Weak Diffusive Stability Induced by High-Order Spectral Degeneracies

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Abstract

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The Lyapunov stability of equilibria in dynamical systems is determined by the interplay between the linearization and the nonlinear terms. In this work, we study the case when the spectrum of the linearization is diffusively stable with high-order spectral degeneracy at the origin. In particular, spatially periodic solutions called roll solutions at the zigzag boundary of the Swift-Hohenberg equation (SHE), typically selected by patterns and defects in numerical simulations, are shown to be nonlinearly stable. This also serves as an example where linear decay weaker than classical diffusive decay, together with quadratic nonlinearity, still gives nonlinear stability of spatially periodic patterns.

The study is conducted on two physical domains: the 2D plane, $\mathbb{R}^2$, and the cylinder, $T_{2\pi} \times \mathbb{R}$. Linear analysis reveals that instead of the classical t^{-1} diffusive decay rate, small localized perturbation of roll solutions with zigzag wavenumbers decay with slower algebraic rates ($t^{-\frac{3}{4}}$ for the 2D plane; $t^{-\frac{1}{4}}$ for the cylindrical domain) due to the high order degeneracy of the translational mode at the origin of the Bloch-Fourier spaces. The nonlinear stability proofs are based on decompositions of the neutral translational mode and the faster decaying modes, and fixed-point arguments, demonstrating the irrelevancy of the nonlinear terms.

DEDICATION

I dedicate this work to my beloved and proud mom, Mary Ofori-Atta. She has given me extreme support throughout my education.

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I would like to express my heartfelt gratitude to my advisor, Professor Qiliang Wu, for his guidance throughout my academic journey. Despite facing difficult challenges, he provided me with far more than I anticipated. In addition to his excellent math guidance, he also helped me improve my writing style. This dissertation could not have taken its current form without his constant motivational talks.

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Lastly, but most importantly, I am deeply grateful to my wife, Kayla. Without her love, understanding, and encouragement, none of this would have been possible.

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List of Notations

The following notations are used in this work.

- The one-dimensional torus of length α is given by $\mathbb{T}_\alpha = \mathbb{R}/\alpha\mathbb{Z}$.
- The standard inner product on $\mathbb{R}^2$ is given by

$$\mathbf{x} \cdot \mathbf{y} := \sum_{j=1}^{2} x_j y_j, \text{ for any } \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2,$$

- The standard inner product on the Hilbert space $L^2(\mathbb{T}_{2\pi})$ is given by

$$\langle u, v \rangle := \int_{\mathbb{T}_{2\pi}} u(\xi)\bar{v}(\xi)\mathrm{d}\xi, \text{ for any } u, v \in L^2(\mathbb{T}_{2\pi})$$

  where $\bar{v}$ denotes the complex conjugate of v.

- The standard inner products on ℓ^2, or the ℓ^p–ℓ^q pairing, is given by

$$\langle\!\langle \underline{u}, \underline{v} \rangle\!\rangle := \sum_{j\in\mathbb{Z}} u_j \bar{v}_j, \text{ for any } \underline{u} = \{u_j\}_{j\in\mathbb{Z}} \in \ell^p, \underline{v} = \{v_j\}_{j\in\mathbb{Z}} \in \ell^q \text{ with } \frac{1}{p}+\frac{1}{q} = 1, 1 \leqslant p, q \leqslant \infty$$

  where $\bar{v}$ denotes the complex conjugate of v.

- For $p \in [1, \infty)$, $n \in \mathbb{N}$, we define the discrete Sobolev space $w^{n,p} := \left\{\underline{u} \,\middle|\, \|u\|_{w^{n,p}} < \infty\right\}$ where the Sobolev norm takes the form

$$\|u\|_{w^{n,p}} := \left(\sum_{i=0}^{n} \left(\sum_{j\in\mathbb{Z}} |j^i u_j|^p \right) \right)^{\frac{1}{p}};$$

  while for $p = \infty$, $n \in \mathbb{N}$, we have $w^{n,\infty} := \left\{\underline{u} \,\middle|\, \|u\|_{w^{n,\infty}} < \infty\right\}$ where

$$\|u\|_{w^{n,\infty}} := \max_{i=0,\cdots,n} \left(\sup_{j\in\mathbb{Z}} |j^i u_j| \right).$$

  We note that $w^{0,p} = \ell^p$.

- For any $u \in L^2(\mathbb{R}^2)$, we use the notations $\mathcal{F}u$ and $\widehat{u}$ interchangeably for its Fourier transform, and $\mathcal{F}^{-1}u$ and $\check{u}$ for its inverse Fourier transform; that is,

$$(\mathcal{F}u)(\boldsymbol{\nu}) = \widehat{u}(\boldsymbol{\nu}) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} u(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\boldsymbol{\nu}}\mathrm{d}\mathbf{x}; \qquad (\mathcal{F}^{-1}u)(\boldsymbol{\nu}) = \check{u}(\boldsymbol{\nu}) := \int_{\mathbb{R}^2} u(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\boldsymbol{\nu}}\mathrm{d}\mathbf{x}.$$

- For $u \in L^2(\mathbb{T}_{2\pi})$, we use the notations $\mathcal{F}_d u$ and $\widehat{\underline{u}}$ interchangeably for its Fourier series; that is,
$$(\mathcal{F}_d u)_j = \widehat{u}_j := \frac{1}{2\pi} \int_{\mathbb{T}_{2\pi}} u(\xi) \mathrm{e}^{-\mathrm{i} j \xi} \mathrm{d}\xi.$$
- The convolution of two functions $u, v : X \to \mathbb{C}$ is defined as
$$u * v(\boldsymbol{\nu}) := \int_X u(x - \widetilde{x}) v(\widetilde{x}) \mathrm{d}\widetilde{x},$$
where we use the Lebesgue measure if X is Euclidean and the counting measure if X is discrete. In addition, we denote
$$u^{*n} := \overbrace{u * \cdots * u}^{n \text{ of } u}.$$

We denote the Euclidean norm in Euclidean spaces as $|\cdot|$, the norm in a general Banach space $\mathscr{X}$ as $\|\cdot\|_{\mathscr{X}}$, and the norm of a linear operator from a Banach space $\mathscr{X}$ to $\mathscr{Y}$ as $|||\cdot|||_{\mathscr{X}\to\mathscr{Y}}$. For the case $\mathscr{Y} = \mathscr{X}$, the last norm notation simply becomes $|||\cdot|||_{\mathscr{X}}$. For $\mathscr{X} = L^p(\mathbb{R}^2), L^p(\mathbb{T}_{2\pi}), L^p(\mathbb{T}_1 \times \mathbb{R})$, or ℓ^p, the second norm notation simply becomes $\|\cdot\|_p$, if there is no ambiguity. At last, we use the universal notation C for positive constants throughout the paper.

List of Acronyms

Swift-Hohenberg Equation- SHE

1 Introduction

1.1 Background and Stability of Patterns

1.1.1 Background of Patterns

Patterns appear in many different systems in the world, including biological, chemical, and physical systems [4],[5],[6]. We can see patterns everywhere in nature, ranging from cloud structures, dessert formation, fingerprints, animal skins, biological tissues, and the counterpart patterns in the laboratory, ranging from fluid convection, chemical reactions, lasers, liquid crystals, and many others. See figure 1.1 for some examples of patterns. Our interest is understanding the rich structure of patterns, where they come from, and how they evolve with time. Understanding patterns could help predict earthquakes, forecast the weather, fight against infectious diseases that spread on human bodies [19], and other non-equilibrium systems. Researchers in many different academic disciplines have studied pattern formation systems for many decades [14],[15]. Still, it was not until Alan Turing's work was published in 1952 that rigorous mathematical analysis was introduced into the study of pattern formation [20]. Since that time, numerous well-known mathematical pattern-forming systems, including the Kuramoto-Sivashinsky equation, the Cahn-Hillard equation, the Ginzburg-Laudau equation, the Swift-Hohenberg equation, the Bousinesseq equation, and many reaction-diffusion models, to name a few, have been used to extensively study spatially periodic patterns. The fact that pattern-forming systems in distinctly different settings can be modeled by the same mathematical model, or by different mathematical models with, say, the same modulated equation near interested patterns, demonstrates the universality of patterns and their dynamics from the perspective of dynamical systems [30]. Given that the patterns we witness in nature are generally resilient and enduring, it is only natural to demand that solutions corresponding to the mathematical pattern formation system exist and exhibit similar dynamical qualities to their physical counterparts, which leads us to study the stability of pattern solutions. Pattern-forming systems giving rise to spatially periodic patterns, called roll solutions, typically accommodate a family of roll solutions parameterized by a continuum of wave numbers. While the wave numbers on the zigzag boundary have been shown to be selected by patterns and their defects in numerical simulations [43], the nonlinear stability of these roll solutions on the zigzag boundary is yet to be proved, and thus the topic of this research work.

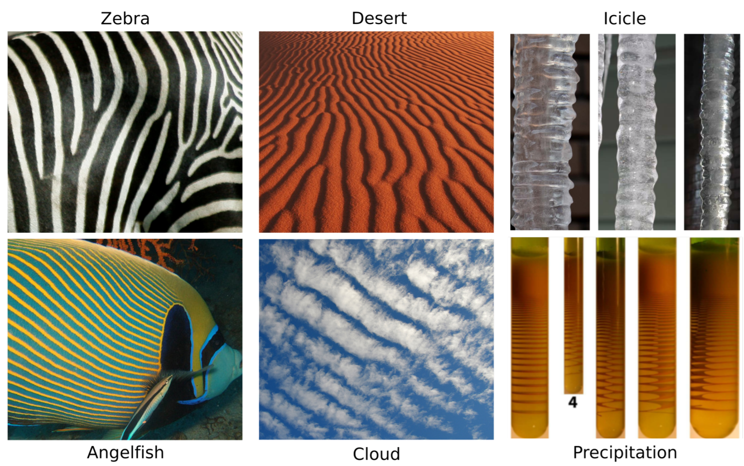

Figure 1.1: Examples of Patterns. Courtesy of Qiliang Wu

1.1.2 Stability of Patterns

In dynamical systems, structural stability and Lyapunov stability are the two traditional conceptions of stability. An equilibrium point of a dynamical system is said to be Lyapunov stable if small perturbations around the equilibrium remain small at all times. An equilibrium point is asymptotically stable if it is Lyapunov stable and small perturbations around the equilibrium go to zero as time goes to infinity. Asymptotic stability often happens in dissipative systems. On the other hand, in dispersive systems, the Lyapunov stability is generically not asymptotic. Instead, it's called neutral or orbital stability due to the existence of conserved quantities of the systems. Extensive studies on the stability of pattern-forming systems have been conducted in both directions in the past half century with the development of two drastically different sets of toolboxes; see [44] and the references therein. The other stability is structural stability; a dynamical structure, say, an invariant manifold, for example, persists under sufficiently small perturbations to the system. Much less attention has been paid to the study of structural stability in pattern-forming systems; see [45–47] for recent progress.

In the context of smooth dissipative systems, we investigate how the connection of weak linear stability and nonlinearity influences the nonlinear asymptotic stability. For further illustration, consider the dynamical system

$$u_t = F(u), \tag{1.1.1}$$

where F is smooth and u_* is an equilibrium such that $F(u_*) = 0$. We then study the Lyapunov stability of u_* via the perturbed system $w = u - u_*$,

$$\begin{cases} w_t = Lw + N(w), \\ w(0) = w_0, \end{cases} \tag{1.1.2}$$

where

$$L := \frac{\partial F}{\partial u}(u_*), \quad N(w) = F(u_* + w) - F(u_*) - Lw.$$

If the spectrum of operator L denoted $\sigma(L)$ lives in the complex plane $\mathbb{C}$ and has negative real parts, then we say u_* is spectrally stable. The equilibrium u_* is said to be linearly stable if $w = 0$ is stable in the linearized flow

$$w_t = Lw,$$

Likewise, if $w = 0$ is stable in the whole nonlinear flow, then we say that u_* is nonlinearly stable. In the case when L is hyperbolic and satisfies certain extra condition(s), linear (in)-stability leads to nonlinear (in)-stability. More specifically, if L is of finite dimensional, or, satisfies some regularity condition such as being sectorial [9], and every member of the spectrum of L, denoted as $\sigma(L)$, admits negative real part (at least one member of $\sigma(L)$ admits positive real part), then u_* is asymptotically stable (unstable). In the case when u_* is asymptotically stable, the decay rate is exponential; that is, there exists some $c > 0$ such that

$$\lim_{t\to\infty} \|\mathrm{e}^{ct} v(t)\| = 0,$$

where $\|\cdot\|$ is a proper norm varying from case to case.

Remark 1.1.1. *Even when the linearized flow is Lyapunov unstable, nonlinear stabilization can take place for C^1 (in the Frechet sense) discrete dynamical systems posed on infinite-dimensional Hilbert space; see [41, 42] for details.*

The nontrivial and most interesting case happens when the spectrum $\sigma(L)$ lies in the left half of the complex plane and touches the imaginary axis; that is,

$$\sigma(L) \subseteq \{a + b\mathrm{i} \mid a \leqslant 0, b \in \mathbb{R}\}, \quad \sigma(L) \cap \mathrm{i}\mathbb{R} \neq \emptyset.$$

The spectrum $\sigma(L)$ can land on the imaginary axis in countless ways: $\sigma(L) \cap \mathrm{i}\mathbb{R} = \{0\}$ as in the linear heat equation; $\sigma(L) \cap \mathrm{i}\mathbb{R} = \{\pm\mathrm{i}\}$ as in the Hopf bifurcation; $\sigma(L) \subseteq \mathrm{i}\mathbb{R}$ as for many dispersive systems. We focus on the simple but interesting case when the neutral-stable spectrum $\sigma(L)$ only touches the imaginary axis at the origin; that is,

$$\sigma(L) \subseteq \{a + b\mathrm{i} \mid a \leqslant 0, b \in \mathbb{R}\}, \quad \sigma(L) \cap \mathrm{i}\mathbb{R} = \{0\}.$$

While previous studies emphasized how different types of nonlinear terms affect the nonlinear dynamics of patterns, we instead focus on the effect of weakening in the linear decay on the nonlinear dynamics of patterns; see [10] for the recent work in this direction.

1.1.3 Heat Equation Stability Example: Relevancy and Irrelevancy of Nonlinear Terms

To illustrate the ideas, we look at the nonlinear heat equation,

$$u_t = \Delta_{\mathbf{x}} u + f(u), \tag{1.1.3}$$

where $u(t;\mathbf{x}) \in \mathbb{R}$ with $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$ and $\mathbf{x} = (x_1, \cdots, x_n)$, and $f(0) = f'(0) = 0$. The equilibrium $u \equiv 0$ is linearly stable. More specifically, we have that

$$\sigma(\Delta_{\mathbf{x}}) = (-\infty, 0],$$

and the linear heat equation $u_t = \Delta_{\mathbf{x}} u$ admits the Gaussian decay estimates

$$\|\partial_{\mathbf{x}}^{\alpha} u(t;\cdot)\|_{L^p(\mathbb{R}^n)} \leqslant C t^{-[\frac{n}{2}(\frac{1}{q}-\frac{1}{p})+\frac{|\alpha|}{2}]} \|u(0;\cdot)\|_{L^q(\mathbb{R}^n)},$$

where $1 \leqslant q \leqslant p \leqslant \infty$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ is the multi-index of partial derivatives with $|\alpha| = \sum_{i=1}^{n} \alpha_i$. In particular, for $p = \infty$, $q = 1$ and $\alpha = 0$, the decay estimate reduces to

$$\|u(t;\cdot)\|_{L^\infty(\mathbb{R}^n)} \leqslant C t^{-\frac{n}{2}} \|u(0;\cdot)\|_{L^1(\mathbb{R}^n)}, \tag{1.1.4}$$

which we refer to as the *diffusive decay estimate* and the algebraic decay rate $t^{-n/2}$ is called the *diffusive decay rate*.

The stability of the equilibrium $u = 0$ in the nonlinear case (1.1.3), however, depends on the type of nonlinearity, $f(u)$. Intuitively, assuming L^1 initial data, we can exploit the Gaussian estimates (1.1.4) to determine whether the diffusion term $\Delta_{\mathbf{x}} u$ or the nonlinear term $f(u)$ is dominant in terms of their temporal decay rates, leading to the classification of nonlinear terms into relevant, irrelevant and critical [33],[34],[35],[36],[37]. More explicitly, we have

$$\|\Delta_{\mathbf{x}} u\|_{L^\infty(\mathbb{R}^n)} \sim t^{-(\frac{n}{2}+1)}, \quad \|f(u)\|_{L^\infty(\mathbb{R}^n)} \sim t^{-k},$$

where the latter estimate is derived based on the Gaussian estimates (1.1.4) on u and its derivatives. The nonlinear term f is called *irrelevant* if $k > n/2 + 1$, *relevant* if

$k < n/2 + 1$, and *critical* if $k = n/2 + 1$. We expect that, given any irrelevant nonlinear term, the equilibrium $u = 0$ is nonlinearly stable with the same diffusive decay rate as the linear case; any relevant nonlinear term makes the local nonlinear dynamics near the equilibrium $u = 0$ different from its linear counterpart; the case for the critical one is undetermined and typically needs to be handled on a case-by-case basis. For example, we let $f(u) = u^m$ and the application of Gaussian estimates leads to

$$\|f(u)\|_{L^\infty(\mathbb{R}^n)} = \|u^m\|_{L^\infty(\mathbb{R}^n)} = \|u\|^m_{L^\infty(\mathbb{R}^n)} \sim t^{-\frac{mn}{2}},$$

which implies that $f(u) = u^m$ is irrelevant if $m > 1 + 2/n$, relevant if $m < 1 + 2/n$ and critical if $m = 1 + 2/n$; see Table 1.1 for more examples. Indeed, Fujita showed in 1966 [33] that, if $m < 1 + 2/n$, then the solution u blows up in finite time; if $m > 1 + 2/n$, then $u = 0$ is asymptotically stable with the decay rate $t^{-n/2}$. The critical case $m = 1 + 2/n$ also admits finite-time blow-up, according to the work by Hayakawa [34].

Classification	Example	Dynamics
Irrelevant ($k > n/2 + 1$)	$f = u\Delta u, \lvert\nabla u\rvert^2, u^m (m > 1 + 2/n)$	$\|u\|_{L^\infty} \sim t^{-n/2}$
Critical ($k = n/2 + 1$)	$f = \pm u^{1+2/n}$	Undetermined
Relevant ($k < n/2 + 1$)	$f = u^m (m < 1 + 2/n)$	Finite time blow up

Table 1.1: Classification of Nonlinear Terms : Irrelevant, Critical, and Relevant.

1.2 Spatially Periodic Patterns: Previous Results and Open Questions

Despite being non-rigorous, this intuitive classification of nonlinear terms still works in the study of spatially periodic patterns, not directly but in a more subtle way, which typically involves a proper change of coordinates based on a decomposition of the neutral modes and stable modes. To fix ideas, we study the prototypical model of spatially periodic patterns–the isotropic Swift-Hohenberg Equation (SHE)

$$u_t = -(1 + \Delta_{\mathbf{x}})^2 u + \mu u - u^3 \tag{1.2.1}$$

where $u(t, \mathbf{x})$ is a real-valued function defined on $[0, \infty) \times \mathbb{R}^n$ and $\mu \in \mathbb{R}$ is the bifurcation parameter. The homogeneous equilibrium $u \equiv 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$. For $0 < \mu \ll 1$, the instability of the homogeneous equilibrium gives rise to a family of even spatially periodic solutions, called roll solutions. More specifically, setting $\mu > 0$ for the rest of the paper and denoting $\varepsilon := \sqrt{\mu}$, we have the following lemma from [38],[39],[1].

Lemma 1.2.1 (Existence of roll solutions)**.** *There exists $0 < \varepsilon_0 \ll 1$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and the wave number $k \in (k^-, k^+)$ with $k^\pm = \sqrt{1 \pm \varepsilon}$, the stationary rescaled one-dimensional SHE,*

$$-(1 + k^2\partial_\xi^2)^2 u + \varepsilon^2 u - u^3 = 0,$$

admits a unique roll solution $u_p(\xi; k)$ which is 2π-periodic and even in ξ with $u_p(0; k) > 0$; see Figure 1.2. The roll solution has the property $u_p(\xi + \pi; k) = -u_p(\xi; k)$ and the leading order expansion

$$u_p(\xi) = a_1 \cos(\xi) + a_3 \cos(3\xi) + O(\widetilde{a}^5), \tag{1.2.2}$$

where

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} u_p(\xi)\cos(\xi)d\xi = \widetilde{a} + \widetilde{a}^3/512 + O(\widetilde{a}^4),$$

$$a_3 = \frac{1}{\pi}\int_0^{2\pi} u_p(\xi)\cos(3\xi)d\xi = -\widetilde{a}^3/256 + O(\widetilde{a}^4),$$

$$\widetilde{a} = \sqrt{\frac{4[\varepsilon^2 - (k^2 - 1)^2]}{3}}.$$

Remark 1.2.2. *We note that the symmetric property $u_p(\xi + \pi; k) = -u_p(\xi; k)$ results from the persistence of translation symmetry $u(\xi) \to u(\xi + \xi_0)$ and the reflection symmetry $u \to -u$ in the construction of roll solutions via the Lyapunov-Schmidt reduction.*

The rotational and translational symmetries of the system (1.2.1) guarantee that

$$u_p(\mathbf{k} \cdot \mathbf{x} + \phi; |\mathbf{k}|)$$

for any $\phi \in \mathbb{R}$, $\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{R}^n$ with $|\mathbf{k}| = \sqrt{\sum_{i=1}^{n} k_i^2} \in (k^-, k^+)$ is also a roll solution.

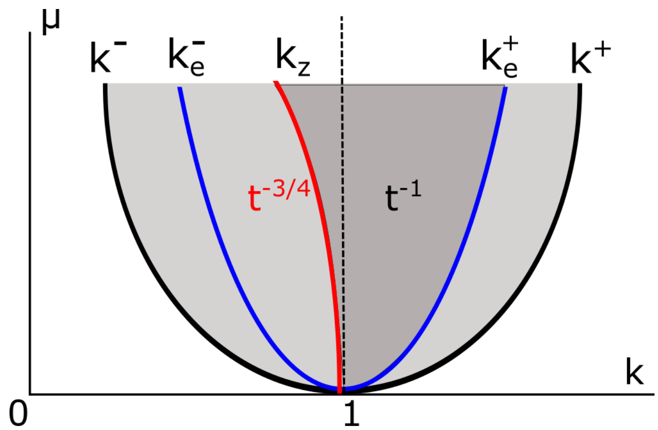

Figure 1.2: Existence and Stability Results of Roll Solutions Shown in the Busse Balloon.

Eckhaus first discovered in 1965 that not all roll solutions are *spectrally* stable, due to sideband instability induced by perturbations with a period close to, but not equal to the period of roll solutions [23].

1.2.1 Stability Results in the Swift-Hohenberg Equation

In [1], a method for studying stability analysis of bifurcating spatially periodic patterns under non-periodic perturbations was developed. In particular, they considered the stability of the roll solution in the 2-dimensional Swift-Hohenberg equation. They came up with a condition depending on the wave number and amplitude of the rolls which is necessary and sufficient to establish stability. After 30 years of Eckhaus instability results at the Eckhaus boundary in 1965 [23], the first nonlinear stability result of the roll solutions was given by Guido Schneider. In [21], the nonlinear stability of the rolls solution of the 1-dimensional Swift-Hohenberg equation at the Eckhaus boundary,

$k \in (k_e^-, k_e^+)$ where $k_e^{\pm} = 1 \pm \sqrt{\frac{\mu}{12}} + O(\mu)$ was investigated under small non-periodic perturbation. The challenge in showing stability for the roll solutions was because linearization around the rolls possesses a continuous spectrum up to zero which has the expansion

$$\lambda(\nu_1; k) = -a_{12}(k)\nu_1^2 + O(\nu_1^3),$$

where ν_1 is the Fourier wave number and $a_{12}(k) > 0$ if and only if $k \in (k_e^-, k_e^+)$. Using renormalization theory, Guido Schneider in [21] showed that the nonlinear terms are irrelevant; thereby, the nonlinear problem behaves asymptotically as the linearized one under a diffusive regime. To be more specific, the perturbed solution $w = u - u_*$ was shown to decay with a rate of one-half, that is,

$$\|w\|_{L^\infty(\mathbb{R})} \sim t^{-\frac{1}{2}}.$$

In [2], the nonlinear stability of roll solutions of the 2-dimensional Swift-Hohenberg equation was investigated for $k > k_z$ where the zigzag boundary is given by $k_z = 1 - \frac{\mu^2}{512} + O(\nu^3)$. This results from the roll solution undergoing secondary instability due to transversal perturbation of roll solutions. The linearization around the roll solution in Bloch wave representation has a continuous spectrum up to 0 with a locally parabolic shape at the critical Bloch vector 0 and has the expansion

$$\lambda(\nu; k) = -a_{12}(k)\nu_1^2 - a_{22}(k)\nu_2^2 + O(|\nu|^3),$$

where $\nu := (\nu_1, \nu_2)$ is the Fourier wave-number vector and $a_{22}(k) > 0$ if and only if $k > k_z$ with $k_z = 1 - \frac{\mu^2}{512} + O(\nu^3)$. Using renormalization theory, Hannes Uecker proved in [2] that the perturbations $w = u - u_*$ of a spectrally stable roll solution u_*, that is sufficiently small in a suitable Banach space, converge diffusively to zero in infinite time with a decay rate of one. That is

$$\|w\|_{L^\infty(\mathbb{R}^2)} \sim t^{-1}.$$

Thirty years ago, similar and more general results were obtained in the Ginzburg-Landau equation [7, 24]. Recently, we have had similar results in viscous conservation laws [26, 27] and in reaction-diffusion systems [28, 29].

In 2018, Guillod *et al.* [10] proved the nonlinear stability of spatially periodic solutions of the Ginzburg Landau equation at the Eckhaus boundary. Putting this result in the same context as the previous ones, the continuation of the zero eigenvalue in [10] takes the expansion

$$\lambda(\nu_1) = -a_{14}\nu_1^4 + O(\nu_1^5),$$

and the perturbation $w = u - u_*$ decays diffusively with decay rate $t^{-\frac{1}{4}}$ as time goes to infinity; that is,

$$\|w\|_{L^\infty(\mathbb{R})} \sim t^{-\frac{1}{4}}.$$

1.2.2 Nonlinear Stability of Periodic Solutions in Abstract System

This study by Guillod *et al.* [10] serves as an illustration of how weakening linear stability and higher order nonlinearity can nevertheless result in the nonlinear stability of spatially periodic patterns, which inevitably raises the following intriguing unanswered questions.

- Does weakening of linear stability of spatially periodic patterns always lead to nonlinear stability? If the answer is no, what is the threshold of the linear decay rate after which nonlinear stability is no longer valid? Also if such a threshold of linear decay rate exists, is it possible to explain the local dynamics of spatially periodic patterns beyond the threshold?

- Is there a general formula demonstrating the relationship between the leading order expansion of the 0 eigenvalue and the nonlinear decay rate in the context where the weakening of linear stability of spatially periodic patterns still leads to nonlinear stability? More specifically, we recall the abstract dynamical system (1.1.1),

$$u_t = F(u),$$

where $u(t, x) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a real-valued function and F is smooth. Suppose u_* is a spatially periodic solution solving $F(u_*) = 0$, and recall that the perturbation $w = u - u_*$ yields the perturbed system (1.1.2)

$$w_t = Lw + N(w),$$

where L is a linear operator and $N(w)$ represents nonlinear terms in w. If the spectrum $\sigma(L)$ lies in the left half of the complex plane and touches the imaginary axis only at the origin; that is,

$$\sigma(L) \subseteq \{a + b\mathrm{i} \mid a \leqslant 0, b \in \mathbb{R}\}, \quad \sigma(L) \cap \mathrm{i}\mathbb{R} = \{0\},$$

and the continuation of the eigenvalue 0 in the Fourier space admits the expansion

$$\lambda(\nu) = -\sum_{i=1}^{n} a_{i(2m_i)} \nu_i^{2m_i} + h.o.t,$$

where ν is the Fourier wave-number vector and $a_{i(2m_i)} > 0$ for all $i = 1, \cdots, n$. We want to investigate if the perturbation v decays algebraically with decay rate $t^{-\sum_{i=1}^{n} 1/2m_i}$ as time goes to infinity; that is,

$$\|w\|_{L^\infty(\mathbb{R}^n)} \sim t^{-\sum_{i=1}^{n} 1/2m_i}.$$

1.3 Main Results: Nonlinear Stability of Zigzag-Rolls in SHE

In this dissertation, we do not claim to provide definitive solutions to the aforementioned unanswered problems; instead, we add two examples where weaker linear stability leads to nonlinear stability. We state the following results about the nonlinear stability of zigzag-roll solutions of the Swift-Hohenberg equation defined on the plane and the cylinder. We employ methods comparable to those in [10], [22], and [11].

Theorem 1.3.1. *For any* $0 < \mu = \varepsilon^2 \ll 1$*,* $u_p(k_z x_1; k_z)$ *is nonlinearly stable in the two-dimensional SHE* (1.2.1)*. Specifically, there exists* $\delta > 0$ *such that for any initial perturbation* $v_0(\mathbf{x}) := u(0,\mathbf{x}) - u_p(k_z x_1; k_z) \in L^2(\mathbb{R}^2)$*, satisfying*

$$\left\|\widehat{v_0}\right\|_{L^1(\mathbb{R}^2)} + \left\|\widehat{v_0}\right\|_{L^\infty(\mathbb{R}^2)} \leqslant \delta,$$

where $\widehat{v_0}$ *represents the Fourier transform of* v_0*, the* L^∞*-norm of the perturbation* $v(t,\mathbf{x}) = u(t,\mathbf{x}) - u_p(k_z x_1; k_z)$ *goes to zero as time goes to infinity. In particular, there exists* $C > 0$ *such that*

$$\left\|v(t,\cdot)\right\|_{L^\infty(\mathbb{R}^2)} \leqslant C \frac{\left\|\widehat{v_0}\right\|_{L^1(\mathbb{R}^2)} + \left\|\widehat{v_0}\right\|_{L^\infty(\mathbb{R}^2)}}{(1+t)^{3/4}}, \quad \forall t > 0.$$

Conjecture 1.3.2. *For any* $0 < \mu = \varepsilon^2 \ll 1$*, the zigzag-rolls* $u_*(k_z x; k_z)$ *of the SHE* (1.2.1) *defined on the cylindrical domain,* $\mathbb{T}_{2\pi} \times \mathbb{R}$ *is nonlinearly stable. Specifically, there exists a* $\delta > 0$ *such that for any initial perturbation* $w_0(\mathbf{x}) := u(0,\mathbf{x}) - u_*(k_z x; k_z) \in L^2(\mathbb{T}_{2\pi} \times \mathbb{R})$*, satisfying*

$$||w_0||_{L^1(\mathbb{T}_{2\pi}\times\mathbb{R})} + ||w_0||_{L^\infty(\mathbb{T}_{2\pi}\times\mathbb{R})} \leqslant \delta,$$

the L^∞*-norm of the perturbation* $w(t,\mathbf{x}) = u(t,\mathbf{x}) - u_*(k_z x_1; k_z)$ *goes to zero as time goes to infinity. In particular, there exists a* $C > 0$ *such that*

$$||w||_{L^\infty(\mathbb{T}_{2\pi}\times\mathbb{R})} \leqslant C \frac{||w_0||_{L^1(\mathbb{T}_{2\pi}\times\mathbb{R})} + ||w_0||_{L^\infty(\mathbb{T}_{2\pi}\times\mathbb{R})}}{(1+t)^{\frac{1}{4}}} \quad \forall t > 0.$$

The rest of this dissertation is organized as follows. In Chapter 2, we will lay down the foundations for the proofs of Theorem 1.3.1 and Conjecture 1.3.2 by discussing spectral analysis and linear decay intuitions. The proof of Theorem 1.3.1 is done in Chapter 3. In Chapter 4, we will discuss the steps to prove Conjecture 1.3.2.

2 Spectral Analysis and Intuitive Decay Estimation

To prove Theorem 1.3.1 and Conjecture 1.3.2, we first discuss the spectral properties arising from linearization around the roll solutions in each case. We also provide intuitive decay estimations based on the spectral properties in this chapter.

2.1 Perturbed SHE on the 2D Plane

Recalling the SHE given in (1.2.1), we fix

$$n = 2, \quad 0 < \varepsilon < \varepsilon_0, \quad k \in (k^-, k^+), \quad \mathbf{x} = (x_1, x_2), \quad \boldsymbol{\nu} = (\nu_1, \nu_2),$$

and denote

$$\kappa := k^2 - 1. \tag{2.1.1}$$

We introduce the rescaling $x_1 \to kx_1$, and study the initial value problem of the rescaled SHE,

$$\begin{cases} u_t = -\left(1 + (1+\kappa)\partial_{x_1}^2 + \partial_{x_2}^2\right)^2 u + \varepsilon^2 u - u^3, \\ u(0, \mathbf{x}) = u_p + v_0, \end{cases} \tag{2.1.2}$$

or equivalently, the perturbation equation of $v := u - u_p$, where u_p is a roll solution.

$$\begin{cases} v_t = \mathcal{L}_p v + \mathcal{N}_p(v), \\ v(0, \mathbf{x}) = v_0(\mathbf{x}), \end{cases} \tag{2.1.3}$$

where

$$\mathcal{L}_p v := -\left(1 + (1+\kappa)\partial_{x_1}^2 + \partial_{x_2}^2\right)^2 v + \varepsilon^2 v - 3u_p^2 v, \quad \mathcal{N}_p(v) = -3u_p v^2 - v^3. \tag{2.1.4}$$

2.1.1 Spectral Analysis of the Linearized Operator $\mathcal{L}_p$ on the 2D Plane

The linearized operator of the stationary SHE at the roll solution u_p is given by

$$\begin{aligned} \mathcal{L}_p : \quad H^4(\mathbb{R}^2) &\longrightarrow L^2(\mathbb{R}^2) \\ v &\longmapsto -\left(1 + (1+\kappa)\partial_{x_1}^2 + \partial_{x_2}^2\right)^2 v + \varepsilon^2 v - 3u_p^2 v. \end{aligned} \tag{2.1.5}$$

As a differential operator, $\mathcal{L}_p$ has coefficients that are 2π-periodic in x_1 and constant in x_2. It is well-known that the spectral analysis of constant-coefficient differential operators can be readily done via the Fourier transform, while the spectral analysis of periodic-coefficient differential operators can be achieved via the Bloch wave decomposition [40]. To analyse the spectrum of $\mathcal{L}_p$, we recall the notation $\mathbb{T}_\alpha = \mathbb{R}/\alpha\mathbb{Z}$, and introduce the Bloch-Fourier transform

$$\begin{array}{cccc} \mathcal{B}: & L^2(\mathbb{R}^2) & \longmapsto & L^2(\mathbb{T}_1 \times \mathbb{R}, L^2(\mathbb{T}_{2\pi})) \\ & v & \longrightarrow & \mathcal{B}w(\boldsymbol{\nu}, \xi) = \sum_{k\in\mathbb{Z}} \widehat{v}(\nu_1 + k, \nu_2)\mathrm{e}^{\mathrm{i}k\xi}, \end{array} \tag{2.1.6}$$

We note that the Bloch-Fourier transform $\mathcal{B}$ is an isomorphism [40] and the linearized operator $\mathcal{L}_p$ is block-diagonalized on the Bloch-Fourier space; that is, $\widehat{L_p} := \mathcal{B} \circ \mathcal{L}_p \circ \mathcal{B}^{-1}$ admits the direct integral form

$$\widehat{\mathcal{L}_p} = \int_{\mathbb{T}_1\times\mathbb{R}} \widehat{\mathcal{L}_p}(\boldsymbol{\nu})\mathrm{d}\boldsymbol{\nu},$$

where

$$\begin{array}{cccc} \widehat{\mathcal{L}_p}(\boldsymbol{\nu}): & H^4(\mathbb{T}_{2\pi}) & \longrightarrow & L^2(\mathbb{T}_{2\pi}) \\ & v(\xi) & \longmapsto & -\left(1 + (1+\kappa)(\partial_\xi + i\nu_1)^2 - \nu_2^2\right)^2 v + \varepsilon^2 v - 3u_p^2(\xi)v. \end{array} \tag{2.1.7}$$

We now introduce a proposition about the spectrum of the linear operators $\mathcal{L}_p$, $\widehat{\mathcal{L}_p}$ and $\widehat{\mathcal{L}_p}(\boldsymbol{\nu})$.

Proposition 2.1.1 (Spectral stability)**.** *For any fixed $\varepsilon \in (0, \varepsilon_0)$, the operator $\mathcal{L}_p$ admits the following spectral properties.*

(i) $\sigma(\mathcal{L}_p) = \sigma(\widehat{\mathcal{L}_p}) = \bigcup_{\boldsymbol{\nu}\in\mathbb{T}_1\times\mathbb{R}} \sigma(\widehat{\mathcal{L}_p}(\boldsymbol{\nu})) \subseteq \mathbb{R}$.

(ii) There exist $\kappa_z(\mu), \kappa_e^+(\mu) \in (k^-, k^+)$ so that $\sigma(\mathcal{L}_p) \subseteq (-\infty, 0]$ if and only if $\kappa \in [k_z, k_e^+]$.

(iii) 0 *is a simple eigenvalue of $\widehat{\mathcal{L}_p}(0)$ with $e_0 := \dfrac{u_p'}{\|u_p'\|_2}$ as its associated normalized eigenfunction. Moreover, there exists $\nu_0 > 0$ such that the eigenpair $(0, e_0)$ admits*

the unique analytic continuation $(\lambda(\boldsymbol{\nu}), e(\boldsymbol{\nu};\xi))$ *for* $|\boldsymbol{\nu}| < \nu_0$ *and* $e(\boldsymbol{\nu};\xi)$ *normalized, satisfying* $\widehat{\mathcal{L}_p}(\boldsymbol{\nu})e(\boldsymbol{\nu}) = \lambda(\boldsymbol{\nu})e(\boldsymbol{\nu})$ *with*

$$\langle e(\nu;\cdot) - e_0, e_0\rangle = 0, \quad e(\boldsymbol{\nu}) - e_0 = \mathcal{O}(|\boldsymbol{\nu}|), \quad e(\boldsymbol{\nu}) = e_r(\boldsymbol{\nu}) + \mathrm{i}\nu_1 e_i(\boldsymbol{\nu}), \tag{2.1.8}$$

where $e_r(\boldsymbol{\nu})$ *is an odd real-valued function and* $e_i(\boldsymbol{\nu})$ *is an even real-valued function. Moreover, we have*

$$\lambda(\boldsymbol{\nu}, \varepsilon^2, \kappa) = a_{20}(\varepsilon^2,\kappa){\nu_1}^2 + a_{02}(\varepsilon^2,\kappa){\nu_2}^2 + a_{04}(\varepsilon^2,\kappa){\nu_2}^4 + \mathcal{O}(\nu_1^4 + \nu_2^6), \tag{2.1.9}$$

where

(a) $a_{20}, a_{02} < 0$ *if and only if* $\kappa \in (k_z, k_e^+)$*;*

(b) For $\kappa = \kappa_z$*,* $a_{20} = -4 + \mathcal{O}(\widetilde{a}^3) < 0$*,* $a_{02} = 0$*,* $a_{04} = -1 + \mathcal{O}(\widetilde{a}^4) < 0$*;*

(c) For $\kappa = k_e^+$*,* $a_{20} = 0$*,* $a_{02} < 0$*.*

Proof. Mielke [1] did the expansion of the eigenvalue $\lambda(\boldsymbol{\nu})$ up to the quadratic order, but we need the expansion up to the quartic order. Our proof is built upon Mielke's proof in [1] and thus delegated to Appendix A. □

2.2 Perturbed SHE on the Cylindrical Domain

Recalling the SHE given in (1.2.1), we fix

$$(x, y) \in \mathbb{T}_{2\pi} \times \mathbb{R}, \quad 0 < \varepsilon < \varepsilon_0 \quad and \quad k \in (k^-, k^+).$$

We introduce the rescaling $\xi := kx$, and study the initial value problem of the rescaled SHE,

$$\begin{cases} u_t = -\left(1 + k^2\partial_\xi^2 + \partial_y^2\right)^2 u + \varepsilon^2 u - u^3, \\ u(0;\xi, y) = u_* + w_0, \end{cases} \tag{2.2.1}$$

or equivalently, the perturbation equation of $w := u - u_*$, where u_* is a roll solution.

$$\begin{cases} w_t = \mathcal{L}_* w + \mathcal{N}(w), \\ w(0,\xi,y) = w_0(\xi,y), \end{cases} \tag{2.2.2}$$

where

$$\mathcal{L}_* w := -\left(1 + k^2\partial_\xi^2 + \partial_y^2\right)^2 w + \varepsilon^2 w - 3u_*^2 w, \quad \mathcal{N}(w) = -3u_* w^2 - w^3. \tag{2.2.3}$$

2.2.1 Spectral Analysis of the Linearized Operator $\mathcal{L}_*$ on the Cylindrical Domain

The linearized operator of the stationary SHE at the roll solution u_* is given by

$$\begin{aligned} \mathcal{L}_* : \quad H^4(\mathbb{T}_{2\pi}\times\mathbb{R}) &\longrightarrow L^2(\mathbb{T}_{2\pi}\times\mathbb{R}) \\ w &\longmapsto -\left(1 + k^2\partial_\xi^2 + \partial_y^2\right)^2 w + \varepsilon^2 w - 3u_*^2 w. \end{aligned} \tag{2.2.4}$$

As a differential operator, $\mathcal{L}_*$ has coefficients that are 2π-periodic in ξ and constant in y. The spectral analysis of $\mathcal{L}_*$ can be achieved via the Fourier transform in the y direction due to the fact that $\xi \in \mathbb{T}_{2\pi}$. We have

$$\begin{aligned} \mathcal{F} : \quad L^2(\mathbb{T}_{2\pi}\times\mathbb{R}) &\longmapsto L^2(\mathbb{R}, L^2(\mathbb{T}_{2\pi})) \\ w &\longrightarrow \mathcal{F}w(\xi,y) = \widehat{w}(\xi,\nu_2), \text{ where } \nu_2 \in \mathbb{R}. \end{aligned} \tag{2.2.5}$$

We note that the Fourier transform $\mathcal{F}$ is an isomorphism and the linearized operator $\mathcal{L}_*$ is diagonalized on the Fourier space; that is, $\widehat{\mathcal{L}_*} := \mathcal{F} \circ \mathcal{L}_* \circ \mathcal{F}^{-1}$ has the direct integral form

$$\widehat{\mathcal{L}_*} = \int_{\mathbb{R}} \widehat{\mathcal{L}_*}(\nu_2)\mathrm{d}\boldsymbol{\nu},$$

where

$$\begin{aligned} \widehat{\mathcal{L}_*}(\nu_2) : \quad H^4(\mathbb{T}_{2\pi}) &\longrightarrow L^2(\mathbb{T}_{2\pi}) \\ w(\xi) &\longmapsto -\left(1 + k^2\partial_\xi^2 - \nu_2^2\right)^2 w + \varepsilon^2 w - 3u_*^2(\xi) w. \end{aligned} \tag{2.2.6}$$

We now introduce a proposition about the spectrum of the linear operators $\mathcal{L}_*$, $\widehat{\mathcal{L}_*}$ and $\widehat{\mathcal{L}_*}(\boldsymbol{\nu})$.

Proposition 2.2.1 (Spectral stability)**.** *For any fixed* $\varepsilon \in (0, \varepsilon_0)$*, the operator* $\mathcal{L}_*$ *admits the following spectral properties.*

(i) $\sigma(\mathcal{L}_*) = \sigma(\widehat{\mathcal{L}_*}) = \bigcup_{\nu_2 \in \mathbb{R}} \sigma(\widehat{\mathcal{L}_*}(\nu_2)) \subseteq \mathbb{R}.$

(ii) There exist the zigzag wavenumber $k_*(\varepsilon^2) := k_z(\varepsilon^2)$ *and the Eckhaus wavenumber* $k_e^{\pm}(\varepsilon^2)$ *so that* $\sigma(\mathcal{L}_p) \subseteq (-\infty, 0]$ *if and only if* $k \in [k_*, k_e^+]$.

(iii) 0 *is a simple eigenvalue of* $\widehat{\mathcal{L}_*}(0)$ *with* $e_0 := u_*'$ *as its associated eigenfunction. Moreover, there exists* $\nu_0 > 0$ *such that the eigenpair* $(0, e_0)$ *admits the unique analytic continuation* $(\lambda(\nu_2), e(\nu_2; \xi))$ *for* $|\nu_2| < \nu_0$*, satisfying* $\widehat{\mathcal{L}_*}(\nu_2) e(\nu_2) = \lambda(\nu_2) e(\nu_2)$ *with*

$$\langle e(\nu_2; \cdot) - e_0, e_0 \rangle = 0. \tag{2.2.7}$$

Moreover, we have

$$\lambda(\nu_2, \varepsilon^2, k) = a_{02}(\varepsilon^2, k) {\nu_2}^2 + a_{04}(\varepsilon^2, k) {\nu_2}^4 + O(\nu_2^6), \tag{2.2.8}$$

where $a_{02}(\varepsilon^2, k_*(\varepsilon^2)) = 0$*,* $a_{04}(\varepsilon^2, k_*(\varepsilon^2) = -1 + O(\widetilde{a}^4) < 0$ *for all* $0 < \varepsilon^2 \ll 1$.

Proof. Recall that $\boldsymbol{\nu} = (\nu_1, \nu_2)$, we have from (2.1.7) that

$$\widehat{\mathcal{L}_p}(\boldsymbol{\nu}) = -\left(1 + (1+\kappa)(\partial_\xi + i\nu_1)^2 - \nu_2^2\right)^2 + \varepsilon^2 - 3u_p^2(\xi).$$

Taking $\nu_1 = 0$ in $\widehat{\mathcal{L}_p}$, we have

$$\widehat{\mathcal{L}_*}(\nu_2) = \widehat{\mathcal{L}_p}(0, \nu_2).$$

Thus, the spectral properties of $\mathcal{L}_*$ are the same as that of $\mathcal{L}_p(0, \nu_2)$. As a result, the proof of this proposition is a direct consequence of the proof of Proposition 2.1.1 by taking $\nu_1 = 0$. □

2.3 Intuitive Decay Estimation

Let's consider the perturbed SHE given in equations (2.1.3) and (2.2.2); that is

$$v_t = \mathcal{L}_p v + \mathcal{N}_p(v) \text{ and } w_t = \mathcal{L}_* w + \mathcal{N}(w).$$

Propositions 2.1.1 and 2.2.1 provide the zero eigenvalue expansion in (2.1.9) and (2.2.8), respectively. By utilizing these expansions and the fact that $\mathcal{L}_p$ or $\mathcal{L}_*$ is a generalized Laplacian, we can draw the following intuitions about the decay estimations in the table below.

Equation	Bloch-Fourier	Continuation of $\lambda = 0$	Decay
$v_t = \partial_x^2 v$	$(\mathcal{B}v)_t = \left(-\nu_2^2\right)\mathcal{B}v$	$\lambda = -\nu_1^2$	$\|u\|_{L^\infty} \sim t^{-\frac{1}{2}}$
$v_t = \mathcal{L}_p v$, $\kappa \in (\kappa_e^-, \kappa_e^+)$	$(\mathcal{B}v)_t = \widehat{\mathcal{L}_p}\mathcal{B}v$	$\lambda = -a_{20}\nu_1^2 + O(\nu_1^4)$	
$v_t = (\partial_x^2 + \partial_y^2)v$	$(\mathcal{B}v)_t = \left[(\partial_\xi + \mathrm{i}\nu_1)^2 - \nu_2^2\right]\mathcal{B}v$	$\lambda = -\nu_1^2 - \nu_2^2$	$\|u\|_{L^\infty} \sim t^{-1}$
$v_t = \mathcal{L}_p v$, $\kappa \in (\kappa_z, \kappa_e^+)$	$(\mathcal{B}v)_t = \widehat{\mathcal{L}_p}\mathcal{B}v$	$\lambda = -a_{20}\nu_1^2 - a_{02}\nu_2^2 + O(\|\nu\|^4)$	
$v_t = (\partial_x^2 + \partial_y^4)v$	$(\mathcal{B}v)_t = \left[(\partial_\xi + \mathrm{i}\nu_1)^2 - \nu_2^4\right]\mathcal{B}v$	$\lambda = -\nu_1^2 - \nu_2^4$	$\|u\|_{L^\infty} \sim t^{-3/4}$
$v_t = \mathcal{L}_p v$, $\kappa = \kappa_z$	$(\mathcal{B}v)_t = \widehat{\mathcal{L}_p}\mathcal{B}v$	$\lambda = -a_{20}\nu_1^2 - a_{04}\nu_2^4 + O(\nu_1^4 + \nu_2^6)$	
$w_t = \partial_y^4 w$	$(\mathcal{B}w)_t = (-\nu_2^4)\mathcal{B}w$	$\lambda = -\nu_2^4$	$\|u\|_{L^\infty} \sim t^{-1/4}$
$w_t = \mathcal{L}_* w$, $\kappa = \kappa_z$	$(\mathcal{B}w)_t = \widehat{\mathcal{L}_*}\mathcal{B}w$	$\lambda = -a_{04}\nu_2^4 + O(\nu_2^6)$	

Based on the table, we expect a $t^{-\frac{3}{4}}$ decay rate for the planar SHE and a $t^{-\frac{1}{4}}$ decay rate for the SHE defined on the cylindrical domain.

3 Nonlinear Stability of Zigzag-Rolls of Planar SHE

In this chapter, we shall discuss the proof of the nonlinear stability of zigzag-roll solutions u_p on the plane $\mathbb{R} \times \mathbb{R}$; that is Theorem 1.3.1. The chapter is organized as follows. Section 3.1 introduces the discrete Bloch-Fourier space and the mode filter decomposition. Here, the irrelevancy of nonlinear terms can be observed intuitively, as in the case of the nonlinear scalar heat equation. The linear semigroup estimations and the nonlinear irrelevancy are done in sections 3.1.3 and 3.1.4, respectively. The rigorous proof of Theorem 1.3.1 is given in section 3.2 via a contraction mapping argument on the variation of constants formula posed on a fine-tuned Banach space. For clarity and conciseness, we relegate to the appendix C and D, the sectorial properties of $\mathcal{L}_p$ in the Bloch-Fourier space and the estimates of various secondary nonlinear terms respectively.

3.0.1 Nonlinear Terms Seem Relevant

From the analogies obtained in Table 2.3, we exploit the intuition we derive from the heat equation to evaluate the temporal decay rates of both linear and nonlinear terms via the linearized flow; that is,

$$||L_p v||_{L^\infty} \sim t^{-7/4}, \quad ||N_p(v)||_{L^\infty} = ||-3u_p v^2 - v^3||_{L^\infty} \sim t^{-3/2},$$

which misleadingly indicates that the nonlinear terms are relevant. This false conclusion results from applying our intuitive reasoning on v, the sum of both neutral and stable modes, instead of the neutral modes. As a result, the remedy here is to study the system in a refined coordinate system where the neutral and stable modes are properly separated via the mode filter decomposition; see [10] for a similar analysis in the Ginzburg-Laudau equation.

3.1 Mode Filter Decomposition and Irrelevancy of Nonlinear Terms

In this section, we split the SHE into neutral and stable modes via a mode filter decomposition in the discrete Bloch-Fourier space. Moreover, we also give refined linear estimates and detailed expressions of nonlinear terms under such a decomposition, from which the irrelevancy of nonlinear terms is followed via the intuitive counting of decay rates analogous to the heat equation case. A rigorous proof of the nonlinear irrelevancy will be given in section 3.1.4.

3.1.1 SHE in the Discrete Bloch-Fourier Space

In the Bloch-Fourier space, the SHE (2.1.3) in terms of the perturbation v takes the form

$$V_t = \widehat{\mathcal{L}_p}(\varepsilon^2, \kappa, \boldsymbol{\nu})V - 3u_p V^{*2} - V^{*3}, \tag{3.1.1}$$

where we introduced the notation $V := \mathcal{B}v$.

Remark 3.1.1. *We exploit the fact that, for $u \in L^2(T_{2\pi}), v_1, v_2 \in L^2(\mathbb{R}^2)$,*

$$\mathcal{B}(uv_1) = u\mathcal{B}(v_1), \qquad \mathcal{B}(v_1 v_2) = \mathcal{B}v_1 * \mathcal{B}v_2, \tag{3.1.2}$$

where the function $(uv_1)(\mathbf{x}) = u(x_1)v_1(\mathbf{x})$; see Appendix B.1 for the proof.

It is typically inevitable to go beyond the L^2 space to general L^p space to perform proper analysis on nonlinear terms. Noting that

$$\|\widehat{v}\|^p_{L^p(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\widehat{v}(\boldsymbol{\nu})|^p \mathrm{d}\boldsymbol{\nu} = \int_{\mathbb{T}_1 \times \mathbb{R}} \left(\sum_{j \in \mathbb{Z}} |\widehat{v}(\nu_1 + j, \nu_2)|^p \right) \mathrm{d}\boldsymbol{\nu} = \int_{\mathbb{T}_1 \times \mathbb{R}} \|\mathcal{F}_d \mathcal{B}v\|^p_{\ell^p} \mathrm{d}\boldsymbol{\nu},$$

we readily see that it is more proper to work in the discrete Bloch-Fourier space $L^p(\mathbb{T}_1 \times \mathbb{R}, \ell^p)$ than its continuous counterpart $L^p(\mathbb{T}_1 \times \mathbb{R}, L^p(\mathbb{T}_{2\pi}))$ for $p \in [1, \infty]$. For convenience, we introduce the discrete Bloch-Fourier transform $\mathcal{B}_d := \mathcal{F}_d\mathcal{B}$; that is,

$$\begin{aligned} \mathcal{B}_d : \quad L^2(\mathbb{R}^2) \quad &\longmapsto \quad L^2(\mathbb{T}_1 \times \mathbb{R}, \ell^2) \\ v \quad &\longrightarrow \quad \mathcal{B}_d v(\boldsymbol{\nu}) = (\mathcal{F}_d \mathcal{B} v)(\boldsymbol{\nu}) = \{\widehat{v}(\nu_1 + j, \nu_2)\}_{j \in \mathbb{Z}}, \end{aligned} \tag{3.1.3}$$

and the discrete version of $\widehat{\mathcal{L}_p}$, denoted as

$$\widehat{\mathcal{L}_d} = \int_{\mathbb{T}_1\times\mathbb{R}} \widehat{\mathcal{L}_d}(\boldsymbol{\nu})\mathrm{d}\boldsymbol{\nu}, \text{ where } \widehat{\mathcal{L}_d}(\boldsymbol{\nu}) := \mathcal{F}_d \circ \widehat{\mathcal{L}_p}(\boldsymbol{\nu}) \circ \mathcal{F}_d^{-1};$$

that is,

$$\begin{aligned} \widehat{\mathcal{L}_d}(\boldsymbol{\nu}) : \qquad w^{4,p} \qquad &\longrightarrow \qquad \ell^p \\ \underline{v} = \{v_j\}_{j\in\mathbb{Z}} \quad &\longmapsto \quad \{\mu_j(\boldsymbol{\nu};\kappa)v_j\}_{j\in\mathbb{Z}} + (\varepsilon^2-\kappa^2)\underline{v} - 3\mathcal{F}_d(u_p^2) * \underline{v}, \end{aligned} \tag{3.1.4}$$

where $\mu_j(\boldsymbol{\nu};\kappa) := -(1-(1+\kappa)(j+\nu_1)^2-\nu_2^2)^2+\kappa^2$. Introducing the notation

$$\underline{V}(\boldsymbol{\nu}) := \mathcal{B}_d v = \{\widehat{v}(\nu_1 + j, \nu_2)\}_{j\in\mathbb{Z}},$$

the SHE with respect to the perturbation in the discrete Bloch-Fourier space takes the form

$$\underline{V}_t = \widehat{\mathcal{L}_d}\underline{V} - 3\underline{\widehat{u_p}} * \underline{V}^{*2} - \underline{V}^{*3}. \tag{3.1.5}$$

Before we introduce the mode filter decomposition, we first prove that the spectral properties of $\widehat{\mathcal{L}_d}(\boldsymbol{\nu})$ are independent of the choice of $p \in [1,\infty]$.

Proposition 3.1.2. *For any $\boldsymbol{\nu} \in \mathbb{T}_1\times\mathbb{R}$ and $p\in[1,\infty]$, the closed operator $\widehat{\mathcal{L}_d}(\boldsymbol{\nu}) : w^{4,p} \to \ell^p$ is sectorial with compact resolvents. More specifically, the sectoriality of $\widehat{\mathcal{L}_d}(\boldsymbol{\nu})$ is independent of the choice of $\boldsymbol{\nu}$ and p; that is, there exist $C>0$, $\omega\in(\pi/2,\pi)$ and $\lambda_0\in\mathbb{R}$, independent of $\boldsymbol{\nu}$ and p, such that*

$$|||(\widehat{\mathcal{L}_d}(\boldsymbol{\nu})-\lambda)^{-1}|||_{\ell^p} \leqslant \frac{C}{|\lambda-\lambda_0|}, \text{ for any } \lambda\in S(\lambda_0,\omega) := \left\{\lambda\in\mathbb{C} \;\middle|\; |\arg(\lambda-\lambda_0)|<\omega, \lambda\neq\lambda_0\right\}.$$

Moreover, the spectrum of $\widehat{\mathcal{L}_d}(\boldsymbol{\nu})$ is independent of the choice of the underlying space ℓ^p and thus denoted as $\sigma(\widehat{\mathcal{L}_d}(\boldsymbol{\nu}))$, consisting only of isolated eigenvalues with finite multiplicities.

Proof. See Appendix C. □

3.1.2 Mode Filter Decomposition

We introduce the notation $\mathbb{X}^p := L^p(T_1 \times \mathbb{R}, \ell^p)$ and define an even smooth cut-off function $\chi : \mathbb{T}_1 \times \mathbb{R} \to [0, \infty)$ as

$$\chi(\nu) = \begin{cases} 1, & |\nu| \leqslant 1, \\ 0, & |\nu| \geqslant 2, \end{cases} \tag{3.1.6}$$

as well as its rescaled version $\chi_\epsilon(\nu) := \chi(\frac{\nu}{\epsilon})$ for any $\epsilon > 0$. We recall from Proposition 2.1.1 that the eigenpair at $\nu = 0$ admits an analytic continuation for $|\nu| < r_0$, and introduce the pseudo-eigenfunction

$$e_c(\nu) := (1 - \chi_{r_1}(\nu))e_0 + \chi_{r_1}(\nu)e(\nu), \tag{3.1.7}$$

where $r_1 := r_0/4$, and the projection mapping

$$\begin{aligned} P: \quad \mathbb{X}^p \quad &\longrightarrow \quad \mathbb{X}^p \\ \underline{V} \quad &\longmapsto \quad \|\underline{\widehat{e_c}}\|_{\ell^2}^{-2} \langle\!\langle \underline{V}, \underline{\widehat{e_c}} \rangle\!\rangle \underline{\widehat{e_c}}. \end{aligned} \tag{3.1.8}$$

We readily see that P is a bounded projection but not commutative with $\widehat{\mathcal{L}_d}$; that is

$$P^2 = P, \qquad P\widehat{\mathcal{L}_d} \neq \widehat{\mathcal{L}_d}P.$$

Introducing $Q := \mathrm{Id} - P$, $\mathbb{X}_c^p := \mathrm{Rg}\,(P)$ and $\mathbb{X}_s^p := \mathrm{Rg}\,(\mathrm{Id} - P)$, we note the subspaces $\mathbb{X}_c^p$ and $\mathbb{X}_s^p$ are closed in $\mathbb{X}$, and

$$\mathbb{X}^p = \mathbb{X}_c^p \bigoplus \mathbb{X}_s^p,$$

in the sense that $\mathbb{X}^p = \mathbb{X}_c^p + \mathbb{X}_s^p$ and $\mathbb{X}_c^p \cap \mathbb{X}_s^p = \{0\}$. Introducing the neutral and stable modes respectively,

$$a(t, \nu) := K\langle\!\langle \underline{V}(t, \nu), \underline{\widehat{e_c}} \rangle\!\rangle, \qquad \underline{V_s}(t, \nu) := Q\underline{V}(t, \nu), \tag{3.1.9}$$

where $K(\nu) := \|\underline{\widehat{e_c}}(\nu)\|_{\ell^2}^{-2} = \|e_c(\nu)\|_{L^2(\mathbb{T}_{2\pi})}^{-2}$. We apply the mode filter decomposition

$$T_{mf}\underline{V} := \begin{pmatrix} a \\ \underline{V_s} \end{pmatrix}, \tag{3.1.10}$$

which is an isomorphism from $\mathbb{X}^p \to L^p(\mathbb{T}_1 \times \mathbb{R}^1) \times \mathbb{X}_s^p$ to the SHE and rewrite the SHE in terms of the mode-filter coordinates $W := (a, \underline{V_s})^T$; that is,

$$W_t = L_{mf} W + N_{mf}(W), \tag{3.1.11}$$

where

$$L_{mf} := \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \qquad N_{mf}(W) := \begin{pmatrix} N_c(W) \\ N_s(W) \end{pmatrix},$$

with

$$\begin{aligned}
L_{11} a &:= K\langle e_c, \mathcal{L}_p e_c \rangle a, \\
L_{12} \underline{V_s} &:= K\langle\!\langle \underline{V_s}, \widehat{\mathcal{L}_d} \underline{\widehat{e_c}} \rangle\!\rangle, \\
L_{21} a &:= \left(\widehat{\mathcal{L}_d} \underline{\widehat{e_c}} - \underline{\widehat{e_c}} L_{11} \right) a, \\
L_{22} \underline{V_s} &:= \left(\widehat{\mathcal{L}_d} - \underline{\widehat{e_c}} L_{12} \right) \underline{V_s}, \\
N_c &:= -K\langle\!\langle 3\underline{\widehat{u_p}} * (a\underline{\widehat{e_c}} + \underline{V_s})^{*2} + (a\underline{\widehat{e_c}} + \underline{V_s})^{*3}, \underline{\widehat{e_c}} \rangle\!\rangle, \\
N_s &:= -3\underline{\widehat{u_p}} * (a\underline{\widehat{e_c}} + \underline{V_s})^{*2} - (a\underline{\widehat{e_c}} + \underline{V_s})^{*3} - N_c \underline{\widehat{e_c}}.
\end{aligned}$$

3.1.3 Linear Semigroup Estimates

We, for now, restrict ourselves to the initial value problem of the linearized flow of (3.1.11); that is,

$$\begin{cases} W_t = L_{mf} W, \\ W(0) = W_0 = (a_0, \underline{V_s}_0)^T, \end{cases}$$

whose solution takes the form

$$W(t) = \mathrm{e}^{L_{mf} t} W_0. \tag{3.1.12}$$

Introducing the notation

$$M := \mathrm{e}^{L_{mf} t} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

we study the temporal decay estimates of this semigroup and its physical derivatives on general L^p spaces. Our analysis is split into two subcases: when $\boldsymbol{\nu}$ is close to zero and when $\boldsymbol{\nu}$ is away from zero. More specifically, we rewrite M as follows.

$$M = \chi_{\frac{r_1}{2}} M + (1 - \chi_{\frac{r_1}{2}})M.$$

For $|\boldsymbol{\nu}| \leqslant r_1$, we denote $L_s(\boldsymbol{\nu}) := \widehat{\mathcal{L}_d}(\boldsymbol{\nu}) \mid_{\langle e(\boldsymbol{\nu})\rangle^{\perp}}$ and have L_{mf} diagonal; that is,

$$L_{mf}(\boldsymbol{\nu}) = \begin{pmatrix} \lambda(\boldsymbol{\nu}) & 0 \\ 0 & L_s(\boldsymbol{\nu}) \end{pmatrix}$$

and thus

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{L_s t} \end{pmatrix}.$$

Moreover, we have the following estimations for $\chi_{\frac{r_1}{2}} M$.

Lemma 3.1.3. *For any $1 \leqslant p \leqslant q \leqslant \infty$, there exists a positive constant C such that the cut-off analytic semigroup $\chi_{\frac{r_1}{2}} M_{11}(t)$ admits the estimates*

$$|||\boldsymbol{\nu}^{\alpha} \chi_{\frac{r_1}{2}} M_{11}(t)|||_{L^q \to L^p} \leqslant C(1+t)^{-[\frac{\alpha_1}{2} + \frac{\alpha_2}{4} + \frac{3}{4}(\frac{1}{p} - \frac{1}{q})]}, \tag{3.1.13}$$

where $\boldsymbol{\nu}^{\alpha} = \nu_1^{\alpha_1} \nu_2^{\alpha_2}$ with $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and the space L^p stands for $L^p(\mathbb{T}_1 \times \mathbb{R})$.

Proof. We recall from (2.1.9) that, for $|\boldsymbol{\nu}| < r_0 = 4r_1$ and $\varepsilon \in (0, \varepsilon_1)$,

$$\lambda(\boldsymbol{\nu}) = -[4 + O(\widetilde{a}^3)]\nu_1^2 - [1 + O(\widetilde{a}^4)]\nu_2^4 + O(\nu_1^4 + \nu_2^6),$$

from which we readily have that for $|\boldsymbol{\nu}| \leqslant r_1$ and $\varepsilon \in (0, \varepsilon_1)$, there exist constants d_1 and d_2, independent of the choice of $\boldsymbol{\nu}$ and ε, such that

$$\lambda(\boldsymbol{\nu}) \leqslant -d_1 \nu_1^2 - d_2 \nu_2^4.$$

As a result, we conclude that, for any $1 \leqslant p \leqslant q \leqslant \infty$,

$$\left\| \boldsymbol{\nu}^{\alpha} \chi_{\frac{r_1}{2}} e^{\lambda t} a \right\|_{L^p} \leqslant \left\| \boldsymbol{\nu}^{\alpha} \chi_{\frac{r_1}{2}} e^{(-d_1 \nu_1^2 - d_2 \nu_2^4)t} \right\|_{L^r} \|a\|_{L^q} \leqslant C(1+t)^{-[\frac{\alpha_1}{2} + \frac{\alpha_2}{4} + \frac{3}{4}(\frac{1}{p} - \frac{1}{q})]} \|a\|_{L^p},$$

where $\frac{1}{r} := \frac{1}{p} - \frac{1}{q}$. □

Lemma 3.1.4. *For any given $p \in [1, +\infty]$, there exists positive constants C and λ_1, independent of the choice of p, such that the cut-off analytic semigroup $\chi_{\frac{r_1}{2}} M_{22}(t)$ admits the estimates*

$$\|\chi_{\frac{r_1}{2}} M_{22}(t)\|_{L^p(\mathbb{T}_1\times\mathbb{R},\mathbb{X}_s^p)} \leqslant Ce^{-\lambda_1 t}. \tag{3.1.14}$$

Proof. For any $|\boldsymbol{\nu}| \leqslant r_1$, we have $M_{22} = \mathrm{e}^{L_s t}$, which also takes the form

$$M_{22} = \mathrm{e}^{L_s t} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{\lambda t}(\lambda - L_s)^{-1} \mathrm{d}\lambda,$$

where Γ is a sectorial curve in the left half of the complex plane so that $\sigma(L_s)$ stays to the left of the Γ. Moreover, we choose Γ independent of $\boldsymbol{\nu}$, and there exist $\lambda_1 > 0$ so that $\sup\{\operatorname{Re}(\lambda) \mid \lambda \in \Gamma\} < -\lambda_1$. A proof similar to Proposition 3.1.2, thus omitted, shows that there exists $C > 0$, independent of $\boldsymbol{\nu}$ and p, so that, for any $\lambda \in \Gamma$,

$$\|(\lambda - L_s)^{-1}\|_{\ell^p} \leqslant \frac{C}{|\lambda - \lambda_1|},$$

which concludes the proof. □

For $\boldsymbol{\nu}$ away from zero, that is $|\boldsymbol{\nu}| > r_1/2$, we have the following estimations for $(1 - \chi_{\frac{r_1}{2}})M$.

Lemma 3.1.5. *For any given $p \in [1, +\infty]$, there exists positive constants C and λ_2, independent of the choice of p such that the analytic semigroup $(1 - \chi_{\frac{r_1}{2}})M$ admits the estimates*

$$\|\boldsymbol{\nu}^\alpha (1 - \chi_{\frac{r_1}{2}}) M_{ij}\|_{L^p\to L^p} \leqslant Ct^{-\frac{\alpha_2}{4}} \mathrm{e}^{-\lambda_2 t}, \tag{3.1.15}$$

for $i, j = 1, 2$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$.

Proof. The operator $L_{mf}(\boldsymbol{\nu})$ is conjugate with $\widehat{\mathcal{L}_d}(\boldsymbol{\nu})$; that is,

$$L_{mf}(\boldsymbol{\nu}) = T_{mf}(\boldsymbol{\nu}) \widehat{\mathcal{L}_d}(\boldsymbol{\nu}) T_{mf}^{-1}(\boldsymbol{\nu}),$$

and thus

$$M(t) = \mathrm{e}^{L_{mf}(\boldsymbol{\nu})t} = T_{mf}(\boldsymbol{\nu})\mathrm{e}^{\widehat{\mathcal{L}_d}(\boldsymbol{\nu})t}T_{mf}^{-1}(\boldsymbol{\nu}).$$

Recalling that the maps T_{mf} defined in (3.1.10) and its inverse are uniformly bounded with respect to $\boldsymbol{\nu}$ and p, we are left to prove that

$$|||\boldsymbol{\nu}^{\alpha}(1-\chi_{\frac{r_1}{2}})\mathrm{e}^{\widehat{\mathcal{L}_d}t}|||_{L^p(\mathbb{T}_1\times\mathbb{R},\ell^p)} \leqslant Ct^{-\frac{\alpha_2}{4}}\mathrm{e}^{-\lambda_2 t}. \tag{3.1.16}$$

To prove the inequality (3.1.16), we first infer from Proposition 3.1.2 that for $|\boldsymbol{\nu}| > \frac{r_1}{2}$ the operator $\widehat{\mathcal{L}_d}(\boldsymbol{\nu})$ is sectorial with

$$\sigma(\widehat{\mathcal{L}_d}(\boldsymbol{\nu})) \subset (-\infty, -\widetilde{\lambda}_2), \tag{3.1.17}$$

for some $\widetilde{\lambda}_2 > 0$, indepedent of p and $\boldsymbol{\nu}$. As a result, a proof similar to the one of Lemma 3.1.4, thus omitted, shows that, for any $p \in [1,\infty]$ and $|\boldsymbol{\nu}| > \frac{r_1}{2}$, there exists $C > 0$, independent of p and $\boldsymbol{\nu}$,

$$|||\mathrm{e}^{\widehat{\mathcal{L}_d}(\boldsymbol{\nu})t}|||_{\ell^p} \leqslant Ce^{-\widetilde{\lambda}_2 t}. \tag{3.1.18}$$

Moreover, we have improved spectral estimates of $\widehat{\mathcal{L}_d}(\boldsymbol{\nu})$ to absorb $\boldsymbol{\nu}^{\alpha}$ for $|\boldsymbol{\nu}| \gg 1$. More specifically, we recall that $\sigma(\widehat{\mathcal{L}_d}(\boldsymbol{\nu})) = \sigma(\widehat{\mathcal{L}_p}(\boldsymbol{\nu}))$ and for any $\lambda \in \sigma(\widehat{\mathcal{L}_d}(\boldsymbol{\nu}))$, there exists $k \in \mathbb{Z}$ such that

$$\lambda = -[\nu_2^2 + (k+\nu_1^2)^2 - 1]^2 + O(\varepsilon^2);$$

see (A.0.2) and its subsequent discussion for details. As a result, we readily see that, for sufficiently large $R \gg 1$ and any $|\boldsymbol{\nu}| > R$, there is a sharper spectral estimate than (3.1.17); that is,

$$\sigma(\widehat{\mathcal{L}_d}(\boldsymbol{\nu})) \subset (-\infty, -\frac{1}{2}(\nu_2^4+1)),$$

where we note that $\frac{1}{2}$ can be replaced with any number in $(0,1)$ by adjusting the size of R. Again, a proof similar to the one of Lemma 3.1.4, thus omitted, shows that, for any $p \in [1,\infty]$ and $|\boldsymbol{\nu}| > R$, there exists $C > 0$, independent of p and $\boldsymbol{\nu}$,

$$|||\boldsymbol{\nu}^{\alpha}\mathrm{e}^{\widehat{\mathcal{L}_d}(\boldsymbol{\nu})t}|||_{\ell^p} \leqslant C|\boldsymbol{\nu}|^{\alpha}\mathrm{e}^{-\frac{1}{2}(\nu_2^4+1)t}. \tag{3.1.19}$$

We now show that the estimate (3.1.16) is true by exploiting the estimates (3.1.18) and (3.1.19); that is,

$$
\begin{aligned}
\left\| \boldsymbol{\nu}^{\alpha}(1-\chi_{\frac{r_1}{2}})e^{\widehat{\mathcal{L}_d}t}\underline{V} \right\|^p_{L^p(\mathbb{T}_1\times\mathbb{R},\ell^p)} \leqslant & \int_{\frac{r_1}{2}\leqslant|\boldsymbol{\nu}|\leqslant R} \left(|\boldsymbol{\nu}|^{\alpha} \left\| e^{\widehat{\mathcal{L}_d}(\boldsymbol{\nu})t}\underline{V} \right\|_{\ell^p} \right)^p \mathrm{d}\boldsymbol{\nu} + \int_{|\boldsymbol{\nu}|>R} \left(\left\| \boldsymbol{\nu}^{\alpha} e^{\widehat{\mathcal{L}_d}(\boldsymbol{\nu})t}\underline{V} \right\|_{\ell^p} \right)^p \mathrm{d}\boldsymbol{\nu} \\
\overset{(3.1.18)}{\leqslant} & C \int_{\frac{r_1}{2}\leqslant|\boldsymbol{\nu}|\leqslant R} \left(|\boldsymbol{\nu}|^{\alpha} e^{-\widetilde{\lambda}_2 t} \left\| \underline{V} \right\|_{\ell^p} \right)^p \mathrm{d}\boldsymbol{\nu} + \int_{|\boldsymbol{\nu}|>R} \left(\left\| \boldsymbol{\nu}^{\alpha} e^{\widehat{\mathcal{L}_d}(\boldsymbol{\nu})t}\underline{V} \right\|_{\ell^p} \right)^p \mathrm{d}\boldsymbol{\nu} \\
\overset{(3.1.19)}{\leqslant} & C \left[e^{-\widetilde{\lambda}_2 pt} \|\underline{V}\|^p_{L^p(\mathbb{T}_1\times\mathbb{R},\ell^p)} + \int_{|\boldsymbol{\nu}|>R} \left(|\boldsymbol{\nu}|^{\alpha} e^{-\frac{1}{2}(\nu_2^4+1)t} \left\| \underline{V} \right\|_{\ell^p} \right)^p \mathrm{d}\boldsymbol{\nu} \right] \\
\leqslant & C \left(e^{-\widetilde{\lambda}_2 pt} + t^{-\frac{\alpha_2 p}{4}} e^{-\frac{p}{2}t} \right) \|\underline{V}\|^p_{L^p(\mathbb{T}_1\times\mathbb{R},\ell^p)} \\
\leqslant & C t^{-\frac{\alpha_2 p}{4}} e^{-\lambda_2 pt} \|\underline{V}\|^p_{L^p(\mathbb{T}_1\times\mathbb{R},\ell^p)},
\end{aligned}
\tag{3.1.20}
$$

where we take $\lambda_2 := \min\{\widetilde{\lambda}_2, \frac{1}{2}\}$ and conclude the proof. □

Taking advantage of Lemmas 3.1.3, 3.1.4 and 3.1.5, we summarize linear estimates results of the linear flow (3.1.12) in the proposition below.

Proposition 3.1.6. *For any $1 \leqslant p \leqslant q \leqslant \infty$, there exists positive constants C,λ_2 and λ_3 independent of the choice of p, such that the linear solution W given in* (3.1.12) *admits the estimates*

$$
\left\| W(t) \right\|_{L^p} = \left\| M(t) \begin{pmatrix} a_0 \\ \underline{V_s}_0 \end{pmatrix} \right\|_{L^p} \leqslant C \begin{pmatrix} (1+t)^{-\frac{3}{4}(\frac{1}{p}-\frac{1}{q})} & e^{-\lambda_2 t} \\ e^{-\lambda_2 t} & e^{-\lambda_3 t} \end{pmatrix} \begin{pmatrix} \|a_0\|_{L^p} + \|a_0\|_{L^q} \\ \left\| \underline{V_s}_0 \right\|_{L^p} \end{pmatrix}, \tag{3.1.21a}
$$

$$
\left\| \boldsymbol{\nu}^{\alpha} a(t) \right\|_{L^p} \leqslant C \left[(1+t)^{-[\frac{\alpha_1}{2}+\frac{\alpha_2}{4}+\frac{3}{4}(\frac{1}{p}-\frac{1}{q})]} \|a_0\|_{L^q} + t^{-\frac{\alpha_2}{4}} e^{-\lambda_2 t} \left(\|a_0\|_{L^p} + \left\| \underline{V_s}_0 \right\|_{L^p} \right) \right] \tag{3.1.21b}
$$

Proof. First, we estimate

$$
\begin{aligned}
\left\| M(t)_{11} a_0 + M(t)_{12} \underline{V_s}_0 \right\|_{L^p} = & \left\| (\chi_{\frac{r_1}{2}} M_{11} + (1-\chi_{\frac{r_1}{2}}) M_{11}) a_0 + (\chi_{\frac{r_1}{2}} M_{12} + (1-\chi_{\frac{r_1}{2}}) M_{12}) \underline{V_s}_0 \right\|_{L^p}, \\
\leqslant & \left\| \chi_{\frac{r_1}{2}} M_{11} a_0 \right\|_{L^p} + \left\| (1-\chi_{\frac{r_1}{2}}) M_{11} a_0 \right\|_{L^p} + \left\| (1-\chi_{\frac{r_1}{2}}) M_{12} \underline{V_s}_0 \right\|_{L^p}, \\
\leqslant & C \left((1+t)^{-\frac{3}{4}(\frac{1}{p}-\frac{1}{q})} \|a_0\|_{L^q} + e^{-\lambda_2 t} \|a_0\|_{L^p} \right) + C e^{-\lambda_2 t} \left\| \underline{V_s}_0 \right\|_{L^p}, \\
\leqslant & C (1+t)^{-\frac{3}{4}(\frac{1}{p}-\frac{1}{q})} \left(\|a_0\|_{L^p} + \|a_0\|_{L^q} \right) + C e^{-\lambda_2 t} \left\| \underline{V_s}_0 \right\|_{L^p}.
\end{aligned}
$$

Next, we estimate

$$\begin{aligned}\left\|M(t)_{21}a_0 + M(t)_{22}\underline{V_s}_0\right\|_{L^p} =&\left\|(\chi_{\frac{r_1}{2}}M_{21} + (1-\chi_{\frac{r_1}{2}})M_{21})a_0 + (\chi_{\frac{r_1}{2}}M_{22} + (1-\chi_{\frac{r_1}{2}})M_{22})\underline{V_s}_0\right\|_{L^p}\\ \leqslant&\left\|(1-\chi_{\frac{r_1}{2}})M_{21}a_0\right\|_{L^p} + \left\|(1-\chi_{\frac{r_1}{2}})M_{22}\underline{V_s}_0\right\|_{L^p} + \left\|(1-\chi_{\frac{r_1}{2}})M_{22}\underline{V_s}_0\right\|_{L^p}\\ \leqslant&C\mathrm{e}^{-\lambda_2 t}\|a_0\|_{L^q} + C\mathrm{e}^{-\lambda_1 t}\left\|\underline{V_s}_0\right\|_{L^p} + C\mathrm{e}^{-\lambda_2 t}\left\|\underline{V_s}_0\right\|_{L^p}\\ \leqslant&C\mathrm{e}^{-\lambda_2 t}\|a_0\|_{L^p} + C\mathrm{e}^{-\lambda_3 t}\left\|\underline{V_s}_0\right\|_{L^p}\end{aligned}$$

where $\lambda_3 = \min\{\lambda_1, \lambda_2\}$. The conclusion of the proposition follows from the above estimations. □

3.1.4 Irrelevancy of Nonlinear Terms

We produce a bound on the slowest dangerous term which allows us to show irrelevance of the nonlinearity with respect to the linear dynamics. Recalling the fact from (3.1.9) that

$$V_s(t,\boldsymbol{\nu};\xi) = V(t,\boldsymbol{\nu};\xi) - a(t,\boldsymbol{\nu})e_c(\boldsymbol{\nu};\xi),$$

we expand the nonlinear terms N_c and N_s from their definitions given in Equation (3.1.11); that is,

$$\begin{aligned}N_c =& -3K\langle u_p(ae_c+V_s)^{*2}, e_c\rangle - K\langle(ae_c+V_s)^{*3}, e_c\rangle,\\ =&\overbrace{-3K\langle u_p(ae_c)^{*2}, e_c\rangle}^{:=N_{c,1}(a,a)}\overbrace{-6K\langle u_p((ae_c)*V_s), e_c\rangle}^{:=N_{c,2}(a,\underline{V_s})}\overbrace{-3K\langle u_p(V_s)^{*2}, e_c\rangle}^{:=N_{c,3}(\underline{V_s},\underline{V_s})}\\ &\overbrace{-K\langle(ae_c)^{*3}, e_c\rangle}^{:=N_{c,4}(a,a,a)}\overbrace{-3K\langle(ae_c)^{*2}*V_s, e_c\rangle}^{:=N_{c,5}(a,a,\underline{V_s})}\overbrace{-3K\langle(ae_c)*(V_s)^{*2}, e_c\rangle}^{:=N_{c,6}(a,\underline{V_s},\underline{V_s})}\overbrace{-K\langle(V_s)^{*3}, e_c\rangle}^{:=N_{c,7}(\underline{V_s},\underline{V_s},\underline{V_s})},\end{aligned}\tag{3.1.22}$$

$$\begin{aligned}N_s =& -3\underline{\widehat{u_p}}*(a\underline{\widehat{e_c}}+\underline{V_s})^{*2} - (a\underline{\widehat{e_c}}+\underline{V_s})^{*3} - N_c\,\underline{\widehat{e_c}},\\ :=&N_{s,1}(a,a) + N_{s,2}(a,V_s) + N_{s,3}(V_s,V_s) +\\ &N_{s,4}(a,a,a) + N_{s,5}(a,a,V_s) + N_{s,6}(a,V_s,V_s) + N_{s,7}(V_s,V_s,V_s),\end{aligned}\tag{3.1.23}$$

where each $N_{c,j}$ is a multilinear operator, and we define $N_{s,j}$ in a similar fashion and omit the details.

We now give intuitions on why the nonlinear terms are all irrelevant. We first note that the linear flow of the $\underline{V_s}$ component decays exponentially and the leading order nonlinear term is $N_{s,1}(a,a)$. As a result, we have that in the nonlinear flow (3.1.11), the $L^1(\mathbb{T}_1\times\mathbb{R},\ell^1)$ norm of $\underline{V_s}(t)$ has the same temporal decay as the $L^1(\mathbb{T}_1\times\mathbb{R})$ norm of a^{*2} as t goes to $+\infty$; that is,

$$\left\|\underline{V_s}(t,\cdot)\right\|_{L^1(\mathbb{T}_1\times\mathbb{R},\ell^1)}\sim\left\|a^{*2}\right\|_{L^1(\mathbb{T}_1\times\mathbb{R})}\sim t^{-\frac{3}{2}}.$$

On the other hand, the temporal decay rate of the $L^1(\mathbb{T}_1\times\mathbb{R})$ norm of the linear terms in the neutral mode equation in (3.1.11) is $t^{-\frac{7}{4}}$ and thus any irrelevant nonlinear terms should have a better temporal decay rate than that. Noting that

$$\|a(t,\cdot)\|_{L^1(\mathbb{T}_1\times\mathbb{R})}\sim t^{-\frac{3}{4}},$$

we readily derive that all nonlinear terms in N_c are irrelevant except for the first term $N_{c,1}(a,a)$. The term $N_{c,1}(a,a)$ is potentially relevant since based on $\left\|N_{c,1}\right\|_{L^1(\mathbb{T}_1\times\mathbb{R})}\sim\left\|a^{*2}\right\|_{L^1(\mathbb{T}_1\times\mathbb{R})}\sim t^{-\frac{3}{2}}$, it seems that $N_{c,1}$ admits a weaker decay than the linear terms. But a careful analysis below reveals a refined structure of $N_{c,1}$, providing extra spatial derivative in the x_1 direction and thus rendering extra $t^{-\frac{1}{2}}$ decay, which in turn shows that $N_{c,1}$ is also irrelevant. More specifically, we have

$$\begin{aligned}
N_{c,1} &= -3K\langle u_p(ae_c)^{*2},e_c\rangle\\
&= -3K\int_{T_{2\pi}} u_p(ae_c)^{*2}\overline{e_c}\mathrm{d}\xi\\
&= -3K\int_{T_{2\pi}} u_p(\xi)\overline{e_c}(\nu;\xi)\left(\int_{\widetilde{\nu}\in T_1\times\mathbb{R}} a(\nu-\widetilde{\nu})e_c(\nu-\widetilde{\nu};\xi)a(\widetilde{\nu})e_c(\widetilde{\nu};\xi)d\widetilde{\nu}\right)\mathrm{d}\xi \qquad (3.1.24)\\
&= \int_{\widetilde{\nu}\in T_1\times\mathbb{R}} \overbrace{-3K\left(\int_{T_{2\pi}} u_p(\xi)\overline{e_c}(\nu;\xi)e_c(\widetilde{\nu};\xi)e_c(\nu-\widetilde{\nu};\xi)\mathrm{d}\xi\right)}^{:=k_1(\nu,\widetilde{\nu},\nu-\widetilde{\nu})} a(\nu-\widetilde{\nu})a(\widetilde{\nu})\mathrm{d}\widetilde{\nu},
\end{aligned}$$

where we note $N_{c,1}$ is a weighted convolution with kernel k_1 and we suppressed t-dependence of a for conveniences. Leveraging the parities in the expansion (2.1.8) of the eigenfunction e, we have the following refined estimate of the kernel k_1.

Lemma 3.1.7. *There exist positive constants $M, C > 0$, independent of the choice of $\boldsymbol{\nu}, \widetilde{\boldsymbol{\nu}} \in \mathbb{T}_1 \times \mathbb{R}$, such that*

$$\left|k_1(\boldsymbol{\nu}, \widetilde{\boldsymbol{\nu}}, \widetilde{\boldsymbol{\nu}} - \boldsymbol{\nu})\right| \leqslant M, \tag{3.1.25a}$$

$$\left|k_1(\boldsymbol{\nu}, \widetilde{\boldsymbol{\nu}}, \widetilde{\boldsymbol{\nu}} - \boldsymbol{\nu})\right| \leqslant C\left(\left|\nu_1 - \widetilde{\nu}_1\right| + \left|\widetilde{\nu}_1\right|.\right). \tag{3.1.25b}$$

Proof. Recalling from (3.1.7) that

$$e_c(\boldsymbol{\nu}; \xi) = (1 - \chi_{r_1}(\boldsymbol{\nu}))e_0(\xi) + \chi_{r_1}(\boldsymbol{\nu})e(\boldsymbol{\nu}; \xi),$$

and the fact that $k_1 = \int_{T_{2\pi}} u_p(\xi)\overline{e_c}(\boldsymbol{\nu}; \xi)e_c(\widetilde{\boldsymbol{\nu}}; \xi)e_c(\boldsymbol{\nu} - \widetilde{\boldsymbol{\nu}}; \xi)\mathrm{d}\xi$, we readily conclude that the integrand of k_1 is uniformly bounded for $\xi \in \mathbb{T}_{2\pi}$, $\boldsymbol{\nu}, \widetilde{\boldsymbol{\nu}} \in \mathbb{T}_1 \times \mathbb{R}$ and thus there exists $M > 0$ such that $|k_1(\boldsymbol{\nu}, \widetilde{\boldsymbol{\nu}}, \widetilde{\boldsymbol{\nu}} - \boldsymbol{\nu})| \leqslant M$. To show the second inequality, we exploit the expansion of $e(\boldsymbol{\nu}) = e_r(\boldsymbol{\nu}) + \mathrm{i}\nu_1 e_i(\boldsymbol{\nu})$ in (2.1.8) to rewrite e_c in (3.1.7); that is,

$$e_c(\boldsymbol{\nu}) = \overbrace{(1 - \chi_{r_1}(\boldsymbol{\nu}))e_0 + \chi_{r_1}(\boldsymbol{\nu})e_r(\boldsymbol{\nu})}^{:=\widetilde{e_r}} + \mathrm{i}\nu_1 \overbrace{\chi_{r_1} e_i}^{:=\widetilde{e_i}}. \tag{3.1.26}$$

where the real-valued functions e_r and e_i are respectively odd and even in ξ for $|\boldsymbol{\nu}| < r_1$. Since e_0 and e_r are odd, it follows that $\widetilde{e_r}$ is odd. Also since e_i is even, so is $\widetilde{e_i}$. Noting that u_p is even in ξ and the integrand $u_p(\xi)\overline{e_c}(\boldsymbol{\nu}; \xi)e_c(\widetilde{\boldsymbol{\nu}}; \xi)e_c(\widetilde{\boldsymbol{\nu}} - \boldsymbol{\nu}; \xi)$ is 2π-periodic in ξ, we readily see that the odd part vanishes under the integration on $T_{2\pi}$, yielding

$$\begin{aligned}
k_1(\boldsymbol{\nu}, \widetilde{\boldsymbol{\nu}}, \widetilde{\boldsymbol{\nu}} - \boldsymbol{\nu}) = &\int_{T_{2\pi}} \mathrm{i}\nu_1\left[3Ku_p\widetilde{e_i}(\boldsymbol{\nu}; \xi)\widetilde{e_r}(\widetilde{\boldsymbol{\nu}}; \xi)\widetilde{e_r}(\boldsymbol{\nu} - \widetilde{\boldsymbol{\nu}}; \xi)\right]\mathrm{d}\xi + \\
&\int_{T_{2\pi}} \mathrm{i}\widetilde{\nu}_1\left[-3Ku_p\widetilde{e_r}(\boldsymbol{\nu}; \xi)\widetilde{e_i}(\widetilde{\boldsymbol{\nu}}; \xi)\widetilde{e_r}(\boldsymbol{\nu} - \widetilde{\boldsymbol{\nu}}; \xi)\right]\mathrm{d}\xi + \\
&\int_{T_{2\pi}} \mathrm{i}(\nu_1 - \widetilde{\nu}_1)\left[-3Ku_p\widetilde{e_r}(\boldsymbol{\nu}; \xi)\widetilde{e_r}(\widetilde{\boldsymbol{\nu}}; \xi)\widetilde{e_i}(\boldsymbol{\nu} - \widetilde{\boldsymbol{\nu}}; \xi)\right]\mathrm{d}\xi + \\
&\int_{T_{2\pi}} \mathrm{i}\nu_1\widetilde{\nu}_1(\nu_1 - \widetilde{\nu}_1)\left[-3Ku_p\widetilde{e_i}(\boldsymbol{\nu}; \xi)\widetilde{e_i}(\widetilde{\boldsymbol{\nu}}; \xi)\widetilde{e_i}(\boldsymbol{\nu} - \widetilde{\boldsymbol{\nu}}; \xi)\right]\mathrm{d}\xi,
\end{aligned}$$

where all the four terms in the brackets are uniformly bounded in $\xi \in \mathbb{T}_{2\pi}$ and $|\boldsymbol{\nu}|, |\widetilde{\boldsymbol{\nu}}| \in \mathbb{T}_1 \times \mathbb{R}$. As a result, there exists a positive constant C independent of $\boldsymbol{\nu}, \widetilde{\boldsymbol{\nu}}$ such that

$$\left|k_1(\boldsymbol{\nu}, \widetilde{\boldsymbol{\nu}}, \widetilde{\boldsymbol{\nu}} - \boldsymbol{\nu})\right| \leqslant C\left(\left|\nu_1 - \widetilde{\nu}_1\right| + \left|\widetilde{\nu}_1\right|\right).$$

□

As a result, we have

$$\left\|N_{c,1}\right\|_{L^1(\mathbb{T}_1\times\mathbb{R})} \sim \left\|a * (\nu_1 a)\right\|_{L^1(\mathbb{T}_1\times\mathbb{R})} \sim t^{-2},$$

which decays faster than the linear terms and is thus irrelevant.

3.2 Proof of Theorem 1.3.1: Nonlinear Stability of Zigzag-Rolls of the Planar SHE

We now give a proof of Theorem 1.3.1 via a fixed point argument on the variation of constants formula. More specifically, the solution to (3.1.11) with the initial condition $W_0 := (a_0, \underline{V_s}_0)^T$ satisfies the variation of constants formula,

$$W(t) = \overbrace{\overbrace{e^{L_{mf}t}W_0}^{:=\mathcal{T}_1(W_0)} + \overbrace{\int_0^t e^{L_{mf}(t-s)}N_{mf}(W(s))\mathrm{d}s}^{:=\mathcal{T}_2(W)}}^{:=\mathcal{T}(W)}. \tag{3.2.1}$$

Our task is to show that $\mathcal{T}$ is a well-defined contraction mapping on a bounded closed set in a proper Banach space, whose norm directly gives rise to the nonlinear weak diffusive decay in Theorem 1.3.1.

3.2.1 The Space $\mathcal{H}$

Based on our intuitive analysis of the irrelevancy of nonlinear terms in Section 2.3, we introduce the Banach space

$$\mathbf{H} := \left\{ W(t, \boldsymbol{\nu}) = \begin{pmatrix} a(t, \boldsymbol{\nu}) \\ \underline{V_s}(t, \boldsymbol{\nu}) \end{pmatrix} \middle| \|W\|_{\mathbf{H}} < +\infty \right\}, \tag{3.2.2}$$

in which we introduce the following norms

$$\|W\|_{\mathbf{H}} := \|a\|_{\mathbf{H}_c} + \left\|\underline{V_s}\right\|_{\mathbf{H}_s}, \tag{3.2.3}$$

$$\|a\|_{\mathbf{H}_c} := \sup_{t\geqslant 0}(1+t)^{3/4}\left\|a(t,\cdot)\right\|_1 + \sup_{t\geqslant 0}\left\|a(t,\cdot)\right\|_\infty + \sup_{t\geqslant 0}(1+t)^{5/4}\left\|\nu_1 a(t,\cdot)\right\|_1, \tag{3.2.4}$$

$$\left\|\underline{V_s}\right\|_{\mathbf{H}_s} := \sup_{t\geqslant 0}(1+t)^{3/2}\left\|\underline{V_s}(t,\cdot)\right\|_1 + \sup_{t\geqslant 0}\left\|\underline{V_s}(t,\cdot)\right\|_\infty, \tag{3.2.5}$$

where $\left\|a(t,\cdot)\right\|_p := \left\|a(t,\cdot)\right\|_{L^p(\mathbb{T}_1\times\mathbb{R})}$ and $\left\|\underline{V_s}(t,\cdot)\right\|_p := \left\|\underline{V_s}(t,\cdot)\right\|_{L^p(\mathbb{T}_1\times\mathbb{R},\ell^p)}$.

3.2.2 Linear Estimates of $\mathcal{T}_1$ in H

We first derive an upper bound of $\mathcal{T}_1(W_0)$ in $\mathbf{H}$.

Proposition 3.2.1. *There exists a positive constant C such that*

$$\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}} = \left\|M(t)W_0\right\|_{\mathbf{H}} \leqslant C\left(\|W_0\|_1 + \|W_0\|_\infty\right). \tag{3.2.6}$$

where $\|W_0\|_1 := \|a_0\|_1 + \left\|\underline{V_s}_0\right\|_1$ *and* $\|W_0\|_\infty := \|a_0\|_\infty + \left\|\underline{V_s}_0\right\|_\infty$.

Proof. By the definition of the $\mathbf{H}$-norm and the notation of the semigroup $M(t)$, we have

$$\begin{aligned}
\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}} &= \left\|M(t)\begin{pmatrix} a_0 \\ \underline{V_s}_0 \end{pmatrix}\right\|_{\mathbf{H}}, \\
&= \left\|M(t)_{11}a_0 + M(t)_{12}\underline{V_s}_0\right\|_{\mathbf{H}_c} + \left\|M(t)_{21}a_0 + M(t)_{22}\underline{V_s}_0\right\|_{\mathbf{H}_s}, \\
&= \overbrace{\sup_{t\geqslant 0}(1+t)^{3/4}\left\|M(t)_{11}a_0 + M(t)_{12}\underline{V_s}_0\right\|_1}^{:=I} + \overbrace{\sup_{t\geqslant 0}\left\|M(t)_{11}a_0 + M(t)_{12}\underline{V_s}_0\right\|_\infty}^{:=II} + \\
&\quad \overbrace{\sup_{t\geqslant 0}(1+t)^{5/4}\left\|\nu_1\left(M(t)_{11}a_0 + M(t)_{12}\underline{V_s}_0\right)\right\|_1}^{:=III} + \overbrace{\sup_{t\geqslant 0}(1+t)^{3/2}\left\|M(t)_{21}a_0 + M(t)_{22}\underline{V_s}_0\right\|_1}^{:=IV} + \\
&\quad \overbrace{\sup_{t\geqslant 0}\left\|M(t)_{21}a_0 + M(t)_{22}\underline{V_s}_0\right\|_\infty}^{:=V}.
\end{aligned}$$

Taking advantage of Proposition 3.1.6, we derive upper bounds of the terms $I - V$ respectively.

(i) We exploit the estimates of M_{11} and M_{12} in (3.1.21a) with $p = 1$ and $q = \infty$, yielding

$$\begin{aligned} I &\leqslant \sup_{t\geqslant 0}(1+t)^{-\frac{3}{4}}\left((1+t)^{\frac{3}{4}}\left(\|a_0\|_1+\|a_0\|_\infty\right)+\mathrm{e}^{-\lambda_2 t}\left\|\underline{V_s}_0\right\|_1\right) \\ &\leqslant C\left(\|a_0\|_1+\|a_0\|_\infty+\left\|\underline{V_s}_0\right\|_1\right). \end{aligned}$$

(ii) We exploit the estimates of M_{11} and M_{12} in (3.1.21a) with $p = q = \infty$, yielding

$$II \leqslant \sup_{t\geqslant 0}\left(\|a_0\|_\infty+\mathrm{e}^{-\lambda_2 t}\left\|\underline{V_s}_0\right\|_\infty\right) \leqslant C\left(\|a_0\|_\infty+\left\|\underline{V_s}_0\right\|_\infty\right).$$

(iii) We exploit the estimate (3.1.21b) with $p = 1$, $q = \infty$ and $\alpha = (1, 0)$, yielding

$$\begin{aligned} III &= \sup_{t\geqslant 0}(1+t)^{\frac{5}{4}}\left\|\nu_1\left(M(t)_{11}a_0+M(t)_{12}\underline{V_s}_0\right)\right\|_1 \\ &\leqslant C\sup_{t\geqslant 0}(1+t)^{-\frac{5}{4}}\left((1+t)^{\frac{5}{4}}\|a_0\|_\infty+\mathrm{e}^{-\lambda_2 t}\|a_0\|_1+\mathrm{e}^{-\lambda_2 t}\left\|\underline{V_s}_0\right\|_1\right) \\ &\leqslant C\left(\|a_0\|_\infty+\|a_0\|_1+\left\|\underline{V_s}_0\right\|_1\right). \end{aligned}$$

(iv) We exploit the estimates of M_{21} and M_{22} in (3.1.21a) with $p = 1$ and $p = \infty$ respectively, yielding

$$\begin{aligned} IV &\leqslant \sup_{t\geqslant 0}(1+t)^{\frac{3}{2}}\left(\mathrm{e}^{-\lambda_2 t}\|a_0\|_1+\mathrm{e}^{-\lambda_3 t}\left\|\underline{V_s}_0\right\|_1\right) \leqslant C\left(\|a_0\|_1+\left\|\underline{V_s}_0\right\|_1\right), \\ V &\leqslant \sup_{t\geqslant 0}\left(\mathrm{e}^{-\lambda_2 t}\|a_0\|_\infty+\mathrm{e}^{-\lambda_3 t}\left\|\underline{V_s}_0\right\|_\infty\right) \leqslant C\left(\|a_0\|_\infty+\left\|\underline{V_s}_0\right\|_\infty\right). \end{aligned}$$

Combining the above estimates concludes the proof. □

3.2.3 Nonlinear Estimates of $\mathcal{T}_2$ in H

We now show that $\mathcal{T}_2(W)$ is bounded in $\mathbf{H}$. More specifically, we have the following proposition.

Proposition 3.2.2. *There exists $C > 0$ such that, for any $W \in \mathbf{H}$,*

$$\left\|\mathcal{T}_2(W)\right\|_{\mathbf{H}} \leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right). \tag{3.2.7}$$

Moreover, $\mathcal{T}_2$ is locally Lipschitz continuous in the sense that there exists $C > 0$ such that, for any $W_1, W_2 \in \mathbf{H}$,

$$\left\|\mathcal{T}_2(W_1) - \mathcal{T}_2(W_2)\right\|_{\mathbf{H}} \leqslant C\|W_1 - W_2\|_{\mathbf{H}} \left(\|W_1\|_{\mathbf{H}} + \|W_2\|_{\mathbf{H}} + \|W_1\|_{\mathbf{H}}^2 + \|W_2\|_{\mathbf{H}}^2\right). \qquad (3.2.8)$$

Remark 3.2.3. *We note that the estimate* (3.2.7) *is only a special case of the Lipschitz continuity estimate* (3.2.8) *when $W_1 = W$ and $W_2 = 0$. The reason why we keep the estimate* (3.2.7) *is two-fold. Firstly, we need to use* (3.2.7) *in our fixed point argument. Secondly, it is more natural to prove* (3.2.7) *first and observe that the more general estimate* (3.2.8) *is a natural consequence of the special estimate* (3.2.7) *by exploiting the fundamental theorem of calculus.*

Proof. To prove the estimate (3.2.7), we first recall that

$$\mathcal{T}_2(W) = \int_0^t e^{L_{mf}(t-s)} N_{mf}(W(s))\mathrm{d}s = \int_0^t \begin{pmatrix} M_{11}(t-s) & M_{12}(t-s) \\ M_{21}(t-s) & M_{22}(t-s) \end{pmatrix} \begin{pmatrix} N_c(W(s)) \\ N_s(W(s)) \end{pmatrix} \mathrm{d}s,$$

where we already have the estimates on the semigroup $M(t)$ in Lemma 3.1.3-3.1.5 and Proposition 3.1.6 and estimates on the nonlinear terms are yet to be given. We claim that the $L^1(\mathbb{T}_1 \times \mathbb{R})$-norm and $L^\infty(\mathbb{T}_1 \times \mathbb{R})$-norm of nonlinear terms N_c and N_s admits the following estimates

$$\left\|N_c(t,\cdot)\right\|_1 \leqslant C(1+t)^{-2}\left(\|W\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^3\right), \qquad \left\|N_c(t,\cdot)\right\|_\infty \leqslant C(1+t)^{-\frac{5}{4}}\left(\|W\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^3\right); \qquad (3.2.9a)$$

$$\left\|N_s(t,\cdot)\right\|_1 \leqslant C(1+t)^{-\frac{3}{2}}\left(\|W\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^3\right), \qquad \left\|N_s(t,\cdot)\right\|_\infty \leqslant C(1+t)^{-\frac{3}{4}}\left(\|W\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^3\right). \qquad (3.2.9b)$$

To prove (3.2.9a), we recall the expansion $N_c = \sum_{j=1}^7 N_{c,j}$ in (3.1.22) and derive L^1 and L^∞ estimates of $N_{c,j}$ respectively. Due to the hidden refined structure of $N_{c,1}$ as shown in (3.1.25b), we derive in detail the estimates of $N_{c,1}$, from which the estimates of the rest

$N_{c,j}$ terms follow naturally. More specifically, using the notations $N_{c,1}(W(t))$ and $N_{c,1}(t,\boldsymbol{\nu})$ interchangeably, we have

$$
\begin{aligned}
|N_{c,1}(t,\boldsymbol{\nu})| =& \left| \int_{\mathbb{T}_1\times\mathbb{R}} k_1(\boldsymbol{\nu},\widetilde{\boldsymbol{\nu}},\boldsymbol{\nu}-\widetilde{\boldsymbol{\nu}}) a(t,\boldsymbol{\nu}-\widetilde{\boldsymbol{\nu}}) a(t,\widetilde{\boldsymbol{\nu}}) \mathrm{d}\widetilde{\boldsymbol{\nu}} \right| \\
\overset{(3.1.25\text{b})}{\leqslant}& C\left(\int_{\mathbb{T}_1\times\mathbb{R}} \left| a(t,\boldsymbol{\nu}-\widetilde{\boldsymbol{\nu}})\right| \left| \nu_1 a(t,\widetilde{\boldsymbol{\nu}})\right| \mathrm{d}\widetilde{\boldsymbol{\nu}} \right) \\
=& C|a| * |\nu_1 a|,
\end{aligned}
\tag{3.2.10}
$$

which, via the Young's inequality for convolution and the definition of $\|\cdot\|_{\mathbf{H}_c}$, yields that

$$
\begin{aligned}
\left\|N_{c,1}(t,\cdot)\right\|_1 \leqslant& C\left\|a(t,\cdot)\right\|_1 \left\|\nu_1 a(t,\cdot)\right\|_1 \leqslant C(1+t)^{-\frac{3}{4}}\|a\|_{\mathbf{H}_c}(1+t)^{-\frac{5}{4}}\|a\|_{\mathbf{H}_c} = C(1+t)^{-2}\|a\|_{\mathbf{H}_c}^2 ; \\
\left\|N_{c,1}(t,\cdot)\right\|_\infty \leqslant& C\left\|a(t,\cdot)\right\|_\infty \left\|\nu_1 a(t,\cdot)\right\|_1 \leqslant C\|a\|_{\mathbf{H}_c}(1+t)^{-\frac{5}{4}}\|a\|_{\mathbf{H}_c} = C(1+t)^{-\frac{5}{4}}\|a\|_{\mathbf{H}_c}^2 .
\end{aligned}
\tag{3.2.11}
$$

For $N_{c,j}$, $2 \leqslant j \leqslant 7$, recalling the definition of $N_{c,j}$ in (3.1.22), similarly to $N_{c,1}$, we can always rewrite $N_{c,j}$ in the form of weighted convolutions of a and V_s with respect to $\boldsymbol{\nu} \in \mathbb{T}_1 \times \mathbb{R}$ by taking integration with respect to $\xi \in \mathbb{T}_{2\pi}$ first. Noting that all the weight functions are uniformly bounded and that the product of 2π-periodic functions in ξ corresponds to the zero Fourier mode of convolution in discrete ℓ^p spaces, we conclude that

$$
|N_{c,j}(t,\boldsymbol{\nu})| \leqslant C|a|^{*m_j} * \left\|\underline{V_s}\right\|_{\ell^1}^{*(n_j-k_j)} * \left\|\underline{V_s}\right\|_{\ell^\infty}^{k_j}, \tag{3.2.12}
$$

where $k_j := \min\{1, n_j\}$, the convolutions are with respect to $\boldsymbol{\nu} \in \mathbb{T}_1 \times \mathbb{R}$ and the nonnegative integers m_j, n_j satisfies that $m_j + n_j = 2$ for $j = 2, 3$; and $m_j + n_j = 3$ for $4 \leqslant j \leqslant 7$. In combination with the Young's inequality for convolution and the definition of $\|\cdot\|_{\mathbf{H}_c}$, (3.2.12) yields that

$$
\begin{aligned}
\left\|N_{c,j}(t,\cdot)\right\|_1 \leqslant& C\left\|a(t,\cdot)\right\|_1^{m_j} \left\|\underline{V_s}(t,\cdot)\right\|_1^{n_j} \leqslant C(1+t)^{-(\frac{3}{4}m_j+\frac{3}{2}n_j)}\|W\|_{\mathbf{H}}^{m_j+n_j} ; \\
\left\|N_{c,j}(t,\cdot)\right\|_\infty \leqslant& C\left\|a(t,\cdot)\right\|_\infty^{\widetilde{k}_j} \left\|\underline{V_s}(t,\cdot)\right\|_\infty^{1-\widetilde{k}_j} \left\|a(t,\cdot)\right\|_1^{m_j-\widetilde{k}_j} \left\|\underline{V_s}(t,\cdot)\right\|_1^{n_j+\widetilde{k}_j-1} \\
\leqslant& C(1+t)^{-\frac{3}{4}(m_j-\widetilde{k}_j)-\frac{3}{2}(n_j+\widetilde{k}_j-1)}\|W\|_{\mathbf{H}}^{m_j+n_j} ,
\end{aligned}
\tag{3.2.13}
$$

where $\widetilde{k_j} := \min\{1, m_j\}$. Noting that for $j > 1$,

$$\frac{3}{4}m_j + \frac{3}{2}n_j \geqslant \frac{9}{4}, \qquad \frac{3}{4}(m_j - \widetilde{k_j}) + \frac{3}{2}(n_j + \widetilde{k_j} - 1) \geqslant \frac{3}{2},$$

we conclude from the estimates (3.2.11) and (3.2.13) that (3.2.9a) is true.

The proof of (3.2.9b) is similar to the one of (3.2.9a). We recall the expansion $N_s = \sum_{j=1}^{7} N_{s,j}$ in (3.1.23) and derive L^1 and L^∞ estimates of $N_{s,j}$ respectively. Based on the above analysis of N_c, we readily see that, for any $1 \leqslant j \leqslant 7$, $t > 0$ and $\boldsymbol{\nu} \in \mathbb{T}_1 \times \mathbb{R}$,

$$\begin{aligned} \left\|N_{s,j}(t,\boldsymbol{\nu})\right\|_{\ell^1} &\leqslant C|a|^{*m_j} * \left\|\underline{V_s}\right\|_{\ell^1}^{*n_j}, \\ \left\|N_{s,j}(t,\boldsymbol{\nu})\right\|_{\ell^\infty} &\leqslant C|a|^{*m_j} * \left\|\underline{V_s}\right\|_{\ell^1}^{*(n_j-k_j)} * \left\|\underline{V_s}\right\|_{\ell^\infty}^{k_j}, \end{aligned} \tag{3.2.14}$$

which, via the Young's inequality for convolution, yields that

$$\begin{aligned} \left\|N_{s,j}(t,\cdot)\right\|_1 &\leqslant C\left\|a(t,\cdot)\right\|_1^{m_j}\left\|\underline{V_s}(t,\cdot)\right\|_1^{n_j} \leqslant C(1+t)^{-(\frac{3}{4}m_j+\frac{3}{2}n_j)}\|W\|_{\mathbf{H}}^{m_j+n_j}; \\ \left\|N_{s,j}(t,\cdot)\right\|_\infty &\leqslant C\left\|a(t,\cdot)\right\|_\infty^{\widetilde{k_j}}\left\|\underline{V_s}(t,\cdot)\right\|_\infty^{1-\widetilde{k_j}}\left\|a(t,\cdot)\right\|_1^{m_j-\widetilde{k_j}}\left\|\underline{V_s}(t,\cdot)\right\|_1^{n_j+\widetilde{k_j}-1} \\ &\leqslant C(1+t)^{-\frac{3}{4}(m_j-\widetilde{k_j})-\frac{3}{2}(n_j+\widetilde{k_j}-1)}\|W\|_{\mathbf{H}}^{m_j+n_j}, \end{aligned} \tag{3.2.15}$$

which in turns leads directly to (3.2.9b).

We now prove (3.2.7). By the definition of the $\mathbf{H}$ norm, we have

$$\begin{aligned} \left\|\mathcal{T}_2(W)\right\|_{\mathbf{H}} =& \left\|\int_0^t \begin{pmatrix} M_{11}(t-s)N_c(W(s)) + M_{12}(t-s)N_s(W(s)) \\ M_{21}(t-s)N_c(W(s)) + M_{22}(t-s)N_s(W(s)) \end{pmatrix} \mathrm{d}s\right\|_{\mathbf{H}} \\ =& \left\|\int_0^t \left(M_{11}(t-s)N_c(W(s)) + M_{12}(t-s)N_s(W(s))\right)\mathrm{d}s\right\|_{\mathbf{H}_c} + \\ & \left\|\int_0^t \left(M_{21}(t-s)N_c(W(s)) + M_{22}(t-s)N_s(W(s))\right)\mathrm{d}s\right\|_{\mathbf{H}_s} \\ \leqslant& \overbrace{\left(\left\|\int_0^t M_{11}(t-s)N_c(W(s))\mathrm{d}s\right\|_{\mathbf{H}_c} + \left\|\int_0^t M_{21}(t-s)N_c(W(s))\mathrm{d}s\right\|_{\mathbf{H}_s}\right)}^{:=I_c} + \\ & \overbrace{\left(\left\|\int_0^t M_{12}(t-s)N_s(W(s))\mathrm{d}s\right\|_{\mathbf{H}_c} + \left\|\int_0^t M_{22}(t-s)N_s(W(s))\mathrm{d}s\right\|_{\mathbf{H}_s}\right)}^{:=I_s} \end{aligned} \tag{3.2.16}$$

<u>*Estimate of I_c*</u> We evaluate I_c for small and large $\boldsymbol{\nu}$ respectively; that is,

$$
\begin{aligned}
I_c =& \left\| \int_0^t M_{11}(t-s) N_c(W(s)) \mathrm{d}s \right\|_{\mathbf{H}_c} + \left\| \int_0^t M_{21}(t-s) N_c(W(s)) \mathrm{d}s \right\|_{\mathbf{H}_s} \\
\leqslant& \overbrace{\left\| \int_0^t \chi_{\frac{r_1}{2}} M_{11}(t-s) N_c(W(s)) \mathrm{d}s \right\|_{\mathbf{H}_c}}^{:=I_{c,1}} + \overbrace{\left\| \int_0^t (1-\chi_{\frac{r_1}{2}}) M_{11}(t-s) N_c(W(s)) \mathrm{d}s \right\|_{\mathbf{H}_c}}^{:=I_{c,2}} + \\
& \overbrace{\left\| \int_0^t (1-\chi_{\frac{r_1}{2}}) M_{21}(t-s) N_c(W(s)) \mathrm{d}s \right\|_{\mathbf{H}_s}}^{:=I_{c,3}},
\end{aligned}
\tag{3.2.17}
$$

where we use the fact that $\chi_{\frac{r_1}{2}} M_{21} = 0$. Moreover, recalling the definition of $\|\cdot\|_{\mathbf{H}_c}$ and $\|\cdot\|_{\mathbf{H}_s}$, we have

$$
\begin{aligned}
I_{c,1} \leqslant& \overbrace{\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}} \int_0^t \left\| \chi_{\frac{r_1}{2}} M_{11}(t-s) N_c(W(s)) \right\|_1 \mathrm{d}s}^{:=A_{c,1}} + \overbrace{\sup_{t\geqslant 0} \int_0^t \left\| \chi_{\frac{r_1}{2}} M_{11}(t-s) N_c(W(s)) \right\|_\infty \mathrm{d}s}^{:=B_{c,1}} + \\
& \overbrace{\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}} \int_0^t \left\| \nu_1 \chi_{\frac{r_1}{2}} M_{11}(t-s) N_c(W(s)) \right\|_1 \mathrm{d}s}^{:=C_{c,1}}, \\
I_{c,2} \leqslant& \overbrace{\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}} \int_0^t \left\| (1-\chi_{\frac{r_1}{2}}) M_{11}(t-s) N_c(W(s)) \right\|_1 \mathrm{d}s}^{:=A_{c,2}} + \\
& \overbrace{\sup_{t\geqslant 0} \int_0^t \left\| (1-\chi_{\frac{r_1}{2}}) M_{11}(t-s) N_c(W(s)) \right\|_\infty \mathrm{d}s}^{:=B_{c,2}} + \\
& \overbrace{\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}} \int_0^t \left\| \nu_1 (1-\chi_{\frac{r_1}{2}}) M_{11}(t-s) N_c(W(s)) \right\|_1 \mathrm{d}s}^{:=C_{c,2}}, \\
I_{c,3} \leqslant& \overbrace{\sup_{t\geqslant 0}(1+t)^{\frac{3}{2}} \int_0^t \left\| (1-\chi_{\frac{r_1}{2}}) M_{21}(t-s) N_c(W(s)) \right\|_1 \mathrm{d}s}^{:=D_{c,3}} + \\
& \overbrace{\sup_{t\geqslant 0} \int_0^t \left\| (1-\chi_{\frac{r_1}{2}}) M_{21}(t-s) N_c(W(s)) \right\|_\infty \mathrm{d}s}^{:=E_{c,3}}.
\end{aligned}
\tag{3.2.18}
$$

In other words, we have

$$I_c \leqslant \sum_{j=1}^{3} I_{c,j} \leqslant \sum_{j=1}^{2} \Big(A_{c,j} + B_{c,j} + C_{c,j}\Big) + D_{c,3} + E_{c,3}. \tag{3.2.19}$$

We are left to estimate all the terms on the right-hand side of (3.2.19). Taking advantage of the neutral mode estimate (3.1.13) and the estimate (3.2.9a) of N_c, we have

$$\begin{aligned}
A_{c,1} =& \sup_{t\geqslant 0}(1+t)^{\frac{3}{4}} \int_0^t \Big\|\chi_{\frac{r_1}{2}} M_{11}(t-s) N_c(W(s))\Big\|_1 \mathrm{d}s \\
\leqslant& \sup_{t\geqslant 0}(1+t)^{\frac{3}{4}} \bigg(\int_0^{t/2} \Big\|\chi_{\frac{r_1}{2}} M_{11}(t-s)\Big\|_{L^\infty\to L^1} \Big\|N_c(W(s))\Big\|_\infty \mathrm{d}s + \\
& \int_{t/2}^t \Big\|\chi_{\frac{r_1}{2}} M_{11}(t-s)\Big\|_{L^1\to L^1} \Big\|N_c(W(s))\Big\|_1 \mathrm{d}s \bigg) \\
\overset{(3.1.13),\,(3.2.9a)}{\leqslant}& C\left(\|W\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^3\right) \sup_{t\geqslant 0}(1+t)^{\frac{3}{4}} \left(\int_0^{t/2} (1+t-s)^{-\frac{3}{4}} (1+s)^{-\frac{5}{4}} \mathrm{d}s + \int_{t/2}^t (1+s)^{-2} \mathrm{d}s \right) \\
\leqslant& C\left(\|W\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^3\right); \\
B_{c,1} =& \sup_{t\geqslant 0} \int_0^t \Big\|\chi_{\frac{r_1}{2}} M_{11}(t-s) N_c(W(s))\Big\|_\infty \mathrm{d}s \\
\leqslant& \sup_{t\geqslant 0} \int_0^t \Big\|\chi_{\frac{r_1}{2}} M_{11}(t-s)\Big\|_{L^\infty\to L^\infty} \Big\|N_c(W(s))\Big\|_\infty \mathrm{d}s \\
\overset{(3.1.13),\,(3.2.9a)}{\leqslant}& C\left(\|W\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^3\right) \sup_{t\geqslant 0} \left(\int_0^t (1+s)^{-5/4} \mathrm{d}s \right) \\
\leqslant& C\left(\|W\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^3\right); \\
C_{c,1} =& \sup_{t\geqslant 0}(1+t)^{\frac{5}{4}} \int_0^t \Big\|\nu_1 \chi_{\frac{r_1}{2}} M_{11}(t-s) N_c(W(s))\Big\|_1 \mathrm{d}s \\
\leqslant& \sup_{t\geqslant 0}(1+t)^{\frac{5}{4}} \bigg(\int_0^{t/2} \Big\|\nu_1 \chi_{\frac{r_1}{2}} M_{11}(t-s)\Big\|_{L^\infty\to L^1} \Big\|N_c(W(s))\Big\|_\infty \mathrm{d}s + \\
& \int_{t/2}^t \Big\|\nu_1 \chi_{\frac{r_1}{2}} M_{11}(t-s)\Big\|_{L^1\to L^1} \Big\|N_c(W(s))\Big\|_1 \mathrm{d}s \bigg) \\
\overset{(3.1.13),\,(3.2.9a)}{\leqslant}& C\left(\|W\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^3\right) \cdot \\
& \sup_{t\geqslant 0}(1+t)^{\frac{5}{4}} \left(\int_0^{t/2} (1+t-s)^{-\frac{5}{4}} (1+s)^{-\frac{5}{4}} \mathrm{d}s + \int_{t/2}^t (1+t-s)^{-\frac{1}{2}} (1+s)^{-2} \mathrm{d}s \right) \\
\leqslant& C\left(\|W\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^3\right),
\end{aligned} \tag{3.2.20}$$

Similarly, taking advantage of the estimates (3.1.15) and (3.2.9a), we have

$$
\begin{aligned}
A_{c,2} &= \sup_{t\geqslant 0}(1+t)^{\frac{3}{4}} \int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{11}(t-s)N_c(W(s))\right\|_1 \mathrm{d}s \\
&\leqslant C \sup_{t\geqslant 0}(1+t)^{\frac{3}{4}} \int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{11}(t-s)\right\|_1 \left\|N_c(W(s))\right\|_1 \mathrm{d}s \\
&\overset{(3.1.15),\,(3.2.9a)}{\leqslant} C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}}\left(\int_0^t \mathrm{e}^{-\lambda_2(t-s)}(1+s)^{-2}\mathrm{d}s\right) \\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}}\left(\mathrm{e}^{-\frac{\lambda_2 t}{2}}\int_0^{t/2}(1+s)^{-2}\mathrm{d}s+(1+t/2)^{-2}\int_{t/2}^t \mathrm{e}^{-\lambda_2(t-s)}\mathrm{d}s\right) \\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right); \\
B_{c,2} &= \sup_{t\geqslant 0}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{11}(t-s)N_c(W(s))\right\|_\infty \mathrm{d}s \\
&\leqslant C \sup_{t\geqslant 0}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{11}(t-s)\right\|_{L^\infty\to L^\infty} \left\|N_c(W(s))\right\|_\infty \mathrm{d}s \\
&\overset{(3.1.15),\,(3.2.9a)}{\leqslant} C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}\left(\int_0^t \mathrm{e}^{-\lambda_2(t-s)}(1+s)^{-\frac{5}{4}}\mathrm{d}s\right) \\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}\left(\mathrm{e}^{-\frac{\lambda_2 t}{2}}\int_0^{t/2}(1+s)^{-\frac{5}{4}}\mathrm{d}s+(1+t/2)^{-\frac{5}{4}}\int_{t/2}^t \mathrm{e}^{-\lambda_2(t-s)}\mathrm{d}s\right) \\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right); \\
C_{c,2} &= \sup_{t\geqslant 0}(1+t)^{\frac{5}{4}} \int_0^t \left\|\nu_1(1-\chi_{\frac{r_1}{2}})M_{11}(t-s)N_c(W(s))\right\|_1 \mathrm{d}s \\
&\leqslant C \sup_{t\geqslant 0}(1+t)^{\frac{5}{4}} \int_0^t \left\|\nu_1(1-\chi_{\frac{r_1}{2}})M_{11}(t-s)\right\|_{L^1\to L^1} \left\|N_c(W(s))\right\|_1 \mathrm{d}s \\
&\overset{(3.1.15),\,(3.2.9a)}{\leqslant} C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}}\left(\int_0^t \mathrm{e}^{-\lambda_2(t-s)}(1+s)^{-2}\mathrm{d}s\right) \\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}}\left(\mathrm{e}^{-\frac{\lambda_2 t}{2}}\int_0^{t/2}(1+s)^{-2}\mathrm{d}s+(1+t/2)^{-2}\int_{t/2}^t \mathrm{e}^{-\lambda_2(t-s)}\mathrm{d}s\right) \\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right),
\end{aligned}
\tag{3.2.21}
$$

At last, taking advantage of the estimates (3.1.15) and (3.2.9a) again, we have

$$
\begin{aligned}
D_{c,3} &= \sup_{t\geqslant 0}(1+t)^{\frac{3}{2}}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{21}(t-s)N_c(W(s))\right\|_1 \mathrm{d}s\\
&\leqslant C\sup_{t\geqslant 0}(1+t)^{\frac{3}{2}}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{21}(t-s)\right\|_{L^1\to L^1}\left\|N_c(W(s))\right\|_1 \mathrm{d}s\\
&\overset{(3.1.15),\,(3.2.9a)}{\leqslant} C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{3}{2}}\left(\int_0^t \mathrm{e}^{-\lambda_2(t-s)}(1+s)^{-2}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{3}{2}}\left(\mathrm{e}^{-\frac{\lambda_2 t}{2}}\int_0^{t/2}(1+s)^{-2}\mathrm{d}s+(1+t/2)^{-2}\int_{t/2}^t \mathrm{e}^{-\lambda_2(t-s)}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right);\\
E_{c,3} &= \sup_{t\geqslant 0}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{21}(t-s)N_c(W(s))\right\|_\infty \mathrm{d}s\\
&\leqslant C\sup_{t\geqslant 0}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{21}(t-s)\right\|_{L^\infty\to L^\infty}\left\|N_c(W(s))\right\|_\infty \mathrm{d}s\\
&\overset{(3.1.15),\,(3.2.9a)}{\leqslant} C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}\left(\int_0^t \mathrm{e}^{-\lambda_2(t-s)}(1+s)^{-\frac{5}{4}}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}\left(\mathrm{e}^{-\frac{\lambda_2 t}{2}}\int_0^{t/2}(1+s)^{-\frac{5}{4}}\mathrm{d}s+(1+t/2)^{-\frac{5}{4}}\int_{t/2}^t \mathrm{e}^{-\lambda_2(t-s)}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right).
\end{aligned}
\tag{3.2.22}
$$

Combining (3.2.19), (3.2.20), (3.2.21) and (3.2.22), we conclude that

$$
I_c \leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right). \tag{3.2.23}
$$

Estimate of I_s By employing similar arguments as in the proof of (3.2.23), we readily exploit (3.1.14),(3.1.15) and (3.2.9b) to conclude that

$$
I_s \leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right). \tag{3.2.24}
$$

We refer interested readers to Appendix D fora detailed proof.

Combining (3.2.16),(3.2.23) and (D.0.7), we conclude the proof of the estimate (3.2.7) of $\mathcal{T}_2(W)$.

To prove (3.2.8), we note that, similar to (3.2.16) for $\mathcal{T}_2(W)$, we have

$$\begin{aligned}&\left\|\mathcal{T}_2(W_1)-\mathcal{T}_2(W_2)\right\|_{\mathbf{H}}\\ \leqslant&\overbrace{\left(\left\|\int_0^t M_{11}(t-s)\Big(N_c(W_1)-N_c(W_2)\Big)(s)\mathrm{d}s\right\|_{\mathbf{H}_c}+\left\|\int_0^t M_{21}(t-s)\Big(N_c(W_1)-N_c(W_2)\Big)(s)\mathrm{d}s\right\|_{\mathbf{H}_s}\right)}^{:=II_c}+\\ &\overbrace{\left(\left\|\int_0^t M_{12}(t-s)\Big(N_s(W_1)-N_s(W_2)\Big)(s)\mathrm{d}s\right\|_{\mathbf{H}_c}+\left\|\int_0^t M_{22}(t-s)\Big(N_s(W_1)-N_s(W_2)\Big)(s)\mathrm{d}s\right\|_{\mathbf{H}_s}\right)}^{:=II_s}\end{aligned} \tag{3.2.25}$$

The proof of estimates of II_c and II_s are exactly the same as the ones for I_c and I_s, except for that we need to replace the estimates (3.2.9) of $N_c(W)$ and $N_s(W)$ with the ones of $N_c(W_1)-N_c(W_2)$ and $N_s(W_1)-N_s(W_2)$; that is,

$$\begin{aligned}\left\|N_c(W_1)-N_c(W_2)\right\|_1 &\leqslant C(1+t)^{-2}\|W_1-W_2\|_{\mathbf{H}}\left(\|W_1\|_{\mathbf{H}}+\|W_2\|_{\mathbf{H}}+\|W_1\|^2_{\mathbf{H}}+\|W_2\|^2_{\mathbf{H}}\right),\\ \left\|N_c(W_1)-N_c(W_2)\right\|_\infty &\leqslant C(1+t)^{-\frac{5}{4}}\|W_1-W_2\|_{\mathbf{H}}\left(\|W_1\|_{\mathbf{H}}+\|W_2\|_{\mathbf{H}}+\|W_1\|^2_{\mathbf{H}}+\|W_2\|^2_{\mathbf{H}}\right),\\ \left\|N_s(W_1)-N_s(W_2)\right\|_1 &\leqslant C(1+t)^{-\frac{3}{2}}\|W_1-W_2\|_{\mathbf{H}}\left(\|W_1\|_{\mathbf{H}}+\|W_2\|_{\mathbf{H}}+\|W_1\|^2_{\mathbf{H}}+\|W_2\|^2_{\mathbf{H}}\right),\\ \left\|N_s(W_1)-N_s(W_2)\right\|_\infty &\leqslant C(1+t)^{-\frac{3}{4}}\|W_1-W_2\|_{\mathbf{H}}\left(\|W_1\|_{\mathbf{H}}+\|W_2\|_{\mathbf{H}}+\|W_1\|^2_{\mathbf{H}}+\|W_2\|^2_{\mathbf{H}}\right).\end{aligned} \tag{3.2.26}$$

We are left to show that (3.2.26) is true, whose proof is again similar to the one of (3.2.9). Therefore, we omit the details except for a brief discussion on the estimate of $N_{c,1}(W_1)-N_{c,1}(W_2)$ for clarity. Noting that

$$\begin{aligned}&|N_{c,1}(W_1(t,\boldsymbol{\nu})-N_{c,1}(W_2(t,\boldsymbol{\nu}))|\\ =&\left|\int_{\mathbb{T}_1\times\mathbb{R}}k_1(\boldsymbol{\nu},\widetilde{\boldsymbol{\nu}},\boldsymbol{\nu}-\widetilde{\boldsymbol{\nu}})\Big(a_1(t,\boldsymbol{\nu}-\widetilde{\boldsymbol{\nu}})a_1(t,\widetilde{\boldsymbol{\nu}})-a_2(t,\boldsymbol{\nu}-\widetilde{\boldsymbol{\nu}})a_2(t,\widetilde{\boldsymbol{\nu}})\Big)\mathrm{d}\widetilde{\boldsymbol{\nu}}\right|\\ \overset{(3.1.25b)}{\leqslant}&C\Big(|a_1|*|\nu_1(a_1-a_2)|+|a_2|*|\nu_1(a_1-a_2)|+|\nu_1a_1|*|(a_1-a_2)|+|\nu_1a_2|*|(a_1-a_2)|\Big).\end{aligned}$$

we readily conclude

$$\begin{aligned}\left\|N_{c,1}(W_1(t))-N_{c,1}(W_2(t))\right\|_1 &\leqslant C(1+t)^{-2}(\|a_1\|_{\mathbf{H}_c}+\|a_2\|_{\mathbf{H}_c})\|a_1-a_2\|_{\mathbf{H}_c};\\ \left\|N_{c,1}(W_1(t))-N_{c,1}(W_2(t))\right\|_\infty &\leqslant C(1+t)^{-\frac{5}{4}}(\|a_1\|_{\mathbf{H}_c}+\|a_2\|_{\mathbf{H}_c})\|a_1-a_2\|_{\mathbf{H}_c},\end{aligned}$$

Lastly, we point out that $N_c(W)$ and $N_s(W)$ consist of quadratic and cubic terms in W, and thus are smooth with respect to W, yielding

$$N_{c\backslash s}(W_1) - N_{c\backslash s}(W_2) = \int_0^1 N'_{c\backslash s}(\tau W_1 + (1-\tau)W_2)(W_1 - W_2)\mathrm{d}\tau,$$

which in turn explains naturally the occurrence of the $\|W_1 - W_2\|_{\mathbf{H}}$ term in the estimate (3.2.8). □

3.2.4 The Variation of Constant Formulation and the Contraction Mapping of $\mathcal{T}$

We aim to show that the map $\mathcal{T}$ is a well-defined contraction in some neighborhood of $\mathcal{T}_1(W_0)$ in the Banach space $\mathcal{H}$ and thus has a fixed point, which corresponds to a solution to the perturbed Swift-Hohenberg equation (2.1.3). Moreover, the **H**-norm implies the nonlinear stability of the roll solution at the zigzag boundary. More specifically, introducing the notation

$$B(W,R) := \{V \in \mathbf{H} \mid \|V - W\|_{\mathbf{H}} \leqslant R\}, \text{ for any } W \in \mathbf{H}, R > 0,$$

we have the following theorem.

Theorem 3.2.4. *There exists $\delta > 0$ such that, given $\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}} < \delta$, the mapping*

$$\mathcal{T} : B(\mathcal{T}_1(W_0), \left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}}) \to B(\mathcal{T}_1(W_0), \left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}})$$

is a well-defined continuous contraction in the sense that

(i) $\mathcal{T}(W) \in B(\mathcal{T}_1(W_0), \left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}})$ *for any* $W \in B(\mathcal{T}_1(W_0), \left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}})$*;*

(ii) $\left\|\mathcal{T}(W_1) - \mathcal{T}(W_2)\right\|_{\mathbf{H}} < \frac{1}{2}\|W_1 - W_2\|_{\mathbf{H}}$ *for any* $W_1, W_2 \in B(\mathcal{T}_1(W_0), \left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}})$.

Proof. We firstly show that there exists $\delta_1 > 0$ such that, if $\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}} < \delta_1$, then (*i*) holds. To prove that, we recall the bound given in (3.2.7) and have that, for all $W \in \mathbf{H}$,

$$\left\|\mathcal{T}(W) - \mathcal{T}_1(W_0)\right\|_{\mathbf{H}} = \left\|\mathcal{T}_2(W)\right\|_{\mathbf{H}} \leqslant C\left(\|W\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^3\right). \tag{3.2.27}$$

Restricting $W \in B(\mathcal{T}_1(W_0), \left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}})$, it follows that $\|W\|_{\mathbf{H}} \leqslant 2\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}}$, which, together with the above estimate (3.2.27) and $\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}} < \delta_1$, yields

$$\begin{aligned}\left\|\mathcal{T}(W) - \mathcal{T}_1(W_0)\right\|_{\mathbf{H}} \leqslant& C\left(4\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}} + 8\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}}^2\right)\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}} \\ \leqslant& 4C(\delta_1 + 2\delta_1^2)\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}} \\ \leqslant& \left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}},\end{aligned}$$

where the last inequality is true as long as we take

$$0 < \delta_1 \leqslant \min\left\{\frac{1}{2}, \frac{1}{8C}\right\}, \tag{3.2.28}$$

which completes our search for δ_1.

We now show that there exists $\delta_2 > 0$ such that, if $\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}} < \delta_2$, then (*ii*) holds. Similarly to the search of δ_1 above, we have, for any $W_1, W_2 \in B(\mathcal{T}_1(W_0), \left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}})$ with $\left\|\mathcal{T}_1(W_0)\right\|_{\mathbf{H}} < \delta_2$,

$$\begin{aligned}\left\|\mathcal{T}(W_1) - \mathcal{T}(W_2)\right\|_{\mathbf{H}} =& \left\|\mathcal{T}_2(W_1) - \mathcal{T}_2(W_2)\right\|_{\mathbf{H}} \\ \overset{(3.2.8)}{\leqslant}& C\left(\|W_1\|_{\mathbf{H}} + \|W_2\|_{\mathbf{H}} + \|W_1\|_{\mathbf{H}}^2 + \|W_2\|_{\mathbf{H}}^2\right)\|W_1 - W_2\|_{\mathbf{H}} \\ \leqslant& 4C(\delta_2 + 2\delta_2^2)\|W_1 - W_2\|_{\mathbf{H}} \\ \leqslant& \frac{1}{2}\|W_1 - W_2\|_{\mathbf{H}},\end{aligned}$$

where the last inequality is true as long as we take

$$0 < \delta_2 \leqslant \min\left\{\frac{1}{2}, \frac{1}{16C}\right\}, \tag{3.2.29}$$

which completes our search for δ_2. We note that the positive constants C in (3.2.28) and (3.2.29) are the distinct constants from (3.2.7) and (3.2.8) respectively but bear the same notation for conveniences.

Finally, we find that if we choose δ such that

$$\delta = \min(\delta_1, \delta_2),$$

then both (*i*) and (*ii*) are true and this concludes the proof of Theorem 3.2.4 is proven. □

We now give the proof of the main theorem.

Proof of Theorem 1.3.1. From Theorem 3.2.4 and the estimate (3.2.6) of $\mathcal{T}_1$, for any

$$\|W_0\|_1 + \|W_0\|_\infty \leqslant \delta_0 := \delta/C,$$

where the positive constant C is the one in (3.2.6), we have that $\mathcal{T}$ is a contraction map, which, by Banach's fixed point theorem, gives rise to a unique fixed point

$$W_*(t,\boldsymbol{\nu}) = \begin{pmatrix} a_*(t,\boldsymbol{\nu}) \\ \underline{V_s}_*(t,\boldsymbol{\nu}) \end{pmatrix} \in \mathbf{H},$$

which in turn solves the initial value problem of the perturbed Swift-Hohenberg equation (3.1.11) with the initial condition $W(0,\boldsymbol{\nu}) = W_0(\boldsymbol{\nu})$. Moreover, recalling the mode filter decomposition T_{mf} in (3.1.10) and the discrete Bloch-Fourier transform $\mathcal{B}_d$ in (3.1.3), we have that

$$v(t,\mathbf{x}) := \left(\mathcal{B}_d^{-1} \circ T_{mf}^{-1} W_*\right)(t,\mathbf{x}) = \mathcal{B}_d^{-1}\left(a_*(t,\boldsymbol{\nu})\underline{\widehat{e_c}}(\boldsymbol{\nu}) + \underline{V_s}_*(t,\boldsymbol{\nu})\right)$$

solves the perturbed Swift-Hohenberg equation (2.1.3)

$$\begin{cases} v_t = \mathcal{L}_p v + \mathcal{N}_p(v), \\ v(0,\mathbf{x}) = v_0(\mathbf{x}) := \left(\mathcal{B}_d^{-1} \circ T_{mf}^{-1} W_0\right)(\mathbf{x}) = \mathcal{B}_d^{-1}\left(a_0(\boldsymbol{\nu})\underline{\widehat{e_c}}(\boldsymbol{\nu}) + \underline{V_s}_0(\boldsymbol{\nu})\right). \end{cases}$$

As a result, we also have

$$\begin{aligned} \left\|v(t,\cdot)\right\|_{L^\infty(\mathbb{R}^2)} \leqslant& \|a_*(t,\cdot)\underline{\widehat{e_c}}(\cdot) + \underline{V_s}_*(t,\cdot)\|_{L^1(\mathbb{T}_1\times\mathbb{R},\ell^1)} \\ \leqslant& C\left(\|a_*(t,\cdot)\|_1 + \|\underline{V_s}_*(t,\cdot)\|_1\right) \\ \leqslant& C(1+t)^{-3/4}\|W\|_{\mathbf{H}} \\ \leqslant& C(1+t)^{-3/4}\left\|\mathcal{T} W_0\right\|_{\mathbf{H}} \\ \leqslant& C(1+t)^{-3/4}\left(\|W_0\|_1 + \|W_0\|_\infty\right) \\ \leqslant& C(1+t)^{-3/4}\left(\|\widehat{v_0}\|_{L^1(\mathbb{R}^2)} + \|\widehat{v_0}\|_{L^\infty(\mathbb{R}^2)}\right), \end{aligned}$$

which concludes the proof. □

4 Nonlinear Stability of Zigzag-Rolls of the SHE Defined on the Cylinder

In this chapter, we shall discuss the steps needed to prove the nonlinear stability of zigzag-roll solutions $u_*(\xi)$ on the cylindrical domain, $\mathbb{T}_{2\pi} \times \mathbb{R}$; that is Conjecture 1.3.2. The chapter is organized as follows. In Section 4.1.1, we split the SHE into neutral and faster modes using a phase modulation decomposition where the neutral mode is represented by a spatial-temporal phase modulation function ψ. The function ψ satisfies the linear diffusion equation given by $\psi_t = -\mathcal{A}_1\psi$, where $\mathcal{A}_1 = \partial_y^4$. The central idea for this decomposition relies on capturing the leading order dynamics of perturbations given by the scheme $u(t;\xi,y) = u_*(\xi + \psi(t,y)) + w(t;\xi + \psi(t,y),y)$. The linear semigroup estimations and the nonlinear irrelevancy are done in sections 4.1.2 and 4.1.3 respectively.

4.0.1 Nonlinear Terms Seem Relevant

From the analogies obtained in the Table in 2.3, we exploit the intuition we derive from the heat equation to evaluate the temporal decay rates of both linear and nonlinear terms via the linearized flow; that is,

$$\|\mathcal{L}_p w\|_{L^\infty} \sim t^{-\frac{5}{4}}, \quad \|N_p(w)\|_{L^\infty} = \|-3u_p w^2 - w^3\|_{L^\infty} \sim t^{-\frac{1}{2}},$$

which misleadingly indicates that the nonlinear terms are relevant. This false conclusion results from the fact that we applied our intuitive reasoning on w, the sum of both neutral and stable modes, instead of the neutral modes. As a result, the remedy here is to study the system in a refined coordinate system where the neutral and stable modes are properly separated via phase modulation decomposition.

4.1 Phase Modulation and Irrelevancy of Nonlinear Terms

In this Section, we split the SHE into neutral and faster modes via a phase modulation scheme in physical space. Under this phase modulation scheme, we derive the detailed expressions of nonlinear terms, from which the irrelevancy of nonlinear terms is intuitively interpreted but the rigorous proof of irrelevancy is given in the next Section.

There are well-known techniques for deriving the normal form of the perturbed system where the neutral and stable modes are well separated in proving nonlinear stability results. We will discuss some of the techniques. In [31] and [32], Johnson et al introduced a phase modulation scheme for the splitting into neutral and stable modes. The phase shift was a function that depended on both time and space and was a sum of well-separated Gaussian waves propagating in opposite directions. In [11], the authors used a mode filter technique for splitting into neutral and stable modes by using a smooth cut-off function and a projection onto the pseudo-eigenfunction space.

The techniques discussed above have their advantages and disadvantages concerning the normal form system that arises after the decomposition into neutral mode and stable modes. In the mode filter decompositions, it is almost inevitable to work in the Bloch-Fourier space, which is mostly good as there are good tools for decay analysis in Fourier space compared to the physical space where phase decomposition is used. Also for the mode filter decompositions, we often have a small number of nonlinear terms that arise from the decompositions, as a result, analyzing them to show they are irrelevant is not as daunting compared to the phase decompositions where we get a high number of nonlinear terms, and often some quasilinear terms, which means a tool like maximal regularity may be needed to show the irrelevancy of nonlinear terms. Nevertheless, the phase decomposition gives us a much more zoomed-in separation of the modes than the mode filter decomposition, where irrelevancy could be easily seen. In this work, we first approached the problem using the mode filter technique as in [11], however we found out it is more advantageous to use the phase modulation technique because the mode filter was not zoomed-in enough and many of the nonlinear terms were either seen as relevant or critical in our analysis. We will discuss the phase modulation technique in Section 4.1.1.

4.1.1 Phase Modulation Decomposition

Our phase modulation scheme stems from the fact that any element sufficiently close to the neutral ring,

$$\{u_*(\cdot + \psi) \mid \psi \in \mathbb{T}_{2\pi}\},$$

can be uniquely parameterized by the phase ψ and the stable mode

$$w \in L^2_\perp(\mathbb{T}_{2\pi}) := \{u \in L^2(\mathbb{T}_{2\pi}) | \langle u, u_*' \rangle = 0\}.$$

More specifically, we have the following lemma.

Lemma 4.1.1. *There exists $\delta > 0$ such that for any*

$$u \in B_\delta(u_*) := \{u \in L^2(\mathbb{T}_{2\pi}) \,\big|\, \|u - u_*\|_{L^2(\mathbb{T}_{2\pi})} < \delta\},$$

u admits the unique phase modulation decomposition

$$u(\xi) = u_*(\xi + \psi) + w(\xi + \psi), \tag{4.1.1}$$

where $w \in L^2_\perp(\mathbb{T}_{2\pi})$, in the sense that there exists a small open neighborhood of the origin in $\mathbb{R} \times L^2_\perp(\mathbb{T}_{2\pi})$, denoted as Ω, such that the mapping

$$\begin{aligned} C: \quad \Omega \quad &\longrightarrow \quad B_\delta(u_*) \\ (\psi, w) \quad &\longmapsto \quad u_*(\xi + \psi) + w(\xi + \psi), \end{aligned} \tag{4.1.2}$$

is a diffeomorphism.

Remark 4.1.2. *The above result can be easily extended to the whole neutral ring, thanks to the compactness of the ring. We only state and prove the result for a neighborhood of u_* since it is sufficient for our work.*

Proof. We introduce the functional

$$\begin{aligned} \mathcal{H}: \quad \mathbb{R}\times L^2_\perp(\mathbb{T}_{2\pi})\times L^2(\mathbb{T}_{2\pi}) \quad &\longrightarrow \quad L^2(\mathbb{T}_{2\pi}) \\ (\psi, w, u) \quad &\longmapsto \quad u_*(\xi+\psi)+w(\xi+\psi)-u(\xi), \end{aligned}$$

and solve

$$\mathcal{H}(\psi, w, u) = 0.$$

We first note that $\mathcal{H}(0,0,u_*) = 0$. Moreover, introducing the notation $W := (\psi, w) \in \mathbb{R}\times L^2_\perp(\mathbb{T}_{2\pi})$ and $\mathcal{H}_0 := \frac{\partial \mathcal{H}}{\partial W}\,|_{W=(0,u_*)}$, we have that

$$\begin{aligned} \mathcal{H}_0: \quad \mathbb{R}\times L^2_\perp(\mathbb{T}_{2\pi}) \quad &\longrightarrow \quad L^2(\mathbb{T}_{2\pi}) \\ (\phi, v) \quad &\longmapsto \quad \phi u_*'(\xi)+v(\xi), \end{aligned}$$

is an isomorphism, which, together with the implicit function theorem, concludes the proof. □

Based on Lemma 4.1.1, we now introduce the nonlinear phase modulation scheme

$$u(t,\xi,y) = u_*(\xi+\psi(t,y)) + w(t,\xi+\psi(t,y),y). \tag{4.1.3}$$

where $w(t,\cdot,y)\in L^2_\perp(\mathbb{T}_{2\pi})$. Defining

$$z := \xi+\psi(t,y), \quad \widetilde{u}(t,z,y) := u(t,\xi,y) = u(t,z-\psi(t,y),y), \tag{4.1.4}$$

we obtain a modified scheme of (4.1.3) given as

$$\widetilde{u}(t,z,y) = u_*(z) + w(t,z,y), \quad \text{with} \;\; \langle w(t,\cdot,y), u_*'(\cdot)\rangle = 0. \tag{4.1.5}$$

Remark 4.1.3. *For systems posted on the whole Euclidean space, the phase variable in the phase modulation scheme [31],[32], typically depends on all spatial and temporal directions and the remaining correction term is not perpendicular to the neutral mode. In this sense, our scheme admits a simpler structure and provides a more explicit geometric and physical interpretation.*

We now derive the SHE in terms of (ψ, w) under the modified independent variables $(t; z, y)$. Noticing from (4.1.4) that

$$\begin{cases} u_t &= \widetilde{u}_t + \psi_t \widetilde{u}_z, \\ u_\xi &= \widetilde{u}_z, \\ u_y &= \widetilde{u}_y + \psi_y \widetilde{u}_z, \end{cases} \tag{4.1.6}$$

we rewrite the SHE in terms of $\widetilde{u}(t; z, y)$; that is,

$$\widetilde{u}_t + \psi_t \widetilde{u}_z = -\left(1 + k_*^2 \partial_z^2 + (\partial_y + \psi_y \partial_z)^2\right)^2 \widetilde{u} + \mu \widetilde{u} - \widetilde{u}^3, \tag{4.1.7}$$

which, plugging in the scheme (4.1.5), $\widetilde{u}(t, z, y) = u_*(z) + w(t, z, y)$, becomes

$$w_t + \psi_t(u_*' + w_z) = \overbrace{-\left(1 + k_*^2 \partial_z^2 + (\partial_y + \psi_y \partial_z)^2\right)^2 (u_* + w) + \mu(u_* + w) - (u_* + w)^3}^{:=\mathcal{G}(\psi, w)}. \tag{4.1.8}$$

Exploiting the orthogonal condition (4.1.5), we further rewrite (4.1.8) in the following form

$$\begin{cases} \psi_t = & \left\langle u_*' + w_z, u_*' \right\rangle^{-1} \left\langle \mathcal{G}(\psi, w), u_*' \right\rangle \\ w_t = & \mathcal{G}(\psi, w) - \left\langle u_*' + w_z, u_*' \right\rangle^{-1} \left\langle \mathcal{G}(\psi, w), u_*' \right\rangle (u_*' + w_z). \end{cases} \tag{4.1.9}$$

We now sort out linear and nonlinear terms with respect to (ψ, w) in (4.1.9). To do that, recalling that u_* is a stationary solution to the rescaled SHE (2.1.2) and independent of y; that is,

$$-\left(1 + k_*^2 \partial_z^2\right)^2 u_* + \mu u_* - u_*^3 = 0, \tag{4.1.10}$$

we first simply the expression of $\mathcal{G}$ in (4.1.8); that is,

$$\begin{aligned}
\mathcal{G}(\psi, w) =& \mathcal{G}(\psi, w) - \left[-\left(1 + k_*^2\partial_z^2\right)^2 u_* + \mu u_* - u_*^3\right] \\
=& \left[-\left(1 + k_*^2\partial_z^2 + \partial_y^2\right)^2 (u_* + w) + \mu(u_* + w) - (u_* + w)^3\right] - \left[-\left(1 + k_*^2\partial_z^2\right)^2 u_* + \mu u_* - u_*^3\right] + \\
& \left[\left(1 + k_*^2\partial_z^2 + \partial_y^2\right)^2 - \left(1 + k_*^2\partial_z^2 + (\partial_y + \psi_y\partial_z)^2\right)^2\right](u_* + w) \\
=& \overbrace{-\left(1 + k_*^2\partial_z^2 + \partial_y^2\right)^2 w + \mu w - 3u_*^2 w}^{:=\mathcal{L}_* w} + \overbrace{\left(-3u_* w^2 - w^3\right)}^{:=\mathcal{N}_1(w)} + \\
& \overbrace{\left[\left(1 + k_*^2\partial_z^2 + \partial_y^2\right)^2 - \left(1 + k_*^2\partial_z^2 + (\partial_y + \psi_y\partial_z)^2\right)^2\right]}^{:=\mathcal{K}(\psi)}(u_* + w).
\end{aligned} \tag{4.1.11}$$

In order to further simply the expression of $\mathcal{G}$, we would like to sort out the linear terms and leading order nonlinear terms in ψ in the differential operator $\mathcal{K}$, where the leading order is in the sense of its temporal decay rate. Noting that intuitively we have the temporal decay rate for the phase modulation ψ as $t^{-1/4}$ and any extra derivative in y gives rise to an extra $t^{-1/4}$ decay, we have

$$\begin{aligned}
\mathcal{K}(\psi) =& \left(1 + k_*^2\partial_z^2 + \partial_y^2\right)^2 - \left[\overbrace{\left(1 + k_*^2\partial_z^2 + \partial_y^2\right)}^{:=\mathcal{J}_1} + \overbrace{\left(\psi_{yy}\partial_z + 2\psi_y\partial_y\partial_z + \psi_y^2\partial_z^2\right)}^{:=\mathcal{J}_2}\right]^2 \\
=& -\left(\mathcal{J}_1\mathcal{J}_2 + \mathcal{J}_2\mathcal{J}_1 + \mathcal{J}_2^2\right) \\
=& -\left(2\mathcal{J}_2\mathcal{J}_1 + \partial_y^2\mathcal{J}_2 - \mathcal{J}_2\partial_y^2 + \mathcal{J}_2^2\right) \\
=& -\left[2\left(\psi_{yy}\partial_z + 2\psi_y\partial_y\partial_z\right)\mathcal{J}_1 + \psi_{yyyy}\partial_z + 4\psi_{yyy}\partial_y\partial_z + 4\psi_{yy}\partial_y^2\partial_z\right] + \\
& \left[(-2)\psi_y^2\partial_z^2(1 + k_*^2\partial_z^2) - 2\psi_y^2\partial_y^2\partial_z^2 - (\psi_y^2)_{yy}\partial_z^2 - 2(\psi_y^2)_y\partial_y\partial_z^2 - \mathcal{J}_2^2\right].
\end{aligned} \tag{4.1.12}$$

As a result, noting that $u_*(z)$ is independent of y and $\mathcal{K}(\psi)w$ admits only nonlinear terms in (ψ, w), we have

$$\begin{aligned}
\mathcal{K}(\psi)u_* &= \left[-u_*'\psi_{yyyy} - 2(u_*' + k_*^2 u_*''')\psi_{yy}\right] + \\
&\quad \overbrace{\left[-2\left(u_*'' + k_z^2 u_*^{(4)}\right)\psi_y^2 - 3u_*''\psi_{yy}^2 - 4u_*''\psi_y\psi_{yyy} - 6u_*'''\psi_y^2\psi_{yy} - u_*^{(4)}\psi_y^4\right]}^{:=\mathcal{N}_2(\psi)}, \qquad (4.1.13)\\
\overbrace{\mathcal{K}(\psi)w}^{:=\mathcal{N}_3(\psi,w)} &= \overbrace{\left[-2\psi_{yy}\left(w_z + k_*^2 w_{zzz}\right) - 4\psi_y\left(w_{zy} + k_*^2 w_{zzzy}\right)\right]}^{:=\mathcal{N}_{3,1}(\psi,w)} + \overbrace{h.o.t.}^{:=\mathcal{N}_{3,2}(\psi,w)},
\end{aligned}$$

where $\mathcal{N}_{3,2}(\psi, w) := \mathcal{K}(\psi)w - \mathcal{N}_{3,1}(\psi, w)$ is higher order in the sense that the temporal decay rate of the L^∞-norm of $\mathcal{N}_{3,2}$ is faster than $\mathcal{N}_{3,1}$. We now introduce the following useful lemma.

Lemma 4.1.4. *At the zigzag boundary $k = k_*$, we have that*

$$\langle u_*' + k_*^2 u_*''', u_*' \rangle = 0. \qquad (4.1.14)$$

Proof. We recall from Proposition 2.2.1 that $(\lambda(\nu_2), e(\nu_2; \xi))$ is an eigenpair of $\widehat{\mathcal{L}_*}(\nu_2)$; that is,

$$\widehat{\mathcal{L}_*}(\nu_2)e(\nu_2; \xi) = \lambda(\nu_2)e(\nu_2; \xi), \qquad (4.1.15)$$

where λ admits the Taylor expansion as given in (2.2.8); that is,

$$\lambda(\nu_2) = a_{02}\nu_2^2 + a_{04}\nu_2^4 + O(\nu_2^6).$$

Noting that $\widehat{\mathcal{L}_*}(\nu_2)$ admits the expansion

$$\widehat{\mathcal{L}_*}(\nu_2) = \widehat{\mathcal{L}_*}(0) + 2\nu_2^2(1 + k_*^2\partial_\xi^2) - \nu_2^4,$$

and introducing the expansion

$$e(\nu_2; \xi) = e_0 + \nu_2^2 e_2 + O(\nu_2^4),$$

where $e_0 = u_*'$, we plug all the expansions of $\widehat{\mathcal{L}_*}$, λ and e into (4.1.15) and have the following identity for all the terms of order $O(\nu_2^2)$,

$$\widehat{\mathcal{L}_*}(0)e_2 + 2(u_*' + k_*^2 u_*''') = a_{02}u_*',$$

whose inner product with u_*', together with the fact that $a_{02}(\mu, k_*(\mu)) = 0$, in turns yields

$$\langle u_*' + k_*^2 u_*''', u_*' \rangle = \frac{1}{2} a_{02} \langle u_*', u_*' \rangle = 0.$$

□

Taking advantage of (4.1.11)and (4.1.13), we have

$$\mathcal{G}(\psi, w) = \mathcal{L}_* w + \left[-u_*' \psi_{yyyy} - 2(u_*' + k_*^2 u_*''')\psi_{yy} \right] + \sum_{i=1}^{3} \mathcal{N}_i, \tag{4.1.16}$$

which, together with (4.1.14), the parity of u_* and $u_* \in \operatorname{Ker}(\mathcal{L}_*)$, yields

$$\langle \mathcal{G}(\psi, w), u_*' \rangle = -\left\| u_*' \right\|_2^2 \psi_{yyyy} + 6 \left\| u_*'' \right\|_2^2 \psi_y^2 \psi_{yy} + \langle \mathcal{N}_1 + \mathcal{N}_3, u_*' \rangle. \tag{4.1.17}$$

Introducing the notations

$$\mathcal{A}_1 := \partial_y^4, \quad \mathcal{A}_2 := -\mathcal{L}_* \mid_{L^2(\mathbb{R}, L^2_\perp(\mathbb{T}_{2\pi}))}, \quad \mathcal{A}_3 := -2(u_*' + k_*^2 u_*''')\partial_y^2, \quad \mathcal{P}(w) := \left\langle u_*' + w_z, u_*' \right\rangle^{-1} - \left\| u_*' \right\|_2^{-2}, \tag{4.1.18}$$

we now plug (4.1.11),(4.1.13) and (4.1.17) into (4.1.9), yielding the SHE in its phase modulation coordinates $W := (\psi, w)^T$ form,

$$W_t = \mathcal{L}_{pm} W + \mathcal{N}_{pm}(W), \tag{4.1.19}$$

where

$$\mathcal{L}_{pm} := \begin{pmatrix} -\mathcal{A}_1 & 0 \\ \mathcal{A}_3 & -\mathcal{A}_2 \end{pmatrix}, \qquad \mathcal{N}_{pm}(W) := \begin{pmatrix} \mathcal{N}^\psi(W) \\ \mathcal{N}^w(W) \end{pmatrix},$$

with

$$\mathcal{N}^{\psi}(\psi, w) = -\mathcal{P}(w)\left\|u_*'\right\|_2^2 \psi_{yyyy} + \left\langle u_*' + w_z, u_*'\right\rangle^{-1}\left[6\left\|u_*''\right\|_2^2 \psi_y^2\psi_{yy} + \langle \mathcal{N}_1 + \mathcal{N}_3, u_*'\rangle\right]$$

$$= -\mathcal{P}(w)\left\|u_*'\right\|_2^2 \psi_{yyyy} + 6\|u_*''\|_2^2\|u_*'\|_2^{-2}\psi_y^2\psi_{yy} + \langle \mathcal{N}_1 + \mathcal{N}_3, u_*'\rangle\|u_*'\|_2^{-2} + 6\mathcal{P}(w)\|u_*''\|_2^2\psi_y^2\psi_{yy} +$$

$$\mathcal{P}(w)\langle \mathcal{N}_1 + \mathcal{N}_3, u_*'\rangle,$$

$$\mathcal{N}^{w}(\psi, w) = \sum_{i=1}^{3} \mathcal{N}_i + \psi_{yyyy}w_z + (u_*' + w_z)\mathcal{N}^{\psi}.$$

Before studying the linear semigroup estimates, it is advantageous to work in the discrete Fourier space. We recall the definitions of the Fourier transforms in 6, and introduce the following

$$\widehat{\mathcal{A}_1} := -\nu_2^4, \quad \widehat{\mathcal{A}_2} := -\widehat{\mathcal{L}_*}\,|_{L^p(\mathbb{R},\ell_\perp^p)}, \quad \widehat{\mathcal{A}_3} := 2\left(\underline{\widehat{u_*'}} + k_*^2\underline{\widehat{u_*'''}}\right)*\nu_2^2, \quad \underline{\widehat{\mathcal{P}(w)}} := \langle\!\langle \underline{\widehat{u_*'}} + \underline{\widehat{w}}_z, \underline{\widehat{u_*'}}\rangle\!\rangle^{-1} - \left\|u_*'\right\|_2^{-2}, \tag{4.1.20}$$

The system (4.1.19) in the discrete Fourier space becomes

$$\widehat{W}_t = \widehat{\mathcal{L}_{pm}}\widehat{W} + \widehat{\mathcal{N}_{pm}}(\widehat{W}), \tag{4.1.21}$$

where

$$\widehat{W}(t,\nu_2) = \begin{pmatrix} \widehat{\psi}(t,\nu_2) \\ \underline{\widehat{w}}(t,\nu_2) \end{pmatrix}, \qquad \widehat{\mathcal{L}_{pm}} := \begin{pmatrix} \nu_2^4 & 0 \\ \widehat{\mathcal{A}_3} & -\widehat{\mathcal{A}_2} \end{pmatrix}, \qquad \widehat{\mathcal{N}_{pm}}(\widehat{W} := \begin{pmatrix} \widehat{\mathcal{N}^{\psi}}(\widehat{W}) \\ \widehat{\mathcal{N}^{w}}(\widehat{W}) \end{pmatrix},$$

with

$$\widehat{\mathcal{N}^{\psi}}(\widehat{\psi}, \underline{\widehat{w}}) = -\|u_*'\|_2^2\underline{\widehat{\mathcal{P}(w)}} * \nu_2^4\widehat{\psi} - 6\|u_*''\|_2^2\|u_*'\|_2^{-2}\left(\mathrm{i}\nu_2\widehat{\psi} * \mathrm{i}\nu_2\widehat{\psi}\right) * \nu_2^2\widehat{\psi} - 6\|u_*''\|_2^2\underline{\widehat{\mathcal{P}(w)}} * \left(\mathrm{i}\nu_2\widehat{\psi} * \mathrm{i}\nu_2\widehat{\psi}\right) * \nu_2^2\widehat{\psi} +$$

$$\langle\!\langle \widehat{\mathcal{N}_1} + \widehat{\mathcal{N}_3}, \underline{\widehat{u_*'}}\rangle\!\rangle\|u_*'\|_2^{-2} + \underline{\widehat{\mathcal{P}(w)}} * \langle\!\langle \widehat{\mathcal{N}_1} + \widehat{\mathcal{N}_3}, \underline{\widehat{u_*'}}\rangle\!\rangle,$$

$$\widehat{\mathcal{N}^{w}}(\widehat{\psi}, \underline{\widehat{w}}) = \sum_{i=1}^{3} \widehat{\mathcal{N}}_i - \nu_2^4\widehat{\psi} * \underline{\widehat{w}}_z + (\underline{\widehat{u_*'}} + \underline{\widehat{w}}_z) * \widehat{\mathcal{N}^{\psi}}.$$

4.1.2 Linear Semigroup Estimates

We consider the initial value problem of the linearized flow of (4.1.21); that is,

$$\begin{cases} \widehat{W}_t = \widehat{\mathcal{L}_{pm}}\widehat{W}, \\ \widehat{W(0)} = \widehat{W_0} = (\widehat{\psi_0}, \widehat{w_0})^T, \end{cases}$$

whose solution takes the form

$$\widehat{W}(t) = \mathrm{e}^{\widehat{\mathcal{L}_{pm}}t}\widehat{W_0}. \tag{4.1.22}$$

Denoting

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix} := \mathrm{e}^{\widehat{\mathcal{L}_{pm}}t} = \begin{pmatrix} \mathrm{e}^{-\widehat{\mathcal{A}_1}t} & 0 \\ \int_0^t \mathrm{e}^{-\widehat{\mathcal{A}_2}(t-s)}\widehat{\mathcal{A}_3}\mathrm{e}^{-\widehat{\mathcal{A}_1}s}\mathrm{d}s & \mathrm{e}^{-\widehat{\mathcal{A}_2}t} \end{pmatrix}, \tag{4.1.23}$$

we state the following temporal decay estimates about the semigroup generated by the operator $\mathcal{M}$.

Lemma 4.1.5. *For any given $1 \leqslant p \leqslant q \leqslant \infty$, there exists $C > 0$ such that the analytic semigroup M_{11} admits the estimate*

$$|||\nu_2^k \mathcal{M}_{11}(t)|||_{q\to p} \leqslant Ct^{-\frac{1}{4}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{k}{4}}, \tag{4.1.24}$$

where $k \in \mathbb{N}$ and $q \to p$ stands for $L^q(\mathbb{R}) \to L^p(\mathbb{R})$.

Proof. For any given $1 \leqslant p \leqslant q \leqslant \infty$, we apply Holders inequality to $\nu_2^k\mathcal{M}_{11}\widehat{\psi_0}$; that is,

$$\begin{aligned} ||\nu_2^k \mathrm{e}^{-\nu_2^4 t}\widehat{\psi_0}||_p &\leqslant ||\nu_2^k \mathrm{e}^{-\nu_2^4 t}||_r ||\widehat{\psi_0}||_q \\ &= t^{-\frac{1}{4r}-\frac{k}{4}}||\widehat{\psi_0}||_q \\ &\leqslant Ct^{-\frac{1}{4}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{k}{4}}||\widehat{\psi_0}||_q, \end{aligned}$$

where r is the constant such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$. □

From Proposition 2.3 in [11], we have that the operator $\widehat{\mathcal{A}_2}$ is sectorial and its spectrum lives strictly in the left half plane bounded away from the imaginary axis. That is, the spectrum of $\widehat{\mathcal{A}_2}$ is contained in the interval $(0, -d)$ for some $d > 0$. As a result, we have the following lemma.

Lemma 4.1.6. *For any fixed $p \in [1, +\infty]$, there exist $C, d > 0$ such that the analytic semigroup $\mathcal{M}_{22} = \mathrm{e}^{-\widehat{\mathcal{A}_2}t}$ admits the estimate*

$$|||\mathcal{M}_{22}|||_{p\to p} \leqslant Ce^{-dt} \tag{4.1.25}$$

where p stands for $L^p(\mathbb{R}, \ell^p_\perp)$).

Proof. The proof is a direct consequence of Lemma 3.1.4 where $\nu_1 = 0$. □

Lemma 4.1.7. *For any fixed $1 \leqslant p \leqslant q \leqslant \infty$, there exist a $C > 0$ such that $\mathcal{M}_{21}$ admits the estimate*

$$|||\mathcal{M}_{21}(t)|||_{q\to p} \leqslant Ct^{-\frac{1}{4}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}, \tag{4.1.26}$$

where $k \in \mathbb{N}$ and $q \to p$ stands for $L^q(\mathbb{R}) \to L^p(\mathbb{R}, \ell^p_\perp)$.

Proof. The estimation of the $L^p(\mathbb{R}, \ell^p_\perp)$ norm of $\mathcal{M}_{21}$ is

$$\begin{aligned}
||\mathcal{M}_{21}\widehat{\psi_0}||_p =& \left\|\int_0^t \mathrm{e}^{-\widehat{\mathcal{A}_2}(t-s)}\widehat{\mathcal{A}_3}\mathrm{e}^{-\widehat{\mathcal{A}_1}s}\widehat{\psi_0}\mathrm{d}s\right\|_p \\
\leqslant& \int_0^t \left\|\mathrm{e}^{-\widehat{\mathcal{A}_2}(t-s)}\widehat{\mathcal{A}_3}\mathrm{e}^{-\widehat{\mathcal{A}_1}s}\widehat{\psi_0}\right\|_p \mathrm{d}s \\
\leqslant& \int_0^t |||e^{-\widehat{\mathcal{A}_2}(t-s)}|||_p||\widehat{\mathcal{A}_3}\mathrm{e}^{-\widehat{\mathcal{A}_1}s}\widehat{\psi_0}||_p\mathrm{d}s \\
\overset{(4.1.25)}{\leqslant} & C\int_0^t e^{-d(t-s)}||\widehat{\mathcal{A}_3}\mathrm{e}^{-\widehat{\mathcal{A}_1}s}\widehat{\psi_0}||_p\mathrm{d}s \\
\overset{(4.1.24)}{\leqslant} & C\int_0^t e^{-d(t-s)}s^{-\frac{1}{4}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}||\widehat{\psi_0}||_q\mathrm{d}s \\
=& C||\widehat{\psi_0}||_q\left(\int_0^{\frac{t}{2}} e^{-d(t-s)}s^{-\frac{1}{4}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\,\mathrm{d}s + \int_{\frac{t}{2}}^t e^{-d(t-s)}s^{-\frac{1}{4}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\mathrm{d}s\right) \\
\leqslant& C||\widehat{\psi_0}||_q\left(e^{-\frac{dt}{2}}\left(\frac{t}{2}\right)^{-\frac{1}{4}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{1}{2}} + \left(\frac{t}{2}\right)^{-\frac{1}{4}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\frac{1}{d}\right) \\
\leqslant& Ct^{-\frac{1}{4}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}||\widehat{\psi_0}||_q.
\end{aligned}$$

□

4.1.3 Irrelevancy of Nonlinear Terms

We provide intuitive details on why the nonlinear terms $\widehat{\mathcal{N}^\psi}$ and $\widehat{\mathcal{N}^w}$ in (4.1.21) are irrelevant with respect to the linear flow. We first recall the linear flow of (4.1.21) and suppress the nonlinear terms; that is

$$\begin{cases} \widehat{W}_t = \widehat{\mathcal{L}_{pm}}\widehat{W}, \\ \widehat{W(0)} = \widehat{W_0} = (\widehat{\psi_0}, \widehat{w_0})^T, \end{cases}$$

and the estimates (4.1.5) and (4.1.7) of Lemmas 4.1.5 and 4.1.7 respectively, we have the following linear decay estimates for $\widehat{W} = (\widehat{\psi}, \widehat{w})$ for $p = 1$ and $q = \infty$; that is

$$\begin{aligned} &\|\nu_2^k\widehat{\psi}(t,\cdot)\|_{L^1(\mathbb{R})} = \|\nu_2^k \mathrm{e}^{-\widehat{\mathcal{A}_1}t}\widehat{\psi_0}\|_{L^1(\mathbb{R})} \leqslant Ct^{-\frac{1}{4}-\frac{k}{4}}\|\widehat{\psi_0}\|_{L^\infty(\mathbb{R})}, \\ &\|\nu_2^k\underline{\widehat{w}}(t,\cdot)\|_{L^1(\mathbb{R},\ell^1_\perp)} = \left\|\nu_2^k\left(\int_0^t \mathrm{e}^{-\widehat{\mathcal{A}_2}(t-s)}\widehat{\mathcal{A}_3}\mathrm{e}^{-\widehat{\mathcal{A}_1}s}\widehat{\psi_0}\mathrm{d}s\right)\right\|_{L^1(\mathbb{R},\ell^p_\perp)} \leqslant Ct^{-\frac{3}{4}-\frac{k}{4}}\|\widehat{\psi_0}\|_{L^\infty(\mathbb{R})}. \end{aligned} \tag{4.1.27}$$

We now consider the full system (4.1.21),

$$\widehat{W}_t = \widehat{\mathcal{L}_{pm}}\widehat{W} + \widehat{\mathcal{N}_{pm}}(\widehat{W}),$$

or equivalently,

$$\begin{cases} \widehat{\psi}_t = -\nu_2^4\widehat{\psi} + \widehat{\mathcal{N}^\psi}(\widehat{\psi}, \underline{\widehat{w}}) \\ \underline{\widehat{w}}_t = -\widehat{\mathcal{A}_2}\widehat{w} + \widehat{\mathcal{A}_3}\mathrm{e}^{-\widehat{\mathcal{A}_1}t}\widehat{\psi_0} + \widehat{\mathcal{N}^w}(\widehat{\psi}, \underline{\widehat{w}}). \end{cases}$$

Based on the estimates in (4.1.27), the linear flow has the following temporal decay estimations in the $L^\infty(\mathbb{R}) \to L^1(\mathbb{R})$ and $L^\infty(\mathbb{R}, \ell^p_\perp) \to L^1(\mathbb{R}, \ell^p_\perp)$ norm respectively as t goes to $+\infty$,

$$\begin{aligned} &\|-\nu_2^4\widehat{\psi}\|_{L^1(\mathbb{R})} \sim t^{-\frac{5}{4}}, \\ &\|-\widehat{\mathcal{A}_2}\widehat{w} + \widehat{\mathcal{A}_3}\mathrm{e}^{-\widehat{\mathcal{A}_1}t}\widehat{\psi_0}\|_{L^1(\mathbb{R},\ell^1_\perp)} \sim t^{-\frac{3}{4}}. \end{aligned}$$

More specifically, we have

$$\begin{cases} \widehat{\psi}_t = \overbrace{-\nu_2^k\widehat{\psi}}^{O(t^{-\frac{5}{4}})} + \widehat{\mathcal{N}^\psi}(\widehat{\psi}, \underline{\widehat{w}}) \\ \underline{\widehat{w}}_t = \overbrace{-\widehat{\mathcal{A}_2}\widehat{w} + \widehat{\mathcal{A}_3}\mathrm{e}^{-\widehat{\mathcal{A}_1}t}\widehat{\psi_0}}^{O(t^{-\frac{3}{4}})} + \widehat{\mathcal{N}^w}(\widehat{\psi}, \underline{\widehat{w}}). \end{cases} \tag{4.1.28}$$

We can classify the nonlinear terms $\widehat{\mathcal{N}^{\psi}}$ and $\widehat{\mathcal{N}^{w}}$ in (4.1.28) as irrelevant as long as they have faster temporal decay rates than their linear flow counterparts.

Recalling the nonlinear terms in (4.1.21) and plugging in the linear estimations of $\widehat{\psi}$ and $\widehat{w}$ in (4.1.27), we have the following temporal decay estimations of the nonlinear terms

$$\widehat{\mathcal{N}^{\psi}}(\widehat{\psi},\widehat{w}) = \overbrace{-\|u_*'\|_2^2\underline{\widehat{\mathcal{P}(w)}} * \nu_2^4\widehat{\psi}}^{O(t^{-2})} + \overbrace{-6\|u_*''\|_2^2\|u_*'\|_2^{-2}\left(\mathrm{i}\nu_2\widehat{\psi} * \mathrm{i}\nu_2\widehat{\psi}\right) * \nu_2^2\widehat{\psi} - 6\|u_*''\|_2^2\underline{\widehat{\mathcal{P}(w)}} * \left(\mathrm{i}\nu_2\widehat{\psi} * \mathrm{i}\nu_2\widehat{\psi}\right) * \nu_2^2\widehat{\psi}}^{O(t^{-\frac{7}{4}})} +$$
$$\overbrace{\langle\!\langle \widehat{\mathcal{N}_1} + \widehat{\mathcal{N}_3}, \underline{\widehat{u_*'}}\rangle\!\rangle\|u_*'\|_2^{-2} + \underline{\widehat{\mathcal{P}(w)}} * \langle\!\langle \widehat{\mathcal{N}_1} + \widehat{\mathcal{N}_3}, \underline{\widehat{u_*'}}\rangle\!\rangle}^{=O(t^{-\frac{3}{2}})},$$
$$\widehat{\mathcal{N}^{w}}(\widehat{\psi},\widehat{w}) = \overbrace{\sum_{i=1}^{3}}^{O(t^{-1})}\widehat{\mathcal{N}}_i - \overbrace{\nu_2^4\widehat{\psi} * \underline{\widehat{w}}_z}^{O(t^{-2})} + \overbrace{(\underline{\widehat{u_*'}} + \underline{\widehat{w}}_z) * \widehat{\mathcal{N}^{\psi}}}^{O(t^{-\frac{3}{2}})},$$
$$(4.1.29)$$

where

$$\underline{\widehat{\mathcal{P}(w)}} = \langle\!\langle \underline{\widehat{u_*'}} + j\underline{\widehat{w}}, \underline{\widehat{u_*'}}\rangle\!\rangle^{-1} - \left\|u_*'\right\|_2^{-2};$$
$$\widehat{\mathcal{N}}_1 = \overbrace{-3\underline{\widehat{u_*'}} * \underline{\widehat{w}}^{*2} - \underline{\widehat{w}}^{*3}}^{O(t^{-\frac{3}{2}})};$$
$$\widehat{\mathcal{N}}_2 = \Bigg[\overbrace{2\left(\underline{\widehat{u_*''}} + k_*^2\underline{\widehat{u_*^{(4)}}}\right) * \left(\mathrm{i}\nu_2\widehat{\psi} * \mathrm{i}\nu_2\widehat{\psi}\right)}^{O(t^{-1})} - \overbrace{\left(3\underline{\widehat{u_*''}}\mathrm{i}\nu_2^2\widehat{\psi}\right) * \mathrm{i}\nu_2^2\widehat{\psi}}^{O(t^{-\frac{3}{2}})} - \overbrace{\left(4\underline{\widehat{u_*''}}\mathrm{i}\nu_2\widehat{\psi}\right) * \mathrm{i}\nu_2^3\widehat{\psi}}^{O(t^{-\frac{3}{2}})} -$$
$$\overbrace{\left(6\underline{\widehat{u_*'''}}\mathrm{i}\nu_2\widehat{\psi} * \mathrm{i}\nu_2\widehat{\psi}\right) * \nu^2\widehat{\psi}}^{O(t^{-\frac{7}{4}})} - \overbrace{\left(\underline{\widehat{u_*^{(4)}}}\mathrm{i}\nu_2\widehat{\psi} * \mathrm{i}\nu_2\widehat{\psi}\right) * \left(\mathrm{i}\nu_2\widehat{\psi} * \mathrm{i}\nu_2\widehat{\psi}\right)}^{O(t^{-2})}\Bigg];$$
$$\widehat{\mathcal{N}}_3 = \overbrace{\left[-2\nu_2^2\widehat{\psi} * \left(\underline{\widehat{w}}_z + k_*^2\underline{\widehat{w}}_{zzz}\right) - 4\mathrm{i}\nu_2\widehat{\psi} * \left(\mathrm{i}\nu_2\underline{\widehat{w}}_z + k_*^2\mathrm{i}\nu_2\underline{\widehat{w}}_{zzz}\right)\right]}^{O(t^{-\frac{3}{2}})} + h.o.t.$$

The temporal decay of the higher order terms denoted *h.o.t*, has faster temporal decay rates than 3/2 in $\widehat{\mathcal{N}}_3$. Refer to the analysis in (4.1.13) for further details. From above, we readily see that the leading order nonlinear term in $\widehat{\mathcal{N}^{\psi}}$ in (4.1.29) has a faster temporal decay rate than the linear temporal decay rate of $t^{-\frac{5}{4}}$; that is

$$\left\|\langle\!\langle \widehat{\mathcal{N}_1} + \widehat{\mathcal{N}_3}, \underline{\widehat{u_*'}}\rangle\!\rangle\|u_*'\|_2^{-2}(t,\cdot)\right\|_{L^1(\mathbb{R})} \sim t^{-\frac{3}{2}} \quad \text{as } t \to +\infty.$$

Similarly, the leading order nonlinear term of $\widehat{\mathcal{N}^w}$ in (4.1.29) is the $\widehat{\mathcal{N}}_2$ term $2\left(\underline{\widehat{u_*''}} + k_*^2\underline{\widehat{u_*^{(4)}}}\right) * \mathrm{i}\nu_2\widehat{\psi} * \mathrm{i}\nu_2\widehat{\psi}$ has a faster decay temporal rate than the linear temporal decay rate of $t^{-\frac{3}{4}}$; that is

$$\left\| 2\left(\underline{\widehat{u_*''}} + k_*^2\underline{\widehat{u_*^{(4)}}}\right) * \left(\mathrm{i}\nu_2\widehat{\psi} * \mathrm{i}\nu_2\widehat{\psi}\right)(t,\cdot)\right\|_{L^1(\mathbb{R},\ell^1_\perp)} \sim t^{-1} \quad \text{as } t \to +\infty.$$

More specifically, we have

$$\begin{cases} \widehat{\psi}_t = \overbrace{-\nu_2^4\widehat{\psi}}^{O(t^{-\frac{5}{4}})} + \overbrace{\widehat{\mathcal{N}^\psi}(\widehat{\psi},\widehat{w})}^{O(t^{-\frac{3}{2}})} \\ \widehat{w}_t = \overbrace{-\widehat{\mathcal{A}_2}\widehat{w} + \widehat{\mathcal{A}_3}\mathrm{e}^{-\widehat{\mathcal{A}_1}t}\widehat{\psi_0}}^{O(t^{-\frac{3}{4}})} + \overbrace{\widehat{\mathcal{N}^w}(\widehat{\psi},\widehat{w})}^{O(t^{-1})}, \end{cases} \tag{4.1.30}$$

as such, we conclude that the nonlinear terms in the phase modulation coordinate $(\widehat{\psi},\underline{\widehat{w}})$ are irrelevant. This means the dynamics of the system (4.1.21) follow the linear flow and consequently give a decay rate of $t^{-\frac{1}{4}}$.

Remark 4.1.8. *We are left to formally prove the nonlinear irrelevancy and the $t^{-\frac{1}{4}}$ decay rate by constructing a well-defined contraction map on a suitable Banach space which, via Banach fixed point theorem, gives rise to the nonlinear stability results we covet. This could be done in the physical space by following the analysis in [31] or in Fourier space by following the analysis in Section 3.2 of Chapter 3. We shall leave this for future work.*

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Appendix A: Proof of Proposition 2.1.1

To prove this proposition, we first note from (1.2.2) in Lemma 1.2.1 that $u_p(\xi+\pi;k) = -u_p(\xi;k)$, and thus it is more convenient to work on the discrete group $\pi\mathbb{Z}$, which gives a refined block diagonalization than $2\pi\mathbb{Z}$. More specifically, we block-diagonalize $\mathcal{L}_p$ via the Bloch-Fourier transform onto $\mathbb{T}_2\times\mathbb{R}$; that is,

$$\begin{aligned} \mathcal{B}_2: \quad L^2(\mathbb{R}^2) &\longmapsto \qquad L^2(\mathbb{T}_2\times\mathbb{R}, L^2(\mathbb{T}_\pi)) \\ v \qquad &\longrightarrow \quad \mathcal{B}_2 v(\boldsymbol{\nu},\xi) = \textstyle\sum_{k\in\mathbb{Z}} \widehat{v}(\nu_1+k,\nu_2)\mathrm{e}^{\mathrm{i}2k\xi}, \end{aligned}$$

yielding

$$\widehat{L_{p,2}} := \mathcal{B}_2\circ\mathcal{L}_p\circ\mathcal{B}_2^{-1} = \int_{\mathbb{T}_2\times\mathbb{R}} \widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu})\mathrm{d}\boldsymbol{\nu},$$

where the Block-Fourier operators $\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu};\varepsilon^2,\kappa): H^4(\mathbb{T}_\pi)\to L^2(\mathbb{T}_\pi)$ is given by

$$\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu};\varepsilon^2,\kappa)U := -(1+(1+\kappa)(\partial_\xi+\mathrm{i}\nu_1)^2-\nu_2^2)^2U+\varepsilon^2U-3u_p^2U,$$

where $\widehat{\mathcal{L}_{p,2}}$ and $\widehat{\mathcal{L}_p}$ are the same operators defined on different domains, and their wave-number vector $\boldsymbol{\nu}$ also lives in different spaces. More specifically, we have the following lemma, which concludes the proof of part (i) and paves the foundation for the proof of the rest parts.

Lemma A.0.1. *The operators $\mathcal{L}_p$, $\widehat{\mathcal{L}_p}$ and $\widehat{\mathcal{L}_{p,2}}$ admit the following spectral property,*

$$\sigma(\mathcal{L}_p) = \sigma(\widehat{\mathcal{L}_p}) = \bigcup_{\boldsymbol{\nu}\in\mathbb{T}_1\times\mathbb{R}} \sigma(\widehat{\mathcal{L}_p}(\boldsymbol{\nu})) = \sigma(\widehat{\mathcal{L}_{p,2}}) = \bigcup_{\boldsymbol{\nu}\in\mathbb{T}_2\times\mathbb{R}} \sigma(\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu})) \subseteq \mathbb{R}. \tag{A.0.1}$$

Moreover, $\widehat{\mathcal{L}_p}(\boldsymbol{\nu})$ is isomorphic to the direct sum of $\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu})$ and $\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu}+\mathbf{e}_1)$, where $\mathbf{e}_1 := (1,0)$.

Proof. The spectral property (A.0.1) is a direct consequence of the self-adjointness of $\mathcal{L}_p$ and the fact that the Bloch-Fourier transform of a periodic-coefficient differential operator is block diagonal; see [40] for details. We also note that for any given

$\boldsymbol{\nu} = (\nu_1, \nu_2) \in \mathbb{T}_1 \times \mathbb{R}$ and $u_1(\xi), u_2(\xi) \in H^4(\mathbb{T}_\pi)$,

$$\widehat{\mathcal{L}_p}(\boldsymbol{\nu})\left(u_1 + \mathrm{e}^{\mathrm{i}\xi} u_2\right) = \widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu}) u_1 + \mathrm{e}^{\mathrm{i}\xi} \widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu} + \mathbf{e}_1) u_2,$$

which, introduces the isomorphism

$$\begin{aligned} M: \quad L^2(\mathbb{T}_\pi) \times L^2(\mathbb{T}_\pi) &\longrightarrow \quad L^2(\mathbb{T}_{2\pi}) \\ (u_1, u_2) \quad &\longmapsto \quad u_1 + \mathrm{e}^{\mathrm{i}\xi} u_2, \end{aligned}$$

can be rewritten as

$$M^{-1} \circ \widehat{\mathcal{L}_p}(\boldsymbol{\nu}) \circ M = \begin{pmatrix} \widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu}) & 0 \\ 0 & \widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu} + \mathbf{e}_1) \end{pmatrix},$$

and we thus conclude the proof. □

In order to investigate spectral properties of $\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu})$, we first prove the following symmetric property.

Lemma A.0.2. *$(\lambda, e(\xi))$ is an eigenpair of $\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu}; \varepsilon^2, \kappa)$ if and only if $(\lambda, e(-\xi))$ is an eigenpair of $\widehat{\mathcal{L}_{p,2}}(-\nu_1, \nu_2; \varepsilon^2, \kappa)$.*

Proof. Denoting $\widetilde{\xi} := -\xi$ and $\widetilde{e}(\xi) := e(-\xi)$, we have

$$\begin{aligned} \left(\widehat{\mathcal{L}_{p,2}}(-\nu_1, \nu_2; \varepsilon^2, \kappa)\widetilde{e}\right)(\xi) &= \left[-\left(1 + (1+\kappa)(\partial_\xi + \mathrm{i}(-\nu_1))^2 - \nu_2^2\right)^2 + \varepsilon^2 - 3u_p^2(\xi)\right]\widetilde{e}(\xi) \\ &= \left[-\left(1 + (1+\kappa)(-\partial_\xi + \mathrm{i}\nu_1)^2 - \nu_2^2\right)^2 + \varepsilon^2 - 3u_p^2(\xi)\right]\widetilde{e}(\xi) \\ &= \left[-\left(1 + (1+\kappa)(\partial_{\widetilde{\xi}} + \mathrm{i}\nu_1)^2 - \nu_2^2\right)^2 + \varepsilon^2 - 3u_p^2(\widetilde{\xi})\right] e(\widetilde{\xi}) \\ &= \left(\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu}; \varepsilon^2, \kappa) e\right)(\widetilde{\xi}) = \lambda e(\widetilde{\xi}) = \lambda \widetilde{e}(\xi). \end{aligned}$$

□

Taking advantage of the symmetric properties from Lemma A.0.2 and that $\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu})$ is even in ν_2, we restrict our analysis to the region $\boldsymbol{\nu} \in [0, 1] \times [0, \infty)$, which is a quarter of

$\mathcal{T}_2 \times \mathbb{R}$. Recalling that $u_p(\xi) = O(\widetilde{a})$, where $\widetilde{a} = \sqrt{4(\varepsilon^2 - \kappa^2)/3}$, and introducing the leading-order constant-coefficient operator

$$\mathcal{N}(\boldsymbol{\nu};\kappa) := \widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu};\kappa^2,\kappa) = -(1 + (1+\kappa)(\partial_\xi + \mathrm{i}\nu_1)^2 - \nu_2^2)^2 + \kappa^2,$$

we now view $\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu})$ as a small perturbation of $\mathcal{N}(\boldsymbol{\nu};\kappa)$ with the estimate

$$\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu};\varepsilon^2,\kappa) - \mathcal{N}(\boldsymbol{\nu};\kappa) = \varepsilon^2 - \kappa^2 - 3u_p^2 = O(\widetilde{a}^2),$$

or, more specifically, to fix ideas, there exists $\varepsilon_1 < \min\{\varepsilon_0, 1\}$ such that, for any $\kappa < \varepsilon < \varepsilon_1$,

$$|||\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu};\varepsilon^2,\kappa) - \mathcal{N}(\boldsymbol{\nu};\kappa)|||_{L^2} \leqslant \frac{9}{2}\widetilde{a}^2 < 6\varepsilon^2. \tag{A.0.2}$$

It is straightforward to see that the spectrum of $\mathcal{N}(\boldsymbol{\nu};\kappa) : H^4(\mathbb{T}_\pi) \to L^2(\mathbb{T}_\pi)$ consists only of eigenvalues

$$\mu_n(\boldsymbol{\nu};\kappa) := -(1 - (1+\kappa)(n+\nu_1)^2 - \nu_2^2)^2 + \kappa^2,$$

with corresponding eigenfunctions $\phi_n := e^{\mathrm{i}n\xi}$, $n \in 2\mathbb{Z}$. Moreover, for any $|\kappa| \leqslant 1/2$, $\boldsymbol{\nu} \in [0,1] \times [0,\infty)$, we have

$$\mu_{max}(\boldsymbol{\nu};\kappa) := \max_{n\in 2\mathbb{Z}}\{\mu_n(\boldsymbol{\nu};\kappa)\} = \max\{\mu_0(\boldsymbol{\nu};\kappa), \mu_{-2}(\boldsymbol{\nu};\kappa)\}; \quad \mu_n(\boldsymbol{\nu};\kappa) \leqslant -\frac{3}{4}, \text{ for } n \neq 0, -2. \tag{A.0.3}$$

As a result, we identify the set where unstable modes stem from; that is,

$$\mathcal{U}_0 := \left\{\boldsymbol{\nu} \in \mathbb{T}_2 \times \mathbb{R} \,\middle|\, \nu_1^2 + \nu_2^2 = 1, \text{ or } (\nu_1 - 2)^2 + \nu_2^2 = 1\right\}. \tag{A.0.4}$$

In other words, if $\boldsymbol{\nu}$ is bounded away from $\mathcal{U}_0$ and ε is sufficiently small, then the spectrum of $\widehat{\mathcal{L}_{p,2}}$ sits strictly in $(-\infty, 0)$. More specifically, to fix ideas, we fix $r_0 \in (0,1)$ and introduce the set

$$\mathcal{S}_{r_0} := \left\{\boldsymbol{\nu} \in [0,1] \times \mathbb{R} \,\middle|\, |\nu_1^2 + \nu_2^2 - 1| > r_0, |(\nu_1 - 2)^2 + \nu_2^2 - 1| > r_0\right\}. \tag{A.0.5}$$

For any $\boldsymbol{\nu} \in \mathcal{S}_{r_0}$ and $\kappa < \varepsilon < r_0/5$, we have

$$\mu_{max} \leqslant -\frac{8}{25}r_0^2. \tag{A.0.6}$$

Combining (A.0.2), (A.0.3) and (A.0.6), we conclude that, for any $\boldsymbol{\nu} \in \mathcal{S}_{r_0}$ and $\kappa < \varepsilon < \varepsilon_2 := \min\{r_0/5, \varepsilon_1\}$,

$$\int_0^{\pi} (\widehat{\mathcal{L}_{p,2}} V)\overline{V} \mathrm{d}\xi \leqslant (-\frac{8}{25} r_0^2 + 6\varepsilon^2)\|V\|^2_{L^2(0,\pi)} \leqslant -\frac{2r_0^2}{25}\|V\|^2_{L^2(0,\pi)}. \tag{A.0.7}$$

We are now left to study the spectrum of $\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu}; \varepsilon^2, \kappa)$ for $\boldsymbol{\nu} \in \Omega := ([0,1] \times \mathbb{R}) \backslash \mathcal{S}_{r_0}$. To this end, we distinguish the two subregions

$$\Omega_1 := \left\{ \boldsymbol{\nu} \in \Omega \,\middle|\, \nu_2 \geqslant \sqrt{r_0} \right\}, \qquad \Omega_2 := \left\{ \boldsymbol{\nu} \in \Omega \,\middle|\, \nu_2 \leqslant \sqrt{r_0} \right\}.$$

Based on estimates (A.0.3) and (A.0.6), we note that $\widehat{\mathcal{L}_{p,2}}$ admits respectively one small eigenvalue and two small eigenvalues in region Ω_1 and Ω_2, and we only need to keep track of these small eigenvalues since all other eigenvalues are stable modes with strictly negative real parts. According to [1], the set of (ε, κ) when $\widehat{\mathcal{L}_{p,2}}$ is spectrally stable when $\boldsymbol{\nu} \in \Omega_2$ is a subset for its counterpart when $\boldsymbol{\nu} \in \Omega_1$ and thus we skip the discussion of the subregion Ω_1 and refer interested readers to [1] for more details. We now restrict ourselves to the subdomain Ω_2, where both neutral modes admit small eigenvalues. We apply the Lyapunov-Schmidt reduction to reduce the eigenvalue problem

$$F(U, \lambda; \boldsymbol{\nu}, \varepsilon^2, \kappa) := (\widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu}; \varepsilon^2, \kappa) - \lambda)U = 0,$$

to a two-dimensional problem in the subspace spanned by the neutral modes, $\mathcal{U}_0 := \mathrm{span}\{\phi_0, \phi_{-2}\}$. More specifically, noting that $L^2(\mathbb{T}_\pi) = \mathcal{U}_0 \bigoplus \mathcal{U}_0^\perp$, we let P be the orthogonal projection from $L^2(\mathbb{T}_\pi)$ onto $\mathcal{U}_0$, $Q := \mathrm{Id} - P$ and introduce the decomposition

$$U = U_0 + U_0^\perp, \quad \text{where } U_0 := PU, \quad U_0^\perp := (\mathrm{Id} - P)U,$$

under which, the eigenvalue problem can be rewritten as

$$\begin{cases} PF(U_0 + U_0^\perp, \lambda; \boldsymbol{\nu}, \varepsilon^2, \kappa) = 0, \\ QF(U_0 + U_0^\perp, \lambda; \boldsymbol{\nu}, \varepsilon^2, \kappa) = 0. \end{cases} \tag{A.0.8}$$

We can apply an implicit-function-theorem argument and the compactness of Ω_2 to solve the second equation in (A.0.8) for $U_0^\perp$ in terms of $\boldsymbol{\nu} \in \Omega_2$ and small λ, U_0, ε, and κ. More specifically, we have $QF(0,0;\boldsymbol{\nu},\varepsilon^2,\kappa) = 0$ and

$$\frac{\partial QF}{\partial U_0^\perp}(0,0;\boldsymbol{\nu},\varepsilon^2,\kappa) = Q\mathcal{N}(\boldsymbol{\nu};\kappa)\,|_{\mathcal{U}_0^\perp} + O(\widetilde{a}^2)$$

which, based on the spectral estimate (A.0.3), is invertible. As a result, for $|\lambda|, \|U_0\|_{L^2(\mathbb{T}_\pi)} \ll 1$,

$$U_0^\perp = \mathcal{V}(\lambda;\boldsymbol{\nu},\varepsilon^2,\kappa)U_0,$$

where

$$\mathcal{V}U := (\mathcal{N}(\boldsymbol{\nu};\kappa) - \lambda + \varepsilon^2 - \kappa^2 - 3Qu_p^2)^{-1}(3Qu_p^2 U) = (\mathcal{N}(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda)^{-1}(3Qu_p^2) + O(\widetilde{a}^4).$$

By substituting $U_0^\perp = \mathcal{V}(\lambda;\boldsymbol{\nu},\varepsilon^2,\kappa)U_0$ into the first equation in (A.0.8), we now analyze the reduced two-dimensional problem $PF(U_0 + \mathcal{V}U_0, \lambda; \boldsymbol{\nu}, \varepsilon^2, \kappa) = 0$, which takes the following explicit form,

$$(\mathcal{N}(\boldsymbol{\nu};\kappa) - \lambda)\,U_0 + \left((\varepsilon^2 - \kappa^2)U_0 - 3P(u_p^2 U_0)\right) - 3P\left(u_p^2 \mathcal{V}U_0\right) = 0. \tag{A.0.9}$$

We now convert the two-dimensional problem (A.0.9) into its matrix form with respect to the basis

$$\left\{U_1 := e^{-i\xi}\cos\xi = \frac{1}{2}(\phi_0 + \phi_{-2}),\ U_2 := e^{-i\xi}\sin\xi = \frac{1}{2i}(\phi_0 - \phi_{-2})\right\}; \tag{A.0.10}$$

that is, introducing $U_0 = c_1 U_1 + c_2 U_2$ with $\mathbf{c} := (c_1, c_2)^T$ and the matrix

$$M(\lambda;\boldsymbol{\nu},\varepsilon^2,\kappa) := \begin{pmatrix} \frac{\langle PF(U_1+\mathcal{V}U_1,\lambda),U_1\rangle_{L^2(\mathbb{T}_\pi)}}{\|U_1\|^2_{L^2(\mathbb{T}_\pi)}} & \frac{\langle PF(U_2+\mathcal{V}U_2,\lambda),U_1\rangle_{L^2(\mathbb{T}_\pi)}}{\|U_1\|^2_{L^2(\mathbb{T}_\pi)}} \\ \frac{\langle PF(U_1+\mathcal{V}U_1,\lambda),U_2\rangle_{L^2(\mathbb{T}_\pi)}}{\|U_2\|^2_{L^2(\mathbb{T}_\pi)}} & \frac{\langle PF(U_2+\mathcal{V}U_2,\lambda),U_2\rangle_{L^2(\mathbb{T}_\pi)}}{\|U_2\|^2_{L^2(\mathbb{T}_\pi)}} \end{pmatrix},$$

the two-dimensional problem (A.0.9) becomes

$$M\mathbf{c} = 0. \tag{A.0.11}$$

Taking advantage of the expression (A.0.9) and the fact that $\|U_1\|^2_{L^2(\mathbb{T}_\pi)} = \|U_2\|^2_{L^2(\mathbb{T}_\pi)} = \frac{\pi}{2}$, We split the matrix M into three parts; that is, $M = M^0 + M^1 + M^2$, where, $M^k = (m^k_{i,j})_{2\times 2}$ for $k = 0, 1, 2$, with

$$
\begin{aligned}
m^0_{i,j} &:= 2 \fint_{T_\pi} \underline{\widehat{U_i}} \left(\mathcal{N}(\boldsymbol{\nu};\kappa) - \lambda\right) U_j \, d\xi, \\
m^1_{i,j} &:= 2 \fint_{T_\pi} \underline{\widehat{U_i}} \left(\varepsilon^2 - \kappa^2 - 3u_p^2\right) U_j \, d\xi, \\
m^2_{i,j} &:= 2 \fint_{T_\pi} \underline{\widehat{U_i}} \left(-3u_p^2\right) \mathcal{V} U_j \, d\xi.
\end{aligned}
$$

A.0.1 Computing M^0

We compute the matrix coefficients of M^0,

$$
m^0_{i,j} = 2 \fint_{T_\pi} \underline{\widehat{U_i}} \left(\mathcal{N}(\boldsymbol{\nu};\kappa) - \lambda\right) U_j \, d\xi.
$$

Noting that

$$
\begin{aligned}
\left(\mathcal{N}(\boldsymbol{\nu};\kappa) - \lambda\right) U_1 &= \frac{1}{2} \left(\mathcal{N}(\boldsymbol{\nu};\kappa) - \lambda\right)(\phi_0 + \phi_{-2}) = \frac{1}{2} \left[(\mu_0(\boldsymbol{\nu};\kappa) - \lambda)\phi_0 + (\mu_{-2}(\boldsymbol{\nu};\kappa) - \lambda)\phi_{-2}\right] \\
\left(\mathcal{N}(\boldsymbol{\nu};\kappa) - \lambda\right) U_2 &= \frac{1}{2i} \left(\mathcal{N}(\boldsymbol{\nu};\kappa) - \lambda\right)(\phi_0 - \phi_{-2}) = \frac{1}{2i} \left[(\mu_0(\boldsymbol{\nu};\kappa) - \lambda)\phi_0 - (\mu_{-2}(\boldsymbol{\nu};\kappa) - \lambda)\phi_{-2}\right]
\end{aligned}
$$

we derive the expression for the entries; that is,

$$
\begin{aligned}
m^0_{1,1} &= \frac{1}{2} \fint_{T_\pi} (\phi_0 + \phi_2) \left[(\mu_0(\boldsymbol{\nu};\kappa) - \lambda)\phi_0 + (\mu_{-2}(\boldsymbol{\nu};\kappa) - \lambda)\phi_{-2}\right] d\xi \\
&= \frac{\mu_0(\boldsymbol{\nu};\kappa) + \mu_{-2}(\boldsymbol{\nu};\kappa)}{2} - \lambda, \\
m^0_{1,2} = \underline{\widehat{m^0_{2,1}}} &= \frac{1}{2\mathrm{i}} \fint_{T_\pi} (\phi_0 + \phi_2) \left[(\mu_0(\boldsymbol{\nu};\kappa) - \lambda)\phi_0 - (\mu_{-2}(\boldsymbol{\nu};\kappa) - \lambda)\phi_{-2}\right] d\xi \\
&= \frac{\mu_0(\boldsymbol{\nu};\kappa) - \mu_{-2}(\boldsymbol{\nu};\kappa)}{2\mathrm{i}}, \\
m^0_{2,2} &= \frac{1}{2} \fint_{T_\pi} (\phi_0 - \phi_2) \left[(\mu_0(\boldsymbol{\nu};\kappa) - \lambda)\phi_0 - (\mu_{-2}(\boldsymbol{\nu};\kappa) - \lambda)\phi_{-2}\right] d\xi \\
&= \frac{\mu_0(\boldsymbol{\nu};\kappa) + \mu_{-2}(\boldsymbol{\nu};\kappa)}{2} - \lambda.
\end{aligned}
$$

Denoting $\rho := (\mu_0(\boldsymbol{\nu};\kappa) + \mu_{-2}(\boldsymbol{\nu};\kappa))/2$ and $\beta := (\mu_0(\boldsymbol{\nu};\kappa) - \mu_{-2}(\boldsymbol{\nu};\kappa))/2$, we have the matrix

$$M^0 = \begin{pmatrix} \rho-\lambda & -\mathrm{i}\beta \\ \mathrm{i}\beta & \rho-\lambda \end{pmatrix}.$$

A.0.2 Computing M^1

We compute the matrix coefficients of M^1,

$$m^1_{i,j} = 2 \fint_{T_\pi} \underline{\widehat{U_i}} \left(\varepsilon^2 - \kappa^2 - 3u_p^2\right) U_j d\xi,$$

which admit the following expressions,

$$\begin{aligned} m^1_{1,1} &= \frac{1}{2} \fint_{T_\pi} (\phi_0 + \phi_2) \left(\varepsilon^2 - \kappa^2 - 3u_p^2\right) (\phi_0 + \phi_{-2}) d\xi \\ &= \varepsilon^2 - \kappa^2 - 6 \fint_{T_\pi} u_p^2 \cos^2(\xi) d\xi, \\ m^1_{1,2} = \underline{\widehat{m^1_{2,1}}} &= \frac{1}{2\mathrm{i}} \fint_{T_\pi} (\phi_0 + \phi_2) \left(\varepsilon^2 - \kappa^2 - 3u_p^2\right) (\phi_0 - \phi_{-2})\, d\xi \\ &= 0, \\ m^1_{2,2} &= \frac{1}{2} \fint_{T_\pi} (\phi_0 - \phi_2) \left(\varepsilon^2 - \kappa^2 - 3u_p^2\right) (\phi_0 - \phi_{-2}) d\xi \\ &= \varepsilon^2 - \kappa^2 - 6 \fint_{T_\pi} u_p^2 \sin^2(\xi) d\xi. \end{aligned}$$

To obtain more explicit expressions of the integrals in $m^1_{1,1}$ and $m^1_{2,2}$, we recall from Lemma 1.2.1 that

$$u_p = a_1 \cos(\xi) + a_3 \cos(3\xi) + O(\widetilde{a}^5),$$

where

$$a_1 = \widetilde{a} + \widetilde{a}^3/512 + O(\widetilde{a}^4), \quad a_3 = -\widetilde{a}^3/256 + O(\widetilde{a}^4), \quad \widetilde{a} = \sqrt{\frac{4[\mu - (k^2-1)^2]}{3}} = \sqrt{\frac{4(\varepsilon^2 - \kappa^2)}{3}}.$$

As a result, we plug the expansion of u_p into the integrals in $m^1_{1,1}$ and $m^1_{2,2}$, yielding

$$\begin{aligned}
⨍_{T_\pi} u_p^2 \cos^2(\xi) d\xi &= ⨍_{T_\pi} \left(a_1^2 \cos^2(\xi) + 2a_1 a_3 \cos(\xi)\cos(3\xi) + O(\widetilde{a}^5)\right)\cos^2(\xi) d\xi \\
&= \frac{3}{8}a_1^2 + \frac{1}{4}a_1 a_3 + O(\widetilde{a}^5), \\
⨍_{T_\pi} u_p^2 \sin^2(\xi) d\xi &= ⨍_{T_\pi} \left(a_1^2 \cos^2(\xi) + 2a_1 a_3 \cos(\xi)\cos(3\xi) + O(\widetilde{a}^5)\right)\sin^2(\xi) d\xi \\
&= \frac{1}{8}a_1^2 - \frac{1}{4}a_1 a_3 + O(\widetilde{a}^5).
\end{aligned}$$

We now conclude that

$$\begin{aligned}
M^1 &= \begin{pmatrix} \varepsilon^2 - \kappa^2 - \frac{9}{4}a_1^2 - \frac{3}{2}a_1 a_3 + O(\widetilde{a}^5) & 0 \\ 0 & \varepsilon^2 - \kappa^2 - \frac{3}{4}a_1^2 + \frac{3}{2}a_1 a_3 + O(\widetilde{a}^5) \end{pmatrix} \\
&= \begin{pmatrix} -\frac{3}{2}\widetilde{a}^2 - \frac{3}{1024}\widetilde{a}^4 + O(\widetilde{a}^5) & 0 \\ 0 & -\frac{9}{1024}\widetilde{a}^4 + O(\widetilde{a}^5) \end{pmatrix}.
\end{aligned} \tag{A.0.12}$$

A.0.3 Computing M^2

We compute the matrix coefficients of M^2,

$$m^2_{i,j} = 2 ⨍_{T_\pi} \underline{\widehat{U_i}} \left(-3u_p^2\right) \mathcal{V} U_j d\xi = 2 ⨍_{T_\pi} \underline{\widehat{U_i}} \left(-3u_p^2\right) \left(\mathcal{N}(\nu;\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1} \left(3Q(u_p^2 U_j)\right) d\xi.$$

Recalling the expansion of u_p^2, we readily conclude that $u_p^2 = \widetilde{a}^2 \cos^2(\xi) + O(\widetilde{a}^4)$ and thus derive the following leading order expansion of $m^2_{i,j}$,

$$m^2_{i,j} = -18\widetilde{a}^4 ⨍_{T_\pi} \left(\underline{\widehat{U_i}} \cos^2(\xi)\right) \left(\mathcal{N}(\nu;\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1} \left(Q\left(U_j \cos^2(\xi)\right)\right) d\xi + O(\widetilde{a}^6).$$

For convenience of computation, we now rewrite $U_j \cos^2(\xi)$ in terms of ϕ_n. More specifically, noting that $\cos^2(\xi) = \frac{1}{2}\phi_0 + \frac{1}{4}(\phi_2 + \phi_{-2})$, we have

$$\begin{aligned}
U_1 \cos^2(\xi) &= \frac{1}{2}(\phi_0 + \phi_{-2})[\frac{1}{2}\phi_0 + \frac{1}{4}(\phi_2 + \phi_{-2})] = \frac{1}{8}\left(\phi_{-4} + 3\phi_{-2} + 3\phi_0 + \phi_2\right), \\
\underline{\widehat{U_1}} \cos^2(\xi) &= \frac{1}{8}\left(\phi_4 + 3\phi_2 + 3\phi_0 + \phi_{-2}\right), \\
U_2 \cos^2(\xi) &= \frac{1}{2i}(\phi_0 - \phi_{-2})[\frac{1}{2}\phi_0 + \frac{1}{4}(\phi_2 + \phi_{-2})] = \frac{1}{8i}\left(\phi_0 + \phi_2 - \phi_{-2} - \phi_{-4}\right), \\
\underline{\widehat{U_2}} \cos^2(\xi) &= \frac{1}{8i}\left(\phi_4 + \phi_2 - \phi_0 - \phi_{-2}\right)
\end{aligned}$$

which, together with the fact that $\mathcal{N}(\boldsymbol{\nu};\kappa)\phi_n = \mu_n^\kappa(\boldsymbol{\nu})\phi_n$, gives the explicit expressions of $m^2_{i,j}$,

$$\begin{aligned}
m^2_{1,1} &= -\frac{9\widetilde{a}^4}{32}\int_{T_\pi} (\phi_4 + 3\phi_2 + 3\phi_0 + \phi_{-2})\left(\mathcal{N}(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1}(\phi_{-4} + \phi_2)\, d\xi + O(\widetilde{a}^6)\\
&= -\frac{9\widetilde{a}^4}{32}\left[\left(\mu_2(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1} + \left(\mu_{-4}(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1}\right] + O(\widetilde{a}^6)\\
&= -\frac{9\widetilde{a}^4}{32}\left[(\mu_2(\boldsymbol{\nu};\kappa) - \lambda)^{-1} + (\mu_{-4}(\boldsymbol{\nu};\kappa) - \lambda)^{-1}\right] + O(\widetilde{a}^6),\\
m^2_{1,2} &= -\frac{9\widetilde{a}^4}{32\mathrm{i}}\int_{T_\pi} (\phi_4 + 3\phi_2 + 3\phi_0 + \phi_{-2})\left(\mathcal{N}(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1}(\phi_2 - \phi_{-4})\, d\xi + O(\widetilde{a}^6)\\
&= -\frac{9\widetilde{a}^4}{32i}\left[\left(\mu_2(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1} - \left(\mu_{-4}(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1}\right] + O(\widetilde{a}^6)\\
&= -\frac{9\widetilde{a}^4}{32i}\left[(\mu_2(\boldsymbol{\nu};\kappa) - \lambda)^{-1} - (\mu_{-4}(\boldsymbol{\nu};\kappa) - \lambda)^{-1}\right] + O(\widetilde{a}^6),\\
m^2_{2,1} &= -\frac{9\widetilde{a}^4}{32\mathrm{i}}\int_{T_\pi} (\phi_4 + \phi_2 - \phi_0 - \phi_{-2})\left(\mathcal{N}(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1}(\phi_{-4} + \phi_2)\, d\xi + O(\widetilde{a}^6)\\
&= \frac{9\widetilde{a}^4}{32i}\left[\left(\mu_2(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1} - \left(\mu_{-4}(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1}\right] + O(\widetilde{a}^6)\\
&= \frac{9\widetilde{a}^4}{32i}\left[(\mu_2(\boldsymbol{\nu};\kappa) - \lambda)^{-1} - (\mu_{-4}(\boldsymbol{\nu};\kappa) - \lambda)^{-1}\right] + O(\widetilde{a}^6),\\
m^2_{2,2} &= \frac{9\widetilde{a}^4}{32}\int_{T_\pi} (\phi_4 + \phi_2 - \phi_0 - \phi_{-2})\left(\mathcal{N}(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1}(\phi_2 - \phi_{-4})\, d\xi + O(\widetilde{a}^6)\\
&= -\frac{9\widetilde{a}^4}{32}\left[\left(\mu_2(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1} + \left(\mu_{-4}(\boldsymbol{\nu};\kappa) + \varepsilon^2 - \kappa^2 - \lambda\right)^{-1}\right] + O(\widetilde{a}^6)\\
&= -\frac{9\widetilde{a}^4}{32}\left[(\mu_2(\boldsymbol{\nu};\kappa) - \lambda)^{-1} + (\mu_{-4}(\boldsymbol{\nu};\kappa) - \lambda)^{-1}\right] + O(\widetilde{a}^6)
\end{aligned}$$

Denoting

$$\eta_\pm := -\frac{9\widetilde{a}^4}{32}\left[(\mu_2(\boldsymbol{\nu};\kappa) - \lambda)^{-1} \pm (\mu_{-4}(\boldsymbol{\nu};\kappa) - \lambda)^{-1}\right],$$

We conclude that

$$M^2 = \begin{pmatrix} \eta_+ & -i\eta_- \\ i\eta_- & \eta_+ \end{pmatrix} + O(\widetilde{a}^6).$$

A.0.4 Determinant of M

We now summarize the above computation and conclude the leading order expansion of the matrix M,

$$M = \begin{pmatrix} \rho + m_{1,1}^1 + \eta_+ - \lambda & -\mathrm{i}(\beta + \eta_-) \\ \mathrm{i}(\beta + \eta_-) & \rho + m_{2,2}^1 + \eta_+ - \lambda \end{pmatrix} + O(\widetilde{a}^6), \tag{A.0.13}$$

from which we now derive the leading order Taylor's expansion of the two small eigenvalues with respect to $\boldsymbol{\nu}$ at $\boldsymbol{\nu} = \mathbf{e}_1 = (1, 0)$ in the subdomain Ω_2.

For the convenience of computation, we introduce the notations

$$\mu_{n,e}(\boldsymbol{\nu}; \kappa) := \frac{\mu_n(\boldsymbol{\nu}; \kappa) + \mu_n(-\boldsymbol{\nu}; \kappa)}{2}; \quad \mu_{n,o}(\boldsymbol{\nu}; \kappa) := \frac{\mu_n(\boldsymbol{\nu}; \kappa) - \mu_n(-\boldsymbol{\nu}; \kappa)}{2}.$$

We also introduce the shift

$$\boldsymbol{\nu} = \widetilde{\boldsymbol{\nu}} + \mathbf{e}_1.$$

Noting that $\mu_n(-\boldsymbol{\nu}; \kappa) = \mu_{-n}(\boldsymbol{\nu}; \kappa)$ and

$$\mu_n(\widetilde{\boldsymbol{\nu}} + \mathbf{e}_1; \kappa) = -(1 - (1 + \kappa)(n + 1 + \nu_1)^2 - \nu_2^2)^2 + \kappa^2 = \mu_{n+1}(\widetilde{\boldsymbol{\nu}}; \kappa),$$

we rewrite ρ, β and $\eta_\pm$ in terms of $\widetilde{\boldsymbol{\nu}}$ and λ; that is,

$$\begin{aligned}
\rho =& (\mu_0(\boldsymbol{\nu}; \kappa) + \mu_{-2}(\boldsymbol{\nu}; \kappa))/2 = (\mu_1(\widetilde{\boldsymbol{\nu}}; \kappa) + \mu_{-1}(\widetilde{\boldsymbol{\nu}}; \kappa))/2 = \mu_{1,e}(\widetilde{\boldsymbol{\nu}}; \kappa) \\
:=& \mu_{1,20}\nu_1^2 + \mu_{1,02}\nu_2^2 + \mu_{1,40}\nu_1^4 + \mu_{1,22}\nu_1^2\nu_2^2 + \mu_{1,04}\nu_2^4, \\
\beta =& (\mu_0(\boldsymbol{\nu}; \kappa) - \mu_{-2}(\boldsymbol{\nu}; \kappa))/2 = (\mu_1(\widetilde{\boldsymbol{\nu}}; \kappa) - \mu_{-1}(\widetilde{\boldsymbol{\nu}}; \kappa))/2 = \mu_{1,o}(\widetilde{\boldsymbol{\nu}}; \kappa) \\
:=& \mu_{1,10}\nu_1 + \mu_{1,30}\nu_1^3 + \mu_{1,12}\nu_1\nu_2^2, \\
\eta_\pm =& -\frac{9\widetilde{a}^4}{32}\left[(\mu_2(\boldsymbol{\nu}; \kappa) - \lambda)^{-1} \pm (\mu_{-4}(\boldsymbol{\nu}; \kappa) - \lambda)^{-1}\right] \\
=& -\frac{9\widetilde{a}^4}{32}\left[\left(\mu_3(\widetilde{\boldsymbol{\nu}}; \kappa) - \lambda\right)^{-1} \pm \left(\mu_{-3}(\widetilde{\boldsymbol{\nu}}; \kappa) - \lambda\right)^{-1}\right] \\
:=& \sum_{n=0}^{\infty} \eta_{\pm,n}\lambda^n = \sum_{n=0}^{\infty}\left(\sum_{j,k}\eta_{\pm,n,jk}\nu_1^j\nu_2^k\right)\lambda^n,
\end{aligned} \tag{A.0.14}$$

where

$$\eta_{\pm,n} := -\frac{9}{32}\widetilde{a}^4 \left[\mu_3(\widetilde{\boldsymbol{\nu}};\kappa)^{-n-1} \pm \mu_{-3}(\widetilde{\boldsymbol{\nu}};\kappa)^{-n-1}\right],$$

and $\mu_{1,jk}$ and $\eta_{\pm,n,jk}$ are respectively the Taylor coefficients of μ_1 and $\eta_{\pm,n}$ at $\boldsymbol{\nu} = 0$.

We first investigate the two small eigenvalues when $\widetilde{\boldsymbol{\nu}} = 0$. Noting that

$$\rho\mid_{\widetilde{\nu}=0} = \mu_{1,00} = \mu_{1,e}(0;\kappa) = 0, \quad \beta\mid_{\widetilde{\nu}=0} = \mu_{1,o}(0;\kappa) = 0.$$

$$\eta_-\mid_{\widetilde{\nu}=0} = 0, \qquad \eta_+\mid_{\widetilde{\nu}=0} = -\frac{9\widetilde{a}^4}{16}\left(\mu_3(0;\kappa) - \lambda\right)^{-1}$$

From the translation symmetry, it's straightforward to see that $(0, \mathrm{e}^{-\mathrm{i}\xi}u_p')$ is an eigenpair of the operator $\widehat{\mathcal{L}_{p,2}}(\mathbf{e}_1;\varepsilon^2,\kappa)$; that is, one of the small eigenvalues, denoted as $\lambda_{s,1}$, is zero. Moreover, recalling that $\{U_1, U_2\}$ defined in (A.0.10) is given as the shifted basis of the center space and noting that

$$\langle \mathrm{e}^{-\mathrm{i}\xi}u_p', U_1\rangle = 0, \qquad \langle \mathrm{e}^{-\mathrm{i}\xi}u_p', U_2\rangle \neq 0,$$

we conclude that 0 is always an eigenvalue of $M(0;0,\varepsilon^2,\kappa)$ with $\mathbf{e}_2 := (0,1)^T$ as its eigenvector. As a result, we readily see that the second column of $M(0;0,\varepsilon^2,\kappa)$ is always trivial. In addition, from the symmetry of the system, we conclude that the off-diagonal elements of M are always complex conjugate to each other, which yields

$$M(0;0,\varepsilon^2,\kappa) = \begin{pmatrix} -\frac{3}{2}\widetilde{a}^2 + O(\widetilde{a}^4) & 0 \\ 0 & 0 \end{pmatrix}.$$

To derive the expansion of the other small eigenvalue, denoted as λ_2^s, we now have the following expansion

$$M(\lambda;0,\varepsilon^2,\kappa) = \begin{pmatrix} -\frac{3}{2}\widetilde{a}^2 - \lambda + O(\widetilde{a}^4 + \widetilde{a}^4|\lambda|) & 0 \\ 0 & -\lambda + O(\widetilde{a}^4|\lambda|) \end{pmatrix},$$

where we use the parity argument to show that the off-diagonal elements are zero. We now conclude that

$$\lambda_{s,2} = -\frac{3}{2}\widetilde{a}^2 + O(\widetilde{a}^4) < 0,$$

when $|\varepsilon|$ and $|\kappa|$ are sufficiently small.

As a result, we conclude that 0 is a simple eigenvalue of $\widehat{\mathcal{L}_{p,2}}(\mathbf{e}_1;\varepsilon^2,\kappa)$ and the spectral stability problem boils down to the continuation of the eigenvalue zero of $\widehat{\mathcal{L}_{p,2}}(\mathbf{e}_1;\varepsilon^2,\kappa)$ with respect to the small wave-number vector perturbation $\widetilde{\boldsymbol{\nu}}$, which can be derived by plugging in the scheme

$$\lambda_{s,1}(\varepsilon^2,\kappa,\widetilde{\boldsymbol{\nu}}) = a_{20}(\varepsilon^2,\kappa)\widetilde{\nu_1}^2 + a_{02}(\varepsilon^2,\kappa)\widetilde{\nu_2}^2 + a_{04}(\varepsilon^2,\kappa)\widetilde{\nu_2}^4 + O(|\widetilde{\boldsymbol{\nu}}|^4), \tag{A.0.15}$$

into the determinant of M. We note that in the leading order expansion of $\lambda_{s,1}$ we single the $\widetilde{\nu}_2^4$ term out of the $O(|\boldsymbol{\nu}|^4)$ terms since later we will focus on the the zigzag boundary case where $a_{02} = 0$ and the $O(|\boldsymbol{\nu}|^4)$ term becomes dominant. Recalling the leading order expansion of M in (A.0.13) and $\eta_\pm$ in (A.0.14), we derive the expansion of the determinant of M in terms of λ; that is,

$$\det(M) = N_0 + N_1\lambda + N_2\lambda^2 + O(\lambda^3), \tag{A.0.16}$$

where the coefficients N_i admits the following expressions,

$$\begin{aligned}
N_0 =& \left(\mu_{1,e} + m^1_{1,1} + \eta_{+,0}\right)\left(\mu_{1,e} + m^1_{2,2} + \eta_{+,0}\right) - \left(\mu_{1,o} + \eta_{-,0}\right)^2 + O(\widetilde{a}^6),\\
N_1 =& -\left(1 - \eta_{+,1}\right)\left(2\mu_{1,e} + 2\eta_{+,0} + m^1_{1,1} + m^1_{2,2}\right) - 2\eta_{-,1}\left(\mu_{1,o} + \eta_{-,0}\right) + O(\widetilde{a}^6),\\
N_2 =& \eta_{+,2}\left(2\mu_{1,e} + 2\eta_{+,0} + m^1_{1,1} + m^1_{2,2}\right) + \left(1 - \eta_{+,1}\right)^2 - 2\eta_{-,2}\left(\mu_{1,o} + \eta_{-,0}\right) - \eta^2_{-,1} + O(\widetilde{a}^6).
\end{aligned}$$

Taking advantage of $\eta_{\pm,n}$ in terms of $\widetilde{a}$ in (A.0.14), $m^1_{1,1}$ and $m^1_{2,2}$ in terms of $\widetilde{a}$ in (A.0.12), we could further simplify the above expansions of N_i's by only keeping terms up to $O(\widetilde{a}^4)$ in the expansions, yielding

$$\begin{aligned}
N_0 =& \mu^2_{1,e} + \left(-\frac{3}{2}\widetilde{a}^2 - \frac{3}{256}\widetilde{a}^4 + 2\eta_{+,0}\right)\mu_{1,e} - \mu^2_{1,o} - 2\eta_{-,0}\mu_{1,o} + O(\widetilde{a}^5),\\
N_1 =& -2\left(1 - \eta_{+,1}\right)\mu_{1,e} + \frac{3}{2}\widetilde{a}^2 + \frac{3}{256}\widetilde{a}^4 - 2\eta_{+,0} - 2\eta_{-,1}\mu_{1,o} + O(\widetilde{a}^5),\\
N_2 =& 1 - 2\eta_{+,1} + 2\eta_{+,2}\mu_{1,e} - 2\eta_{-,2}\mu_{1,o} + O(\widetilde{a}^5).
\end{aligned} \tag{A.0.17}$$

Moreover, we recall the expansion of $\mu_{1,e}$, $\mu_{1,o}$, $\eta_{\pm,n}$ in terms of $\boldsymbol{\nu}$ in (A.0.14) and derive the expansions of N_i, $i = 1, 2, 3$, in terms of $\boldsymbol{\nu}$; that is,

$$\begin{aligned} N_0 &= A\widetilde{\nu_1}^2 + B\widetilde{\nu_2}^2 + C\widetilde{\nu_2}^4 + O(|\widetilde{\nu_1}|^3 + |\widetilde{\nu_2}|^5 + \widetilde{a}^5), \\ N_1 &= D + E\widetilde{\nu_1}^2 + F\widetilde{\nu_2}^2 + O(|\boldsymbol{\nu}|^3 + \widetilde{a}^5), \\ N_2 &= G + O(|\widetilde{\boldsymbol{\nu}}|^2 + \widetilde{a}^5), \end{aligned} \tag{A.0.18}$$

where the coefficients admits the following expressions,

$$\begin{aligned} A &= \left(-\frac{3}{2}\widetilde{a}^2 - \frac{3}{256}\widetilde{a}^4 + 2\eta_{+,0,00}\right)\mu_{1,20} - \mu_{1,10}^2 - 2\eta_{-,0,10}\mu_{1,10}, \\ B &= \left(-\frac{3}{2}\widetilde{a}^2 - \frac{3}{256}\widetilde{a}^4 + 2\eta_{+,0,00}\right)\mu_{1,02}, \\ C &= \mu_{1,02}^2 + 2\eta_{+,0,02}\mu_{1,02} + \left(-\frac{3}{2}\widetilde{a}^2 - \frac{3}{256}\widetilde{a}^4 + 2\eta_{+,0,00}\right)\mu_{1,04}, \\ D &= -\left(-\frac{3}{2}\widetilde{a}^2 - \frac{3}{256}\widetilde{a}^4 + 2\eta_{+,0,00}\right), \\ E &= -2(1 - \eta_{+,1,00})\mu_{1,20} - 2\eta_{+,0,20} - 2\eta_{-,1,10}\mu_{1,10}, \\ F &= -2(1 - \eta_{+,1,00})\mu_{1,02} - 2\eta_{+,0,02}, \\ G &= 1 - 2\eta_{+,1,00}. \end{aligned} \tag{A.0.19}$$

A.0.5 Coefficients of the Expansion of λ

We plug the scheme (A.0.15) into (A.0.16) and obtain

$$0 = N_0 + N_1(a_{20}\widetilde{\nu_1}^2 + a_{02}\widetilde{\nu_2}^2 + a_{04}\widetilde{\nu_2}^4) + N_2(a_{20}\widetilde{\nu_1}^2 + a_{02}\widetilde{\nu_2}^2 + a_{04}\widetilde{\nu_2}^4)^2 + O(|\boldsymbol{\nu}|^6), \tag{A.0.20}$$

where, collecting up to leading order terms of $\widetilde{\nu_1}^2$, $\widetilde{\nu_2}^2$ and $\widetilde{\nu_2}^4$ in (A.0.20) via expansions in (A.0.18), yields

$$\begin{aligned} &O(\widetilde{\nu_1}^2) : A + Da_{20} + O(\widetilde{a}^5) = 0 \implies a_{20} = -\frac{A + O(\widetilde{a}^5)}{D}, \\ &O(\widetilde{\nu_2}^2) : B + Da_{02} + O(\widetilde{a}^5) = 0 \implies a_{02} = -\frac{B + O(\widetilde{a}^5)}{D}, \\ &O(\widetilde{\nu_2}^4) : C + Fa_{02} + Da_{04} + Ga_{02}^2 + O(\widetilde{a}^5) = 0 \implies a_{04} = -\frac{-CD^2 + FBD - GB^2 + O(\widetilde{a}^5)}{D^3}. \end{aligned}$$

We find by plugging the expressions of A, B, C, D, E, F and G found in (A.0.19) that,

$$a_{20} = \mu_{1,20} - \frac{\mu_{1,10}^2 + 2\eta_{-,0,10}\mu_{1,10} + O(\widetilde{a}^5)}{\left(-\frac{3}{2}\widetilde{a}^2 - \frac{3}{256}\widetilde{a}^4 + 2\eta_{+,0,00}\right)}$$

$$a_{02} = \mu_{1,02} + O(\widetilde{a}^3)$$

$$a_{04} = \mu_{1,04} + \frac{\mu_{1,02}^2 + 2\eta_{+,0,02}\mu_{1,02}}{\left(-\frac{3}{2}\widetilde{a}^2 - \frac{3}{256}\widetilde{a}^4 + 2\eta_{+,0,00}\right)} - \frac{2(1-\eta_{+,1,00})\mu_{1,02}^2 + 2\eta_{+,0,02}\mu_{1,02}}{\left(-\frac{3}{2}\widetilde{a}^2 - \frac{3}{256}\widetilde{a}^4 + 2\eta_{+,0,00}\right)} - \frac{(1-2\eta_{+,1,00})\mu_{1,02}^2}{\left(-\frac{3}{2}\widetilde{a}^2 - \frac{3}{256}\widetilde{a}^4 + 2\eta_{+,0,00}\right)}$$

where a lengthy but straightforward calculation shows the exact expressions of $\mu_{1,jk}$ and $\eta_{\pm,n,jk}$ which are respectively the Taylor coefficients of μ_1 and $\eta_{\pm,n}$ at $\boldsymbol{\nu} = 0$ as

$$\begin{aligned}
\mu_{1,20} &= -2(1+\kappa)(2+3\kappa), \\
\mu_{1,02} &= -2\kappa, \\
\mu_{1,04} &= -1, \\
\mu_{1,10} &= -4\kappa(1+\kappa), \\
\eta_{+,0,00} &= -\frac{9\widetilde{a}^4}{16(\varepsilon^2 - (8+9\kappa)^2)}, \\
\eta_{-,0,10} &= -\frac{27\widetilde{a}^4(1+\kappa)(8+9\kappa)}{4(\varepsilon^2 - (8+9\kappa)^2)^2}, \\
\eta_{-,1,10} &= -\frac{27\widetilde{a}^4(1+\kappa)(8+9\kappa)}{2(\varepsilon^2 - (8+9\kappa)^2)^3}, \\
\eta_{+,0,02} &= -\frac{9\widetilde{a}^4(8+9\kappa)}{8(\varepsilon^2 - (8+9\kappa)^2)^2}, \\
\eta_{+,0,20} &= -\frac{9\widetilde{a}^4(1+\kappa)\left(\varepsilon^2(26+27\kappa) + (8+9\kappa)^2(46+45\kappa)\right)}{8(\varepsilon^2 - (8+9\kappa)^2)^3}, \\
\eta_{+,1,00} &= -\frac{9\widetilde{a}^4}{16(\varepsilon^2 - (8+9\kappa)^2)^2}.
\end{aligned}$$

We then readily compute the Eckhaus boundary by setting $a_{20} = 0$, yielding

$$\kappa_e^{\pm} = \pm\frac{\varepsilon}{\sqrt{3}} + h.o.t.,$$

where $a_{02}(\kappa_e^-) = -2\kappa_e^- + h.o.t > 0$ and $a_{02}(\kappa_e^+) = -2\kappa_e^+ + h.o.t. < 0$. In addition, $a_{20} < 0$ for $\kappa \in (\kappa_e^-, \kappa_e^+)$. On the other hand, the zigzag boundary is given by $a_{02} = 0$, yielding the implicit scheme

$$\kappa_z = O(\widetilde{a}^3),$$

which is sufficient to show that

$$a_{20} = -4 + O(\widetilde{a}^3), \qquad a_{04}(\kappa_z) = -1 + O(\widetilde{a}^4) < 0,$$

which in turns concludes the proof of (2.1.9).

Recalling that 0 is a simple eigenvalue of $\widehat{\mathcal{L}_{p,2}}(\mathbf{e}_1; \varepsilon^2, \kappa)$ and that $\widehat{\mathcal{L}_p}(\boldsymbol{\nu}) = \widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu}) \bigoplus \widehat{\mathcal{L}_{p,2}}(\boldsymbol{\nu} + \mathbf{e}_1)$ as shown in Lemma (A.0.1), we conclude that 0 is a simple eigenvalue of $\widehat{\mathcal{L}_p}(0)$. It is straightforward to see that $\widehat{\mathcal{L}_p}(\mathbf{0})u_p' = 0$ and conclude that $(0, e_0)$ is an eigenpair of $\widehat{\mathcal{L}_p}(0)$, where we recall that $e_0 := \dfrac{u_p'}{\|u_p'\|_2}$. Introducing the eigen-problem functional

$$\begin{array}{rccc} F(\lambda, e, \boldsymbol{\nu}): & \mathbb{R} \times L^2(\mathbb{T}_{2\pi}) \times \mathbb{T}_1 \times \mathbb{R} & \longrightarrow & L^2(\mathbb{T}_{2\pi}) \\ & (\lambda, e, \boldsymbol{\nu}) & \longmapsto & \widehat{\mathcal{L}_p}(\boldsymbol{\nu})e - \lambda e, \end{array}$$

we readily see that

$$F(0, e_0, \mathbf{0}) = 0, \quad \partial_{(\lambda,e)} F(0, e_0, \mathbf{0}) = (-e_0, \widehat{\mathcal{L}}(\mathbf{0})) \text{ is invertible.}$$

As a result, we conclude from the implicit function theorem that there exists $r_0 > 0$ such that for all $|\boldsymbol{\nu}| < r_0$, the eigenpair $(0, e_0)$ admits a unique analytic continuation $(\lambda(\boldsymbol{\nu}), e(\boldsymbol{\nu}; \xi))$ with $e(\boldsymbol{\nu}; \xi) - e_0 = O(|\boldsymbol{\nu}|)$ and $\langle e(\nu; \cdot) - e_0, e_0 \rangle = 0$. We are left to show the parity properties in (2.1.8). We readily see from their Taylor's expansions that the eigen-pair takes the following forms

$$\lambda(\boldsymbol{\nu}) = \lambda_r(\boldsymbol{\nu}) + \mathrm{i}\nu_1 \lambda_i(\boldsymbol{\nu}), \quad e(\boldsymbol{\nu}; \varepsilon) = e_r(\boldsymbol{\nu}; \varepsilon) + \mathrm{i}\nu_1 e_i(\boldsymbol{\nu}; \varepsilon), \tag{A.0.21}$$

where $\lambda_{r\backslash i}(\boldsymbol{\nu})$ and $e_{r\backslash i}(\boldsymbol{\nu})$ are all real-valued and even in ν_1 and ν_2 respectively. Noting that $\lambda(\boldsymbol{\nu}) \in \sigma(\widehat{\mathcal{L}_p}(\boldsymbol{\nu})) \subseteq \mathbb{R}$, we conclude that $\lambda_i(\boldsymbol{\nu}) \equiv 0$ and

$$\lambda(-\nu_1, \nu_2) = \lambda(\nu_1, \nu_2) = \lambda_r(\boldsymbol{\nu}),$$

which, together with Lemma A.0.2 and the uniqueness of the eigen-pair, leads to

$$e(-\nu_1, \nu_2; -\xi) = e(\nu_1, \nu_2; \xi).$$

We rewrite this equality in terms of the real and imaginary formulation of e in (A.0.21), yielding

$$e_r(\nu_1, \nu_2; -\xi) - \mathrm{i}\nu_1 e_i(\nu_1, \nu_2; -\xi) = e_r(\nu_1, \nu_2; \xi) + \mathrm{i}\nu_1 e_i(\nu_1, \nu_2; \xi),$$

that is;

$$e_r(\nu_1, \nu_2; \xi) = e_r(\nu_1, \nu_2; -\xi), \quad e_i(\nu_1, \nu_2; \xi) = -e_i(\nu_1, \nu_2; -\xi),$$

which concludes the proof of the proposition.

Appendix B: Properties of the Bloch Transform

B.1 Proof of Remark 3.1.2

Proof. Let $v_1, v_2 \in L^2(\mathbb{R}^2)$, we will first show that $\mathcal{B}(v_1 v_2) = \mathcal{B}v_1 * \mathcal{B}v_2$. By the definition of the inverse Fourier transform, we have

$$\begin{aligned}
v_1 v_2 &= \left(\int_{\mathbb{R}^2} \widehat{v_1}(\boldsymbol{\nu}) \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\boldsymbol{\nu}} \mathrm{d}\boldsymbol{\nu}\right)\left(\int_{\mathbb{R}^2} \widehat{v_2}(\boldsymbol{\nu}) \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\boldsymbol{\nu}} \mathrm{d}\boldsymbol{\nu}\right) \\
&= \left(\int_{\mathbb{R}^2} \widehat{v_1}(\boldsymbol{\nu} - \widetilde{\boldsymbol{\nu}}) \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot(\boldsymbol{\nu}-\widetilde{\boldsymbol{\nu}})} \mathrm{d}\boldsymbol{\nu}\right)\left(\int_{\mathbb{R}^2} \widehat{v_2}(\widetilde{\boldsymbol{\nu}}) \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot(\widetilde{\boldsymbol{\nu}})} \mathrm{d}\boldsymbol{\nu}\right) \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \widehat{v_1}(\boldsymbol{\nu} - \widetilde{\boldsymbol{\nu}}) \widehat{v_2}(\widetilde{\boldsymbol{\nu}}) d\widetilde{\boldsymbol{\nu}}\right) \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\boldsymbol{\nu}} \mathrm{d}\boldsymbol{\nu} \\
&= \int_{\mathbb{R}^2} \widehat{v_1} * \widehat{v_2}(\boldsymbol{\nu}) \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\boldsymbol{\nu}} \mathrm{d}\boldsymbol{\nu} \\
&= \overbrace{\widehat{v_1} * \widehat{v_2}},
\end{aligned}$$

which, combined with the definition of the Bloch transform of $v_1 v_2$, yields

$$\begin{aligned}
\mathcal{B}(v_1 v_2)(\boldsymbol{\nu}) &= \sum_{k\in\mathbb{Z}} \widehat{(v_1 v_2)}(\nu_1 + k, \nu_2) \mathrm{e}^{\mathrm{i}k\xi} = \sum_{k\in\mathbb{Z}} (\widehat{v_1} * \widehat{v_2})(\nu_1 + k, \nu_2) \mathrm{e}^{\mathrm{i}k\xi} \\
&= \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \int_{\mathbb{T}_1\times\mathbb{R}} \widehat{v_1}(\nu_1 + k - \omega_1 - j, \nu_2 - \omega_2) \widehat{v_2}(\omega_1 + j, \omega_2) \mathrm{d}\boldsymbol{\omega} \right) \mathrm{e}^{\mathrm{i}(k-j)\xi} \mathrm{e}^{\mathrm{i}j\xi} \\
&= \int_{\mathbb{T}_1\times\mathbb{R}} \left[\sum_{j\in\mathbb{Z}} \widehat{v_2}(\omega_1 + j, \omega_2) \mathrm{e}^{\mathrm{i}\xi\cdot j} \left(\sum_{k\in\mathbb{Z}} \widehat{v_1}(\nu_1 + k - \omega_1 - j, \nu_2 - \omega_2) \mathrm{e}^{\mathrm{i}\xi\cdot(k-j)} \right) \right] \mathrm{d}\boldsymbol{\omega} \\
&= \int_{\mathbb{T}_1\times\mathbb{R}} (\mathcal{B}v_1)(\boldsymbol{\nu} - \boldsymbol{\omega})(\mathcal{B}v_2)(\boldsymbol{\omega}) \mathrm{d}\boldsymbol{\omega} \\
&= (\mathcal{B}v_1 * \mathcal{B}v_2)(\boldsymbol{\nu}).
\end{aligned}$$

Next, we will show that $\mathcal{B}(uv_1) = u\mathcal{B}v_1$ for any $u \in L^2(\mathbb{T}_{2\pi})$, $v_1 \in L^2(\mathbb{R}^2)$. By definition of the Fourier transform, we have

$$\begin{aligned}
\widehat{uv_1}(\boldsymbol{\nu}) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} u(x_1)v_1(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\boldsymbol{\nu}}\mathrm{d}\mathbf{x} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} (\sum_{j\in\mathbb{Z}} u_j \mathrm{e}^{\mathrm{i}jx_1})v_1(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\boldsymbol{\nu}}\mathrm{d}\mathbf{x} \\
&= \frac{1}{(2\pi)^2}\sum_{j\in\mathbb{Z}} u_j \int_{\mathbb{R}^2} v_1(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\boldsymbol{\nu}-j\mathbf{e}_1)}\mathrm{d}\mathbf{x} \\
&= \sum_{j\in\mathbb{Z}} u_j \widehat{v_1}(\boldsymbol{\nu} - j) = (\widehat{\underline{u}} * \widehat{v_1})(\boldsymbol{\nu}).
\end{aligned}$$

Now by the definition of Bloch transform of uv_1,

$$\begin{aligned}
\mathcal{B}(uv_1)(\boldsymbol{\nu}) &= \sum_{k\in\mathbb{Z}} (\widehat{uv_1})(\nu_1 + k, \nu_2)\mathrm{e}^{\mathrm{i}k\xi} \\
&= \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} u_j \widehat{v_1}(\nu_1 + k - j, \nu_2)\mathrm{e}^{\mathrm{i}(k-j+j)\xi} \right) \\
&= \sum_{j\in\mathbb{Z}} u_j \mathrm{e}^{\mathrm{i}j\xi} \left(\sum_{k\in\mathbb{Z}} \widehat{v_1}(\nu_1 + k - j, \nu_2)\mathrm{e}^{\mathrm{i}(k-j)\xi} \right) \\
&= u\mathcal{B}(v_1)(\boldsymbol{\nu}).
\end{aligned}$$

□

Appendix C: Proof of Proposition 3.1.2: Sectoriality of the Linearized Operator $\widehat{\mathcal{L}_d}$ in Discrete Bloch-Fourier Space

We recall $\mu_j(\boldsymbol{\nu};\kappa) = -(1-(j+\nu_1)^2-\nu_2^2)^2+\kappa^2$ as in (3.1.4), denote $h := -3u_p^2$, and rewrite the operator in the form

$$\widehat{\mathcal{L}_d} = L_0 + H,$$

where

$$\begin{aligned} L_0 : \quad w^{4,p} &\longrightarrow \ell^p \\ \underline{u} &\longmapsto \{\left(\mu_j(\boldsymbol{\nu};\kappa)+\varepsilon^2-\kappa^2\right)u_j\}_{j\in\mathbb{Z}}, \end{aligned}$$

and

$$\begin{aligned} H : \quad \ell^p &\longrightarrow \ell^p \\ \underline{u} &\longmapsto \underline{\widehat{h}} * \underline{u}. \end{aligned}$$

We only need to show that the proposition holds for L_0 and adding H does not alter these properties. The closedness of L_0 follows from the fact that the $w^{4,p}$ norm and the graph norm of L_0 are equivalent. Noting that L_0 is a multiplication operator, we have the spectrum of L_0 independent of p; that is,

$$\sigma(L_0) = \{\mu_j(\boldsymbol{\nu};\kappa)+\varepsilon^2-\kappa^2\}_{j\in\mathbb{Z}},$$

and, for any $\lambda \in \rho(L_0) = \mathbb{C}\backslash\sigma(L_0)$,

$$|||(L_0-\lambda)^{-1}|||_{\ell^p} \leqslant \frac{1}{\operatorname{dist}(\lambda,\sigma(L_0))}, \quad \text{for any } p \in [1,+\infty].$$

In addition, $(L_0-\lambda)^{-1} : \ell^p \to w^{4,p}$ is bounded and the inclusion $w^{4,p} \hookrightarrow \ell^p$ is compact, so the resolvent of L_0 is always compact and thus the spectrum of L_0 only consists of eigenvalues. Denoting $\mu_{max} := \sup_{\boldsymbol{\nu}\in\mathbb{T}_1\times\mathbb{R}} \max_{j\in\mathbb{Z}}\{\mu_j(\boldsymbol{\nu};\kappa)+\varepsilon^2-\kappa^2\}$ and fixing $\omega \in (\pi/2,\pi)$, we introduce the sector

$$S(\mu_{max},\omega) := \left\{\lambda\in\mathbb{C}\;\middle|\;|\arg(\lambda-\mu_{max})| < \omega, \lambda \neq \mu_{max}\right\},$$

and readily derive that, for any $\lambda \in S(\mu_{max}, \omega)$,

$$|||(L_0 - \lambda)^{-1}|||_{\ell^p} \leqslant \frac{1}{(\sin\omega)|\lambda - \mu_{max}|}, \quad \text{for any } p \in [1, +\infty].$$

As a result, we conclude that all properties in the proposition hold for L_0.

On the other hand, we note that $H : \ell^p \to \ell^p$ is bounded, uniformly with respect to $p \in [1, \infty]$; that is,

$$\left\|H\underline{u}\right\|_{\ell^p} \leqslant \left\|\widehat{\underline{h}}\right\|_{\ell^1} \left\|\underline{u}\right\|_{\ell^p}, \quad \text{for any } \underline{u} \in \ell^p, p \in [1, +\infty].$$

Choosing

$$\lambda_0 := \mu_{max} + \frac{2\|\widehat{\underline{h}}\|_{\ell^1}}{\sin\omega},$$

we have that, for any $\lambda \in S(\lambda_0, \omega) := \{\lambda \in \mathbb{C} \,\big|\, |\arg(\lambda - \lambda_0)| < \omega, \lambda \neq \lambda_0\} \subset S(\mu_{max}, \omega)$,

$$|||H(L_0 - \lambda)^{-1}|||_{\ell^p} \leqslant |||H|||_{\ell^p} |||(L_0 - \lambda)^{-1}|||_{\ell^p} \leqslant \|\widehat{\underline{h}}\|_{\ell^1} \frac{1}{2\|\widehat{\underline{h}}\|_{\ell^1}} = \frac{1}{2}, \quad \text{for any } p \in [1, +\infty],$$

and thus $\widehat{\mathcal{L}_d} - \lambda$ is invertible with compact resolvent whose operator norm admits the following estimate.

$$|||(\widehat{\mathcal{L}_d} - \lambda)^{-1}|||_{\ell^p} = |||(L_0 - \lambda)^{-1}\left(I + H(L_0 - \lambda)^{-1}\right)^{-1}|||_{\ell^p} \leqslant \frac{2}{(\sin\omega)|\lambda - \lambda_0|}, \quad \text{for any } p \in [1, +\infty].$$

We are left to show that the spectrum of $\widehat{\mathcal{L}_d} : \ell^p \to \ell^p$, denoted for now as $\sigma(\widehat{\mathcal{L}_d}, p)$, is independent of the choice of p. For any $p, q \in [1, \infty]$, if $\lambda_* \in \sigma(\widehat{\mathcal{L}_d}, p)$, then λ is an eigenvalue for $\widehat{\mathcal{L}_d} : \ell^p \to \ell^p$ and admits an eigenvector $\underline{u}_* \in \mathcal{D}(\widehat{\mathcal{L}_d}) = w^{4,p} \subset \ell^q$. Moreover, given any $\lambda \in S(\lambda_0, \omega)$, we have

$$\underline{u}_* = (\lambda_* - \lambda)(\widehat{\mathcal{L}_d} - \lambda)^{-1}\underline{u}_* \in w^{4,q},$$

and thus $\lambda_* \in \sigma(\widehat{\mathcal{L}_d}, q)$ with $\underline{u}_*$ as its eigenfunction. As a result, we have $\sigma(\widehat{\mathcal{L}_d}, p) \subseteq \sigma(\widehat{\mathcal{L}_d}, q)$, for any $p, q \in [1, \infty]$; that is, equivalently, $\sigma(\widehat{\mathcal{L}_d}, p) = \sigma(\widehat{\mathcal{L}_d}, q)$, for any $p, q \in [1, \infty]$, which concludes the proof.

APPENDIX D: ESTIMATES OF I_s

In the proof of the estimate (3.2.7) of $\mathcal{T}_2(W)$ in Propsition 3.2.2, we exploited the estimate (3.2.16); that is,

$$\|\mathcal{T}_2(W)\|_{\mathbf{H}} \leqslant \overbrace{\left(\left\|\int_0^t M_{11}(t-s)N_c(W(s))\mathrm{d}s\right\|_{\mathbf{H}_c} + \left\|\int_0^t M_{21}(t-s)N_c(W(s))\mathrm{d}s\right\|_{\mathbf{H}_s}\right)}^{:=I_c} +$$
$$\overbrace{\left(\left\|\int_0^t M_{12}(t-s)N_s(W(s))\mathrm{d}s\right\|_{\mathbf{H}_c} + \left\|\int_0^t M_{22}(t-s)N_s(W(s))\mathrm{d}s\right\|_{\mathbf{H}_s}\right)}^{:=I_s},$$

where we discuss the derivation of the estimate (3.2.23) of I_c in details. We give the estimates of I_s in this section.

<u>*Estimate of I_s*</u> We evaluate I_s for small and large ν respectively; that is,

$$\begin{aligned} I_s =& \left\|\int_0^t M_{12}(t-s)N_s(W(s))\mathrm{d}s\right\|_{\mathbf{H}_c} + \left\|\int_0^t M_{22}(t-s)N_s(W(s))\mathrm{d}s\right\|_{\mathbf{H}_s} \\ \leqslant& \overbrace{\left\|\int_0^t (1-\chi_{\frac{r_1}{2}})M_{12}(t-s)N_s(W(s))\mathrm{d}s\right\|_{\mathbf{H}_c}}^{:=I_{s,1}} + \overbrace{\left\|\int_0^t \chi_{\frac{r_1}{2}}M_{22}(t-s)N_s(W(s))\mathrm{d}s\right\|_{\mathbf{H}_c}}^{:=I_{s,2}} + \\ & \overbrace{\left\|\int_0^t (1-\chi_{\frac{r_1}{2}})M_{22}(t-s)N_s(W(s))\mathrm{d}s\right\|_{\mathbf{H}_s}}^{:=I_{s,3}}. \end{aligned} \tag{D.0.1}$$

where we use the fact that $\chi_{\frac{r_1}{2}} M_{12} = 0$. Moreover, recalling the definition of $\|\cdot\|_{\mathbf{H}_c}$ and $\|\cdot\|_{\mathbf{H}_s}$, we have

$$
\begin{aligned}
I_{s,1} &\leqslant \overbrace{\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}} \int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{12}(t-s)N_s(W(s))\right\|_1 \mathrm{d}s}^{:=A_{s,1}} + \overbrace{\sup_{t\geqslant 0} \int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{12}(t-s)N_s(W(s))\right\|_\infty \mathrm{d}s}^{:=B_{s,1}} + \\
&\quad \overbrace{\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}} \int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{12}(t-s)N_s(W(s))\right\|_1 \mathrm{d}s}^{:=C_{s,1}}, \\
I_{s,2} &\leqslant \overbrace{\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}} \int_0^t \left\|\chi_{\frac{r_1}{2}} M_{22}(t-s)N_s(W(s))\right\|_1 \mathrm{d}s}^{:=A_{s,2}} + \\
&\quad \overbrace{\sup_{t\geqslant 0} \int_0^t \left\|\chi_{\frac{r_1}{2}} M_{22}(t-s)N_s(W(s))\right\|_\infty \mathrm{d}s}^{:=B_{s,2}} + \\
&\quad \overbrace{\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}} \int_0^t \left\|\nu_1\chi_{\frac{r_1}{2}} M_{22}(t-s)N_s(W(s))\right\|_1 \mathrm{d}s}^{:=C_{s,2}}, \\
I_{s,3} &\leqslant \overbrace{\sup_{t\geqslant 0}(1+t)^{\frac{3}{2}} \int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{22}(t-s)N_c(W(s))\right\|_1 \mathrm{d}s}^{:=D_{s,3}} + \\
&\quad \overbrace{\sup_{t\geqslant 0} \int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{22}(t-s)N_s(W(s))\right\|_\infty \mathrm{d}s}^{:=E_{s,3}}.
\end{aligned}
\tag{D.0.2}
$$

In other words, we have

$$
I_s \leqslant \sum_{j=1}^{3} I_{s,j} \leqslant \sum_{j=1}^{2} \left(A_{s,j} + B_{s,j} + C_{s,j}\right) + D_{s,3} + E_{s,3}. \tag{D.0.3}
$$

We are left to estimate all the terms in the right hand side of (D.0.3). Taking advantage of the neutral mode estimate (3.1.15) and the estimate (3.2.9b) of N_s, we have

$$
\begin{aligned}
A_{s,1} =& \sup_{t\geqslant 0}(1+t)^{\frac{3}{4}} \int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{12}(t-s)N_s(W(s))\right\|_1 \mathrm{d}s \\
\leqslant& C\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}} \int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{12}(t-s)\right\|_{L^1\to L^1} \left\|N_s(W(s))\right\|_1 \mathrm{d}s \\
\overset{(3.1.15),\,(3.2.9b)}{\leqslant}& C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}}\left(\int_0^t \mathrm{e}^{-\lambda_2(t-s)}(1+s)^{-\frac{3}{2}}\mathrm{d}s\right) \\
\leqslant& C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}}\left(\mathrm{e}^{-\frac{\lambda_2 t}{2}}\int_0^{t/2}(1+s)^{-\frac{3}{2}}\mathrm{d}s+(1+t/2)^{-\frac{3}{2}}\int_{t/2}^t \mathrm{e}^{-\lambda_2(t-s)}\mathrm{d}s\right) \\
\leqslant& C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right); \\
B_{s,1} =& \sup_{t\geqslant 0}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{12}(t-s)N_s(W(s))\right\|_\infty \mathrm{d}s \\
\leqslant& \sup_{t\geqslant 0}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{12}(t-s)\right\|_{L^\infty\to L^\infty}\left\|N_s(W(s))\right\|_\infty \mathrm{d}s \\
\overset{(3.1.15),\,(3.2.9b)}{\leqslant}& C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}\left(\int_0^t \mathrm{e}^{-\lambda_2(t-s)}(1+s)^{-\frac{3}{4}}\mathrm{d}s\right) \\
\leqslant& C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}\left(\mathrm{e}^{-\frac{\lambda_2 t}{2}}\int_0^{t/2}(1+s)^{-\frac{3}{4}}\mathrm{d}s+(1+t/2)^{-\frac{3}{4}}\int_{t/2}^t \mathrm{e}^{-\lambda_2(t-s)}\mathrm{d}s\right) \\
\leqslant& C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right); \\
C_{s,1} =& \sup_{t\geqslant 0}(1+t)^{\frac{5}{4}} \int_0^t \left\|\nu_1(1-\chi_{\frac{r_1}{2}})M_{12}(t-s)N_c(W(s))\right\|_1 \mathrm{d}s \\
\leqslant& C\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}} \int_0^t \left\|\nu_1(1-\chi_{\frac{r_1}{2}})M_{12}(t-s)\right\|_{L^1\to L^1}\left\|N_s(W(s))\right\|_1 \mathrm{d}s \\
\overset{(3.1.15),\,(3.2.9b)}{\leqslant}& C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}}\left(\int_0^t \mathrm{e}^{-\lambda_2(t-s)}(1+s)^{-\frac{3}{2}}\mathrm{d}s\right) \\
\leqslant& C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}}\left(\mathrm{e}^{-\frac{\lambda_2 t}{2}}\int_0^{t/2}(1+s)^{-\frac{3}{2}}\mathrm{d}s+(1+t/2)^{-\frac{3}{2}}\int_{t/2}^t \mathrm{e}^{-\lambda_2(t-s)}\mathrm{d}s\right) \\
\leqslant& C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right),
\end{aligned}
\tag{D.0.4}
$$

Similarly, taking advantage of the estimates (3.1.14) and (3.2.9b), we have

$$
\begin{aligned}
A_{s,2} &= \sup_{t\geqslant 0}(1+t)^{\frac{3}{4}}\int_0^t \left\|\chi_{\frac{r_1}{2}} M_{22}(t-s)N_s(W(s))\right\|_1 \mathrm{d}s\\
&\leqslant C\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}}\int_0^t \left\|\chi_{\frac{r_1}{2}} M_{22}(t-s)\right\|_{L^1\to L^1}\left\|N_s(W(s))\right\|_1 \mathrm{d}s\\
&\overset{(3.1.14),\,(3.2.9b)}{\leqslant} C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}}\left(\int_0^t \mathrm{e}^{-\lambda_1(t-s)}(1+s)^{-\frac{3}{2}}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{3}{4}}\left(\mathrm{e}^{-\frac{\lambda_1 t}{2}}\int_0^{t/2}(1+s)^{-\frac{3}{2}}\mathrm{d}s+(1+t/2)^{-\frac{3}{2}}\int_{t/2}^t \mathrm{e}^{-\lambda_1(t-s)}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right);\\
B_{s,2} &= \sup_{t\geqslant 0}\int_0^t \left\|\chi_{\frac{r_1}{2}} M_{22}(t-s)N_s(W(s))\right\|_\infty \mathrm{d}s\\
&\leqslant \sup_{t\geqslant 0}\int_0^t \left\|\chi_{\frac{r_1}{2}} M_{22}(t-s)\right\|_{L^\infty\to L^\infty}\left\|N_s(W(s))\right\|_\infty \mathrm{d}s\\
&\overset{(3.1.14),\,(3.2.9b)}{\leqslant} C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}\left(\int_0^t \mathrm{e}^{-\lambda_1(t-s)}(1+s)^{-\frac{3}{4}}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}\left(\mathrm{e}^{-\frac{\lambda_1 t}{2}}\int_0^{t/2}(1+s)^{-\frac{3}{4}}\mathrm{d}s+(1+t/2)^{-\frac{3}{4}}\int_{t/2}^t \mathrm{e}^{-\lambda_1(t-s)}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right);\\
C_{s,2} &= \sup_{t\geqslant 0}(1+t)^{\frac{5}{4}}\int_0^t \left\|\nu_1\chi_{\frac{r_1}{2}} M_{22}(t-s)N_c(W(s))\right\|_1 \mathrm{d}s\\
&\leqslant C\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}}\int_0^t \left\|\nu_1\chi_{\frac{r_1}{2}} M_{22}(t-s)\right\|_{L^1\to L^1}\left\|N_s(W(s))\right\|_1 \mathrm{d}s\\
&\overset{(3.1.14),\,(3.2.9b)}{\leqslant} C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}}\left(\int_0^t \mathrm{e}^{-\lambda_1(t-s)}(1+s)^{-\frac{3}{2}}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{5}{4}}\left(\mathrm{e}^{-\frac{\lambda_1 t}{2}}\int_0^{t/2}(1+s)^{-\frac{3}{2}}\mathrm{d}s+(1+t/2)^{-\frac{3}{2}}\int_{t/2}^t \mathrm{e}^{-\lambda_1(t-s)}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right),
\end{aligned}
\tag{D.0.5}
$$

At last, taking advantage of the estimates (3.1.15) and (3.2.9b) again, we have

$$
\begin{aligned}
D_{s,3} &= \sup_{t\geqslant 0}(1+t)^{\frac{3}{2}}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{22}(t-s)N_s(W(s))\right\|_1 \mathrm{d}s\\
&\leqslant C\sup_{t\geqslant 0}(1+t)^{\frac{3}{2}}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{22}(t-s)\right\|_{L^1\to L^1}\left\|N_s(W(s))\right\|_1 \mathrm{d}s\\
&\overset{(3.1.15),\,(3.2.9b)}{\leqslant} C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{3}{2}}\left(\int_0^t \mathrm{e}^{-\lambda_2(t-s)}(1+s)^{-\frac{3}{2}}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}(1+t)^{\frac{3}{2}}\left(\mathrm{e}^{-\frac{\lambda_2 t}{2}}\int_0^{t/2}(1+s)^{-\frac{3}{2}}\mathrm{d}s+(1+t/2)^{-\frac{3}{2}}\int_{t/2}^t \mathrm{e}^{-\lambda_2(t-s)}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right);\\
E_{s,3} &= \sup_{t\geqslant 0}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{22}(t-s)N_s(W(s))\right\|_\infty \mathrm{d}s\\
&\leqslant C\sup_{t\geqslant 0}\int_0^t \left\|(1-\chi_{\frac{r_1}{2}})M_{22}(t-s)\right\|_{L^\infty\to L^\infty}\left\|N_s(W(s))\right\|_\infty \mathrm{d}s\\
&\overset{(3.1.14),\,(3.2.9b)}{\leqslant} C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}\left(\int_0^t \mathrm{e}^{-\lambda_2(t-s)}(1+s)^{-\frac{3}{4}}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right)\sup_{t\geqslant 0}\left(\mathrm{e}^{-\frac{\lambda_2 t}{2}}\int_0^{t/2}(1+s)^{-\frac{3}{4}}\mathrm{d}s+(1+t/2)^{-\frac{3}{4}}\int_{t/2}^t \mathrm{e}^{-\lambda_2(t-s)}\mathrm{d}s\right)\\
&\leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right).
\end{aligned}
\tag{D.0.6}
$$

Combining (D.0.3), (D.0.4), (D.0.5) and (D.0.6), we conclude that

$$
I_s \leqslant C\left(\|W\|_{\mathbf{H}}^2+\|W\|_{\mathbf{H}}^3\right). \tag{D.0.7}
$$

Appendix E: Maximal Regularity Estimates

In this section, we develop the tools of maximal regularity to find optimal bounds for the $\mathcal{N}(W)$ terms involving the quasilinear term ψ_{yyyy}.

We consider the IVP problem

$$\begin{cases} u_t(t,y) = \mathcal{A}_1 u(t,y) + f(t,y), \\ u(0,y) = u_0. \end{cases} \tag{E.0.1}$$

where $\mathcal{A}_1 = \partial_y^4$, $(t,y) \in [0,\infty) \times \mathbb{R}$ and $u \in \mathbb{R}$. The solution to this problem is given by the variation of constant formula

$$u_s(t,y) = e^{-\mathcal{A}_1 t} u_0 + \int_0^t e^{-\mathcal{A}_1 (t-s)} f(s,y) \mathrm{d}s, \tag{E.0.2}$$

where the nonlinear convolutional solution is given by

$$u(t,y) = \int_0^t e^{-\mathcal{A}_1 (t-s)} f(s,y) \mathrm{d}s, \tag{E.0.3}$$

We have the following maximal regularity results about the solution u.

Lemma E.0.1. *Consider the IVP problem* (E.0.1)*, for any $r \in (1,\infty)$ and $t \geqslant 0$, if $f \in L^r\left((0,t), W^{4,2}(\mathbb{R})\right)$ and $u(t,y)$ is given as in* (E.0.3)*, then there exist a $C > 0$ such that*

$$\|\mathcal{A}_1 u\|_{L^r\left((0,t),L^2(\mathbb{R})\right)} \leqslant C\|f\|_{L^r\left((0,t),W^{4,2}(\mathbb{R})\right)}. \tag{E.0.4}$$

Proof. It suffices to show that $\mathcal{A}_1$ has maximal L^2 regularity property, and conclude from Proposition 2.4 in Monniaux [12] that it will be true for L^r for any $r \in (1,\infty)$. We note that the operator $\mathcal{A}_1$ is a generalized Laplacian and its spectrum is given by

$$\sigma(\mathcal{A}_1) = (-\infty, 0].$$

There exists an $M > 0$ such that the resolvent operator

$$\|R(\lambda, A)\| \leqslant \frac{M}{\lambda},$$

for some λ in the resolvent set. The operator $\mathcal{A}_1$ is a densely defined since its domain $W^{4,2}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. closed operator. Moreover, $\mathcal{A}_1$ is a closed operator since its graph norm is norm equivalent to $W^{4,2}(\mathbb{R})$ norm. From the above properties of $\mathcal{A}_1$, we infer from the Hille-Yosida semigroup theorem to conclude that $-\mathcal{A}_1$ generates a bounded analytic semigroup.

Let $f \in L^2\Big((0,t), L^2(\mathbb{R})\Big)$ and u(t,y) be given by the convolutional term in (E.0.3). Then we have

$$\mathcal{A}_1 u(t,y) = \int_0^t \mathcal{A}_1 e^{-\mathcal{A}_1(t-s)} f(s,y)\mathrm{d}s. \tag{E.0.5}$$

We extend the integral in (E.0.5) to $\mathbb{R}$ by letting

$$\begin{cases} f(t,y) = 0 \quad \text{if } t < 0, \\ l(t) = \mathcal{A}_1 e^{-\mathcal{A}_1 t} \quad \text{if } t > 0, \\ l(t) = 0 \quad \text{if } t \leqslant 0. \end{cases}$$

With this extension, (E.0.5) becomes

$$\mathcal{A}_1 u(t,y) = \int_{\mathbb{R}} l(t-s) f(s,y)\mathrm{d}s. \tag{E.0.6}$$

We take the Fourier transform in t and have for any $\lambda \in \mathbb{R}$

$$\begin{aligned} \widehat{\mathcal{A}_1 u(\lambda, y)} &= \int_{\mathbb{R}} e^{-it\lambda} \mathcal{A}_1 u(t,y) dt \\ &\overset{(E.0.6)}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it\lambda} l(t-s) f(s,y)\mathrm{d}s\mathrm{d}t \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(t+s)\lambda} l(t) f(s,y)\mathrm{d}s\mathrm{d}t \\ &= \left(\int_0^\infty e^{-it\lambda} e^{-\mathcal{A}_1 t}\mathrm{d}t\right)\left(\int_{\mathbb{R}} e^{-is\lambda}\mathcal{A}_1 f(s,y)\mathrm{d}s\right) \\ &= (i\lambda + \mathcal{A}_1)^{-1}\mathcal{A}_1 \widehat{f(\lambda, y)}. \end{aligned}$$

Since $-\mathcal{A}_1$ generates a bounded analytic semigroup, we have

$$\sup_{\lambda \in \mathbb{R}} \|(i\lambda + \mathcal{A}_1)^{-1}\mathcal{A}_1\| < \infty.$$

Hence

$$\|\widehat{\mathcal{A}_1 u}\|_{L^2} \leqslant C\|\widehat{f}\|_{L^2}.$$

We infer from Plancherel's theorem that

$$\|\mathcal{A}_1 u\|_{L^2} \leqslant C\|f\|_{L^2}.$$

This proves that $\mathcal{A}$ has maximal regularity for $L^2\Big((0,t), W^{4,2}(\mathbb{R})\Big)$ and hence maximal regularity for $L^r\Big((0,t), W^{4,2}(\mathbb{R})\Big)$.

□

Lemma E.0.2. *For any $r_0 \in \mathbb{R}$ and $r \in (1,\infty)$, If*

$$u(t,y) = \int_{t-1}^{t} e^{-\mathcal{A}_1(t-s)} f(s,y)\mathrm{d}s \ \ \textit{for all } t \geqslant 1,$$

then there exists $C > 0$ such that

$$\int_1^\infty (1+t)^{r_0}\|\mathcal{A}_1 u\|^r_{L^2(\mathbb{R})} dt \leqslant C\int_0^\infty (1+t)^{r_0}\|f\|^r_{L^2(\mathbb{R})}\mathrm{d}t \quad \textit{holds} . \tag{E.0.7}$$

Proof. Suppose u is as given in (E.0.7). Then for $t \in [n, n+1)$ and $n \geqslant 1$,

$$\begin{aligned} u(t,y) &= \Big(\int_{n-1}^{t} - \int_{n-1}^{t-1}\Big)\mathrm{e}^{-\mathcal{A}_1(t-s)} f(s,y)\mathrm{d}s, \\ &= \Big(\int_0^{t-n+1} - \int_0^{t-n}\Big)\mathrm{e}^{-\mathcal{A}_1(t-n+1-s)} f(s+n-1,y)\mathrm{d}s, \\ &= u_1(t,y) - u_2(t,y). \end{aligned} \tag{E.0.8}$$

We claim applying the estimate (E.0.4) in Lemma E.0.1 to (E.0.8) gives

$$\int_n^{n+1} \|\mathcal{A}_1 u\|^r_{L^2(\mathbb{R})} dt \leqslant C\int_{n-1}^{n+1} (\|f\|_{L^2(\mathbb{R})})^r \mathrm{d}t. \tag{E.0.9}$$

To show this claim, we take advantage of the fact the solution $u_1(t,y)$ in (E.0.8) can be written in terms of convolution as

$$u_1(t,y) = G *_{t,y} f_{n-1}(t)$$

where

$$f_{n-1}(t) := f(t+n-1, y) \ \text{ and } \ G(t) := \mathrm{e}^{-\mathcal{A}_1(t-n+1)}.$$

With this form of $u_1(t, y)$, we have

$$\begin{aligned}
\int_n^{n+1} \|\mathcal{A}_1 u_1\|_{L^2}^r dt &= \int_n^{n+1} \left\|\mathcal{A}_1(G *_{t,y} f_{n-1})(t-n+1, y)\right\|_{L^2}^r \mathrm{d}t \\
&\leqslant \int_{n-1}^{n+1} \left\|\mathcal{A}_1(G *_{t,y} f_{n-1})(t-n+1, y)\right\|_{L^2}^r \mathrm{d}t \\
&\overset{\tilde{t}=t-n-1}{=} \int_0^2 \left\|\mathcal{A}_1(G *_{\tilde{t}+n-1,y} f_{n-1})(\tilde{t}, y)\right\|_{L^2}^r d\tilde{t} \\
&\overset{(E.0.4)}{\leqslant} C \int_0^2 \|f_{n-1}(\tilde{t}, \cdot)\|_{L^2}^r d\tilde{t} \\
&= C \int_{n-1}^{n+1} \|f(t, \cdot)\|_{L^2}^r \mathrm{d}t.
\end{aligned}$$

This proves the claim is true for u_1.

Next we look at $u_2(t, y)$ which is given in (E.0.8) as

$$u_2(t, y) = \int_0^{t-n} e^{-\mathcal{A}_1(t-n+1-s)} f(s+n-1, y) \mathrm{d}s.$$

We let $h(s, y) = \mathrm{e}^{-\mathcal{A}_1} f(s+n-1, y)$ and write

$$u_2(t, y) = \int_0^{t-n} \mathrm{e}^{-\mathcal{A}_1(t-n-s)} h(s, y) \mathrm{d}s.$$

We rewrite the solution in the convolutional form

$$u_2(t, y) = G *_{t,y} h(t-n, y), \quad \text{where } G(t) := \mathrm{e}^{-\mathcal{A}_1(t-n)}.$$

With this form of $u_2(t,y)$, we have

$$\begin{aligned}\int_n^{n+1} \|\mathcal{A}_1 u_1\|_{L^2}^r dt &= \int_n^{n+1} \|\mathcal{A}_1(G *_{t,y} h)(t-n,y)\|_{L^2}^r \mathrm{d}t \\ &\overset{\tau=t-n}{=} \int_0^1 \|\mathcal{A}_1(G *_{\tau+n,y} h)(\tau,y)\|_{L^2}^r \mathrm{d}\tau \\ &\overset{(E.0.4)}{\leqslant} C\int_0^1 \|h(\tau,\cdot)\|_{L^2}^r \mathrm{d}\tau \\ &= \int_n^{n+1} \|e^{\mathcal{A}_1} f(t-n+n-1,\cdot)\|_{L^2}^r \mathrm{d}t \\ &\leqslant C\int_n^{n+1} \|f(t-1,\cdot)\|_{L^2}^r \mathrm{d}t \\ &\leqslant C\int_{n-1}^{n+1} \|f(t,\cdot)\|_{L^2}^r \mathrm{d}t.\end{aligned}$$

This proves the claim is true for u_2 and hence u.

Now if we multiply both sides of the claim in (E.0.9) by $n^{r_0} \sim (1+t)^{r_0}$ and sum over all $n \geqslant 1$, we get the results we desire

$$\int_1^\infty (1+t)^{r_0}\|\mathcal{A}_1 u\|_{L^2(\mathbb{R})}^r dt \leqslant C\int_0^\infty (1+t)^{r_0}(\|f\|_{L^2(\mathbb{R})})^r \mathrm{d}t.$$

□