_h^3 \tau_\alpha - (b_3 + q_3)\Delta_h^2 \tau_\alpha - (b_2 + q_2)\Delta_h \tau_\alpha - (b_1 + q_1)\tau_\alpha \tag{3.114}$$

□

Remark 3.4. *It is to be noted that $\mathcal{L}_e$ is formally close to $\mathcal{L}_0$ for small ε since $|b_n(x)| \le C|e^{kx}|\varepsilon^3 \delta_0^{-3-1/k} \ln^2(\delta_0/h)$ for $x \in \mathcal{D}^-$, $n = 1, 2, 3$ and $\mathcal{L}_0$ corresponds to $b_n = 0$.*

In order to find fundamental set of solutions to $\mathcal{L}_0[v] = 0$, for $\alpha = \pm 1$ we rewrite

$$\mathcal{L}_e[\tau](x) = \{\mathcal{L}_e - \mathcal{L}_0\}[]tau](x) \tag{3.115}$$

Using fundamental set of solutions $\{\tau_{e,\alpha}\}_{\alpha=-1,0,1}$ to define $\mathcal{L}_{e,-}^{-1}$, we get

$$\mathcal{L}_e^{-1}[g] := \sum_{\alpha=-1,0,1} \tau_{e,\alpha}\Delta_-^{-1}[\frac{M_{e,\alpha}}{W_e}g] \tag{3.116}$$

where $W_e(x)$ is the difference Wronskian of $\{\tau_{e,\alpha}\}_{\alpha=-1,0,1}$ and $M_{e,\alpha}$ is the cofactor of the last element in the α column of $D_e(x)$.

$$\tau_{0,\alpha}(x) := \tau_{e,\alpha}(x) + \mathcal{L}_e^{-1}\left[\{\mathcal{L}_e - \mathcal{L}_0\}[\tau_{0,\alpha}]\right](y) \tag{3.117}$$

We will apply Lemma 3.4 to the above equation and the following Proposition is useful to show that the conditions (3.74) and (3.75) holds. Let $\eta_{0,\alpha}(x) := \frac{\tau_{0,\alpha}(x)-\tau_{e,\alpha}(x)}{\tau_{e,\alpha}(x)}$.

Proposition 3.10. *For $\eta, \zeta \in \mathcal{B}$, $x \in \mathcal{D}^-$, $y \geq 0$, we have*

$$|\tau_{e,\alpha}^{-1}(x)\{\mathcal{L}_e - \mathcal{L}_0\}[\tau_{e,\alpha}(1+\eta)](x-y)| \leq K(x,y)(1+\|\eta\|_\infty)\varepsilon^3|e^{-(k+\ln\Lambda_\alpha/h)y}| \tag{3.118}$$

$$\begin{aligned} & |\tau_{e,\alpha}^{-1}(x)(\{\mathcal{L}_e - \mathcal{L}_0\}[\tau_{e,\alpha}(x)(1+\eta)](x-y) - \{\mathcal{L}_e - \mathcal{L}_0\}[\tau_{e,\alpha}(1+\zeta)](x-y))| \\ \leq\ & K(x,y)(1+\|\eta\|_\infty)\varepsilon^3\|\eta-\zeta\|_\infty|e^{-(k+\ln\Lambda_\alpha/h)y}| \end{aligned} \tag{3.119}$$

where $K(x,y) = C|x-X_s|^{-\Re\beta_\alpha}|\ |x-y-X_s|^{-3-1/k+\Re\beta_\alpha}$ if $x \in \mathcal{D}_1^-(d_0,\delta_0,\pm)$ and $K(x,y) := C$ if $x \in \mathcal{D}_2(d_0,\delta_0)$. C is a constant in dependent of ε and δ.

Proof. (3.103), (3.105) and (3.109)-(3.112) imply that for $x \in \mathcal{D}^-$, we have $|b_n(x)| \leq \tilde{K}(x)|e^{kx}|$, $n = 1,2,3$, where $\tilde{K}(x) = |x-X_s|^{-2-1/k}\varepsilon^2$ if $x \in \mathcal{D}_1(d_0,\delta_0,\pm)$, and $\tilde{K}(x) = C$ if $x \in \mathcal{D}_2(d_0,\delta_0)$.

$$|(\mathcal{L}_e - \mathcal{L}_0)[\tau](x)| = -b_3(x)\Delta_h^2[\tau](x) - b_2(x)\Delta_h[\tau](x) - b_1(x)\tau(x). \tag{3.120}$$

Noticing that

$$\left|\tau^{-1}(x)\Delta_h[\tau](x)\right| = \left|\frac{\tau(x+h)}{\tau(x)} - \frac{\tau(x)}{\tau(x)}\right| \le 2\max_{m=0,1}\left|\frac{\tau(x+mh)}{\tau(x)}\right|, \tag{3.121}$$

we get

$$\begin{aligned} & |\tau_{e,\alpha}^{-1}(x)\{\mathcal{L}_e - \mathcal{L}_0\}[\tau_{e,\alpha}(1+\eta)](x-y)| \\ \le\ & (1+\|\eta\|)\left(\max_{n=1,2,3}|b_n(x-y)|\right)\left(\max_{m=0,1,2}\left|\frac{\tau_{e,\alpha}(x+mh-y)}{\tau_{e,\alpha}(x)}\right|\right) \\ \le\ & K(x,y)(1+\|\eta\|_\infty)\varepsilon^2|e^{-(k+\ln\Lambda_\alpha/h)y}| \end{aligned} \tag{3.122}$$

Since both $\mathcal{L}_0$ and $\mathcal{L}_e$ are linear, we have

$$\begin{aligned} & \{\mathcal{L}_e - \mathcal{L}_0\}[\tau_{e,\alpha}(1+\eta)](x) - \{\mathcal{L}_e - \mathcal{L}_0\}[\tau_{e,\alpha}(1+\zeta)](x)) \\ =\ & \{\mathcal{L}_e - \mathcal{L}_0\}[\tau_{e,\alpha}(\eta-\zeta)](x). \end{aligned} \tag{3.123}$$

from which (3.119) follows. □

Lemma 3.11. *There exists a fundamental set of solution $\{\tau_{0,\alpha}\}_{\alpha=-1,0,1}$ to the linear homogenous difference equation*

$$\mathcal{L}_0[\tau] = 0, \tag{3.124}$$

such that

$$\tau_{0,\alpha} = \tau_{e,\alpha}(1+\mathcal{O}(\varepsilon^2\delta_0^{-2-1/k}\ln^2(\delta_0/h))) \tag{3.125}$$

Proof. Proposition 3.10 shows that condition of Lemma 3.4 holds. Applying the Lemma, we get the existence and the estimate for $\tau_{0,\alpha}$. □

Remark 3.5.

$$\tau_{0,-1}(x) = \overline{\tau_{0,1}(\bar{x})} \tag{3.126}$$

Now we are ready to invert the operator $\mathcal{L}_0$ on the left of Equation (3.28). Define

$$\mathcal{L}_0^{-1}[g] := \sum_{\alpha=-1,0,1} \tau_{0,\alpha}\Delta_-^{-1}[\frac{M_{0,\mu}}{W_0}g] \tag{3.127}$$

where $W_0(x)$ is the difference Wronskian of $\tau_{0,\alpha}$ and $M_{0,\alpha}$ is the cofactor of the last element in the α column of Wronskian matrix $D_0(x)$.

Proposition 3.12. *$v = z_-(x) - p(x)$ satisfies*

$$v = \varrho_1\tau_{0,0} + \mathcal{L}_0^{-1}[f] + \mathcal{L}_0^{-1}[\mathcal{L}_1[v] + \mathcal{N}_1[v]] \tag{3.128}$$

$$\textit{where} \qquad \varrho_1 := e^{-\varrho\varepsilon\ln h} - 1 + \varrho\varepsilon\ln h = \mathcal{O}(\varepsilon^2\ln^2 h) \tag{3.129}$$

Proof. By Lemma 2.16, general solution to $\mathcal{L}_0[v] = g$ is represented by

$$v = v_0(x) + \mathcal{L}_0^{-1}[f] + \mathcal{L}_0^{-1}[\mathcal{L}_1[v] + \mathcal{N}_1[v]] \tag{3.130}$$

where $v_0(x) := \sum_{\alpha=-1,0,1} c_\alpha(x)\tau_{0,\alpha}$, and $c_\alpha(x), \alpha = -1, 0, 1$ are periodic function of period h. Since $z_-(x)$ satisfies initial condition (3.24), we get $v(x)$ satisfies

$$\lim_{x\to-\infty} \frac{v(x)}{e^x} = \varrho_1 \tag{3.131}$$

Hence $v \in \mathcal{S}_{\mu_1}(\mathcal{D}^-)$, where $\mu_1 = 1$. By definition of $\mathcal{N}_1[v]$, it is easy to see $\mathcal{N}_1[v] \in \mathcal{S}_{\mu_2}(\mathcal{D}^-)$, with $\mu_2 = k + 1 > 1$ therefore $\mathcal{L}_0^{-1}[\mathcal{N}_1[v]] \in \mathcal{S}_{\mu_2}(\mathcal{D}^-)$.

By Taylor expansion for large $|x|$ and small ε we get for $x \in \mathcal{D}^-$, $|e^{-(k+1)x}f(x)| \leq C\varepsilon^3$, which implies $f \in \mathcal{S}_{\mu_2}(\mathcal{D}^-)$, therefore $\mathcal{L}_0^{-1}[f] \in \mathcal{S}_{\mu_2}(\mathcal{D}^-)$. Dividing by e^x and letting $x \to \infty$ on both sides of (3.130), we get

$$\varrho_1 = \lim_{x\to-\infty} e^{-x}v_0(x) \tag{3.132}$$

$$(c_{-1}(x), c_0(x), c_1(x))^T = D_0^{-1}(x)(v_0(x), v_0(x+h), v_0(x+2h))^T \tag{3.133}$$

where $D_0^{-1}(x)$ is the inverse of matrix Wronskian of $\{\tau_{0,\alpha}\}_{\alpha=-1,0,1}$. Estimating $D_0^{-1}(x)$ shows $c_\alpha(x) \sim K\frac{v_0(x)}{\tau_{0,\alpha}}$ as $x \to \infty$ where K is a constant. Using (3.132), we get $\lim_{x\to-\infty} c_{\pm 1}(x) = 0$, and $\lim_{x\to-\infty} c_0(x) = C$ where C is a constant. But $c_\alpha(x)$ are periodic functions, hence $c_{\pm 1} = 0$ and $c_0(x) = C$ is a constant. From (3.132) we get $c_0(x) = \varrho_1$ and the lemma follows. □

Let $\eta(x) := \frac{v(x)}{z_0(x)}$. It is to be noted that $z_0(x) \neq 0$ for $x \in \mathbb{C}$. η satisfies

$$\eta = \mathcal{J}_2[\eta] := \varrho_1 z_0^{-1}\tau_{0,0} + z_0^{-1}\mathcal{L}_0^{-1}[f + \mathcal{L}_1[z_0\eta] + \mathcal{N}_1[z_0\eta]] \tag{3.134}$$

Let $\mathcal{B}_2$ be a ball of size δ in $\mathcal{S}_0(\mathcal{D}^-)$.

Proposition 3.13. *$\mathcal{J}_2$ is a contraction in $\mathcal{B}_2$ with $\delta = C\varepsilon^2\delta_0^{-2-1/k}\ln^2\left(\frac{\delta_0}{h}\right)$, for ε small enough and $\delta_0 \gg \varepsilon$ where C is some constant independent of δ_0, ε.*

Proof. We estimate each term in (3.134). Obviously, $\|\varrho_1 z_0^{-1}\tau_{0,0}\|_\infty \leq C(\varepsilon \ln h)^2\delta_0^{-1}$.

By Taylor expansion for small ε with care taken to estimate a lower bound to distance from complex singularity of $z_0(x)$, we get for $x \in \mathcal{D}_2(d_0, \delta_0)$

$$|e^{-(k+1)x} f(x)| \leq C\varepsilon^3 d_0^{-3-1/k} \ln^2\left(\frac{d_0}{h}\right) \tag{3.135}$$

and $f(x)$ satisfies (3.54) and (3.55) with $\gamma = -3 - 1/k$, $K = C\ln^2\left(\frac{\delta_0}{h}\right)$. For $x \in \mathcal{D}_2(d_0, \delta_0)$, we have

$$\left|\frac{1}{z_0(x)}\mathcal{L}_0^{-1}[f]\right| \leq C_1 \left|\frac{1}{z_0(x)} e^{-\mu x}\mathcal{L}_0^{-1}[f]\right| \leq C_2 d_0^{-1-1/k}\|\mathcal{L}_0^{-1} f\|_{\mu_2} \leq C\varepsilon^2 \tag{3.136}$$

where $\mu_2 = k+1$, $k \geq 1$. For $x \in \mathcal{D}_1(d_0, \delta_0, \pm)$, by Remark 3.2,

$$\begin{aligned}
\left| \frac{1}{z_0(x)} \mathcal{L}_0^{-1}[f] \right| &\leq C\varepsilon^2 |x - X_s|^{1/k} \frac{\ln^2\left(\frac{\delta_0}{h}\right) \delta_0^{3+1/k}}{|x - X_s|^{2+1/k}} \|f\|_{\mu_2} \\
&\leq C\varepsilon^2 \delta_0^{-2} \ln^2\left(\frac{\delta_0}{h}\right) \qquad (3.137)
\end{aligned}$$

We used $\|f\|_{\mu_2} \leq C\delta_0^{-3-1/k}$. Let $q(x) := (\ln h) z_1(x) + z_2(x)$. From (3.30), we get

$$\mathcal{L}_1[z_0 \eta] = \sum_{l=1}^{k} \varepsilon^{l+1} [q^{k+1-l} z_0^{l+1} \eta](x + 2h) \qquad (3.138)$$

For $x \in \mathcal{D}_2(d_0, \delta_0)$

$$\begin{aligned}
|e^{-\mu_2 x}[q^l z_0^{k+2-l} \eta](x+2h)| &\leq C|e^{-x} q(x+2h)|^l |e^{-x} z(x+2h)|^{k+2-l} \|\eta\|_\infty \\
&\leq C\|\eta\|_\infty \qquad (3.139)
\end{aligned}$$

$q^{k+1-l} z_0^{l+1}$ satisfies (3.54) and (3.55) with $\gamma = l+1+1/k > 2+1/k$ and $K = C\ln^l\left(\frac{\delta_0}{h}\right)$. Hence for $x \in \mathcal{D}_2(d_0, \delta_0)$, we have

$$\begin{aligned}
\left| \frac{1}{z_0} \mathcal{L}_0^{-1}[\varepsilon^{l+1}[q^{k+1-l} z_0^{l+1} \eta](x+2h)] \right| &\leq C\varepsilon^{l+1} \|\mathcal{L}_0^{-1} q^{k+1-l} z_0^{l+1} \eta\|_{\mu_2} \\
&\leq C\varepsilon^l \|\eta\|_\infty \qquad (3.140)
\end{aligned}$$

For $x \in \mathcal{D}_1(d_0, \delta_0, \pm)$, we have

$$\begin{aligned}
&\left| \frac{\varepsilon^{l+1}}{z_0} \mathcal{L}_0^{-1}[[q^{k+1-l} z_0^{l+1} \eta](x+2h)] \right| \\
&\leq C\varepsilon^l |x - X_s|^{1/k} \frac{\delta_0^{l+1+1/k} \ln^l\left(\frac{\delta_0}{h}\right)}{|x - X_s|^{l+1/k}} \|q^{k+1-l} z_0^{l+1}\|_{\mu_2} \|\eta\|_\infty \\
&\leq C\varepsilon^l \delta_0^{-l} \ln^l\left(\frac{\delta_0}{h}\right) \|\eta\|_\infty \qquad (3.141)
\end{aligned}$$

(3.138),(3.141) and (3.140) imply

$$\|z_0^{-1} \mathcal{L}_0^{-1}[z_0 \eta]\|_\infty \leq C\varepsilon^2 \delta_0^{-2} \ln^2\left(\frac{\delta_0}{h}\right) \|\eta\|_\infty \qquad (3.142)$$

Since $\mathcal{L}_1$ is linear, for $\tilde{\eta} \in \mathcal{B}$, we have

$$\|z_0^{-1}\mathcal{L}_0^{-1}[z_0\eta] - z_0^{-1}\mathcal{L}_0^{-1}[z_0\tilde{\eta}]\|_\infty \le C\varepsilon^2\delta_0^{-2}\ln^2\left(\frac{\delta_0}{h}\right)\|\eta - \tilde{\eta}\|_\infty \tag{3.143}$$

Expanding terms in $\mathcal{N}_1[z_0\eta]$ and similar with the proof of (3.142) and (3.143), we get

$$\|z_0^{-1}\mathcal{N}_1^{-1}[z_0\eta]\|_\infty \le C\delta_0^{1/k}\ln(\delta_0^{-1})\|\eta\|_\infty^2 \tag{3.144}$$

$$\|z_0^{-1}\mathcal{L}_0^{-1}[z_0\eta] - z_0^{-1}\mathcal{L}_0^{-1}[z_0\tilde{\eta}]\|_\infty \le C\delta_0^{1/k}\ln(\delta_0^{-1})\delta\|\eta - \tilde{\eta}\|_\infty \tag{3.145}$$

From (3.136), (3.137), (3.143), (3.142), (3.144) and (3.145), we get $\mathcal{J}_2$ is a contraction in $\mathcal{B}_2$ with $\delta = C\varepsilon^2\delta_0^{-2}\ln^2\left(\frac{\delta_0}{h}\right)$. ☐

Proof. (Proof of Lemma 3.1) As a corollary of Proposition 3.13, $v(x) = z_0(x)\eta(x)$ satisfies the bound $|e^{-x}v(x)| \le K\varepsilon^2\delta_0^{-2-1/k}\ln^2\left(\frac{\delta_0}{h}\right)$.

(3.12) is essentially the same type of equation with (3.10). Starting from (3.12), follow the same procedure as we did for proving the assertions for z_-, we get the result for $z_+(x)$. ☐

3.2 Difference between Stable and Unstable Manifolds

In the outer region, consider the difference between $z_-(x)$ and $z_+(x)$

$$\tau(x) := z_-(x) - z_+(x). \tag{3.146}$$

Lemma 3.1 implies that $\tau(x) = \mathcal{O}(\varepsilon^2\delta_0^{-2-1/k}\ln^2(\delta_0/h))$ for $x \in \mathcal{D}_0$. However, we will show that there exists a reparametrization of the unstable manifold $\tilde{z}_-(x)$ for which the closest distance between $\tilde{\mathbf{Z}}_-(x) = (\tilde{z}_-(x), \tilde{z}_-(x+h), \tilde{z}_-(x+2h))$ and $\tilde{\mathbf{Z}}_+(x) = (z_+(x), z_+(x+h), z_+(x+2h))$ is exponentially small. $\tau(x)$ satisfies a homogenous equation:

$$\mathcal{L}[\tau](x) = \mathcal{N}[\tau](x) \tag{3.147}$$

where

$$\mathcal{L}[\tau](x) = \tau(x+3h) - \tau(x) - \varepsilon g'[z_-(x+2h)]\tau(x+2h) \tag{3.148}$$

$$\mathcal{N}[\tau](x) = \varepsilon \sum_{l=2}^{k+1} \binom{k+1}{l} \tau^l(x+2h) z_-^{k+1-l}(x+2h) \tag{3.149}$$

Remark 3.6. *In (3.147) we linearized the equation about z_-. The advantage of this is that $\psi(x) = \tilde{z}_-(x) - z_-(x)$ satisfies the same equation as $\tau(x)$.*

To find solution to (3.147), we need a set of fundamental solutions to the linear equation

$$\mathcal{L}[\tau] = 0 \tag{3.150}$$

Since Lemma 3.1 states that $z_-(x)$ is close to $z_0(x)$ in $\mathcal{D}^-$, in the following lemma, we will prove that there exists a fundamental set of solutions of (3.147) τ_α close to $\tau_{0,\alpha}$ for $\alpha = -1, 0, 1$

Lemma 3.14. *There exists a fundamental set of solution τ_α to the linear homogenous difference equation (3.150), such that*

$$\tau_\alpha(x) = \tau_{0,\alpha}(x)(1 + \mathcal{O}(\varepsilon\delta_0^{-2-2/k} \ln(\delta_0/h))) \tag{3.151}$$

Proof. Rewrite equation (3.150) as

$$\mathcal{L}_0[\tau_\alpha](x) = \{\mathcal{L}_0 - \mathcal{L}\}[\tau_\alpha](x) \tag{3.152}$$

$$\tau_\alpha(x) := \tau_{0,\alpha}(x) + \mathcal{L}_0^{-1}\left[\frac{M_{0,\mu}}{W_0}\{\mathcal{L}_0 - \mathcal{L}\}[\tau_\alpha]\right](x) \tag{3.153}$$

where $\mathcal{L}_0^{-1}$ is defined in (3.127) . Let

$$\eta_\alpha(x) := \frac{\tau_\alpha(x) - \tau_{0,\alpha}(x)}{\tau_{0,\alpha}(x)}. \tag{3.154}$$

$$\{\mathcal{L}_0 - \mathcal{L}\}[\tau_\alpha] \quad = \quad \varepsilon(k+1)\left\{z_0^k(x+2h) - z_-^k(x+2h)\right\}\tau_\alpha(x+2h), \tag{3.155}$$

By Lemma 3.1, for $x \in \mathcal{D}^-$,

$$\varepsilon|(k+1)z_0^k(x+2h) - (k+1)z_-^k(x+2h)| \le \varepsilon^2\tilde{K}(x)|e^{kx}|. \tag{3.156}$$

where $\tilde{K}(x) = C$ if $x \in \mathcal{D}_2(d_0,\delta_0)$; $\tilde{K}(x) = C|x - X_s|^{-2}\ln(\delta_0/h)$ if $x \in \mathcal{D}_1(d_0,\delta_0,\pm)$. It follows that

$$\begin{aligned} &\left|\tau_{0,a}^{-1}(x)\{\mathcal{L}_0 - \mathcal{L}\}[\tau_{0,\alpha}(1+\eta)](x-y)\right| \\ \le\ & \varepsilon^2 K(x,y)|e^{-(k+\frac{\ln\Lambda_\alpha}{h})y}|(\|\eta\|_\infty + 1), \end{aligned} \tag{3.157}$$

$$\begin{aligned} &\left|\tau_{0,a}^{-1}(x)\left(\{\mathcal{L}_0 - \mathcal{L}\}[\tau_{0,\alpha}(1+\eta)](x-y) - \{\mathcal{L}_0 - \mathcal{L}\}[\tau_{0,\alpha}(1+\zeta)](x-y)\right)\right| \\ \le\ & \varepsilon^2 K(x,y)|e^{-(k+\frac{\ln\Lambda_\alpha}{h})y}|(\|\eta - \zeta\|_\infty) \end{aligned} \tag{3.158}$$

where $K(x,y) = C$ if $x \in \mathcal{D}_2(d_0,\delta_0)$; $K(x,y) = C|x - X_s|^{-\Re\beta_\alpha}|x-y-X_s|^{-2+\Re\beta_\alpha}\ln(\delta_0/h)$ if $x \in \mathcal{D}_1(d_0,\delta_0,\pm)$. (3.151) follows from Lemma 3.4. □

Define

$$\mathcal{L}^{-1}[g] = \sum_{\alpha=-1,0,1}\tau_\alpha\Delta_h^{-1}[\frac{M_\alpha}{W}g], \tag{3.159}$$

where Δ_h^{-1} is defined on strips $\mathcal{D}^{(n)}$, and $W(x)$ is the difference Wronskian of $\{\tau_{0,\alpha}\}_{\alpha=-1,0,1}$ and M_α is the cofactor of the last element in the α column of $D(x)$.

By Lemma 2.18 and Remark 2.9, for $x \in \mathcal{D}^{(n)}$, there exist $\phi(x)$

$$\phi^{(n)}(x) := \sum_{\alpha=-1,0,1} b_\alpha^{(n)}(x)\tau_\alpha(x), \tag{3.160}$$

where $b_\alpha^{(n)}(x)$ are analytic in $\mathcal{D}^-$ periodic with period h, $b_\alpha^{(n)}(x) = \sum_{m=-\infty}^{\infty} b_{\alpha,m}^{(n)} \exp\{\frac{2m\pi ix}{h}\}$, such that $\tau(x)$ satisfies

$$\tau(x) = \mathcal{J}_3[\tau] := \phi^{(n)} + \mathcal{L}^{-1}[\mathcal{N}[\tau]](x) \tag{3.161}$$

For simplicity of symbols, we drop the superscripts (n).

Proposition 3.15. *$\phi(x) = \mathcal{O}(\varepsilon^2\delta_n^{-4-1/k})$ for $1 \gg \delta_0 \gg \varepsilon$.*

Proof. By Lemma 3.5, and (3.149), we have

$$\begin{aligned} |\mathcal{L}^{-1}[\mathcal{N}[\tau]](x)| &\leq \|\mathcal{L}^{-1}\|\|\mathcal{N}[\tau]\| \leq C\delta_0^{-1-1/k}\varepsilon^{-1}\varepsilon\|\tau\|^2\delta_0^{-1+1/k} \\ &\leq C\varepsilon^4\delta_0^{-6-1/k}\ln^4(\delta_0/h) \end{aligned} \tag{3.162}$$

Using $\varepsilon^2\delta_0^{-2}\ln^4(\delta_0/h) \ll 1$ for $1 \gg \delta_0 \gg \varepsilon$, we get

$$\phi(x) = \tau(x) - \mathcal{L}^{-1}[\mathcal{N}[\tau]](x) = \mathcal{O}(\varepsilon^2\delta_0^{-4-1/k}). \tag{3.163}$$

□

Now consider a reparameterization of the unstable manifold $\tilde{z}_-(x) = z_-(\xi(x))$, where $\xi(x)$ is a analytic function and $\tilde{z}_-$ also satisfies (3.148).

$$\psi(x) := \tilde{z}_- - z_-(x) \tag{3.164}$$

We seek for $\tilde{z}_-$ such that $|\tilde{\mathbf{Z}}_- - \mathbf{Z}_+| \le |\mathbf{Z}_- - \mathbf{Z}_+|$, therefore we require $\psi(x) = \mathcal{O}(\varepsilon^2 \delta_0^{-2-1/k} \ln^2(\delta_0/h))$. Again by Lemma 2.18, there exist $\tilde{\phi}(x)$ such that $\tau(x)$ satisfies (3.161) with τ replaced by ψ and ϕ replace by $\tilde{\phi}$, where

$$\tilde{\phi}(x) := \sum_{\alpha=-1,0,1} \tilde{b}_\alpha(x)\tau_\alpha(x), \tag{3.165}$$

$\tilde{b}_\alpha(x)$ are analytic in $\mathcal{D}^-$ with period h. Similar to Proposition 3.15, we have

$$\tilde{\phi}(x) = \mathcal{O}(\varepsilon^2 \delta_0^{-2-1/k} \ln^2(\delta_0/h)) \tag{3.166}$$

Let $\mathcal{B}$ be a ball of size δ centered at ϕ in $\mathcal{S}_0(\mathcal{D}^{(n)})$.

Lemma 3.16. *Given $\phi(x) = \sum_{\alpha=-1,0,1} b_\alpha(x)\tau_\alpha(x)$, where b_α analytic in $\mathcal{D}_0$ with period h, and ϕ satisfying*

$$\|\phi\|_n = \sup_{x\in\mathcal{D}^{(n)}} |\phi(x)| \le K\varepsilon^2\delta_n^{-4-1/k} \tag{3.167}$$

where K is a constant , then for ε small enough there exists a unique solution of (3.161) in $\mathcal{B}$ for small enough δ, and it satisfies

$$\|\tau(x) - \phi(x)\|_n \le C\delta_n^{-2}\|\phi\|^2, \tag{3.168}$$

where $\|\cdot\|_n$ is sup norm in $\mathcal{D}^{(n)}$ defined in (3.84), C is a constant independent of B, ε, δ_0.

Proof. (3.149) and (3.167) implies that for small enough δ,

$$|\mathcal{N}[\tau](x)| \le C_1\delta_n^{-1+1/k}\varepsilon(\|\phi\|_n + \delta)^2 \tag{3.169}$$

for some constant $C_1 > 0$. By Lemma 3.5,

$$|\mathcal{J}_3[\tau] - \phi| \leq \|\mathcal{L}^{-1}\| \|\mathcal{N}[\tau](x)\|_n \leq C_1 \delta_n^{-2} (\|\phi\|_n + \delta)^2 \tag{3.170}$$

For ε small enough, choose $\delta = 2C_1\delta_n^{-2}\|\phi\|_n^2$, then we have $\|\mathcal{J}_3[\tau] - \phi\|_n \leq \delta$. Similarly, for $\tau, \tilde{\tau} \in \mathcal{B}$, we have

$$|\mathcal{N}[\tau](x) - \mathcal{N}[\tilde{\tau}](x)| \leq C\delta_n^{-1+1/k} \delta \|\tau - \tilde{\tau}\|_n \tag{3.171}$$

$$\|\mathcal{J}_3[\tau] - \mathcal{J}_3[\tilde{\tau}]\|_n \leq C\delta_n^{-2} \delta \|\tau - \tilde{\tau}\|_n \tag{3.172}$$

Therefore $\mathcal{J}_3$ is a contraction in $\mathcal{B}$ for small enough ε with $\delta = 2C\delta_n^{-2}\|\phi\|_n^2$. Therefore the lemma follows. ☐

Remark 3.7. *Since* $\frac{d}{dx} z_-(x) = e^x(1 + e^{kx})^{-1-1/k} + \mathcal{O}(\varepsilon \delta_0^{-1-1/k})$, *and* $\tau_0(x) = \tau_{e,0}(1 + \varepsilon^2 \delta_0^{-2-2/k} \ln(\delta_0/h))$, $b_0(x)$ *in decomposing* $\tau(x) = \sum_{\alpha=-1,0,1} b_\alpha(x)\tau_\alpha(x)$ *describes 'tangential' difference between* $z_-(x)$ *and* $z_+(x)$. *In the following Lemmas, we will show that if two solutions of (3.168) have different* ϕ, *but with the same* $b_0(x)$, *(the coefficient of* $\tau_0(x)$ *in the decomposition (3.160)), then the difference between the two solutions is exponentially small in the real domain.*

Define $\rho(x) := \psi(x) - \tau(x)$.

Proposition 3.17. $\rho(x)$ *satisfies*

$$\rho(x) = \mathcal{J}_4[\rho](x) := \rho_0(x) + \mathcal{L}^{-1}\mathcal{N}_2[\rho](x), \tag{3.173}$$

$$\textit{where} \quad \rho_0(x) = \sum_{\alpha=-1,0,1} (\tilde{b}_\alpha(x) - b_\alpha(x))\tau_\alpha(x), \tag{3.174}$$

$$\mathcal{N}_2[\rho, \tau](x) := \varepsilon \left[\rho \sum_{l=2}^{k+1} \binom{k+1}{l} z_-^{k+1-l} \sum_{n=0}^{l-1} [\rho + \tau]^{l-1-n} \tau^n \right] (x + 2h) \tag{3.175}$$

Proof.

$$\mathcal{N}[\tau+\rho](x)-\mathcal{N}[\tau](x)=\mathcal{N}_1[\rho,\tau](x) \tag{3.176}$$

Since both τ and $\psi=\tau+\rho$ satisfies (3.161), we have that

$$\rho(x)=\rho_0(x)+\mathcal{L}^{-1}\mathcal{N}[\tau+\rho](x)-\mathcal{L}^{-1}\mathcal{N}[\tau](x). \tag{3.177}$$

Noticing that operator $\mathcal{L}^{-1}$ is linear, after explicit calculation we get that

$$\rho(x)=\rho_0(x)+\mathcal{L}^{-1}\{\mathcal{N}[\tau+\rho]-\mathcal{N}[\tau]\}(x)=\rho_0(x)+\mathcal{L}^{-1}\mathcal{N}_2[\rho,\tau](x) \tag{3.178}$$

□

Remark 3.8. *The ρ_0 in (3.174) depends on the particular inversion, and it is different for each $\mathcal{D}^{(n)}$, denote it by $\rho_0^{(n)}$. However, they are related.*

Lemma 3.18. *If ρ_0 satisfies*

$$\|\rho_0\|_n=\sup_{x\in\mathcal{D}^{(n)}}|\rho_0(x)|\le K\varepsilon^2\delta_n^{-4-1/k}, \tag{3.179}$$

where K is a constant independent of ε, then in region $\mathcal{D}_0$ we have that in each $\mathcal{D}^{(n)}$ with sup norm on $\mathcal{D}^{(n)}$.

$$\|\rho-\rho_0\|_n\le C\varepsilon^2\delta_n^{-4-1/k}\|\rho_0\|_n \tag{3.180}$$

Proof. The proof is similar to proof of Lemma 3.16. Let $\mathcal{B}_4$ be a ball of size δ centered at ρ_0 in $\mathcal{S}_0(\mathcal{D}^{(n)})$. (3.175) and (3.179) implies that for $\rho,\rho_1\in\mathcal{B}_4$,

$$\begin{aligned}|\mathcal{N}_2[\rho,\tau]&\le C\varepsilon\delta_0^{-1+1/k}(\|\tau\|_n+\|\rho\|_n)\|\rho\|_n\\ &\le C_1\varepsilon\delta_0^{-1+1/k}(\varepsilon^2\delta_0^{-2-1/k}\ln^2(\delta_0/h)+\|\rho_0\|_n+\delta)(\|\rho_0\|_n+\delta)\end{aligned} \tag{3.181}$$

$$\|\mathcal{J}_4[\rho]-\rho_0\|_n \leq \|\mathcal{L}^{-1}\|\|\mathcal{N}_2[\rho,\tau]\|_n$$
$$\leq \quad K\delta_0^{-2}(\varepsilon^2\delta_0^{-2-1/k}\ln^2(\delta_0/h)+\|\rho_0\|_n+\delta)(\|\rho_0\|_n+\delta)) \tag{3.182}$$

$$\begin{aligned}|\mathcal{N}_2[\rho,\tau]-\mathcal{N}_2[\rho_1,\tau]| &\leq C\varepsilon\delta_0^{-1+1/k}\|\tau\|_n\|\rho-\rho_1\|_n \\ &\leq C\varepsilon^3\delta_0^{-3}\ln^2(\delta_0/h)\|\rho-\rho_1\|_n\end{aligned} \tag{3.183}$$

$$\|\mathcal{J}_4[\rho]-\mathcal{J}_4[\rho_1]\|_n \leq C\varepsilon^2\delta_0^{-4-\frac{1}{k}}\ln^2(\delta_0/h)\|\rho-\rho_1\|_n \tag{3.184}$$

So we have $\mathcal{J}_4$ is a contraction in $\mathcal{B}_4$ for $\delta = 2K\varepsilon^2\delta_0^{-6-1/k}\|\rho_0\|$ and ε small enough. □

Lemma 3.19. *Suppose $\phi_-(x) = d_0(x)\tau_0(x)$, and with sup norm in $\mathcal{D}^-$, $\|\phi_-\|_\infty = \mathcal{O}(\varepsilon^2\delta_0^{-4-1/k})$ and ϕ_- is analytic in $\mathcal{D}^-$, periodic with period h and ψ satisfies*

$$\psi(x) = \mathcal{J}[\psi] = \phi_-(x) + \mathcal{L}_-^{-1}[\mathcal{N}[\psi]] \tag{3.185}$$

Then $\phi_- \in \mathcal{S}_{\mu_1}(\mathcal{D}^-)$ with $\mu_1 = 1$, Let $\mathcal{B}$ be a ball of size $\delta = \mathcal{O}(\delta_0^{-2}\|\phi_-\|_{\mu_1})$ in $\mathcal{S}_{\mu_1}(\mathcal{D}^-)$ centered at ϕ_-, then there is a unique solution $\psi(x)$ in $\mathcal{B}$ and

$$\|\psi-\phi_-\|_{\mu_1} = \mathcal{O}(\varepsilon^2\delta_0^{-4-1/k})\|\phi_-\|_{\mu_1} \tag{3.186}$$

Proof. $\|\phi_-\|_\infty = \mathcal{O}(\varepsilon^2\delta_0^{-4-1/k})$. since $d_0(x)$ is periodic, and $\tau_0(x) = \mathcal{O}(1)$ for $1 \leq \Re(x) \leq 1+h$, we have $\|d_0(x)\|_\infty = \mathcal{O}(\varepsilon^2\delta_0^{-4-1/k})$. Therefore $|e^{-x}\phi_-(x)| = \mathcal{O}(\varepsilon^2\delta_0^{-4-1/k})$. That is , $\|\phi_-\|_{\mu_1} \leq K\varepsilon^2\delta_0^{-4-1/k}$.

Let $\mu = 2$. Recall that $\mathcal{L}_-^{-1} : \mathcal{S}_\mu \mapsto \mathcal{S}_\mu$. By (3.149), for $\psi \in \mathcal{B}$ and $x \in \mathcal{D}^-$, we have that $|e^{-2x}\mathcal{N}[\psi]| \leq C\delta_0^{-1+1/k}\varepsilon\|\psi\|_{\mu_1}^2$, which implies $\|\mathcal{N}[\psi]\|_{\mu_2} \leq C\delta_0^{-1+1/k}\varepsilon\delta^2$.

For $\tilde{\psi} \in \mathcal{B}$

$$|e^{-2x}(\mathcal{N}[\psi]-\mathcal{N}[\tilde{\psi}])| \quad \leq \quad C\varepsilon\|\psi-\tilde{\psi}\|_{\mu_1}(\|\psi\|_{\mu_1}+\|\tilde{\psi}\| \tag{3.187}$$
$$\|\mathcal{N}[\psi]-\mathcal{N}[\tilde{\psi}]\|_{\mu_2} \quad \leq \quad C\varepsilon\delta_0^{-1+1/k}\delta\|\psi-\tilde{\psi}\|_{\mu_1} \tag{3.188}$$

By Lemma 3.3 , we have

$$|e^{-x}|\ |(\mathcal{J}[\psi](x)-\phi_-(x)| \le |e^x \|\mathcal{L}_-^{-1}\| \|\mathcal{N}[\psi]\|_{\mu_2} \le C\delta_0^{-2}(\|\phi_-\|_{\mu_1}+\delta)^2 \quad (3.189)$$

$$|e^{-x}|\ |\mathcal{J}[\psi]-\mathcal{J}[\tilde{\psi}]| \le C_1\delta_0^{-2}\delta|e^x|\|\psi-\tilde{\psi}\|_{\mu_1} \le C\delta_0^{-2}\delta\|\psi-\tilde{\psi}\|_{\mu_1} \quad (3.190)$$

Therefore, for ε small enough, choosing $\delta = 2C\delta_0^{-2}\|\phi_-\|_{\mu_1}^2$, we have $\mathcal{J}$ is a contraction in $\mathcal{B}$. □

Lemma 3.20. *For any ψ analytic in $\mathcal{D}^-$, continuous in $\overline{\mathcal{D}^-}$, and satisfying (3.185) with $\phi = \phi_- = d_0(x)\tau_0(x)$ where $d_0(x) = \mathcal{O}(\varepsilon)$, are analytic in $\mathcal{D}^-$, periodic with period h, then $\tilde{z}_-(x) := z_-(x) + \psi(x)$ is a reparametrization of the unstable manifold $z_-(x)$.*

Proof. $\psi(x)$satisfies (3.185) implies that $\tilde{z}_-$ satisfies (3.16).

$$\lim_{x\to-\infty} \tilde{z}_-(x) = \lim_{x\to-\infty}(z_-(x)+\psi(x)) = \lim_{x\to-\infty} \phi_-(x) + \mathcal{L}_-^{-1}[\mathcal{N}[\psi]] \quad (3.191)$$

By definition of $\mathcal{L}_-^{-1}$, it is clear that the limit of second term in the last equation equals 0, while for the first term,

$$\lim_{x\to-\infty} \phi_-(x) = \lim_{x\to-\infty} d_0(x)\tau_0(x) = 0 \quad (3.192)$$

Therefore $\lim_{x\to-\infty} \tilde{z}_-(x) = 0$. Hence $\tilde{z}_-(x)$ is a reparametrization of the unstable manifold $z_-(x)$. □

Lemma 3.21. *For any $\hat{z}_-(x)$, reparametrization of the unstable manifold that satisfies (3.16) , there exists $p_0(x)$ periodic with period h, such that $\hat{z}_-(x) = z_-(x+p_0(x))$*

Proof. $\hat{z}_-(x)$ is a reparametrization of the unstable manifold. So there exists $q(x)$ such that $z_-(q(x)) = \hat{z}_-(x)$ where $x \in \mathbb{R}$.

$$(z_-(q(x)), z_-(q(x)+h), z_-(q(x)+2h)) = (\hat{z}_-(x), \hat{z}_-(x+h), \hat{z}_-(x+2h)) \tag{3.193}$$

Hence $z_-(q(x)+h) = \hat{z}_-(x+h)$. On the other hand, $\hat{z}_-(x+h) = z_-(q(x+h))$. Since by Lemma 3.1, $z_-(x)$ is monotonically increasing function for $x \leq A$ and ε small enough , we get $q(x+h) = q(x) + h$. Therefore there exists $p_0(x)$, periodic with period h, such that $q(x) - x = p_0(x)$. □

Lemma 3.22. *For any $p_0(x) = \mathcal{O}(\varepsilon^2 \delta_0^{-4-1/k})$ analytic in $\mathcal{D}^-$, periodic with period h, the reparametrization of the unstable manifold $\hat{z}_-(x) = z(x + c_0(x))$ satisfies (3.16). $\psi = \hat{z}_-(x) - z_-(x)$ satisfies (3.185) with $\phi = d_0(x)\tau_0(x)$ where $d_0(x)$ is some analytic function in $\mathcal{D}^-$, periodic with period h.*

Proof. Substituting $\hat{z}_-(x) = z(x + p_0(x))$ into (3.16) we can easily see it is a solution. To show that $\mathcal{L}_-^{-1}[\mathcal{N}[\psi]]$ exists, noticing Lemma 3.1 implies that $|e^x z_-(x)| \leq K$ for $x \in \mathcal{D}^-$, we have

$$|e^{-x}\psi(x)| = |e^{p_0(x)} e^{-x-p_0(x)} z(x + p_0(x)) - e^{-x} z_-(x)| \leq C \tag{3.194}$$

since $p_0(x)$ is periodic. Therefore $\mathcal{N}[\psi] \in \mathcal{S}_\mu$ with $\mu = 2$. So $\mathcal{L}_-^{-1}[\mathcal{N}[\psi]]$ exists and

$$\mathcal{L}_-^{-1}[\mathcal{N}[\psi]] \in \mathcal{S}_\mu(\mathcal{D}^-) \quad \text{where } \mu = 2. \tag{3.195}$$

By Lemma 2.18, there exists $d_\alpha(x)$, $\alpha = -1, 0, 1$, analytic $\mathcal{D}^-$, periodic with period h such that ψ satisfying (3.185) with $\phi = \sum_{\alpha=-1,0,1} d_\alpha(x)\tau_\alpha(x)$. Since $z_-(x)$ satisfies initial conditions (3.24), we have $\hat{z}(x) = z_-(x + p_0(x))$ satisfies $\lim_{x\to-\infty} \frac{\hat{z}(x)}{e^{x+p_0(x)}} = e^{-\varrho\varepsilon \ln h}$ Hence

$$\lim_{x\to-\infty} \frac{z(x) + \sum_{\alpha=-1,0,1} d_\alpha(x)\tau_\alpha(x) + \mathcal{L}_-^{-1}[\mathcal{N}[\psi]](x)}{e^{x+p_0(x)}} = e^{-\varrho\varepsilon \ln h} \tag{3.196}$$

(3.195) and $p_0(x) = \mathcal{O}(\varepsilon)$ implies $\lim_{x\to-\infty} e^{-(x+p_0(x))}\mathcal{L}_-^{-1}[\mathcal{N}[\psi]](x) = 0$. Since $d_\alpha(x)$ are periodic and $\tau_{\pm1}$ grows exponentially as $x \to -\infty$, we get $d_{\pm1}(x) = 0$. $\square$

$\psi(x) = \tilde{z}_-(x) - z_-(x)$ satisfies two equations

$$\psi = \phi_- + \mathcal{L}_-^{-1}[\mathcal{N}[\psi]] \quad \text{where } \phi_- = d_0(x)\tau_0(x) \tag{3.197}$$

$$\psi = \phi + \mathcal{L}^{-1}[\mathcal{N}[\psi]] \quad \text{where } \phi = \sum_{\alpha=-1,0,1} \tilde{b}_\alpha(x)\tau_\alpha(x) \tag{3.198}$$

Consider the above two equations where x restricted to $\mathcal{D}_3 =$, where

$$\mathcal{D}_3 := \mathcal{D}(b_{n_0}, h), \text{ where } \mathcal{D}^{(n)} \text{ contains the real axis} \tag{3.199}$$

It is convenient to define

$$\mathcal{D}_3^- := \{x \in \mathcal{D}^- \ : \ b_{n_0} < \Im x < b_{n_0+1}\} \tag{3.200}$$

Given $b_0(x)$ from decomposition of $\tau(x)$, next we want to show there exist a ψ that satisfies both (3.197) and (3.198) with $\tilde{b}_0(x) = b_0(x)$ as given, and for some d_0, $\tilde{b}_{\pm1}$ to be determined.

It follows from Lemma 3.19 and 3.16 that (3.197) and (3.198) define unique ψ determined by ϕ_- and ϕ respectively, denoted by $\psi_-(x; \phi_-)$ and $\psi(x; \phi)$.

Proposition 3.23. *There exists a constant C such that restricted in $\mathcal{D}_3$ or $\mathcal{D}_3^-$ respectively, we have*

$$\|\psi(x;\phi) - \phi\|_{n_0} \leq C\varepsilon \|\phi\|_{n_0} \tag{3.201}$$

$$\|\psi_-(x;\phi_-) - \phi_-\|_{\mu 1} \leq C\varepsilon\delta_0^{-2}\|\phi_-\|_{\mu 1} \tag{3.202}$$

Proof. (3.201) and (3.202) follows from Lemma 3.16 and 3.19 respectively. ☐

Let $\mathbf{U} := (b_{-1}, d_0, b_1)(x)$,and $\mathbf{F} = (0, b_0(x), 0)$. Then $\mathbf{U}$ satisfies

$$\mathbf{U} = \mathcal{R}[\mathbf{U}] := \mathbf{F} + \mathcal{R}_0[\mathbf{U}], \tag{3.203}$$

where

$$\mathcal{R}_0[\mathbf{U}]_\alpha := \mathcal{R}_1[\mathbf{U}]_\alpha - \mathcal{R}_2[\mathbf{U}]_\alpha \tag{3.204}$$

$$\mathcal{R}_1[\mathbf{U}]_\alpha := (-1)^\alpha \tau_\alpha^{-1} \mathcal{L}_{-,\alpha}^{-1}[\mathcal{N}[\psi_-(x;\phi_-)]] \tag{3.205}$$

$$\mathcal{R}_2[\mathbf{U}]_\alpha := (-1)^\alpha \tau_\alpha^{-1} \mathcal{L}_{\alpha}^{-1}[\mathcal{N}[\psi_-(x;\phi_-)]]. \tag{3.206}$$

Proposition 3.24. *For $g \in \mathcal{S}_\mu(\mathcal{D}_3^-)$ then we have that $\tau_\alpha^{-1}(\mathcal{L}_{-,\alpha}^{-1}[g] - \mathcal{L}_\alpha^{-1}[g])$ is periodic with period h.*

Proof. Since for $g \in \mathcal{S}_\mu(\mathcal{D}_3)$, g must have finite sup norm on $\mathcal{D}_3^-$, hence $g \in \mathcal{S}_0(\mathcal{D}_3)$. It is easy to see that $\Delta[\Delta_{h,-}^{-1}[g] - \Delta_h^{-1}[g]](x) = g(x) - g(x) = 0$. Hence the $\Delta_{h,-}^{-1}[g] - \Delta_h^{-1}[g]$ is periodic with period h. Then proposition follows from definition of $\mathcal{L}_\alpha^{-1}$ and $\mathcal{L}_{-,\alpha}^{-1}$. ☐

Consider vector function space

$$\mathcal{S} := \{\mathbf{V}(x) := (v_{-1}, v_0, v_1)(x) \quad | \quad v_\alpha \text{ analytic in } \mathcal{D}_3,\ \text{continuous in } \overline{\mathcal{D}}_3 \\ v_\alpha(x) \text{ periodic with period } h\} \tag{3.207}$$

Equipped with the norm $\|\cdot\|_v$, where

$$\|\mathbf{V}\|_v := \sum_{\alpha=-1,0,1} \sup_{x\in\mathcal{D}_3} |v_\alpha(x)\tau_\alpha(x)|, \tag{3.208}$$

$\mathcal{S}$ forms a Banach space. Let $\mathcal{B}$ be a ball of size δ in $\mathcal{S}$ centered at $\mathbf{F}$.

Proposition 3.25. *There is a constant C independent of n such that*

$$\sup_{x\in\mathcal{D}_3} |e^{-x} v_0(x)\tau_0(x)| \le C\|\mathbf{V}\|_v \tag{3.209}$$

Proof. Noticing that $\lim_{x\to-\infty} e^{-x}\tau_0(x) = 1+\mathcal{O}(\varepsilon)$, there exists D independent of $\mathbf{V}$, such that $\Re(x) < -D$,

$$|e^{-x}v_0(x)\tau_0(x)| \le 2|v_0(x)| \le 2C_1\|\mathbf{V}\|_v \tag{3.210}$$

The last step follows from v_0 is periodic and τ_0 is $\mathcal{O}(1)$ in some vertical strip of width h. For $-D \le \Re(x) \le A$ it is clear that e^{-x} is bounded, hence $|e^{-x}v_0(x)\tau_0(x)| < C\|\mathbf{V}\|_v$ Therefore, the lemma follows. □

Lemma 3.26. *For $\|\mathbf{F}\|_v = \mathcal{O}(\varepsilon^2)$, $\mathcal{R}$ is a contraction in $\mathcal{B}$ for $\delta = 2\|\mathbf{F}\|_v$ and ε small enough.*

Proof. By Proposition 3.23 and 3.25, noticing $\mathcal{D}_3$ is constant away from $X_s^{\pm}$, for $\mathbf{U}\in\mathcal{B}$ we have

$$\begin{aligned} |e^{-2x}\mathcal{N}[\psi_-(x;\phi_-)]| &\le C_1\varepsilon|e^{-2x}\psi_-(x;\phi_-)|^2 \\ &\le C_1\varepsilon|e^{-2x}\phi_-^2(x)|(1+\varepsilon^2) \\ &\le C_3\varepsilon|e^{-2x}d_0^2(x)\tau_0^2(x)| \le C_4\varepsilon\|\mathbf{U}\|_v^2 \end{aligned} \tag{3.211}$$

For $\tilde{\mathbf{U}} := (b_{-1}, d_0, b_1)(x) \in \mathcal{B}$,

$$\begin{aligned} |e^{-2x}(\mathcal{N}[\psi_-(x;\phi_-)] - \mathcal{N}[\psi_-(x;\tilde{\phi_-})])| &= \varepsilon|e^{-2x}\mathcal{N}_2[d_0 - \tilde{d}_0, d_0](x)| \\ &\leq C_1\varepsilon\|\mathbf{U} - \tilde{\mathbf{U}}\|_v\|\mathbf{U}\|_v \qquad (3.212) \end{aligned}$$

Hence from Lemma 3.3, with $\mu = 2$ we have

$$\begin{aligned} \|\mathcal{L}_{-,\alpha}^{-1}[\mathcal{N}[\psi_-(x;\phi_-)]]\|_\mu &\leq \sum_{\alpha=-1,0,1} \|\mathcal{L}_\alpha^{-1}\|\|\mathcal{N}[\psi_-(x;\phi_-)]\|_\mu \\ &\leq C\|\mathbf{U}\|_v^2 \leq C_1\delta^2 \qquad (3.213) \end{aligned}$$

For $x \in \mathcal{D}(b_n, h)$, we have $\Re(x) < A$, hence

$$|\tau_\alpha(x)\mathcal{R}_1[\mathbf{U}]_\alpha(x)| \leq \sum_{\alpha=-1,0,1} |e^{2x}|\|\mathcal{L}_{-,\alpha}^{-1}[\mathcal{N}[\psi_-(x;\phi_-)]]\|_\mu \leq C\delta^2 \qquad (3.214)$$

$$\begin{aligned} |\tau_\alpha(x)(\mathcal{R}_1[\mathbf{U}]_\alpha(x) - \mathcal{R}_1[\tilde{\mathbf{U}}]_\alpha(x))| &\leq \sum_{\alpha=-1,0,1} \left\|\mathcal{L}_\alpha^{-1}[\mathcal{N}[\psi_-(x;\phi_-)]] - \mathcal{L}_\alpha^{-1}[\mathcal{N}[\psi_-(x;\tilde{\phi_-})]]\right\| \\ &\leq C\delta\|\mathbf{U} - \tilde{\mathbf{U}}\|_v \qquad (3.215) \end{aligned}$$

Similarly,

$$|\mathcal{N}[\psi_-(x;\phi_-)](x)| \leq \varepsilon C_1\|\phi\|^2(1 + C_2\varepsilon) \qquad (3.216)$$

$$|\mathcal{N}[\psi_-(x;\phi_-)](x) - \mathcal{N}[\psi(x;\tilde{\phi}_-)]| \leq C\varepsilon\delta\|\phi - \tilde{\phi}_-\| \qquad (3.217)$$

$$\begin{aligned} |\tau_\alpha(x)\mathcal{R}_2[\mathbf{U}]_\alpha| &= \|\mathcal{L}_\alpha^{-1}[\mathcal{N}[\psi_-(x;\phi_-)]]\| \leq \sum_{\alpha=-1,0,1} \|\mathcal{L}_\alpha^{-1}\|\|\mathcal{N}[\psi_-(x;\phi_-)]\| \\ &\leq C\|\mathbf{U}\|_v^2 \leq C\delta^2 \qquad (3.218) \end{aligned}$$

$$\begin{aligned} |\tau_\alpha(x)(\mathcal{R}_2[\mathbf{U}]_\alpha - \mathcal{R}_2[\tilde{\mathbf{U}}]_\alpha)| &\leq \sum_{\alpha=-1,0,1} \left(\|\mathcal{L}_\alpha^{-1}[\mathcal{N}[\psi_-(x;\phi_-)] - \mathcal{N}[\psi(x;\tilde{\phi}_-)]]\|\right) \\ &\leq C\delta\|\mathbf{U} - \tilde{\mathbf{U}}\|_v \qquad (3.219) \end{aligned}$$

(3.213),(3.215),(3.218) and (3.219) implies that $\mathcal{R}$ is a contraction for small enough δ. $\square$

Let $b_0(x)$ be the periodic function determined by $\tau(x)$ through (3.160),(3.161).

Corollary 3.27. *Given $b_0(x) = \mathcal{O}(\varepsilon^2)$ analytic in $\mathcal{D}^-$ with period of h, there exists unique $\tilde{z}_-$ such that $\psi(x) = \tilde{z}_-(x) - z_-(x)$ satisfies (3.198) with $\tilde{b}_0(x) = b_0(x)$, for some $\tilde{b}_{-pm}(x)\tau_\pm = \mathcal{O}(\varepsilon^2)$ and it satisfies (3.197) with some $d_0(x) = \mathcal{O}(\varepsilon^2)$*

Let $\tilde{z}_-(x) := z_-(x) + \psi(x)$ where $\psi(x)$ is the unique function determined in Corollary 3.27. The estimate in Proposition 3.28 is useful also in matching in chapter 4.

Proposition 3.28. *$\tilde{z}_-(x)$ satisfies*

$$\tilde{z}_-(x) = z_0(x) + (\varepsilon \ln h) z_1(x) + \varepsilon z_2(x) + \mathcal{O}(\varepsilon^2 \delta_0^{-2-1/k} \ln^2(\frac{\delta_0}{h})) \tag{3.220}$$

Proof. $\tilde{z}_-(x) = z_-(x) + \psi(x)$ satisfies

$$\lim_{x \to -\infty} \frac{\hat{z}(x)}{e^{x+\varpi(x)+\varrho}} = 1 \tag{3.221}$$

where $\varpi(x) = -\ln(1 + d_0(x)/e^\varrho)$. It is to be noted that $\varphi(x)$ is analytic in $\mathcal{D}^-$ and periodic with period h.

Let $\tilde{v}(x) = \tilde{z}_-(x) - p(x)$, then $\tilde{v}$ satisfies

$$\tilde{v}(x) = [d_0(x) + \varrho]\tau_{0,0}(x) + \mathcal{L}_0^{-1}[f + \mathcal{L}_1[\tilde{v}] + \mathcal{N}_0[\tilde{v}]](x) \tag{3.222}$$

Follow the proof of Lemma 3.1, noticing $|d_0(x)\tau_{0,0}(x)| \leq C|d_0(x)\tau_0(x)| = \mathcal{O}(\varepsilon^2\delta_0^{-1-1/k})$, we get there exists a unique solution of (3.222) in $\mathcal{O}(\varepsilon^2\delta_0^{-2-1/k}\ln^2(\frac{\delta_0}{h}))$. Hence the lemma follows. $\square$

Now we show that the difference between $\tilde{z}_-(x)$ and $z_+(x)$ is exponentially small in the real domain.

Let $\rho := \psi - \tau$, where $\psi := \tilde{z}_- - z_-$, and $\tau = z_+ - z_-$. By Lemma 3.17, ρ satisfies (3.173) with $\rho_0^{(n)} = \sum_{\alpha=-1,0,1} c_\alpha^{(n)}(x)\tau_\alpha(x)$, where $c_\alpha^{(n)}$ are analytic in $\mathcal{D}^{(n)}$ and periodic with period h, hence

$$c_\alpha^{(n)}(x) = \sum_{m=-\infty}^{\infty} c_{\alpha,m}^{(n)} e^{2\pi mix/h} \tag{3.223}$$

By Corollary 3.27, $c_0^{(0)}(x) = 0$. ρ satisfies the following equation:

$$\rho = \mathcal{J}_e[\rho] := \tilde{\rho}_0 + \mathcal{L}^{-1}[\mathcal{N}_2[\rho, \tau]] \tag{3.224}$$

where

$$\tilde{\rho}_0 := c_{-1}^{(t)}\tau_{-1} + c_1^{(b)}\tau_1 + \sum_\alpha \tau_\alpha d_\alpha[\rho] \tag{3.225}$$

$d_\alpha, \alpha = -1, 0, 1$ are analytic periodic function with period h defined first in $\mathcal{D}^{(t)}, \mathcal{D}^{(b)}, \mathcal{D}^{(0)}$ then analytically extended to $\mathcal{D}_0$.

$$d_{-1}[\rho] := \frac{1}{\tau_{-1}}\{\mathcal{L}_{-1}^{-1}[\mathcal{N}_2[\rho, \tau]] - \mathcal{L}_{-1,(t)}^{-1}[\mathcal{N}_2[\rho, \tau]]\}, \quad x \in \mathcal{D}^{(t)} \tag{3.226}$$

$$d_1[\rho] := \frac{1}{\tau_1}\{\mathcal{L}_1^{-1}[\mathcal{N}_2[\rho, \tau]] - \mathcal{L}_{1,(b)}^{-1}[\mathcal{N}_2[\rho, \tau]]\}, \quad x \in \mathcal{D}^{(b)} \tag{3.227}$$

$$d_0[\rho] := \frac{1}{\tau_0}\{\mathcal{L}_0^{-1}[\mathcal{N}_2[\rho, \tau]] - \mathcal{L}_{0,(0)}^{-1}[\mathcal{N}_2[\rho, \tau]]\}, \quad x \in \mathcal{D}^{(0)} \tag{3.228}$$

For $s \in \mathbb{R}$, define

$$\nu(s) := \sup_{\substack{\Im t = \Im s, \\ t \in \mathcal{D}_0}} \frac{[\tau_{-1} + \tau_1](\frac{\pi i}{k} - \delta_0 i)}{([\tau_{-1} + \tau_1]^2(t) + [\tau_{-1} + \tau_1]^2(t+h) + [\tau_{-1} + \tau_1]^2(t+2h))^{1/2}}, \tag{3.229}$$

$$\|f\|_e = \sup_{x \in \mathcal{D}_0} |\nu(\Im x) f(x)| \tag{3.230}$$

Let $\mathcal{S}_e$ be the space of functions analytic in $\mathcal{D}_0$, continuous up to the boundary and has finite $\|\cdot\|_e$ norm, and $\mathcal{B}_e$ be a ball of size δ in $\mathcal{S}_e$ center at ρ_0 where

$$\rho_0 = \sum_{\alpha=\pm 1} c_\alpha \tau_\alpha, \quad c_{-1} := c_{-1}^{(t)}, \quad c_1 := c_1^{(b)} \tau_1 \tag{3.231}$$

Proposition 3.29. *$\mathcal{J}_e$ is a contraction in $\mathcal{B}_e$ with $\delta = K\varepsilon^2$ for ε small enough where K is a constant.*

Proof. By Lemma 3.1, $\tau(x) = z_-(x) - z_+(x) = \mathcal{O}(\varepsilon^2 \delta_0^{-2-1/k} \ln^2(\delta_0/h))$. From (3.175) and (3.79), for $\rho \in \mathcal{B}_e$, we have

$$|\nu(\Im x)\tau_\alpha \Delta_{h,\alpha} \frac{M_\alpha}{W} \mathcal{N}_2[\rho,\tau]| \le C_1(\|\Delta_{h,+}\| + \|\Delta_{h,-}\|)\varepsilon^3 \|\rho\|_e \le C\varepsilon^2 \|\rho\|_e \tag{3.232}$$

$$|\nu(\Im x)\mathcal{L}_{-1,\alpha}^{-1}[\mathcal{N}_2[\rho,\tau]]| \le C\varepsilon^2 \|\rho\|_e \tag{3.233}$$

Similarly, for $\tilde{\rho} \in \mathcal{B}_e$,

$$|\nu(\Im x)(\mathcal{L}_{-1,\alpha}^{-1}[\mathcal{N}_2[\rho,\tau]] - \mathcal{L}_{-1,\alpha}^{-1}[\mathcal{N}_2[\tilde{\rho},\tau]])| \le C\varepsilon^2 \|\rho - \tilde{\rho}\|_e \tag{3.234}$$

Hence,

$$\|\mathcal{L}^{-1}[\rho]\tau_\alpha\|_e \le C\varepsilon^2 \|\rho\|_e \tag{3.235}$$

From (3.233) we also get

$$|\nu(\Im x)\tau_\alpha d_\alpha[\rho]| \le C\varepsilon^2 \|\rho\|_e \tag{3.236}$$

for $x \in \mathcal{D}^{(t)}, \mathcal{D}^{(0)}, \mathcal{D}^{(b)}$ respectively when $\alpha = -1, 0, 1$. Noticing that by symmetry $d_\alpha(x) = d_{-\alpha}(\bar{x})$, we get

$$\|\tau_\alpha d_\alpha[\rho]\|_e \le C\varepsilon^2 \|\rho\|_e \tag{3.237}$$

(3.235) and (3.237) implies

$$\|\mathcal{J}_e[\rho]\|_e \leq C\varepsilon^2\|\rho\|_e \tag{3.238}$$

Similarly,

$$\|\mathcal{J}_e[\rho]\tau_\alpha - \mathcal{J}_e[\tilde{\rho}]\tau_\alpha\|_e \leq C\varepsilon^2\|\rho-\tilde{\rho}\|_e \tag{3.239}$$

Hence $\mathcal{J}_e$ is a contraction in $\mathcal{B}_e$. □

Corollary 3.30. *$|\rho_0(x)| \leq C\varepsilon^2\delta_0^{-2-1/k}\ln^2(\delta_0/h)$ for $x \in \mathcal{D}_0$.*

Proof. Noticing $|\nu(\Im x)| \leq C$ for $x \in \mathcal{D}_0$, we get this corollary from (3.224) and Proposition 3.29 □

Lemma 3.31.

$$c_{\alpha,0} = \mathcal{O}(\varepsilon^2\delta_0^{-2-1/k}\ln^2(\delta_0/h)\exp\{\delta_0\frac{2\pi}{\varepsilon}\}\exp\{-\frac{2\pi^2}{k\varepsilon}\}) \tag{3.240}$$

$$c_{\alpha,n} = \mathcal{O}(\varepsilon^2\delta_0^{-2-1/k}\ln^2(\delta_0/h)\exp\{\delta_0\frac{(6|n|+2\alpha\frac{|n|}{n})\pi}{\varepsilon}\}\exp\{-\frac{2\pi^2}{k\varepsilon}(3|n|+\alpha\frac{|n|}{n})\}) \tag{3.241}$$

Proof. By Corollary 3.30, $\rho_0(x) = \sum_{\alpha=\pm1} c_\alpha(x)\tau_\alpha(x) = \mathcal{O}(\varepsilon^2\delta_0^{-2-1/k}\ln^2(\delta_0/h))$ as $\varepsilon \to 0^+$ for $x \in \mathcal{D}_0$. Let $r = -2-1/k$, $\rho_0(x) = \mathcal{O}(\varepsilon^2\delta_0^r\ln^2(\delta_0/h))$ for $x \in \mathcal{D}_0$, therefore

$$c_\alpha(x)\tau_\alpha(x) = \alpha\frac{-R_\alpha(x+h)\rho_0(x)+\rho_0(x+h)}{R_1(x+h)-R_{-1}(x+h)} = \mathcal{O}(\varepsilon^2\delta_0^{-r}\ln^2(\delta_0/h)) \tag{3.242}$$

where $R_\alpha(x) = \frac{\tau_\alpha(x+h)}{\tau_\alpha(x)} = \mathcal{O}(1)$. By Fourier expansion

$$\begin{aligned} |c_{\alpha,n}| &= \left|\frac{1}{h}\int_0^h c_\alpha(x+t)e^{2\pi i(x+t)/h}dt\right| \\ &\leq \frac{K}{h}\int_0^h |\varepsilon\delta_0^{-r}e^{-(x+t)\frac{2\pi n+\ln\Lambda_\alpha}{h}}(1+e^{k(x+t)})^{-\frac{k+1}{k}\Omega^{2\alpha}}|\, d|t| \end{aligned} \tag{3.243}$$

where $x \in \mathcal{D}_0$. Using (3.15), we get

$$|c_{\alpha,0}| \leq C\varepsilon^2 \delta_0^{-r} \ln^2(\delta_0/h) \exp\{\delta_0 \frac{2\pi}{\varepsilon}\} \exp\{-\frac{2\pi^2}{k\varepsilon}\} \tag{3.244}$$

$$|c_{\alpha,n}| \leq C\varepsilon^2 \delta_0^{-r} \ln^2(\delta_0/h) \exp\{\delta_0 \frac{(6|n| + 2\alpha\frac{|n|}{n})\pi}{\varepsilon}\} \exp\{-\frac{2\pi^2}{k\varepsilon}(3|n| + \alpha\frac{|n|}{n})\} \tag{3.245}$$

for $n \neq 0$. □

Theorem 3.32. *There exist a parameterization of the unstable manifold $\tilde{z}_-(x)$ such that as $\varepsilon \to 0^+$, $x \in \mathbb{R}$,*

$$\tilde{z}_-(x) - z_+(x) = \rho_0(x) + \mathcal{O}(\varepsilon^2 \rho_0(x)) \tag{3.246}$$

where

$$\rho_0(x) = \sum_{\alpha=\pm 1} c_\alpha(x)\tau_\alpha(x), \tag{3.247}$$

$c_{\pm 1}(x)$ are analytic in $\mathcal{D}_0$ and period with period h, and the Fourier coefficients satisfies

$$c_{\alpha,0} = \mathcal{O}(\varepsilon^2 \delta_0^{-2-1/k} \ln^2(\delta_0/h) \exp\{\delta_0 \frac{2\pi}{\varepsilon}\} \exp\{-\frac{2\pi^2}{k\varepsilon}\}) \tag{3.248}$$

$$c_{\alpha,n} = \mathcal{O}(\varepsilon^2 \delta_0^{-2-1/k} \ln^2(\delta_0/h) \exp\{\delta_0 \frac{(6|n| + 2\alpha\frac{|n|}{n})\pi}{\varepsilon}\} \exp\{-\frac{2\pi^2}{k\varepsilon}(3|n| + \alpha\frac{|n|}{n})\}) \tag{3.249}$$

Proof. Lemma 3.27 implies that there exists $\tilde{z}_-(x)$ such that $\rho(x) := \psi(x) - \tau(x)$ satisfies (3.173)where $\psi(x) := \tilde{z}_-(x) - z_-(x)$,$\tau(x) := z_+(x) - z_-(x)$.

By Lemma 3.1 we have $\tau(x) = \mathcal{O}(\varepsilon^2 \delta_0^{-2-1/k} \ln^2(\delta_0/h))$. By Corollary 3.30, we have $\rho_0(x) = \mathcal{O}(\varepsilon^2 \delta_0^{-2-1/k} \ln^2(\delta_0/h))$. (3.246) follows (3.230), and Lemma 3.29. (3.248) and (3.249) follows from Lemma 3.31. □

CHAPTER 4

INNER PROBLEM ANALYSIS

4.1 Leading Order Equation

As shown earlier in Chapter 3, $z_0(x)$ fails to approximate the unstable separatrix $z_+(x)$ for the values of x close to the singularities $\frac{(2m+1)}{k}\pi i$ of $z_0(x)$. Near the singularity $X_x \equiv \frac{\pi}{k}i$, that is closest [1] to the real axis,we consider the inner equation. We introduce scaled variables

$$s = \frac{x - X_s}{\ln \Lambda}, \qquad w_\varepsilon(s) = \epsilon^{\frac{1}{k}} z(x(s+2)) \tag{4.1}$$

, then the equation (3.28) becomes

$$w_\varepsilon(s+1) = w_\varepsilon(s-2) - w_\varepsilon(s)^{k+1} + \epsilon w_\varepsilon(s+2) \tag{4.2}$$

As $\varepsilon \to 0^+$, the leading order inner equation is given by

$$w_0(s+1) = w_0(s-2) - w_0(s)^{k+1}. \tag{4.3}$$

Let $\tilde{y}(s)$ be a formal series solution that satisfies

$$\tilde{y}(s+1) = \tilde{y}(s-2) - \tilde{y}(s)^{k+1} \tag{4.4}$$

[1] There are actually two closest singularities to the real axis at $\pm\frac{\pi}{k}i$. However, since the solution is real valued for x real, it suffices to consider only the region around $\frac{\pi}{k}i$.

In order that the behavior of $\tilde{y}(s)$ for large s is consistent with leading order outer solution $z_0(x)$, it is necessary for $\tilde{y}(s) \sim \frac{a_{00}}{s^{\frac{1}{k}}}$ for $-\pi < \arg(s) < -\delta_a$ for matching with $z^+(x)$, or $-\pi + \delta_a < \arg(s) < 0$ for matching with $z^-(x)$, where $a_{00} := (\frac{3}{k})^{\frac{1}{k}}$. This is consistent with a dominant balance argument. Continuing the dominant balance argument we find formally

$$\tilde{y}(s) \sim \frac{a_{00}}{s^{\frac{1}{k}}} + \sum_{m=1}^{\infty} \sum_{n=0}^{m} a_{mn} \frac{(\ln s)^n}{s^{m+\frac{1}{k}}} \tag{4.5}$$

for $m \geq 1$, $a_{m,n}$ are uniquely determined in term of a_{10} by substituting the right hand side of equation (4.5) into Equation (4.4) and equate like power of $\ln(s)$ and $\frac{1}{s}$. Only a_{10} remains undetermined at this stage. As shall be seen later, matching with outer solution determines a_{10}. The main result of this section is the following lemma.

Lemma 4.1. *Given a_{10}, there exists $w_0^-(s)$ and $w_0^+(s)$ satisfying the following conditions:*

- *$w_0^{\pm}(s)$ satisfy leading order inner equation (4.3).*
- *$w_0^{\pm}(s)$ are Borel transformable*
- $$w_0^-(s) \sim \frac{a_{00}}{s^{\frac{1}{k}}} + \frac{a_{11} \ln s}{s^{1+\frac{1}{k}}} + \frac{a_{10}}{s^{1+1/k}} \text{ as } s \to \infty, \arg s \in (-2\pi, 0) \tag{4.6}$$
  $$w_0^+(s) \sim \frac{a_{00}}{s^{\frac{1}{k}}} + \frac{a_{11} \ln s}{s^{1+\frac{1}{k}}} + \frac{a_{10}}{s^{1+1/k}} \text{ as } s \to \infty, \arg s \in (-\pi, \pi) \tag{4.7}$$

Proof of Lemma 4.1 is at the end of this section.

Remark 4.1. *In fact, $w_0^{\pm}(s)$ satisfying the three conditions in Lemma 4.1 is unique.*

It is convenient to define

$$f(s) = \sum_{m=0}^{3} \sum_{n=0}^{m} a_{mn} \frac{(\ln s)^n}{s^{m+\frac{1}{k}}}, \tag{4.8}$$

and $y(s) = \tilde{y}(s) - f(s)$. By subtracting out the first few terms of $f(s)$, it is now possible to look for solution $y(s)$ whose Borel transform has nice properties. Equation(4.4) is transformed into

$$\begin{aligned} y(s+1) &= y(s-2) - (k+1)f(s)^k y(s) \\ &\quad - \sum_{l=2}^{k+1} \binom{k+1}{l} y(s)^l f(s)^{k+1-l} - g(s) \end{aligned} \tag{4.9}$$

where

$$g(s) = f(s+1) - f(s-2) - f(s)^{k+1} \tag{4.10}$$

Remark 4.2. *Since a_{mn} are determined to cancel out powers of $\frac{1}{s}$, by construction, we have that $g(s) = o(s^{-4-\frac{1}{k}})$ as $s \to \infty$.*

Formally applying Borel transform $\mathbf{B}$ to Equation (4.9) , we have

$$(e^{-p} - e^{2p})Y(p) + 3\left(1 + \frac{1}{k}\right) 1 * Y = G(p) + \mathcal{M}_1[Y](p) + \mathcal{M}_2[Y](p), \tag{4.11}$$

where $Y(p) := \mathbf{B}[y](p)$, where $\mathbf{B}$ is the Borel transform $G(p) := \mathbf{B}[g](p)$, $F(p) := \mathbf{B}[f](p)$ and

$$\mathcal{M}_1[Y] := -(k+1)(F^{*k} - a_{00}) * Y \tag{4.12}$$

$$\mathcal{M}_2[Y] := -\sum_{l=2}^{k} \binom{k+1}{l} F^{*(k+1-l)} * Y^{*l} + Y^{*(k+1)} \tag{4.13}$$

Remark 4.3. *Since simple calculations shows*

$$\mathcal{L}[p^m (\ln p)^n] = \frac{1}{s^{m+1}} \hat{P}_{mn}(\ln s) \tag{4.14}$$

for some nth order polynomial $\hat{P}_{mn}$, it follows that

$$\mathcal{L}^{-1}[\frac{1}{s^{m+1}} \ln s] = p^m P_{mn}(\ln p) \tag{4.15}$$

for some other nth order polynomial P_{mn}. Hence (4.8) implies that

$$F(p) = \sum_{m=0}^{3} \sum_{n=0}^{m} F_{mn} p^{m-1+\frac{1}{k}} (\ln p)^n, \tag{4.16}$$

where F_{mn} are determined by a_{ij} with $i \leq m$ and $j \leq n$.

$$\begin{aligned} F^{*k} - a_{00} &= \mathcal{L}^{-1}[f^k] - a_{00} \\ &= \mathcal{L}^{-1} \left[\frac{a_{00}}{s} + \left(\sum_{m=2}^{3} \sum_{n=0}^{m} \frac{(\ln s)^n}{s^m} \sum_{|\vec{i}|=m, |\vec{j}|=n} \prod_{q=1}^{k+1} a_{i_q, j_q} \right) \right] - a_{00} \\ &= \sum_{m=1}^{3k} \sum_{n=0}^{m} \tilde{F}_{mn} p^m (\ln p)^n, \end{aligned} \tag{4.17}$$

Similarly, $F^{*(k+1-l)}$ is finite double series:

$$F^{*(k+1-l)} = p^{(-l+1)/k} \sum_{m=0}^{3(k+1-l)} \sum_{n=0}^{m} \tilde{F}_{m,n,k+1-l} p^m \ln^n p \tag{4.18}$$

Remark 4.4. *Applying above results to (4.10), we get $G(p) = p^{\frac{1}{k}} G_1(p, p \ln p)$ where $G_1(p, \xi)$ is analytic in both variables with $G_1(p, \xi) = o(p^3, \xi^3)$.*

Differentiating both sides of (4.11), after some manipulation, we derived

$$\begin{aligned} Y'(p) &- \frac{1}{kp} Y(p) = -\left(\frac{3 + \frac{3}{k} - e^{-p} - 2e^{2p}}{e^{-p} - e^{2p}} + \frac{1}{kp} \right) Y(p) \\ &+ \frac{1}{e^{-p} - e^{2p}} \left(G'(p) + \frac{d}{dp} \mathcal{M}_1[Y](p) + \frac{d}{dp} \mathcal{M}_2[Y](p) \right) \end{aligned} \tag{4.19}$$

Multiply by integrating factor on both sides, integrating and using $Y(p) = o(p^{\frac{1}{k}})$ as $p \to 0$, we have

$$Y = \tilde{\mathcal{P}}[Y] := \tilde{G} + \tilde{\mathcal{U}}_1[Y] + \tilde{\mathcal{U}}_2[Y], \tag{4.20}$$

where

$$\tilde{G}(p) := \frac{G(p)}{e^{-p} - e^{2p}} + p^{\frac{1}{k}} \int_0^p q^{-\frac{1}{k}} A_2(q) G(q) dq \tag{4.21}$$

$$\begin{aligned} \tilde{\mathcal{U}}_1[Y] &:= p^{\frac{1}{k}} \int_0^p A_1(q) q^{-\frac{1}{k}} Y(q) dq + \frac{\mathcal{M}_1[Y](p)}{e^{-p} - e^{2p}} \\ &+ p^{\frac{1}{k}} \int_0^p q^{-\frac{1}{k}} A_2(q) \mathcal{M}_1[Y](q) dq \end{aligned} \tag{4.22}$$

$$\tilde{\mathcal{U}}_2[Y] := \frac{\mathcal{M}_2[Y](p)}{e^{-p} - e^{2p}} + p^{\frac{1}{k}} \int_0^p q^{-\frac{1}{k}} A_2(q) \mathcal{M}_2[Y](q) dq \tag{4.23}$$

$$A_1(p) := -\frac{3 + \frac{3}{k} - e^{-p} - 2e^{2p}}{e^{-p} - e^{2p}} - \frac{1}{kp}, \tag{4.24}$$

$$A_2(p) := \frac{(e^{-p} + 2e^{2p})}{(e^{-p} - e^{2p})^2} + \frac{1}{kp(e^{-p} - e^{2p})}. \tag{4.25}$$

Let

$$\mathcal{D}_B := \{p \in \mathbb{C} \ : \ |p| < \frac{2\pi}{3} - \nu \text{ and } -\frac{\pi}{2} < \arg(p) < \frac{3\pi}{2}\}, \tag{4.26}$$

$$\mathcal{D}_p^+ := \{p \in \mathbb{C} \ : \ -\frac{\pi}{2} + \delta_1 < \arg(p) < \frac{\pi}{2} - \delta_1\} \tag{4.27}$$

and let $\mathcal{D}_p^-$ be the reflection of $\mathcal{D}_p^-$ about the imaginary axis,

$$\mathcal{D}_p^- := \{p \in \mathbb{C} \ : \ \frac{\pi}{2} + \delta_1 < \arg(p) < \frac{3\pi}{2} - \delta_1\} \tag{4.28}$$

where $\nu, \delta_1 > 0$. Let region $\mathcal{D}_p := \mathcal{D}_B \cup \mathcal{D}_p^- \cup \mathcal{D}_p^+$. See Figure 4.1.

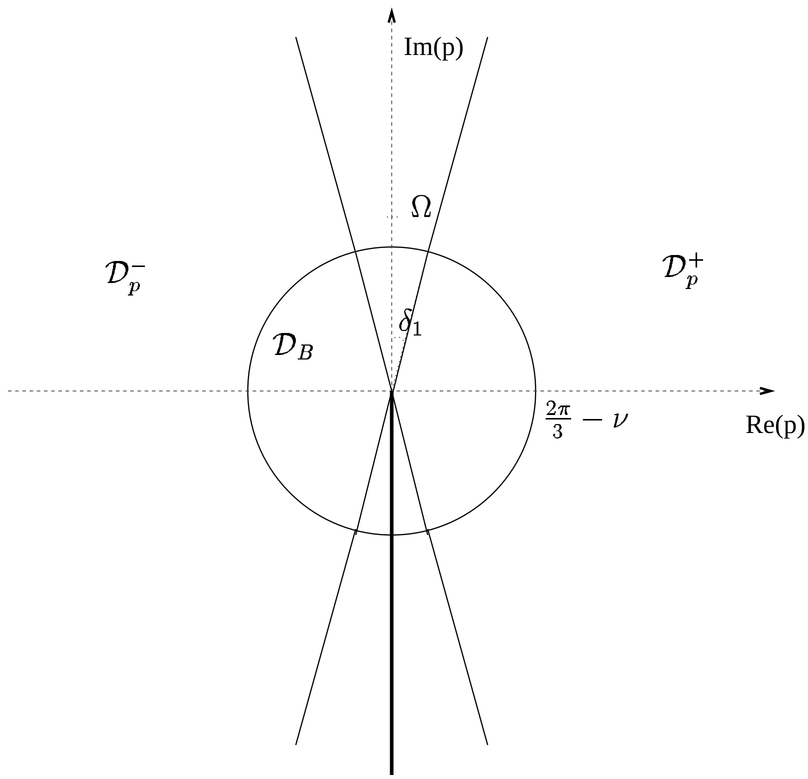

Figure 4.1: Region $\mathcal{D}_p$

$\mathcal{D}_B$ is the disk. $\mathcal{D}_p^+$ and $\mathcal{D}_p^+$ are angular region.

Remark 4.5. *$A_1(p)$ is analytic at $p = 0$ and bounded in $\mathcal{D}_p$,*

$$|A_1(p)| \leq K_1; \tag{4.29}$$

But $A_2(p)$ has a double pole at $p = 0$ and there exists some constant K_2 such that for $p \in \mathcal{D}_p$,

$$\left|p^2 A_2(p)\right| \leq K_2 \tag{4.30}$$

Proposition 4.2. *$\tilde{G}(p) = o\left(p^{2+\frac{1}{k}}\right)$ as $p \to 0$ and polynomially bounded as $p \to \infty$.*

$\tilde{G}(p)$ is polynomially bounded as $p \to \infty$ in complex plane excluding imaginary axis.

Proof. Equation (4.10) and Remark 4.2 implies that $G(p) = o\left(p^{j-2+\frac{1}{k}}\right)$. Noticing that $\frac{1}{e^{-p}-e^{2p}} \sim -\frac{1}{3p}$, $\tilde{G}(p) = o\left(p^{2+\frac{1}{k}}\right)$ as $p \to 0$. follows easily from (4.33). Since $\tilde{G}(p) = p^{\frac{1}{k}} G_1(p, p \ln p)$ where G_1 is a polynomial and $\frac{1}{e^{-p}-e^{2p}}$ is bounded in a complex plane domain that excludes a compact neighborhood of $\frac{2n\pi i}{3}$ the conclusion about bounds at infinity follows.

□

Let $V := p^{-\frac{1}{k}} Y(p)$, then V satisfies

$$V = \mathcal{P}[V] := \breve{G} + \mathcal{U}_1[V] + \mathcal{U}_2[Y], \tag{4.31}$$

where

$$\breve{G} = p^{-\frac{1}{k}} \tilde{G} \qquad \mathcal{U}_i[V] := p^{-\frac{1}{k}} \tilde{\mathcal{U}}_i[p^{\frac{1}{k}} V] \quad i = 1, 2. \tag{4.32}$$

$$\breve{G}(p) := \frac{p^{-\frac{1}{k}} G(p)}{e^{-p} - e^{2p}} + \int_0^p q^{-\frac{1}{k}} A_2(q) G(q) dq \tag{4.33}$$

$$\begin{aligned} \mathcal{U}_1[V] \ &:= \ \int_0^p A_1(q) V(q) dq + \frac{\mathcal{M}_1[p^{\frac{1}{k}} V](p)}{p^{\frac{1}{k}}(e^{-p} - e^{2p})} \\ &+ \ \int_0^p q^{-\frac{1}{k}} A_2(q) \mathcal{M}_1[p^{\frac{1}{k}} V](q) dq \end{aligned} \tag{4.34}$$

$$\mathcal{U}_2[V] \ := \ \frac{\mathcal{M}_2[p^{\frac{1}{k}} V](p)}{p^{\frac{1}{k}}(e^{-p} - e^{2p})} + \int_0^p q^{-\frac{1}{k}} A_2(q) \mathcal{M}_2[p^{\frac{1}{k}} V](q) dq \tag{4.35}$$

Consider the function space $\mathcal{S}$ defined as

$$\mathcal{S}_p := \{W \ : \ W \text{ analytic for } p \in \mathcal{D}_p, \quad \text{continuous in} \overline{\mathcal{D}}_p\} \tag{4.36}$$

Here we consider a norm $\|\cdot\|_b$ introduced in [4], where

$$\|W\|_b = M_0 \sup_{p \in \mathcal{D}_p} (1+|p|^2) e^{-b|p|} |W(p)|, \tag{4.37}$$

and

$$M_0 = \sup_{s \geq 0} \left\{ \frac{2(1+s^2)\{\ln(1+s^2) + s \arctan s\}}{s(s^2+4)} \right\} = 3.76 \ldots.$$

then $\mathcal{S}_p$ equipped with norm $\|\cdot\|_b$ is a Banach space.

Let $\mathcal{B}_\delta$ be a ball of radius δ in $\mathcal{S}_p$, $\mathcal{B}_\delta := \{W \in \mathcal{S}_p \ : \ \|W\|_b \leq \delta\}$. We have the following lemma.

Lemma 4.3. *The norm $\|\cdot\|_b$ has the following property.*

- *1. For any $w \in \mathcal{S}_p$ with the property $\sup_{p \in \mathcal{D}_p} |p^{-r} e^{-\rho|p|} w(p)| \leq K$ for some $\rho > 0$, some $r > 0$ and constant K, then we have $\|w\|_b \leq C \frac{2^r}{b^r} \Gamma(r+1)$, for b large enough, where C depends on w, but not b, ρ and r.*

- *2. For any $T_1, T_2 \in \mathcal{S}_p$, we have $\|T_1 * T_2\|_b \leq \|T_1\|_b \|T_2\|_b$.*

- *3. For any $T \in \mathcal{S}_p$, we have $\|T * (p^{\alpha-1} \ln^n p)\|_b \leq C \frac{\Gamma(\alpha-\nu)}{b^\alpha} \|T\|_b$ where $\alpha > 0$, $n \geq 0$ is an integer, $0 < \nu < \alpha$.*

- *4. For any $T_1, T_2 \in \mathcal{S}_p$, $r_1, r_2 \geq 0$, we have*

$$\|p^{-(r_1+r_2)} [(p^{r_1} T_1) * (p^{r_2} T_2)]\|_b \leq \| |T_1| * |T_2| \|_b \tag{4.38}$$

- *5.For any $m \in \mathbb{N}$,$r > 0$, we have $\|p^r(\ln p)^m\|_b \leq C\frac{2^{r-\nu}}{b^{r-\nu}}\Gamma(r-\nu+1)$ for some small $\nu > 0$.*

Proof. Proof of (1). For $|p| < \delta < 1$, we have

$$\begin{aligned}\sup_{|p|<\delta}(1+|p|^2)e^{-b|p|}|w(p)| &\leq 2K(b-\rho)^{-r}\sup_{|p|<\delta}e^{-(b-\rho)|p|}|p|^r(b-\rho)^r \\ &\leq \frac{2K\Gamma(r+1)}{(b-\rho)^r} \leq C\frac{2^r}{b^r}\Gamma(r+1) \end{aligned} \tag{4.39}$$

where the last step follows from the maximum value of function $e^{-x}x^r$ occurs at $x = r$. For $|p| \geq \delta$, $(1+|p|^2)e^{-b|p|}|w(p)|$ is exponentially decaying for b large.

Proof of (2).

$$\begin{aligned}|[T_1 * T_2](p)| &\leq \left|\int_0^p T_1(p-t)T_2(t)dt\right| \\ &\leq e^{b|p|}\|T_1\|_b\|T_2\|_b M_0^{-2}\int_0^p \frac{d\tilde{t}}{\left(1+(|p|-\tilde{t})^2\right)\left(1+\tilde{t}^2\right)} \\ &\leq \frac{e^{b|p|}}{M_0(1+|p|^2)}\|T_1\|_b\|T_2\|_b \end{aligned} \tag{4.40}$$

Therefore we have (2).

Proof of (3). First prove for $n = 0$ case.

$$\begin{aligned} &|(1+|p|^2)e^{-b|p|}\int_0^p T(p-t)(t)^{\alpha-1}dt| \\ \leq\ & C_1(1+|p|^2)\|T\|_b\int_0^{|p|}\frac{e^{-b|t|}}{1+|p-t|^2}|t|^{\alpha-1}d|t| \end{aligned} \tag{4.41}$$

For $|p| \leq 1$,$\frac{1}{1+|p-t|^2} \leq 1$

$$\begin{aligned}(1+|p|^2)e^{-b|p|}|T * p^{\alpha-1}| &\leq C_2\|T\|_b b^{-\alpha}\int_0^\infty e^{-bs}(bs)^{\alpha-1}d(bs) \\ &\leq C_2\frac{\Gamma(\alpha)}{b^\alpha}\|T\|_b \end{aligned} \tag{4.42}$$

For $|p| > 1$, by Watson's Lemma, as $b \to \infty$,

$$\int_0^\infty \frac{e^{-b|t|}}{1+|p-t|^2}|t|^{\alpha-1}d|t| \sim \frac{1}{1+|p|^2}b^{-\alpha}\Gamma(\alpha) \tag{4.43}$$

Therefore there exists constant C, such that

$$\|T * p^{\alpha-1}\|_b \leq C\frac{\Gamma(\alpha)}{b^\alpha}\|T\|_b \tag{4.44}$$

If $n > 0$, noticing for $|p| < 1$, $|p^{-\alpha+1}\ln(p)| \leq K_1 p^{-\alpha+1-\nu}$ and for $|p| > 1$, $|p^{-\alpha+1}\ln(p)| \leq K_1 p^{-\alpha+1+\nu}$ we get (3).

Proof of (4). Noticing that

$$\begin{aligned} & \left|p^{-(r_1+r_2)}\int_0^p t^{r_1}T_1(t)(p-t)^{r_2}T_2(p-t)dt\right| \\ \leq & \int_0^p \left|\frac{t}{p}\right|^{r_1}|T_1(t)|\left|\frac{p-t}{p}\right|^{r_2}|T_2(p-t)|\,|dt| \\ \leq & \int_0^p |T_1(t)|\,|T_2(p-t)|\,|dt| = |T_1| * |T_2|, \end{aligned} \tag{4.45}$$

we have that (4.38) holds.

Proof of (5). Since there exist a constant K and ν, $0 < \nu < r$ such that

$$\sup_{p\in\mathcal{D}_p}|p^{-r+\nu}e^{-\nu|p|}p^r(\ln p)^m| \leq K \tag{4.46}$$

By 1 of Lemma 4.3, we have

$$\|p^r(\ln p)^m\|_b \leq C\frac{2^{r-\nu+1}}{b^{r-\nu+1}}\Gamma(r-\nu+1) \tag{4.47}$$

□

Proposition 4.4. *For $V_1, V_2 \in \mathcal{B}_\delta$ and $l \geq 2$, we have that*

$$\|V_1^{*l} - V_2^{*l}\|_b \leq C\delta^{l-1}\|V_1 - V_2\| \tag{4.48}$$

Proof.

$$|V_1^{*l}(p) - V_2^{*l}(p)| \leq |V_1(p) - V_2(p)| * \sum_{n=0}^{l-1} |V_1^{*(l-1-n)} * V_2^{*n}| \tag{4.49}$$

Therefore

$$\begin{aligned} \|V_1^{*l}(p) - V_2^{*l}(p)\|_b &\leq \|V_1(p) - V_2(p)\|_b \sum_{n=0}^{l-1} \|V_1^{*(l-1-n)} * V_2^{*n}\|_b \\ &\leq C\delta^{l-1}\|V_1(p) - V_2(p)\|_b \end{aligned} \tag{4.50}$$

□

Proposition 4.5. *For $V \in \mathcal{S}_p$, $2 \leq l \leq k$, we have*

$$\left\| p^{-1-\frac{1}{k}}[(p^{\frac{1}{k}}V)^{*l} * p^{(1-l)/k}] \right\|_b \leq \frac{C}{b^{1+(l-1)/k}} \|V\|_b^l \tag{4.51}$$

where C is a constant independent of V and b.

$$\|p^{-2-1/k}[p \ln^n(p) * p^{1/k}V]\|_b \leq \frac{C}{b^{1-\nu}} \|V\|_b \tag{4.52}$$

where $n = 0, 1,$ and $0 < \nu < 1$.

Proof.

$$\begin{aligned} I(p) &:= \left| p^{-1-\frac{1}{k}}[(p^{\frac{1}{k}}V)^{*l} * p^{(1-l)/k} \right| \leq \left| p^{\frac{l-1}{k}-1}[|V|^{*l} * |p^{(1-l)/k}|\right| \\ &\leq \int_0^1 s^{\frac{1-l}{k}}\{|V|^{*l}(|p|(1-s))\}ds \\ &\leq \|V\|_b^l e^{b|p|} \int_0^1 \frac{e^{-b|p|s}}{1+|p|^2(1-s)^2} s^{\frac{1-l}{k}} ds \end{aligned} \tag{4.53}$$

If $|p| < 1$, then for large enough b, we have

$$I(p) \leq 2\|V\|_b^l \frac{e^{b|p|}}{1+|p|^2} \int_0^1 s^{\frac{1-l}{k}} ds \leq 2\|V\|_b^l \frac{e^{b|p|}}{1+|p|^2} \tag{4.54}$$

By Watson's lemma, for large $|p|$, we have

$$\int_0^1 \frac{e^{-b|p|s}}{1+|p|^2(1-s)^2} s^{\frac{1-l}{k}} ds \sim \frac{1}{1+|p|^2} b^{-1+\frac{l-1}{k}} \Gamma(1-\frac{l-1}{k}) \tag{4.55}$$

Therefore we get (4.51). Proof of (4.52) is similar. □

Proposition 4.6. *For $V \in \mathcal{S}_p$, $2 \le l \le k$, we have*

$$\left\| \int_0^p t^{-2-\frac{1}{k}} [(t^{\frac{1}{k}} V)^{*l} * t^{(1-l)/k}] dt \right\|_b \le C \|V\|_b^l \tag{4.56}$$

$$\left\| \int_0^p t^{-2-\frac{1}{k}} [(t^{\frac{1}{k}} V)^{*l} * t^{(1-l)/k+1} \ln t] dt \right\|_b \le C \|V\|_b^l \tag{4.57}$$

where C is a constant independent of V and b.

Proof.

$$\begin{aligned} & |\int_0^p t^{-2-\frac{1}{k}} [(t^{\frac{1}{k}} V)^{*l} * t^{(1-l)/k}] dt| \\ \le & \left(\int_0^{\tilde{p}} + \int_{\tilde{p}}^p \right) |t|^{-2+(l-1)/k} [|V|^{*l} * |t|^{(1-l)/k}] d|t|, \end{aligned} \tag{4.58}$$

where $\tilde{p} = \frac{p}{\max\{1,|p|\}}$.

The first integral can be bounded by

$$\begin{aligned} & \left| \int_0^{\tilde{p}} |t|^{-2+(l-1)/k} [t(t^{-1} V^{*l}) * t^{(1-l)/k}] dt \right| \\ \le & C_1 \sup_{|t| \le \tilde{p}, t \in \mathcal{D}_p} \left\{ \left| \frac{V^{*l}(t)}{t} \right| \right\} \left| \int_0^{\tilde{p}} |t|^{-2+(l-1)/k} [|t| * |t|^{(1-l)/k}] dt \right| \\ \le & C_2 \sup_{|t| \le \tilde{p}, t \in \mathcal{D}_p} \left\{ \left| \frac{V^{*l}(t)}{t} \right| \right\} \end{aligned} \tag{4.59}$$

But

$$
\begin{aligned}
\left|\frac{V^{*l}(t)}{t}\right| &\leq C_2 \frac{1}{|t|}\int_0^{|t|} |V^{*(l-1)}||V(t-s)|\,|ds| \\
&\leq \frac{1}{|t|}\int_0^{|t|} \frac{\|V^{*(l-1)}\|_b\|V\|_b e^{|t|b}}{(1+|s|^2)(1+|t-s|^2)} d|s| \\
&\leq 2\|V\|_b^l \frac{e^{b|t|}}{1+|t|^2} \\
&\leq 2\|V\|_b^l \frac{e^{b|p|}}{1+|t|^2} \quad \text{for large enough } b
\end{aligned} \tag{4.60}
$$

For $|p| \leq 1$, the second integral is zero, otherwise

$$
\begin{aligned}
\left|\int_{\tilde{p}}^{p} |t|^{-2+(l-1)/k}[|V|^{*l} * |t|^{(1-l)/k}]\,|dt|\right| &\leq \int_0^{|p|} |V|^{*l} * |t|^{(1-l)/k}\,|dt| \\
&\leq 1 * |V|^{*l} * |t|^{(1-l)/k} \\
&\leq \frac{C}{b^{(k+1-l)/k}}\|V\|_b^l \frac{e^{b|p|}}{1+|p|^2}
\end{aligned} \tag{4.61}
$$

Combining (4.60) and (4.61) we get (4.56). □

Lemma 4.7. *For $V, \tilde{V} \in \mathcal{B}_\delta$, for some small $\nu > 0$, we have that*

$$\|\mathcal{U}_1[V]\|_b \leq \frac{C}{b^{1-\nu}}\|V\|_b \tag{4.62}$$

$$\|\mathcal{U}_1[V] - \mathcal{U}_1[\tilde{V}]\|_b \leq \frac{C}{b^{1-\nu}}\|V - \tilde{V}\|_b \tag{4.63}$$

where C is a constant.The proof of (4.57) is similar.

Proof. By Remark 4.5, the norm of the first term in $\mathcal{U}_1[V]$ defined in (4.62) can be bounded by $K_1\|1 * |V|\|_b = \frac{C}{b}\|V\|_b$.

Noticing that $\frac{p}{e^{-p}-e^{2p}}$ is bounded by some constant K for $p \in \mathcal{D}_p$, using remark 4.3 and Lemma 4.3 for some small $\nu > 0$, we get

$$\begin{aligned}
\left\| \frac{\mathcal{M}_1[p^{1/k}V]}{p^{\frac{1}{k}}(e^{-p}-e^{2p})} \right\|_b &\leq K \sum_{m=1}^{3} \sum_{n=0}^{m} |\tilde{F}_{mn}| \; \| |p|^{-1-\frac{1}{k}} [|p^m (\ln p)^n| * |p|^{\frac{1}{k}} |V|] \|_b \\
&\leq \sum_{m=1}^{3} \sum_{n=0}^{m} |\tilde{F}_{mn}| \; \| |p|^{-1} [|p^m (\ln p)^n| * |V|] \|_b \\
&\leq \frac{C}{b^{1-\nu}} \|V\|_b \qquad (4.64)
\end{aligned}$$

Using 4.30 of Remark 4.5, we get

$$\begin{aligned}
&\left| \int_0^p q^{-\frac{1}{k}} \left(\frac{(e^{-q}+2e^{2q})}{(e^{-q}-e^{2q})^2} + \frac{1/(kq)}{e^{-q}-e^{2q}} \right) \mathcal{M}_1[p^{\frac{1}{k}}V](q) dq \right| \\
\leq\;& K_2 \int_0^{|p|} \sum_{m=1}^{3} \sum_{n=0}^{m} |\tilde{F}_{mn}| \; \left| q^{-2-\frac{1}{k}} [q^m \ln^n q * q^{\frac{1}{k}} V](q) \right| d|q| \\
\leq\;& K_2 \int_0^{|p|} \sum_{m=1}^{3} \sum_{n=0}^{m} \tilde{F}_{mn} [\; |q^{m-2} \ln^n q| * |V| \;](q) dq \\
=\;& \frac{C_1}{b^{1-\nu}} \|V\|_b + K_2 \sum_{m=1}^{2} \sum_{n=1}^{m} \tilde{F}_{m+1,n} \left(1 * |p^{m-1} \ln^n p| * |V| \right) \\
=\;& \frac{C_1}{b^{1-\nu}} \|V\|_b + K_2 \left(\sum_{m=1}^{2} \sum_{n=0}^{m+1} \tilde{F}_{m+1,n} (1 * |p^{m-1} \ln^n p|) \right) * |V| \\
\leq\;& \frac{C_1}{b^{1-\nu}} \|V\|_b + C \|V\|_b \sum_{m=1}^{2} \sum_{n=0}^{m+1} \frac{2^{m+\nu} \Gamma(m+\nu)}{b^{m+1-\nu}} \leq C \frac{\|V\|_b}{b^{1-\nu}} \qquad (4.65)
\end{aligned}$$

for some small $\nu > 0$. Therefore, for large b we have (4.62). Noticing that $\mathcal{U}_1$ is linear, we get (4.62) implies (4.63). □

Lemma 4.8. *For $V, \tilde{V} \in \mathcal{B}_\delta$, we have that*

$$\|\mathcal{U}_2[V]\|_b \leq C\delta^2 \qquad (4.66)$$

$$\|\mathcal{U}_2[V]-\mathcal{U}_2[\tilde{V}]\|_b \le C\delta\|V-\tilde{V}\|_b \tag{4.67}$$

Proof. For $2 \le l \le k$, by Remark 4.3, Lemma 4.3 Proposition 4.5 , we have that

$$\begin{aligned}
&\|p^{-1-\frac{1}{k}}[F^{*(k+1-l)} * (p^{\frac{1}{k}}V)^{*l}]\|_b \\
\le\ & C_1\|p^{-1-\frac{1}{k}}[(p^{\frac{1}{k}}V)^{*l} * p^{(1-l)/k}]\|_b + \sum_{m=1}^{3(k+1-l)}\sum_{n=0}^{m}|\tilde{F}_{m,n,k+1-l}|\ \||p^{m-1}\ln^n p| * |V|^{*l}\|_b \\
\le\ & C_2\|V\|_b^l + \sum_{m=1}^{3(k+1-l)}\sum_{n=0}^{m}|\tilde{F}_{m,n,k+1-l}|\frac{C_{mn}}{b^{m+\nu}}\|V^{*l}\|_b \\
\le\ & C\|V\|_b^l
\end{aligned} \tag{4.68}$$

Therefore, by Equation (4.13), Remark 4.3, we have that

$$\begin{aligned}
&\left\|\frac{\mathcal{M}_2[p^{\frac{1}{k}}V](p)}{p^{\frac{1}{k}}(e^{-p}-e^{2p})}\right\|_b \\
\le\ & K_2\sum_{l=2}^{k}\binom{k+1}{l}\|p^{-1-\frac{1}{k}}[|F^{*(k+1-l)}| * |p^{\frac{1}{k}}V|\ ]\|_b + K_2\|p^{-1-\frac{1}{k}}|p^{\frac{1}{k}}V|^{*(k+1)}\|_b \\
\le\ & C\|V\|_b^2
\end{aligned} \tag{4.69}$$

for some constant C.

$$\begin{aligned}
&|\int_0^p q^{-2-\frac{1}{k}}[F^{*(k+1-l)} * q^{\frac{1}{k}}V]dq| \\
\le\ & \sum_{m=0}^{3}\sum_{n=0}^{m}|\tilde{F}_{m,n,k+1-l}|\left\|\int_0^{|p|}|q|^{-2-\frac{1}{k}}[|q^{m-(l-1)/k}\ln^n q| * |q^{\frac{1}{k}}V|^{*l}](q)d|q|\right\|_b \\
\le\ & C_1\|\int_0^{|p|}|q|^{-2-\frac{1}{k}}[|q^{-(l-1)/k}| * |q^{\frac{1}{k}}V|^{*l}](q)d|q|\|_b \\
&+C_2\sum_{m=1}^{3}\sum_{n=0}^{m}|\tilde{F}_{m,n,k+1-l}|\left\|1 * |q^{m-1}\ln^n q| * |V|^{*l}\right\|_b \\
\le\ & C_3\|V\|_b^l + C_4\sum_{m=1}^{3}\sum_{n=0}^{m}\frac{2^{m-1+\nu}}{b^{m+\nu}}\Gamma(m-1+\nu)\|V\|_b^l \\
\le\ & C\|V\|_b^l
\end{aligned} \tag{4.70}$$

$$
\begin{aligned}
& \left\| \int_0^p q^{-\frac{1}{k}} \left(\frac{(e^{-q}+2e^{2q})}{(e^{-q}-e^{2q})^2} + \frac{1/(kq)}{e^{-q}-e^{2q}} \right) \mathcal{M}_2[p^{\frac{1}{k}}V](q)dq \right\|_b \\
\leq \; & K_2 \int_0^{|p|} \sum_{l=2}^{k} \binom{k+1}{l} q^{-1-\frac{1}{k}} [|F^{*(k+1-l)}| * |q^{\frac{1}{k}}V|] d|q| \\
& + K_2 \int_0^{|p|} q^{-1-\frac{1}{k}} |q^{\frac{1}{k}}V|^{*(k+1)} d|q| \\
\leq \; & C\|V\|_b^2 + \frac{C}{b}\|V\|_b^{k+1} \leq C\|V\|_b^2 \qquad (4.71)
\end{aligned}
$$

Combining (4.69) and (4.71), we get (4.66). Using Proposition 4.4 the proof of (4.67) is similar.

□

Lemma 4.9. *For large enough b, equation (4.31) has a unique solution $V(p)$ in $\mathcal{B}_\delta$.*

Proof. Using definition of $\breve{G}$ (4.32), Proposition 4.2, we have that

$$
\|\breve{G}\|_b = \mathcal{O}(b^{-1+\frac{1}{k}-\mu}) \qquad (4.72)
$$

for small $\mu > 0$, and in particular, $\breve{G} \in \mathcal{B}_\delta$. This conclusion together with (4.62) and (4.66) imply for $V \in \mathcal{B}_\delta$,

$$
\|\mathcal{P}[V]\|_b \leq C(\frac{1}{b}+\delta)\delta \qquad (4.73)
$$

By (4.63) and (4.67) we have for $V_1, V_2 \in \mathcal{B}_\delta$,

$$
\|\mathcal{P}[V_1]-\mathcal{P}[V_2]\|_b \leq C(\frac{1}{b}+\delta)\|V_1-V_2\|_b \qquad (4.74)
$$

Thus $\mathcal{P}$ is a contraction in $\mathcal{B}_\delta$, and the lemma follows.

□

Lemma 4.10. *$V(p) = o(p)$ as $p \to 0$, where $V(p)$ is the unique solution of Equation(4.31) found in Lemma 4.9.*

Proof. Consider the linear operator $\hat{\mathcal{P}} : \breve{\mathcal{S}}_p \to \breve{\mathcal{S}}_p$ deduced from $\mathcal{P}$ by replacing all but one unknown function V's, in the nonlinear convolution terms of $\mathcal{P}[V]$ by the unique solution of (4.31) found in Lemma 4.9, which is denoted here by V_s. For example, V^{*l} is replace by $V * V_s^{*(l-1)}$. Hence in terms of $\tilde{\mathcal{R}}$, Equation (4.31) becomes

$$V = \hat{\mathcal{P}}[V] = \mathcal{U}[V] + \breve{G}, \tag{4.75}$$

where $\mathcal{U}[V] = \mathcal{U}_1[V] + \hat{\mathcal{U}}_2[V]$, and $\hat{\mathcal{U}}_2$ is deduced from $\mathcal{U}_2$ by the same procedure mentioned above, where $\mathcal{U}_2[V]$, $\breve{G}$ are defined in (4.32).

Noticing that convergence in $\|\cdot\|_b$ implies convergence in $\|\cdot\|_{\infty,\varepsilon}$, where

$$\|V\|_{\infty,\varepsilon} := \sup_{p\in\mathcal{D}_\varepsilon} |V(p)|, \tag{4.76}$$

we get $\|V_s\|_{\infty,\varepsilon} \to 0$ as $\varepsilon \to 0$. Using the property of the uniform norm $\|\cdot\|$ on $\tilde{\mathcal{D}}_\varepsilon$ $\|T_1 * T_2\|_{u,\varepsilon} \leq \varepsilon \|T_1\|_{u,\varepsilon} \|T_2\|_{u,\varepsilon}$. Similar with proof of Lemma 4.9, we have $\|\mathcal{U}[V]\|_{\infty,\varepsilon} \leq C\varepsilon\|V\|_{\infty,\varepsilon}$,where C is a constant independent of ε, therefore $\|\mathcal{U}\| \to 0$ as $\varepsilon \to 0$. Equation (4.75) has a unique solution in $\breve{\mathcal{B}}_\delta$ for small enough δ and ε. Since V_s is a solution in $\breve{\mathcal{B}}_\delta$, it is the unique solution. We conclude that the linear operator $(\mathcal{I}_d - \mathcal{U})$ is invertible, where $\mathcal{I}_d$ is the identity map. Further, we have that

$$\|V\|_{\infty,\varepsilon} \leq \frac{1}{1 - \|\mathcal{U}\|} \|\breve{G}\|_{\infty,\varepsilon} \leq 2\|\breve{G}\|_{\infty,\varepsilon} \tag{4.77}$$

for sufficiently small ε, and Equation (4.75) has a unique solution in $\breve{\mathcal{B}}_\delta$ for small enough δ and ε. Since V_s is a solution in $\breve{\mathcal{B}}_\delta$, it is the unique solution. (4.10), (4.32) imply

$$\lim_{\varepsilon\to 0}\frac{\|\breve{G}\|_{\infty,\varepsilon}}{\varepsilon}=0 \tag{4.78}$$

Since this argument can be repeated for any ε on a ball of radius $|p|=\varepsilon$, $V=V_s=o(p)$.

□

Proof. (Proof of Lemma 4.1) $\|V(p)\|_b$ is finite implies that $V(p)$ is exponentially bounded and in $\mathcal{D}_p$, which in turn implies that $Y(p)=p^{1/k}V(p)$ is Laplace transformable. By Lemma 4.10, $Y(p)=p^{1/k}V(p)=o(p^{1+1/k})$. By Watson's Lemma, we have

$$\mathbf{L}_\theta[Y](s)=o(s^{-2-1/k}) \tag{4.79}$$

where $\theta\in\left(-\frac{\pi}{2}+\delta_1,\frac{\pi}{2}-\delta_1\right)$ or $\theta\in\left(\frac{\pi}{2}+\delta_1,\frac{3}{2}\pi-\delta_1\right)$. Since δ_1 is arbitrary, (4.79) is true for $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\theta\in\left(\frac{\pi}{2},\frac{3}{2}\pi\right)$.

Let $w_0^-(s):=f(s)+\mathbf{L}_{\theta^-}Y(p)$ where $\theta^-\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $f(s)$ is defined in (4.8) and $w_0^+(s):=f(s)+\mathbf{L}_{\theta^+}Y(p)$ where $\theta^+\in\left(\frac{\pi}{2},\frac{3}{2}\pi\right)$. Hence $w_0^\pm(s)$ satisfies the asymptotic relation (4.6) and (4.6) respectively.

The proof that $w_0^\pm(x)$ satisfies the difference equation (4.3) follows from that $\mathbf{L}_\theta[e^{-p}Y]=y(x+1)$ and $\mathbf{L}_\theta[Y*Y]=y_0^2$ as follows from using Fubini's theorem in the convolution, knowing a priori that the functions are integrable. □

4.2 Ramified Analytic Structure at Zero

$V(p)$ is not analytic at $p = 0$ since the inhomogeneous term $\breve{G}$ in (4.31) involves powers of $\in p$. Next we will state the ramified analytic structure of $V(p)$ at $p = 0$.

Let $\mathcal{S}_1$ be the function space of complex function of two variables p and ξ, analytic in the open set

$$\mathcal{D}_1 := \{(p, \xi) \in \mathbb{C}^2 \mid |p| < \rho_1 \text{ and } |\xi| < \rho_2 \} \tag{4.80}$$

where we choose $\rho_1 = \rho_2/3$,and continuous on $\overline{\mathcal{S}}_1$. Define function space $\mathcal{S}_2$ as

$$\mathcal{S}_2 := \{ \ v(p) \mid \exists w \in \mathcal{S}_1 \text{ s.t. } v(p) = w(p, p \ln p)\}. \tag{4.81}$$

Remark 4.6. *It is clear that for $v \in \mathcal{S}_2$, there is a unique $w \in \mathcal{S}_1$ such that $v(p) = w(p, p \ln p)$ for $p \in \mathcal{D}_1$.*

Consider projection

$$\mathcal{T} : \mathcal{S}_2 \to \mathcal{S}_1 \quad v \to w, \tag{4.82}$$

where v, w satisfies $v(p) = w(p, p \ln p)$. Definition of $\mathcal{S}_1$, $\mathcal{S}_2$ and Proposition 4.6 imply that $\mathcal{T}$ is bijection.

Define norm $\| \cdot \|_u$ of v as

$$\|v\|_u := \sup_{|p|<\rho_1, |\xi|<\rho_2} |\mathcal{T}[v](p, \xi)| \tag{4.83}$$

recall that we choose $\rho_1 = \rho_2/3$. $\mathcal{S}_2$ equipped with sup norm $\|v\|_u$ forms a Banach space. Let $\mathcal{B}_2$ be a ball of size δ_2 centered at 0 in $\mathcal{S}_2$.

Let $W(p) := pV(p)$. Rewrite Equation (4.31) in terms of W, we get

$$W(p) = \mathcal{Q}[W](p) := p \ \breve{G}(p) + p \ \mathcal{U}_1[p^{-1}W](p) + p \ \mathcal{U}_2[p^{-1}W](p) \tag{4.84}$$

Define $\mathcal{B}_2$ as

$$\mathcal{B}_2 := \{v \in \mathcal{S}_2 \ : \|v\|_u \leq \delta, v(p) = o(p) \text{ as } p \to 0\} \tag{4.85}$$

Lemma 4.11. *Let $V(p)$ be the unique solution of equation (4.31) found in Lemma 4.9, then $pV(p) \in \mathcal{S}_2$.*

We will prove the lemma by showing that $\mathcal{Q}$ is contraction in $\mathcal{B}_2$ at the end of this section. The following propositions and lemmas are preparation for proof of the above lemma.

Proposition 4.12. *If $v_1, v_2 \in \mathcal{S}_2$, then $v_1 * v_2 \in \mathcal{S}_2$ and $\|v_1 * v_2\|_u \leq d_1 \|v_1\|_u \ \|v_2\|_u$.*

Proof. Considering $|p| < \rho_1$ then

$$\begin{aligned}
& [v_1 * v_2](p) = \int_0^p v_1(t), v_2(p-t)dt \\
= & \int_0^p \mathcal{T}[v_1](t, t\ln t)\mathcal{T}[v_2](p-t, (p-t)\ln(p-t))dt \\
= & \ p\int_0^1 \mathcal{T}[v_1](ps, (ps)\ln(ps))\mathcal{T}[v_2](p(1-s), p(1-s)\ln(p(1-s)))dt \\
= & \ p\int_0^1 \mathcal{T}[v_1](ps, ps\ln s + s\xi) \\
& \cdot\mathcal{T}[v_2](p(1-s), p(1-s)\ln(1-s) + \xi(1-s))ds
\end{aligned} \tag{4.86}$$

where $\xi = p\ln p$.

For $t = sp$, $0 \leq s \leq 1$, we want to show that $|ps\ln s + s\xi| \leq \rho_2$. Since $|s\ln s|$ has maximal value of $1/e$ for $0 \leq s \leq 1$, so for $0 \leq s \leq 1/2$, we have $|ps\ln s + s\xi| \leq \rho_1/e + \rho_2/2 < \rho_2$ for $d_1 < d_2/2$. For $1/2 \leq s \leq 1$, noticing that then $|ln(s)| = |ln(1-(1-s))| < 2(1-s)$, we get

$$|ps\ln s + s\xi| < \rho_1 2(1-s) + \rho_2 s < \rho_2. \tag{4.87}$$

The right hand side of the last equation of (4.86), viewed as function of two variable (p,ξ), is an integral of analytic function in p and ξ, integrating with respect to an $\mathbb{L}^1$ measure, therefore is analytic in p and ξ for $(p,\xi)\in\mathcal{D}_1$. Thus we showed $v_1 * v_2 \in \mathcal{S}_2$.

$$\begin{aligned}\|v_1 * v_2\|_u &= \sup_{|p|<\rho_1,|\xi|<\rho_2} |\mathcal{T}[v_1 * v_2](p,\xi)| \le \sup_{|p|<\rho_1,|\xi|<\rho_2} \left|\int_0^p \|v_1\|_u \|v_2\|_u dt\right| \\ &\le \rho_1 \|v_1\|_u \|v_2\|_u \end{aligned} \tag{4.88}$$

□

Proposition 4.13. *For $v_1, v_2 \in \mathcal{S}_2$, $r_1 > -1$ and $r_2 > 0$,we have*

$$p^{-(r_1+r_2)}[p^{r_1} v_1 * p^{-1+r_2} v_2] \in \mathcal{S}_2 \tag{4.89}$$

and

$$\|p^{-(r_1+r_2)}[p^{r_1} v_1 * p^{-1+r_2} v_2]\|_u \le C\|v_1\|_u \cdot \|v_2\|_u. \tag{4.90}$$

More generally, for $n \in \mathbb{N}$ $n \ge 2$, $v_i \in \mathcal{S}_2$ where $i = 1\cdots n$, $r_1 \ge 0$ and $r_i > 0$ where $i = 2\cdots n$, we have

$$p^{-\sum_{i=1}^n r_i}[p^{r_1} v_1 * p^{-1+r_2} v_2 * \cdots * p^{-1+r_n} v_n] \in \mathcal{S}_2 \tag{4.91}$$

and

$$\|p^{-\sum_{i=1}^n r_i}[p^{r_1} v_1 * p^{-1+r_2} v_2 * \cdots * p^{-1+r_n} v_n]\|_u \le C\prod_{i=1}^n \|v_i\|_u, \tag{4.92}$$

where C only depends on r_i, not on v_i, $i = 1, 2, \cdots, n$.

Proof.

$$\begin{aligned} & p^{-(r_1+r_2)}[p^{r_1}v_1 * p^{-1+r_2}v_2] \\ = \;& p^{-(r_1+r_2)}\int_0^p t^{r_1}v_1(t)(p-t)^{-1+r_2}v_(p-t)dt \\ = \;& \int_0^1 s^{r_1}(1-s)^{-1+r_2}\mathcal{T}[v_1](ps, ps\ln s + s\xi) \\ & \cdot\mathcal{T}[v_2](p(1-s), p(1-s)\ln(1-s) + \xi(1-s))ds \end{aligned} \tag{4.93}$$

where $\xi = p\ln p$. Similar with proof of (4.12), we have that $p^{-(r_1+r_2)}[p^{r_1}v_1 * p^{-1+r_2}v_2] \in \mathcal{S}_2$. For $|p| < \rho_1$ and $|p\ln p| < \rho_2$,

$$\begin{aligned} |p^{-(r_1+r_2)}[p^{r_1}v_1 * p^{-1+r_2}v_2]| &\leq \|v_1\|_u \cdot \|v_2\|_u |p|^{-r_1-r_2}\int_0^p |t|^{r_1}|p-t|^{-1+r_2}d|t| \\ &\leq \|v_1\|_u \cdot \|v_2\|_u \int_0^1 s^{r_1}(1-s)^{-1+r_2}ds \\ &\leq C\|v_1\|_u \cdot \|v_2\|_u \end{aligned} \tag{4.94}$$

Noticing that if we define $\tilde{v} := p^{-(r_1+r_2)}[p^{r_1}v_1 * p^{-1+r_2}v_2]$ then $\tilde{v} \in \mathcal{S}_2$ and $p^{r_1}v_1 * p^{-1+r_2}v_2$ can be written as $p^{r_1+r_2}\tilde{v_1}$, we can easily get(4.91) and (4.92) by induction on n.

□

Proposition 4.14. *For $m, n \in \mathbb{Z}$, and $m \geq -2$, $n \geq 0$, $m+n \geq 0$ we have*

$$\int_0^p q^m (q\ln q)^n dq = \sum_{j=0}^n b_{m,n,j}p^{m+n+1-j}(p\ln p)^j \tag{4.95}$$

where $|b_{m,n,j}| \leq 1$ for $m \geq -1$, and $|b_{m,n,j}| \leq 2$ for $m = -2$.

Proof. It is trivial to verify (4.95) holds for $n = 0$ case. Noticing

$$\int_0^p q^{\tilde{m}} \ln q \; dq = \frac{1}{\tilde{m}+1}p^{\tilde{m}+1}\ln p - \frac{1}{(\tilde{m}+1)^2}p^{\tilde{m}+1}. \tag{4.96}$$

and

$$\int_0^p q^{\tilde{m}} \ln^n q \; dq = \frac{1}{\tilde{m}+1} p^{\tilde{m}+1} \ln^n p - \frac{n}{\tilde{m}+1} \int_0^p q^{\tilde{m}} \ln^{n-1} q \; dq, \tag{4.97}$$

by induction on n we get that

$$\int_0^p q^{\tilde{m}} (\ln q)^n \; dq = \sum_{j=0}^{n} \tilde{b}_{\tilde{m},n,j} p^{\tilde{m}+1} \ln^j p. \tag{4.98}$$

and $|\tilde{b}_{\tilde{m},n,j}| \leq 1$ when $n \leq \tilde{m}+1$; and $|\tilde{b}_{\tilde{m},n,j}| \leq \frac{n}{\tilde{m}+1} \leq 2$ when $n = \tilde{m}+2$. (4.95) is get by let $\tilde{m} = m+n$ and $\tilde{b}_{\tilde{m},n,j} = b_{m,n,j}$. □

Lemma 4.15. *If $v \in \mathcal{S}_2$ and $v(p) = o(p)$ as $p \to 0$, has the expansion*

$$\mathcal{T}[v](p,\xi) = \sum_{m=2}^{\infty} \sum_{n=0}^{\infty} w_{m,n} p^m \xi^n \tag{4.99}$$

then we have $p \int_0^p q^{-2} v(q) dq \in \mathcal{S}_2$ and

$$\|p \int_0^p q^{-2} v(q) dq\|_u \leq C \ln \rho_1 \|v\|_u. \tag{4.100}$$

where C is a constant.

Proof. Let $w(p,\xi) = \mathcal{T}[v](p,\xi)$, $v(p) = o(p)$ as $p \to 0$ and $w(p,\xi)$ is analytic in $(p,\xi) \in \mathcal{D}_1$ imply that $w(p,\xi)$ has the expansion

$$w(p,\xi) = \sum_{m\geq 0, n \geq 0, m+n \geq 2} w_{m,n} p^m \xi^n \tag{4.101}$$

for any small positive γ there exist a C K, such that $|w_{m,n}| \le K(\rho_1-\gamma)^{-(m+n)}$, where $0<\gamma<\rho_1$. Using Proposition 4.14, we have

$$\begin{aligned} & p\int_0^p q^{-2}v(q)\ dq \\ = & \sum_{m\ge 0, n\ge 0, m+n\ge 2} w_{m,n}\int_0^p q^{m-2}(q\ln q)^n dq \\ = & \sum_{m\ge 0, n\ge 0, m+n\ge 2} w_{m,n}\sum_{j=0}^{n} b_{m,n,j}p^{m+n-j-1}(p\ln p)^j \\ = & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p^i(p\ln p)^j\left(\sum_{m=0}^{i+j} w_{m,i+j-m}b_{m-2,i+j-m,j}\right) \end{aligned} \tag{4.102}$$

Let $\tilde{w}_{i,j} := \sum_{m=1}^{i} w_{m,i+j-m}b_{m-2,i+j-m,j}$, then

$$|\tilde{w}_{i,j}| \le \sum_{m=1}^{i+j} K(\rho_1-\gamma)^{-(i+j)} \le 2K(i+j)(\rho_2-\gamma)^{-(i+j)} \tag{4.103}$$

Therefore the righthand side of 4.102 converges in both p, and $\xi = p\ln p$ for $(p,\xi)\in \mathcal{D}_1$.Hence we have $p\int_0^p q^{-2}v(q)dq \in \mathcal{S}_2$.

To show (4.100), break up the integral ,

$$p\int_0^p q^{-2}v(q)dq = p\int_0^{p/2} q^{-2}v(q)dq + p\int_{p/2}^{p} q^{-2}v(q)dq \tag{4.104}$$

The absolute value of the second term can be bounded by $C\|v\|_u$ where C is a constant. To bound the first term, let $w(p,\xi) = \mathcal{T}[v](p,\xi)$ is analytic in $\mathcal{D}_1$, for $p<\rho_1/2$, $|\xi|<\rho_2/2$, by Cauchy's integral,

$$|w_p(p,\xi)| = \left|\frac{1}{2\pi i}\int_{|p|=\rho_1}\frac{w(q,\xi)}{(q-p)^2}dq\right| \le 4\rho_1^{-1}\|v\|_u \tag{4.105}$$

Similarly, we have the following bounds on partial derivative of w,

$$|w_\xi| \le \frac{C\|v\|_u}{\rho_2} \quad |w_{pp}| \le \frac{C\|v\|_u}{\rho_1^2}; \quad |w_{p\xi}| \le \frac{C\|v\|_u}{\rho_1\rho_2} \quad |w_{\xi\xi}| \le \frac{C\|v\|_u}{\rho_2^2} \tag{4.106}$$

where C is a constant. Hence using integration by parts, we have for $p < \rho_1/2$, $\xi < \rho_2/2$, and ρ_1, ρ_2 small enough.

$$\begin{aligned}
& p\int_0^{\frac{p}{2}} q^{-2}v(q)dq \\
= \quad & \frac{v(p)}{2} + p\int_0^{\frac{p}{2}} \frac{1}{q}\{w_p + (1+\ln q)w_\xi\}dq \\
= \quad & \frac{v(p)}{2} + \frac{p}{2}\ln\frac{p}{2}\{w_p + w_\xi\}|_{\frac{p}{2}} + p\int_0^{\frac{p}{2}} \frac{\ln q}{q} w_\xi dq \\
& -p\int_0^{\frac{p}{2}} \ln q\{w_{pp} + (2+\ln q)w_{\xi q} + (1+\ln q)w_{\xi\xi}\}dq \\
= \quad & \frac{v(p)}{2} + +\frac{p}{2}\ln\frac{p}{2}\{w_p + w_\xi\}|_{\frac{p}{2}} + \frac{1}{2}\frac{p}{2}\ln^2\frac{p}{2}\ w_\xi|_{\frac{p}{2}} - p\int_0^{\frac{p}{2}} \ln^2 q\{w_{\xi p} + (1+\ln q)w_{\xi\xi}\}dq \\
& -p\int_0^{\frac{p}{2}} \ln q\{w_{pp} + (2+\ln q)w_{\xi q} + (1+\ln q)w_{\xi\xi}\}dq \qquad (4.107)
\end{aligned}$$

Estimate absolute value of each term in (4.107) using (4.105) and (4.106), noticing $\rho_1 = \rho_2/3$, for example

$$\begin{aligned}
|p\int_0^{\frac{p}{2}} \ln q w_{pp} dq| \quad & \leq \quad C_1\frac{\|v\|_u}{\rho_1^2}|p|\int_0^{\frac{|p|}{2}} |\ln q|d|q| \\
& \leq \quad C_1\frac{\|v\|_u}{\rho_1^2}(|p^2\ln p| + |p|^2) \leq C\|v\|_u \qquad (4.108)
\end{aligned}$$

Similar calculation shows that all other terms in (4.107) can be bounded by $C\|v\|_u$ except for $\frac{p}{4}\ln^2(\frac{p}{2})\ w_\xi|_{\frac{p}{2}} \leq C\ln\rho_1\|v\|_u$.

Therefore the first term in (4.104) can be bounded by $C\ln\rho_1\|v\|_u$. Hence the (4.100) holds. □

Lemma 4.16. *If $v \in \mathcal{S}_2$ satisfies $v(0) = 0$, then we have $p\int_0^p q^{-1}v(q)dq \in \mathcal{S}_2$ and*

$$\|p\int_0^p q^{-1}v(q)dq\|_u \leq C\rho_1\|v\|_u. \qquad (4.109)$$

Proof. $v \in \mathcal{S} - 2$ and $v(0) = 0$ implies $pv(p)$ satisfies condition of Lemma 4.15, therefore

$$p \int_0^p q^{-1} v(q) dq = p \int_0^p q^{-1} \{qv(q)\} dq \in \mathcal{S}_2 \tag{4.110}$$

To show (4.109), similar with proof of Lemma 4.15, we break up the integral

$$p \int_0^p q^{-1} v(q) dq = p \left(\int_0^{\frac{p}{2}} + \int_{\frac{p}{2}}^p \right) q^{-1} v(q) dq \tag{4.111}$$

The second term can be bounded by $C\|v\|$, using integration by parts, (4.106), (4.105), we get (4.109). ☐

Proposition 4.17. *For $W, \tilde{W} \in \mathcal{B}_2$, and ρ_1 small enough, we have $p\mathcal{U}_1[p^{-1}W] \in \mathcal{B}_2$ and there exist a constant C , such that*

$$\|p\mathcal{U}_1[p^{-1}W] - p\mathcal{U}_1[p^{-1}\tilde{W}]\|_u \leq C\rho_1 \|W - \tilde{W}\|_u \tag{4.112}$$

Proof. By (4.29) of Remark 4.5, $A_1(p)$ is analytic and bounded in $\mathcal{D}_p$, hence $\|A_1\|_u \leq K_1$. By Lemma 4.16, $p\int_0^p q^{-1} A_1(q) W(q)\ dq \in \mathcal{S}_2$ and

$$\|p \int_0^p A_1(q) q^{-1} W(q)\ dq\|_u \leq \rho_1 \|A_1 W\|_u \leq \rho_1 K_1 \|W\|_u \tag{4.113}$$

For the second term in $\mathcal{U}_1[p^{\frac{1}{k}-1}W]$, by (4.12) we have

$$|p^{-1-\frac{1}{k}} \mathcal{M}_1[p^{\frac{1}{k}-1}W](p)| = p^{-1-\frac{1}{k}}[p\{(k+1)p^{-1}(F^{*k} - a_0 0)\} * p^{\frac{1}{k}-1}W] \tag{4.114}$$

Remark 4.3 shows that $(k+1)p^{-1}(F^{*k} - a_0 0) \in \mathcal{S}_2$, by Proposition 4.13 with $r_1 = 1, r_2 = \frac{1}{k}$, we get

$$p^{-1-\frac{1}{k}} \mathcal{M}_1[p^{\frac{1}{k}-1}W] \in \mathcal{S}_2 \tag{4.115}$$

Since $\frac{p}{e^{-p}-e^{2p}}$ is analytic and bounded in $\mathcal{D}_p$, we get

$$\frac{\mathcal{M}_1[p^{\frac{1}{k}-1}W]}{p^{\frac{1}{k}}(e^{-p}-e^{2p})} \in \mathcal{S}_2 \tag{4.116}$$

and

$$\begin{aligned}
\left\| \frac{\mathcal{M}_1[p^{\frac{1}{k}-1}W]}{p^{\frac{1}{k}}(e^{-p}-e^{2p})} \right\| &\leq C_1 \| p^{-1-\frac{1}{k}} \mathcal{M}_1[p^{\frac{1}{k}-1}W](p) \|_u \\
&\leq C_2 \rho_1 \| (k+1) p^{-1} (F^{*k} - a_0 0) \|_u \| W \|_u \\
&\leq C \rho_1 \| W \|_u
\end{aligned} \tag{4.117}$$

Using Remark 4.29, $q^2 A_2(q)$ is analytic and bounded in $\mathcal{D}_p$, by (4.116) and Lemma 4.16 we have

$$\begin{aligned}
& p \int_0^p q^{-\frac{1}{k}} A_2(q) \mathcal{M}_1[q^{\frac{1}{k}-1}W](q)\ dq \\
= \ & p \int_0^p q^{-1} \{ A_2(q) q^{-1-\frac{1}{k}} \mathcal{M}_1[q^{\frac{1}{k}-1}W](q) \}\ dq \in \mathcal{S}_2
\end{aligned} \tag{4.118}$$

Furthermore,

$$\begin{aligned}
& \left\| p \int_0^p q^{-\frac{1}{k}} A_2(q) \mathcal{M}_1[q^{\frac{1}{k}-1}W](q)\ dq \right\|_u \\
\leq \ & K_2 \left\| p \int_0^p q^{-1} \{ q^{-1-\frac{1}{k}} \mathcal{M}_1[q^{\frac{1}{k}-1}W](q) \}\ dq \right\|_u \\
\leq \ & C_1 \rho_1 \left\| \frac{\mathcal{M}_1[q^{\frac{1}{k}-1}W](q)}{q^{\frac{1}{k}}(e^{-q}-e^{2q})} \right\|_u \leq C \rho_1 \| W \|_u
\end{aligned} \tag{4.119}$$

It is easy to check that if $W = o(p)$ then $p\mathcal{U}_1[p^{-1}] = o(p)$ as $p \to 0$. Together with (4.113), (4.117) and (4.119), we get$p\mathcal{U}_1[p^{-1}W] \in \mathcal{B}_2$ for ρ_1 small enough. Since $\mathcal{U}_1$ is linear, (4.112) follows immediately.

□

Proposition 4.18. *For $W, \tilde{W} \in \mathcal{B}_2$, and ρ_1 small enough, we have $p\mathcal{U}_2[p^{-1}W] \in \mathcal{S}_2$;and there exist a constant C , such that*

$$\|p\mathcal{U}_2[p^{-1}W]\|_u \leq C \ln \rho_1 \|W\|_u^2 \tag{4.120}$$

$$\|p\mathcal{U}_2[p^{-1}W] - p\mathcal{U}_2[p^{-1}\tilde{W}]\|_u \leq C \ln \rho_1 \delta_2 \|W - \tilde{W}\|_u \tag{4.121}$$

Proof. By (4.18), $p^{-(-l+1)/k} F^{*(k+1-l)} \in \mathcal{S}_2$. Using Lemma 4.13, with $r_1 = (l-1)/k, r_2 = r_3 = \cdots = r_{l+1} = \frac{1}{k}$, we get that

$$p^{-\frac{1}{k}}\{F^{*k+1-l} * (p^{\frac{1}{k}-1}W)^{*l}\} \in \mathcal{S}_2 \tag{4.122}$$

and

$$\begin{aligned} &\|p^{-\frac{1}{k}}\{F^{*k+1-l} * (p^{\frac{1}{k}-1}W)^{*l}\}\|_u \\ \leq\ & C_1 \|W\|_u^l \|p^{-(-l+1)/k} F^{*(k+1-l)}\|_u \leq C\|W\|_u^l \end{aligned} \tag{4.123}$$

Again by Lemma 4.13 with $r_1 = -1 + \frac{1}{k}, r_2 = r_3 = \cdots = r_{k+1} = \frac{1}{k}$, we get $p^{-\frac{1}{k}}(p^{\frac{1}{k}-1}W)^{*(k+1)} \in \mathcal{S}_2$ and

$$\|p^{-\frac{1}{k}}(p^{\frac{1}{k}-1}W)^{*(k+1)}\|_u \leq C\|W\|_u^{k+1} \tag{4.124}$$

Therefore by (4.13), we get

$$p^{-\frac{1}{k}} \mathcal{M}_2[p^{\frac{1}{k}-1}W](p) \in \mathcal{S}_2 \tag{4.125}$$

and

$$\|p^{-\frac{1}{k}} \mathcal{M}_2[p^{\frac{1}{k}-1}W]\|_u \leq C \ln \rho_1 \|W\|_u^2 \tag{4.126}$$

It follows from (4.125), (4.126), Remark 4.5 and Lemma 4.15 that

$$p\int_0^p q^{-\frac{1}{k}}A_2(q)\mathcal{M}_2[p^{\frac{1}{k}-1}W](q)dq \in \mathcal{S}_2 \tag{4.127}$$

and

$$\left\| p\int_0^p q^{-\frac{1}{k}}A_2(q)\mathcal{M}_2[p^{\frac{1}{k}-1}W](q)dq \right\| \leq C\ln\rho_1\|W\|_u^2 \tag{4.128}$$

Using

$$W^{*l}-\tilde{W}^{*l}=(W-\tilde{W})\sum_{j=0}^{l-1}W^{*(l-1-j)}\tilde{W}^{*j}, \tag{4.129}$$

similar with above we get (4.121). □

Proof. [Proof of Lemma 4.11]

Then $\mathcal{Q}$ is contraction in $\mathcal{B}_2$. To show this, first we will show that for $v\in\mathcal{B}_2$, we have $\mathcal{T}[v]\in\mathcal{B}_2$.

Remark 4.4 implies that $p^{-\frac{1}{k}}G(p)\in\mathcal{S}_2$, by Proposition 4.13, we have that $p\,\breve{G}(p)\in\mathcal{S}_2$.

Then by Proposition 4.17 and 4.18 , we get that

$$\|\mathcal{Q}[W]\|_u \leq C(\delta_1+\rho_2)\delta_2 \tag{4.130}$$

and for $W_1, W_2\in\mathcal{B}_2$,

$$\|\mathcal{Q}[W_1]-\mathcal{Q}[W_2]\|_u \leq C(\delta_1+\rho_2)\|W_1-W_2\|_u \tag{4.131}$$

Therefore $\mathcal{Q}$ is a contraction in $\mathcal{B}_2$. The uniqueness of solution of (4.31) and (4.84) implies that $pV(p)=W(p)\in\mathcal{S}_2$. □

4.3 Singularity Analysis of Leading Order Equation

To determine the behavior of Y near the singular points $P = \frac{2\pi}{3}i$, here we use a method analogous to the one used before by Costin [1]. For $\nu \in (0, |P|/3)$, define a truncation of $Y(p)$,

$$H(p) := \begin{cases} Y(p) & \text{if } p \in \mathcal{D}_p \text{ and } |p| < \frac{2}{3}\pi - \nu \\ 0 & \text{otherwise.} \end{cases} \tag{4.132}$$

Let $h(P-p) := Y(p) - H(p)$. Let $z := P - p$ For $z \in \mathcal{D}_v$, where

$$\mathcal{D}_\nu := \{z \in \mathbb{C} \mid |z| < \nu, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\} \tag{4.133}$$

In terms of $h(z)$, equation (4.11) reads:

$$\begin{aligned} &(e^{-(P-z)} - e^{2(P-z)})h(z) - 3(1+\frac{1}{k})\int_{\nu i}^{z} h(s)ds \\ = \; & G_1(z) + \mathcal{M}_1[H+h](P-z) + \mathcal{M}_2[H+h](P-z), \end{aligned} \tag{4.134}$$

where

$$G_1(z) := G(P-z) - 3(1+\frac{1}{k})\int_0^{P-\nu i} H(s)ds \tag{4.135}$$

Lemma 4.19. *For $l \in \mathbb{N}$, $H^{*l}(P-z)$ can be extended from a function of z from $(0, P-\nu]$ to a holomorphic function in the region $\mathcal{D}_\nu$ with continuous boundary values on the circle $|z| = \nu$. The same holds for $D_l(P-z)$. and $G_1(z)$.*

Proof. Prove by induction on l. For $l = 1$, by definition, $H(p)$ extend to zero analytic function. Suppose that $H^{*l}(p)$ extends to analytic function in $P + \mathcal{D}_\nu$, first consider purely imaginary p , $|p| \in (|P| - \nu, |P| + \nu)$.

$H(p)$ is analytic in an ν neighborhood of $[\nu, |P| - 2\nu]i$, therefore so is $H^{*l}(p)$.

$$\begin{aligned} H^{*(l+1)}(p) &= \int_0^{p-(P-\nu i)} H(t)H^{*l}(p-t)dt \\ &= \int_0^{p-(P-\nu i)} H(t)H^{*l}(p-t)dt + \int_{p-(P-\nu i)}^{P-\nu i} H(p-s)H^{*k}(s)ds \end{aligned} \tag{4.136}$$

For $t \in (0, p - (P - \nu i))$, we have $p - t \in (P - \nu i, P + \nu i)$, therefore $H^{*l}(p - t)$ as a function of p is analytic in a neighborhood of $(P - \nu i, P + \nu i)$ by induction assumption. $H(p - (P - \nu i))$ is analytic in $(P - \nu i, P + \nu i)$ Hence the first integral in (4.136) is analytic in $(P - \nu i, P + \nu i)$. For the second integral, since $s \in (p - (P - \nu i), P - \nu i)$ we have $p - s \in (0, P - \nu i)$, therefore $H(p - s)$ is analytic in $(P - \nu i, P + \nu i)$. Since $p - (P - \nu i) \in (0, 2\nu i)$, so $H^{*l}(p - (P - \nu i))$ is analytic in $(P - \nu i, P + \nu i)$. Therefore the second integral in (4.136) is also analytic in a neighborhood of $(P - \nu i, P + \nu i)$. For $p = |p|e^{i\theta}$,

$$H^{*(l+1)}(p) = \int_0^{P-\nu e^{i\theta}} H(s)H^{*k}(p-s)ds \tag{4.137}$$

Breaking up the integral into $(0, p - (P - \nu e^{i\theta}))$ and $(p - (P - \nu e^{i\theta}), P - \nu e^{i\theta})$. Following the same argument, we get the lemma.

This completes the induction.

□

Lemma 4.20. *For $p \in (P - \nu i, P)$, and $\nu < |P|/4$, we have $h * h(p) = 0$ and analytically extends to the zero analytic function in $\mathcal{D}_v$.*

Proof.

$$h * h(p) = \int_{P-\nu i}^{p} h(s)h(p-s)ds = \int_{0}^{p-P+\nu i} h(s)h(p-s)ds = 0 \tag{4.138}$$

The last step is because $p - P + \nu i \in (0, \nu i)$ so $|p - P + \nu i| < \nu < |P| - \nu$ hence $h(s) = 0$ for s on the integration path. □

Lemma 4.21. *Equation(4.134) can be written as*

$$\begin{aligned} & zh(z) - (1 + \frac{1}{k})e^{-2P} \int_{\nu i}^{z} h(s)ds \\ = \; & G_2(z) - D_1(z)h(z) - \int_{\nu i}^{z} D_2(s-z)h(s)ds \end{aligned} \tag{4.139}$$

where

$$D_1(z) \;:=\; -\frac{e^{-2P}}{3}(e^{-(P-z)} - e^{2(P-z)} - 3z) \tag{4.140}$$

$$\begin{aligned} D_2(z) \;:=\; & \frac{e^{-2P}}{3} \sum_{l=2}^{k} l[F^{*(k+1-l)} * H^{*(l-1)}](z) + \frac{e^{-2P}}{3}(k+1)H^{*k}(z) \\ & -(k+1)(F^{*k} - a_{00})(z) \end{aligned} \tag{4.141}$$

and

$$G_2(z) = \frac{1}{3}e^{-2P} \left\{ G_1(z) + \mathcal{M}_1[H](P-z) + \mathcal{M}_2[H](P-z) \right\} \tag{4.142}$$

Proof. Noticing Lemma 4.19 and Lemma 4.20, equation (4.139) is straight forward calculation from (4.134). □

Remark 4.7. $D_1(z) = z^2\tilde{D}_1(z)$ *where* $\tilde{D}_1$ *is an analytic function.*

Let $Q(z) := \int_{\nu i}^{z} h(v)dv$. Using integration by parts, Equation(4.139) is transformed into a linear first order differential equation:

$$\begin{aligned} zQ'(z) + \beta Q(z) &= D_1(z)Q'(z) + D_2(\nu i - z)Q(z) \\ &+ \int_{\nu i}^{z} Q(v)D_2'(v-z)dv + G_2(z) \end{aligned} \tag{4.143}$$

where

$$\beta := -(1+\frac{1}{k})e^{-2P}. \tag{4.144}$$

Multiplying both sides by the integration factor for the operator on the left hand side of the above equation, integrating both sides and using integration by parts, we get

$$Q(z) = \mathcal{R}_1[Q](z) := Q_0(z) + G_3(z) + \mathcal{R}[Q](z) \tag{4.145}$$

where

$$Q_0(z) := S_0 z^{-\beta}, \quad S_0 := \int_{\nu i}^{0} v^{\beta-1}G_2(v)dv \tag{4.146}$$

$$G_3(z) := z^{-\beta}\int_0^z v^{\beta-1}G_2(v)dv \tag{4.147}$$

$$\begin{aligned} \mathcal{R}[Q](z) \;:=\; & z^{-1}D_1(z)Q(z) - z^{-\beta}\int_{\nu i}^{z} \frac{d}{ds}\left[s^{\beta-1}D_1(s)\right]Q(s)ds \\ & +z^{-\beta}\int_{\nu i}^{z} s^{\beta-1}D_2(\nu i - s)Q(s)ds \\ & +z^{-\beta}\int_{\nu i}^{z} s^{\beta-1}\left(\int_{\nu i}^{s} D_2(w-s)Q(w)dw\right)ds \end{aligned} \tag{4.148}$$

Proposition 4.22. *For $\alpha > 0$, and $l, n \in \mathbb{N}$, we have*

$$\int_0^{P-z} (P-z-s)^{\alpha}\ln^n(P-z-s)H^{*l}(s)ds \tag{4.149}$$

is analytic in $z \in \mathcal{D}_\nu$.

Proof.

$$\begin{aligned}
&\int_0^{P-z} (P-z-s)^\alpha \ln^n(P-z-s) H^{*l}(s) ds \\
= &\int_0^{P-\nu i} (P-z-s)^\alpha \ln^n(P-z-s) H^{*l}(s) ds \\
&+ \int_0^{\nu i - z} s^\alpha \ln^n(s) H^{*l}(P-z-s) ds \qquad (4.150)
\end{aligned}$$

The first integral in righthand side of (4.150) is an integral of a function analytic in $z \in \mathcal{D}_\nu$ and $\mathbb{L}^1$ in s, therefore analytic in z. In the second term, take the integration path to be the line segment, by Lemma 4.19, $H^{*l}(P-z-s)$ is analytic in a neighborhood of integration path, while and the upper limit depends analytically on z, therefore the integral is analytic in $\mathcal{D}_\nu$. □

Proposition 4.23. *$G_2(z)$ are analytic in $\mathcal{D}_\nu$.*

Proof. It is clear that $G_1(z)$ is analytic in $\mathcal{D}_\nu$. $G_2(z)$ analytic follows from Proposition 4.22, (4.12),(4.13) and (4.142). □

Proposition 4.24. *$D_2(z)$ is analytic in $\mathcal{D}_\mu$ where $\mu = 2\nu$,*

$$D_2(z) = \mathcal{O}(z \ln z) \quad as \quad p \to 0 \qquad (4.151)$$

$$\int_{C_1 \cup C_2} \left| \frac{D_2(s-z)}{s} \right| d|s| \leq C\nu |\ln \nu| \qquad (4.152)$$

where the integration path $C_1 \cup C_2$, $C1 := \{|s| = \nu : -\frac{\pi}{2} \leq \arg(s) \leq \arg(z)\}$, $C_2 := \{s = te^{i\arg(z)} : |z| \leq t \leq \nu\}$.

Proof. By Remark 4.3, $F^{*k} - a_{00}$ is analytic in $\mathcal{D}_\mu$, and $[F^{*k} - a_{00}](p) = \mathcal{O}(p \ln p)$.

Since $H(p)^{*l}, F(p)$ are analytic in $p \in \mathcal{D}_\mu$ we get $D_2(z)$ is analytic there. By Lemma 4.10 , for $p \in \mathcal{D}_\mu$,

$$H(p) = Y(p) = p^{\frac{1}{k}} V(p) = o(p^{1+\frac{1}{k}}) \tag{4.153}$$

Remark 4.3 indicates that $F^{*(k+1-l)} = \mathcal{O}(p^{(1-l)/k})$, therefore $[F^{*(k+1-l)} * H^{*(l-1)}](p) = o(p)$. By (4.141) we get $D_2(z) = \mathcal{O}(zlnz)$ as $z \to 0$. So for $s \in \mathcal{D}_\mu$, we have $\left|\frac{D_2(s-z)}{(s-z)\ln(s-z)}\right| \leq K$ for some constant K.

$$\begin{aligned}\int_{C_1 \cup C_2} \left|\frac{D_2(s-z)}{s}\right| d|s| &\leq \int_{C_1 \cup C_2} \left|\frac{D_2(s-z)}{(s-z)\ln(s-z)} \frac{(s-z)\ln(s-z)}{s}\right| d|s| \\ &\leq K \int_{C_1 \cup C_2} \left|\frac{(s-z)\ln(s-z)}{s}\right| d|s| \leq \nu \ln \nu \end{aligned} \tag{4.154}$$

□

Define norm

$$\|q\|_r := \sup_{z \in \mathcal{D}_\nu} \{|z^\beta q(z)|\} \tag{4.155}$$

Consider function space

$$\mathcal{S}_3 := \{q(z) \mid \text{q(z) analytic in } \mathcal{D}_3 \text{ continuous in } \bar{\mathcal{D}}_3\} \tag{4.156}$$

equipped with norm $\|\cdot\|_r$. Let $\mathcal{B}_3$ be a ball of size δ centered at the Q_0 in $\mathcal{S}_3$. Define norm $\|\cdot\|_\infty$ as

$$\|f\|_\infty := \sup_{z \in \mathcal{D}_\nu} |f|. \tag{4.157}$$

Lemma 4.25. *For $Q(z) \in \mathcal{S}_3$ and A bounded in $\mathcal{D}_\nu$, we have*

$$\left\| z^{-\beta} \int_{\nu i}^{z} s^\beta Q(s) A(s) ds \right\|_r \leq C \cdot \nu \|Q\|_r \|A\|_\infty \tag{4.158}$$

Proof. For $z \in \mathcal{D}_\nu$,

$$\left| \int_{\nu i}^{z} s^{\beta} q(s) A(s) ds \right| \leq \int_{\nu i}^{z} \|q\|_r |A(s)| d|s| \leq 2\nu \|q\|_r \|A\|_\infty \tag{4.159}$$

□

Proposition 4.26. *For $Q \in \mathcal{B}_3$, we have*

$$\left\| z^{-\beta} \int_{\nu i}^{z} s^{\beta-1} D_2(s-z) Q(s) ds \right\|_r \leq C\nu |\ln \nu| \, \|Q\|_r \tag{4.160}$$

$$\left\| z^{-\beta} \int_{\nu i}^{z} s^{\beta-1} \left(\int_{\nu i}^{s} D_2(w-s) Q(w) dw \right) ds \right\|_r \leq C\nu \|Q\|_r \tag{4.161}$$

Proof. For $z \in \mathcal{D}_\nu$, applying

$$\begin{aligned} \left| \int_{\nu i}^{z} s^{\beta-1} D_2(s-z) Q(s) ds \right| &\leq \|Q\|_r \int_{C_1 \cup C_2} \left| \frac{D_2(s-z)}{s} \right| d|s| \\ &\leq C\nu |\ln \nu| \, \|Q\|_r \end{aligned} \tag{4.162}$$

Hence (4.160) follows. By (4.151) D_2 is bounded in $\mathcal{D}_\nu$, therefore

$$\left| \int_{\nu i}^{z} s^{\beta-1} \left(\int_{\nu i}^{s} D_2(w-s) Q(w) dw \right) ds \right|$$

$$\leq \|Q\|_r \left(\int_0^{\nu i} + \int_0^{z} \right) |s^{\beta-1}| \left\{ \left(\int_0^{\nu i} + \int_0^{s} \right) D_2(w-s) |w|^{-\beta} d|w| \right\} d|s| \tag{4.163}$$

$$\leq C_1 \|Q\|_r \nu^{-Re(\beta)+1} \left(\int_0^{\nu i} + \int_0^{s} \right) |s^{\beta-1}| \; |ds|$$

$$\leq C_2 \nu \|Q\|_r \tag{4.164}$$

□

Lemma 4.27. *For small enough δ, $\mathcal{R}_1$ is a contraction in $\mathcal{B}_3$.*

Proof. By proposition 4.23, for $z \in \mathcal{D}_\nu$, $|G_2(z)| \le K$ for some constant K.

$$|z^\beta G_3(z)| \le K \int_0^z |s^{\beta-1}|\,|ds| \le C\nu^{\Re(\beta)} \tag{4.165}$$

For $Q \in \mathcal{S}_3$, $|z^{\beta-1}D_1(z)Q(z)| \le \|z^{-1}D_1\|_\infty \|Q\|_r$ By Remark 4.7, there exist a C K_1, such that $\|z^{-1}D_1\|_\infty \le K_1\nu$. Hence

$$\|z^{-1}D_1(z)Q(z)\|_r \le C\cdot\nu\|Q\|_r \tag{4.166}$$

By Remark 4.7, $\frac{d}{ds}[s^{\beta-1}D_1(s)] = s^\beta \tilde{D}_1$, where $\tilde{D}_1$ is analytic in D_ν. Lemma 4.25 implies

$$\left\| z^{-\beta}\int_{\nu i}^z \frac{d}{ds}\left[s^{\beta-1}D_1(s)\right] Q(s)ds \right\|_r \le \nu\|\tilde{D}_1\|_\infty \|Q\|_r \tag{4.167}$$

(4.166),(4.167) and Proposition 4.26 implies that

$$\|\mathcal{R}[Q]\|_r \le C\cdot\nu|\ln\nu|\ \|Q\|_r \tag{4.168}$$

Since $\mathcal{R}_1$ is linear, (4.165) and (4.168) $\mathcal{R}_1$ is a contraction in $\mathcal{B}_3$ and the ball size can be choose as $\delta = \mathcal{O}(\nu^{\Re(\beta)})$. ☐

Lemma 4.28.

$$Y(P-z) = S_1(P-z)^{-\beta-1} + o(z^{-\beta}), \tag{4.169}$$

where

$$\begin{aligned} S_1 :\ &= S_0 + \int_{\nu i}^0 \frac{d}{ds}\left[s^{\beta-1}D_1(s)\right] Q(s)ds + \int_{\nu i}^0 s^{\beta-1}D_2(\nu i - s)Q(s)ds \\ &+ \int_{\nu i}^0 s^{\beta-1}\left(\int_{\nu i}^s D_2(w-s)Q(w)dw\right)ds \end{aligned} \tag{4.170}$$

where β is given by (4.144).

Proof. Let Q be the unique solution in $\mathcal{B}_3$ implies by Lemma 4.27. $Q \in \mathcal{B}_3$ implies that $Q_\beta := z^\beta Q(z)$ is continuous at $z = 0$ for $z \in \overline{\mathcal{D}}_\nu$. It is easy to check that $S_1 = Q_\beta(0)$. Therefore,

$$\lim_{z\to 0} \frac{Q(z) - S_1 z^{-\beta}}{z^{-\beta}} = \lim_{z\to 0} Q_\beta(z) - Q_\beta(0) = 0 \tag{4.171}$$

Hence $Q(z) = S_1 z^{-\beta} + o(z^{-\beta})$. For $z \in \mathcal{D}_\nu$,

$$Y(P - z) = h(z) = Q'(z) = S_1 z^{-\beta-1} + o(z^{-\beta-1}) \tag{4.172}$$

□

Remark 4.8. *The dependence of S_1 on the ν from the expression (4.170) is an illusion. We showed that there is a unique $Y(p)$ analytic in $\mathcal{D}$ For any ν choosing δ_1 small enough, we have an overlapping region between $\mathcal{D}$ and $\mathcal{D}_\nu + P$. The uniqueness and analyticity of $Y(p)$ in $\mathcal{D}$ guarantees $h(z)$ is the same for different ν. Hence the singularity cannot depends on ν.*

$Y(p)$ has a singularity at $p = P = \frac{2\pi}{3}i$, therefore Equation(4.9) has Stokes line at $\arg s = \frac{\pi}{2}$. We seek for the exponential small term on the Stokes line. Let $\theta^- \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\theta^+ \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$.

Lemma 4.29.

$$w_0^+(s) - w_0^-(s) = \mathcal{L}_{\theta^+}[Y](s) - \mathcal{L}_{\theta^-}[Y](s) \sim -2i\sin(\beta\pi) S_1 \Gamma(-\beta) s^\beta e^{-s\frac{2\pi i}{3}} \tag{4.173}$$

as $s \to \infty$ in the sector $\arg s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Proof.

$$
\begin{aligned}
w_0^+(s) - w_0^-(s) = \mathcal{L}_{\theta+}[Y](s) - \mathcal{L}_{\theta-}[Y](s) &= \left(\int_0^{\infty e^{i\theta^+}} - \int_0^{\infty e^{i\theta^-}} \right) Y(p) e^{-ps} dp \\
&= \int_{\mathrm{C}} Y(p) e^{-ps} dp \qquad (4.174)
\end{aligned}
$$

where C is the deformed contour in figure 4.2.

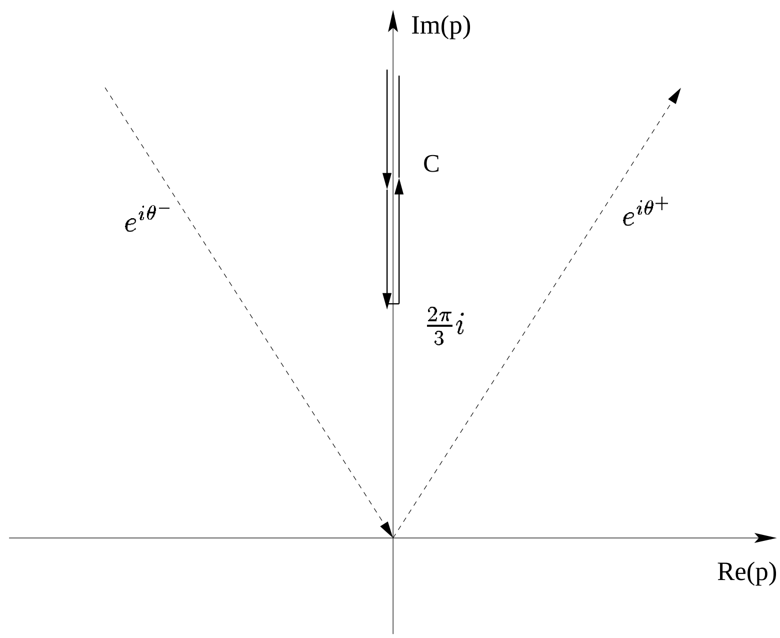

Figure 4.2: Deformed integration path C.

The dashed line is the original integration path. The thick line is the deformed one.

By Lemma 4.28 and Watson's Lemma,

$$\int_C Y(p)e^{-ps}dp \quad \sim \quad -2i\sin(\beta\pi)S_1\Gamma(-\beta)s^{\beta}e^{-s\frac{2\pi i}{3}} \tag{4.175}$$

as $s \to \infty$, and $\arg s \in (-\frac{\pi}{2}, \frac{\pi}{2})$. □

Remark 4.9. *S_1 is related to the stokes constant σ by $\sigma = -2i\sin(\beta\pi)S_1$*

4.4 Full Inner Problem Analysis

4.4.1 Operator $\mathcal{H}$

In this subsection, we will establish some property of the leading order equation for preparation of the next section.

Given a_{10}, let the unique solution of (4.3) in $\mathcal{D}(B, \pm)$ that satisfies conditions of Lemma 4.1 be $w_0^{\pm}(s; a_{10})$ respectively.

Let $w(s)$ be any solution of (4.3), and let $v(s) := w(s) - w_0^-(s)$. Then $v(s)$ satisfies

$$\mathcal{H}[v] = \mathcal{N}_0[v] \tag{4.176}$$

where

$$\mathcal{H}[v](s) := v(s+1) - v(s-2) - (k+1)w_0^-(s; a_{10})^k v(s). \tag{4.177}$$

$$\mathcal{N}_0[v] := \sum_{l=2}^{k+1} \binom{k+1}{l} v^l(s) w_0^+(s; a_{10})^{k+1-l} \tag{4.178}$$

We omit the dependence on a_{10} when there is no confusion.

Similar with the outer equation case, we first seek a fundament set of solutions to

$$\mathcal{H}[v] = 0 \tag{4.179}$$

by aids of approximate solutions.

Noticing $w_0^-(s) \sim \frac{a_{00}}{s^{\frac{1}{k}}}$ as $s \to \infty$ in region $\mathcal{D}(B,-)$, substituting $\phi(s) = s^\beta \Omega^s$ into (4.179), expanding terms as power series in terms of $\frac{1}{s}$, then setting the constant term and coefficient of $1/s$ to 0, we get that there are 3 independent formal approximate solutions to (4.179):

$$\phi_{e,\alpha}(s) = s^{\beta_\alpha} \Omega_\alpha^s, \qquad \alpha = -1, 0, 1 \tag{4.180}$$

where $\beta_\alpha = -(\frac{1+k}{k})\Omega_\alpha^2$ and $\Omega_\alpha = e^{\frac{2\pi i \alpha}{3}}$.

Proposition 4.30. *There exist an linear operator $\mathcal{H}_e$ such that $\{\phi_{e,\alpha}\}_{\alpha=-1,0,1}$ is a fundamental set of solutions to $\mathcal{H}_e[\phi] = 0$ and if $\phi(s) \neq 0$, $s \in \mathcal{D}(B,-)$, we have*

$$|\phi^{-1}(s)(\mathcal{H}_e[\phi](s-n) - \mathcal{H}[\phi](s-n))| \leq \frac{C|\ln(s-n)|}{|s-n|^2} \sup_{s \in \mathcal{D}(B,\pm), n \in \mathbb{N}} |\frac{\phi(s)}{\phi(s-n)}| \tag{4.181}$$

Proof. Define $m_{2,\alpha} = \frac{\Delta \phi_{e,\alpha}}{\phi_{e,\alpha}}$, $m_{3,\alpha} = \frac{\Delta^2 \phi_{e,\alpha}}{\phi_{e,\alpha}}$ and

$$R_\alpha := \frac{\mathcal{H}[\phi_{e,\alpha}]}{\phi_{e,\alpha}}. \tag{4.182}$$

Using the same procedure as in proof of Proposition 3.9, with $h = 1$, we can show the existence of $\mathcal{H}_e$. Estimating residual $\mathcal{H}[\phi_{e,\alpha}](x)$ for large s, it is seen that

$$R_\alpha = \mathcal{O}\left(\frac{\ln s}{s^2}\right). \tag{4.183}$$

(3.105), (4.182) , and (3.109)-(3.112) imply that $b_\alpha = \mathcal{O}(s^{-2} \ln s)$.

$$|\mathcal{H}_e[\phi](s) - \mathcal{H}[\phi](s)| = -b_3 \Delta^2 \phi - b_2 \Delta \phi - b_1 \phi. \tag{4.184}$$

Noticing that

$$|\Delta[\phi](s)| = |\phi(s+1) - \phi(s)| \leq 2 \sup_{\substack{s \in \mathcal{D}(B,\pm) \\ \Re t = \Re s}} |\phi(t)|, \tag{4.185}$$

from (4.184) we get the estimate (4.181). ☐

Lemma 4.31.

$$\mathcal{H}[\phi] = 0 \tag{4.186}$$

has a fundamental set of solutions $\phi_\alpha(s)$, $\alpha = -1, 0, 1$ *such that*

$$\phi_\alpha(s) = \phi_{e,\alpha}(s)(1 + \mathcal{O}(s^{-1+r})) \text{ for } s \in \mathcal{D}(B, \pm) \tag{4.187}$$

where $0 < r < 1$ *is a constant.*

Proof. It is easy to check that $\phi_0 := -ka_{00}^{-1}\frac{d}{ds}w_0^-(s)$ is a solution to (4.186), and $\phi_0(s) = \phi_{e,0}(1 + \mathcal{O}(\frac{\ln s}{s}))$ follows from $w_0^-(s) = a_{0,0}s^{-\frac{1}{k}} + \mathcal{O}(s^{-1-1/k}\ln s)$ for $s \in \mathcal{D}(B, -)$. Hence the assertion is true for $\alpha = 0$.

For $\alpha = 1$, rewrite (4.179) as

$$\mathcal{H}_e[\phi] = \mathcal{H}_e[p] - \mathcal{H}[\phi] \tag{4.188}$$

Using $\phi_{e,\alpha}$ to invert the operator $\mathcal{H}_e$, we define

$$\mathcal{H}_e^{-1}[g] := \sum_{\alpha=-1,0,1} \phi_{e,\alpha}\Delta_-^{-1}[\frac{M_{e,\alpha}}{W_e}g], \tag{4.189}$$

where $W_e(s)$ is the difference Wronskian of $\phi_{e,\alpha}$ and $M_{e,\alpha}$ is the cofactor of the last element in the α column of $W_e(s)$.

Let $\zeta = \frac{\phi_1 - \phi_{e,1}}{\phi_{e,1}}$. ζ satisfies

$$\zeta = \mathcal{J}[\zeta] := \phi_{e,1}^{-1}\mathcal{H}_e^{-1}[\{\mathcal{H}_e - \mathcal{H}\}[\phi_{e,1}(1+\zeta)]] \tag{4.190}$$

Let $\mathcal{B}_4$ be a ball of size δ in $\Upsilon_0(\mathcal{D}(B, +))$ with radius δ centered at zero. For

$\zeta \in \mathcal{B}_4$, direct calculation shows that $\frac{M_{e,\alpha}(s)}{W_e(s)} = \phi_{e,\alpha}^{-1}(s)(1+\mathcal{O}(s^{-1}))$. By (4.181), there exists a constant K, such that

$$\begin{aligned} & \left|\phi_{e,1}(s)\phi_{e,\alpha}^{-1}(s)\frac{M_{e,\alpha}(s)}{W_e(s)}\{\mathcal{H}_e - \mathcal{H}\}[\phi_{e,1}(1+\zeta)](s-n)\right| \\ \leq \; & K(1+\|\zeta\|_\infty)\frac{|\ln(s-n)|}{|s-n|^2}| \sup_{s\in\mathcal{D}(B,-)} \left|\frac{\phi_{e,\alpha}(s)\phi_{e,1}(s-n)}{\phi_{e,\alpha}(s-n)\phi_{e,1}(s)}\right| \end{aligned} \tag{4.191}$$

By (4.31), for $x \in \mathcal{D}(B,-)$,

$$\left|\frac{\phi_{e,\alpha}(s)\phi_{e,1}(s-n)}{\phi_{e,\alpha}(s-n)\phi_{e,1}(s)}\right| \leq C_1 \left|\frac{s}{s-n}\right|^{-\Re\beta_1+\Re\beta_\alpha} \leq C \tag{4.192}$$

We get that

$$|\phi_{e,1}^{-1}(s)\Delta_-^{-1}\{\mathcal{H}_e - \mathcal{H}\}[\phi_{e,1}(1+\zeta)](s)| \leq CB^{-1+r}(1+\|\zeta\|_\infty) \tag{4.193}$$

Hence

$$\begin{aligned} |\mathcal{J}[\zeta](s)| & \leq |\phi_{e,\alpha}\Delta_-^{-1}\{\mathcal{H}_e - \mathcal{H}\}[\phi_{e,1}(1+\zeta)](s)|| \\ & \leq C\ B^{-1+r}(1+\|\zeta\|_\infty) \end{aligned} \tag{4.194}$$

Since $\mathcal{J}$ is linear, we also have that for $\zeta_1 \in \mathcal{B}_4$,

$$\|\mathcal{J}[\zeta] - \mathcal{J}[\zeta_1]\|_\infty = \|\mathcal{J}[\zeta - \zeta_1]\|_\infty \leq CB^{-1+r}\|\zeta - \zeta_1\|_\infty \tag{4.195}$$

Therefore $\mathcal{J}$ is a contraction in $\mathcal{B}_4$ for any B large enough and the ball size $\delta = \mathcal{O}(B^{-1+r})$. Therefore there is a unique solution of (4.190) in $\mathcal{B}_4$, which implies that the assertion in the Lemma 4.31 is true for $\alpha = 1$. The proof for $\alpha = -1$ is similar. □

4.4.2 Full inner equation and matching of inner and outer solutions

In this subsection we will show in terms of the inner variable the difference between the stable and unstable manifolds turns to difference of solution of leading order inner equation $w_0^- - w_0^+$ in certain region.

Let $w_\varepsilon^-(s) := \varepsilon^{\frac{1}{k}}\tilde{z}_-(x(s+2))$ and $w_\varepsilon^+(s) := \varepsilon^{\frac{1}{k}} z_+(x(s+2))$. By Lemma 3.1 and Proposition 3.28, $z_+(x) = z_0(x) + (\varepsilon \ln h) z_1(x) + \varepsilon z_2(x) + \mathcal{O}(\varepsilon^2 \delta_0^{-2-1/k} \ln^2(\frac{\delta_0}{h}))$ for $x \in \mathcal{D}^+$ and $\tilde{z}_-(x) = z_0(x) + (\varepsilon \ln h) z_1(x) + \varepsilon z_2(x) + \mathcal{O}(\varepsilon^2 \delta_0^{-2-1/k} \ln^2(\frac{\delta_0}{h}))$ for $x \in \mathcal{D}^-$.

Rewritten in terms of the inner variable s, for $0 < r < 1$,

$$\varepsilon \ll \delta_0 \ll 1, \quad \frac{\delta_0}{\varepsilon} \ll s \ll \left[\left(\frac{\varepsilon}{\delta_0}\right)^{2+1/k} \ln\left(\frac{\delta_0}{h}\right)\right]^{-\frac{k}{k+1-r}} \text{ and } s \ll \varepsilon^{-1/2}, \tag{4.196}$$

we have

$$w_\varepsilon^\pm(s) = \frac{a_{00}}{s^{\frac{1}{k}}} + \frac{a_{11}\ln s}{s^{1+1/k}} + \frac{a_{10}}{s^{1+1/k}} + \mathcal{O}(\varepsilon s^{1-1/k}, s^{-1-1/k-r}) \tag{4.197}$$

where $s \in \mathcal{D}(B, \pm)$ and

$$a_{00} = (3/k)^{1/k}, \quad a_{11} := \frac{1}{6}(k+1)k^{1/k}, \quad a_{10} = \frac{1}{6}(k+1)k^{1/k}\ln(-k). \tag{4.198}$$

where $\ln(-k) = \ln k + \pi i$. Since (4.197) holds for $\varepsilon \ll \delta_0 \ll 1$ uniformly, we get (4.197) is true for $1 \ll s \ll \varepsilon^{-1/2}$.

The main result in this subsection is the following Lemma.

Lemma 4.32. *As* $\varepsilon \to 0^+$,

$$w_\varepsilon^-(s) \sim w_0^-(s) + \mathcal{O}(s^{\beta_1} e^{-\frac{4\pi s i}{3}}), \qquad \text{as } \varepsilon \to 0^+, s \in \mathcal{D}(B,-), |\Im s| \ll \varepsilon^{-1} \tag{4.199}$$

$$w_\varepsilon^+(s) \sim w_0^+(s) + \mathcal{O}(s^{\beta_1} e^{-\frac{4\pi s i}{3}}), \qquad as\ \varepsilon \to 0^+, s \in \mathcal{D}(B,+), |\Im s| \ll \varepsilon^{-1} \tag{4.200}$$

$$w_\varepsilon^+(s) - w_\varepsilon^-(s) \to w_0^+(s) - w_0^-(s) + \mathcal{O}(s^{\beta_1} e^{-\frac{4\pi s i}{3}}) \tag{4.201}$$

for $s \in \mathcal{D}(B,+) \cap \mathcal{D}(B,-)$ *where* $1 \ll |\Im s| \ll \varepsilon^{-\kappa}$ *and* $w_0^\pm(s) := w_0^\pm(s; a_{10})$*, and* a_{10} *is given by (4.198),* κ *is a constant satisfying* $0 < \kappa < 1/2$.

The proof is at the end of this subsection. The idea of the proof is that we will show that $w_\varepsilon^-(s) \sim w_0^-(s)$ as $\varepsilon \to 0^+$ in some region depending on ε, away from the origin respectively, then in a region that contains $\Re s = \mathcal{O}(1)$, we show $w_\varepsilon^- \sim w_0^- + q_{-1,0}^\pm \phi_{-1}$, where $q_{-1,0}^-$ is constants depending on ε . By comparing to the two asymptotic relation, we conclude $\lim_{\varepsilon\to 0^+} q_{-1,0} = 0$ and thereby prove Lemma 4.32. The proof of the + case is similar.

Define

$$\mathcal{D}(B,-) = \{s \in \mathbb{C} : \Re s < B_0, \Im s < -B\} \tag{4.202}$$

where $B = \tilde{\delta}_0/h$; B_0 is a constant independent of ε.

$$\mathcal{D}_5^- := \{s \in \mathbb{C} : -B - R < \Im s < -B, -R < \Re s < -W\} \tag{4.203}$$

We choose B large but independent of ε; $R(\varepsilon) = \varepsilon^{-\kappa}$, where $0 < \kappa < 1/2$ is a constant; $W(\varepsilon) = \varepsilon^{-\kappa_1}$ where $0 < \kappa_1 < \kappa$. It is to be noted that $\mathcal{D}_5^- \subset \mathcal{D}(B, \pm)$. Define

$$\mathcal{D}_6^- := \{s \in \mathcal{D}(B,-) : -B - R < \Im s < -B, -R < \Re s < B_0\}, \tag{4.204}$$

(See Figure 4.3).

$\mathcal{D}(B,+)$, $\mathcal{D}_5^+$, $\mathcal{D}_6^+$ are the reflection of $\mathcal{D}(B,-)$ $\mathcal{D}_5^-$ and $\mathcal{D}_6^-$ about the imaginary axis respectively. Clearly for any $W > 0$, $\mathcal{D}_5^- \subset \mathcal{D}_6^-$.

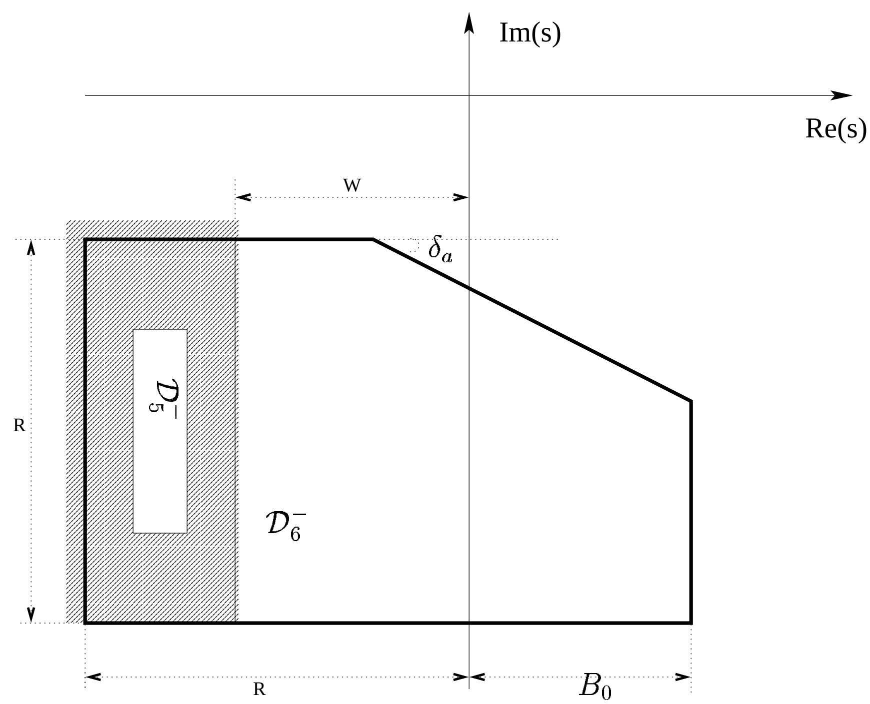

Figure 4.3: $\mathcal{D}_5^-$ and $\mathcal{D}_6^-$

$\mathcal{D}_5^-$ is the shaded region. $\mathcal{D}_6^-$ is the region bounded by thick lines. $\mathcal{D}_5^- \subset \mathcal{D}_6^-$

Remark 4.10. *It is to be noted that for $s \in \mathcal{D}_6^-$ we have*

$$w_0^-(s) = \frac{a_{00}}{s^{1/k}} + \frac{a_{11}\ln s}{s^{1+1/k}} + \frac{a_{10}}{s^{1+1/k}} + \mathcal{O}(s^{-2-1/k}\ln s) \tag{4.205}$$

We consider $\varsigma^-(s;\varepsilon) := w_\varepsilon^-(s;\varepsilon) - w_0^-(s)$ in $\mathcal{D}_5^-$ and $\tilde{\varsigma}^-(s;\varepsilon) := w_\varepsilon^-(s;\varepsilon) - w_0^-(s)$ in $\mathcal{D}_6^-$. Since many of the lemmas for ς^- and $\tilde{\varsigma}^-$ are similar and in parallel, we present those lemmas together using ς to represent any of them and point out the difference if there is any. We also drop $-$ in the representation of the region in such case. But for those lemmas that has significant difference between ς and $\tilde{\varsigma}$, we specify clearly.

By Remark 4.10 and (4.197),

$$\varsigma(s) = \mathcal{O}(\varepsilon s^{1-1/k}, s^{-1-1/k-r}) \tag{4.206}$$

for $s \in \mathcal{D}(B,-)$, and $1 \ll s \ll \varepsilon^{-1/2}$. By (4.2) and (4.3), we have $\varsigma(s)$ satisfies

$$\mathcal{H}[\varsigma] = \mathcal{N}_\varepsilon[\varsigma] \tag{4.207}$$

where $\mathcal{H}$ is defined in (4.177), and

$$\mathcal{N}_\varepsilon[\varsigma] := \varepsilon w_0(s) + \varepsilon\varsigma(s) - \sum_{l=2}^{k+1} \binom{k+1}{l} w_0(s)^{k+1-l}\varsigma(s)^l \tag{4.208}$$

By Remark 2.8, we define inverse of $\mathcal{H}$ in region $\mathcal{D}_5^-$ using horizontal strips of width 1 (except the lowest strip may have width less than 1) $\mathcal{D}_5(B_n) := \mathcal{D}_5 \cap \mathbb{R} \times (-B_n - 1, -B_n)i$, $n = 1, \cdots \lceil R \rceil$, where $B_1 := B$, and $B_{n+1} = B_n + 1$,$n = 1, \cdots \lceil R \rceil$.

$$\mathcal{H}^{-1} : \Upsilon_0(\mathcal{D}_5^-) \mapsto \Upsilon_0(\mathcal{D}_5^-) \qquad \mathcal{H}^{-1}[g](s) := \sum_{\alpha=-1,0,1} \phi_\alpha \tilde{\Delta}^{-1}\left[\frac{M_\alpha}{D} g\right](s). \tag{4.209}$$

where

$$\tilde{\Delta} = \Delta_+^{-1}\mathcal{J}_{+,\alpha}^{-1}\mathcal{P}_+\mathcal{J}_\alpha + \Delta_-^{-1}\mathcal{J}_{-,\alpha}^{-1}\mathcal{P}_-\mathcal{J}_\alpha \tag{4.210}$$

where

$$\mathcal{J}_\alpha : \Upsilon_0(\mathcal{D}) \mapsto \Upsilon_0(\mathcal{D}) \quad \mathcal{J}_0[g](s) = s^{1+\gamma}\phi_\alpha(s)g(s); \tag{4.211}$$

$$\mathcal{J}_{\pm,\alpha} : \Upsilon_0(\mathcal{D}(B,\pm)) \mapsto \Upsilon_{1+\gamma}(\mathcal{D}(B,\pm)) \quad \mathcal{J}_\pm[g](s) = s^{1+\gamma}\phi_\alpha(s)g(s) \tag{4.212}$$

Lemma 4.33.

$$\|\mathcal{H}^{-1}\| \leq KR^{1+\gamma}B^{-\gamma}\ln R \tag{4.213}$$

where K is a constant independent of B_n, R.

Proof. By definition of $\mathcal{D}_5$ and (4.180), for $s \in \mathcal{D}_5(B_n)$,

$$\|\phi_{e,\alpha}\| \leq C B_n^{\Re(\beta_\alpha)} e^{\alpha \frac{2}{3}\pi B_n} \qquad \left\|\frac{1}{\phi_{e,\alpha}}\right\| \leq C_1 B_n^{-\Re(\beta_\alpha)} e^{-\alpha\frac{2}{3}\pi B_n} \tag{4.214}$$

where C, C_1 are constants independent of B_n and the norm is the sup norm in $\mathcal{D}_5(B_n)$.

Proposition 4.30 implies that $\|\phi_{0,\alpha}\| \leq C_1 \|\phi_{e,\alpha}\|$ and

$$\frac{M_{0,\alpha}}{W_0} = \frac{M_{e,\alpha}}{W_e}(1 + \mathcal{O}(s^{-1})) = \phi_{e,\alpha}^{-1}(K_\alpha + \mathcal{O}(s^{-1})) \tag{4.215}$$

So $|\frac{M_{0,0}}{W_\alpha}\phi_\alpha(s)| \leq C$. By (4.209) and Lemma 2.14,

$$\|\mathcal{H}^{-1}\| \leq C \frac{(B_n + R)^{\gamma+1}}{B_n^\gamma} \ln(R) \tag{4.216}$$

By definition of region $\mathcal{D}_5$, $B_n \leq R$. So the proposition follows. □

Proposition 4.34. *For $\varsigma(s)$ satisfying (4.207) and (4.206), there exists $q(s)$, such that for s in $\mathcal{D}_5^-$ ($\mathcal{D}_6^-$), $\varsigma(s)$ satisfies*

$$\varsigma = \mathcal{J}[\varsigma] := q + \mathcal{H}^{-1}\left[\mathcal{N}_\varepsilon[\varsigma]\right] \tag{4.217}$$

where $q(s) = \sum_{\alpha=-1,0,1} q_\alpha(s)\phi_\alpha(s)$ and $q_\alpha(s)$, $\alpha = -1, 0, 1$ are analytic in $\mathcal{D}_5^-$ ($\mathcal{D}_6^-$) and periodic with period 1.

Proof. Since $\varsigma(s)$ satisfies (4.206), $\mathcal{H}^{-1}\mathcal{N}_\varepsilon[\varsigma]$ is well defined. The existence of q follows from Lemma 2.18. □

Remark 4.11. *Lemma 4.33, Proposition 4.34 also hold for $\mathcal{H}^{-1}$ defined on strips of $\mathcal{D}_6^{-1}$.*

Proposition 4.35. *$q(s)$ found in Proposition 4.34 satisfies*

$$q(s) = \mathcal{O}((|\Im s| + W(\varepsilon))^{-1-1/k-r}, \varepsilon^{\lambda}) \qquad (4.218)$$

for $s \in \mathcal{D}_5^-$, ε small enough, where $\lambda > 1/2$ is a constant.

Proof. From (4.206), using $1 \ll R(\varepsilon) \ll \varepsilon^{-\kappa}$ and $1 \ll W$, we get

$$\|\varsigma\| \leq C(B_n + W)^{-1-\frac{1}{k}-r} + C\varepsilon(B_n + W + R)^{1-\frac{1}{k}} \leq C((B_n + W)^{-1-\frac{1}{k}-r} + \varepsilon^{\lambda}) \quad (4.219)$$

where we choose $\lambda > 1 - \kappa(1 - \frac{1}{k}) > 1/2$.

$\omega_0^-(s) \sim \frac{a_{00}}{s^{\frac{1}{k}}}$, $s \to \infty$ for $s \in \mathcal{D}(B, -)$, we have $|\omega_0^-(s)| \leq C(B_n + W)^{-\frac{1}{k}}$ for $s \in \mathcal{D}_5(B, R(\varepsilon), W(\varepsilon))$.

By (4.208),

$$|\mathcal{N}_\varepsilon[\varsigma](s)| \leq C(\varepsilon(B_n + W)^{-\frac{1}{k}} + \varepsilon\|\varsigma\| + \|\varsigma\|^2) \qquad (4.220)$$

By Lemma 4.33,

$$\begin{aligned} |q(s;\varepsilon)| &\leq |\varsigma(s;\varepsilon)| + |\mathcal{H}^{-1}\mathcal{N}_\varepsilon[\varsigma]| \\ &\leq \|\varsigma\| + C(B_n + R)^{1+\gamma} \ln R(\varepsilon(B_n + W)^{-\frac{1}{k}} + \varepsilon\|\varsigma\| + \|\varsigma\|^2) \\ &\leq 2\|\varsigma\| + C_1\varepsilon^{\lambda} \qquad (4.221) \end{aligned}$$

Hence from (4.219) we get (4.218). ☐

Remark 4.12. *Proposition 4.35 is also true for ς replaced by $\tilde{\varsigma}$, p replace by $\tilde{p}$ and W replaced by 0 in (4.218).*

By definition, $q(s;\varepsilon) = \sum_{\alpha=-1,0,1} q_\alpha(s;\varepsilon)\phi_\alpha(s)$, where $q_\alpha(s;\varepsilon)$ are periodic function with period 1. Let the Fourier expansion of $q_\alpha(s;\varepsilon)$ be $q_\alpha(s;\varepsilon) = \sum_{n=-\infty}^{\infty} q_{\alpha,n}(\varepsilon)e^{2\pi ins}$ Let

$$p(s;\varepsilon) := \sum_{\alpha=-1,0,1} \phi_\alpha(s) \sum_{\alpha+3n\geq 0} q_{\alpha,n}(\varepsilon)e^{2\pi ins}, \quad r(s;\varepsilon) := q(s;\varepsilon) - p(s;\varepsilon) \tag{4.222}$$

so for ε fixed, $r(s;\varepsilon) = q_{-1,0}(\varepsilon)\phi_{-1}(s) + \mathcal{O}(\phi_1(s)e^{-2\pi is}$. Next we will use the bound on $|q(s;\varepsilon)|$ to show that the coefficients of exponentially large terms are small as $\varepsilon \to 0^+$ and so is $q_{0,0}$.

Proposition 4.36. *For ε small enough there exists constant $\rho, K > 0$, independent of ε and s, such that*

$$|p(s;\varepsilon)| \leq K\varepsilon^\rho \tag{4.223}$$

for $p^-(s;\varepsilon)$ where $s \in \mathcal{D}_5^-$.

Proof. Let $\mathbf{W}(s)$ be the matrix Wronskian of $\{\phi_\alpha\}_{\alpha=-1,0,1}$, and $\mathbf{W}^{-1}(s)$ be inverse of $\mathbf{W}(s)$, by (4.187), we have that $\mathbf{W}^{-1}[i,j](s) = A_{i,j}\phi_{e,\alpha}^{-1}(s)(1+\mathcal{O}(\frac{1}{s}))$, where $A_{i,j} \neq 0$ are constants. Therefore

$$q_\alpha(s) = \sum_{j=-1,0,1} q(s+j)\mathbf{W}^{-1}[\alpha,j](s) = C\phi_{e,\alpha}^{-1}(s) \sum_{j=-1,0,1} q(s+j)(1+\mathcal{O}\left(\frac{1}{s}\right)) \tag{4.224}$$

$$\begin{aligned} |q_{\alpha,n}| &= \int_0^1 |q_\alpha(s+t)e^{-2i\pi n(s+t)}|dt \\ &\leq \int_0^1 C(s+t)^{-\beta_\alpha} e^{-i\frac{2\alpha+6n}{3}\pi(s+t)} \sum_{j=-1,0,1} |q(s+j)|dt \end{aligned} \tag{4.225}$$

For $\alpha + 3n \geq 0$, using the bound (4.218), at $s = -\varepsilon^{-\kappa}i$, we get

$$|q_{\alpha,n}| = \mathcal{O}(\varepsilon^\rho \exp\{-\frac{(2\alpha+6n)\pi}{3\varepsilon^\kappa}\}, \varepsilon^\lambda) \tag{4.226}$$

$$\text{where } \rho := (1 + 1/k + r + \Re\beta_\alpha)\kappa \geq r\kappa > 0 \tag{4.227}$$

The bounds given on $q_{\alpha,n}$ implies (4.223), for $s \in \mathcal{D}_5^-$. $\square$

Remark 4.13. *Lemma 4.36 also holds for p replaced by $\tilde{p}$ and $\mathcal{D}_5^-$ by $\mathcal{D}_6^-$ respectively.*

Proposition 4.37. *For $s \in \mathcal{D}_5^-$, there exist a constant ρ_1 and C independent of s and ε, such that $|r^-(s;\varepsilon)| \leq C\varepsilon^{\rho_1}$. In particular,*

$$q_{-1,0}^-(\varepsilon) \leq C\varepsilon^{\rho_1} \tag{4.228}$$

Proof. For $\alpha + 3n < 0$, using the bound (4.218), estimating the integral in (4.225) at $s = -W - 4 - Bi$, we get

$$|q_{\alpha,n}^-| \leq KW^{-\rho}e^{(2\alpha+6n)\pi B/3} \tag{4.229}$$

Recall we choose $W(\varepsilon) = \varepsilon^{-\kappa_1}$, $0 < \kappa_1 < \kappa$, let $\rho_1 := \rho\kappa_1$ we get the proposition. $\square$

Let $\varsigma_0(s;\varepsilon)$, $\tilde{\varsigma}_0(s;\varepsilon)$ satisfies

$$\varsigma_0(s;\varepsilon) = \mathcal{J}_7[\varsigma_0](s;\varepsilon) := r(s;\varepsilon) + \mathcal{H}^{-1}[\mathcal{N}_0[\varsigma_0(s;\varepsilon)]] \tag{4.230}$$

we have the following lemma showing ς_0 is exponentially small in s.

Define region $\mathcal{D}_7^-$ as $\mathcal{D}_7^- \subset \mathcal{D}_5^-$ is a region satisfying $s \in \mathcal{D}_7^- \Rightarrow$

$$\forall \Im s' = \Im s, \{|\phi_{-1}(s')| \leq \varepsilon^{\lambda_1}\} \text{ and } |\Im s| \gg 1 \tag{4.231}$$

where λ_1 , λ_2 are constants satisfying $\frac{1}{2} < \lambda_1 < \lambda_2$.

$\mathcal{D}_8^- \subset \mathcal{D}_6^-$ is the region satisfying $s \in \mathcal{D}_8^- \Rightarrow$ (4.231).

Proposition 4.38.

$$\varsigma_0(s;\varepsilon) = q_{-1,0}(\varepsilon)\phi_{-1}(s) + \mathcal{O}(\phi_1(s)e^{-2\pi i s}, \varepsilon^{\lambda_2}) \tag{4.232}$$

for $s \in \mathcal{D}_7^-$

(4.232) also holds for $s \in \mathcal{D}_8^-$.

Proof. We will only show for $\mathcal{D}_7^-$ case since the other case are similar. Let $\mathcal{B}_7$ be a ball of size δ centered at $r(s;\varepsilon)$ in $\mathcal{S}_0(\mathcal{D}_7^-)$. We will show $\mathcal{J}_7$ is a contraction in $\mathcal{B}_7$ for $\delta = K\varepsilon^{\lambda_2}$ where K is some constant.

By (4.231), $\|r\| := \sup_{s\in\mathcal{D}_7^-} |r(s;\varepsilon)| \le C\varepsilon^{\lambda_1}$ By (4.230) for δ small enough, we have

$$\begin{aligned} |\varsigma_0(s;\varepsilon) - r(s;\varepsilon)| &\le \|\mathcal{H}^{-1}[\mathcal{N}_0[\varsigma_0]]\| \\ &\le C_1 R^{1+\gamma} B_n^{-\gamma} \ln R(\|r\|^2 + 2\delta\|r\| + \delta^2)| \\ &\le C_2\varepsilon^{-\lambda_3}(\varepsilon^{2\lambda_1} + \varepsilon^{\lambda_1}\delta + \delta^2) \end{aligned} \tag{4.233}$$

where $1/2 > \lambda_3 > \kappa(1+\gamma)$. Therefore we choose λ_2 such that $\lambda_1 < \lambda_2 < 2\lambda_1 - \lambda_3$, then for ε small enough, $\|\mathcal{J}[\varsigma_0]\| \le \delta$. Similarly for $\varsigma_1 \in \mathcal{B}_7$,

$$|\mathcal{J}[\varsigma_0] - \mathcal{J}[\varsigma_1] \le C_2\varepsilon^{-\lambda_3}\|\varsigma_0 - \varsigma_1\|\delta \tag{4.234}$$

Hence $\mathcal{J}$ is a contraction in $\mathcal{B}_7$ for $\delta = K\varepsilon^{\lambda_2}$. Noticing that $r(s;\varepsilon) = q_{-1,0}(\varepsilon)\phi_{-1}(s) + \mathcal{O}(\phi_1 e^{-2\pi i s})$, we get the proposition. □

Lemma 4.39.

$$w_\varepsilon^-(s;\varepsilon) = w_0^-(s) + \mathcal{O}(\varepsilon^{\rho_1}\phi_{-1}(s), \phi_1(s)e^{-2\pi i s}, \varepsilon^{\lambda_2}), s \in \mathcal{D}_7^- \tag{4.235}$$

$$w_\varepsilon^+(s;\varepsilon) \sim w_0^+(s) + \mathcal{O}(\varepsilon^{\rho_1}\phi_{-1}(s), \phi_1(s)e^{-2\pi i s}, \varepsilon^{\lambda_2}), \ \ as\ \varepsilon \to 0^+, s \in \mathcal{D}_7^+ \tag{4.236}$$

Proof. As an immediate consequence of Proposition 4.37, and 4.38, we have

$$\varsigma^-(s;\varepsilon) = \mathcal{O}(\varepsilon^{\rho_1}\phi_{-1}(s), \phi_{-1}^2(s), \varepsilon^{\lambda_2}) \tag{4.237}$$

for $s \in \mathcal{D}_7^-$ By similarly with proof of $-$ case, we get (4.236). □

The following lemma shows that for fixed s the difference between $\varsigma(s;\varepsilon)$ and $\varsigma_0(s;\varepsilon)$ goes to zero as $\varepsilon \to 0^+$. Let $\eta(s;\varepsilon) := \varsigma(s;\varepsilon) - \varsigma_0(s;\varepsilon)$

Proposition 4.40. *$\eta(s;\varepsilon) = \mathcal{O}(\varepsilon^\lambda)$ where $\lambda > 1/2$ is a constant, and $s \in \mathcal{D}_5^-$ or $s \in \mathcal{D}_6^-$*

Proof. Then $\eta(s;\varepsilon)$ satisfies

$$\eta(s;\varepsilon) = \mathcal{J}_6[\eta](s;\varepsilon) := p(s;\varepsilon) + \mathcal{H}^{-1}[\varepsilon\varsigma_\varepsilon] + \mathcal{H}^{-1}[\mathcal{N}_1[\eta,\varsigma_\varepsilon]] \tag{4.238}$$

where

$$\mathcal{N}_1[\eta,\varsigma_\varepsilon](s;\varepsilon) := -\sum_{l=2}^{k+1}\binom{k+1}{l} w_0^-(s)^{k+1-l}\eta(s;\varepsilon)\left(\sum_{n=0}^{l-1}\varsigma^{l-n}(\eta+\varsigma)^n\right)(s;\varepsilon) \tag{4.239}$$

By Proposition 4.36, $|p(s;\varepsilon)| \le C\varepsilon^\rho$, where $\rho > 0$ for $s \in \mathcal{D}_7^-$ or $s \in \mathcal{D}_8^-$. Similar with proof of Proposition , we can show $\mathcal{J}_6$ is a contraction in ball of size $\mathcal{O}(\varepsilon^\lambda)$. □

Proof. (Proof of Lemma 4.32). For $s \in \mathcal{D}_8^-$, restricted in a region where $s^{\beta_1}e^{-4\pi i s/3} \gg \varepsilon^{\lambda_2}$, by Proposition 4.38, 4.40, and (4.231), we get

$$w_\varepsilon^-(s) = w_0^- + q_{-1,0}^-(\varepsilon)\phi_{-1,0}(s) + \mathcal{O}(s^{\beta_1}e^{-4\pi i s/3}) \tag{4.240}$$

For $s \in \mathcal{D}_7^-$, restricted in a region where $s^{\beta_1}e^{-4\pi i s/3} \gg \varepsilon^{\lambda_2}$, Proposition 4.39 we have

$$w_\varepsilon^-(s) = w_0^- + \mathcal{O}(s^{\beta_1}e^{-\frac{4\pi i s}{3}}) \tag{4.241}$$

Comparing (4.240) and (4.241) in $\mathcal{D}_5$, by Lemma 4.29 we get

$$\lim_{\varepsilon\to 0^+} q^-_{-1,0}(\varepsilon) = 0 \tag{4.242}$$

Similarly compare w^+_ε with w^+_0 in corresponding region gives $\lim_{\varepsilon\to 0^+} q^+_{-1,0} = 0$ Since $q^\pm_{-1,0}$ is independent of s, for $s \in \mathcal{D}^-_8 \cap \mathcal{D}^+_8$, we find

$$\lim_{\varepsilon\to 0^+}(w^-_\varepsilon - \omega^+_\varepsilon) = \sigma\phi_{-1} + \mathcal{O}(s^{\beta_1} e^{-\frac{4\pi i s}{3}}) \tag{4.243}$$

□

4.5 Matching of $\tilde{z}_-(x) - z_+(x)$ to $w_-(s) - w_+(s)$

Lemma 4.41.

$$\inf_{x\in\mathbb{R}} |\tilde{\mathbf{Z}}_-(x) - \mathbf{Z}_+(x)| \sim |\sigma|\ \sqrt{6}\Theta\varepsilon^{-\gamma_1} e^{-\gamma/\varepsilon} \ \textit{as}\ \varepsilon \to 0^+ \tag{4.244}$$

where σ is the Stokes constant of the leading order equation (4.3).

$$\gamma := 2\pi^2/k, \qquad \gamma_1 := \frac{3}{2k} + \frac{1}{2}, \tag{4.245}$$

$$\Theta := |e^{-(\ln k - \pi i)\beta_{-1}}|3^{\Re\beta_{-1}}(1 + 1/k)^{\frac{k+1}{2}} k^{1/(2k)} \tag{4.246}$$

Proof. By Theorem 3.32, for $x \in \mathcal{D}(b_n, h)$, $\tilde{z}_-(x) - z_+(x) = \rho_0(x) + \mathcal{O}(\varepsilon\delta_0^{-4-2/k}\rho_0(x))$ where $\rho_0(x) = \sum_{\alpha=\pm1} c_\alpha(x)\tau_\alpha(x)$, $c_\alpha(x)$ is analytic in $\mathcal{D}$ and periodic with period 1.

In the inner variable s, we get $\sum_{\alpha=-1,0,1} c_\alpha(x)\tau_\alpha(x)$

$$w^-_\varepsilon(s) - w^+_\varepsilon(s) \ = \ \sum_{\alpha=\pm1}\sum_{n=-\infty}^{\infty} \hat{c}^{(t)}_{\alpha,n}\tau_\alpha(x(s)) + \mathcal{O}(\varepsilon^{1+1/k}\delta_0^{-4-1/k}\|\phi\|) \tag{4.247}$$

$$\text{where } \hat{c}_{\alpha,n} := c^{(t)}_{\alpha,n}(-k)^{\beta_\alpha}\varepsilon^{\beta_\alpha+1/k}e^{\frac{2}{3h}\pi(\alpha+3n)iX_s}, \quad \alpha = \pm 1 \tag{4.248}$$

Comparing (4.247) and (4.201), we get

$$\lim_{\varepsilon\to 0^+} \hat{c}^{(t)}_{-1,0} = \sigma \tag{4.249}$$

$$\lim_{\varepsilon\to 0^+} \hat{c}^{(t)}_{\alpha,n} = 0, \quad \alpha = \pm 1, \alpha + 3n \geq 0 \tag{4.250}$$

Therefore

$$c^{(t)}_{-1,0} = \sigma 3^{\beta_{-1}} e^{-(\ln k+\pi i)\beta_{-1}}\varepsilon^{-\beta_{-1}-\frac{1}{k}}e^{-\frac{2\pi^2}{k\varepsilon}}(1+o(1)) \tag{4.251}$$

We get

$$c^{(t)}_{-1,0} = \sigma 3^{\beta_{-1}} e^{-(\ln k+\pi i)\beta_{-1}}\varepsilon^{-\beta_{-1}-\frac{1}{k}}e^{-\frac{2\pi^2}{k\varepsilon}}(1+o(1)) \tag{4.252}$$

Since the solution $\tilde{z}_-(x)$, $z_+(x)$ are real on the real axis, and $c^{(t)}_{1,0}$ is the complex conjugate of $c^{(b)}_{-1,0}$, we get

$$\lim_{\varepsilon\to 0^+} c^{(b)}_{1,0} = \bar{\sigma} 3^{\beta_1} e^{-(\ln k-\pi i)\beta_1}\varepsilon^{-\beta_1-\frac{1}{k}}e^{-\frac{2\pi^2}{k\varepsilon}}(1+o(1)) \tag{4.253}$$

Therefore as $\varepsilon \to 0^+$,

$$|\tilde{\mathbf{Z}}_-(x) - \mathbf{Z}_+(x)| \quad \sim \quad |c^{(b)}_{-1,0}| \left(\sum_{n=0,1,2} \{2|\tau_{-1}(x+nh)|\cos\theta(x)\}^2 \right) \tag{4.254}$$

where $\theta(x) := \arg(c^{(b)}_{-1,0}) + \arg(\tau_{-1,0}(x))$. Since $\arg(\tau_{-1,0}(x)) = \frac{2\pi x}{3h}(1+\mathcal{O}(h))$, we get

$$|\tilde{\mathbf{Z}}_-(x) - \mathbf{Z}_+(x)| \quad \sim \quad |\sigma|\sqrt{6}\Theta\varepsilon^{-\frac{1}{2}-\frac{3}{2k}}e^{-\frac{2\pi^2}{k\varepsilon}} \tag{4.255}$$

where

$$\Theta := 3^{\Re\beta_{-1}}|e^{-(\ln k-\pi i)\beta_{-1}}|(1+1/k)^{\frac{k+1}{2}}k^{1/(2k)} \tag{4.256}$$

□

Lemma 4.42. *If $\sigma \neq 0$ then*

$$\lim_{x \to -\infty} z_+(x) \neq 0 \tag{4.257}$$

$$\lim_{x \to \infty} z_-(x) \neq 1 \tag{4.258}$$

Proof. We prove (4.257) by contradiction. Suppose $\sigma \neq 0$, and $\lim_{x \to -\infty} z_+(x) = 0$ Let $\hat{z}_-(x) = z_+(x)$, then $\hat{z}_-(x)$ is a reparameterization of the unstable manifold. Since $z_+(x) - z_-(x) = \hat{z}_-(x) - z_-(x) = \mathcal{O}(\varepsilon^2 \delta_0^{-2-1/k} \ln(\delta_0/h))$, by Lemma 3.27 there exists unique $\tilde{z}_-(x)$ such that $\tilde{z}_-$ satisfies 3.198 with given $b_0(x)$ and $\tilde{z}_-$ satisfies 3.197. $\hat{z}_-(x)$ satisfies the above two conditions, therefore $\tilde{z}_-(x) = \hat{z}_-(x) = z_+(x)$ contradict with Lemma 4.32 if $\sigma \neq 0$.

Uniqueness of the stable manifolds at **1** and (4.257) imply (4.258). □

Theorem 4.43. *Let $\hat{z}_-(x)$ be any reparameterization of $z_-(x)$*

$$\inf_{x \in \mathbb{R}, \hat{\mathbf{Z}}} |\hat{\mathbf{Z}}_-(x) - \mathbf{Z}_+(x)| \sim |\sigma| \Theta \varepsilon^{-\gamma_1} e^{-\gamma/\varepsilon} \text{ as } \varepsilon \to 0^+ \tag{4.259}$$

where

$$\gamma := 2\pi^2/k, \qquad \gamma_1 := \frac{3}{2k} + \frac{1}{2}, \tag{4.260}$$

$$\Theta := 3^{\frac{k+1}{2k}} |e^{-(\ln k - \pi i)\beta_{-1}}| (1 + 1/k)^{\frac{k+1}{2}} k^{1/(2k)}, \quad \beta_{-1} = -\frac{k+1}{k} e^{-4\pi i/3} \tag{4.261}$$

and σ is the Stokes constant of the leading order equation (4.3), hence σ is parameter free number which can be obtained, and estimated numerically.

Proof. We seek for $\hat{z}_-(x)$, reparameterization of $z_-(x)$, such that

$$\inf_{x \in \mathbb{R}} |\hat{\mathbf{Z}}_-(x) - \mathbf{Z}_+(x)| < \inf_{x \in \mathbb{R}} |\tilde{\mathbf{Z}}_-(x) - \mathbf{Z}_+(x)| \tag{4.262}$$

$$\hat{z}_-(x) - z_+(x) = (\hat{z}_-(x) - \tilde{z}_-(x)) - (\tilde{z}_-(x) - z_+(x)) \tag{4.263}$$

By Lemma 3.19, there exists $\tilde{\phi}_-(x) := \tilde{d}_0(x)\tau_0(x)$, such that $\tilde{\psi}(x) := \tilde{z}_-(x) - z_-(x)$ satisfies (3.185) and $\|\tilde{\psi} - \tilde{\phi}_-\|_{\mu_1} = \mathcal{O}(\varepsilon\delta_0^{-2})\|\tilde{\phi}_-\|_{\mu_1}$ where $\mu_1 = 1$; and there exists $\hat{\phi}_-(x) := \hat{d}_0(x)\tau_0(x)$, such that $\hat{\psi}(x) := \hat{z}_-(x) - z_-(x)$ satisfies (3.185) and $\|\hat{\psi} - \hat{\phi}_-\|_{\mu_1} = \mathcal{O}(\varepsilon\delta_0^{-2})\|\hat{\phi}_-\|_{\mu_1}$.

Hence $\|(\hat{z}_- - \tilde{z}_-) - (\hat{\phi} - \tilde{\phi})\| \leq \varepsilon\delta_0^{-2}\|\tilde{\phi} - \hat{\phi}\|$.

So the task of seeking desired $\hat{z}$ is simplified to seeking $\hat{\phi}$ that minimizes $\|\hat{\phi} - \tilde{\phi} + \tilde{Z}_- - Z_+\|$. Let $d_0(x) := \hat{d}_0(x) - \tilde{d}_0(x)$, then $d_0(x)$ is periodic. To the leading order, we have $\tau_0(x + nh) = \tau_0(x)(1 + \mathcal{O}(\varepsilon))$, $n = 1, 2$.

Since for $x \in \mathbb{R}$, $\hat{d}_{0,n}$ are multiplied to $e^{2\pi nix/h}$ which has period of $\frac{h}{|n|}$, while $\tau_0(x + nh) = \tau(x)(1 + \mathcal{O}(h))$ for n small, it suffice to consider $\hat{d}_0(x) = \hat{d}$ is a constant independent of x. Similarly, for real x, $\tilde{d}_0(x) = \tilde{d}$ is a constant independent of x.

By Lemma 4.41, $\tilde{z}_-(x) - z_+(x) = \mathcal{O}(\varepsilon^{\gamma_1}e^{\gamma})$ for $x \in \mathcal{R}$. If $\hat{d} - \tilde{d}$ is chosen to be higher order in ε than $\tilde{z}_-(x) - z_+(x)$, then the smallest distance will not change in the leading order of ε. If it is chosen to be lower order in ε, then the smallest distance will change to lower order in ε. Hence we choose $\hat{d} - \tilde{d} = K|\sigma|\Theta\varepsilon^{\gamma_1}e^{\gamma}$. Plugging into (4.263),we get

$$\begin{aligned}
\inf_{x\in\mathbb{R}} |\tilde{\mathbf{Z}}_-(x) - \mathbf{Z}_+(x)| &\geq |\sigma|\Theta\varepsilon^{\gamma_1}e^{\gamma} \sum_{n=0,1,2} [K\tau_0(x + nh) + 2|\tau_{-1}(x + nh)| \cos(\theta(x))]^2 \\
&\sim |\sigma|\Theta\varepsilon^{\gamma_1}e^{\gamma} \left\{[K\tau_0(x) + 2|\tau_{-1}(x)|]^2 + 2[K\tau_0(x) - |\tau_{-1}(x)|]^2\right\}^{1/2} \\
&\sim |\sigma|\Theta\varepsilon^{\gamma_1}e^{\gamma}\{2[K\tau_0(x)]^2 + 6[|\tau_{-1}(x)|]^2\}^{1/2}
\end{aligned} \tag{4.264}$$

$\tau_0(x)$ is real for real x. Clearly, the righthand of (4.264) get the minimum at $K = 0$. Hence the theorem follows. □

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