THREE DIMENSIONAL FC ARTIN GROUPS ARE CAT(0)

DISSERTATION

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By

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ABSTRACT

Following earlier work of T. Brady, we construct locally CAT(0) classifying spaces for those Artin groups which are three dimensional and which satisfy the FC (flag complex) condition. The approach is to verify the "link condition" by applying gluing arguments for CAT(1) spaces and by using curvature testing techniques as suggested by the work of M. Elder and J. McCammond.

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CHAPTER 1

INTRODUCTION

It remains an open problem to decide whether or not every Artin group acts geometrically on a CAT(0) space. The answer, in fact, is not even known for the classical braid groups on more than four strings. In addition to providing a rich class of examples, an affirmative answer to this question would give a geometric proof of a number of group-theoretic properties which conjecturally hold for all Artin groups, including solvable word and conjugacy problems. We begin our investigation of this problem with an overview of "CAT(0) groups".

Let G be a group and let (X, d) be a geodesic metric space. Recall that a *geodesic metric space* is a metric space in which every pair of points x_1, x_2 can be joined by a *geodesic path*; i.e., there is a path $\gamma : [0, d(x_1, x_2)] \to X$ from x_1 to x_2 which is an isometric embedding. The image of such a path is called a *geodesic segment* and, by abuse of notation, is denoted by $[x_1, x_2]$. We say that G acts *geometrically* on X if the action is properly discontinous, cocompact, and by isometries.

Let $\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ be triangle in X, and consider a *comparison triangle*, $\bar{\Delta} = [\bar{x}_1, \bar{x}_2] \cup [\bar{x}_2, \bar{x}_3] \cup [\bar{x}_3, \bar{x}_1]$, in the Euclidean plane $\mathbb{E}^2$. By definition, this is a Euclidean triangle with vertices $\bar{x}_i$ and the same corresponding side lengths: $|\bar{x}_i - \bar{x}_{i+1}| = d(x_i, x_{i+1})$ for $i = 1, 2, 3$, indices read mod 3. We say that Δ satisfies the

CAT(0) inequality if for each $p, q \in \Delta$, $d(p,q) \leq |\bar{p} - \bar{q}|$, where $\bar{p}$ and $\bar{q}$ are the points in $\bar{\Delta}$ that are the same corresponding distance from the vertices: $d(p, x_i) = |\bar{p} - \bar{x}_i|$ and $d(q, x_i) = |\bar{q} - \bar{x}_i|$ for $i = 1, 2, 3$. If every triangle in X satisfies the CAT(0) inequality, we say that X is a *CAT(0) space*. If a group G acts geometrically on a $CAT(0)$ space X, then we say that G is a *CAT(0) group*.

CAT(0) spaces are necessarily uniquely geodesic and contractible. So, for instance, the Euclidean and hyperbolic planes are CAT(0) spaces, but a sphere (with any metric) is not a CAT(0) space. If a group G acts geometrically on a CAT(0) space X, then X is a classifying space for G. The CAT(0) groups generalize the class of groups which which are the fundamental group of a compact, non-positively curved Riemannian manifold. We refer the reader to the book by M. Bridson and A. Haefliger [BH] for a systematic study CAT(0) groups. The terminology and notation used herein is generally consistent with their book.

Returning to the open problem, we may re-phrase it as follows:

Open Problem. *Is every Artin group a CAT(0) group?*

There are some partial answers to this question. R. Charney and M. Davis [CD] have shown that each Artin group acts geometrically its “Salvetti complex”. This is a piecewise Euclidean cube complex; the complex is CAT(0) if and only if the Artin group is “right-angled”. Some (trivial) examples of right angled Artin groups are finitely generated free groups and finitely generated free abelian groups. Each of these is easily seen to be a CAT(0) group by considering its action, respectively, on a metric tree or on Euclidean space.

T. Brady and J. McCammond [BM] recently approached the problem in a novel way. They discovered new finite presentations for "two dimensional" Artin groups. Then they showed that many of the associated presentation 2-complexes are locally CAT(0). It follows from the Cartan-Hadamard theorem for locally CAT(0) spaces that the universal cover of such a complex is (globally) CAT(0); hence, the fundamental group is acting geometrically on a CAT(0) space via deck transformations. Thus, many of the two-dimensional Artin groups are CAT(0); in particular, their techniques show that "two dimensional FC" Artin groups are CAT(0).

T. Brady [Br1] continued this line of investigation for the finite type Artin groups with three generators. These are precisely the Artin groups whose associated Coxeter group is an essential finite reflection group on $\mathbb{R}^3$; there are only three such Coxeter groups which do not split as a direct product: the full symmetry groups of the tetrahedron, the cube, and the dodecahedron. For each such Artin group, G, Brady constructed a three dimensional, connected, piecewise Euclidean complex K (with a single vertex v_0) so that $\pi_1(K, v_0) \cong G$. He then showed that K is a locally CAT(0) space by cleverly verifying the *link condition*, i.e. that the (geometric) link of v_0 in K is a CAT(1) space. As a corollary, he concludes that G acts geometrically via deck transformations on the universal cover of K. Inspired by this last result, we will prove the following:

Main Theorem. *Every three dimensional FC Artin group is CAT(0).*

The complex we will consider is an amalgamation of the spaces considered by Brady. However, unlike Brady's complex, the link of v_0 does not split as a join of

CAT(1) spaces. In his case, the link is a (spherical) suspension of a 1-complex. As the spherical suspension of a CAT(1) space is CAT(1), it sufficed to check that a certain 1-complex was CAT(1); this is essentially a combinatorial condition. Whereas in the complexes we will consider, the link is not a suspension. The difficulty, then, is to check that a given piecewise spherical 2-complex is CAT(1). With the exceptions of Gromov's "all-right" criterion for cubical complexes [G] and Moussong's Lemma for complexes with polyhedral cells of "size $\geq \pi/2$" [M], there are no known combinatorial characterizations of CAT(1) 2-complexes. We are able to overcome this difficulty by using gluing arguments for CAT(1) spaces and by using "curvature testing" techniques as in the recent work of M. Elder & J .McCammond [EM]. Combined with some deep results of B. Bowditch on locally CAT(1) spaces [Bow], we demonstate that curvature testing is an effective way to study piecewise spherical 2-complexes.

The structure of the complex we will consider is closely related to the structure of special subgroups in Coxeter groups. Thus, we begin with an introduction to Artin groups and their associated Coxeter groups.

CHAPTER 2

ARTIN GROUPS AND COXETER GROUPS

2.1 Overview

Definition. Let S be a finite set of cardinality n. A *Coxeter matrix* for S is an $n \times n$ symmetric matrix with entries $m_{ij} \in \{1, 2, \ldots, \infty\}$ such that $m_{ij} = 1$ if and only if $i = j$. Fix a Coxeter matrix M, and let A be the group given by the following presentation:

$$A = \langle S \mid \langle s_i, s_j \rangle^{m_{ij}} = \langle s_j, s_i \rangle^{m_{ij}} \rangle,$$

where $\langle s_i, s_j \rangle^{m_{ij}}$ means the string $s_i s_j s_i \cdots$ having m_{ij} letters when $m_{ij} < \infty$. Such a relation will be called an *Artin relation* of length m_{ij}. If $m_{ij} = \infty$, the relation $\langle s_i, s_j \rangle^{m_{ij}} = \langle s_j, s_i \rangle^{m_{ij}}$ is omitted from the presentation. The pair (A, S) is called an *Artin system* and the group A is called an *Artin group*.

Similarly, we define a group W by the following presentation:

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where, again, we omit the relation if $m_{ij} = \infty$. In particular, we note that if $i = j$, then we have the relation $s_i^2 = 1$. The pair (W, S) is called a *Coxeter system*, and the group W is called a *Coxeter group*.

Given an Artin system (A, S), we refer to (W, S) as the *associated* Coxeter system. If the associated Coxeter group W is finite, we say that the Artin system is *spherical*. Similarly, by a *spherical* Coxeter system, we mean that the Coxeter group is finite. If the associated Coxeter group is infinite, then we say the Artin or Coxeter system is of *infinite type*. Note that it is common in the literature to use the term "finite type Artin group" instead of "spherical Artin group".

Example. Let M be the $n \times n$ Coxeter matrix with entries $m_{ij} = 2$, for $|i-j| > 2$, $m_{i,i+1} = 3$, and $m_{ii} = 1$. Then the presentation for A is exactly the usual presentation of the braid group on $n+1$ strings. The generator s_i represents the braid which crosses the i-th string over the $(i+1)$-st string and which leaves the other $n-1$ strings fixed. The braid relations are exactly those appearing in the presentation for A, namely s_i and s_j commute if $|i-j| > 2$ and s_i and s_{i+1} satisfy the relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. The associated Coxeter group W is the symmetric group on $n+1$ letters. As the symmetric group is a finite group, the braid group is an example of a spherical Artin group.

Note that the associated Coxeter group W is naturally a quotient of its Artin group. We have a surjective homomorphism $\pi : A \to W$ sending each generator $s \in S \subset A$ to the generator denoted by the same letter in W. The kernel of this map is referred to as the *pure Artin group.* In the example above, the kernel of the map from Braid(n+1) to Sym(n+1) is the pure braid group— those braids whose strings begin and end at the same node.

Definition. Given a (possibly empty) subset $T \subseteq S$. Define A_T and W_T to be the subgroups of A and, respectively, W generated by T. These subgroups are called *special subgroups*. The spherical special subgroups (or in the case of Coxeter groups, the finite special subgroups) will be referred to as *spherical subgroups*. They are indexed by the *spherical subsets* of S:

$$\mathcal{S} = \{\, T \subseteq S \mid W_T \text{ is finite} \,\}.$$

For any subset $T \subseteq S$, one can define a new Artin system, $(A(T), T)$, or a new Coxeter system, $(W(T), T)$, by forming the Coxeter matrix M_T whose entries are those entries of M indexed by pairs $(i, j) \in T \times T$. There are obvious homomorphisms $A(T) \longrightarrow A_T$ and $W(T) \longrightarrow W_T$. In fact, these maps are isomorphisms. Also, it is a fact that $A_T \cap A_{T'} = A_{T \cap T'}$ and $W_T \cap W_{T'} = W_{T \cap T'}$. The proofs for Coxeter groups are in Bourbaki [Bo]. The proofs for Artin groups are in van der Lek's Ph.D. thesis [L].

Definition. There is another, more visual, way to define an Artin system or Coxeter system: let S be a set of cardinality n and let M be a Coxeter matrix for S. Let $\mathcal{G}$ be the labeled graph with vertex set S having a single edge labeled m_{ij} joining s_i to s_j whenever $1 < m_{ij} < \infty$. The graph $\mathcal{G}$ is called a *Coxeter graph*. Clearly, a Coxeter graph contains precisely the same information as a Coxeter matrix.

Remark. Note that what we call a Coxeter graph is not the same as the “Dynkin diagrams” encountered in the study of Lie algebras. In Lie theory, the graph has, again, S as its vertex set; but its edges join those vertices for which $m_{ij} > 2$. Such an edge is labeled by m_{ij} if $m_{ij} > 3$ and no label is given if $m_{ij} = 3$.

Definition. Let Γ have vertex set S. Say that a nonempty set of vertices $T \subseteq S$ spans a simplex in Γ whenever $T \in \mathcal{S}$. We will refer to Γ as the *link complex* of the Artin or Coxeter system. As the only two generator spherical Coxeter groups are the finite dihedral groups ($m_{ij} < \infty$), the graph $\mathcal{G}$ (without labels) is precisely the 1-skeleton of Γ.

Definition. The *dimension* of an Artin system (A, S) is $\max\{|T| : T \in \mathcal{S}\}$. It follows that $\dim(A, S) = \dim\Gamma + 1$, where Γ is the link complex of (A, S). When the context is clear, we say that $\dim(A, S)$ is the dimension of the Artin group.

In particular, if (A, S) is a three dimensional Artin system and W is the associated Coxeter group, then there is some $T \subseteq S$ of cardinality three such that the special subgroup W_T is finite and such that any other subset $T \subseteq S$, having four or more elements, generates an infinite subgroup of W.

Remark. It is conjectured and, in many cases, known that the dimension of an Artin system (A, S) is equal to the cohomological dimension of A [CD]. To each Artin system, there is an associated complexified hyperplane complement, Q. The Artin group acts freely on the universal cover $\widetilde{Q}$ and the quotient space is conjectured to be a $K(A, 1)$ space. Those familiar with this work will recognize the cone on the link complex as a chamber of the *Deligne complex*— a piecewise Euclidean cell complex which is homotopy equivalent to $\widetilde{Q}$. For many Artin groups, it is known that the Deligne complex is CAT(0). However, this does not answer the CAT(0) question for Artin groups— the groups do not act properly on this complex.

Definition. An Artin system is said to satisfy the *FC condition* if for each $T \subseteq S$ we have $T \in \mathcal{S}$ if and only if $m_{ij} < \infty$ for all $s_i, s_j \in T$. This is equivalent to the requirement that the link complex Γ be a flag complex (FC), i.e. a subset $T \subseteq S$ spans a simplex of Γ if and only if every distinct pair of vertices $s_i, s_j \in T$ spans an edge. When the context is clear, we say that the group is FC.

Example. Let (A, S) be the Artin system with $S = \{s_1, s_2, s_3\}$ and with $m_{ij} = 3$ for $i \neq j$. Geometrically, the associated Coxeter group, W, can be realized as the subgroup of isometries of the Euclidean plane generated by the affine reflections across three lines which meet pairwise, forming an equilateral triangle. The product of two such reflections is a rotation by $2\pi/3$. So each special subgroup with two generators is a dihedral group of order six. But, the group W is not finite: the W-orbit of any equilateral triangle covers the entire plane. So, in terms of the complex Γ, we have that each distinct pair $\{s_i, s_j\}$ spans a simplex; but $\{s_1, s_2, s_3\}$ does not— the subgroup, namely all of W, is not finite. Thus (A, S) is not FC.

We briefly recall some of the theory of Coxeter groups. We refer the reader to the books by N. Bourbaki [Bo], K. Brown [Bro], and J. Humphreys [H] for an introduction to Coxeter groups.

Definition. Let (W, S) be a Coxeter system. The *reflections* of (W, S) are the elements of the set $R = \{\, wsw^{-1} \in W \mid w \in W, s \in S\,\}$. Given $1 \neq w \in W$, define its *reflection length* or simply its *length*, denoted by $\ell(w)$, to be the smallest $k \in \{\,1, 2, \ldots\,\}$ such that $w = r_1 \ldots r_k$, where each r_i is a reflection. By convention, $\ell(1) = 0$. Similarly, given $T \subset S$, we define $R_T := \{\, wtw^{-1} \in W \mid w \in W_T, t \in T\,\}$. If

$w \in W_T$, we denote its reflection length with respect to R_T by $\ell_T(w)$. Let $\mathcal{R}$ denote the union of all R_T such that $T \in \mathcal{S}$. We refer to these reflections as the *reflections of spherical type.*

Remark. In the standard references on Coxeter groups, $\ell(w)$ denotes the length of w with respect to the Coxeter generating set S. As we will have no need to use this length function, there should be no confusion.

The term "reflection" is justified by the following fact about Coxeter groups.

Theorem. *(Geometric Representation) Let (W, S) be a Coxeter system. Let V be a vector space of dimension $|S|$. Then there is a canonical faithful linear representation $\sigma : W \to GL(V)$.*

A proof of this theorem can be found in any of the references on Coxeter groups. As in [Bo], the geometric representation is used to study the relationship between W and its special subgroups W_T. Passing to the contragredient representation of σ, W acts on the dual space V^*. From here, it can be shown that there is a polyhedral cone $\bar{C}$ which is a strict fundamental domain for the action of W on a W-invariant subset $U \subset V^*$ called the Tits cone. The stabilzer of a point in an open face C_T of $\bar{C}$ is precisely the special subgroup W_T. The maximal open face, $C = C_\emptyset$ is called the fundamental chamber.

If (W, S) is a spherical Coxeter system, then W can be faithfully represented as a discrete subgroup of $O(V)$, where $O(V)$ is the subgroup of $GL(V)$ preserving a positive definite bilinear form. Moreover, the matrix C which represents the form with respect to the standard basis is $C = (\cos(\pi/m_{ij}))$, where $M = (m_{ij})$ is the Coxeter matrix of (W, S). The set of elements of W which act as orthogonal reflections with respect to this form is precisely the set of reflections as defined above. In fact, the finite Coxeter groups are precisely the finite subgroups of $GL(n, \mathbb{R})$ generated by reflections. For this reason, the finite Coxeter groups are often called finite reflection groups.

When (W, S) is an infinite type Coxeter system, the form is no longer positive definite. Still, each reflection $r \in R$ acts by a "reflection" in the sense that $\sigma(r)$ fixes a codimension 1 hyperplane in V, has a simple (-1) eigenvalue, and $\sigma(r)^2 = 1$.

Coxeter groups admit other interesting geometric interpretations too. Every Coxeter group W acts geometrically on its $Coxeter - Davis$ complex X (see [DM] for a good survey). It was shown by Moussong [M] that X admits a CAT(0) metric in a very natural way. Moreover, the elements of $\mathcal{R}$ act by reflections in the "walls" of X.

Definition. Let (W, S) be a Coxeter system, where S has cardinality n. An element $x \in W$ of the form $x = s_{i_1} \dots s_{i_n}$, where $\{ i_1, \dots, i_n \}$ is a permutation of $\{ 1, \dots, n \}$, is called a *Coxeter element*.

Our construction of a non-positively curved $K(A, 1)$ space for each three dimensional FC Artin group is directly related to a partial ordering of the associated Coxeter group with respect to a family of Coxeter elements chosen for each spherical subgroup $W_T \subseteq W$. We describe this partial ordering in the next section.

2.2 Allowable elements and allowable expressions

Definition. Let (W, S) be a spherical Coxeter system. Let R be the set of reflections. Then the reflection length ℓ defines a relation, $\leq$, on W as follows:

$$w \leq w' \text{ if an only if } \ell(w) + \ell(w^{-1}w') = \ell(w').$$

Regarding R as a generating set for W, we say a word, $r_1 \dots r_k$, is *reduced* if $\ell(r_1 \dots r_k) = k$. A *prefix* of a reduced word $r_1 \dots r_k$ is a word of the form $r_1 \dots r_i$ for some $i, 1 \leq i \leq k$. The empty word is also considered a prefix.

Lemma 2.1. *Let (W, S) be a spherical Coxeter system, let R be the set of reflections, and let $\leq$ be the relation as above. Suppose $w, w' \in W$. Then $w \leq w'$ if and only if w is prefix of a reduced word in R representing w'. Thus, the relation, $\leq$, defines a partial order on W.*

Proof. This follows from the following more general observation:

If G is group with a finite generating set S, then G is a poset via the relation $g \leq g'$ if and only if there is a geodesic from 1 to g' passing through g in the Cayley graph $\Gamma(G, S)$. The relation is clearly reflexive. If $g \leq g'$ and $g' \leq g$, then the geodesics have the same end vertex: $g = g'$. So, the relation is anti-symmetric. Finally, by possibly replacing a geodesic subpath from 1 to g' with one passing through g, we see that $g \leq g'$ and $g' \leq g''$ imply that $g \leq g''$. So, $\leq$ defines a partial order.

The reflection length of $w \in W$ is evidently the same as the distance from the identity to w in the Cayley graph $\Gamma(W, R)$. And prefixes are just geodesic subpaths beginning at the identity. This proves the lemma. $\square$

We will use $<$ to denote the strict partial order on W. Thus, $w < w'$ if and only if $w \leq w'$ and $w \neq w'$ if and only if $w \leq w'$ and $\ell(w) < \ell(w')$.

Now let (W, S) be an arbitrary Coxeter system. For each $T \in \mathcal{S}$, there is a partial order $\leq_T$ on W_T with respect to the reflections R_T. We will show that these partial orders agree on the intersection of any collection of spherical subgroups.

A first step in this direction is the well known fact that $R \cap W_T = R_T$. Here is a quick proof: (See K. Brown's book for the definitions of walls and galleries [Bro].) Given $r \in R \cap W_T$, choose an S-reduced word for r. By the solution to the word problem for W, this word only involves letters in T. If we consider the resulting minimal gallery from C to rC, we see that it must cross the wall corresponding to r. Thus, $r = wtw^{-1}$ for some $w \in W_T$ and $t \in T$. The other inclusion is obvious.

We will use the following theorem due to R. Carter; refer to Lemma 2.8 in [Ca] for a proof. The theorem in its stated form is due to D. Bessis [Be]. Also, see Proposition 2.2 in [BW] for an independent proof.

Theorem. *(Carter's Lemma) Let (W, S) be a finite Coxeter system with reflections R and reflection length function ℓ. Suppose $\rho : W \to GL(V)$ is a faithful linear representation of W on a finite dimensional vector space $V \cong \mathbb{R}^n$ such that, for every $w \in W$, $codim(ker(\rho(w) - Id)) = 1$ if and only if $w \in R$. Suppose $w \in W$. Then the reflection length of w is equal to the codimension of its fixed subspace: $\ell(w) = codim(ker(\rho(w) - Id))$.*

Remark. For any Coxeter group, the geometric representation $\sigma : W \to GL(V)$ has the stated property: a non-trivial $w \in W$ fixes a codimension one hyperplane in V

if and only if w is a reflection. However, the conclusion of Carter's Lemma does not hold for arbitrary infinite Coxeter groups. For instance, the Coxeter group W with the Coxeter graph consisting of three disjoint vertices acts on $\mathbb{R}^3$ via its geometric representation. But the square of the product of the three generators has reflection length four. However, Carter's Lemma does hold for the $\tilde{A}_n$ Coxeter groups, and likely holds for all Euclidean and, perhaps, many hyperbolic Coxeter groups.

The following theorem is due to R. Charney and the author. A similar result for finite Coxeter groups can be found in [CP].

Theorem 2.1. *Let (W, S) be a Coxeter system and let R be the set of reflections. Suppose that $w = r_1 \dots r_k$ is R-reduced. If $w \in W_T$ and $T \in \mathcal{S}$, then $r_i \in R_T$ for all i.*

Proof. Let $n = |S|$ and consider the geometric representation $\sigma : W \to GL(V)$. Pass to the contragredient representation, so that W acts on V^*. Each reflection of W acts by a reflection in a hyperplane of $V^* \cong \mathbb{R}^n$. Let $w \in W_T$ and write $w = r_1 \dots r_k$ as an R-reduced word. Assume that $T \in \mathcal{S}$ so that W_T is a finite Coxeter group. We can also write w as an R_T-reduced word: $w = q_1 \dots q_l$, where each $q_i \in R_T$. Necessarily, $k \leq l$. Let $F := \cap_{i=1}^{k} H_i$, where each H_i is the codimension one hyperplane fixed by r_i. Let $Fix(w) := \{ v \in V^* : w.v = v \}$. Carter's Lemma, applied to σ restricted to W_T, tells us that $l = \ell_T(w)$ is equal to the codimension of $Fix(w) \subset V^*$. On the other hand, w fixes the subspace F; so $F \subset Fix(w)$. As $\operatorname{codim}(F) \leq k \leq l$, we must have equality: $F = Fix(w)$. In particular, each r_i fixes every point in $Fix(w)$. Choose a point $x \in C_T$, the open face of the fundamental chamber C. The stabilizer of x is

W_T. So, $x \in Fix(w) = F$, and, hence, each r_i fixes x. Thus, each r_i belongs to W_T; and so, each $r_i \in R \cap W_T = R_T$. $\square$

The following is an immediate corollary:

Corollary 2.1. *Let (W, S) be a Coxeter system, let R be the set of reflections, and let ℓ denote the reflection length. If $w \in W_T$ and $T \in \mathcal{S}$, then $\ell(w) = \ell_T(w)$. In particular, the reflection length functions, ℓ_T, and the partial orders, $\leq_T$, agree on the intersection of spherical subgroups.*

Definition. Let (W, S) be a Coxeter system together with a total ordering $\prec$. Set $x_T := t_1 \dots t_k$ where $T = \{t_1 \prec \cdots \prec t_k\}$. Thus, the total ordering chooses a Coxeter element x_T for each Coxeter system (W_T, T).

For each $T \in \mathcal{S}$, let $\leq_T$ be the partial order on W_T with respect to the reflections R_T. Define the *allowable elements* of W_T thus:

$$Allow(x_T) := \{\, w \in W_T \,|\, 1 \neq w \leq_T x_T \,\}.$$

Define the allowable elements of W to be the set

$$Allow(W) := \bigcup_{T \in \mathcal{S}} Allow(x_T)$$

Each set $Allow(x_T)$ is a subset of W; so the union is understood to be a union of subsets in W.

By Lemma 2.1, the x_T-allowable elements are precisely the nontrivial elements of W_T which can be represented as a prefix of an R_T-reduced expression of x_T. In particular, by repeated application of the move $x = r_1 r_2 \dots r_n = r_2(r_2^{-1} r_1 r_2) r_3 \dots r_n$, it is easy to deduce that $T \subset Allow(x_T)$. In fact, as we shall prove later, every reflection in R_T is allowable, i.e. $R_T \subset Allow(x_T)$.

The posets $(W_T, \leq_T)$ are subposets of a larger poset $(\mathcal{W}, \leq)$.

Theorem 2.2. *Let (W, S) be a Coxeter system and let ℓ denote the reflection length with respect to R. Define $\mathcal{W} := \bigcup_{T \in \mathcal{S}} W_T$. Then $\mathcal{W}$ is a poset via the relation $w \leq w'$ if and only if $w, w' \in W_T$ and $w \leq_T w'$ for some $T \in \mathcal{S}$. Moreover, $w \leq w'$ if and only if w is a prefix of some R-reduced expression for w'.*

Proof. It is easy to see that the relation is reflexive and anti-symmetric. Suppose $w \leq w'$ and $w' \leq w''$, then there exists $T, T' \in \mathcal{S}$ such that $w \leq_T w' \in W_T$ and $w' \leq_{T'} w'' in W_{T'}$. By Lemma 2.1, w is an R_T reduced prefix of w'. But $w' \in W_{T'}$, so, by Theorem 2.1, each reflection appearing in this reduced word also belongs to $R_{T'}$. And, as $\ell_T(w) = \ell_{T'}(w)$, the word is $R_{T'}$-reduced. Substituting this word for the prefix representing w' in an $R_{T'}$-reduced word for w'', we see that $w \leq_{T''} w''$. Thus, the relation is transitive.

Now suppose $w \leq w'$. So, $w \leq_T w'$ for some $T \in \mathcal{S}$ such that $w, w' \in W_T$. Lemma 2.1 says that w is a prefix of and R_T-reduced word for w'. By Theorem 2.1, this word is R-reduced. On the other hand, if w is a prefix of an R-reduced word for $w' \in \mathcal{W}$. Then, by Theorem 2.1, the word is R_T-reduced, where $w' \in W_T$ for some $T \in \mathcal{S}$. Hence, $w \leq_T w'$. $\square$

The following is an easy corollary:

Corollary 2.2. *Let (W, S) be a Coxeter system together with a total ordering of S and let $(\mathcal{W}, \leq)$ be the poset as above. Then*

$$Allow(W) = \{w \in \mathcal{W} : 1 < w \leq x_T \text{ for some } T \in \mathcal{S}\}.$$

Proof. Suppose $w \in \mathcal{W}$ and $1 < w \leq x_T$ for some $T \in \mathcal{S}$. Then, by Theorem 2.2, w is a prefix of an R-reduced expression of x_T. We can write x_T as a R_T-reduced and, hence, an R-reduced expression in W_T. Thus, $1 < w \leq_T x_T$. So, $w \in Allow(x_T)$. The opposite inclusion is immediate from the definitions. □

Proposition 2.1. *Let (W, S) be a Coxeter system together with a total ordering of S. Suppose $T, T' \in \mathcal{S}$ are such that $T \subset T'$. Then $Allow(x_T) \subset Allow(x_{T'})$ is an inclusion of posets.*

Proof. We have that $x_T \leq_{T'} x_{T'}$ and $x_T, x_{T'} \in W_{T'}$. So, $x_T \leq x_{T'}$. Now, observe the following: if $r_1 \ldots r_k$ is an R-reduced expression, then we may perform a shift move: $r_i(r_i r_1 r_i) \ldots (r_i r_{i-1} r_i) r_{i+1} \ldots r_k$ is also R-reduced. Suppose $T = \{t_{i(1)} \prec \cdots \prec t_{i(m)}\} \subset T' = \{t_1 \prec \cdots \prec t_k\}$. Apply the shift move to $x_{T'} = t_1 \ldots t_k$ until x_T appears as a prefix of a reduced word. This is possible because the total ordering of S gave a consistent choice of Coxeter elements. □

To summarize, the set of allowable elements, $Allow(W)$ is a poset under the relation $\leq$. Two elements are related if and only if one is a prefix of a reduced expression for the other. Moreover, the notions of R-reduced and R_T-reduced are the same on W_T. In particular, every allowable element w has a well defined length $\ell(w)$.

Remark. These partial orders have been studied by a number of other researchers in the case where (W, S) is a spherical Coxeter system, cf. D. Bessis [Be]; D. Bessis, F. Digne, J. Michelle [BDM]; J. Birman, K. Ko, & J. Lee [BKL]; T. Brady [Br2]; T. Brady & C. Watts [BW]; and M. Picantin [P]. Our notation is consistent with [Be]. In each of these articles, the object of study is a "dual braid monoid". D. Bessis, building on the other authors' partial results, proves that if the Coxeter element x_S is correctly chosen, then the group of fractions of the dual braid monoid is isomorphic to the associated Artin group. However, it is unclear how or if this result generalizes to an infinite type Artin group.

Definition. We say a sequence of allowable elements $(w_1, \ldots, w_k)$ in $Allow(x_T)$ defines an *allowable expression* of length k, $k > 0$, if the product $w_1 \cdots w_k \in Allow(x_T)$ and $\sum_{i=1}^{k} \ell(w_i) = \ell(w_1 \ldots w_k)$. Denote the allowable expressions of length k by $Expr(x_T; k)$ and all the allowable expressions by $Expr(x_T)$.

We define allowable expressions in W to be the set

$$Expr(W) = \bigcup_{T \in \mathcal{S}} Expr(x_T).$$

$Expr(W)$ is understood to be a union of sequences in W. Note that $Allow(W) = Expr(W; 1)$.

In particular, the allowable expressions of length $|T|$ in $Expr(x_T)$ correspond to all the R-reduced words which represent x_T. It also noteworthy that, equivalently, an allowable expression is a sequence $(w_1, \ldots, w_k)$ of elements of $Allow(W)$ such that $\Sigma_{i=1}^{k} \ell(w_i) = \ell(w_1 \ldots w_k)$, and $w_1 \ldots w_k \in Allow(W)$. For, by definition $w_1 \ldots w_k \leq$

x_T for some $T \in \mathcal{S}$. It follows that each $w_i \leq x_T$, as well. Thus, $(w_1, \ldots, w_k) \in Expr(x_T; k)$.

Proposition 2.2. *If $T \subset T' \in \mathcal{S}$, then $Expr(x_T) \subset Expr(x_{T'})$.*

Proof. Use the fact that $Allow(x_T) \subset Allow(x_{T'})$ (Proposition 2.1). If $(w_1, \ldots, w_k) \in Expr(x_T; k)$, then each w_i belongs to $Allow(x_{T'})$ and so does the product $w_1 \ldots w_k$. The length condition, $\Sigma_{i=1}^k \ell(w_i) = \ell(w_1 \ldots w_k)$, is stated independent of T or T'. $\square$

Lemma 2.2. *Let (W, S) be a Coxeter system of dimension ≤ 3 together with a total ordering of S. Let $T, T' \in \mathcal{S}$ and let $w \in \mathcal{W}$. Suppose $w \leq x_T, x_{T'}$. If $\ell(w) = |T \cap T'|$, then $w = x_{T \cap T'}$. Moreover, for each nonempty $T \in \mathcal{S}$, there is a unique allowable element w of length $|T|$ belonging to W_T.*

Proof. First, observe that if $w \leq x_T, x_{T'}$, then $w \in W_T \cap W_{T'} = W_{T \cap T'}$. If $T \subset T'$, then $\ell(w) = |T \cap T'| = |T|$. But $\ell(w) + \ell(w^{-1} x_T) = \ell(x_T) = |T|$. So, $w = x_T = x_{T \cap T'}$. Everything so far is true regardless of the dimension.

Now suppose that neither T nor T' is a subset of the other. If the intersection has cardinality one, then w belongs to $W_{T \cap T'}$— a group with only one non-trivial element. So $w \in T \cap T'$ and the conclusion holds. The only remaining possibility is that w has length two and T and T' have have three elements. Suppose $T = \{t_1 \prec t_2 \prec t_3\}$ and $T \cap T' = \{t_i \prec t_j\}$. Then, by shifting, we can write $x_T = t_i t_j u$, reduced, for some $u \in R$. But u cannot belong to $R_{T \cap T'}$, because $x_T \notin W_{T \cap T'}$. Now $w \leq x_T$ has length two, so there exists a $q \in R$ such that $r_1 r_2 q = x_T = t_i t_j u$. So, $uq = t_j t_i r_1 r_2 \in W_{T \cap T'}$. By Theorem 2.1, uq cannot be reduced— u would belong to $R_{T \cap T'}$. So, $uq = 1$. Thus, $w = r_1 r_2 = t_i t_j = x_{T \cap T'}$.

For the second statement, suppose $w \in \mathit{Allow}(W; |T|) \cap W_T$, where $T \in \mathcal{S}$ is nonempty. If $|T| = 1$, there is only one nontrivial element in W_T. So, $w = x_T$. If $|T| = 2$, suppose $x \leq x_{T'}$. Because $\ell(w) = 2$, we must have that $T \subset T'$. From the above, we may assume that $|T'| = 3$. So, there is an R-reduced expression $x_{T'} = wq = t_i t_j u$, where q and u are reflections and $T = \{t_i \prec t_j\} \subset T'$. As above, we must have that $uq = 1$. So, $w = t_i t_j = x_T$. Finally, if $|T| = 3$, suppose $w \leq x_{T'}$. Again, $\ell(w) = 3$ forces $T \subset T'$ (every element of a dihedral group is a product of at most two reflections). But, then, W is three dimensional, so $T = T'$ and, hence, $w = x_T$. $\square$

A more unified statement is the following:

Proposition 2.3. *Let (W, S) be a Coxeter system of dimension ≤ 3 together with a total ordering of S. Suppose $T \in \mathcal{S}$. Then $\mathit{Allow}(W) \cap W_T = \mathit{Allow}(x_T)$ and $\mathit{Expr}(W; k) \cap (W_T)^k = \mathit{Expr}(W_T; k)$ for each k.*

Proof. Suppose $w \in \mathit{Allow}(W) \cap W_T$. If $\ell(w) = 1$, then w is a reflection in R_T. Corollary 3.1 (proof in the next chapter) implies that $w \in R_T \subset \mathit{Allow}(x_T)$. Suppose $\ell(w) = 2$ and $w \leq x_{T'}$ for some $T' \in \mathcal{S}$. By Proposition 2.1, we may assume T is minimal. So, $T \subset T'$. Either $|T| = 2$, forcing $y = x_T$, or $T = T'$. In either case, $w \in \mathit{Allow}(x_T)$. Finally if $\ell(w) = 3$, then $w = x_T$. Thus, $\mathit{Allow}(W) \cap W_T \subset \mathit{Allow}(x_T)$. The opposite inclusion is immediate. Similarly, there is only one nontrivial inclusion in the second statement. Suppose $(w_1, \ldots, w_k) \in \mathit{Expr}(W; k) \cap (W_T)^k$. Then, each $w_i \in \mathit{Allow}(W) \cap W_T = \mathit{Allow}(x_T)$. By subgroup closure, the product $w_1 \ldots w_k$ belongs to $\mathit{Allow}(W) \cap W_T$. It follows that $(w_1, \ldots, w_k) \in \mathit{Expr}(x_T; k)$. $\square$

Remark. Lemma 2.2 and Proposition 2.3 probably admit generalizations to all dimensions. However, the brute force arguments given here provide little insight. The work of D. Bessis [Be] is particularly recommended to those who may want to generalize these statements.

We are now ready to define the proposed non-positively curved $K(A, 1)$ space for a three dimensional FC Artin group A. This complex is defined purely in terms of a three dimensional Coxeter system together with a total ordering of its generating set.

CHAPTER 3

BRADY'S COMPLEX

3.1 The construction

Let Γ be a link complex of dimension together with a total ordering of its vertices. So, the complex Γ defines a Coxeter system (W, S) together with a total ordering of its generating set S. To emphasize its origin, we may denote the Coxeter group by W_Γ. Similarly, the Artin group defined by Γ may be denoted by A_Γ.

Assume that Γ is a simplicial complex of dimension ≤ 2. Thus, Coxeter group $W = W_\Gamma$ has dimension ≤ 3. Let $K = K_\Gamma$ be the cell complex (*Brady's complex*) defined as follows:

- 0-skeleton $K^{(0)}$: a single vertex labeled v_0.

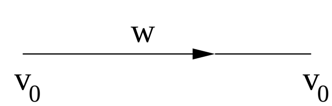

Figure 3.1: A labelled, oriented 1-cell.

- 1-skeleton $K^{(1)}$: a labelled, oriented edge for each allowable element, $w \in Allow(W)$, with both ends attached at v_0.

- 2-skeleton $K^{(2)}$: a two simplex for each allowable expression of length two: $(w_1, w_2) \in Expr(W; 2)$. The edges of the simplex are labeled by w_1, w_2, and $w_1 w_2$, viewed as elements of W, not as words. The boundary of this 2-simplex is attached to the 1-skeleton according to its labeling and orientation. Refer to Figure 3.2.

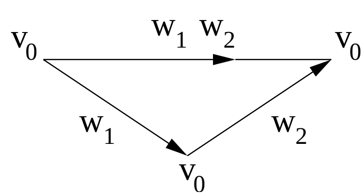

Figure 3.2: A 2-cell.

- 3-skeleton $K^{(3)}$: a three simplex for each allowable expressions of length three: $(w_1, w_2, w_3) \in Expr(W; 3)$. It's six edges are labeled by the following elements of W:

$$w_1, w_1 w_2, w_2, w_2 w_3, w_3, w_1 w_2 w_3.$$

Their orientation and adjacency is shown in Figure 3.3. The boundary of this 3-simplex is attached to the 2-skeleton accordingly.

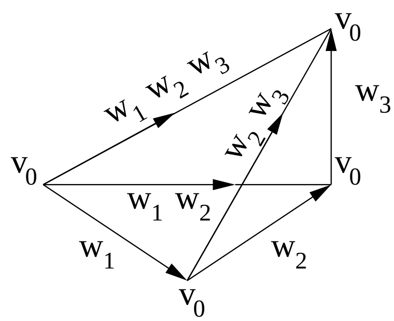

Figure 3.3: A typical 3-cell in K.

From the description of the the 2-skeleton, it follows that the fundamental group of K is presented thus:

$$\pi_1(K, v_0) = \langle \{[w] : w \in Allow(W)\} \mid \{[w_1][w_2] = [w_1w_2] : (w_1, w_2) \in Expr(W; 2)\}\rangle,$$

or, informally, $\pi_1(K) = \langle Allow(W) \mid Expr(W; 2)\rangle$. The generator $[w]$ is called a *lift* of the allowable element $w \in W$. We use these brackets to distinguish the element of the fundamental group from the element of the Coxeter group.

Suppose Γ defines Artin system of dimension ≤ 3 together with a total ordering of its generating set. We may form the complex K_Γ by considering the associated Coxeter group. If the A_Γ is a spherical Artin group with generators $S = \{b \prec a \prec c\}$ such that $m_{ac} = 2$, then K is exactly the complex considered by T. Brady [Br1]. If A_Γ is a spherical Artin group with two generators, this is the 2-complex considered by T. Brady and J. McCammond in [BM]. And, if A_Γ has only one generator, then K_Γ is just a single oriented, labelled edge with its vertices identified.

If $T \in \mathcal{S}$ is a spherical subset, then we have an associated Brady complex. Define K_T to be the Brady complex associated to the Coxeter system (W_T, T) together with the total ordering of S restricted to T. Thus, K_T is a subcomplex of K_Γ. Also, if $T, T' \in \mathcal{S}$, then it follows from Proposition 2.1 and Proposition 2.2 that $K_{T \cap T'}$ is a subcomplex of K_T and K'_T.

The goal of the next few sections is to prove that $\pi_1(K_\Gamma, v_0) \cong A_\Gamma$.

3.2 The fundamental group of K_Γ

If A_Γ is a spherical Artin group of dimension ≤ 3 and W_Γ is its associated Coxeter group, then, regardless of the choice of total ordering, the cell complex K_Γ always has the same fundamental group. More precisely, we prove the following:

Theorem 3.1. *Let (W, S) be a spherical Coxeter system. Let x and y be two Coxeter elements for (W, S). Then there is an element $w_0 \in W$ such that $w_0 y w_0^{-1} = x$ in W. The automorphism of W, ϕ, mapping $w \mapsto w_0 w w_0^{-1}$ restricts to a bijection between Allow(y) and Allow(x). This, in turn, gives a bijection between Expr(y) and Expr(x). Moreover, if W has three or fewer generators, ϕ induces an isomorphism $\Phi : \pi_1(K_y) \to \pi_1(K_x)$, by setting $\Phi([a]) := [\phi(a)]$, where K_y and K_x are the Brady complexes defined by their respective Coxeter elements.*

Proof. The first statement, that all Coxeter elements in a finite Coxeter group are conjugate, is classical [Bo]. That ϕ is a bijection follows from the fact that it permutes the reflections and preserves the reflection length of any element of W.

In the special case of length two expressions, we see that the relation $[\phi(a_1)][\phi(a_2)] = [\phi(a_1a_2)]$ holds in $\pi_1(K_x)$. Hence,

$$\Phi([a_1][a_2]) = [\phi(a_1)][\phi(a_2)] = [\phi(a_1a_2)] = \Phi([a_1a_2]).$$

Thus, Φ takes relators to relators, and so, Φ is a well-defined group homomorphism. Moreover, Φ is clearly invertible and, hence, is an isomorphism. $\square$

As a corollary, we deduce that all the reflections in a spherical Coxeter system are allowable, regardless of the choice of Coxeter element.

Corollary 3.1. *Let (W, S) be a spherical Coxeter system and let x be a Coxeter element. Then every reflection is x-allowable.*

Proof. The hard work has already been done by D. Bessis. He proves that every spherical Coxeter system admits a "chromatic" Coxeter element y for which the set of reflections are y-allowable (use Lemma 1.3.4 in [Be]). Now, apply the bijection between $Allow(x)$ and $Allow(y)$ from the theorem above. The bijection preserves the set of reflections. Hence, every reflection is x-allowable. $\square$

To establish an isomorphism between an (infinite type) Artin group A_Γ and the fundamendal group of its Brady complex K_Γ, we seek compatible isomophisms $A_T \to \pi_1(K_T, v_0)$ for each $T \in \mathcal{S}$. As A_Γ will be assumed to have dimension ≤ 3, the spherical subgroups will have at most three generators. We consider each case separately. Recall that $\pi : A \to W$ denotes the natural map.

If $A = A_T$ is a spherical Artin group with one generator, s, and W is its associated Coxeter group, then:

1. $A = \langle\, s \mid \quad \rangle \cong \mathbb{Z}$,

2. $W = \langle\, s \mid s^2 = 1\,\rangle \cong \mathbb{Z}/2\mathbb{Z}$, and

3. $\pi_1(K) = \langle\, [s] \mid \quad \rangle \cong \mathbb{Z}$.

So, the map $\beta : s \to [\pi(s)]$ defines an isomorphism $A \to \pi_1(K)$.

If $A = A_T$ is a spherical Artin group with two generators, s_1 and s_2, and W is its associated Coxeter group, then there is an integer $m > 1$ such that

1. $A = \langle\, s_1, s_2 \mid < s_1, s_2 >^m = < s_2, s_1 >^m \,\rangle$,

2. $W = \langle\, a, b \mid (ab)^m = 1\,\rangle$, a dihedral group of order $2m$, and

3. $\pi_1(K) = \langle\, [x], [r_1], \ldots, [r_m] \mid [x] = [r_1][r_2], \ldots, [x] = [r_m][r_1]\,\rangle$.

The last statement appears in [BM]. For clarity and for purposes of introducing notation, we repeat the proof:

Proposition 3.1. *(T. Brady, J. McCammond) Let (A, S) be the two generator spherical Artin group with generating set $S = \{s_1,\, s_2\}$, and let (W, S) be the associated Coxeter system. Fix a Coxeter element: either $x = s_1 s_2$ or $x = s_2 s_1$. In either case, we have $\pi_1(K) = \langle\, [x], [r_1], \ldots, [r_m] \mid [x] = [r_1][r_2], \ldots, [x] = [r_m][r_1]\,\rangle$, where K is the Brady complex of (W, S) with respect to x.*

Proof. By Corollary 3.1, every reflection is x-allowable; and, clearly, x is the only allowable element of length 2. Let $r_1 = s_1, r_2 = s_2$, and let $r_i := r_{i-2}r_{i-1}r_{i-2}^{-1}$ for $i = 1, \ldots, m$. This enumerates the set of reflections. Geometrically, the reflections act by reflection in hyperplanes in $\mathbb{R}^2$. The hyperplanes of r_i and r_{i+1} meet with dihedral angle π/m. The Coxeter element, x, acts by rotation by $2\pi/m$. From the complete description of the dihedral group of order $2m$, it is easy to deduce that if $x = s_1 s_2$, then the allowable expressions of length two are precisely $(r_1, r_2), (r_2, r_3), \ldots, (r_{m-1}, r_m)$, and (r_m, r_1). If $x = s_2 s_1$, then we set $r_1 = s_2$ and $r_2 = s_1$ and the list of allowable expressions is the same. Thus, $\pi_1(K)$ has the desired presentation. $\square$

By performing Tietze transformations on this presentation, we can eliminate the generator $[x]$ to obtain an alternate presentation of the fundamental group:

Corollary 3.2. *Let Γ, together with a total ordering $\prec$ of the vertices, define a two generator spherical Artin group with generating set $S = \{s_1 \prec s_2\}$. Let W be the associated Coxeter group, and let $R = \{r_1, \ldots, r_m\}$ be the set of reflections. Then $\pi_1(K_\Gamma = \langle \{[r_i] : r_i \in R\} \mid [r_1][r_2] = [r_2][r_3], \ldots, [r_1][r_2] = [r_m][r_1] \rangle$.*

As in the one generator case, the fundamental group of the Brady complex is naturally isomorphic to the Artin group:

Proposition 3.2. *(T. Brady, J. McCammond [BM]) Let (A, S) be the two generator spherical Artin group with generating set $S = \{s_1,\ s_2\}$, and let (W, S) be the associated Coxter system. Then, for either choice of Coxeter element, the map β, which takes each generator $s_i \in S$ to the allowable element $[\pi(s_i)]$, defines an isomorphism of A onto $\pi_1(K, v_0)$.*

Refer to [BM] for a proof of Proposition 3.2. Bascially, consider the cases of m odd or m even separately and direcly verify that the desired relations hold in both presentations.

As in the case of spherical Artin groups with one or two generators, we want to have a complete understanding of the fundamental group of the Brady complex with respect to any Coxeter element in a three generator Coxeter group. Specifically, we will prove

Theorem. *Let (A, S) be a spherical three generator Artin group and let W be its associated Coxeter group. Then, regardless of the choice of Coxeter element in W, the map $\beta : A \to \pi_1(K, v_0)$ sending each generator s to $[\pi(s)]$ is an isomorphism.*

We will prove this theorem in the next section (Theorem 3.3). Assuming this result, we prove that the fundamental group of K_Γ is isomorphic to A_Γ. We begin by deriving an alternate presentation for the fundamental group of Brady's complex in the case of a three generator finite Coxeter group.

Lemma 3.1. *Let W_Γ be a finite three generator Coxeter group together with a total ordering,$\prec$, of the generating set $S = \{s_1 \prec s_2 \prec s_3\}$. Let $x := s_1 s_2 s_3$ denote the Coxeter element, and let R be the set of reflections in W. Then*

$$\pi_1(K, v_0) = \langle \{[r] : r \in R \mid \{[s_1][s_2][s_3] = [r_1][r_2][r_3] : (r_1, r_2, r_3) \in Expr(x; 3)\}\rangle.$$

Proof. We perform the following Tietze transformations on the presentation:

First, for every allowable element y of length two, there are exactly two allowable expressions involving y: (y, t) and (q, y), where q and t are reflections. So, we have exactly two defining relations, $[x] = [y][t]$ and $[x] = [q][y]$, which involve $[y]$. There are also defining relations of the form $[y] = [r][s]$ for each pair of reflections r and s such that $y = rs$ in W. Thus, the following are consequences of the defining relations:

$[x] = [r][s][t]$ and $[x] = [q][r][s]$. Add all such relations to the given presentation. In particular, we have added the relation $[x] = [s_1][s_2][s_3]$.

Second, we remove all the relations we first considered; that is, we remove all relations of the form $[x] = [y][t]$ and $[x] = [q][y]$. This is permissible because each such relation is a consequence. (Use the relations $[x] = [r][s][t], [x] = [q][r][s]$, and $[y] = [r][s]$.)

Third, fix a relation of the form $[y] = [r][s]$ for each allowable element $[y]$ of length two. Any other relation $[y] = [r'][s']$ is a consequence of the relations $[x] = [r][s][t], [x] = [r'][s'][t]$, and $[y] = [r][s]$. So we can remove those other relations.

Fourth, remove each generator $[y]$ and its fixed relation $[y] = [r][s]$ from the presentation. This is permissible, as this is the only relation remaining which involves $[y]$.

The generating set which remains consists of lifts of allowable elements of length 1 (reflections) and the lift of the Coxeter element. The relations which remain correspond precisely to the allowable expressions of length three. For, clearly all the remaining relations may be viewed as allowable expressions of length three; and, conversely, given an allowable expression (r, s, t), then $y := rs$ is allowable of length two and so the initial defining relations $[x] = [y][t]$ and $[y] = [r][s]$ would give rise to the relation $[x] = [r][s][t]$.

Finally, we can eliminate the generator $[x]$ by first adding a relation $[r_1][r_2][r_3] = [s_1][s_2][s_2]$ for each allowable expression (r_1, r_2, r_3) and by then removing $[x]$ and all relations involving $[x]$. $\square$

For the general complex K_Γ associated to a Coxeter group W_Γ of dimension ≤ 3, we may proceed as above and deduce that

$$\pi_1(K_\Gamma, v_0) = \langle \{[r] : r \in \mathcal{R}\} \mid \{\, [s_{i_1}] \dots [s_{i_k}] = [r_1] \dots [r_k] \,\} \rangle,$$

where the relations are indexed by $\{(r_1, \dots, r_k) \in Expr(x_T; |T|), T \in \mathcal{S}, |T| \geq 2\}$. Here, $T = \{s_{i_1} \prec \cdots \prec s_{i_k}\}$ and $\mathcal{R}$ is the set of spherical reflections. Informally, we may say that

$$\pi_1(K) \cong \langle \mathcal{R} \mid \text{expressions of each } x_T \text{ as a product of reflections } \rangle.$$

Theorem 3.2. *Let Γ define an Artin group of dimension ≤ 3 together with a total ordering of the generating set. Then $\pi_1(K_\Gamma, v_0) \cong A_\Gamma$.*

Proof. Let (A, S) be the Artin system defined by Γ and let (W, S) be the associated Coxter system. Suppose $T \subsetneq T' \in \mathcal{S}$. Let $i : A_T \to A_{T'}$ be the map induced by the inclusion $T \subset T'$. Then i is injective. (This is result is due to van der Lek [L] in general, though inclusions of spherical Artin groups were known to be injective by earlier work of Brieksorn & Saito [BS] and Deligne [De].) Let $j : \pi_1(K_T) \to \pi_1(K_{T'})$ be defined by mapping $[r] \to [r]$. Here we are using the alternate presentations of the fundamental groups as generated by lifts of reflections. The map j is well-defined because x_T is $x_{T'}$-allowable. If $T = \{r_1 \prec r_2\}$, then the following are both x_T and $x_{T'}$-allowable expressions: $(r_1, r_2), \dots, (r_m, r_1)$. So, in particular, the image of each defining relation $[r_1][r_2] = [r_i][r_{i+1}]$ in $\pi_1(K_T)$ is a consequence of the following relation in $\pi_1(K_{T'})$: $[r_1][r_2][q] = [r_i][r_{i+1}][q]$, where $(x_T, q) \in Expr(x_{T'}; 2)$ for some $q \in R_{T'}$. We have such a relation because x_T is a prefix of a reduced word representing $x_{T'}$.

Now consider the following diagram:

$$\begin{array}{ccc} A_T & \longrightarrow & A_{T'} \\ \Big\downarrow & & \Big\downarrow \\ \pi_1(K_T) & \longrightarrow & \pi_1(K_{T'}) \end{array}$$

The horizontal maps are the maps i and j above; the vertical maps are the maps β_T and $\beta_{T'}$ which map $t \mapsto [\pi(t)]$.

We have shown that β_T and $\beta_{T'}$ are isomorphisms when $|T'| \leq 2$. By assuming Theorem 3.3, we have that these maps are also isomorphisms when $|T'| = 3$. To check the diagram commutes, it suffices to chase the generating set T. By construction, these maps restrict to the identity on T. So, the diagram commutes. As the vertical maps are isomorphisms, it follows that the bottom horizontal map is also injective. Taking colimits over $T \in \mathcal{S}$, we get an isomorphism

$$\operatorname*{colim}_{T\in\mathcal{S}} A_T \cong \operatorname*{colim}_{T\in\mathcal{S}} \pi_1(K_T).$$

By examining the alternate presentations of the fundamental groups as generated by lifts of reflections, we see that we have defined the desired isomorphism. $\square$

3.3 Three generator spherical Artin groups

Let (A, S) be an Artin system. Let W be the associated Coxeter group, and let $R \subset W$ be the set of reflections. Let $\pi : A \to W$ be the natural map.

Definition. Given $S' \subset A$, we say that S' is *equivalent* to S if

- S' generates A
- there is a bijection $S \leftrightarrow S'$, and
- the corresponding Artin relations, $< s'_i, s'_j >^{m_{ij}} = < s'_j, s'_i >^{m_{ij}}$, are a set of defining relators for A.

We specialize to the case where (A, S) a three generator spherical Artin group. The Coxeter graph which defines A is thus:

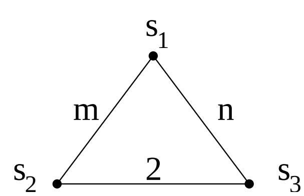

Figure 3.4: The Coxeter graph for A.

The only cases for which the group is spherical is when $m = 2$ and $2 \leq n < \infty$ or when $m = 3$, and $n \in \{3, 4, 5\}$.

Suppose $S' = \{s'_1, s'_2, s'_3\} \subset A$ generates A as an Artin group, where $s_i \leftrightarrow s'_i$. Let $x' = s'_1 s'_2 s'_3$. We emphasize that x' is the product of the elements of the Artin generating set in a particular order; specifically, the first two generators satisfy an Artin relation of length m and the last two generators commute. Denote the corresponding

Coxeter element by $y' := \pi(x')$. We may then construct Brady's complex; we denote it by $K_{y'}$. In particular, we can describe the fundamental group as follows:

$$\pi_1(K_{y'}) = \langle \{[y']\} \cup \{[r] : r \in R\} \mid \{[y'] = [r_1][r_2][r_3] : r_1 r_2 r_3 =_W y', r_i \in R\} \rangle.$$

In [Br1], T. Brady proves that the map $\beta_{S',x'} : A \to \pi_1(K_{y'})$ taking $s' \mapsto [\pi(s')]$ is an isomorphism. The inverse map is obtained by noting that the set of reflections is the closure of $\pi(S') \subset W$ under the action of conjugation by y'. Given $r \in R$, we can thus write

$$r = (y')^{-k} \pi(s'_i)(y')^k = \pi((x')^{-k} s'_i (x')^k)$$

for some integer k. The inverse map sends $[r] \mapsto (x')^{-k} s'_i (x')^k \in A$. That this map is well-defined is the main content of Brady's proof. We wish to extend this result to the case where K_y is defined by an arbitrary Coxeter element $y \in W$.

Remark. Given an arbitrary Coxeter element y in a spherical Coxeter system (W, S), it need not be true that the reflections are the closure of S under the action of conjugation by y. D. Bessis [Be] has shown that this is the case provided y is a "chromatic" Coxeter element, i.e. 2-color the Coxeter *diagram* of W (the Coxeter diagram of a spherical Coxeter group is a forest with at most one vertex of valence three in any component) by say 'red' and 'blue'; then a Coxeter element which is a product of the 'red' generators followed by the product of the 'blue' generators is called *chromatic*. However, in the case of an infinite type Coxeter system, it is not always possible to make a consistent choice of chromatic Coxeter elements for each spherical subgroup. Nonetheless, as we argue below, given y, one can find an alternate Coxeter generating set for which y is chromatic. We only have a case by case proof for the three generator spherical Coxeter groups. It is reasonable to expect that this could generalized in a unified way by, perhaps, viewing alternate generating sets as different chambers in the Coxeter-Davis complex.

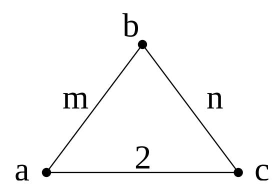

Figure 3.5: The Coxeter graph of (W, S).

Let $S = \{a, b, c\}$ so that the Coxeter graph is as shown in Figure 3.5. Let $x = abc \in A$, and let $y := \pi(x) \in W$ be the Coxeter element. Define $S' := aSa^{-1} = \{a' := a, b' := aba^{-1}, c' = c\} \subset A$. Then, as conjugation by a is an automormism of A and $c' = aca^{-1}$, we see that S' is equivalent to S. Let $x' := b'a'c'$ and $y' := \pi(x')$. Then (S', x', y') has the form of Brady's setup, namely x' is defined to be the product of the elements of an Artin generating set in the prefered order. So, the map $\beta_{S',x'} : A \to \pi_1(K_{y'})$ is an isomorphism. To simplify our notation, we use the following convention: if $g \in A$, then $[g] := [\pi(g)]$.

Lemma 3.2. *The map $\beta_{S,x} : A \to \pi_1(K_y)$ sending $s \mapsto [s]$ is an isomorphism. In fact, $\beta_{S,x} = \beta_{S',x'}$.*

Proof. The second statement makes sense because $\pi_1(K_{y'}) = \pi_1(K_y)$. The choice of x' is such that $x' = (aba^{-1})(a)(c) = abc = x$. So, $y' = y$ and the presentations of the fundamental groups are identical. It suffices to prove that $\beta_{S',x'}(s') = \beta_{S,x}(s')$ for every $s' \in S'$. If $s' = a'$ or c', there is nothing to prove. If $s' = b'$, then

$$\beta_{S,x}(b') = \beta_{S,x}(aba^{-1}) = [a][b][a]^{-1}.$$

On the other hand, as $y = \pi(abc) = \pi(aba^{-1})\pi(a)\pi(c)$, the relation

$$[a][b][c] = [b'][a][c]$$

holds in $\pi_1(K_y)$. Solving the equation in the group, we find that

$$[b'] = [a][b][a]^{-1}.$$

Hence, $\beta_{S',x'}(b') = \beta_{S,x}(b')$, as claimed. □

The cases where the Coxeter elements are defined by other permutations of the set $S = \{a, b, c\}$ are similar. Given $x =$ permutation of S, we define a corresponding element $g \in A$:

- if $x = bac$ or bca, let $g = 1$
- if $x = abc$, let $g = a$ (this is the case above)
- if $x = cba$, let $g = c$
- if $x = acb$ or cab, let $g = ac$

Let $S' := gSg^{-1}$. Then $a' := gag^{-1} = a, b' := gbg^{-1}$, $c' := gcg^{-1} = c$ and S' is equivalent to S. Let $x' = b'a'c'$. Then $x' = gbacg^{-1} = x$. In particular, x and x' define the same Coxeter element. Moreover, (S', x') takes the form of Brady's theorem.

The Brady complexes have identical fundamental groups. To show that $\beta_{S',x'} = \beta_{S,x}$, we check that this holds for each generator $s' \in S'$. As $a' = a$ and $c' = c$, we need only check that $\beta_{S',x'}(b') = \beta_{S,x}(b)$. So, we write b' in terms of the generating set S: $b' = gbg^{-1}$. So, $\beta_{S,x}(b') = (\beta_{S,x}(g))[b](\beta_{S,x}(g))^{-1}$. To see that this equals $\beta_{S',x'}(b') = [b']$, we only need to verify that that the relation $(\beta_{S,x}(g))[b] = [b'](\beta_{S,x}(g))$ holds in $\pi_1(K)$. We return to cases:

- if $g = 1$, then $\beta_{S,x}(g) = 1$ and the relation follows from $b' = b$.

- if $g = a$, this is the case above

- if $g = c$, then $\beta_{S,x}(g) = [c]$ and the relation follows from

$$[x] = [c][b][a] = [cbc^{-1}][c][a] = [b'][c][a]$$

- if $g = ac$, then $\beta_{S,x}(g) = [a][c]$ and the relation follows from

$$[x] = [a][c][b] = [acbc^{-1}a^{-1}][a][c] = [b'][a][c]$$

We conclude that

Theorem 3.3. *Given a three generator spherical Artin system (A, S) and given a Coxeter element $y \in W$ chosen by any total ordering of S, the map $\beta : A \to \pi_1(K_y)$ sending $s \mapsto [\pi(s)]$ is an isomorphism.*

CHAPTER 4

THE GEOMETRY OF BRADY'S COMPLEX

4.1 The metric on K_Γ

Let Γ define an Artin group of dimension ≤ 3 together with a total ordering of the generating set. Define a piecewise Euclidean structure on K by assigning a length of $\sqrt{k}$ to each edge labelled by an allowable element of length k. The metric on each cell is then determined. The model polyhedral 3-cell is shown below.

Above is the 3-cell corresponding to the allowable expression (w_1, w_2, w_3). It is isometric to the tetrahedron $\{(x, y, z) \in \mathbb{R}^3 \,|\, 0 \leqslant z \leqslant y \leqslant x \leqslant 1\}$. In Figure 4.1, the allowable rotations (length two allowable elements) are denoted by $w_{12} := w_1 w_2$ and $w_{23} := w_2 w_3$; a Coxeter element is denoted by $w_{123} := w_1 w_2 w_3$.

We will study the geometry of K within the formal framework of M_κ- polyhedral and simplicial complexes. Let M^n_κ denote the complete simply connected Riemannian manifold of constant curvature κ and dimension n. So, M^n_0 is Euclidean n-space, M^n_1 is the unit n-sphere, and M^n_{-1} is the hyperbolic n-space.

Roughly speaking, an M_0 or an M_1-complex is cell complexes whose cells are metrized, respectively, as convex Euclidean or convex spherical polyhedra. The reader

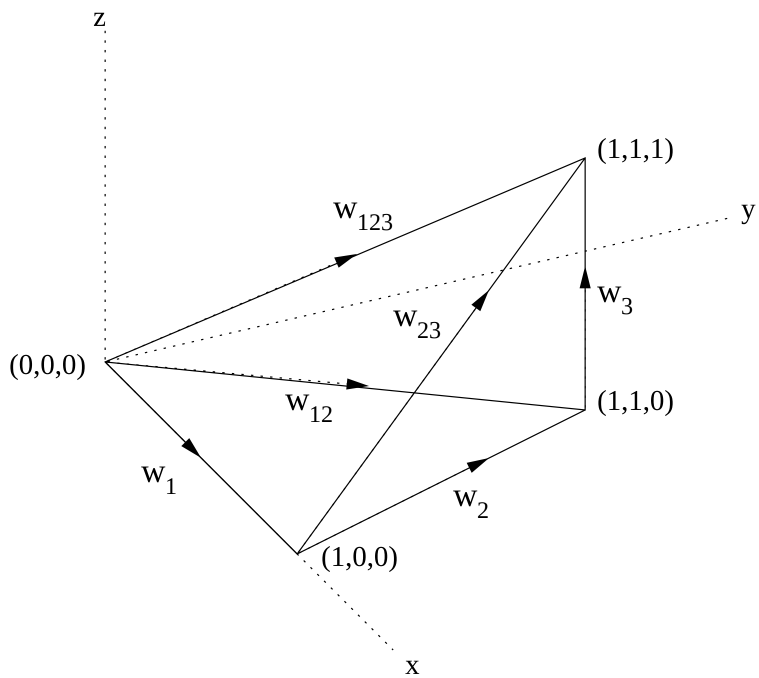

Figure 4.1: A model metric 3-cell in K.

is refered to our standard reference [BH]. But to facilitate the notation, we state the formal definitions in the following supplement:

4.2 Supplement: M_κ-complexes

Definition. A *convex M_κ-polyhedral cell* C is the convex hull of a finite set of points in M_κ^n. If $\kappa > 0$, we require that these points lie in an open hemisphere of M_κ^n. The dimension of C is the dimension the smallest geodesic hyperplane containing it. The *faces* of C are those nonempty subsets $F \subseteq C$ of the form $F = H \cap C$, where $H \subset M_\kappa^n$

is a hyperplane such that C lies entirely in one of the closed half-spaces determined by H. A face is *proper* if it is not all of C. Note that every face is itself a geodesic m-cell for some $m \leq \dim(C)$. The 0-dimensional faces are called *vertices*. Given a face $F \subseteq C$, we define the *interior of* F to be the set $int(F) := F - \{\text{proper faces of } F\}$. The interiors of faces are called *open faces*. Given $x \in C$, we denote by $\text{supp}(x)$ its support in C, i.e. the unique face F such that $x \in int(F)$. Note that unless $x \in C$ is a vertex, $int(\text{supp}(x))$ is the interior in the usual topological sense.

Definition. An M_κ*-polyhedral complex* is a metric space K arising as follows: Let $\{C_\lambda | \lambda \in \Lambda\}$ be a family of convex M_κ-polyhedral cells. Let $Y = \bigcup_{\lambda \in \Lambda}(C_\lambda \times \{\lambda\})$. Let $\sim$ be an equivalence relation on Y and let $K = Y/\sim$. Let $q : Y \to K$ be the quotient map. For each λ we have a map $q_\lambda : S_\lambda \to K$ defined by restriction. The maps $(q_\lambda)_{\lambda \in \Lambda}$ are required to satisfy the following compatibility conditions:

(1) For every $\lambda \in \Lambda$, the restriction of q_λ to the interior of C_λ is injective, and

(2) For all $\lambda_1, \lambda_2 \in \Lambda$ and $x_1 \in C_{\lambda_1}$, $x_2 \in C_{\lambda_2}$, if $q_{\lambda_1}(x_1) = q_{\lambda_2}(x_2)$, then there is an isometry $h : \text{supp}(x_1) \to \text{supp}(x_2)$ such that $q_{\lambda_1}(x) = q_{\lambda_2}(h(x))$ for all $x \in \text{supp}(x_1)$.

The metric on K is given by the *intrinsic psuedometric.* That is to say, the distance between points $x, y \in K$ is defined to be the infimum of lengths of paths which join x to y.

Definition. An M_κ-polyhedral complex K is called an M_κ *simplicial complex* if the following conditions also hold:

(3) Every convex polyhedral n-cell is a simplex, i.e. it is the convex hull of $n+1$ points in general position,

(4) For every $\lambda \in \Lambda$, q_λ is injective, and

(5) If $p(C_{\lambda_1}) \cap p(C_{\lambda_2}) \neq \emptyset$, then there is an isometry h from a face F_1 of C_{λ_1} to a face F_2 of C_{λ_2} such that $p(x_1, \lambda_1) = p(x_2, \lambda_2)$ if and only if $x_2 = h(x_1)$.

(Note that conditions (4) and (5) imply conditions (1) and (2).)

The basic theorem about an M_κ-polyhedral complex or simplicial complex K is that if there are only finitely many isometry types of cells in K (referred to as *finite shapes*), then intrinsic psuedometric is, in fact, a metric. Moreover, K is a complete geodesic metric space (Theorem I.7.50 of [BH]).

M_0- and M_1-polyhedral complexes are often called, respectively, *piecewise Euclidean* and *piecewise spherical* cell complexes.

The following is left as an exercise in using the above definitions:

Proposition 4.1. *Let $K = K_\Gamma$ be assigned the piecewise Euclidean metric as above. Then K is a M_0-polyhedral complex. Moreover, as the complex is finite, K is compact.*

We now recall the definition of the *geometric link* of a vertex of K. Let $\mathrm{Vert}(S_\lambda)$ denote the set of vertices of C_λ. Then, by definition, the vertices of K are

$$\mathrm{Vert}(K) := \bigcup_{\lambda \in \Lambda} q_\lambda(\mathrm{Vert}(C_\lambda)).$$

Let v be a vertex of K. So, $v = q_\lambda(v_\lambda)$ for some vertex $v_\lambda \in C_\lambda$, for some $\lambda \in \Lambda$. We say that v_λ *covers* v. Generally, a vertex v may be covered by vertices belonging to the same cell. However, if K is a simplicial complex, then the vertices which cover v must belong to different cells. The geometric link of v_λ in the polyhedral cell C_λ, denoted by $Lk(v_\lambda, C_\lambda)$, is the set of initial unit tangent vectors of locally geodesic rays in $C_\lambda \subset M^n_\kappa$ based at v_λ. The geometric link of v in K is then defined to be

$$Lk(v, K) := \coprod_{\{v_\lambda \text{ covers } v\}} Lk(v_\lambda, S_\lambda) \quad / \sim,$$

where $\sim$ is equivalence relation generated by the adjacency of cells in K. More precisely, if t_1 and t_2 are the initial unit tangent vectors of locally geodesic paths $[v_1, x_1] \subset C_{\lambda_1}$ and $[v_2, x_2] \subset C_{\lambda_2}$ which are glued in K, then the tangent vectors are glued in L.

4.3 The metric on L_Γ

Let $L = L_\Gamma := Lk(v_0, K_\Gamma)$ be the geometric link of v_0 in K. This is naturally a M_1-polyhedral complex. Each cell of the link is given a spherical metric by identifying the set of unit tangent vectors in $C_\lambda \subset M^n_0$ at the vertex v with the convex polyhedral cell described on the unit tangent sphere $S_v C_\lambda \subset S_v M^n_0 \cong M^{n-1}_1$.

We describe the vertices of L. Each 1-cell of K contributes exactly two vertices to L. Suppose that the the 1-cell C^1_λ is oriented from the vertex v_1 to v_2 and labelled by the allowable element w. The attaching map q_λ maps both vertices to v_0 in K. Thus, $Lk(v_0, C^1_\lambda)$ consists of two vertices—one for the initial tangent vector of the geodesic path from v_1 to v_2 and one for the initial tangent vector of the reverse path. We label the vertex in the link contributed by the path from v_1 to v_2 by $(w, 1)$; the other vertex is labelled $(w, -1)$. (Refer to Figure 4.2.) As every vertex of L must arise in this way, we see that the vertices of L are in bijective correspondence with the set $Allow(W) \times \{\pm 1\}$. Given a vertex (w, ϵ) of L, we say it has *length* $\ell(w)$ and *sign* ϵ.

Remark. The set $Allow(W) \times \{\pm 1\}$ becomes a poset by via the reverse lexicographic order: $(w_1, \epsilon_1) \leq (w_2, \epsilon_2)$ iff $\epsilon_1 < \epsilon_2$ or $\epsilon_1 = \epsilon_2$ and $w_1 \leq w_2$. We can use this to label each cell of L purely in terms of its vertices.

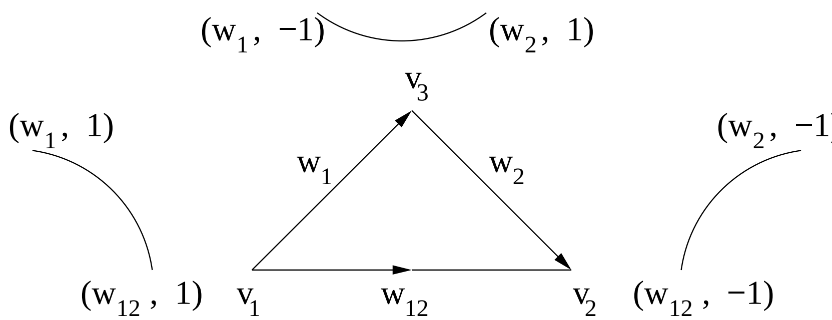

Figure 4.2: Each vertex contributes a 1-cell in L.

Next, we describe the 1-cells of L. Suppose C^2_λ is a 2-cell in K. So, C^2_λ is isometric to a Euclidean triangle and indexed by an allowable expression $\lambda := (w_1, w_2)$ of length 2. Let the vertices of C^2_λ be v_1, v_2, and v_3, and let the directed edge from v_i to v_{i+1} be labelled by w_i, $i = 1, 2$. Label the directed edge from v_1 to v_3 by $w_{12} := w_1 w_2 \in W$. The attaching map q_λ maps all of the vertices to v_0 and maps each directed edge to the 1-cell of K with the same label and orientation. Thus, the link of of a 2-cell of K consists three disjoint arcs:

$$Lk(v_0, q_\lambda(C^2_\lambda)) = \bigsqcup_{i=1,2,3} Lk(v_i, C^2_\lambda).$$

Notice that the vertices of a 1-cell in L are necessarily related by the reverse lexicographic ordering on $Allow(W) \times \{\pm\epsilon\}$. Making the convention that vertices are listed in ascending order, we can list the 1-cells according to their vertex set as follows:

$$[(w_1, 1), (w_{12}, 1)],\ [(w_1, -1), (w_2, 1)],\ \text{and}\ [(w_2, -1), (w_{12}, -1)].$$

This is a complete list if we range over all ordered pairs $(w_1, w_2) \in Expr(W; 2)$.

We observe that the following algebraic properties characterize those vertices which span a 1-cell in L:

Proposition 4.2. *Let $w_1, w_2 \in Allow(W)$. Then,*

1. *the vertices $\{(w_1, 1), (w_2, 1)\}$ or $\{(w_1, -1), (w_2, -1)\}$, span a 1-cell in L if and only if $w_1, w_2 \in Allow(x_T)$, for some $T \in \mathcal{S}$ and either $w_1 < w_2$ or $w_2 < w_1$*
2. *the vertices $\{(w_1, -1), (w_2, 1)\}$ span a 1-cell in L if and only if $(w_1, w_2) \in Expr(W; 2)$.*

We now list all the 1-cells of L up to the length and sign of their vertices. We will also compute the (spherical) length of each cell.

Notation. Henceforth, we will use the convention that the reflections will be indicated by the letters p, q, r, s or t. Likewise, the *rotations* (i.e. elements in W of length two) will be indicated by the letters y or z. Lastly, we will reserve the letter x or x_T for elements of length three in W.

Below are the three different oriented, metric 2-cells of K. They correspond to expressions of the form $(r, s), (y, t)$, and (q, y) in $Expr(x; 2)$. Recall that the lengths of the edges are 1 for a reflection, $\sqrt{2}$ for a rotation, and $\sqrt{3}$ for and element of length three. The first triangle is an isoceles right triangle. The angles in the second two triangles are indicated, where $\alpha = \arctan(\sqrt{2})$ and $\beta = \arctan(1/\sqrt{2})$. Thus, $0 < \beta < \pi/4 < \alpha < \pi/2$.

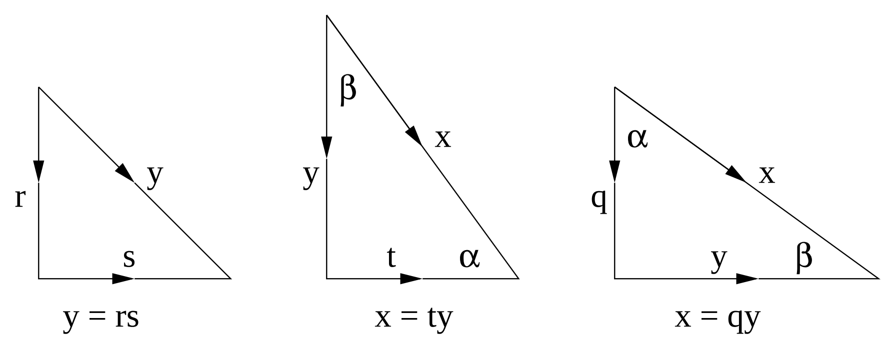

Figure 4.3: The metric 2-cells of K.

From the left 2-cell, we get the following contributions to the link:

- $[(r, 1), (y, 1)]$ of length $\pi/4$; (algebraically: $r < y$)
- $[(r, -1), (s, 1)]$ of length $\pi/2$; ($rs = y$ is a reduced expression)

- $[(s,-1),(y,-1)]$ of length $\pi/4$; $(s < y)$

The middle 2-cell contributes:

- $[(y,1),(x,1)]$ of length β; $(y < x)$
- $[(y,-1),(t,1)]$ of length $\pi/2$; ($yt = x$ is reduced)
- $[(t,-1),(x,-1)]$ of length α; $(t < x)$

And the right 2-cell contributes:

- $[(q,1),(x,1)]$ of length α; $(q < x)$
- $[(q,-1),(y,1)]$ of length $\pi/2$; ($qy = x$ is reduced)
- $[(y,-1),(x,-1)]$ of length β; $(y < x)$

So, we see that if we consider all unordered pairs of vertices $\{\,(w,\epsilon),(w',\epsilon')\,\}$ up to their length and signs $\{\,(\ell(w),\epsilon),(\ell(w'),\epsilon')\,\}$, we get exactly nine different 1-cells in L. This list is complete because every 1-cell in L necessarily arises from a link of one of the three different oriented, metric 2-cells of K as in Figure 4.3 .

We now repeat this analysis for the 2-cells of L. Each 2-cell of L is a simplex. Each such simplex arises by looking at the link of v_0 in a fixed 3-cell of K.

The 3-cells of K are in one to one correspondence with the allowable expressions of length three. For each allowable expression $\lambda := (r,s,t)$, we get four 2-cells in L by considering $Lk(v_0, q_\lambda(C^3_\lambda)) = \bigsqcup_{i=1,\dots,4} Lk(v_i, C^3_\lambda)$. We study the 2-cells of L by considering the links of each of these vertices in turn as in Figure 4.4.

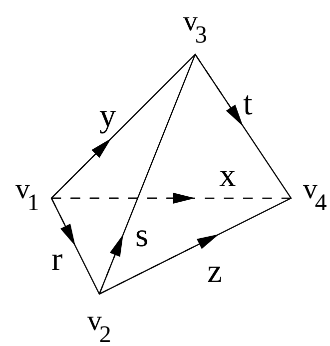

Figure 4.4: Each vertex of the above 3-cell contributes a 2 cell to the link.

We get the following 2-cells in L (refer to Figure 4.5): The counter-clockwise from the upper left corner, these are, respectively, the links of v_1, v_3, v_4, and v_2. They are illustrated in the order shown so as to suggest how the 2-cells will fit together in L. Two 2-cells are glued along a face if and only if they have the same vertices (labelled by the same allowable element and sign). In particular, such vertices must have the same length. We have illustrated the length of a vertex as follows: a reflection is symbolized by a solid circle, a rotation by an open circle, and a element of length three by a solid triangle. When needed, we may indicate the sign of the vertex by adding a $\pm$ symbol to the diagram, as in the list to the right.

In the right column, we list the lengths of the edges of these 2-cells. Recall that $\beta < \pi/4 < \alpha < \pi/2$. Below each vertex is listed either a $+$ or a $-$ sign. We recover our complete listing of 1-cells in L (up to length and sign of the vertices) if we change all the $+$ signs to $-$ signs. Note that the bottom 1-cell does not give rise to a new

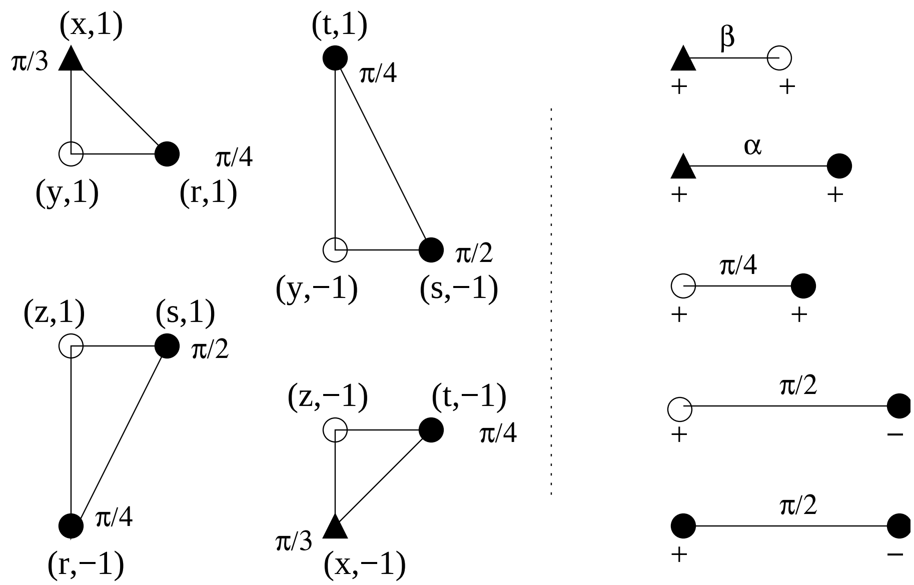

Figure 4.5: The metric 2-cells of L.

1-cell if we change the signs— it is characterized (up to the length and signs of its vertices) as a pair of vertices of length one with opposite signs.

The 2-cells are spherical triangles. From the spherical law of cosines or by considering the dihedral angles between the faces of the model polyhedral 3-cell of K, we can compute their angles. The measures of the angles in each spherical triangle is indicated to the side of the vertex (in the left column). The unlabelled angles are understood to be $\pi/2$.

We note that we can list the vertices of each 2-cells in ascending order with respect to the ordering on $Allow(W) \times \{\pm 1\}$:

- $Lk(v_1, C^3) = [(r, 1), (y, 1), (x, 1)]$. The vertices are related as follows: $r < y < x$.
- $Lk(v_2, C^3) = [(r, -1), (s, 1), (z, 1)]$. The vertices satisfy $rz = x$ and $s < z$.
- $Lk(v_3, C^3) = [(s, -1), (y, -1), (t, 1)]$. The vertices satisfy $s < y$ and $yt = x$.
- $Lk(v_4, C^3) = [(t, -1), (z, -1), (x, -1)]$. The vertices satisfy $t < z < x$.

We can characterize the 2-cells in L algebraically according to their vertex set as follows:

Proposition 4.3. *Given vertices* $\{(w_1, \epsilon_1), \ldots, (w_3, \epsilon_3)\}$. *These vertices span a* 2*-cell in* L *if and only if*

1. *all the vertices have the same sign and the vertices are totally ordered:* $w_i \leq w_j \leq w_k$ *for some permutation* (i, j, k) *of* $(1, 2, 3)$*, or*
2. *exactly two vertices,* $w_i \leq w_j$*, are positive and* $w_k w_j = x_T$ *for some* $x_T \in Allow(W; 3)$*, or*
3. *exactly two vertices,* $w_i \leq w_j$*, are negative and* $w_j w_k = x_T$ *for some* $x_T \in Allow(W; 3)$.

In each of the last two, the negative vertex right multiplied by the positive vertex gives an allowable element.

Proposition 4.4. L *is an* M_1*-simplicial complex.*

Proof. We have seen that every cell of L is a simplex. The simplices are glued according to the labelling of their vertices. Each simplex injects into L as none of the faces of a given simplex have the same vertex set. From the classification of all such cells, we see that the intersection of two simplices is always another simplex. Moreover, the metric on each simplex is compatible along the intersection. $\square$

We have the following corollary:

Corollary 4.1. *Let $T \in \mathcal{S}$ and let K_T be the Brady complex of (W_T, T) together with the total ordering of S restricted to T. Denote the link of v_0 by L_T. Then L_T is subcomplex of L; moreover, the inclusion $L_T \to L$ restricts to an isometry on each face.*

CHAPTER 5

LOCAL CURVATURE OF L_Γ

5.1 Review of CAT(0) spaces and the link condition

Let κ be a real number. Let $D_\kappa := \pi/\kappa$ if $\kappa > 0$ and $D_\kappa = \infty$ if $\kappa \leq 0$. A metric space, (X, d), is D_κ-*geodesic* if every two points $x, y \in X$ of distance less than D_κ may be joined by a geodesic segment $[x, y]$. Note that we will allow for the possibility that there are points $x, y \in X$ are such that $d(x, y) = \infty$.

Let M_κ^n denote the unique complete, simply connected, n-dimensional Riemannian manifold with constant sectional curvature κ. In particular, M_1^n is the unit sphere, M_0^n is Euclidean space, and M_{-1}^n is hyperbolic space. These manifolds are D_κ-geodesic metric spaces with respect to the intrinsic length metric.

Definition. Let (X, d) be a D_κ-geodesic metric space. A triangle $\Delta = [x, y] \cup [y, z] \cup [x, z]$ satisfies the *CAT(κ) inequality* if for each point p in the arc $[y, z]$, $d(x, p) \leqq |\bar{x} - \bar{p}|$, where $\bar{x}$ and $\bar{p}$ are the comparison points on a comparison triangle $\bar{\Delta} \subset M_\kappa^2$. If every triangle in X of perimeter $< 2D_\kappa$ satisfies the CAT(κ) inequality, we say that X is a *CAT(κ) space.*

A geodesic metric space (X, d) is said to bee *locally CAT(κ)* if each point has a open neighborhood in which all triangles satisfy the CAT(κ) inequality. Locally CAT(κ) spaces are said to have *curvature* $\leq \kappa$.

In particular, any finite 1-complex is a locally CAT(1) space if we assign positive lengths to the edges and consider the intrinsic length metric.

By the comparison theorems of Riemannian geometry, if X is a Riemannian manifold of non-positive sectional curvature and d is the intrinsic length metric, then X is a locally CAT(0) space. The study of $CAT(0)$ spaces, thus, extends the study of non-positive curvature to the more general setting of metric spaces. However, we emphasize that the complexes we are considering are not manifolds.

We will show that the universal covering space of K_Γ is a CAT(0) space whenever Γ defines a three dimensional FC Artin group. So, $A_\Gamma \cong \pi_1(K_\Gamma)$ will act geometrically on $\widetilde{K_\Gamma}$ by deck transformations.

There are two key theorems which allow one to construct piecewise Euclidean cell complexes which are CAT(0) spaces. The first is the following generalization of the Cartan-Hadamard Theorem of Riemannian geometry: if X is a complete, connected, locally CAT(0) space, then the universal cover of X, equipped with the pullback metric, is (globally) a CAT(0) space. This local to global theorem is, perhaps, the most remarkable fact about non-positive curvature. In the case of positive curvature bounded above by a constant, there is an analogous local to global result: if X is a cocompact, proper, geodesic metric space of curvature $\leq \kappa$ which has no isometrically embedded circle of length $< 2D_\kappa$, then X is (globally) CAT(κ).

The second theorem is called the "Link Condition". Let X be an M_κ-polyhedral complex. If the link of every vertex $p \in X$ is a CAT(1) space, then X is locally CAT(κ).

As both of these theorems are pivotal to the arguments herein, we state them formally:

Theorem. *(Local to Global) A M_κ-polyhedral complex K, with Shapes(K) finite, is (globally) CAT(κ) if and only if K is locally CAT(κ) and contains no isometrically embedded circles of length less than $2D_\kappa$. In particular, such a M_0-polyhedral complex is CAT(0) if and only if it is locally CAT(0) and simply connected.*

Theorem. *(Link Condition) A M_κ-polyhedral complex K, with Shapes(K) finite, is a locally CAT(κ) space if and only if for every vertex v of K, the geometric link, $Lk(v, K)$, is CAT(1) space.*

We refer the reader to [BH] for proofs of the above theorems, as well for a discussion of a more general class of metric complexes. The condition of Shapes(K) finite (i.e. K is modelled on finitely many isometry types of cells) is satisfied for both the complexes K_Γ and L_Γ. Refer to the appendix for more on M_κ-complexes.

Together, these theorems reduce the question of whether an M_0-polyhedral complex is locally CAT(0) to the question of whether the links of the vertices in the complex are CAT(1). The difficulty is ruling out the existence of isometrically embedded circles of length $< 2\pi$ in the link. Such loops are defined by closed (local) geodesics:

Definition. Let X be a geodesic metric space. A path $\gamma : [a, b] \to X$ is a *local geodesic* if it is locally an isometric embedding. This path defines a *closed local* *geodesic* if $\gamma(a) = \gamma(b)$ and the induced map from $[a, b]/(a \sim b) \to X$ defines a local isometric embedding with respect to the quotient metric. A closed local geodesic of length $< 2\pi$ will be referred to as a *short loop*.

We will rule out the existence of short loops in L. In the case of a finite M_κ-complex X, a path defines a local geodesic if and only if for each $a \leq t \leq b$, the distance in $Lk(\gamma(t), X)$ between the incoming and outgoing unit vectors is $\geq \pi$. This is a practical way to decide if a given path is locally geodesic because such a path must necessarily "look" like a geodesic in M_κ^n when restricted to an open cell. So, in practice, we only need to study the links of the finite number of points along the path which are limit points of more than one open cell.

Remark. When the context is clear, we will think of the path $\gamma : [a, b] \to X$ as a subset of X. So, for instance, given a subspace $Y \subseteq X$, we will say that γ "intersects" or "meets" Y instead of referring to the image of γ. Similarly, we will write $\gamma \cap Y = \emptyset$, instead of $\gamma([a, b]) \cap Y = \emptyset$.

5.2 Outline of the proof of the Main Theorem

Here is an outline of the proof of the Main Theorem. Let Γ define a three dimensional FC Artin group. Let $K = K_\Gamma$ be its Brady complex. We will show that K is a locally CAT(0) space. We prove this by verifying the link condition for sole vertex, v_0. The link, L_Γ will be shown to be locally CAT(1) by verifying the link condition. Then we will verify that L_Γ does not contain any short loops.

The rest of the proof is immediate from the link condition and the local to global properties: First, by the Local to Global theorem, locally CAT(1) and no short loops implies that L_Γ is CAT(1). Next, the Link Condition implies that K is locally CAT(0). Finally, by the Local to Global theorem, the universal cover of K is (globally) CAT(0). So, the Artin group, which is isomorphic to $\pi_1(K)$, acts geometrically on the universal cover of K by deck transformations. Thus, A_Γ is a CAT(0) group.

As noted before, the difficulty is to understand the link L. We begin by studying its local curvature.

5.3 Links of edges

Let $L = L_\Gamma$ be the link of the Brady complex defined by an FC Artin group A_Γ of dimension ≤ 3. . To show that L is locally CAT(1), the link condition tells us to consider the links of each of the vertices in L. Recall that the vertices of L are of the form $(w, \epsilon) \in Allow(W) \times \{\pm 1\}$.

Suppose (w, ϵ) is a vertex in L. Recall that (w, ϵ) is the initial unit tangent vector of the locally geodesic path along an edge of K labelled by the allowable element w. The sign tells us whether or not the unit vector is in the same or opposite direction of the oriented edge.

Lemma 5.1. *Let $(w, \epsilon) \in L$ and let E_w be the edge in K labelled by w. Then $Lk((w, \epsilon), L)$ is isometric to $Lk(E_w, K)$, the geometric link of the edge E_w.*

The geometric link of an edge in an M_0-complex, K is, by definition, the M_1-complex determined by all initial unit tangent vectors of geodesics in K which are orthogonal to the edge. More formally, it is the M_1-complex whose cells are defined as follows: Suppose C_λ is convex polyhedral cell of K which contains an edge E_λ that is attatched to E. Then there is a (possibly empty) cell, $C_\lambda^\perp$, of $Lk(E, K)$ which is, by defintion, the the convex M_1-cell determined by the inital unit tangent vectors of geodesic rays from $x \in int(E_\lambda)$ into C_λ which are perpendicular (in M_0^n) to E_λ. Here, we choose x to be some interior point of E_λ.

All the cells of $Lk(E, K)$ arise in this way. Two such cells $C_{\lambda(1)}^\perp$ and $C_{\lambda(2)}^\perp$ are glued along the common face $(C_{\lambda 1} \cap C_{\lambda(2)})^\perp$ whenever $C_{\lambda(1)}$ and $C_{\lambda(2)}$ are both attached to the edge $E \subset K$. The metric on $Lk(E, K)$ is defined by the intrinsic pseudometric. Now for the proof of the lemma:

Proof. Let $(w, 1) \in L$. Let $E_w \subset K$ be the edge of K labelled by the allowable element w. The initial unit tangent vector of a geodesic ray based at v_0 that traverses E_w with the same orientation ($\epsilon = 1$) defines $(w, 1)$. Choose a point x in the interior E_w which lies in the initial segment of the geodesic ray. Then the points of $Lk(E_w, K)$ correspond to initial unit tangent vectors of rays based at x. If x is sufficiently close to v_0, then we may view these vectors as being tangent to the spheres of radius $d(v_0, x)$ at v_0 in each cell that attaches to the edge E_w. Dilating the spheres till they have radius one, we can identify these vectors, these points of $Lk(E_w, K)$, with the initial unit tangent vectors of geodesics in $Lk((w, 1), L)$ based at $(w, 1)$. This defines a continuous map $Lk(E_w, K) \to Lk((w, 1), L)$. This map is easily seen to have a continuous inverse by reversing the process— given an initial tangent vector of a geodesic ray in $Lk((w, 1), L)$, for each cell that attaches to E_w, we can view the vector as being tangent to the unit tangent sphere of vertex. Then shrink the unit tangent sphere till it meets an interior point of the edge attaching to E_w. The scaled vector, regardless of how much it is scaled, determines the same point in $Lk(E_w, K)$.

To see that this map is an isometry, it suffices to check that corresponding edges are assigned the same (spherical) length. A pair of adjacent vertices in $Lk(E_w, K)$ occurs whenever there are two faces F_1 and F_2 of cell C which are attached at E_w: $F_1 \cap F_2$ is glued to E_w. The distance in C between these two vertices is defined to be the dihedral angle between the faces. On the other hand, if we consider the hyperplanes supported by each F_i ($C \subset M_0^n$) and consider their intersection with the appropriate unit tangent sphere, we can see that the corresponding vertices of $Lk((w, 1), L)$ are separated by a distance equal to this dihedral angle. Thus, the metrics agree on every edge, and because these are piecewise spherical simplicial complexes, they metrics agree on every face. So, the intrinsic psuedometrics are the same; and, hence, the map is an isometry. The argument for $\epsilon = -1$ is basically the same. In particular, we have shown that $Lk((w, 1), L) \cong Lk(E_w, K) \cong Lk((w, -1), L)$. $\square$

Notation. The link L is a simplicial complex; so, the vertices in $Lk((w, \epsilon), L)$ are naturally labelled by another vertex in L, i.e. they can be labelled by an element $(w, \epsilon) \in Allow(W) \times \{\pm 1\}$. This, in turn, defines a labelling of the vertices in the link of an edge of K: Given $E_w \subset K$, the label of a vertex of $Lk(E_w, K)$ is chosen to be the same as the label on the corresponding vertex of $Lk((w, 1), K)$. We illustrate this in Figure 5.1.

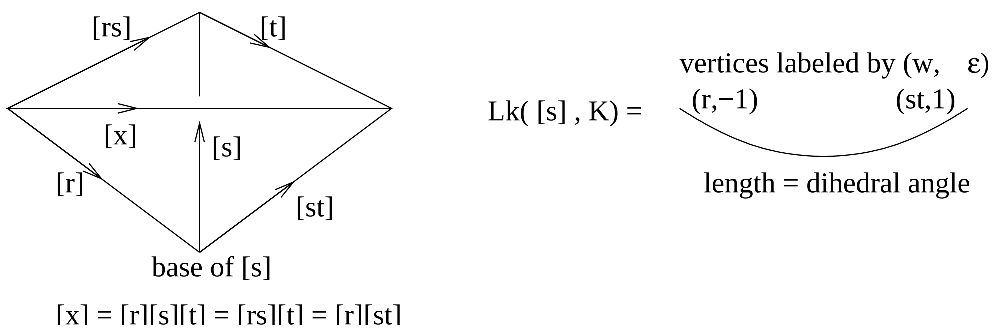

Figure 5.1: Labeling the vertices in the link of an edge of K.

Now we begin an analysis of the link of an edge E_w. There are three cases according to the length of w.

Case 1: Let $w \in \mathit{Allow}(W)$ have length one. So, w is a reflection; let $r := w$. We can enumerate the vertices of $Lk(E_r, K)$ by finding all the two cells which are attached to E_r. These two cells correspond to precisely those allowable expressions of length two (w_1, w_2) for which $r = w_1$ or w_2. Thus, using the notation as above, (r, w_2) corresponds to the vertex $(rw_2, 1) \in Lk(E_r, K)$ and (w_1, r) corresponds to the vertex $(w_1, -1) \in Lk(E_r, K)$. So

1. $(w, 1) \in Lk(E_r, K)$ if and only if $r < w$, and

2. $(w, -1) \in Lk(E_r, K)$ if and only if wr is allowable.

Similarly, we can enumerate all the edges of $Lk(E_r, K)$ by considering all three cells which are attached to E_r. These correspond to allowable expressions (w_1, w_2, w_3) in which $r = w_i$ for some i. As we will be most interested in which edges share a common vertex, we will use a more suitable notation. The allowable expression (w_1, w_2, w_3) is a product of three reflections, one of which is r. Their product is an allowable element $x = x_T$, for some $T \in \mathcal{S}$. To see the general picture, it suffices to consider three overlapping R-reduced words: $x = pqr = qrs = rst$. These words define allowable expressions (p, q, r), etc. The edges in $Lk(E_r, L)$ corresponding to these expressions are listed below. Refer to Figure 5.2. Notice the similarities between vertices of these edges and the vertices which define 2-cells in L.

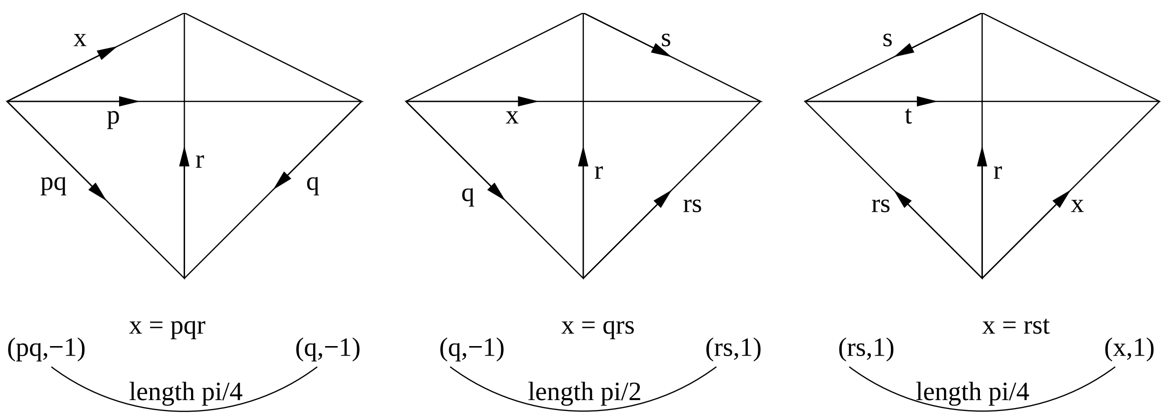

Figure 5.2: The edges of $Lk(E_r, L)$.

1. (r, s, t) corresponds to an edge from $(rs, 1)$ to $(rst, 1)$ in the link of E_r, i.e. an edge from $(y, 1)$ to $(x, 1)$, where $r < y < x$. This edge has length $\pi/4$.

2. (q, r, s) corresponds to an edge from $(q, -1)$ to $(rs, 1)$ in the link of E_r, i.e. an edge from $(q, -1)$ to $(y, 1)$, where $qy = x$ and $r < y$. This edge has length $\pi/2$.

3. (p, q, r) corresponds to an edge from $(pq, -1)$ to $(q, -1)$ in the link of E_r, i.e. an edge from $(xr, -1)$ to $(q, -1)$, where $q < xr$. This edge has length $\pi/4$.

From this, we observe a simple description of the link of E_r in the subcomplex K_T:

Lemma 5.2. *Let (W, S) be three dimensional Coxeter system with a total ordering S. Let $T \in \mathcal{S}$ have cardinality three. Suppose $r < x_T$. Then $Lk(E_r, K_T)$ is isometric to the spherical suspension $\{(x_T, 1), (xr, -1)\} * \{p_1, \ldots, p_k\}$, where each p_i corresponds to a unique allowable rotation $y \in Allow(x_T; 2)$ such that $r < y$.*

$X * Y$ denotes the spherical join of two M_1-complexes. Refer to [BH] for precise definitions. When X has just two elements this is called the spherical suspension. The basic fact is that the spherical join of CAT(1) spaces in CAT(1). So, as a corollary, we have

Corollary 5.1. *Let (W, S) be a three dimensional Coxeter system with a total ordering of S and let $T \in \mathcal{S}$. Then $Lk(E_r, K_T)$ is CAT(1).*

Now for the proof of Lemma 5.2:

Proof. The vertices of $Lk(E_r, K_T)$ are labelled by elements of $Allow(x_T) \times \{\pm 1\}$. There is a unique allowable element of length three, namely $x = x_T$; and there is a unique allowable element of length two y such that $yr = x$, namely $y = xr$.

From the list of vertices in $Lk(E_r, K_T)$, we conclude that $(x, 1)$ is the only vertex of length three and $(xr, -1)$ is the only negative vertex of length two. There are no positive vertices of length one. So, the remaining vertices are of the form $(y, 1)$ where $r < y < x$ or $(q, -1)$ where $qr \in Allow(x_T)$. Given $r < y < x$, we can write $x = rst$, reduced, where $y = rs$. By shifting, we get a reduced word $x = (rstsr)(r)(s)$; let $q = rstsr = xy^{-1}$. So, these these vertices arise in pairs. From the reduced expressions $x = rst = qrs = pqr$, where $pq = xr$, we obtain edges in $Lk(E_r, K_T)$ from $(x, 1)$ to $(y, 1)$ to $(q, -1)$ to $(xr, -1)$. (Refer to the characterization of the edges and to Figure 5.3. In this figure, we have illustrated an alternate point of view: the link of the link). On the other hand, these are all the edges. If (w, ϵ) is adjacent to $(x, 1)$, then, by the characterization above, $r < w < x$ and $\epsilon = 1$. A vertex (w, ϵ) adjacent to $(y, 1)$, where $r < y$, must satisfy $y < w$ if $\epsilon = 1$ and must satisfy $wy = x$ if $\epsilon = -1$. So, w is, respectively, x or xy^{-1}. The other two cases are similar.

Now, let p_y be the midpoint of the edge from $(y, 1)$ to $(xy^{-1}, -1)$. The distance from p_y to $(x, 1)$ or to $(xr, -1)$ is exactly $\pi/2$. Thus, $Lk(E_r, K_T)$ is the spherical suspension over these points with poles $(x, 1)$ and $(xr, -1)$.

□

To describe the link of E_r in the entire complex K, we consider the intersections of subcomplexes.

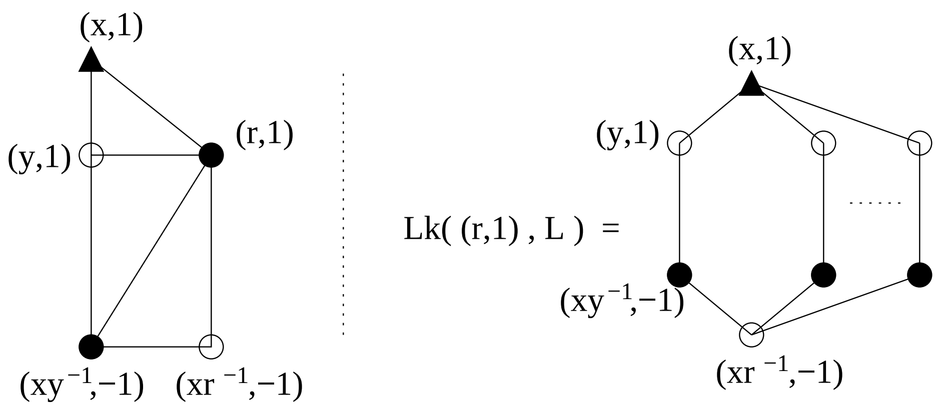

Figure 5.3: The link of the link is isometric to the link of the edge.

Lemma 5.3. *Let $T, T' \in \mathcal{S}$ be distinct and of cardinality three. Suppose $r < x_T, x_{T'}$. Then $Lk(E_r, K_T) \cap Lk(E_r, K_T) = \{(x_{T\cap T'}, 1), (x_{T\cap T'}r, -1)\}$. In particular, the intersection is empty if $|T \cap T'| < 2$. If there are two common points, then they lie distance π apart in each of the links.*

Proof. Suppose $(w, 1)$ belongs to each link. Because $x_T \neq x_{T'}$, w has length less than three. By the characterization of vertices, we see that $r < w$ and $w < x_T, x_{T'}$. But Lemma 2.2 tells us that $w = x_{T\cap T'}$. On the other hand, if $(w, -1)$ is a common vertex, then $x_T r \neq x_{T'} r$ combined with the characterization of vertices, we see that wr is a rotation and $wr < x_T, x_{T'}$. Again, apply Lemma 2.2.

To see that these two common points lie distance π apart in each link, simply observe that they are not joined by an edge (the product $x_{T\cap T'} r x_{T\cap T'}$ would have to equal both x_T and $x_{T'}$). Thus, the two points lie on different great arcs in the suspension. From its description, it is easy to see that they are joined by a geodesic of length π. $\square$

Theorem 5.1. *Let (W, S) be a three dimensional FC Coxeter system together with a total ordering of S. Suppose $r \in Allow(W; 1)$ is an allowable reflection. Then $Lk(E_r, K)$ is CAT(1).*

Proof. As $Lk(E_r, K)$ is a 1-complex, it suffices to show that it contains no short loops (length $< 2\pi$). Suppose we are given a geodesic loop γ. If γ is contained in a subcomplex $Lk(E_r, K_T)$ for some $T \in \mathcal{S}$, then γ is either constant ($|T| = 2$) or has length equal to 2π. The first statement is just taking into account the dimension of the subcomplex; the second statement follows from Corollary 5.1. So, γ must meet two or more subcomplexes of the form K_T, $|T| = 3$, along a subspace which contains an edge. The path γ consists of two or more *segments* which join *common* vertices. A segment is, by definition, an edge path which joins two vertices which could possibly belong to two distinct complexes of the form $Lk(E_r, K_T)$ with $|T| = 3$ and $T \in \mathcal{S}$. A common vertex is, by definition, a vertex which is common to two or more such subcomplexes. We have already characterized the common vertices. The segments, on the other hand, have one of the following three forms:

1. an edge path from $(y, 1)$ to $(x_T, 1)$ to $(y', 1)$
2. an edge (path) from $(y, 1)$ to $(x_T y^{-1}, -1)$
3. an edge path from $(x_T y^{-1}, -1)$ to $(x_T r, -1)$ to $(x_T y'^{-1}, -1)$.

All of these edge paths have length $\pi/2$. So, we only need to rule out loops consisting of two or three segments. (One segment cannot form a loop— the path would not be geodesic.)

An edge path of two segments must consist of segments of the same form, and these segments must lie in a distinct subcomplexes. The first and third are not possible (use Lemma 2.2— the common vertices would be the same). The second would not be geodesic— it traverses the same edge twice.

An edge path of three segments must lie in either two or three distinct subcomplexes. In the first case, let the complexes be indexed by T and T'. There are two common vertices along the path, and each is labelled by an element of $W_{T\cap T'}$. The vertices cannot have the same sign (as before, use Lemma 2.2). Then the negative vertex right multiplied by the positive vertex defines an allowable element of length three. This is impossible— the product belongs to $W_{T\cap T'}$.

In the second case, there are three spherical subsets, T, T', and T'', and there are three common vertices. These subsets intersect pairwise in subsets of cardinality two (Lemma 5.3). In light of the FC condition and the hypothesis of dimension three, the link complex must look like three triangles with a common edge. (Refer to Figure 5.4.) So, $|T \cap T' \cap T''| = 2$. An edge loop which consists entirely of segments of the first type or entirely of segments of the second type gives rise to three common rotations: $x_{T\cap T'}, x_{T'\cap T'}, x_{T\cap T'}$. But the indices are all the same. So, the path cannot be geodesic. And edge loop which uses different segments must join either two positive and one negative common vertex or two negative and one positive vertex. In the case of two positive common vertices, they are each labelled by some

$x_{T(i)\cap T(j)}$; but $|T \cap T' \cap T''| = 2$, so they are the same. So, the path is not geodesic. In the other case, the vertices are labelled by elements of the form $x_{T(i)\cap T(j)}r$. Again, they are the same, and the path cannot be geodesic.

Thus, every edge loop must consist of at least four segments, i.e. every closed geodesic has length $\geq 2\pi$. So, by the Local to Global theorem, $Lk(E_r, K)$ is CAT(1).

$\square$

Remark. Observe that if we omit the FC condition, the K, with its given metric, need not be locally CAT(0). For instance, let W_Γ be defined by a link complex Γ which is the boundary of a three simplex. There is a short loop of length $3\pi/2$ in $Lk(E_r, K)$ which consists of segments of the first form.

Case 2: Now we turn to the second type of edge in K, where E_w is labelled by an allowable rotation. Here, the situation is much simpler. We write $y := w$. As before, we characterize the vertices and edges in the complex $Lk(E_y, K)$. Each vertex corresponds to a 2-cell labelled by an allowable expression (w_1, w_2) which is attached to E_y. Thus, $y = w_1, w_2$, or $w_1 w_2$. The 2-cell labelled by (y, w_2) contributes the vertex $(yw_2, 1) \in Lk(E_y, K)$. Similarly, (w_1, y) contributes $(w_1, -1)$, and (w_1, w_2), where $y = w_1 w_2$ contributes $(w_1, 1)$. So, the vertices fall into exactly one of the following categories:

1. $(x, 1) \in Lk(E_y, K)$ if and only if $y < x$,

2. $(q, -1) \in Lk(E_y, K)$ if and only if $qy \in Allow(W; 3)$.

3. $(r, 1) \in LK$ if and only if $r < y$.

As usual, the letters q, r, s, and t represent reflections and x represents an allowable element of length three (Coxeter element for some $T \in \mathcal{S}$ such that $|T| = 3$).

Edges correspond to allowable expressions (w_1, w_2, w_3) of length three for which $y = w_1 w_2$ or $y = w_2 w_3$. As before, we use a more suggestive notation to enumerate the edges. It suffices to consider two R-reduced expressions of length three: $x = qrs = rst$, where $y = rs$. The lengths of the edges are determined by the dihedral angles between faces. The corresponding 3-cells in K give rise to the following edges in $Lk(E_r, K)$

1. A reduced expression $x = qrs = qy$ corresponds to an edge from $(q, -1)$ to $(r, 1)$, i.e. and edge from $(xy^{-1}, -1)$ to $(r, 1)$ where $r < y < x$. This edge has length $\pi/2$.

2. A reduced expression $x = rst = yt$ corresponds to an edge from $(r, 1)$ to $(x, 1)$, where $r < y < x$. This edge has length $\pi/2$.

Lemma 5.4. *Let (W, S) be a three dimensional Coxeter system with a total ordering of S. Let $T \in \mathcal{S}$ have cardinality three. Suppose $y < x_T$ is an allowable rotation. Then $Lk(E_y, K_T)$ is isometric to the spherical suspension $\{(x_T, 1), (x_T y^{-1}, -1)\} * \{(r, 1) : r < y\}$. Thus, $Lk(E_y, K_T)$ is CAT(1).*

Proof. $(w, 1) \in Lk(E_y, K_T)$ if and only if $w < y$ or $y < w$. Only $w = x_T$ satisfies the second condition, and only reflections $w = r < y$ satisfies the first. $(w, -1) \in Lk(E_y, K_T)$ if and only if (w, y) is an allowable expression of length three if and only if $w = x_T y^{-1}$; so this vertex is unique. We have already enumerated all the edges between such vertices: there is an edge of length $\pi/2$ from $(x_T, 1)$ to each $(r, 1)$ such that $r < y$ and there is an edge of length $\pi/2$ from each $(r, 1)$ to $(x_T y^{-1}$ because $r < y < x$. Thus, we have described $Lk(E_y, K_T)$ as a spherical suspension with poles $(x_T, 1)$ and $(x_T y^{-1}, -1)$. It is the suspension of two CAT(1) spaces (discrete sets); so it is CAT(1). $\square$

The singular points in $Lk(E_y, K)$ are easy to describe:

Lemma 5.5. *Let (W, S) be a three dimensional Coxeter system together with a total ordering of S. Let $T, T' \in \mathcal{S}$ be distinct and of cardinality three. Suppose $y < x_T, x_{T'}$. Then $Lk(E_y, K_T) \cap Lk(E_y, K_{T'}) = \{(r, 1) : r < y\} = R_{T \cap T'}$. These common points lie distance π apart in each of the links.*

Proof. As $x_T \neq x_{T'}$, the common vertices must have the form $(r, 1)$ where $r < y$. These points lie at the equator of the suspensions; thus, two such points are always distance π apart. In fact, by Lemma 2.2, we have that $y = x_{T \cap T"}$. Every reflection in $R_{T \cap T'}$ is y-allowable. $\square$

Theorem 5.2. *Let (W, S) be a three dimensional Coxeter system together with a total ordering of S. Suppose $y \in Allow(W; 2)$ is an allowable rotation. If y is a maximal element ($\nexists\, w \in Allow(W)$ such that $w > y$), then $Lk(E_y, K)$ is the discrete set $\{(r, 1) : r < y\}$ and $y = x_T$ for some $T \in \mathcal{S}$ of cardinality two. If y is not a maximal element, then $Lk(E_y, K) \cong \{(x_T, 1), (x_T y^{-1}, -1) : y < x_T\} * \{(r, 1) : r < y\}$. In either case, $Lk(E_y, K)$ is CAT(1).*

Proof. The case of y maximal is obvious from the characterization of vertices. A discrete space is CAT(1). If y is not maximal, then there are subcomplexes $Lk(E_y, K_T)$

such that $y < x_T$. The union of these subcomplexes is $Lk(E_y, K)$. Their common intersection is precisely the set $\{r < y : r \in R_T\}$. The description of $Lk(E_y, K)$ as a spherical suspension follows. In particular, the complex is CAT(1). $\square$

Remark. We have shown that if y is not maximal, then $Lk(E_y, K)$ has diameter π. That is, every pair of points in the complex lie distance $\leq \pi$ apart. This follows from the fact that the complex is a spherical suspension. If y is maximal, then $Lk(E_y, K)$ is discrete. This will be useful later on for describing the paths in L which are locally geodesic at the vertex $(y, \epsilon) \in L$.

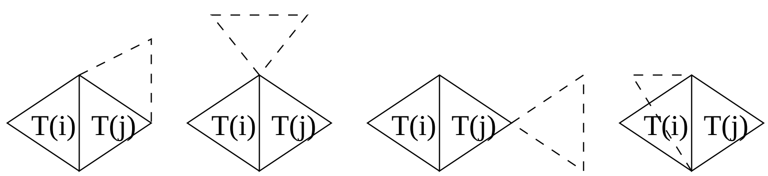

Figure 5.4: How three (maximal) spherical subsets can define a three dimensional FC Artin group.

Case 3: Finally, consider the third type of edge in K, where E_w is labelled by an allowable element of length three, i.e. $w = x_T$ for some spherical subset of cardinality three. Let $x := x_T$. The vertices of $Lk(E_x, K)$ correspond to allowable expression of length two (w_1, w_2) such that $w_1 w_2 = x$. So, the vertices have the form $(w_1, 1)$, where $w_1 < x$. These vertices are labelled by either a reflection $r < x$ or a rotation $y < x$.

There is an edge from $(r,1)$ to $(y,1)$ if and only if $r < y < x$; the length of this edge is $\pi/3$. Observe that $Lk(E_x, K) = Lk(E_x, K_T)$.

Theorem 5.3. *Let (W,S) define a three dimensional FC Coxeter system together with a total ordering of S. Let $x = x_T \in Allow(W;3)$. Then $Lk(E_x, K)$ is CAT(1).*

Proof. As the link is equal to the subcomplex $Lk(E_x, K_T)$, the argument is identical to T. Brady's argument in [Br1]. Brady's observation is that the complex is bipartite graph all of whose edges have length $\pi/3$. So, it suffices to eliminate short loops which are made up of two or four edges. A loop of two edges would not be geodesic. A loop of four edges corresponds to the algebraic problem $r, q < y, z < x$. The reflections, r and q, and the rotations, y and z, belong to the finite three generator Coxeter group W_T. Considering geometric representation $\sigma : W_T \to O(\mathbb{R}^3)$, we see that the intersection of the hyperplanes corresponding to r and q is the axis of rotation for both y and z. There is only one such allowable rotation; so $y = z$ and the path is not geodesic. Thus, every loop is made up of at least six edges, and, hence, has length $\geq 2\pi$. □

Thus, we have verified that the link of each edge in K is a CAT(1) space. By Lemma 5.1 and the Link Condition, we conclude that

Theorem 5.4. *Suppose Γ defines a three dimensional FC Artin group together with a total ordering of the generating set. Then L_Γ is locally CAT(1).*

The remainder of the paper is dedicated to proving that L_Γ is globally CAT(1) whenever Γ defines a three dimensional FC Artin group. First, we show that there do not exist short loops inside the 1-skeleton of L. Next, we will show that if γ is an isometrically embedded loop of minimal length in L, then it can be "rotated", preserving it's length, into the 1-skeleton of L. Thus, the analysis of the 1-skeleton of L is sufficient to rule out all short loops that might occur in L.

5.4 Basic gluing of CAT(1) spaces

Recall that a subspace Y of a geodesic metric space (X, d) is *r-convex* if every pair of points $x, y \in Y \subset X$ such that $d(x, y) < r$ may be joined by a geodesic segment, and, moreover, every such segment lies in Y.

Theorem. *(Basic Gluing of CAT(1) Spaces) Let X_1 and X_2 be CAT(1) spaces and let Y be a complete metric space. Suppose we are given π-convex subspaces $A_i \subset X_i$ and isometries $\phi_i : Y \to Y_i \subset X_i$ for $i = 1, 2$. Then the space obtained by gluing X_1 and X_2 along Y, denoted by $X := X_1 \sqcup_Y X_2$, is CAT(1).*

The proof may be found in [BH]. The idea is to use Aleksandrov's Lemma. Basically, the lemma says that if two triangles satisfy the CAT(1) inequality, then so does the triangle obtained gluing the two given triangles together along an isometric edge. Then one gives sufficient hypotheses to guarantee that every triangle in X of perimeter $< 2\pi$ may be decomposed into two triangles which each lie in either X_1 or X_2.

By applying the basic gluing lemma, we will prove that L_Γ is CAT(1) whenever the Coxeter graph Γ is sufficiently simple. For such a Coxeter graph, it will be relatively easy to decide that the subcomplexes Y_i are π-convex. More precisely, successive application of either of the following two lemmas will apply:

Lemma 5.6. *Let X_i be CAT(1) and let $Y_i \subset X_i$, $i = 1, 2$. Assume that Y_1 and Y_2 are finite discrete sets of the same cardinality. If every pair of points $x, y \in Y_i$ are distance $\geq \pi$ apart in X_i, then Basic Gluing applies to $X_1 \sqcup_{Y_1 = Y_2} X_2$.*

Proof. Each Y_i is trivially a π-convex subspace of X_i. The proof of the Basic Gluing Lemma is based on Aleksandrov's lemma that says that if two triangles sharing a common edge satisfy the CAT(κ) inequality, then so does the bigger triangle. The only debate is whether a triangle which meets both X_1 and X_2 satisfies the CAT(1) inequality. Usually, one proceeds by decomposing such a triangle into two smaller triangles which, individually, lie entirely in X_1 or X_2, but which share a common edge in the identified subspaces $Y_1 \sim Y_2$. But, as every pair of points in each Y_i are distance $\geq \pi$ apart, any such triangle already has perimeter $\geq 2\pi$. $\square$

Lemma 5.7. *Let Y and X_1 be connected CAT(1) spaces. Suppose we are given a continuous bijection $\phi_1 : Y \to Y_1 \subset X_1$ which takes local geodesics to local geodesics. If Y has diameter $\leq \pi$ then ϕ_1 is an isometry and Y_1 is a π-convex subspace of X_1.*

Proof. Let $x, y \in Y$. Let λ parameterize a geodesic segment from x to y. Then $\phi \circ \lambda$ parameterizes a locally geodesic segment in X of length $\leq \pi$. As X is CAT(1), this segment is, in fact, a geodesic. Hence, ϕ is an isometry. So, every pair of points in Y_1 may be joined by a geodesic which lies in Y_1. As geodesics in a CAT(1) space of length $< \pi$ are unique, every geodesic, which joins a pair of points in Y_1 which are distance $< \pi$ apart in X_1, is contained in Y_1. Hence, Y_1 is a π-convex subspace of X_1. $\square$

5.5 Simple three dimensional FC Artin systems

We will prove that if Γ is sufficiently simple, then L_Γ is CAT(1). We begin by recalling what is known about the (global) curvature of L_T, the link of the Brady complex K_T associated to a spherical Coxeter group W_T.

Theorem. *(T. Brady, J. McCammond) If (W_T, T) defines a spherical Coxeter group with one, two, or three generators, then L_T is CAT(1).*

The one generator case is trivial, the two generator case was studied by T. Brady and J. McCammond in [BM], and the three generator case is the subject of T. Brady's article [Br1]. Each link decomposes as a spherical suspension:

If $|T| = 2$, then $L_T \cong \{ (y,1), (y,-1) \} * \{ p_r : r < y \}$, where $y = x_T$. Thus, L_T is CAT(1). The point p_r is the midpoint of the edge $[(yr,-1),(r,1)]$. The longitudinal arcs in the suspension are the union of three edges: $[(y,-1),(yr,-1)] \cup [(yr,-1),(r,1)] \cup [(r,1),(y,1)]$. In particular, L_T has diameter π.

If $|T| = 3$, then $L_T \cong \{(x,1),(x-1)\} * Lk(E_x, K_T)$, where $x = x_T$. Thus, L_T is CAT(1).

Now, consider the intersection of such subcomplexes in L:

Lemma 5.8. *Let (W,S) be a Coxeter system of dimension ≤ 3. Suppose $T, T' \in \mathcal{S}$ are distinct and $T \cap T' \neq \emptyset$. Then the vertices common to L_T and $L_{T'}$ are the elements of Allow$(x_{T\cap T'} \times \{\pm 1\}$.*

Proof. Suppose (w, ϵ) is a vertex of L_T and $L_{T'}$. Then $w \leq x_T, x_{T'}$. If $\ell(w) = |T \cap T'|$, then, by Lemma 2.2, $w = x_{T \cap T'}$. Otherwise, $\ell(w) < |T \cap T'| \leq 2$; so w is a reflection in $R_T \cap R_{T'} = R_{T \cap T'}$. Every such reflection is $x_{T \cap T'}$-allowable. □

Lemma 5.9. *Let (W, S) be a Coxeter system of dimension ≤ 3. Suppose $T, T' \in \mathcal{S}$ are distinct and $|T \cap T'| = 2$. Then the edges common to L_T and $L_{T'}$ are precisely the edges of $L_{T \cap T'}$. If $|T \cap T'| < 2$, then there are no common edges.*

Proof. The second statement is just a dimension count. For the first statement, we only need to show that there are no more edges than those in $L_{T \cap T'}$. If there is an edge $[(w, \epsilon), (w', \epsilon)]$ in common, then $w, w' \leq x_{T \cap T'}$. So, each is either a reflection in $R_{T \cap T'}$ or equal to $x_{T \cap T'}$. Let $y := x_{T \cap T'}$. According to the characterization of edges, there are, naively, five possible types:

$$(r, -1), (y, -1)], \quad [(r, -1), (q, 1)], \quad [(r, 1), (y, 1)], \quad [(y, -1), (q, 1)], \quad [(r, -1), (y, 1)],$$

where $r, q \leq y$. The first three (left to right) necessarily lie in $L_{T \cap T'}$ and the final two are not possible. (The product of the negative multiplied on the right by the positive defines an allowable element of length three; but the product belongs to $W_{T \cap T'}$.) □

Combining the above two lemmas, we conclude that the subcomplex $L_T \cap L_{T'}$, with its intrinsic metric, is isometric to $L_{T \cap T'}$.

Lemma 5.10. *Let (W, S) be an FC Coxeter system of dimesion ≤ 3. Totally order S, and let K be the Brady complex and L its link. Let $T \in \mathcal{S}$ and let L_T be the link of K_T. Assume that L is CAT(1).*

1. *If $|T| = 1$, then L_T is a finite discrete set and every pair of points $x, y \in L_T \subset L$ are distance $\geq \pi$ apart.*

2. *If $|T| = 2$, then L_T has diameter $\leq \pi$ and the inclusion $L_T \to L$ takes local geodesics to local geodesics.*

The two cases in the Lemma 5.10 correspond to the two cases where we can apply basic gluing (Lemmas 5.6 and 5.7). At first glance, Lemma 5.10 is only applicable in the base cases where (W, S) is a spherical Coxeter system of three or fewer generators; for, only in these, cases do we know that the links are CAT(1). But, given two such links, L_{T_1} and L_{T_2} with $|T_1 \cap T_2| = 1$ or 2, we can apply Basic Gluing along the subcomplex $L_{T_1 \cap T_2}$. Thus, we deduce that $L_{T_1 \cup T_2}$ is CAT(1). This process can continue, but only if we glue along a common spherical subset with one or two generators. In terms of link complexes, if we know that L_{Γ_1} and L_{Γ_2} are CAT(1) and $\Gamma_1 \cap \Gamma_2$ is a single vertex or a single edge, then we can deduce that $L_{\Gamma_1 \cup \Gamma_2}$ is CAT(1). The link complex of an arbitrary three dimensional FC Artin group cannot be so easily described; for instance, the link complex might be the cone on a loop with four or more edges. But this method will be sufficient to detect all short loops in L.

Definition. Let (W, S) be a Coxeter system. We say that $T \in \mathcal{S}$ is a *maximal* spherical subset if it is not a proper subset of any $T' \in \mathcal{S}$. A subcomplex indexed by a maximal spherical subset is called a *maximal subcomplex.* This descriptor may be applied to the complexes L_T or $Lk(E_w, K_T)$.

Proof. (of Lemma 5.10) Let L be the link of the Brady complex of an FC Coxeter system (W, S) of dimension ≤ 3. Let $T \in \mathcal{S}$ have a single element: $T = \{r\}$. Then,

$L_T = \{(r,1),(r,-1)\}$. We need to show that these points lie distance $\geq \pi$ apart from one another inside L. As we are assuming that L is CAT(1) and as every local geodesic of length $\leq \pi$ is a (global) geodesic, it suffices to find a locally geodesic segment from $(r,1)$ to $(r,-1)$. If T is maximal, then $(r,1)$ and $(r,-1)$ are connected components of L, and, so, they lie distance $\geq \pi$ apart. Suppose $T \subset T' = \{r,s\}$. Let $y := x_{T'}$.

If T' is maximal, then the connected component of $L_{T'}$ is a 1-complex. Every path along the edges which does not double back on itself is a local geodesic. We find a path γ consisting of the following adjacent edges:

$$[(r,1),(y,1)] \cup [(ry,1),(y,1)] \cup [(r,-1),(ry,1)].$$

This path has length $\pi/4 + \pi/4 + \pi/2 = \pi$. Note that as $r < y$, $\ell(ry)$. (Either $y = rs$ or $y = sr$ according to the total order; thus, $ry = s$ or $ry = rsr$.)

If T' is not maximal, then we consider $T'' = \{r,s,t\}$. The connected component of $L_{T''}$ is a piecewise spherical 2-complex. In fact, the same path above remains locally geodesic. We only need to verify that the path γ is locally geodesic at vertices $(y,1)$ and $(ry,1)$. We appeal to our analysis of the links $Lk((y,1),L) \cong Lk(E_y,K)$ and $Lk((ry,1),L) \cong Lk(E_{ry},K)$:

There is a (locally geodesic) path in $Lk(E_y,K)$ of length π which joins $(r,1)$ to $(ry,1)$. As $Lk(E_y,K)$ is CAT(1), this path is a geodesic. Thus, the distance between $(r,1)$ and $(ry,1)$ is equal to π. Thus, γ is locally geodesic at $(y,1)$.

Likewise, there is a locally geodesic path in $Lk(E_{ry},K)$ of length π which joins $(y,1)$ to $(r,-1)$. As $Lk(E_{ry},1)$ is CAT(1), this path is a geodesic. Thus, the distance between $(y, 1)$ and $(r, -1)$ is equal to π. Thus, γ is locally geodesic at $(ry, 1)$. We conclude that γ is a geodesic in L; and, hence, $(r, 1)$ and $(r, -1)$ are distance π apart in L.

Now suppose that $T \in \mathcal{S}$ has two elements: $T = \{r, s\}$. As we have already remarked above, L_T has diameter π. To show that the inclusion $L_T \to L$ takes local geodesics to local geodesics, it suffices to compute distances in the links of the link or, equivalently, in the links of the edges. Let (w, ϵ) be a vertex of L_T. We want to show that if (w', ϵ') and (w'', ϵ'') are vertices adjacent to (w, ϵ) in L_T, then the distance between (w', ϵ') and (w'', ϵ'') in $Lk(E_w, K)$ is equal to π. As $Lk(E_w, K)$ is CAT(1), it suffices to find a locally geodesic path from (w', ϵ') to (w'', ϵ'') of length π. There are several cases, but the computations are straightforward. $\square$

Definition. Let Γ define a three dimensional FC Artin group A_Γ together with a total ordering of the generating set S. We say that Γ is *simple* if there are at most three maximal spherical subsets in $\mathcal{S}$.

Theorem 5.5. *Let Γ define an FC Artin group of dimension ≤ 3. If Γ is simple, then L_Γ is CAT(1).*

Proof. Let $T_1, \ldots, T_k$ be the maximal spherical subsets. As Γ is simple, $k \leq 3$. Each maximal spherical subset defines a top dimensional simplex in Γ. As Γ has dimension ≤ 2, these subsets define either a vertex, an edge, or a 2-simplex. If every simplex is a vertex, then L is discrete and, hence, trivially CAT(1). Suppose that Γ has least one simplex of dimension > 0 and assume that Γ is connected. We observe that $L_\Gamma = \bigcup L_{T_i}$. Moreover, L_Γ may be constructed by gluing (in some order) the T_i's so that each gluing occurs along either a single vertex or a single edge. This follows from the FC hypothesis. Refer to Figure 5.4 to visualize all such possibilities. There are at most two such stages of gluing, and at each stage, the pieces which are glued are of the form L_{T_i} or $L_{T_i \cup T_j}$. These are glued along intersections of the form $L_{T_i \cap T_j}$ or $L_{T_1 \cap T_2 \cap T_3}$. The links arise from Coxeter systems of dimension ≤ 3 with at most two maximal spherical subsets. The intersections arise from spherical Coxter groups with one or two generators. Thus, we may apply Lemma 5.10 at each stage of the gluing. The resulting complex, namely L_Γ, is CAT(1). If Γ has more than one connected component, then the links of these components are disjoint in L_Γ. A disjoint union of CAT(1) spaces is obviously CAT(1). $\square$

5.6 Subcomplexes of CAT(1) M_1-simplical complexes

Lemma 5.11. *If L and L' are M_1-simplicial complexes and $L' \subset L$ is a subcomplex, then every closed geodesic $\gamma : S^1 \to L$ such that $\gamma(S^1) \subset L'$ defines a closed geodesic in L' with respect to its intrinsic metric. Therefore, if L' is CAT(1) with respect to its intrinsic metric, then γ has length $\geq 2\pi$.*

Proof. A path in $L' \subset L$ which minimizes the distance in L between points in its image clearly minimizes the distance between these points in L'. Thus, a geodesic in L with image in L' defines a geodesic in L'. In particular, if L' is CAT(1) with respect to its intrinsic metric, then the length of this geodesic must be $\geq 2\pi$. $\square$

We will use this lemma to rule out the existence of short loops in L_Γ. Applying Lemma 5.11 in conjuction with Theorem 5.5, we see that no short loop can be contained in any subcomplex $L_{\Gamma'} \subset L_\Gamma$ such that Γ' is simple.

CHAPTER 6

GLOBAL CURVATURE OF L_Γ

6.1 Locally Geodesic Edge Loops in L_Γ

By a *locally geodesic edge loop*, we mean the image of a locally isometrically embedded loop which lies entirely in the 1-skeleton of L. We begin by proving that certain certain edge paths in $L^{(1)}$ are not locally geodesic.

Recall that a path $\gamma : (a, b) \to L$ is locally geodesic at $\gamma(t_0)$, $a < t_0 < b$, if and only if the distance between $-\gamma'(t_0)$ and $\gamma'(t_0)$ is greater than or equal to π in $Lk(\gamma(t_0), L)$. Here $-\gamma$ denotes the reverse path, and $\gamma'(t_0)$ denotes the unit tangent vector at $t = t_0$.

Lemma 6.1. *A locally geodesic edge path in L cannot contain a subpath of the form*

$$x_{T(1)} \to y_1 \to x_{T(2)} \to y_2 \to x_{T(3)}.$$

The signs of the vertices of this path are all the same; strictly speaking, we mean $(x_{T(1)}, \epsilon)$, etc. In the proof, we assume that all the vertices have positive sign. Thus, we have only labelled the vertices by their allowable element. The argument for the case where the vertices have a negative sign is essentially the same. As usual, the letter x denotes an allowable element of length three and the letter y denotes an allowable element of length two (rotation).

Proof. For this path to be locally geodesic we must have that $x_{T(1)} \neq x_{T(2)}$ and $x_{T(2)} \neq x_{T(3)}$. We do not exclude $T_1 = T_3$. (L is a simplicial complex. There are unique edges joining these vertices; and, clearly, a local geodesic cannot "double back" along an edge just traversed.) Using the same reasoning, we see that y_1 and y_2 must be distinct. By Lemma 2.2, $y_1 = x_{T(1)\cap T(2)}$ and $y_2 = x_{T(2)\cap T(3)}$.

Suppose that $T(2) = \{a \prec b \prec c\}$ and $x_{T(2)} = abc$. Then each y_i must be one of ab, bc or ac. These fit together to make the following 2-cell in $L_{T(2)}$:

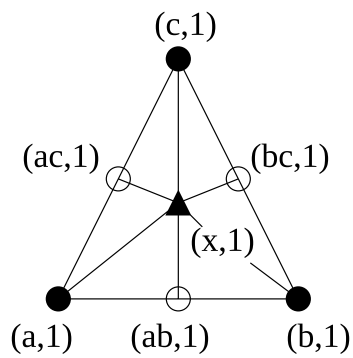

Figure 6.1: The 2-cells of L form an all-right spherical triangle.

y_1 and y_2 must be different allowable rotations of length two from among ab, bc, and ac. Regardless of which particular rotations they are in W_{T_2}, the path $y_1 \to x_{T(2)} \to y_2$ makes an angle of $2\pi/3$ at $x_{T(2)}$; and so, this path is not locally geodesic. So, this configuration cannot appear as a subpath of a locally geodesic edge path in L. $\square$

We seek a list of all potentially short edge loops in L. As in the analysis of the links of edges in K, we consider locally geodesic edge paths which join vertices in common to the subcomplexes L_T and $L_{T'}$, where $T, T' \in \mathcal{S}$. (Refer to Lemmas 5.8 and 5.9.) We can refine the search by using the following observation:

Lemma 6.2. *Let $L = L_\Gamma$ where Γ defines a three dimensional Coxeter system together with a total ordering of its generating set. Suppose γ is a local geodesic in L which passes through a vertex (y, ϵ), where y is an allowable rotation. Then either y is not contained in any W_T with $T \in \mathcal{S}$ and $|T| = 3$, or the distance between incoming and outgoing unit tangent vectors is equal to π.*

Proof. This is just a restatement of the observation that $Lk(E_y, K)$ is a suspension. The link has diameter π or is discrete (Theorem 5.2). $\square$

In particular, Lemma 6.2 implies that every local geodesic in L which extends an edge from (w, ϵ) to (y, ϵ') (and through this vertex) must first traverse another edge in L. Moreover, if w is a reflection and $\epsilon = \epsilon'$, then the other edge lies in the same subcomplex, L_T, which contained the initial edge. (See Figure (insert picture).) In particular, an edge path of the form $[(r, \epsilon), (y, \epsilon)] \cup [(r', \epsilon), (y, \epsilon)]$ stays within any subcomplex L_T, where $y \leq x_T$. Thus, in some sense, the vertices in L of length two are not "singular".

Remark. There are several different ways to extend a path geodesically through a vertex (r, ϵ) of length one or a vertex (x, ϵ) of length three; for, the links $Lk(E_r, K)$ and $Lk(E_x, K)$ do not have diameter π. These vertices will be called *singular vertices.*

By a *segment* in L, we mean one of the geodesic edge paths appearing in Figure 6.2. Refer to the explanation below.

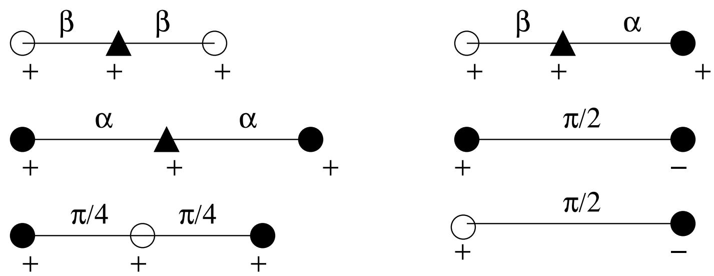

Figure 6.2: The six segments in $L^{(1)}$.

The black squares denote vertices of length one (reflections), the white squares denote vertices of length two (rotations), and the white triangles denote vertices of length three. We have displayed the sign of the vertices. Four of the segments have vertices with the same sign; two have vertices of opposite sign. These polarities may be reversed, changing all positive signs to negative signs.

The lengths of the segments in the left column are (from top to bottom) $2\beta, 2\alpha$, and $\pi/2$. The lengths of every segment in the right column is $\pi/2$. Note that $\alpha+\beta = \pi/2$. It is also helpful to keep in mind the following estimates: $\beta < \pi/4 < \alpha < \pi/2$, $4\beta > \pi/2$, and $\alpha + \beta = \pi/2$.

Proposition 6.1. *Every locally geodesic edge loop in L can be decomposed into segments which meet only at vertices.*

Proof. Every possible sequence of two adjacent edges can be formed by using the list of segments except for the following two:

1. $\bullet - \circ - \triangle$; e.g. $[(r,1),(y,1)] \cup [(y,1),(x,1)]$. All the vertices in this configuration have the same sign. The two edges make a right angle at the center vertex. This is due to the fact that the path belongs to the boundary of a 2-cell in L.

2. $\bullet - \circ - - \bullet$; e.g. $[(r,1),(y,1)] \cup [(q,-1),(y,1)]$. The two ends have opposite signs; the double dash denotes an edge of length $\pi/2$. The two edges make a right angle at the center vertex. This is due to the fact that the edges belong to the boundary of a 2-cell.

So, neither of these paths is locally geodesic. $\square$

We now try to list all locally geodesic edge loops which have length $< 2\pi$. Observe that each segment in Figure 6.2 which contains a vertex x of length three is entirely contained in the subcomplex L_T, where $x = x_T$. Similarly, the segment $\bullet - \circ - \bullet$ where the middle vertex is the allowable rotation y, is entirely contained in the subcomplex $L_{T(y)}$, where $T(y)$ is the smallest spherical subset $T \in \mathcal{S}$ such that $y \in W_T$. Finally, the remaining two segments are labelled by vertices of opposite sign. Suppose the negative vertex is labelled by the allowable element w_1, and the positive vertex is labelled by the allowable element w_2. Then $w := w_1 w_2$ is allowable and the segment is contained in the subcomplex $L_{T(w)}$. Thus, given a locally geodesic edge loop γ, we can study the subcomplexes it meets along (at least) an edge by decomposing it into segments.

Applying Lemma 5.11 in conjuction with Theorem 5.5, we observe that γ cannot be a short loop in L unless it consists of at least four segments. If there are three or fewer segments, then the Artin system defined by the union of the subsets T corresponding to the segments would define a simple Artin system. So, this leaves only geodesic edge loops consisting of four or more segements. Considering their lengths, at least on one of the segments must be of the form $\circ - \blacktriangle - \circ$. A path three or more of these segements is forbidden by Lemma 6.1. $|T| = 3, T \in \mathcal{S}$. So, in fact, every short loop can contain at most four segments.

But now consider what happens along the path γ at the end vertices of $\circ - \blacktriangle - \circ$. Suppose the middle vertex of length three is labelled by x_T and the end vertices are labelled by y_1 and y_2. At each end vertex, (y_i, ϵ), either the next segment of the path lies in the same maximal subcomplex L_T or it lies in a distinct maximal subcomplex $L_{T'}$, where $T' \in \mathcal{S}$ and $|T'| = 3$. In the first case, γ is contained in a subcomplex involving three or fewer maximal subcomplexes; So, by Lemma 5.11 and Theorem 5.5, γ would have length $\geq 2\pi$. In the second case, we may assume that both y_1 and y_2 belong to maximal subcomplexes distinct from L_T. But, then we can apply the arguments of Lemma 6.1: the vertices $(y_1, \epsilon), (y_2, \epsilon)$, and (x_T, ϵ) belong to a 2-cell as in Figure 6.1. Thus, such a path can not be locally geodesic at the vertex (x_T, ϵ). This completes the proof of the following theorem:

Theorem 6.1. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set. Let L_Γ be the link of Brady's complex K_Γ. Then L_Γ does not contain any short loops in its 1-skeleton.*

6.2 Developing galleries onto the sphere

The following definitions are due to M. Elder and J. McCammond [EM] and [EM2].

Definition. Let $\gamma : [a,b] \to X$ define a local geodesic in an M_1-simplicial complex X. Let $(\sigma_1, \ldots, \sigma_k)$ be the sequence of closed simplices $\sigma \subset X$ such that $\mathring{\sigma} \cap \gamma \neq \emptyset$ (if σ is a vertex, then we define $\mathring{\sigma} = \sigma$). These simplices are ordered according to the order in which γ meets each one. Let $\mathcal{G}(\gamma)$ denote the M_1-simplicial complex defined by gluing σ_i to σ_j if σ_i is a proper face of σ_j and $j = i-1$ or $i+1$, where $1 \leq i, j \leq k$. This complex is called the *linear gallery* determined by γ.

If γ defines a closed local geodesic, then we give the sequence of closed simplices $(\sigma_1, \ldots, \sigma_k)$ a cyclic ordering and define an M_1-complex as before but allow the first and last cells to be glued along their common face. Such a gallery is called a *circular gallery.*

To each circular gallery $\mathcal{G}$ there is an associated linear gallery $\mathcal{G}'$ obtained by "cutting open $\mathcal{G}$ along σ_i". Consider the cyclically ordered sequence of closed cells, $(\sigma_1, \ldots, \sigma_k)$. Choose a fixed σ_i, and consider the sequence $(\sigma_i, \ldots, \sigma_k, \sigma_1, \ldots, \sigma_{i-1}, \sigma_i)$. We define $\mathcal{G}'$ to be the linear gallery defined by this sequence.

For each gallery $\mathcal{G}$, there is a unique locally geodesic path (or loop) defined by gluing the restrictions of γ to each closed cell. This path is called the *lift* of γ.

Theorem 6.2. *(M. Elder, J. McCammond) Let γ be a local geodesic path (or loop) in a 2-dimensional M_1-complex X. Then the interior of the linear (or circular) gallery γ immerses into X and retracts onto the lift of γ.*

The proof can be found in [EM2].

We want to measure the length of a closed local geodesic γ in the link L by developing the it onto the unit sphere M_1^2. In fact, we develop the entire gallery determined by γ onto M_1^2:

Definition. (Developing a circular gallery onto the sphere) Let $\gamma : [0, h] \to L$ define a closed local geodesic. Choose a parametrization so that $\gamma(0)$ belongs to an edge or vertex of L. Let $\mathcal{G}$ be the gallery determined by γ. Let $\mathcal{G}'$ be the linear gallery obtained by cutting open $\mathcal{G}$ along the closed cell containing $\gamma(0)$ in its interior. Let $\hat{\gamma}$ denote the lift of γ. Fix a point p (pole) in the unit sphere and fix an oriented great arc from p to the antipodal point $-p$. Let $\phi : \mathcal{G}' \to M_1^2$ be defined by mapping $\hat{\gamma}(0)$ onto the midpoint of the oriented great arc. If we insist that the image of the $\hat{\gamma}$ define a local geodesic which makes an (oriented) angle of 90 degrees with the oriented great arc, then the map ϕ is uniquely determined. ϕ is called a *developing map.* We say that ϕ *develops* $\mathcal{G}$ onto the sphere.

The key fact about a the developing map is that, by construction, the lift of γ is mapped to a great arc (or circle) on the unit sphere. In particular, if $\phi(\hat{\gamma})$ meets any other great arc in two points, then γ must have length $\geq \pi$.

Local geodesics in L develop in a very special way onto the 2-sphere. Let $\mathbb{S}^2$ denote the unit 2-sphere, M_1^2, together with the following simplicial structrure: First divide the sphere into eight spherical triangles by intersecting with the usual coordinate planes. Each of these triangles is a spherical triangle with all lengths and angles measuring 2π. Such a triangle is called an *all-right* triangle. Then, secondly, take the barycentric subdivision of this complex. The resulting M_1-simplicial complex has 48 spherical triangles is denoted by $\mathbb{S}^2$.

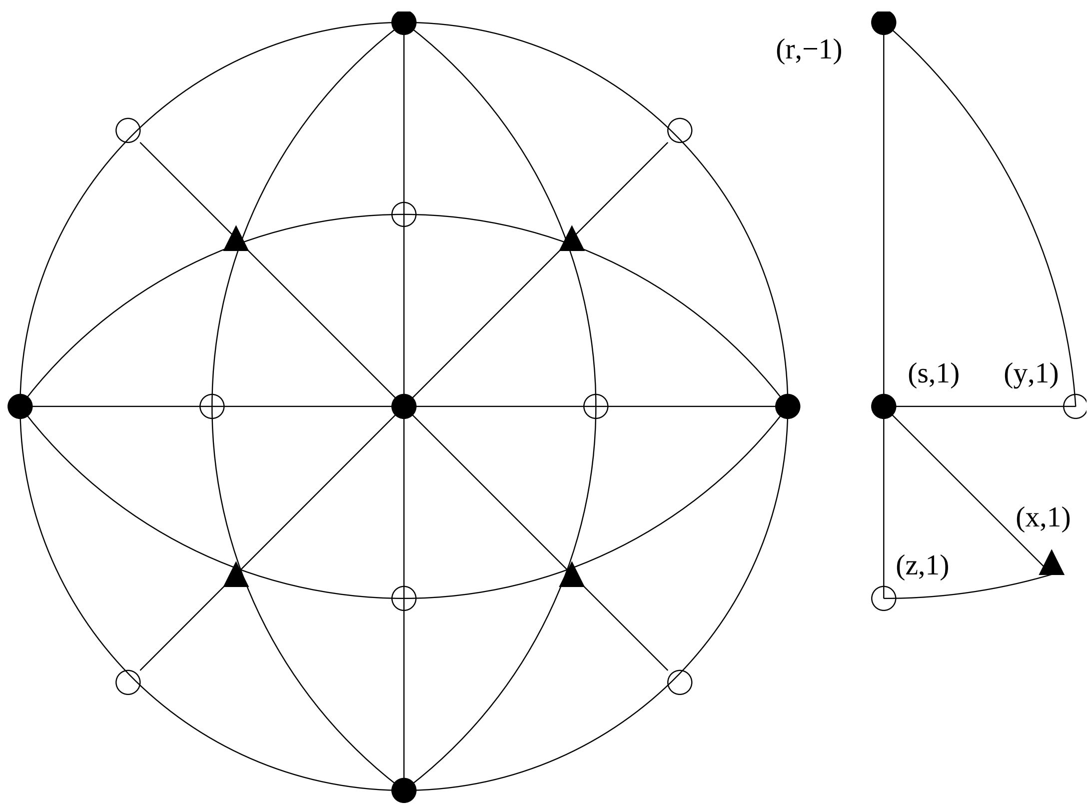

Figure 6.3: A top view of the simplicial complex $\mathbb{S}^2$. To the right are shown the two types of spherical 2-simplices which occur in L. Each simplex is isometric to a subcomplex of $\mathbb{S}^2$

Each of the 48 spherical triangles is isometric to the 2-cell of L with edge lengths $\beta, \pi/4$, and α. Such 2-cells are labelled by allowable elements satisfying $r < y < x$.

The other 2-cells of L are isometric to subcomplexes of $\mathbb{S}^2$. The 2-cells with edge lengths $\pi/4, \pi/2, \pi/2$ are isometric to one half of an all-right triangle. Such a 2-cell is isometric to a subcomplex consisting of three adjacent 2-simplices in $\mathbb{S}^2$. Refer to Figure 6.3.

Recall that the vertices of L of length one or three are called singular vertices. The link of these vertices in L have diameter $> \pi$.

Proposition 6.2. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set, and let $L = L_\Gamma$ be the link of Brady's complex K_Γ. Suppose $\gamma : [0, h] \to L$ is a local geodesic. Assume that either γ does not meet any singular vertices or that such vertices only occur at its endpoints. Then γ determines a gallery which develops onto a subcomplex of $\mathbb{S}^2$.*

Proof. Let $\mathcal{G}$ be the gallery determined by γ. If γ is closed, then let $\mathcal{G}'$ be the linear gallery obtained by cutting open $\mathcal{G}$ along the closed cell containing $\gamma(0)$ in its interior. Choose a point p (pole) in $\mathbb{S}^2$ and an oriented great arc from p to the antipodal point $-p$ so that the initial cell of the gallery maps to a simplex. We may adjust the choice of pole and oriented arc so that the first top dimensional cell crossed by γ develops onto an isometric simplex in $\mathbb{S}^2$. (The dimensions of the cells in $\mathcal{G}$ alternate going up and down. Insist that the larger of the first two cells be mapped to a simplex).

Once this choice is made, the developing map is determined by the condition that the lift of γ be geodesic in $\mathbb{S}^2$. But as γ does not meet any singular points between time 0 and h, the gallery $\mathcal{G}'$ develops onto a subcomplex of $\mathbb{S}^2$. This follows from the fact that the link of any non-singular point in L has components which are discrete or of diameter π. We have already observed that the link of a vertex labelled by an allowable rotation has this property (Theorem 5.2). And it is easy to see that the link of an interior point of an edge or 2-cell of L has this property.

On the other hand, if γ meets a singular vertex in its interior, it may well make an angle larger than π at this vertex. The resulting cells in gallery near this vertex need not develop onto a subcomplex of $\mathbb{S}^2$. $\square$

In particular, given a local geodesic γ in L, we view its gallery as subcomplex of typical galleries determined by the geodesics in $\mathbb{S}^2$.

We can refine the developing map further in the case of a closed local geodesic which does not lie entirely in the 1-skeleton of L. As L is simplicial, every such geodesic, admits a parametrization so that it begins in at least one of the following *general positions*:

1. There exists a $\delta > 0$ such that $\gamma(0)$ is a vertex of length three and $\gamma(t) \notin L^{(1)}$ for all $0 < t < \delta$.

2. There exists a $\delta > 0$ such that $\gamma(0)$ belongs to a segment of type $\bullet - \circ - \bullet$ or $\bullet - -\bullet$ and $\gamma(t) \notin L^{(1)}$ for all $0 < t < \delta$.

In Figure 6.4, we have sketched the intial few cells of galleries (cut-open and developed onto $\mathbb{S}^2$) determined by local geodesics of L beginning in general position.

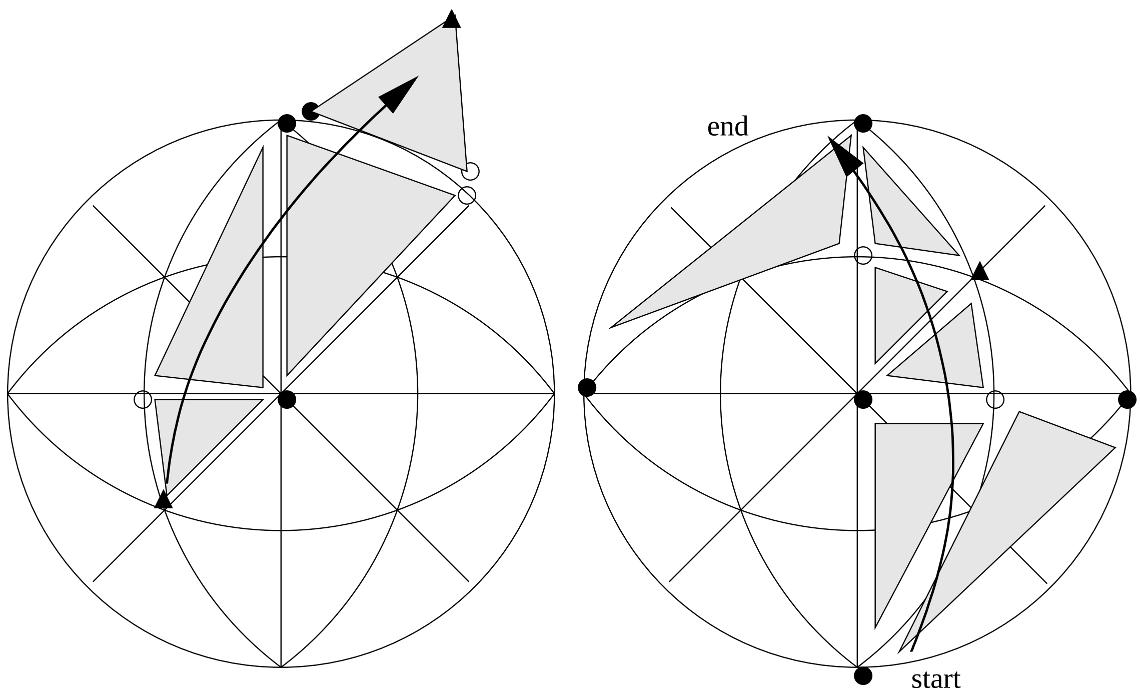

Figure 6.4: Typical galleries of local geodesics in general position.

6.3 Extra-short loops

Definition. A closed local geodesic is called an *extra-short loop* if it has length $\leqq \pi$.

In practice, given a closed local geodesic γ, which does not lie entirely in the 1-skeleton, we can choose a parametrization so that its cut-open and developed gallery develops onto a subcomplex of $\mathbb{S}^2$ which begins at one of the general positions. In terms of Figure 6.3, either we develop the local geodesic beginning at a vertex of length three or we develop beginning at one of the points in the large boundary arc (a great arc of the sphere) as in Figure 6.4.

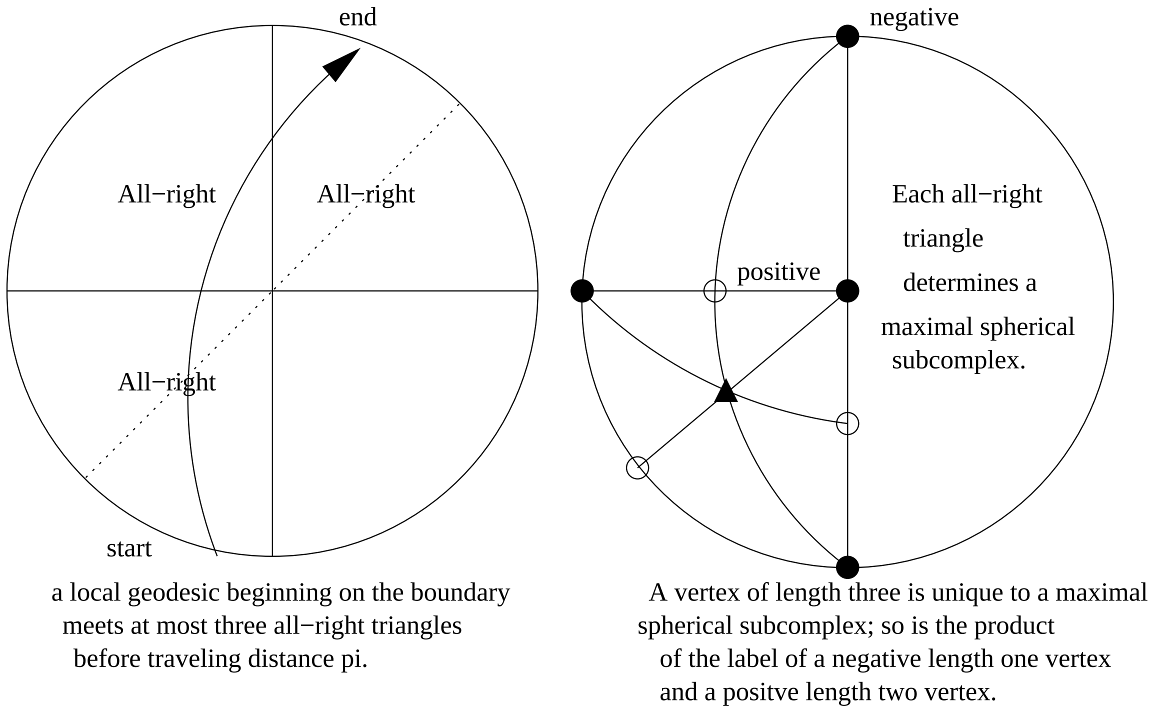

Figure 6.5: These all-right triangles encode subcomplexes L_T which contain the local geodesic γ.

Observe that the gallery of such a closed local geodesic develops onto at most three all-right triangles as depicted in Figure 6.5. Fix one such all-right triangle Δ. Then, the simplices in L which develop onto Δ all belong to the same maximal subcomplex L_T for some $T \in \mathcal{S}$. This follows from the fact that the only edges which are common to two distinct maximal subcomplexes belong to a segment of type $\bullet - \circ - \bullet$ or $\bullet - - \bullet$. (Use Lemma 5.9.) The spherical subset T is determined by either a vertex of length three or by the product of a negative and a positive vertex: one of length one, one of length two, the product of length three. (Refer two the two types of 2-cells in L as in Figure 4.5.) Using this technique, we prove the following:

Proposition 6.3. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set, and let $L = L_\Gamma$ be the link of Brady's complex K_Γ. Suppose γ is a closed local geodesic which does not lie entirely in the 1-skeleton of L. Then, γ has length $> \pi$.*

Proof. Choose a parametrization of γ so that it is in one of the general positions. As in the above discussion, either $\gamma(0)$ is a vertex of length three or $\gamma(0)$ belongs to an edge of the form $\bullet - \circ - \bullet$ or $\bullet - - \bullet$. Then cut-open and develop the gallery onto $\mathbb{S}^2$. If γ had length $\leq \pi$, then, by inspection of the all-right triangles in $\mathbb{S}^2$, we find that γ lies in the union of at most three maximal spherical subcomplexes. But with respect to its intrinsic metric, this complex is CAT(1). (Theorem 5.5.) So, by Lemma 5.11, γ has length $\geq 2\pi$. Contradiction. $\square$

Thus, we conclude that there are no extra short loops in L_Γ:

Theorem 6.3. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set, and let $L = L_\Gamma$ be the link of Brady's complex K_Γ. Suppose $\gamma : [0, h] \to L$ is a closed local geodesic. Then, γ has length $> \pi$. In other words, L_Γ does not contain any extra-short loops.*

Proof. By Theorem 6.1, we may assume that γ does not lie entirely in the 1-skeleton of L. Thus, Proposition 6.3 applies. $\square$

Using the same arguments as in Proposition 6.3, we also have the following:

Lemma 6.3. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set, and let $L = L_\Gamma$ be the link of Brady's complex K_Γ. Suppose γ is a local geodesic which joins two singular vertices in L. If γ is not an edge path, then it has length $\geq \pi$.*

Proof. We are assuming that $\gamma(0)$ is a singular vertex and that γ is not an edge path. Thus, γ begins in general position. Choose a parametrization so that γ begins in general position and develop its gallery onto $\mathbb{S}^2$. Observe, using Figure 6.3, that the only geodesics which join (potentially) singular vertices are either edge paths or have length $= \pi$. We are using the fact that the lift of γ must be a great arc in $\mathbb{S}^2$. $\square$

6.4 Shrinking and rotating local geodesics

It remains to show that L does not contain any isometrically embedded circles of length $< 2\pi$ which do not lie entirely in the 1-skeleton. The arguments are inspired by an alternate characterization of CAT(1) spaces due to B. Bowditch [Bow]. We would like to thank J. McCammond for first bringing Bowditch's work to our attention. The actual implementation of Bowditch's ideas are in the spirit of the curvature testing techniques in [EM] and, especially, the more recent paper by M. Elder, J. McCammond, and J.Meier [EMM].

The following theorems of B. Bowditch are of interest. Refer to the original article for proofs [Bow]. In each of the below, we apply the theorems to $X = L_\Gamma$.

Theorem 6.4. *(Bowditch) Let X be a compact locally CAT(1) space. If X is not CAT(1), then there exists a minimal length closed geodesic of length $< 2\pi$. Moreover, a closed local geodesic of minimal length is, in fact, a closed geodesic.*

Theorem 6.5. *(Bowditch) Let X be a compact locally CAT(1) space. If γ is a closed local geodesic in X of length $< 2\pi$, then γ may be freely homotoped via non-length increasing paths to constant loop. γ is said to be* shrinkable.

Theorem 6.6. *(Bowditch) Let X be a compact locally CAT(1) space. If γ is a loop in X, then either γ is shrinkable or γ freely homotoped via non-length increasing paths to closed geodesic α.*

Remark. The above three theorems use a reformulation of the locally CAT(1) condition in terms of the length of a minimal closed geodesic. He defines a space to be ϵ-CAT(1) if every triangle of perimeter $< 2\epsilon$ satisfies the CAT(1) inequality. (So, π-CAT(1) is the same thing as CAT(1).) Then he proves that for a compact, locally CAT(1) space there is a unique ϵ, $0 < \epsilon < \pi$, such that X is ϵ-CAT(1), and X contains an isometrically embedded circle of length 2ϵ. The analogous 2ϵ in differential geometry is the systole.

Theorems 6.5 and 6.6 use the Birkhoff curve shortening process. This process takes a closed loop and iterates the process by which we subdivide the loop into segments, join the midpoints of adjacent segments by geodesics, and consider this new loop as the next input. The difficult problem is to decide when this process converges.

We apply Bowditch's theorems to a minimal length local geodesic γ in L_Γ. If L_Γ is CAT(1), we are done; otherwise, by Theorem 6.4, such minimal closed geodesic exists and has length $< 2\pi$. We will derive a contradiction.

Definition. Let γ be a closed local geodesic of length $\geq \pi$ in L_Γ. Suppose $\alpha \subset \gamma$ is an arc of length equal to π. A *rotation* of α is a constant length homotopy of α which leaves endpoints fixed. The loop γ' obtained by removing the arc α and replacing it by the rotated arc is said to be *obtained by rotating the arc* α. In particular, γ and γ' have the same length.

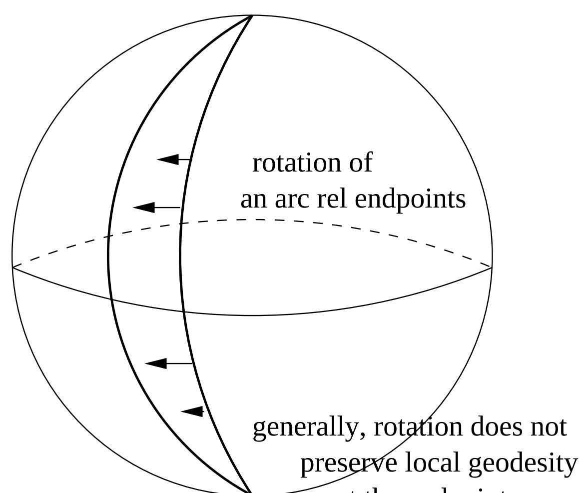

Figure 6.6: An arc $\alpha \subset \gamma$ of length π may be rotated.

(Rotation of Geodesics): Let L_Γ be the link of a Brady complex K_Γ for a three dimensional FC Artin group. Suppose γ is a minimal length closed geodesic of length $< 2\pi$. By Theorem 6.1, we may assume γ is not contained in the 1-skeleton of L. By Theorem 6.3, we may assume that the length of γ is greater than π. Choose an arc α in γ of length π which is not contained in the 1-skeleton. This is possible because the singular vertices are distance at least π apart. The singular vertices are the only points in L where a locally geodesic path may switch from an arc contained in the 1-skeleton to an arc which is an edge path. Moreover, we may choose an arc α which admits a parametrization which begins in general position.

If the endpoints of α are not singular points, then we may rotate the arc α by a small amount. In practice, this is done by developing the gallery determined by α onto the two sphere. Then rotate the lift of α by a small amount within the developed gallery. This is possible because the lift of α is, by construction, a great arc in $\mathbb{S}^2$. This rotation induces a rotation of α in L. We are using the fact that the interior of the gallery determined by α immerses into $\mathbb{S}^2$ (Elder and McCammond's Theorem 6.2).

But, the loop γ' obtained by rotation now fails to be geodesic at the endpoints of α. (This need not be the case if the endpoints were singular!) Choose small balls about each endpoint so that γ' meets each ball in two points. Then join each pair of points by geodesics. The resulting loop has length strictly less than the length of γ'. Moreover, we may realize this reduction by a sequence of homotopies which do not increase length (these are local computations in unit 2-sphere). But then, by composing these homotopies, we see that we have homotoped γ through non-length increasing paths to a path of strictly smaller length. This contradicts the minimality of γ.

On the other hand, if we have chosen an arc α in γ which joins singular points, then the path may be rotated into the 1-skeleton. Use Figure 6.3 to help with visualization. If $\alpha(0)$ is a vertex of length one, then we may rotate α into the central line or the boundary arc, whichever is closer. (By general position, the lift of α begins at a vertex in the boundary.) Both the central line and the boundary arc in $\mathbb{S}^2$ correspond to an edge in L. Figure 6.4 is particularly instructive. As the cells of the gallery are developed onto the sphere the, central arc lies in the image. There are other cases besides the gallery shown there. However, by symmetry, the arc begins along the boundary in the region labelled “start”. Because the arc is great arc, it must terminate in the opposite region of the boundary labelled “end”. If $\alpha(0)$ is a vertex of length three, then we may rotate α into either of the arcs which bound the typical gallery as shown in Figure 6.4.

The only possible obstruction to continuing to rotate an arc occurs when a rotated arc meets a vertex. If the rotated arc is already an edge path, there is no need to rotate further. We never have need to rotate through a singular vertex; for we have already observed that the only paths which join singular vertices which are less than π apart are in fact edge paths. In the present case, $\alpha(0)$ is singular; if the rotation α' ever met a singular point in its interior, then all of α' would be an edge path. So, we would stop rotating.

On the other hand, it is possible that a rotated arc meet a vertex of length two. This is okay, however— the link is either discrete or a suspension. In the first case, the arc must be an edge path; in the second case, it is clear that rotation may be continued (see Figure 6.7).

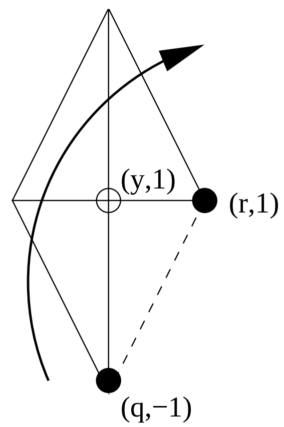

Figure 6.7: Rotating a locally geodesic arc through a vertex of length two.

Now consider the rotated loop γ'. If γ' is not geodesic at the endpoints, is may be shortened, contradicting minimality. So, instead we have obtained a geodesic loop which now contains an arc of length π which is contained in the 1-skeleton of L. If there were an arc of γ' which was not contained in the 1-skeleton, then the arc must join singular vertices. But, then γ' must have length at least 2π. On the other hand, if γ' is contained in the 1-skeleton, it must have length at least 2π. Thus, we have proved that L_Γ is CAT(1):

Theorem 6.7. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set. Then the link L_Γ is CAT(1).*

CHAPTER 7

RESULTS AND CONCLUDING REMARKS

The proof of Theorem 6.7 is the final ingredient in the outline of the proof of the Main Theorem:

Main Theorem. *Let* Γ *define a 3 dimensional FC Artin system, and fix a total ordering of* $S = vert(\Gamma)$*. Then the link* $L_\Gamma = Lk(K_\Gamma, v_0)$ *is CAT(1); and, moreover, the Artin group,* $A_\Gamma \cong \pi_1(K_\Gamma, v_0)$*, is CAT(0): it acts geometrically on the universal cover of* K_Γ *by deck transformations.*

The proof given also works for FC Artin systems of dimension less than three. Two dimensional FC Artin groups were shown to be CAT(0) by T. Brady and J. McCammond in [BM]. Brady and McCammond used the same cell complex K, but with a different metric: every edge was assigned length one, so that boundary of every 2-cell was an equilateral triangle. When considering the link L, a simplicial 1-complex, checking the link condition became equivalent to deciding if L contained any edge loops of fewer than six edges (in the link, the edges have length $\pi/3$). It would be worthwhile to investigate to what extent the three dimensional complexes K_Γ considered herein admit flexiblity (if any) in a choice of locally CAT(0) metric. However, this would require further analysis of T. Brady's original treatment of the three generator spherical Artin groups in [Br1].

The partial ordering on by reflection length on $\mathcal{W}$ left many open questions. Related to these questions is the idea of a "dual" theory of Coxeter groups. The question of whether Coxeter groups admit a classification purely in terms of reflection length was asked by D. Bessis in [Be]. For finite Coxeter groups, he defines an *abstract finite reflection group* to be a group W together a generating set R and a faithful linear representation $\rho : W \to V \cong \mathbb{R}^n$. The generating set R (reflections) is characterized as precisely those elements $w \in W$ such that $\mathrm{codim}(\ker(\rho(w) - \mathrm{Id})) = 1$. But, it is unclear what is the correct definition of an infinite abstract reflection group. The natural question to ask in the present context, is whether such a clarification is related to finding $K(\pi, 1)$ spaces for Artin groups.

In spite of these prospects for future research, the question of whether Artin groups are CAT(0) remains open.

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