Algebraic Structures on the Set of all Binary Operations over a Fixed Set

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Abstract

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The word magma is often used to designate a pair of the form $(S, *)$ where $*$ is a binary operation on the set S. We use the notation $\mathcal{M}(S)$ (the magma of S) to denote the set of all binary operations on the set S (i.e. all magmas with underlying set S.) Our work on this set was motivated initially by an intention to better understand the distributivity relation among operations over a fixed set; however, our research has yielded structural and combinatoric questions that are interesting in their own right.

Given a set S, its (left, right, two-sided) hierarchy graph is the directed graph that has $\mathcal{M}(S)$ as its set of vertices and such that there is an edge from an operation $*$ to another one $\circ$ if $*$ distributes over $\circ$ (on the left, right, or both sides.) The graph-theoretic setting allows us to describe easily various interesting algebraic scenarios. We study several combinatorial properties of hierarchy graphs including the length of a longest cycle-free path in the hierarchy graph and the largest possible cardinality of a complete subgraph. Of particular interest in this study are the collections of operations defined next: given $* \in \mathcal{M}(S)$, the outset of $*$, is the set $\text{out}(*) = \{\circ \in \mathcal{M}(S) | * \text{ distributes over } \circ\}$.

We define an operation $\triangleleft$ that makes $\mathcal{M}(S)$ a monoid in such a way that each outset $(out(*))$ is a submonoid of $(\mathcal{M}(S), \triangleleft)$. This endowment gives us a possibility to explore the properties of an element $* \in \mathcal{M}(S)$ in terms of the structure of its outset monoid $(out(*), \triangleleft)$. Other connections between the $\triangleleft$ operation and the hierarchy graph are studied.

Among the properties of the $\triangleleft$ operation that are considered are various submonoids, one-sided, and two-sided ideals. A complete description of its group of units is given and its group of automorphisms is studied. In addition, multiple choices for an additive binary

operations + on $\mathcal{M}(S)$ are given so that $\triangleleft$ may be considered the multiplicative operation at the bottom of a nearring $(\mathcal{M}(S), +, \triangleleft)$.

Our study of magmas that enjoy special behaviors in the context of the magma monoid led us to the discussion of certain operations induced by graphs; at least two of them are of special note: the one-value magmas and the two-value magmas.

Our study was carried through the traditional strategies of mathematical inquiry but aided at times with experimentation with the symbolic computation software package MAGMA.

DEDICATION

I would like to dedicate this dissertation to my children, Maame Nyarko and Akwasi.

Its been a pleasure having you kids with me.

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List of Acronyms

DCC —Distinct Constant Column Operations

DCR —Distinct Constant Row Operations

$\mathcal{M}(S)$ —Set of All Binary Operations on S

1 Introduction and Preliminaries.

1.1 Introduction.

Algebra is, fundamentally, the study of sets endowed with operations which satisfy specific properties. The most common basic setting is when only one set is involved having only one binary operation. Semigroups, monoids and groups are examples of this type of structure. When one is not making any assumptions regarding properties that a binary operation $* : S \times S \to S$ on a set S satisfies one simply refers to the pair $(S, *)$ as a magma.

This notion of a most basic, all encompassing, type of operation can be extended to sets with two operations $(S, *, \circ)$ which would also be denoted a magma (or a bimagma if emphasis is needed) to indicate that no further properties of $*$ or $\circ$ are assumed. For the important case of bimagmas $(S, *, \circ)$ where $\circ$ distributes over $*$ (on the left, right, or both sides) we use the expression $(S, *, \circ)$ *is a (left, right, two-sided) distributive magma.* Rings, semirings, nearrings, fields, etc. are examples of distributive magmas.

In order to facilitate the discussion of distributive properties, the so-called (left, right, two-sided) hierarchy graphs of a set S were introduced in [13]. Aided by the readily available graph-theoretic terminology, many natural questions regarding hierarchy graphs can be expressed. The existence of loops, the possible lengths of cycles, and the longest length of cycle-free paths are natural questions to investigate; some of them are explored in this dissertation.

Remark 1.1.1. *A loop in a hierarchy graph pertains an operation that is distributive over itself, i.e. a distributive magma* $(S, \circ, \circ)$*. There are remarkable instances of this phenomenon in the study of quandles ([8],[10]), braids [6], and distributive groupoids [7] etc. These notions play important roles in knot theory and mathematical physics.*

Significant emphasis in this dissertation is placed on considering the outset of a fixed operation in as much as its properties may reflect those of the operation itself.

1.2 Magmas, Bimagmas and Distributive Magmas.

Throughout, S will denote a non empty set (usually a finite one). Following the use of the expression in the literature (e.g. [13], [3]), by a magma we mean a pair of the form $(S, *)$, where $*$ is a binary operation on the set S when we do not intend to impose any algebraic property (associativity, commutativity, identity, group etc) on S.

Definition 1.2.1. Let S be an arbitrary set.

A pair of the form $(S, *)$ where $* : S \times S \to S$ is a binary operation on S will be called a magma. The collection of all binary operations on the set S will be denoted as $\mathcal{M}(S)$ and will be referred to as *the magma of S*.

Notice that when S is finite with $|S| = n$ then $|\mathcal{M}(S)| = n^{n^2}$, therefore, the cardinality of magma sets increases very rapidly; this makes computational exploration of even the smallest finite cases to be challenging. For instance, while when $n = 2$, $|\mathcal{M}(S)| = 2^{2\times 2} = 16$, when S has three elements we have $|\mathcal{M}(S)| = 3^{3\times 3} = 19,683$. Dealing with a hierarchy graph with so many vertices can be difficult even while using a computer. For that reason, our computational explorations are mostly limited to the case $n = 2$.

Considering that a binary operation on a nonempty set S is a mapping from $S \times S$ into S. One can actually proceed, as they do in universal algebra, with larger generality and consider, for any positive integer n, maps $f : S^n \to S$ where $S^n = S \times S \times ... \times S$ to be n-ary operations on S. It is indeed a tantalizing challenge to extend the considerations from this work to that larger setting. We defer such considerations to future projects.

Given $\circ_1, \circ_2 \in \mathcal{M}(S)$, $\circ_1$ is said to distribute over $\circ_2$ on the left if $a \circ_1 (b \circ_2 c) = (a \circ_1 b) \circ_2 (a \circ_1 c), (\forall a, b, c \in S,)$ or on the right if

$(b \circ_2 c) \circ_1 a = (b \circ_1 a) \circ_2 (c \circ_1 a), (\forall a, b, c \in S.)$ If $\circ_1$ is both left- and right- distributive over $\circ_2$ then $\circ_1$ is said to be two-sided distributive over $\circ_2$

Definition 1.2.2. 1. If $\circ, * \in \mathcal{M}(S)$, the relation "$*$ distributes over $\circ$" (be it on the left, on the right, or both) will be denoted symbolically by $* \rightsquigarrow \circ$; the exact meaning will either be clear from the context or will be stated explicitly as needed.

2. A set with two binary operations $(S, *, \circ)$ is said to be a (left, right, two-sided) distributive magma if $\circ$ distributes over $*$ (on the left, on the right, on both sides.) The notational device of writing $(S, *, \circ)$ in that order when $\circ \rightsquigarrow *$ is consistent with usual conventions, for example, in the case of a ring $(\mathcal{R}, +, \cdot)$ where $\cdot \rightsquigarrow +$.

3. If $(S, \circ, *)$ is (left, right or two-sided)- distributive magma, we say that the pair $(\circ, *)$ is a (left, right) distributive pair. We also use the expression that $*$ is at the top or $\circ$ is at the bottom of the distributive pair $(\circ, *)$ and that $*$ is above $\circ$ (eq. $\circ$ is below $*$) in the distributive hierarchy.

One can also, just as well, talk about twistributivity hierarchies(Refer to Section 1.7. Our results in Subsection 2.2 prove that the twistributive hierarchy of a set S with n elements has a complete subgraph with $2n!$ elements; this is better than the best result known (to us) about distributive hierarchies, where the record is n.

Definition 1.2.3. Given $* \in \mathcal{M}(S)$, the set $\text{out}(*) = \{\circ \in \mathcal{M}(S) | * \rightsquigarrow \circ\}$ is called the *outset* of $*$. In other words, the outset of $*$ consists of all operations $\circ$ that are below $*$ in a distributive pair $(*, \circ)$.

Note that the definition above really encompasses (at least) three notions: left outset, right outset and two-sided outset; once again, the meaning should be clear from the context.

Natural questions when dealing with outsets include wondering about their possible cardinalities. We will introduce later a monoid operation $\triangleleft$ on $\mathcal{M}(S)$ that has all outsets as submonoids; this structure opens the door to consider the possibility of exploring connections between the algebraic properties of an operation $* \in \mathcal{M}(S)$ and those of its monoid $(\mathrm{out}(*), \triangleleft)$.

The notion of outsets is instrumental in our project; a dual notion, the inset, seems just as natural but our efforts in its direction have not been as fruitful. A type of operation which can be described on certain insets is given in [9] but it does not seem likely that one can consider those structures as restrictions to insets of a monoidal structure on $\mathcal{M}(S)$. More on this subject in subsection 2.3.

1.3 Representations of Operations over a Finite Set, Palindromic Operations.

We opted to begin exploring some ideas in the context of operations over a finite set. As we looked for a user friendly way to represent multiple operations on such sets, we came up with something we have not seen anywhere in the literature before; we think that it is a very natural way approach which, in fact, shows that all binary operations on a finite set S may be interpreted as representations of numbers in a base $|S| = n$. There is no loss of generality to identify S with $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. The representation of operations we introduce next is such that applying the operation to a given input pair $(i, j) \in S \times S$ simply yields the value in said representation that corresponds to the entry with S-ary representation ij. The idea will become clear to the reader after only a couple of examples.

Let $S = \{0, 1\}$ and, for example, consider the usual operation of addition modulo 2. Construct an addition table in the usual way except that you make sure that the rows and columns are labeled in the descending order (first 1, then 0.)

+	1	0
1	0	1
0	1	0

Reading the entries in the inside of the table (without the labeling first row and column) one row at a time, left to right, from the top, we obtain the sequence 0110, which is the binary representation of the number 6. The rightmost digit is said to be the 0^{th} digit (because it is the coefficient of 2^0), the next one from the right is the first coefficient, and so on.

We contend that 6 is a good name for + since, without necessarily using a table, one can do all calculations in the following way: the input pair (i, j) is the binary representation ij of a number k between 0 and 3; the value of $i + j$ is the $k - th$ digit in 0110. So the following translation occurs:

$0 + 0 = 0$	0 is the 0^{th} digit of 6 and 0 is represented as 00 in base 2.
$0 + 1 = 1$	1 is the 1^{st} digit of 6 and 1 is represented as 01 in base 2.
$1 + 0 = 1$	1 is the 2^{nd} digit of 6 and 2 is represented as 10 in base 2.
$1 + 1 = 0$	0 is the 3^{rd} digit of 6 and 3 is represented as 11 in base 2.

The motivated reader might want to confirm that "multiplication modulo 2" is represented by number 8.

Likewise, you should see that operation 11 has the following multiplication table, where the redundant labeling row and column are omitted for brevity:

1	0
1	1

Do not forget, though, because they are omitted, that our convention is that the labeling row and column would appear in descending order 1 0.

Two final illustrations, which are operations on the set $S = \{0, 1, 2\}$. Operations on S can be numbered between 0 and $3^9 - 1 = 19,682$. First, the three by three table representing operation 58, considering that $(58)_3 = 000002011$, is as follows:

58	2	1	0
2	0	0	0
1	0	0	2
0	0	1	1

On the other hand, addition modulo 3 corresponds to number $8,229$. So the rule "$1 + 2 = 0(mod 3)$" translates into "the fifth digit of $8,229$ in base 3 is 0" while $2 + 0 = 2(mod 3)$ because the sixth digit in its representation is 2.

8229	2	1	0
2	1	0	2
1	0	2	1
0	2	1	0

Example 1.3.1. Palindromes: An operation $\circ \in \mathcal{M}(S)$ is a palindrome if, and only if $\forall i, j \in S, i \circ j = (n - 1 - i) \circ (n - 1 - j)$. For $n = 2$, $0 = 0000, 6 = 0110, 9 = 1001, 15 = 1111$ are palindromes. For $n = 3$ any operation with representation abcdedcba, where $a, b, c, d, e \in S = \{0, 1, 2\}$ is a palindrome, example 3009 represented by 012010210 is a palindrome.

Considering our intention to introduce algebraic structures on $\mathcal{M}(S)$ we can now, for the finite case, provide a modest start.

Definition 1.3.2. We define the operation $\oplus$ on $\mathcal{M}(S)$ as follows:

$$a(* \oplus \circ)b = (a * b) + (a \circ b)$$

for every a, b in S and $*, \circ$ in $\mathcal{M}(S)$, where $+$ is entry-wise addition modulo n on S.

A biproduct of the notation for operations on finite sets introduced here and definition 1.3.2, above, we introduce the following straightforward result.

Proposition 1.3.3. *If $|S| = n$, then $(\mathcal{M}(S), \oplus) \simeq ((\mathbb{Z}_n)^{n^2}, +)$, where $+$ is vector addition modulo n.*

1.4 Nearrings

We will see later, in subsection 2.4 that the operation $\triangleleft$ on $\mathcal{M}(S)$ can be considered as the multiplicative part of many nearring structures. In this section, we introduce the basic notions of the theory of nearrings, as presented in [5].

Definition 1.4.1. A set S with two binary operations $\circ$ and $*$ is a (left) nearring if

1. $(S, *)$ is a a group
2. $(S, \circ)$ is a semigroup
3. $a \circ (b * c) = (a \circ b) * (a \circ c)$, $\forall a, b, c \in S$

A right nearing is a right distributive magma ,$(S, *, \circ)$ satisfying conditions $1, 2$.

Given an element $x \in S$, if $x \circ (b * c) = (x \circ b) * (x \circ c), \forall b, c \in S$ and $(b * c) \circ x = (b \circ x) * (c \circ x), \forall b, c \in S$, then x is a distributive element in S. If every element in S is a distributive element, then the nearring $(S, \circ, *)$ is a distributive nearring.

Example 1.4.2. Let G be an additive group. The sets

1. $\mathcal{N}(G)$ of maps from G to G,
2. $\mathcal{N}_o(G)$ of maps $f : G \to G$ such that $f(0) = 0$,
3. $\mathcal{N}_c(G)$ of constant functions from G to G

under pointwise sum and (the opposite of) the ordinary composition product is a right (left) nearring .

Proposition 1.4.3. *Every nearring N is a subnearring of some $\mathcal{N}(G)$, and so it is embedded in a nearring with identity*

1.5 Graphs

Graphs will play at least two different roles in this dissertation: first of all, the hierarchy graphs mentioned before provide a natural language in which to describe natural questions pertaining the relations among operations that constitute distributive pairs; that is the subject of Subsection 3.3. Secondly, we will investigate by mimicking a construction from [11] ways in which graphs may be used to induce interesting operations on arbitrary sets; this is the subject of section 4. These graph-induced operations have become important in the research of amenable bases over infinite dimensional algebras as studied in [1] and [14]. All the graphs that will be used in this work are simple directed graphs (no parallel edges), some introductory definitions that will be used in this work are given below.

Definition 1.5.1. A directed graph (or a digraph) , $G = (V, E)$ consists of a nonempty set V, the **vertices** or **nodes** of the graph, and a set E whose elements are called **directed edges** of the graph. Each edge $e \in E$ is associated with a set of two (not necessarily different) vertices, v and v^1.

For $v, v^1 \in V$ a directed edge $e = (v, v^1)$ is denoted by vv^1 and that means that the edge e is directed from initiall vertex v to the terminal vertex v^1

Definition 1.5.2. In a directed graph, the number of edges incident out of a vertex v is called the **Out-degree** of v, and the number of edges incident into a vertex v is called the **In-degree** of v.

A **path** between the vertices v and v^1 is a sequence of edges, written as $e_1e_2...e_r$, which is such that v is incident on e_1 , v^1 is incident on e_r, and successive edges e_i and e_{i+1} meet at a vertex, that is, they are incident on the same vertex. A nontrivial closed trail in a graph $\mathcal{G}$ is cycle if its origin and internal vertices are distinct. A graph is connected if there is a path between any two of its vertices, and it is disconnected otherwise. A largest connected subgraph is called a connected component. A *null graph* is a graph without edges, and a *complete graph* is a graph with an edge between any two vertices. On a set of vertices with n elements, the null graph is denoted N_n and the complete graph K_n.

Natural questions for a hierarchy graph on a set S with n vertices came to mind as we pursued this study; they include:

1. How long a cycle-free path can be created on the hierarchy graph of a set ?
2. How large can $m < n$ be such that the hierarchy graph has a full complete subgraph isomorphic to K_n?
3. What are necessary and sufficient conditions for the outsets of two operations to have the same number of elements or to be isomorphic as $\triangleleft$ -monoids?

The first question mentioned above undoubtedly relate to the graph-theoretic notions of *diameter* and *girth*; since our emphasis is on the algebraic structures rather than on the graph-theoretic notions involved, we have not made, for the time being, any extra efforts to clarify the connections. For completeness, we include those two definitions here. Question two also relates to the size of a connected component but the expectation there is stronger than just having connectednes and asks for complete connectedness instead.

Definition 1.5.3. The greatest distance between any two vertices of a connected graph $\mathcal{G}$ is called the **diameter** of $\mathcal{G}$. This is mostly denoted by diam($\mathcal{G}$).

The length of a smallest cycle in a graph is referred to as its **girth**.

1.6 Various Operations on $\mathcal{M}(S)$

1.6.1 Various Forms of $\triangleleft$

A central focus on this dissertation is the study of a monoid structure $\triangleleft$ on $\mathcal{M}(S)$ that has all outsets as submonoids. After the definition we realized the operation can similarly be defined in four different forms but the other forms are either isomorphic or anti-isomorphic as illustrated in the Remark 1.6.1, in view of this we will be using the form of $\triangleleft$ given in Definition 2.1.2 throughout this dissertation.

Remark 1.6.1. *We defined four operations on $\mathcal{M}(S)$ and show that all of them are isomorphic or anti-isomorphic:*

For any $a, b \in S$ and $\circ_1$ and $\circ_2$ in $\mathcal{M}(S)$;

$$a(\circ_1 \triangleleft_1 \circ_2)b = (a \circ_1 b) \circ_2 (b \circ_1 a)$$

$$a(\circ_1 \triangleleft_2 \circ_2)b = (b \circ_1 a) \circ_2 (a \circ_1 b)$$

$$a(\circ_1 \triangleright_1 \circ_2)b = (a \circ_2 b) \circ_1 (b \circ_2 a)$$

$$a(\circ_1 \triangleright_2 \circ_2)b = (b \circ_2 a) \circ_1 (a \circ_2 b).$$

$(\mathcal{M}(S), \triangleleft_1)$ and $(\mathcal{M}(S), \triangleleft_2)$ are isomorphic under the isomorphism below:

$\varphi : (\mathcal{M}(S), \triangleleft_1) \longrightarrow (\mathcal{M}(S), \triangleleft_2)$ such that for every $\circ \in \mathcal{M}(S)$, $a\varphi(\circ)b = b \circ a$.

Similarly, $(\mathcal{M}, \triangleright_1)$ and $(\mathcal{M}(S), \triangleright_2)$ are isomorphic under the same isomorphism as above.

$(\mathcal{M}(S), \triangleleft_1)$ and $(\mathcal{M}(S), \triangleright_1)$ are anti isomorphic, under the identity map as an anti-isomorphism.

1.6.2 An Operation on Insets

. Our literature search only yielded one case of an operation defined on insets. That operation seems to only work on insets of abelian groups.

Remark 1.6.2. *In [9], Fuchs defines an operation over $in(+)$ where $(S, +)$ is an abelian group in the following way: For every $*_1$ and $*_2$ in $in(+)$ and a and b in S, $a(*_1 + *_2)b = (a *_1 b) + (a *_2 b)$. We show that if $*_1$ and $*_2$ are in $in(+)$ then $*_1 + *_2$ is in $in(+)$:*

Left distributivity: for every a, b and c in S we have

$a(*_1 + *_2)(b+c) = [a *_1 (b+c)] + [a *_2 (b+c)] = [(a *_1 b) + (a *_1 c)] + [(a *_2 b) + (a *_2 c)] = [(a *_1 b) + (a *_2 b)] + [(a *_1 c) + (a *_2 c)] = (a *_1 + *_2 b) + (a *_1 + *_2 c).$

A similar argument is used to show right distibutivity.

Motivated by our seemingly more successful scenario for outsets, we propose the following question as a potential topic for future research:

Question 1.6.3. *Can we define a similar structure on $in(+)$ when $(S, +)$ is a magma without some its the nice properties of an abelian group?*

1.6.3 A Lower Bound for Complete Subgraphs of the Distributive Hierarchy Graph

In the process of answering this question, we chance on a work done by [18] and [16], which provides a lower bound (n) to the question, in the follow up we discuss their approach to the Question 3.3.23

Proposition 1.6.4. *[18] $(\mathcal{M}(S), \Box)$ has a monoidal structure with $(\circ_1 \Box \circ_2)$ as the composition defined by $x(\circ_1 \Box \circ_2)y = (x \circ_1 y) \circ_2 y \; \forall \circ_1, \circ_2 \in \mathcal{M}(S)$ and $\forall x, y \in S$ and the identity is π_1 where $\pi_1(x, y) = x, \forall x, y \in S$*

Definition 1.6.5. A set $D \subset \mathcal{M}(S)$ is a distributive set if for all pairs $\circ_1, \circ_2 \in \mathcal{M}(S)$ are right distributive, i.e. $(x \circ_1 y) \circ_2 z = (x \circ_2 z) \circ_1 (y \circ_2 z)$

Proposition 1.6.6. *[18]*

1. *If D is a distributive set and $\tau \in D(S)$ is invertible, then $D \cup \{\bar{\tau}\}$ is also a distributive set.*

2. *If D is a distributive set and $(D(S), \Box)$ is the monoid generated by D, then $D(S)$ is a distributive monoid .*

3. *If D is a distributive set of invertible operations and $(G(S), \Box)$ is the group generated by D, then $G(S)$ is a distributive group.*

Theorem 1.6.7. *Every group G embeds in $M(G)$. This embedding(monomorphism) $\phi^{reg} : G \to M(G)$ sends g to $\circ_g$ where $a \circ_g b = ab^{-1}gb$*

Proof. Refer to [16] □

Remark 1.6.8. *Theorem 1.6.7 set n as the lower bound for Question3.3.23.*

1.7 Twings

In this section we introduce a notion that naturally arise from the various definition of distributivity. This will not be our main concern in this dissertation, it became relevant when we tried to answer Question 3.3.23. We will illustrate our definitions and present the only example we have. It will be a possible area for future research.

Given $\circ_1, \circ_2 \in \mathcal{M}(S)$, $\circ_1$ is said to be left-right distributive over $\circ_2$ if $a \circ_1 (b \circ_2 c) = (b \circ_1 a) \circ_2 (c \circ_1 a), (\forall a, b, c \in S,)$ or right-left distributive if $(b \circ_2 c) \circ_1 a = (a \circ_1 b) \circ_2 (a \circ_1 ca), (\forall a, b, c \in S.)$ We will indulge and refer to left-right distributivivity as *left twistributivity* and to right-left distributivity as *right twistributivity*. Then, if $\circ_1$ is both left-right and right-left distributive over $\circ_2$ then we can simply say that $\circ_1$ is (two-sided) twistributive over $\circ_2$.

Definition 1.7.1. Given a set $\mathcal{R}$, with $|\mathcal{R}| = n, n \geq 1$, $(\mathcal{R}, +, .)$ is a **Twing** if $(\mathcal{R}, +)$ is a group, $(\mathcal{R}, .)$ is a semigroup and . twistribute over + on both sides.

Proposition 1.7.2. *If $(\mathcal{R}, +, .)$is a twing then $(\mathcal{R}, .)$ is commutative.*

Proof. Given $a, b \in \mathcal{R}$ we show that $ab = ba$

Step 1: We show that $a0 = -(0a)$

$$a0 = a(0 + 0) = 0a + 0a$$

$$0a = (0 + 0)a = a0 + 0a$$

$$\implies a0 = a0 + a0 + 0a$$

$$\implies 0 = a0 + 0a$$

Step 2: We show that $a0 = 0b(\forall a, b)$

$$ab = a(b + 0) = ba + 0a$$

$$ab = (a + 0)b = ba + b0$$

$$\implies 0a = b0$$

From Step 2 we can see that $0a = a0$

Step 3

$$ab = a(b + 0 + 0) = ba + 0a + 0a = ba + 0a + a0 = ba(\forall a, b)$$

□

From Proposition 1.7.2, if $(\mathcal{R}, +)$ is an abelian group then $(\mathcal{R}, +, .)$ is commutative ring.

2 A Monoid Structure on $\mathcal{M}(S)$

In this chapter we will define an operation, $\triangleleft$, that makes $\mathcal{M}(S)$ a monoid in such a way that each outset of $\mathcal{M}(S)$ is a submonoid. This endowment gives us a possibility to compare the algebraic properties of the various elements of $\mathcal{M}(S)$ with those of the monoid structure of their outsets. Various properties of $\triangleleft$ are considered, including multiple (additive) structures on $\mathcal{M}(S)$ that have $\triangleleft$ as the multiplicative part of a nearring with underlying set S.

2.1 Outsets, Monoid and Submonoid Structure on $\mathcal{M}(S)$

Remember the definitions of outsets and insets.

Definition 2.1.1.

1. For every operation $*$ in $\mathcal{M}(S)$, the outset of $*$, denoted by $out(*)$, is defined as the set $out(*) = \{\circ \in \mathcal{M}(S) |\ * \text{ distributes over } \circ\}$.
2. We define the inset of $*$, denoted by $in(*)$, $\forall * \in \mathcal{M}(S)$, as the set $in(*) = \{\circ \in \mathcal{M}(S) |\ \circ \text{ distributes over } *\}$

Definition 2.1.2. We define an operation $\triangleleft$ on $\mathcal{M}(S)$ as follows.

$$\triangleleft(*, \circ)(a, b) = \circ(*(a, b), *(b, a)) \tag{2.1.1}$$

for every $*, \circ \in \mathcal{M}(S)$ and $a, b \in S$.

As is customary and depending on the circumstances, we will use $* \triangleleft \circ$ and $\triangleleft(*, \circ)$ interchangeably.

Theorem 2.1.3. *For every set S, $(\mathcal{M}(S), \triangleleft)$ is a non-commutative monoid.*

Proof. We show that the operation is associative and has an identity.

1. Associativity: For any $\circ_1, \circ_2, \circ_3 \in \mathcal{M}(S)$ and $a, b \in S$ we have

$$a(\circ_1 \triangleleft (\circ_2 \triangleleft \circ_3))b = (a \circ_1 b)(\circ_2 \triangleleft \circ_3)(b \circ_1 a) =$$

$$[(a \circ_1 b) \circ_2 (b \circ_1 a)] \circ_3 [(b \circ_1 a) \circ_2 (a \circ_1 b)] = (a \circ_1 \triangleleft \circ_2 b) \circ_3 (b \circ_1 \triangleleft \circ_2 a) =$$

$$a((\circ_1 \triangleleft \circ_2) \triangleleft \circ_3)b.$$

2. Identity: Let π_1 be the projection map onto the first component,that is $\pi_1(a,b) = a \quad \forall a,b \in S$, then $(\circ \triangleleft \pi_1)(a,b) = \pi_1(\circ(a,b), \circ(b,a)) = \circ(a,b)$ and $(\pi_1 \triangleleft \circ)(a,b) = \circ(\pi_1(a,b), \pi_1(b,a)) = \circ(a,b)$. So π_1 is the identity of the operation $\triangleleft$.

3. Non-commutativity: since we only need to show an example of two elements that do not commute, we defer it to the following remark 2.1.7 where we show, in general, that distinct constant operations do not commute with one another.

□

The following result is central to this dissertation and is the first thing that caught our attention about operation $\triangleleft$.

Proposition 2.1.4. *For every operation $* \in \mathcal{M}(S)$, the (left, right, two-sided) outset $out(*)$ is a submonoid of $(\mathcal{M}(S), \triangleleft)$ with the same identity π_1.*

Proof. Note that $* \rightsquigarrow \pi_1$ since for all $a,b,c \in S$, $a * (b\pi_1 c) = a * b = (a * c)\pi_1(b * c)$. Also, $(b\pi_1 c) * a = b * a = (b * a)\pi_1(c * a)$.

It only rests to check that $out(*)$ is closed with respect to $\triangleleft$. For every $\circ_1$ and $\circ_2$ in $out(*)$ and a, b and c in S we have

$a * (b(\circ_1 \triangleleft \circ_2)c) = a * [(b \circ_1 c) \circ_2 (c \circ_1 b)] = [a * (b \circ_1 c)] \circ_2 [a * (c \circ_1 b)] =$ $[(a * b) \circ_1 (a * c)] \circ_2 [(a * c) \circ_1 (a * b)] = (a * b)(\circ_1 \triangleleft \circ_2)(a * c)$. A similar calculation shows that the result holds also for right distributivity. □

Definition 2.1.5. A nonempty subset $\mathcal{I}$ of a $\mathcal{M}(S)$ is said to be an left ideal of $\mathcal{M}(S)$ if $* \triangleleft \circ \in \mathcal{I}$, for all $* \in \mathcal{M}(S)$, and $\circ \in \mathcal{I}$. Right ideals are defined similarly. When $\mathcal{I}$ is both a left and right ideal then it is said to be a two-sided ideal or simply an ideal.

The constant operations that we define next constitute an ideal, as shown in Remark 2.1.7. Other examples of ideals are deferred until subsection 2.3, which is devoted solely to the topic. Constant operations are further characterized in Lemma 2.1.8 as being precisely the *right annihilators* in the monoid $(\mathcal{M}(S), \triangleleft)$.

Definition 2.1.6. Let $i \in S$. The binary operation $C_i \in \mathcal{M}(S)$ given by $C_i(x, y) = i$, $\forall x, y \in S$ is said to be *the constant operation induced by* i.

Remark 2.1.7. *1. Let $\mathcal{K} = \{C_i | i \in S\}$ be the set of all constant operations in $\mathcal{M}(S)$. Then $(\mathcal{K}, \triangleleft)$ is a two-sided ideal of the monoid $(\mathcal{M}(S), \triangleleft)$. For every $* \in \mathcal{M}(S)$, $* \triangleleft C_i = C_i$ and $C_i \triangleleft * = C_{i*i}$. In particular $\mathcal{K}$ is a subsemigroup of $\mathcal{M}(S)$.*

2. *The fact that the operation $\triangleleft$ is not commutative can be dramatically illustrated in the context of the subsemigroup $\mathcal{K}$ where the operation $\triangleleft$ acts like projection on the second entry. In fact, if $i \neq j$ $\forall i, j \in S$, then $C_i \triangleleft C_j = C_j$ and $C_j \triangleleft C_i = C_i$.*

The set of constant operations can be characterized by the interesting behavior they exhibit under the $\triangleleft$ operation.

Lemma 2.1.8. *Given $\circ \in \mathcal{M}(S)$, $\circ \in \mathcal{K}$ if and only if $\forall * \in \mathcal{M}(S), * \triangleleft \circ = \circ$*

Proof. If $\circ = C_a$ then $\forall x, y \in S$ $\circ(x, y) = a$. Choose arbitrary $* \in \mathcal{M}(S)$, $(* \triangleleft \circ)(x, y) = \circ(*(x, y), *(y, x)) = a$, $\forall x, y \in S$, and the results follows from definition.

Conversely, if $\forall * \in \mathcal{M}(S), * \triangleleft \circ = \circ$. Assume that $\circ$ is not a constant operation, then choose $x \neq y, w \neq z \in S$ such that $\circ(w, z) \neq \circ(x, y)$, define $* \in \mathcal{M}(S)$ by $*(x, y) = w$, and $*(y, x) = z$ then we have $(* \triangleleft \circ)(x, y) = \circ(*(x, y), *(y, x)) = \circ(w, z) \neq \circ(x, y)$.

If we choose $x \neq z \in S$ such that $\circ(z, z) \neq \circ(x, x)$ define $* \in \mathcal{M}(S)$ by $*(x, x) = z$, and $*(x, y) = j \forall x, y \in S, x \neq y$ then we have $(* \triangleleft \circ)(x, x) = \circ(*(x, x), *(x, x)) = \circ(z, z) \neq \circ(x, x)$.

□

Following from Remark 2.1.7 and Lemma 2.1.8, $(\mathcal{M}(S), \triangleleft)$ is never a group; indeed since the set of constant operations absorbs all multiples under $\triangleleft$, $\implies \nexists * \in \mathcal{M}(S)$ such that $\circ_i \triangleleft * = \pi_1$. The group of units in $\mathcal{M}(S)$ is discussed in the next section.

2.2 Units in $(\mathcal{M}(S), \triangleleft)$

The set of units of a monoid constitutes a group under the monoid operation. It interesting to characterize the group of units $U(\mathcal{M}(S))$ of $(\mathcal{M}(S), \triangleleft)$ for an arbitrary set S. One step in that direction would be determining, for a finite set S with cardinality n, the order of $(U(\mathcal{M}(S)), \triangleleft)$. We achieve that goal and give a characterization of $U(\mathcal{M}(S))$ as a subset of $\mathcal{M}(S)$. We prove, in subsection 2.2.2 that

$$|U(\mathcal{M}(S))| = \binom{n}{k}!n!2^{\binom{n}{2}}$$

and show that, consequently,

$$\lim_{n\to\infty} \frac{|U(\mathcal{M}(S)|}{|\mathcal{M}(S)|} = 0.$$

2.2.1 The subgroup of Distinct Constant Row and Distinct Constant Column Operations

Before tackling the characterization in general, we warm up by introducing two disjoint families of operations whose union is a subgroup of $(U(\mathcal{M}(S)), \triangleleft)$.

Definition 2.2.1. An operation $* \in \mathcal{M}(S)$ is said to be a distinct constant rows operation if there exists a one-to-one and onto map $\varphi : S \to S$ such that, for all $a, b \in S$, $a * b = \varphi(a)$. If there exists a one-to-one and onto map $\psi : S \to S$ such that, for all $a, b \in S$, $a * b = \psi(b)$, $*$ is said to be a distinct constant columns operation.

In particular, when S is finite and $|S| = n$, the following definition is equivalent to Definition 2.2.1

Definition 2.2.2. Let $\mathcal{S}_n$ be the symmetric group of the set $S = \{1, 2, \ldots n\}$. For each $\sigma, \eta \in \mathcal{S}_n$ we can define operations $*_\sigma, *^\eta \in \mathcal{M}(S)$ via $*_\sigma(i, j) = \sigma(i)$ and $*^\eta(i, j) = \eta(j)$

$\forall i, j \in S$ We say that $*_\sigma$ is a *distinct constant row operation* and denote DCR the set $\{*_\sigma | \sigma \in \mathcal{S}_n\}$. Likewise, we say that $*^\eta$ is a *distinct constant column operation* and denote DCC the set $\{*^\eta | \eta \in \mathcal{S}_n\}$

The distinct constant row operations receive that name because, when represented with a table, the rows are constant and distinct. A similar explanation holds for distinct constant column operations. The following results will culminate with Theorem 2.2.5 where it is shown that $(DCC \cup DCR, \triangleleft)$ is a group isomorphic to $\mathcal{S}_n \times \mathbb{Z}_2$.

Lemma 2.2.3. *1. If $* \in DCC \cup DCR$ then $* \in U(\mathcal{M}(S))$,that is, $*$ is a unit.* *$(DCC \cup DCR \subset U(\mathcal{M}(S))$.*

2. *Distinct constant row binary operations are closed under $\triangleleft$,that is, if $\circ_\sigma, \circ_\eta \in DCR$ then $\circ_\sigma \triangleleft \circ_\eta \in DCR$*

Proof. 1. Suppose $\circ_\sigma \in DCR$, so $\circ_\sigma(i, j) = \sigma(i)$ $\forall i, j \in S$, choose a $\circ_{\sigma^{-1}} \in \mathcal{M}(S)$, where σ^{-1} is the inverse of σ in S_n then we show that $\circ_\sigma \triangleleft \circ_{\sigma^{-1}} = \circ_{\sigma^{-1}} \triangleleft \circ_\sigma = \pi_1$.
$\forall i, j \in S$ $(\circ_\sigma \triangleleft \circ_{\sigma^{-1}})(i, j) = \circ_{\sigma^{-1}}(\sigma(i), \sigma(j)) = \sigma^{-1}(\sigma(i)) = i$
Similarly, $\forall i, j \in S$, $(\circ_{\sigma^{-1}} \triangleleft \circ_\sigma)(i, j) = \circ_\sigma(\sigma^{-1}(i), \sigma^{-1}(j)) = \sigma(\sigma^{-1}(i)) = i$
Suppose $\circ^\eta \in DCC$, so $\circ^\eta(i, j) = \eta(j)$ $\forall i, j \in S$, choose a $\circ^{\eta^{-1}} \in \mathcal{M}(S)$, where η^{-1} is the inverse of η in S_n then we show that $\circ^\eta \triangleleft \circ^{\eta^{-1}} = \circ^{\eta^{-1}} \triangleleft \circ^\eta = \pi_1$. $\forall i, j \in S$
$(\circ^\eta \triangleleft \circ^{\eta^{-1}})(i, j) = \circ^{\eta^{-1}}(\eta(j), \eta(i)) = \eta^{-1}(\eta(i)) = i$
Similarly, $\forall i, j \in S$, $(\circ^{\eta^{-1}} \triangleleft \circ^\eta)(i, j) = \circ^\eta(\eta^{-1}(j), \eta^{-1}(i)) = \sigma(\sigma^{-1}(i)) = i$

2. Suppose $\circ_\sigma$ and $\circ_\eta$ are distinct constant row operations for each $\sigma, \eta, \tau \in S_n$, where $\eta\sigma = \tau$ and for every $i, j \in S$, $\circ_\sigma(i, j) = \sigma(i)$ and $\circ_\eta(i, j) = \eta(i)$. Given $i \in S$, for every $j \in S$, $i(\circ_\sigma \triangleleft \circ_\eta)j = \circ_\eta(\circ_\sigma(i, j), \circ_\sigma(j, i)) = \circ_\eta(\sigma(i), \sigma(j)) = \eta\sigma(i) = \tau(i)$ which shows $\circ_\sigma \triangleleft \circ_\eta \in DCR$
Let $*^\sigma$ and $*^\eta$ be distinct constant column operations and for every $a, b, c \in S$, $*^\sigma(a, b) = \sigma(b)$ and $*^\eta(a, b) = \eta(b)$. Given $b \in S$, for every $a, c \in S$,

$a(*^{\sigma} \triangleleft *^{\eta})b = *^{\eta}(*^{\sigma}(a,b), *^{\sigma}(b,a)) = *^{\eta}(\sigma(b), \sigma(a)) = \eta\sigma(a) = \tau(a)$ and

$c(*^{\sigma} \triangleleft *^{\eta})b = *^{\eta}(*^{\sigma}(c,b), *^{\sigma}(b,c)) = *^{\eta}(\sigma(b), \sigma(c)) = \eta\sigma(c) = \tau(c)$ which shows that

$a(*^{\sigma} \triangleleft *^{\eta})b = c(*^{\sigma} \triangleleft *^{\eta})b$ thus $*^{\sigma} \triangleleft *^{\eta} \notin DCC$.

□

Remark 2.2.4. *Following from Lemma 2.2.3 we can show that for every* $\circ_{\sigma}, \circ_{\eta} \in DCC$ *and* $*^{\sigma}, *^{\eta} \in DCR$, $\circ_{\sigma} \triangleleft \circ_{\eta} \in DCR$, $*^{\eta} \triangleleft *^{\eta} \in DCR$, $\circ_{\sigma} \triangleleft *^{\sigma} \in DCC$ *and* $*^{\eta} \triangleleft \circ_{\sigma} \in DCC$

Now the following result holds:

Theorem 2.2.5. *If* $\mathcal{G}(S) = DCC \cup DCR$, *then* $(\mathcal{G}(S), \triangleleft)$ *is a group, and* $\mathcal{G}(S) \cong S_n \times \mathbb{Z}_2$

Proof. This follows from Theorem 2.1.3, Lemma 2.2.3 and Remark 2.2.4

Define a map $\mu : S_n \times \mathbb{Z}_2 \to \mathcal{G}(S)$ by taking $(\sigma, 0) \longmapsto *_{\sigma}$ and $(\eta, 1) \longmapsto *^{\eta}$. Then μ is one-to-one and onto. □

Corollary 2.2.6. *Given* $|S| = n$ *for each* $n \geq 2$ *and* $DCR, \mathcal{G}(S) \subset \mathcal{M}(S)$, *DCR is a normal subgroup of* $\mathcal{G}(S)$

Proof. Given $\circ_{\sigma} \in DCR$ and for every $*^{\eta} \in G(S)$, $(*^{\eta} \triangleleft (\circ_{\sigma} \triangleleft *^{\eta^{-1}})) \in DCR$ follows from Remark 2.2.4

□

Example 2.2.7. Given $S = \{0, 1\}$ if we let $\mathcal{X}(S) = \{\mathbf{3,5,10,12}\}$

then $(\mathcal{X}(S), \triangleleft)$ forms a group and $(\mathcal{X}(S), \triangleleft) \cong Z_2 \times Z_2$ or D_2

(Check Appendix B for full representation of these numbers and Appendix D for all the calculations of $(\mathcal{M}(S), \triangleleft)$ for $n = 2$)

Example 2.2.8. If $S = \{0, 1, 2\}$ then the following operations can generate the following groups

$$D_3 = \{< \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} >\} \text{ and } D_6 = \{< \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} >\}$$

(Check Appendix A for full representation of these numbers and Appendix F for the codes used in the calculations of $(\mathcal{M}(S), \triangleleft)$ for $n = 3$)

2.2.2 The Magma Group

While $\mathcal{G}(S)$ is always a subgroup of $U(\mathcal{M}(S)$, equality is only achieved when $|S| = 2$. In fact, we conclude this section with a general characterization of the units of $(\mathcal{M}(S), \triangleleft)$, a group $(U(\mathcal{M}(S)), \triangleleft)$ we will call *the magma group* of S. We prove that

$$|U(\mathcal{M}(S))| = \binom{n}{k}!n!2^{\binom{n}{2}}$$

and that, consequently,

$$\lim_{n\to\infty} \frac{|U(\mathcal{M}(S)|}{|\mathcal{M}(S)|} = 0.$$

So, while the growth of $|U(\mathcal{M}(S)|$ is rapid, it pales in comparison with that of $|\mathcal{M}(S)|$.

We start with a necessary definition which will allow us to take a closer look at the structure of the $\triangleleft$ operation.

Definition 2.2.9. For each set S and $*, \circ \in \mathcal{M}(S)$, define a map $(* \times \circ\zeta) : S \times S \to S \times S$ by $(* \times \circ\zeta)(a, b) = (a * b, b \circ a)$. In particular, $(* \times *\zeta)(a, b) = (a * b, b * a)$.

Notice that, with this notation, from Definition 2.2.9 and Definition 2.1.2, $\circ \triangleleft *$ is the composition of functions $\circ(* \times *\zeta)$.

Theorem 2.2.10. *Given* $* \in \mathcal{M}(S)$, $* \in U(\mathcal{M}(S)) \iff * \times *\zeta$ *is one-to-one and onto.*

Proof. Given $* \in \mathcal{M}(S)$.

$\Longrightarrow$ Assume $*^{-1}$ is the inverse of $*$ then by definition

$(* \triangleleft *^{-1})(a, b) = (*^{-1} \triangleleft *)(a, b) = \pi_1(a, b) = a \ \forall a, b \in S$.

If $(a, b) \neq (c, d) \forall a, b, c, d \in S$ then

$$(* \times *\zeta)(a, b) = (* \times *\zeta)(c, d) \tag{2.2.1}$$

$$(a * b, b * a) = (c * d, d * c) \tag{2.2.2}$$

$$*^{-1}(a * b, b * a) = *^{-1}(c * d, d * c) \tag{2.2.3}$$

$$a = c \tag{2.2.4}$$

Similarly,

$$(* \times *\zeta)(b, a) = (* \times *\zeta)(d, c) \tag{2.2.5}$$

$$(b * a, a * b) = (d * c, c * d) \tag{2.2.6}$$

$$*^{-1}(b * a, a * b) = *^{-1}(d * c, c * d) \tag{2.2.7}$$

$$b = d \tag{2.2.8}$$

This implies $(a, b) = (c, d)$ and $* \times *\zeta$ is one-to-one.

Given $(c, d) \in S \times S$ we show that $\exists (a, b) \in S \times S$ such that $(* \times *\zeta)(a, b) = (c, d)$. Let $a = *^{-1}(c, d)$ and $b = *^{-1}(d, c)$, then

$(* \times *\zeta)(a, b) = (a * b, b * a) = (*(c *^{-1} d, d *^{-1} c), *(d *^{-1} c, c *^{-1} d) = (c, d)$.

Thus $* \times *\zeta$ is onto.

$\Longleftarrow$ If $* \times *\zeta$ is one-to-one and onto, then given $(a, b) \neq (c, d) \forall a, b, c, d \in S$, let $(* \times *\zeta)(a, b) = (c, d)$ define $*^{-1}(c, d) = a$ and $*^{-1}(d, c) = b$ then

$*^{-1}(* \times *\zeta)(a, b) = *^{-1}(a * b, b * a) = (*^{-1} \triangleleft *)(a, b) = *^{-1}(c, d) = a$.

Implying that $*$ has a left inverse.

Using the left invertibility of $*$, we can see that

$(*^{-1} \times *^{-1}\zeta)(a, b) = (a *^{-1} b, b *^{-1} a) = (*^{-1}(c * d, d * c), *^{-1}(d * c, c * d)) = (c, d)$

$*(*^{-1} \times *^{-1}\zeta)(a, b) = *(c, d) = a$

Thus $* \triangleleft *^{-1} = * \triangleleft *^{-1} = \pi_1$

$\square$

Theorem 2.2.11. $|U(\mathcal{M}(S))| = \binom{n}{2}!n!2^{\binom{n}{2}}$.

Proof. It follows from the characterization of units given above that, when a unit operation $*$ is represented in a table, the diagonal elements of the table constitute a permutation of the n elements of S. Likewise, one can see that, given a pair $\{c, d\}$ of distinct elements in S, there exists exactly one subset $\{a, b\} \subset S$ such that the entries in the table consisting to (a, b) and (b, a) are precisely $\{c, d\}$ in some order. In order to construct a table corresponding to a unit, one begins by filling out the diagonal entries; there are $n!$ ways to do that. After that, one fills out a first pair of non-diagonal entries (labeled (a, b) and (b, a), say. There are $2\binom{n}{2}$ ways to do this (because there are $\binom{n}{2}$ sets of choices and for each choice $\{c, d\}$ there are two ways in which the entries can be filled in. Filling out a second pair of symmetrically placed non-diagonal entries, there will be one less choice of a subset of S with two elements; the two values chosen, once again, can be placed in two different ways. So, this step may be performed in $2(\binom{n}{2} - 1)$ ways. The next step would yield $2(\binom{n}{2} - 2)$ possibilities, and so on. Since there are a total of $n^2 - n$ places to fill out after completing the diagonal, there are $\frac{n(n-1)}{2} = \binom{n}{2}$ steps to perform after the preliminary step of filling the diagonal. It follows, hence, by the multiplication principle of combinatorics, that $|U(\mathcal{M}(S))| = \binom{n}{2}!n!2^{\binom{n}{2}}$, as claimed. □

While the previous theorem shows that the order of the magma groups for finite sets increase rapidly, their growth is eclipsed by that of the magma monoids themselves; that is the purpose of the following theorem.

We would like to thank Pedro Fernando Morales-Almazán for helpful conversations to figure out the proof of the next theorem, which starts with a lemma:

Lemma 2.2.12. *For all $n \geq 2$, $(n(n-1))! > 2^{\binom{n}{2}}\binom{n}{2}!$*

Proof. Notice that $n(n-1) = 2\binom{n}{2}$. Therefore,

$$(n(n-1))! = (2\binom{n}{2})! = 2\binom{n}{2}(2\binom{n}{2} - 1)(2\binom{n}{2} - 2)\cdots >$$

$$(2\binom{n}{2})(2\binom{n}{2}-2)(2\binom{n}{2}-4)\cdots = 2^{\binom{n}{2}}\binom{n}{2}!$$

□

Theorem 2.2.13. $\lim_{n\to\infty} \frac{|U(\mathcal{M}(S)|}{|\mathcal{M}(S)|} = 0.$

Proof. Recall that

$$|U(\mathcal{M}(S)| = \binom{n}{2}!n!2^{\binom{n}{2}},$$

and

$$|\mathcal{M}(S)| = n^{n^2}.$$

Consider the sequences $\{a_n\}, \{b_n\}$, and $\{c_n\}$, defined as follows.

$$a_n = \frac{2^{\frac{n(n-1)}{2}}(\frac{n(n-1)}{2})!n!}{n^{n^2}},$$

$$b_n = \frac{(n(n-1))!^{\frac{1}{2}}!n!}{n^{n^2}},$$

and

$$c_n = \frac{(n(n-1))!(n!)^2}{(n^2)^{n^2}}.$$

Note that $\frac{|U(\mathcal{M}(S))|}{|\mathcal{M}(S)|} = a_n < b_n$, by Lemma 2.2.12. Also, $b_n = \sqrt{c_n}$. Therefore, the theorem will be proven if we show that $c_n \to 0$. To that avail, rewrite c_n

$$c_n = \frac{(n(n-1))!}{(n)^{2n^2-2n}}\left(\frac{n!}{n^n}\right)^2 \tag{2.2.9}$$

$$= \frac{(n(n-1))!}{(n^2)^{n(n-1)}}\left(\frac{n!}{n^n}\right)^2 \tag{2.2.10}$$

As $n \to \infty$, $\frac{n!}{n^n} \to 0$ and $\frac{(n(n-1))!}{(n^2)^{n(n-1)}} \to 0$, and therefore $\lim_{n\to\infty} c_n = 0$.

□

Example 2.2.14. As illustration, we close this section with a couple examples of units and their inverses (for the case when $n = 4$.)

$*$	3	2	1	0
3	2	3	1	3
2	2	1	1	0
1	0	3	0	2
0	0	2	1	3

$*^{-1}$	3	2	1	0
3	0	3	1	3
2	2	3	1	0
1	2	0	2	3
0	0	2	1	1

$\circ$	3	2	1	0
3	1	3	0	3
2	2	0	3	2
1	1	1	3	1
0	0	0	2	2

$\circ^{-1}$	3	2	1	0
3	1	3	2	3
2	2	0	0	2
1	1	1	3	1
0	0	0	3	2

The MAGMA codes used in generating these inverses are in Appendix G

2.2.3 Procedure for Finding Inverse of Units in $(\mathcal{M}(S), \triangleleft)$

The following is a procedure to find the inverse of an operation $* \in U(\mathcal{M}(S))$. The code for a magma program to carry this procedure may be found in Appendix G.

Let $x_1, x_2, ..., x_n \in S$

1. To get elements for the diagonals, read for each $i \in \{1, 2, ..., n\}$, $x_i * x_i$, if $*(x_i, x_i) = x_j$ then $*^{-1}(x_j, x_j) = x_i$
2. Given $i \neq j$ and $m \neq n$, read $((*(x_i, x_j), *(x_j, x_i))$, assume $((*(x_i, x_j), *(x_j, x_i)) = (x_n, x_m)$ then $*^{-1}(x_n, x_m) = x_i$ and $*^{-1}(x_m, x_n) = x_j$
3. Go through Step 2 varying x_i, x_j till all the cells in $*^{-1}$ are filled up.

Example 2.2.15. Given

$$* = \begin{array}{c|cccc} & 3 & 2 & 1 & 0 \\ \hline 3 & 2 & 3 & 1 & 3 \\ 2 & 2 & 1 & 1 & 0 \\ 1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 1 & 3 \end{array}$$

we follow the above procedure to find its inverse.

Step 1: Reading from $*$. $*(3,3) = 2 \implies *^{-1}(2,2) = 3$ following up we can see that $*^{-1}(3,3) = 0, *^{-1}(1,1) = 2$ and $*^{-1}(0,0) = 1$

Step 2 and 3: Reading from $*$;

$(*(3,2), *(2,3)) = (3,2) \implies *^{-1}(3,2) = 3, *^{-1}(2,3) = 2$

$(*(3,1), *(1,3)) = (1,0) \implies *^{-1}(1,0) = 3, *^{-1}(0,1) = 1$

$(*(3,0), *(0,3)) = (3,0) \implies *^{-1}(3,0) = 3, *^{-1}(0,3) = 0$

$(*(2,1), *(1,2)) = (1,3) \implies *^{-1}(1,3) = 2, *^{-1}(3,1) = 1$

$(*(2,0), *(0,2)) = (0,2) \implies *^{-1}(0,2) = 2, *^{-1}(2,0) = 0$

$(*(1,0), *(0,1)) = (2,1) \implies *^{-1}(2,1) = 1, *^{-1}(1,2) = 0$

Following from the procedure we have

$$*^{-1} = \begin{array}{c|cccc} & 3 & 2 & 1 & 0 \\ \hline 3 & 0 & 3 & 1 & 3 \\ 2 & 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 2 & 1 & 1 \end{array}$$

The MAGMA code to verify that $* \triangleleft *^{-1} = *^{-1} \triangleleft * = \pi_1$ is in Appendix F.

2.3 Ideals on $(\mathcal{M}(S), \triangleleft)$

The notion of ideals of $(\mathcal{M}(S), \triangleleft)$ was introduced in Definition 2.1.5; we devote this section to determining some left, right, and two-sided ideals.

Proposition 2.3.1. *For all* $\circ_1, \circ_2 \in \mathcal{M}(S)$ $in(\circ_1) \bigcap in(\circ_2) \subseteq in(\circ_1 \triangleleft \circ_2)$

Proof. Given $* \in \mathcal{M}(S)$ if $* \in in(\circ_1) \bigcap in(\circ_2)$ then $\circ_1, \circ_2 \in out(*)$ and $\circ_1 \triangleleft \circ_2 \in out(*)$ therefore $* \in in(\circ_1 \triangleleft \circ_2)$ □

Following from Proposition 2.3.1 we can state the following corollary.

Corollary 2.3.2. *For all* $\circ_1, \circ_2 \in \mathcal{M}(S)$*, if* $\circ_1 \triangleleft \circ_2$ *is an isolation then* $in(\circ_1) \bigcap in(\circ_2) = in(\circ_1 \triangleleft \circ_2) = \{\}$

Proposition 2.3.3. *For a given set S and* $\circ_1 \in \mathcal{M}(S)$ *the following are equivalent*

1. $\circ_1 \in \bigcap_{* \in \mathcal{M}(S)} out(*)$

2. $\forall \circ_2 \in \mathcal{M}(S)$ *and* $\circ_2 \triangleleft \circ_1 = \circ_3$ *if* $*$ *distributes over* $\circ_2$ *then* $*$ *distributes over* $\circ_3$*.*

Definition 2.3.4. A binary operation $\circ$ is said to be a unique square operation if $\circ(a, a) = \circ(b, b), \forall a, b \in S, .$

Proposition 2.3.5. *The following sets are ideals in* $(\mathcal{M}(S), \triangleleft)$

1. *The set of all unique square binary operations on S*

2. *The set of all commutative binary operations on S*

3. *The set of all constant operations on S*

Proof. 1. Let X be the set of all unique square binary operations on S, suppose $\circ \in X$ such that $a \circ a = A$ for every $a \in S$. For every $* \in \mathcal{M}(S)$ and $s \in S$ we have

$s(* \triangleleft \circ)s = (s * s) \circ (s * s) = A$, which shows X is the right ideal, and we have $s(\circ \triangleleft *)s = (s \circ s) * (s \circ s) = A * A$, which means $\circ \triangleleft *$ is in X, so X is the right ideal too.

2. Let X be the set of all commutative binary operations on S, for every $\circ \in X$, $* \in \mathcal{M}(S)$ and $a, b \in S$ we have:
   $a(\circ \triangleleft *)b = (a \circ b) * (b \circ a)$ and $b(\circ \triangleleft *)a = (b \circ a) * (a \circ b)$.
   Since $\circ$ is commutative, both of them are same which means X is the right ideal of $(\mathcal{M}(S), \triangleleft)$, there is a same way to show that X is the left ideal too.

3. Follows from Remark 2.1.7 (i)

□

Corollary 2.3.6. *For $n = 2$, the set of Palindromic operations are ideals in $(\mathcal{M}(S), \triangleleft)$*

Proof. For $n = 2$ Palindromic operations have unique square since $\circ(0, 0) = \circ(1, 1)$ and so by Proposition 2.3.5 and Example 1.3.1, Palindromic operations are ideals in $(\mathcal{M}(S), \triangleleft)$ □

2.4 Nearring Structures having $\triangleleft$ as Multiplication

Definition 2.4.1. Suppose S is an arbitrary set and $\mathcal{M}(S)$ is the set of all binary operations on S, then we can define a new set of operations on $\mathcal{M}(S)$, denoted by $M(\mathcal{M}(S))$. This new set contains operations that operate on binary operations on the set S.

In generalizing the vector modulo operation defined in Definition 1.3.2 and 2.4.1 we can define a map $\odot : \mathcal{M}(S) \to M(\mathcal{M}(S))$ by $* \mapsto \circledast$. Where $a(\circ_1 \circledast \circ_2)b = (a \circ_1 b) * (a \circ_2 b)$ for all $\circ_1, \circ_2 \in \mathcal{M}(S)$ and $a, b \in S$.

Remark 2.4.2. *For $a, b \in S$ and $*, \circ \in \mathcal{M}(S)$, we choose $\circ_1, \circ_2 \in \mathcal{M}(S)$ arbitrary, then*

$$(* \triangleleft \circ)(\circ_1, \circ_2)(a, b) = \circ(*(\circ_1, \circ_2)(a, b), *(\circ_2, \circ_1)(a, b)).$$

Similarly,

$\circledcirc(\circledast(\circ_1, \circ_2)(a.b), \circledast(\circ_2, \circ_1)(a, b)) = \circ(*(\circ_1, \circ_2)(a, b), *(\circ_2, \circ_1)(a, b)).$

This shows that $* \triangleleft \circ = \circledast \triangleleft \circledcirc$ *and* $\odot$ *is a* $\triangleleft-$*homomorphism*

Proposition 2.4.3. *If* $(S, \circ)$ *distributes over* $(S, *)$ *then* $(\mathcal{M}(S), \circledcirc)$ *distributes over* $(\mathcal{M}(S), \circledast)$

Proof. If $(S, \circ)$ left distributes over $(S, *)$ by definition we can choose $a, b, c \in S$, such that $a \circ (b * c) = (a \circ b) * (a \circ c)$ then for arbitrary $\circ_1, \circ_2$, and $\circ_3 \in \mathcal{M}(S)$, $a(\circ_1 \circledcirc (\circ_2 \circledast \circ_3)b = (a \circ_1 b) \circ a(\circ_2 \circledast \circ_3)b = (a \circ_1 b) \circ \{(a \circ_2 b) * (a \circ_3 b)\} = \{(a \circ_1 b) \circ (a \circ_2 b)\} * \{(a \circ_1 b) \circ (a \circ_3 b)\} = a(\circ_1 \circledcirc \circ_2)b * a(\circ_1 \circledcirc \circ_3)b = a((\circ_1 \circledcirc \circ_2) \circledast (\circ_1 \circledcirc \circ_3))b$. We can similarly show for right distributive.

□

Proposition 2.4.4. $(\mathcal{M}(S), \oplus, \triangleleft)$ *is a left nearring.*

Proof. By Proposition 1.3.3, $(\mathcal{M}(S), \oplus)$ is an abelian group, and by Theorem 2.1.3 $(\mathcal{M}(S), \triangleleft)$ is a monoid, $\triangleleft$ distributes over $\oplus$ from the left since for every $a, b \in S$ and $\circ_1, \circ_2, \circ_3 \in \mathcal{M}(S)$, $a(\circ_1 \triangleleft (\circ_2 \oplus \circ_3))b = \circ_2 \oplus \circ_3(\circ_1(a, b), \circ_1(b, a)) = \circ_2(\circ_1(a, b), \circ_1(b, a)) + \circ_3(\circ_1(a, b), \circ_1(b, a)) = a((\circ_1 \triangleleft \circ_2) \oplus (\circ_1 \triangleleft \circ_3))b$.

□

The following theorem generalizes Proposition 2.4.4

Theorem 2.4.5. *If* $(S, *)$ *is an Abelian group then* $(\mathcal{M}(S), \circledast)$ *is an Abelian group and* $(\mathcal{M}(S), \circledast, \triangleleft)$ *is a left nearring.*

Proof. First we show that the operation $\circledast$ satisfies five conditions, closure, associativity, identity, invertibility and commutativity, which means $(\mathcal{M}(S), \circledast)$ is an abelian group. $(\mathcal{M}(S), \circledast)$ is closed by definition.

1. Associativity: by the definition we have $\forall \circ_1, \circ_2, \circ_3 \in \mathcal{M}(S)$ and $a, b \in S$

   $\circledast(\circledast(\circ_1, \circ_2), \circ_3)(a, b) = *(\circledast(\circ_1, \circ_2)(a, b), \circ_3(a, b)) = *(*(\circ_1(a, b), \circ_2(a, b)), \circ_3(a, b))$

   $= *(\circ_1(a, b), *(\circ_2(a, b), \circ_3(a, b))) = \circledast(\circ_1, \circledast(\circ_2, \circ_3))(a, b)$

   Since $*$ is associative, $\circledast$ has the associative property.

2. Identity: Suppose the identity of $(S, *)$ is e, we show that the operation $\circ_e$ with $\circ_e(a, b) = e$ for every $a, b \in S$ is an identity for $(\mathcal{M}(S), \circledast)$.

   For every $\star \in \mathcal{M}(S)$ and $a, b \in S$ we have

   $\circledast(\circ_e, \star)(a, b) = *(\circ_e(a, b), \star(a, b)) = *(e, \star(a, b)) = \star(a, b)$

   Similarly we can show that $\circ_e$ is the right identity.

3. Invertibility: For every $\star \in \mathcal{M}(S)$, define $\star^{-1}$ as follows. $\star^{-1}(a, b) = \star(a, b)^{-1}$, for every $(a, b) \in S$. It is easy to show that $\star^{-1}$ is the inverse of $\star$ in $(\mathcal{M}(S), \circledast)$.

4. Commutativity: For every $\circ, \star \in \mathcal{M}(S)$ and $(a, b) \in S$,

   $\circledast(\circ, \star)(a, b) = *(\circ(a, b), \star(a, b))$.

   Since $*$ is commutative the last statement is equal to

   $*(\star(a, b), \circ(a, b)) = \circledast(\star, \circ)(a, b)$.

Now we show that $\triangleleft$ distributes over $\circledast$ from the left, for this purpose we show that for every $\circ_1, \circ_2, \circ_3 \in \mathcal{M}(S)$ and $(a, b) \in S$,

$\triangleleft(\circ_1, \circledast(\circ_2, \circ_3))(a, b) = \circledast(\triangleleft(\circ_1, \circ_2), \triangleleft(\circ_1, \circ_3))(a, b)$.

From the left hand side we have

$\circledast(\circ_2, \circ_3)(\circ_1(a, b), \circ_1(b, a)) = *(\circ_2(\circ_1(a, b), \circ_1(b, a)), \circ_3(\circ_1(a, b), \circ_1(b, a)))$

From the right hand side we have

$*(\triangleleft(\circ_1, \circ_2)(a, b), \triangleleft(\circ_1, \circ_3)(a, b)) = *(\circ_2(\circ_1(a, b), \circ_1(b, a)), \circ_3(\circ_1(a, b), \circ_1(b, a)))$.

$\square$

2.5 The Group of Automorphisms of the Magma Monoid

One of the driving forces of this project has been to understand subsets of $\mathcal{M}(S)$ of the form $out(*)$. The $\triangleleft$ operation allows us to ponder the possibility that the following question may have a positive answer.

Question 2.5.1. *Is it the case that* $(out(*), \triangleleft) \cong (out(\star), \triangleleft) \Leftrightarrow (S, *) \cong (S, \star)$

Seeking a proof leads us to take a closer look at $\triangleleft$ homomorphisms in general and to the group of automorphisms of $\mathcal{M}(S)$ in particular. The first of the following three subsections includes a positive answer to one of the implications of the question above; the second subsection develops terminology and results needed to tackle the proof of the converse. The third subsection illustrates the strategy and points out the results that would be needed to prove that converse. Those results have remained elusive and will be the subject of future research.

2.5.1 A Group of Automorphisms of $\mathcal{M}(S)$ Induced by Permutations of S

Definition 2.5.2. Let $|S| \geq 1$ and $\mathcal{P}(S)$ the symmetric group on S elements i.e ($\mathcal{P}(S) = \{f : S \to S | f$ is bijective$\}$), define the mapping $\hat{\ } : \mathcal{P}(S) \to \mathcal{M}(S)^{\mathcal{M}(S)}$ such that for each $\sigma \in \mathcal{P}(S)$, $\hat{\sigma} : \mathcal{M}(S) \to \mathcal{M}(S)$ and for each $* \in \mathcal{M}(S)$
$\hat{\sigma}(*)(a, b) = \sigma(\sigma^{-1}(a) * \sigma^{-1}(b)) \forall a, b \in S$

Proposition 2.5.3. $\hat{\ }$ *respects composition.* $\forall \sigma, \tau \in \mathcal{P}(S)$, $\widehat{\sigma\tau} = \hat{\sigma}(\hat{\tau})$

Proof. $\forall a, b \in S$, $* \in \mathcal{M}(S)$ and $\sigma, \tau \in \mathcal{P}(S)$,

$$\widehat{\sigma\tau}(*)(a, b) = \sigma\tau(\tau^{-1}\sigma^{-1}(a) * \tau^{-1}\sigma^{-1}(b)) = \sigma(\sigma^{-1}(a)\hat{\tau}(*)\sigma^{-1}(b)) = \hat{\sigma}(\hat{\tau}(*))(a, b).$$

□

Following from definition 2.5.2, $\hat{\sigma}^{-1}(*)(a, b) = \sigma^{-1}(\sigma(a) * \sigma(b))$, and so $\hat{\sigma}\hat{\sigma}^{-1} = \hat{\mathbf{1}}_s$ where $\hat{\mathbf{1}}_s(*)(a, b) = a * b$.

It can be seen that $\hat{\sigma}$ is one-to-one and onto.

Proposition 2.5.4. *Given $(\mathcal{M}(S), \triangleleft)$ and the mapping ,*

$\hat{\ } : P(S) \to Aut(\mathcal{M}(S), \triangleleft)$ *then* $\hat{\sigma}(\star \triangleleft *) = \hat{\sigma}(\star) \triangleleft \hat{\sigma}(*)$ *for all* $\sigma \in P(S)$ *and* $\star, * \in M(S)$. *This implies that* $\hat{\sigma}$ *is* $\triangleleft-$*homomorphism,* $\forall \sigma \in \mathcal{P}(S)$.

Proof. We show that $[\hat{\sigma}(\star \triangleleft *)](a, b) = [\hat{\sigma}(\star) \triangleleft \hat{\sigma}(*)](a, b)$

LHS : $[\hat{\sigma}(\star \triangleleft *)](a, b) = \sigma(\sigma^{-1}(a)[\star \triangleleft *]\sigma^{-1}(b)) =$
$\sigma((\sigma^{-1}(a) \star \sigma^{-1}(b)) * (\sigma^{-1}(b) \star \sigma^{-1}(a)))$

RHS: $[\hat{\sigma}(\star) \triangleleft \hat{\sigma}(*)](a, b) = [\hat{\sigma}(\star)(a, b)]\hat{\sigma}(*)[\hat{\sigma}(\star)(b, a)] =$
$\sigma((\sigma^{-1}(a) \star \sigma^{-1}(b)) * (\sigma^{-1}(b) \star \sigma^{-1}(a)))$

□

Corollary 2.5.5. *Let $\mathcal{S}(\mathcal{M}(S))$ be the permutation of elements in $\mathcal{M}(S)$ then* $\hat{\ } : P(S) \to \mathcal{S}(\mathcal{M}(S))$ *is one-to-one and* $(\mathcal{S}(\mathcal{M}(S)), o) \subset (\mathcal{M}(S)^{\mathcal{M}(S)}, o)$

Proof. Given $\sigma \in \mathcal{P}(S), * \in \mathcal{M}(S)$ and $\forall a, b \in S$ we show that if $a(\hat{\sigma}(*))b = a * b$ $\implies \sigma = \mathbf{1}$.

$\sigma(\sigma^{-1}(a) * \sigma^{-1}(b)) = a * b \forall * \in \mathcal{M}(S), \forall a, b \in S$

$\implies \sigma^{-1}(a) * \sigma^{-1}(b) = \sigma^{-1}(a * b) \forall a, b \in S$

$\sigma^{-1}$is a homomorphism $\forall * \in \mathcal{M}(S)$ and so $\sigma(a * b) = \sigma(a) * \sigma(b) \forall * \in \mathcal{M}(S)$

In particular for $* = C_c, \sigma(c) = c \forall c \in S$ and $\sigma = \mathbf{1}$ □

Let $\sigma, \eta \in P(S)$, if $\sigma \neq \eta$ then there exist $a \in S$ such that $\sigma(a) \neq \eta(a)$, let $C_a \in \mathcal{M}(S)$ and $C_a(x, y) = a, (\forall x, y \in S)$. Then $\hat{\sigma}(C_a)(x, y) = \sigma(a) \neq \hat{\eta}(C_a)(x, y) = \eta(a)$. This means that $\hat{\ }$ is a one-to-one function.

We close this section with a partial answer to question 2.5.1, the answer is affirmative for one of the two directions.

Proposition 2.5.6. *If $\lambda : (S, \star) \to (S, *)$ is an isomorphism then* $\hat{\lambda} : (out(\star), \triangleleft) \to (out(*), \triangleleft)$ *is an isomorphism.*

Proof. If λ is an isomorphism then $\lambda \in P(S)$, and this implies that $\forall \alpha_1, \alpha_2 \in \mathcal{M}(S)$ $\hat{\lambda}(\alpha_1 \triangleleft \alpha_2) = \hat{\lambda}(\alpha_1) \triangleleft \hat{\lambda}(\alpha_2)$ by Proposition 2.5.4.

Suppose $\Box \in out(\star)$ then for all $a, b \in S$, $a \star (b \Box c) = (a \star b) \Box (a \star c)$ we show that $a * (b \hat{\lambda}(\Box) c) = (a * b) \hat{\lambda}(\Box)(a * c)$

LHS : $a * (b \hat{\lambda}(\Box) c) = a * \lambda(\lambda^{-1}(b) \Box \lambda^{-1}(c))$

RHS: $(a * b) \hat{\lambda}(\Box)(a * c) = \lambda(\lambda^{-1}(a * b) \Box \lambda^{-1}(a * c)) = \lambda(\lambda^{-1}(a) \star \lambda^{-1}(b) \Box \lambda^{-1}(a) \star \lambda^{-1}(c))$
$= \lambda(\lambda^{-1}(a) \star (\lambda^{-1}(b) \Box \lambda^{-1}(c)) = \lambda(\lambda^{-1}(a)) * \lambda(\lambda^{-1}(b) \Box \lambda^{-1}(c)) = a * \lambda(\lambda^{-1}(b) \Box \lambda^{-1}(c))$

□

The following subsection shows a strategy to prove the converse of this proposition; as you will see, the strategy relies strongly on the possiblity of obtaining an affirmative answer for the following question:

Question 2.5.7. *Is* $\hat{\cdot} : P(S) \to Aut(\mathcal{M}(S), \triangleleft)$ *onto?*

2.5.2 Subsets of the Magma Monoid that are Invariant Over Automorphisms

As seen in Remark 2.1.7, the set $\mathcal{K}$ of constant operations on S is an ideal of $M(S)$; we show next that it is invariant under automorphisms. As an illustration, the following remark shows that $\mathcal{K}$ is invariant under elements in the range of $\hat{\cdot}$. The general proof follows immediately after.

Remark 2.5.8. *The set* $\hat{\mathcal{K}} = \{C_i | i \in S\}$ *is equal to* $\mathcal{K}$.

Proof. $\hat{\sigma}(C_a)(x, y) = \sigma(\sigma^{-1}(x) C_a \sigma^{-1}(y)) = \sigma(a) \implies \hat{\sigma}(C_a) = C_{\sigma(a)} \forall C_a \in \mathcal{K}$ and $\forall a \in S$. □

Proposition 2.5.9. *For all* $\theta \in Aut(\mathcal{M}(S), \triangleleft)$ *there exist* $\sigma \in P(S)$ *such that* $\theta\big|_{\mathcal{K}} = \hat{\sigma}\big|_{\mathcal{K}}$

For all $* \in M(S)$, $* \triangleleft C_i = C_i$ by Lemma 2.1.8 and by Remark 2.5.8 $\theta(C_i) \in \mathcal{K} \forall i \in S$.

$$* \triangleleft \theta(C_i) = \theta(C_i) \tag{2.5.1}$$

$$\theta^{-1}(* \triangleleft \theta(C_i)) = C_i \tag{2.5.2}$$

$$\theta^{-1}(*) \triangleleft C_i = C_i \tag{2.5.3}$$

Define a map $i : \mathcal{K} \to S$ by $C_a \mapsto a$, then for $\theta : \mathcal{K} \to \mathcal{K}$, we can find λ that makes the following diagram commutes.

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "\theta"] \arrow[d, "i"] & \mathcal{K} \arrow[d, "i"] \\
S \arrow[r, "\lambda"] & S
\end{tikzcd}$$

Where $\lambda = io\theta oi^{-1}$ (" $\circ$ " is the composition symbol)

Lemma 2.1.8 allows us to show that Remark 2.5.8 does not hold only for permutation-induced automorphisms:

Lemma 2.5.10. *Let $\phi \in Aut(\mathcal{M}(S), \triangleleft)$ then for each $* \in \mathcal{K}$, $\phi(*) \in \mathcal{K}$*

Proof. Let $C_i \in \mathcal{K}$ for $i \in \{1, 2, 3, ..., n\}$ and choose $\kappa \in \mathcal{M}(S)$ and let $\phi(C_i) = \kappa$, then we show that κ is a constant operation.

For each $\eta \in \mathcal{M}(S), \phi(\eta) \in \mathcal{M}(S)$ then by definition

$\phi(\eta \triangleleft C_i) = \phi(\eta) \triangleleft \phi(C_i)$

$\implies \kappa = \phi(\eta) \triangleleft \kappa \forall \eta \in \mathcal{M}(S)$

Then by Lemma 2.1.8, κ is a constant operation. $\square$

If $|S| = n$, then $|\mathcal{K}| = n$ and the above lemma then indicates that every automorphism of the magma monomial induces a permutation in S_n.

The following Lemma may be informally expressed by saying that the diagonal values of the images are determined by the diagonal values of the input and the images of the constant operations. More precisely, we state:

Lemma 2.5.11. *For any automorphism $\phi \in Aut(\mathcal{M}(S), \triangleleft))$ of the magma monoid, and for any $* \in \mathcal{M}(S), a \in S$, $\phi(*)(a, a)$ is determined by the images of the constant operations under ϕ and the values of $*(b, b)$ for $b \in S$.*

Proof. Let $\sigma \in S_n$ and for a given $\phi \in Aut(\mathcal{M}(S), \triangleleft)$ we define $\phi(C_i) = C_{\sigma(i)} \forall i \in S$ then for arbitrary $* \in \mathcal{M}(S)$ let $*(i, i) = j$.

Then by definition

$$\phi(C_i \triangleleft *) = \phi(C_i) \triangleleft \phi(*)$$

$$\phi(C_{*(i,i)}) = C_{\sigma(i)} \triangleleft \phi(*)$$

$$\phi(C_{*(i,i)}) = C_{\phi(*)(\sigma(i),\sigma(i))}$$

$$C_{\sigma(*(i,i))} = C_{\phi(*)(\sigma(i),\sigma(i))}$$

This implies that $\phi(*)(\sigma(i), \sigma(i)) = \sigma(*(i, i))$, running through each $i \forall i \in S$ we can get $\phi(*)(\sigma(i), \sigma(i))$ □

Definition 2.5.12. An operation $* \in \mathcal{M}(S)$ is said to be an idempotent operation with respect to $\triangleleft$ if $(* \triangleleft *)(a, b) = *(a, b), \forall a, b \in S$

For the remainder of this section, let us denote by $\mathcal{D}$ the set of all idempotent operations in $\mathcal{M}(S)$.

Lemma 2.5.13. *Let $\phi \in Aut(\mathcal{M}(S), \triangleleft)$ then for each $* \in \mathcal{D}$, $\phi(*) \in \mathcal{D}$.*

Proof. The proof is straightforward.

□

Proposition 2.5.14. *If $\circ, * \in \mathcal{U}(\mathcal{M}(S))$ and $(S, *) \cong (S, \circ)$ then $(S, *^{-1}) \cong (S, \circ^{-1})$*

Proof. Let $\mu : (S, *) \longmapsto (S, \circ)$ and take $a, b, c \in S$.

$$a *^{-1} b = c \implies \exists d \in S \text{ such that } c * d = a \text{ and } d * c = b$$

$$\iff \exists d \in S \text{ such that } \mu(c) \circ \mu(d) = \mu(a) \text{ and } \mu(d) \circ \mu(c) = \mu(b)$$

$$\iff \mu(a) \circ^{-1} \mu(b) = \mu(c) \text{ and } \mu(b) \circ^{-1} \mu(a) = \mu(d)$$

□

Notation 2.5.15. *For each* $i \in \{1, 2, 3, 4, 5, 6\}$ *, define* A_i *by the following*

$A_1 = \{$ *Units in* $\mathcal{M}(S)\}$

$A_2 = \{$ *Constant Operations in* $\mathcal{M}(S)\}$

$A_3 = \{$ *Square root of Constant Operations in* $\mathcal{M}(S)\}$

$A_4 = \{$ *Idempotent Operations in* $\mathcal{M}(S)\}$

$A_5 = \{$ *Square root of Idempotent Operations in* $\mathcal{M}(S)\}$

$A_6 = \{$ *4th root of Constant Operations in* $\mathcal{M}(S)\}$

These classifications are arranged in order of precedence, so elements in A_1 are selected first, followed by elements in A_2 in that order. This is done to make the classification disjoint.

Proposition 2.5.16. *For* $n = 2$*,* $|Aut(\mathcal{M}(S), \triangleleft)| = 2$ *that is* $\hat{\ } : P(S) \to Aut(\mathcal{M}(S), \triangleleft)$ *is an isomorphism for* $|S| = 2$*.*

Proof. For $n = 2$, $|\mathcal{M}(S)| = 16$, and so $1 \leq |Aut(\mathcal{M}(S), \triangleleft)| \leq 16!$ to prove that $|Aut(\mathcal{M}(S), \triangleleft)| = 2$ we divide all the elements of $\mathcal{M}(S)$ into 5 subsets namely: Constant operations which contain $\{\mathbf{0}, \mathbf{15}\}$, Square root of constant operations, $\{\mathbf{6}, \mathbf{9}\}$, Units, $\{\mathbf{3}, \mathbf{5}, \mathbf{10}, \mathbf{12}\}$, Idempotent operations, $\{\mathbf{4}, \mathbf{8}, \mathbf{13}, \mathbf{14}\}$, Square root of idempotent operations, $\{\mathbf{1}, \mathbf{2}, \mathbf{7}, \mathbf{11}\}$

Following from Lemma 2.5.10 and Lemma 2.5.13 we can show that each sets is invariant under the automorphism. The calculations needed to find the automorphism for $n = 2$ can be found in Appendix D and the two maps are displayed in the table in Appendix E. From the table in Appendix E, it is clear that $Aut(\mathcal{M}(S), \triangleleft) \cong \hat{\sigma}\ \ \forall \sigma \in S_2$ where $\hat{\sigma}$ is defined in Definition 2.5.2

□

2.5.3 Comparing Outsets

Remark 2.5.17. *From Appendix C we can see that* $(out(14), \triangleleft) = (out(15), \triangleleft)$ *but* 15 *is a constant operation and* 14 *is an idempotent operation, implying that* $14 \cong 15$.

The converse of Prop 2.5.6 from Remark 2.5.17 is not generally true and so we propose a conjecture that will provide a limited positive answer to Question 2.5.1.

Conjecture 2.5.18. *If* $(out(\star), \triangleleft) \simeq (out(*), \triangleleft)$ *and there exist* $\phi \in Aut(\mathcal{M}(S), \triangleleft)$ *such that* $\phi(\star) = *$ *then* $(S, \star) \simeq (S, *)$

3 Distributive Magma

3.1 Introduction and Motivation

In [13], left, right, and two-sided distributivity hierarchy graphs of a set are introduced. Given a set S, its (left, right, two-sided) hierarchy graph (lH(S), rH(S), H(S)) has $\mathcal{M}(S)$ as vertices and there is an edge from one operation $*$ to another one $\circ$ if $*$ distributes over $\circ$, respectively, on the left, on the right, or on both sides. Even if one focuses on finite sets, the complexity of these hierarchy graphs grows very rapidly.

3.2 Universal Distributors

Definition 3.2.1. A binary operation on a nonempty set S is a mapping from $S \times S$ into S. More generally, for any positive integer, n a mapping $f : S^n \to S$ where $S^n = S \times S \times \ldots \times S$ is called an n-ary operation on S.

Definition 3.2.2. A pair of the form $(S, *)$ where $a * b \in S$ for all $a, b \in S$ will be called a **magma** and denoted as $\mathcal{M}(S)$.

A triple $(S, *, \circ)$ where $(S, *)$ and $(S, \circ)$ are magmas is also called a magma.

Definition 3.2.3. Given $\circ_1, \circ_2 \in \mathcal{M}(S)$, $\circ_1$ is said to distribute over $\circ_2$ on the left if:

$$a \circ_1 (b \circ_2 c) = (a \circ_1 b) \circ_2 (a \circ_1 c), \forall a, b, c \in S \tag{3.2.1}$$

or on the right, if:

$$(b \circ_2 c) \circ_1 a = (b \circ_1 a) \circ_2 (c \circ_1 a), \forall a, b, c \in S \tag{3.2.2}$$

If $\circ_1$ is both left- and right- distributive over $\circ_2$, then $\circ_1$ is said to be two-sided distributive over $\circ_2$

Definition 3.2.4. A magma $(S, \circ, *)$ is called a **left (resp. right)** distributve magma if $*$ left (resp. right) distribute over $\circ$.

A magma is simply called a distributive magma if its **two-sided** distributive (both left and right). If $(S, \circ, *)$ is (left, right or two-sided)- distributive magma, we say that the pair $(\circ, *)$ is a (left, right) distributive pair, and that $*$ is at the top and $\circ$ is at the bottom of $(S, \circ, *)$.

Definition 3.2.5. 1. For every operation $*$ in $\mathcal{M}(S)$, the outset of $*$, is denoted by $out(*)$, and is defined as the set $out(*) = \{\circ \in \mathcal{M}(S) | * \text{ distributes over } \circ\}$.

2. We define the inset of $*$, denoted by $in(*)$, $\forall * \in \mathcal{M}(S)$, as $in(*) = \{\circ \in \mathcal{M}(S) | \circ \text{ distributes over } *\}$

Question 3.2.6. *1. For an arbitrary set S, does there exist an operation $*$ that is a universal distributor; that is such that for all $\circ \in \mathcal{M}(S)$, $(S, *, \circ)$ is a distributive magma*

2. For an arbitrary set S, does there exist an operation $*$, that is never at a bottom for all operations in $\mathcal{M}(S)$; that is $\forall \circ \in \mathcal{M}(S)$, $(S, *, \circ)$ is not a distributive magma

The first question was asked in [13], Question 9 and a universal distributor in the form of the right zero band which was introduced in [4] was given as an example of a universal left distributor. In this work we prove that the left and right zero band are the only universal distributors. We also construct operations called almost- constant operations and show that no operation in $\mathcal{M}(S)$ distributes over them.The answer to the two above questions is the subject of Theorem 3.2.19.

We divide the proof of Theorem 3.2.19 into the following five lemmas.

Definition 3.2.7. $\forall * \in \mathcal{M}(S)$, we define $*^{op} \in \mathcal{M}(S)$ as $a *^{op} b = b * a$, and called $*^{op}$ of $*$. If $*$ is commutative then $* = *^{op}$.

Lemma 3.2.8. *$\forall * \in \mathcal{M}(S)$, $\{\pi_1, \pi_2\} \in \bigcap_{* \in \mathcal{M}(S)} out(*)$ where $\pi_1(i, j) = i$ and $\pi_2(i, j) = j$, $\forall i, j \in S$*

Proof. $\forall * \in \mathcal{M}(S)$, and $a, b, c \in S$ $a * (b\pi_1 c) = a * b = (a * b)\pi_1(a * c)$.

Also, $a * (b\pi_2 c) = a * c = (a * b)\pi_2(a * c)$.

Similar argument holds for the right distributes. □

Lemma 3.2.9. *Given* $\circ \in \mathcal{M}(S)$, $\circ \in \bigcap_{*\in\mathcal{M}(S)} out_l(*)$ *if, and only if* $\circ \in \bigcap_{*\in\mathcal{M}(S)} out_r(*)$

Proof. Let $* \in \mathcal{M}(S)$, if $\circ \in \mathcal{M}(S)$, and $\circ \in \bigcap_{*\in\mathcal{M}(S)} out_r(*)$ then $\forall a, b, c \in S$,

$a *^{op} (b \circ c) = (b \circ c) * a = (b * a) \circ (c * a) = (a *^{op} b) \circ (a *^{op} c)$.

So $\circ \in \bigcap_{*\in\mathcal{M}(S)} out_l(*)$

Similar argument holds for the converse. □

Lemma 3.2.10. *If* $\circ \in \bigcap_{*\in\mathcal{M}(S)} out(*)$ *then* $\circ$ *is idempotent. ie.* $(\circ(a, a) = a, \forall a \in S)$

Proof. Choose $\pi_1 \in \mathcal{M}(S)$, if $\circ \in \bigcap_{*\in\mathcal{M}(S)} out(*)$, then $\forall a, b, c \in S$

$a\pi_1(b \circ c) = (a\pi_1 b) \circ (a\pi_1 c) \longleftrightarrow a = a \circ a$ □

Lemma 3.2.11. *If* $\circ \in \bigcap_{*\in\mathcal{M}(S)} out(*)$ *then* $\circ(a, b) \in \{a, b\}$, $\forall a, b \in S$. *In this certain* $\circ$ *is said to be a 2-value magma.*

Proof. Assume $\circ(a, b) = c$ where $c \notin \{a, b\}$. Let $* \in \mathcal{M}(S)$ and defined $*$ such that $*(a, a) = a, *(a, b) = b, *(a, c) = a$ and $\forall x, y \in S, *(x, y) = a$

$$a * (a \circ b) = (a * a) \circ (a * b) \tag{3.2.3}$$

$$a * c = (a * a) \circ (a * b) \tag{3.2.4}$$

$$a = a \circ b = c \tag{3.2.5}$$

Since $a \neq c$. There is a contradiction. Hence $\circ(a, b) \in \{a, b\}$ □

Lemma 3.2.12. *Given* $\circ \in \mathcal{M}(S)$, $\circ \in \bigcap_{*\in\mathcal{M}(S)} out(*)$ *if, and only if given* $a, b, c, d \in S$, $\circ \in \mathcal{M}(S)$ *if* $a \circ b = a$ *and* $c \circ d = d$ *then either* $a = b$ *or* $c = d$

Proof. Suppose $a \circ b = a$, $c \circ d = d$ $c \neq d$ Let $* \in \mathcal{M}(S)$ and defined $*$ such that $*(a, d) = b, *(a, c) = a$ and $\forall x, y \in S, *(x, y) = b$

$$a * (c \circ d) = (a * c) \circ (a * d) \tag{3.2.6}$$

$$a * d = (a * c) \circ (a * d) \tag{3.2.7}$$

$$b = a \circ b = a \tag{3.2.8}$$

□

Definition 3.2.13. An operation $C_i \in \mathcal{M}(S)$ for each $i \in S$ is said to be a constant operation if and only if $C_i(a, b) = i$ for each $a, b \in S$

Definition 3.2.14. Given $* \in \mathcal{M}(S)$, $*$ is said to be an isolated operation if no operation distributes over it, that is($\circ \nrightarrow *$ for all $\circ \in \mathcal{M}(S)$)

Question 3.2.15. *How are the isolated operations described? Is there any general way of describing them? Do they exist for $\mathcal{M}(S)$ for every set, S with $|S| \geq 2$*

Notation 3.2.16. *For a constant operation C_i, an operation $C_i(a)$ is said to be a* ***almost-constant*** *operation if* $C_i(a)(x, y) = \begin{cases} a & if \{x, y\} = \{i\} \\ i & otherwise \end{cases}$

Example 3.2.17. For $|S| = 3$, the following operations are almost constant operations.

1	2	1	0
2	0	0	0
1	0	0	0
0	0	0	1

13121	2	1	0
2	1	2	2
1	2	2	2
0	2	2	2

9760	2	1	0
2	1	1	1
1	1	0	1
0	1	1	1

(3.2.9)

Proposition 3.2.18. *Every almost-constant operation is an isolated operation in $\mathcal{M}(S)$*

Proof. Following from Notation 3.2.16 let $\circ = C_i(a)$ for each $a, i \in S$ and $a \neq i$.

Assume that there exist an $* \in \mathcal{M}(S)$ that distributes over $\circ$.

Step 1: We show that $i * i = i$

$i * (i \circ a) = (i * i) \circ (i * a)$

$i * i = (i * i) \circ (i * a)$.

Since $i * i \neq a, \forall a \in S$ this implies that $i * i = i$.

Step 2: We show that $a * i = i, \forall a \in S\, a \neq i$

$a * (a \circ i) = (a * a) \circ (a * i)$

$a * i = (a * a) \circ (a * i)$

Following from similar argument in Step 1, $a * i = i$

Step 3:

$(a \circ a) * i = (a * i) \circ (a * i)$

$i * i = (a * i) \circ (a * i)$

Following from step 1 and Step 2;

$i = i * i = (a * i) \circ (a * i) = i \circ i = a$

This is a contradiction. Since $a \neq i$

Hence $*$ does not exist and $C_i(a)$ is an isolated operation.

□

Theorem 3.2.19. *$\bigcap_{* \in \mathcal{M}(S)} out(*) = \{\pi_1, \pi_2\}$ where $\pi_1(i, j) = i$ and $\pi_2(i, j) = j$ $\forall i, j \in S, \pi_1, \pi_2 \in \mathcal{M}(S)$ and $\bigcup_{* \in \mathcal{M}(S)} out(*) \subsetneq \mathcal{M}(S)$*

Proof. Follows from Lemma 3.2.10, 3.2.11, 3.2.12 and Proposition 3.2.18 □

Notation 3.2.20. *Given $n \in \mathbb{Z}$ we define the number theoretic function $\Pi(n)$ by $\Pi(n) = Max\ \{|out(*)|, \forall * \in \mathcal{M}(S)\}$ where $|S| = n$. For left and right distributes $\Pi_R(n) = \Pi_L(n) = n^{n \times n}$ since $\pi_1(i, j) = i$ right distributes over every operation in $\mathcal{M}(S)$ and $\pi_2(i, j) = j$ left distributes over every operation in $\mathcal{M}(S)$*

Proposition 3.2.21. *1. If $|S| = n$,then for any constant operation $C_i \in \mathcal{M}(S)$,the out degree of C_i for each i , $d_o(C_i) := |\{w | C_i \rightsquigarrow w\}| = n^{n^2-1}$ (ie $|out(C_i)| = n^{n^2-1}$)*

2. *For all $* \in \mathcal{M}(S)$, $*$ can distribute on at most one constant operation*

3. *$\{\pi_1, \pi_2\}$ distributes over every idempotent operations.*

4. *Given $k \in S$, if we define an operation $* \in S$ such that $*_k(i,k) = k = *_k(k,i) \forall i \in S$ then $*_k$ distributes over the constant operation C_k*

Proof. 1. Given $\circ \in \mathcal{M}(S)$, $a, b, c \in S$. Let $C_i(a,b) = i \forall a, b \in S$, if C_i distributes on $\circ$ then $aC_i(b \circ c) = (aC_ib) \circ (aC_ic) \Leftrightarrow i = i \circ i$. Thus C_i will distribute over every operation with idempotent i. Number of operations with idempotent i has cardinality n^{n^2-1}.

2. Given a constant operation C_i and $\circ \in \mathcal{M}(S)$. If $\circ \rightsquigarrow C_i$ then for all $a, b, c \in S$. $a \circ (bC_ic) = (a \circ b)C_i(a \circ c) \Longrightarrow a \circ i = i$. Similarly, $(bC_ic) \circ a = (b \circ a)C_i(c \circ a) \Longrightarrow i \circ a = i$. Thus any operation $\circ \in \mathcal{M}(S)$ with $a \circ i = i \circ a = i, \forall a \in S$ distributes over the constant operation C_i. Assume that $\circ$ distributes over the constants C_i, C_j where $j \neq i$. Then, there is a contradiction, $i = j \circ i = j$. Hence $\circ$ can distribute over only one constant operation.

3. π_1 right-distributes over every operation in $\mathcal{M}(S)$. $\forall a, b, c \in S$ and $* \in \mathcal{M}(S)$, $a\pi_1(b * c) = a$ and $(a\pi_1 b) * (a\pi_1 c) = a * a$. This implies that $\pi_1 \rightsquigarrow *$ if $a = a * a \forall a \in S$. Thus π_1 distributes over every idempotent. We can make similar argument for π_2.

4. We show that $*_k \rightsquigarrow C_k$. $\forall a, b, c \in S$ $a * (bC_kc) = (a * b)C_k(a * c) \implies a *_k k = k$. On the right-distributive: $\forall a, b, c \in S$ $(bC_kc) * a = (b * a)C_k(c * a) \implies k * a = k$. Which is true if $* = *_k$

$\square$

Remark 3.2.22. *Following from Proposition 3.2.21 and Lemma 3.2.18, we can see that $\forall * \in \mathcal{M}(S), |out(*)| < n^{n^2}$.*

Example 3.2.23. For $n = 2$, $d_0(C_1) = 2^3 = 8$

The constant operation $C_1 :=$ $\begin{array}{c|cc} \mathbf{15} & 1 & 0 \\ \hline 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}$ is intial for any operation with first entry 1

That is : $C_1 \to w$ if $w :=$ $\begin{array}{c|cc} & 1 & 0 \\ \hline 1 & 1 & * \\ 0 & * & * \end{array}$

Similarly, for $|S| = 3$, $C_1 :=$ $\begin{array}{c|ccc} & 2 & 1 & 0 \\ \hline 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}$ is initial for any binary operation of the form

$$\begin{array}{c|ccc} & 2 & 1 & 0 \\ \hline 2 & * & * & * \\ 1 & * & 1 & * \\ 0 & * & * & * \end{array}$$

Example 3.2.24. For $n = 2$, $\Pi(2) = 8$ This is achieved by the constant operations.

We now give the bound for the maximum *outsets* of $*$, $\forall * \in \mathcal{M}(S)$.

Proposition 3.2.25. *For all* $|S| = n \geq 2$, $n^{n^2-1} \leq \Pi(n) < n^{n^2}$.

3.3 Hierarchy Graph

3.3.1 Distributive Hierarchy Graph

Definition 3.3.1. The (left, right, two-sided) **Distributive Hierarchy Graph** $\mathcal{H}(S)$ of the set S is a directed graph which has vertices = { all binary operations on S } and edges : $\circ \to *$ i.e ($\circ$ is (left, right, two-sided) distributive over $*$). If $(S, *, \circ)$ is a (Right, Left or 2-sided) distributive magma, then : $\circ \to *$ and so we can say that a simple distributive hierarchy graph exist for every distributive magma.

The outset of an operation $*$, out($*$), is the neighborhood of the operation $*$, if $*$ is self distributive then out($*$) is a closed neighborhood.

Notation 3.3.2. *Given $n \in \mathbb{Z}$ and $n > 1$ we define the number theoretic function $O(n) =$ largest cycle-free path in $\mathcal{H}(S)$ where $|S| = n$. We denote $O_L(n), O_R(n)$ for left and right distributive chains respectively and $O(n)$ for two-sided distributive chains .*

$O(n)$ cannot be used to determine the order of an outset of an operation since distributive property in general is not transitive. We can only established the existence or nonexistence of outset by using the distributive hierarchy graph on the conditions stated in Remark 3.3.4

Question 3.3.3. *Whats is $O(n)$ for $n \geq 2$ for left, right and two-sided distributive operations in $\mathcal{M}(S)$*

Remark 3.3.4. *If $O(n) = 0$, then $(S, \circ, *)$ does not exist, and if $O(n) = n^{n \times n}$, then for each $\circ \in \mathcal{M}(S)$ there exist $* \in \mathcal{M}(S)$ such that $(S, \circ, *)$ is a distributive magma*

Notation 3.3.5. *Let $N = \{0, 1, 2, ..., n-1\}$ then given $\star \in M(N)$ we define a new operation $\hat{\star} \in M(N+1)$ by*

$$
\begin{array}{c|c|cccc}
\hat{\star} & n & n-1 & \cdots & 1 & 0 \\
\hline
n & n & n & \cdots & n & n \\
\hline
n-1 & n & & & & \\
\vdots & \vdots & & \bigstar & & \\
1 & n & & & & \\
0 & n & & & &
\end{array}
$$

Proposition 3.3.6. *Given $\star, * \in \mathcal{M}(S)$ if $\star$ distributes over $*$ ($\star \rightsquigarrow *$) then $\hat{\star}$ distributes over $\hat{*}$ ($\hat{\star} \rightsquigarrow \hat{*}$).*

Proof. If $\star \rightsquigarrow *$ then $\forall a, b, c \in \{0, 1, ..., n-1\}$,
$a \hat{\star} (b \hat{*} c) = a \star (b * c) = (a \star b) * (a \star c) = (a \hat{\star} b) \hat{*} (a \hat{\star} c)$. If we let one for the variables be n, then $a \hat{\star} (b \hat{*} c) = n = (a \hat{\star} b) \hat{*} (a \hat{\star} c)$

□

Remark 3.3.7. *Following from Proposition 3.3.6 it can be inferred that $\forall n$, every path in $\mathcal{H}(n)$ there is a related path in $\mathcal{H}(n+1)$ $\forall n \geq 2$ by extending each operation using Notation 3.3.5*

Definition 3.3.8. An operation $\star \in \mathcal{M}(S)$ is said to be simple if $\star \rightsquigarrow \circ$ then $\circ \in \{\pi_1, \pi_2\}$

Question 3.3.9. *For each $|S| = n$, how many simple operations are in $\mathcal{M}(S)$ $\forall n \geq 2$*

For $n = 2$ there are 8 simple operations.

Remark 3.3.10. *1. If $|S| = n$ and $i = 1, 2$, the out-degree of π_i ,*
$d_o(\pi_i) := |\{w | \pi_i \to w\}| = n^{n(n-1)}$

2. *If $|S| = n$, then for a constant operation C_i ,the in-degree of C_i ,*
$d_i(C_i) := |\{w | w \to C_i\}| = n^{(n-1)\times(n-1)}$

3. $*_k \rightsquigarrow C_k \rightsquigarrow \star_k \rightsquigarrow \pi_1 \rightsquigarrow \mathcal{I}_i \rightsquigarrow \pi_2 \rightsquigarrow \mathcal{I}_j$ *is path in $\mathcal{H}(S)$ $\forall n$ where $\mathcal{I}_i, \mathcal{I}_j$ are distinct idempotent operations and $\star_k$ is an operation with $k \star k = k, k \in S$*

Example 3.3.11. For $n = 2$, $d_i(C_1) = 2^{(2-1)\times(2-1)} = 2$

If $|S| = 2$ then $C_0 := \begin{array}{c|cc} & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ is terminal for any operation w if $w := \begin{array}{c|cc} & 1 & 0 \\ \hline 1 & * & 0 \\ 0 & 0 & 0 \end{array}$

For $|S| = 3$, $C_1 := \begin{array}{c|ccc} & 2 & 1 & 0 \\ \hline 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}$ is terminal for any binary operation w if $w :=$

$$\begin{array}{c|ccc} & 2 & 1 & 0 \\ \hline 2 & * & 1 & * \\ 1 & 1 & 1 & 1 \\ 0 & * & 1 & * \end{array}$$

Example 3.3.12. For $n = 2$, $d_o(10) = 2^{2(2-1)} = 4$, $d_o(12) = 2^{2(2-1)} = 4$

For $n = 3$, $d_o(19305) = 3^{3(3-1)} = 3^6$, $d_o(15897) = 3^{3(3-1)} = 3^6$

For $10 := \begin{array}{c|cc} & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}$, $10 \to w$ if $w := \begin{array}{c|cc} & 1 & 0 \\ \hline 1 & 1 & * \\ 0 & * & 0 \end{array}$

If $|S| = 3$ then $15897 =: \begin{array}{c|ccc} & 2 & 1 & 0 \\ \hline 2 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}$, $15897 \to w$ if $w := \begin{array}{c|ccc} & 2 & 1 & 0 \\ \hline 2 & 2 & * & * \\ 1 & * & 1 & * \\ 0 & * & * & 0 \end{array}$

For $12 := \begin{array}{c|cc} & 1 & 0 \\ \hline 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}$, $12 \to w$ if $w := \begin{array}{c|cc} & 1 & 0 \\ \hline 1 & 1 & * \\ 0 & * & 0 \end{array}$

If $|S| = 3$ then $19305 \to w$ if $w := \begin{array}{c|ccc} & 2 & 1 & 0 \\ \hline 2 & 2 & * & * \\ 1 & * & 1 & * \\ 0 & * & * & 0 \end{array}$

For $n = 2$, the hierarchy graph has 16 elements and it is still feasible to tackle the problem on a trial and error basis; that is the way in which the following examples were obtained. Gabriel A. López-Matthews kindly coded program [12] to aid us with that endeavor. The results obtained from the codes were used to complete the Table in Appendix C

Example 3.3.13. 1. For left-distributive magma $\mathcal{O}_L(2) = 14$

2. $\mathcal{O}(2)$ is unique but the representation is not unique

3. Every binary operation distributes over **10** and **12**

Example 3.3.14. Examples of largest paths in $\mathcal{O}_L(2)$ for left distributive

1. $0 \to 2 \to 4 \to 12 \to 8 \to 6 \to 3 \to 14 \to 9 \to 5 \to 10 \to 11 \to 15 \to 13$

Remark 3.3.15. *For 2-sided-distributive magma* $\mathcal{O}(2) = 11$

Example 3.3.16. largest paths in $|S| = 2$, for 2-sided distributive magma

1. $2 \to 10 \to 8 \to 0 \to 6 \to 3 \to 12 \to 14 \to 15 \to 9 \to 3$

2. $1 \to 12 \to 14 \to 15 \to 9 \to 3 \to 10 \to 8 \to 0 \to 6 \to 5$

Notation 3.3.17. *Let $B(n)_i$ denote an operation in $\mathcal{M}(S)$ such that in each operation i is idempotent for each $i \in S$, and $jB_i(n)$ denote an opration in $\mathcal{M}(S)$ such that in each operation i is idempotent and $\forall k \in S$ and $\star \in \mathcal{M}(S)$ $\star(k, j) = \star(j, k) = j$*

Let $D(n)_i$ be the longest chain $\mathcal{H}(S)$ generated by all operations of the form $B(n)_i$, and $jD_i(n)$ be the longest in $\mathcal{H}(S)$ be the longest chain $\mathcal{H}(S)$ generated be all operations of the form $jB(n)_i$

Example 3.3.18. For $n = 2$

$$B(2)_0 = \begin{array}{c|cc} & 1 & 0 \\ \hline 1 & * & * \\ 0 & * & 0 \end{array} \quad \text{where } * \in S$$

These are represented by all the even operations in $\mathcal{M}(S)$, so for all $i \in S$,

$B_i(2) = \{0, 2, 4, 6, 8, 10, 12, 14\}$ and $|D_0(2)| = 6$ which is given by

$8 \to 0 \to 2 \to 12 \to 14 \to 10$

$$1B_0(2) = \begin{array}{c|cc} & 1 & 0 \\ \hline 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}$$

For $n = 3$

$$B(3)_0 = \begin{array}{c|ccc} & 2 & 1 & 0 \\ \hline 2 & * & * & * \\ 1 & * & * & * \\ 0 & * & * & 0 \end{array} \quad \text{where } * \in S$$

$$2B_0(3) = \begin{array}{c|ccc} & 2 & 1 & 0 \\ \hline 2 & 2 & 2 & 2 \\ 1 & 2 & * & * \\ 0 & 2 & * & 0 \end{array}$$

Proposition 3.3.19. *For a given set S with $|S| = n > 2$,*

$n\bar{O}(n-1) + n + 5 \le \bar{O}(n) \le n^{n^2} - (k-1)$.

where k is number of isolated vertices.

Following from Proposition 3.3.19 we can see that;

$$\begin{aligned}
O(n) &\geqslant O(n-1) + (n-1)D(n-1) + n + 4 \\
O(n-1) &\geqslant O(n-2) + (n-2)D(n-2) + n - 1 + 4 \\
O(n-2) &\geqslant O(n-3) + (n-3)D(n-3) + n - 2 + 4 \\
&\vdots \\
O(3) &\geqslant O(2) + (2)D(2) + 3 + 4
\end{aligned}$$

Substituting $O(n-1), O(n-2), \ldots O(3)$ into Proposition 3.3.19, we get:

$O(n) \geqslant O(2) + \sum_{i=1}^{n-2}(n-i)D(n-i) + \sum_{i=0}^{n-3}(n-i) + 4(n-2)$

Proposition 3.3.20. *For a given set S with $|S| = n > 2$,*

$O(2) + \sum_{i=1}^{n-2}(n-i)D(n-i) + \sum_{i=0}^{n-3}(n-i) + 4(n-2) \leq O(n) \leq n^{n^2} - k.$

where k is number of isolated vertices.

Question 3.3.21. . *If $(A, \circ)$ is a magma then what should be the characteristics of $(A, *)$ such that $(A, \circ, *)$ is a distributive magma.*

This question has been studied for when $(A, \circ)$ is a group in [13]. It has not yet been generalized, thats when $(A, \circ)$ is a magma. We answer when $(A, \circ)$ is a constant operation

Proposition 3.3.22. *If $\circ_i, i \in A$ is a constant operation then $(A, \circ_i, *)$ is a distributive magma if and only if we define $*$ such that $i * i = i$, for all $i \in A$.*

Question 3.3.23. *What is the largest cardinality of a set of vertices $X(S) \subset M((S)$ such that the full subgraph of $\mathcal{H}(S)$ having $X(S)$ as its set of vertices is complete*

Proposition 3.3.24. *If $*_\sigma \in DCR$, then $*_\sigma$ is self distributive and also distributes over its inverse.*

Proof. $\forall \eta, \sigma \in S_n$ and $\forall a, b, c \in S$ if $(a \circ_\eta b) \circ_\sigma c = (a \circ_\sigma c) \circ_\eta (b \circ_\sigma c)$ then

$$\eta(a) \circ_\sigma c = \sigma(a) \circ_\eta \sigma(b) \tag{3.3.1}$$

$$\sigma(\eta(a)) = \eta(\sigma(a)) \tag{3.3.2}$$

From the last equation, $\sigma(\eta(a)) = \eta(\sigma(a)) \iff \eta(a) = \sigma(a) \forall a \in S$ or

$\iff \eta\sigma = \sigma\eta = Id..$ $\square$

4 Operations Induced by Graphs

4.1 Graph Magmas

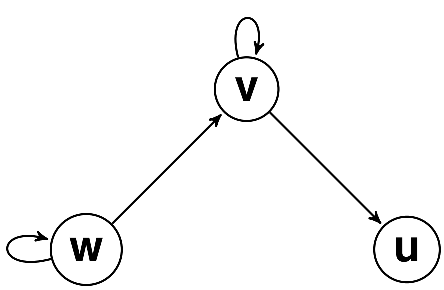
Definition 4.1.1. The graph algebra, $\mathrm{Alg}(G)$, of a graph $G = (V, E)$ is the set $V \cup \{\infty\}$ endowed with the multiplication $ab = a$ if $(a, b) \in E$ and $ab = \infty$ otherwise, and $a\infty = \infty a = \infty\infty = \infty\ \forall a, b \in V$

Definition 4.1.2. Given a graph $G = (S, E)$ we let the set $G\mathcal{M}(S) = \{* \in \mathcal{M}(S) | \exists 0 \in S,$ such that $\forall a \in S, a * 0 = 0 = 0 * a$ and $\forall a, b \in S; a * b \in \{a, 0\}$ where $a * b = a$ if $(a, b) \in E$ and $a * b = 0$ otherwise $\}$. Then we call $G\mathcal{M}(S)$ a **graph magma**

If we take $V = S \setminus \{0\}$ and define $a \to b \iff a * b = a$. Then every graph induces an operation and $G(\mathcal{M}(S)) \subseteq \mathcal{M}(S)\ \ \forall n \geq 2$.

Example 4.1.3. We can represent the following graph as an operation in the table below:

Figure 4.1: Related Graph of $*$

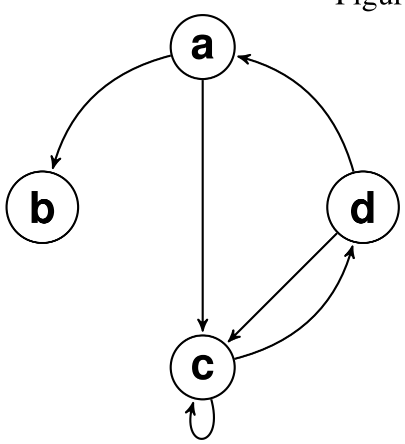


$*$	0	u	v	w
0	0	0	0	0
u	0	0	0	0
v	0	v	v	0
w	0	0	w	w

Example 4.1.4. We can represent the following operation table in graph form as follows:

$\star$	0	a	b	c	d
0	0	0	0	0	0
a	0	0	a	a	0
b	0	0	0	0	0
c	0	0	0	c	c
d	0	d	0	d	0

Figure 4.2: Graph Representation of $\star$



Proposition 4.1.5. *For any directed graph $G = (V, E)$, the following conditions are equivalent:*

1. *The graph algebra is associative*
2. *For all $(a, b) \in E$ and $c \in V$*

   $(a, c) \in E \iff (b, c) \in E$
3. *each connected component of G is isomorphic to $\mathcal{N}_1$, null graph, or a complete graph or a direct sum of a null graph and a complete graph.*

Proof. Refer to [11] $\square$

We describe the set of elements with the same 0 elements.

Proposition 4.1.6. *If* $GM_0(S) := \{* \in G\mathcal{M}(S)|0 \in S$ *is the zero element of* $*\}$, *then*

1. $(GM_0(S), \triangleleft)$ *is closed*

2. *if* $G(*), G(\circ)$ *are associative then* $G(* \triangleleft \circ)$ *is associative, where* $G(*) = (GM_0(S), *)$

Proof. 1. Given $*, \circ \in GM_0(S)$ and $a, b \in S$ then by definition,
$a(* \triangleleft \circ)b = (a * b) \circ (b * a)$. If either a or b is 0, then $a(* \triangleleft \circ)b = 0$. Suppose $\{a, b\} \neq \{0\}$ then, $a(* \triangleleft \circ)b \in \{a, 0\}$.

2. By Proposition 4.1.5, we need to show that for all $a(* \triangleleft \circ)b = a$ and $c \in V$,
$a(* \triangleleft \circ)c = a \iff b(* \triangleleft \circ)c = b$
Let $a(* \triangleleft \circ)b = a$, then by definition $*(a, b) = a, *(b, a) = b$ and $\circ(a, b) = a$, given that $c \in V$ if we assume that $a(* \triangleleft \circ)c = a$ then $*(a, c) = a, *(c, a) = c$ and $\circ(a, c) = a$, by the associativity of both $*$ and $\circ$, $*(b, c) = b, *(c, b) = c$ and $\circ(b, c) = b$ and so $b(* \triangleleft \circ)c = (b * c) \circ (c * b) = b \circ c = b$. We can use similar argument to show that $b(* \triangleleft \circ)c = b \implies a(* \triangleleft \circ)c = a$.
Thus $G(* \triangleleft \circ)$ is associative

□

4.2 Two-Value Graph Magma

Definition 4.2.1. Given $G = (V, E)$, we define the set $2V(S)$ by
$2V(S) = \{* \in \mathcal{M}(S)|\forall a, b \in S \quad a * b \in \{a, b\}$ and $a * b = a, if(a, b) \in E$ and $a * b = b, if(a, b) \notin E\}$. $2V(S)$ is called **Two-Value Graph Magma**. G and $*$ are said to be the related graph and the related binary operation of the two-value graph magma respectively.

Let $*, \circ \in 2V(S)$, then $\forall a, b \in S$, $a * \triangleleft \circ b = (a * b) \circ (b * a) \in \{a, b\}$. Thus, $(2V(S), \triangleleft)$ is closed.

By writing the definition of $a(* \triangleleft \circ)b, \quad \forall a, b \in S$ for $2V(S)$ and considering all cases we can have the following remark

Remark 4.2.2. *Given $*, \circ \in 2V(S)$, if $a \neq b \ \ \forall a, b \in S$, then $a(* \triangleleft \circ)b = a$ is equivalent to one of the following conditions:*

A) $a * b = b * a = a$;

B) $a * b = a, b * a = b$ and $a \circ b = a$;

C) $a * b = b, b * a = a$ and $b \circ a = a$.

Notice that, another possibility for $*$ is $a * b = b * a = b$ which by this we have $a(* \triangleleft \circ)b = b$.

A graph G is said to be a Two-Value Graph Magma if, its related binary operation is Two-Value Graph Magma, also a graph is Associative Two-Value Graph Magma (A2V(S)) if its related binary operation is an associative Two-Value Graph Magma

Question 4.2.3. *If $* \in 2V(S)$, Is $*$ associative?*

Remark 4.2.4. *If a graph is not trivial (not a null graph) ($\exists a \neq b$ such that $a \to b$) and it has two components then $*$ is not associative.*

Lemma 4.2.5. *If G is the related graph of an associative 2V(S) then G is transitive.*

Proof. Suppose $*$ is the related binary operation of G and $(a, b), (b, c) \in E(G)$ then because G is associative $a = a * (b * c) = (a * b) * c = a * c$ which means $(a, c) \in E(G)$. □

Lemma 4.2.6. *If G is the related graph of an associative 2V(S) then, G is a Null graph or every edge $(a, b) \in G$ is connected with all vertices in G.*

Proof. Suppose $*$ is the related binary operation of G; If G is Null then for every three vertices a, b and c, $a * (b * c) = c = (a * b) * c$. Now, suppose $(a, b) \in E(G)$ and there exists a vertex c which is not connected to (a, b) then, $(a * c) * b = c * b = b$ but $a * (c * b) = a * b = a$ which contradicts associativity. □

Lemma 4.2.7. *If G is the related graph of an associative 2V(S) then for every three vertices $a, b, c \in V$ if $(a, b) \in E$ then $(a, c) \in E$ or $(c, b) \in E$.*

Proof. Suppose $(a, b) \in E(G)$ but neither (a, c) nor (c, b) are in $E(G)$ then, by lemma 4.2.6 and 4.2.5 $(b, c) \in E$ which implies $(a, c) \in E$ or $(c, a) \in E$ which implies $(c, b) \in E$; which both contradicts the assumption. □

Theorem 4.2.8. *For any directed graph $D = (V, E)$, Two-Value Graph Magma is associative if and only if its related graph is transitive and for all $(a, b) \in E$ and $c \in V$, either $(a, c) \in E$ or $(c, b) \in E$.*

Proof. If a Two-Value Graph Magma is associative then by Lemma 4.2.5 and 4.2.7 its related graph, G, is transitive and for all $(a, b) \in E$ and $c \in V$, either $(a, c) \in E$ or $(c, b) \in E$. Conversely, suppose $*$ is the related binary operation of G. We consider two cases:

1. Suppose $(a, b) \in E$. Take any vertex $c \in V$, if $(a, c) \in E$ then $(a * b) * c = a * (b * c) = a$; if, however, $(a, c) \notin E$ but, $(c, b) \in E$, then transitivity condition implies that $(b, c) \notin E$, so, $(a * b) * c = a * (b * c) = c$.

2. Suppose $(a, b) \notin E$, then $(a * b) * c = b * c$; now, we consider two possibilities; first, $(b, c) \in E$, then $b * c = b$ and $a * (b * c) = a * b = b$. Secondly, $(b, c) \notin E$, which implies $(a, c) \notin E$ by transitivity and Lemma 4.2.6; so, $b * c = c$ and $a * (b * c) = a * c = c$.

□

Corollary 4.2.9. *The set of all associative 2V(S) are closed under $\triangleleft$, that is $* \triangleleft \star$ is an associative $2V(S)$, $\forall *, \star \in$ associative $2V(S)$,*

Proof. Suppose $*$ and $\circ$ are associative 2V(S), and let $\mathcal{G}, \mathcal{H}, \mathcal{GH}$ be graphs induced by $*$,$\circ$ and $* \triangleleft \circ$ respectively. By using Theorem 4.2.8, in two steps we show that $* \triangleleft \circ$ is associative 2V(S).

First, we show that $\mathcal{GH}$ is transitive;

By Remark 4.2.2, we can define when $b(* \triangleleft \circ)c) = b$ and $a(* \triangleleft \circ)c = a$ by using the same labeling;

$b(* \triangleleft \circ)c) = b$ is equivalent to one of the following conditions:

A) $b * c = c * b = b$;

B) $b * c = b, c * b = c$ and $b \circ c = b$;

C) $b * c = c, c * b = b$ and $c \circ b = b$.

Similarly, we have $a(* \triangleleft \circ)c) = a$ if one of the following conditions occur;

A) $a * c = c * a = a$;

B) $a * c = a, c * a = c$ and $a \circ c = a$;

C) $a * c = c, c * a = a$ and $c \circ a = a$.

There are nine cases for $*$ which we should consider, you can find all of them in the table below, we will only discuss one of these cases, similar arguments can be followed to verify the other cases.

Suppose (a, b) and (b, c) are edges in $\mathcal{GH}$, using Remark 4.2.2, supoose we have AB (from the table, we pick A from the first column and B from the second column) which means $(a, b) \in E(\mathcal{G})$, $(b, a) \notin E(\mathcal{G})$ and $(b, c), (c, b) \in E(\mathcal{G})$, since $*$ is associative, it implies $(a, c) \in E(\mathcal{G})$ and (c, a) does not belong to $E(\mathcal{G})$ which implies $(a, c) \in E(\mathcal{GH})$. So, AB implies A.

Secondly, if $(a, b) \in E(\mathcal{GH})$ then, for every vertice $c \in \mathcal{GH}$, either $(a, c) \in E(\mathcal{GH}$)or $(c, b) \in E(\mathcal{GH})$. Suppose otherwise, that is, there exist $c \in V(\mathcal{GH})$ such that $(a, c), (c, b) \notin E(\mathcal{GH}$, using Remark 4.2.2, there are six cases to consider in this situations, we discuss the first one, the other cases can be verified using similar argument:

i) We laCase D, D, which means (a, c) and (c, b) do not belong to $E(* \triangleleft \circ)$ because of situation D in both cases. But if it happens in $*$ we have $(c, a), (b, c) \in E(*)$ and

Table 4.1: Nine Cases for $*$

Transitivity		
Relationship between a, b in $*$	Relationship between b, c in $*$	Result for a, c in $*$
A	A	A
A	B	A
A	C	A
B	A	A
B	B	B
B	C	impossible
C	A	A
C	B	impossible
C	C	C

$(a, c), (c, b) \notin E(*)$ which by transitivity we have $(b, a) \in E(*)$ and since $(a, b) \in E(* \triangleleft \circ)$ by Remark 4.2.2 we should have case number A, which means $(a, b) \in E(*)$ which by transitivity it implies $(c, b) \in E(*)$ which contradicts assumption D.

The methods for all other cases are similar, we only list the cases.

ii) $D, \sim B$ ($\sim B$ for vertices a and c means $(a, c), (c, a) \in E(*)$ and (a, c) does not belong to $E(\circ)$).

iii) $D, \sim C$(C for vertices a and c means $(a, c), (c, a)$ do not belong to $E(*)$ and $(c, a) \in E(\circ)$).

iv) $\sim B, \sim B$

v) $\sim B, \sim C$

vi) $\sim C, \sim C$. □

Definition 4.2.10. For all $*_1, *_2 \in \mathcal{M}(S)$ we let

$< *_1, *_2 >= \{* \in \mathcal{M}(S) | \forall a, b \in S, *(a,b) \in \{*_1(a,b), *_2(a,b)\}\}$ We call $< *_1, *_2 >$ for each $*_1, *_2 \in \mathcal{M}(S)$ **graph magma generators**

Example 4.2.11. 1. $< \pi_1, C_0 > \supseteq GA(S)$

2. $< \pi_1, \pi_2 >= 2V(S)$

where $C_0 : S \times S \to S$, is a constant function, $\pi_1(a,b) = a, \pi_2(a,b) = b\ \forall a, b \in S$. For each n, $< C_0, C_1, ..., C_n >= \mathcal{M}(S)$

Question 4.2.12. *Can we characterize pairs $*_1, *_2$ such that $< *_1, *_2 >$ is a semigroup.*

Conjecture 4.2.13. *Suppose $\circ \in \mathcal{M}(S)$ and $\hat{\circ}(a,b) = \circ(b,a)$ such that $a \circ a = a$, $\forall a \in S$ and $< \circ, \hat{\circ} >$ is closed under $\triangleleft$ then $< \circ, \hat{\circ} >$ is a semigroup, and if $\circ$ is not commutative then $\circ = \pi_1$ or $\circ = \pi_2$ or $\circ = \eta_{\pi_1}$ or $\circ = \eta_{\pi_2}$*

Example 4.2.14. For $n = 3$ if $\circ$ is not commutative then $\circ$ is equal to any one of these operations

$$
\begin{array}{c|ccc} \pi_1 & 2 & 1 & 0 \\ \hline 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}
\qquad
\begin{array}{c|ccc} \pi_2 & 2 & 1 & 0 \\ \hline 2 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}
\tag{4.2.1}
$$

$$
\begin{array}{c|ccc} \eta_{\pi_1} & 2 & 1 & 0 \\ \hline 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 0 & 2 & 0 & 0 \end{array}
\qquad
\begin{array}{c|ccc} \eta_{\pi_2} & 2 & 1 & 0 \\ \hline 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}
\tag{4.2.2}
$$

4.3 New Algebraic Structures

We describe other set of structures on $\mathcal{M}(S)$.

Definition 4.3.1. Given $(M, *)$ and $(N, \circ)$ we define a new magma $(M \bigsqcup N, \star)$ called "$(M, *)$ augmented by multidentities by $(N, \circ)$ and $\star = * \bigsqcup \circ$"such that

$\forall\, m_1, m_2 \in M, m_1 \star m_2 = m_1 * m_2$

$\forall\, n_1, n_2 \in N\; n_1 \star n_2 = n_1 \circ n_2$ and

$\forall\, m \in M$ and $n \in N\; m \star n = n \star m = m$

Remark 4.3.2. *If $*$ and $\circ$ are associative then $\star$ is also associative*

Proposition 4.3.3. *Given $(\mathcal{M}(S) \times M(T), \triangleleft)$ we define $(*_1, \circ_1) \triangleleft (*_2, \circ_2) = (*_1 \triangleleft *_2, \circ_1 \triangleleft \circ_2)$ then the map*

$L(S) \times M(T) \to M(S \bigsqcup T)$ defined by $(*, \circ) \mapsto * \bigsqcup \circ$ is $\triangleleft-$homomorphism,

where $L(T) = \{* \in M(T) | \forall a \in T, a * a = a\}$

We let $\Gamma \in \mathcal{G}(S)$ be a simple, directed graph defined by

$\mathcal{G}(S) = \{f : S \times S \to \{0, 1\}, f(a, b) = 1 \Leftrightarrow a \to b \;\text{ and }\; f(a, b) = 0 \Leftrightarrow a \nrightarrow b\}$

Definition 4.3.4. Let the map $\varphi : \mathcal{G}(S) \to M(\mathcal{M}(S))$ be defined by

$$[\varphi(\Gamma)(*, \circ)](a, b) = \left\{ \begin{array}{ll} *(a, b) & if \;\; a \to b \\ \circ(a, b) & if \;\; a \nrightarrow b \end{array} \right.$$

Following from definition 4.3.4 we can use the following notations to represent the defined map. $*\Gamma \circ (a, b) = \left\{ \begin{array}{ll} *(a, b) & if \;\; a \to b \\ \circ(a, b) & if \;\; a \nrightarrow b \end{array} \right.$ and $\mathcal{G}(*, \circ) = \{*\Gamma \circ | \Gamma \in \mathcal{G}(S)\}$

If Γ acts on Γ_1, Γ_2, how do we describe $[*\Gamma_1 \circ] \;\; \Gamma \;\; [\Gamma_2(*, \circ)]$

Definition 4.3.5. Let the map $\Psi : \mathcal{G}(S) \to M(\mathcal{G}(S))$ be defined by

$$\Psi(\Gamma)(\Gamma_1, \Gamma_2)(a, b) = \left\{ \begin{array}{ll} \Gamma_1(a, b) & if \;\; \Gamma(a, b) = 1 \\ \Gamma_2(a, b) & if \;\; \Gamma(a, b) = 0 \end{array} \right.$$

Question 4.3.6. *What is $\Gamma_1 \triangleleft \Gamma_2$? Is it the case that $\Gamma_1(*, \circ) \triangleleft \Gamma_2(*, \circ) = (\Gamma_1 \triangleleft \Gamma_2)(*, \circ)$*

Following from definition 2.1.2 and 4.3.5, we have

$$(\Gamma_1 \triangleleft \Gamma_2)(\gamma_1, \gamma_2)(a,b) = \{(\gamma_1\Gamma_1\gamma_2)\Gamma_2(\gamma_2\Gamma_1\gamma_1)\}(a,b) \tag{4.3.1}$$

$$= \{ \begin{array}{ll} (\gamma_1\Gamma_1\gamma_2)(a,b) & if \;\; \Gamma_2(a,b) = 1 \\ (\gamma_2\Gamma_1\gamma_1)(a,b) & if \;\; \Gamma_2(a,b) = 0 \end{array} \tag{4.3.2}$$

$$(\gamma_1\Gamma_1\gamma_2)(a,b) = \{ \begin{array}{ll} \gamma_1(a,b) & if \;\; \Gamma_1(a,b) = 1 \\ \gamma_2(a,b) & if \;\; \Gamma_1(a,b) = 0 \end{array} \tag{4.3.3}$$

$$(\gamma_2\Gamma_1\gamma_1)(a,b) = \{ \begin{array}{ll} \gamma_2(a,b) & if \;\; \Gamma_1(a,b) = 1 \\ \gamma_1(a,b) & if \;\; \Gamma_1(a,b) = 0 \end{array} \tag{4.3.4}$$

Definition 4.3.7. Given $\Gamma_1, \Gamma_2 \in M(G)S))$,

$$(\Gamma_1 \triangleleft \Gamma_2)(a,b) = 1 - |\Gamma_1(a,b) - \Gamma_2(a,b)|$$

Lemma 4.3.8. $(M(G(S)), \triangleleft)$ *is associative*

Proof. Given $\Gamma_1, \Gamma_2, \Gamma_3 \in M(G(S))$ and $a, b \in S$. We show that $(\Gamma_1 \triangleleft \Gamma_2) \triangleleft \Gamma_3 =$ $\Gamma_1 \triangleleft (\Gamma_2 \triangleleft \Gamma_3)$. We consider three cases.

Case 1. When $\Gamma_1(a,b) = \Gamma_2(a,b) = \Gamma_3(a,b) = 0$. Then

$(\Gamma_1 \triangleleft \Gamma_2) \triangleleft \Gamma_3(a,b) = 1 - |(\Gamma_1 \triangleleft \Gamma_2)(a,b) - \Gamma_3(a,b)| = 1 - |1 - |\Gamma_1(a,b) - \Gamma_2(a,b)|| = 0$

$\Gamma_1 \triangleleft (\Gamma_2 \triangleleft \Gamma_3)(a,b) = 1 - |(\Gamma_1(a,b) - (\Gamma_2 \triangleleft \Gamma_3)(a,b)| = 1 - ||\Gamma_2(a,b) - \Gamma_3(a,b) - 1|| = 0$

Case 2: When $\Gamma_1(a,b) = \Gamma_2(a,b) = \Gamma_3(a,b) = 1$. Then

$(\Gamma_1 \triangleleft \Gamma_2) \triangleleft \Gamma_3(a,b) = 1 - |(\Gamma_1 \triangleleft \Gamma_2)(a,b) - \Gamma_3(a,b)| = 1 - |1 - |\Gamma_1(a,b) - \Gamma_2(a,b)| - 1|| = 1$

$\Gamma_1 \triangleleft (\Gamma_2 \triangleleft \Gamma_3)(a,b) = 1 - |(\Gamma_1(a,b) - (\Gamma_2 \triangleleft \Gamma_3)(a,b)| = 1 - |1 - |1 - |\Gamma_2(a,b) - \Gamma_3(a,b)||| = 1$

Case 3: When at least one of them is 1, Assume $\Gamma_1 = 1$, but both $\Gamma_2, \Gamma_3 = 0$. Then

$(\Gamma_1 \triangleleft \Gamma_2) \triangleleft \Gamma_3(a,b) = 1 - |(\Gamma_1 \triangleleft \Gamma_2)(a,b) - \Gamma_3(a,b)| = 1 - |1 - |\Gamma_1(a,b) - \Gamma_2(a,b)|| = 1$

$\Gamma_1 \triangleleft (\Gamma_2 \triangleleft \Gamma_3)(a,b) = 1 - |(\Gamma_1(a,b) - (\Gamma_2 \triangleleft \Gamma_3)(a,b)| = 1 - |1 - |1 - |\Gamma_2(a,b) - \Gamma_3(a,b)||| = 1$

We can similarly show when two of them are equal to 1.

Thus from the three cases we can see that $(\Gamma_1 \triangleleft \Gamma_2) \triangleleft \Gamma_3 = \Gamma_1 \triangleleft (\Gamma_2 \triangleleft \Gamma_3)$ for all $\Gamma_1, \Gamma_2, \Gamma_3 \in M(G(S))$ □

Theorem 4.3.9. $(M(G(S)), \triangleleft)$ *is an abelian group*

Proof.

1. Closure, commutativity and associativity are clear from Definition 4.3.5, 4.3.7 and Lemma 4.3.8.

2. We show that the complete graph is the identity in $M(G(S))$. Let $\Gamma_1 \in M(G(S))$ be the complete graph then by definition $\Gamma_1(a,b) = 1 \ \ \forall a, b \in S$. Given $\Gamma \in M(G(S))$,
$$(\Gamma_1 \triangleleft \Gamma)(a,b) = 1 - |\Gamma_1(a,b) - \Gamma(a,b)| = 1 - |1 - \Gamma(a,b)| = \begin{cases} 1 & if \ \ \Gamma_1(a,b) = 1 \\ 0 & if \ \ \Gamma_1(a,b) = 0 \end{cases}$$
Thus $(\Gamma_1 \triangleleft \Gamma)(a,b) = (\Gamma \triangleleft \Gamma_1)(a,b) = \Gamma(a,b)$.

3. We show that each graph in $M(G(S))$ is self-invertible. That is $\Gamma \triangleleft \Gamma = \Gamma_1$.
$(\Gamma \triangleleft \Gamma)(a,b) = 1 - |\Gamma(a,b) - \Gamma(a,b)| = 1$, in all cases.

□

5 Future Work

5.1 Furthering the Study of Distributivity, Twistributivity and Distributive Triples

There is still a lot of work to be done in the study of one-sided and two-sided distributivity. On the other hand, as it was seen here, twistributivity (one-sided and two-sided, both) emerges naturally as a notion to be dealt with. In fact, the quantitative evidence at this moment suggests that (for some reason we are not anywhere near to explain) twistributivity seems to be more abundant than distributivity. It is certainly too soon to tell but the question is tantalizing nonetheless and we cannot resist the temptation to introduce some appropriate terminology here.

A generalization of distributivity and twistributivity comes to mind: a triple of operations $(*, \circ, \star) \in (\mathcal{M}(S))^3$ is said to be a *left distributive triple* if, for all $a, b, c \in S$, $a * (b \circ c) = (a \star b) \circ (a \star c)$. We suggest an expression like $*$ *distributes over* $\circ$ *through* $\star$.

With this terminology, $(\circ, \star)$ is a distributive pair if and only if $(\star, \circ, \star)$ is a distributive triple and $(\circ, \star)$ is a twistributive pair if and only if $(\star, \circ, \star^{op})$ is a distributive triple.

5.2 Concepts on Poor and Rich Module Relations

Definition 5.2.1. If $\mathcal{W}, \mathcal{C}$ are two classes and $\rho \in \mathcal{W} \times \mathcal{C}$ is a relation between them. Given $c \in \mathcal{C}$, $\rho-$domain of c is defined as follows: $dom_\rho(c) = \{w \in \mathcal{W} | w\rho c\}$ and for $w \in \mathcal{W}$, $\rho-$range of w is defined as follows $ran_\rho(w) = \{c \in \mathcal{C} | w\rho c\}$.

For each $(w, c) \in \rho$ we denote $w\rho c$

Definition 5.2.2. Given two assignments $dom_\rho : \mathcal{C} \to \rho(\mathcal{W})$ and $ran_\rho : \mathcal{W} \to \rho(\mathcal{C})$

1. $c \in \mathcal{C}$ is said to be wealthy or rich $\iff dom(c) = \bigcup_{x \in \mathcal{C}} dom(x)$
2. $c \in \mathcal{C}$ is said to be poor $\iff dom(c) = \bigcap_{x \in \mathcal{C}} dom(x)$
3. $w \in \mathcal{W}$ is affordable if $w \in \bigcap_{c \in \mathcal{C}} dom(c)$

4. $w \in \mathcal{W}$ is said to be exclusive if $w\rho c \implies c$ is rich.

Example 5.2.3. Let $\mathcal{W} = Mod - \mathcal{R} = C$. We define the relation $N\rho M$ if M is $N - inj$. Then $dom(M) = In^{-1}(M)$

Then If N is an injective module then N is rich

Example 5.2.4. Let $\mathcal{W} = \mathcal{M}(S) = C$. For each $*, \circ \in \mathcal{M}(S)$ We define the relation $*\rho\circ$ if $* \rightsquigarrow \circ$ ($*$ distributes over $\circ$) and $dom(\circ) = in(\circ)$ and $ran(*) = out(*)$

Then

1. $* \in \mathcal{M}(S)$ is said to be wealthy if $dom(*) = \bigcup_{\circ \in \mathcal{M}(S)} in(\circ) = \mathcal{M}(S)$. Since $in(\pi_1) = in(\pi_2) = \mathcal{M}(S) \implies$ wealthy $= \{ \pi_1, \pi_2 \}$. See [15]

2. $* \in \mathcal{M}(S)$ is said to be poor if $dom(*) = \bigcap_{\circ \in \mathcal{M}(S)} in(\circ) = \{\}$. By [15], $in(C_i(a)) = \{\}$ $\implies$ poor = isolations.

3. $* \in \mathcal{M}(S)$ is said to be exclusive if, and only if $*$ distributes over only $\{\pi_1, \pi_2\}$

4. $* \in \mathcal{M}(S)$ is said to be affordable if $* \in \bigcap_{* \in \mathcal{M}(S)} in(*)$, which is empty. So by definition, no operation in $\mathcal{M}(S)$ is affordable.

For $|S| = 2$, wealthy = {10, 12}, poor = {1, 7}, exclusive = {1, 3, 5, 7, 11, 13 } and affordable = { }

Definition 5.2.5. For all $A, B \subseteq \mathcal{M}(S)$, (A, B) is said to be a balanced pair if and only if $dom(B) = A$ and $ran(A) = B$, where $dom(B) = \bigcap_{* \in B} dom(*)$ and $ran(A) = \bigcap_{* \in A} ran(*)$

Example 5.2.6. 1. $(\mathcal{M}(S), \{\pi_1, \pi_2\})$ is a balanced pair by Example 5.2.4

2. $(\emptyset, \mathcal{M}(S))$ is a balanced pair

3. $((\mathcal{M}(S), \mathcal{M}(S))$ is not a balanced pair since $dom(\mathcal{M}(S)) = \emptyset$

Proposition 5.2.7. *If (A, B) is a balanced pair then either B is isolation free or A is empty*

Proposition 5.2.8. *Let* $(A, B) \in \mathcal{P}_p$*, then* $(B, \triangleleft)$ *is a submonoid of* $(\mathcal{M}(S), \triangleleft)$ *where* $\mathcal{P}_p = \{A, B \subseteq \mathcal{M}(S) | (A, B)$ *is a balanced pair* $\}$.

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Appendix A: DCC, DCR for $n = 3$

$$
\begin{array}{c|ccc}
\mathbf{377} & 2 & 1 & 0 \\ \hline
2 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 2 & 2 & 2
\end{array}
\quad
\begin{array}{c|ccc}
\mathbf{715} & 2 & 1 & 0 \\ \hline
2 & 0 & 0 & 0 \\
1 & 2 & 2 & 2 \\
0 & 1 & 1 & 1
\end{array}
\quad
\begin{array}{c|ccc}
\mathbf{9503} & 2 & 1 & 0 \\ \hline
2 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 2 & 2 & 2
\end{array}
\quad
\begin{array}{c|ccc}
\mathbf{10179} & 2 & 1 & 0 \\ \hline
2 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
0 & 0 & 0 & 0
\end{array},
$$

$$
\begin{array}{c|ccc}
\mathbf{18967} & 2 & 1 & 0 \\ \hline
2 & 2 & 2 & 2 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1
\end{array}
\quad
\begin{array}{c|ccc}
\mathbf{19305} & 2 & 1 & 0 \\ \hline
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{array},
\quad
\begin{array}{c|ccc}
\mathbf{3785} & 2 & 1 & 0 \\ \hline
2 & 0 & 1 & 2 \\
1 & 0 & 1 & 2 \\
0 & 0 & 1 & 2
\end{array}
\quad
\begin{array}{c|ccc}
\mathbf{5299} & 2 & 1 & 0 \\ \hline
2 & 0 & 2 & 1 \\
1 & 0 & 2 & 1 \\
0 & 0 & 2 & 1
\end{array},
$$

$$
\begin{array}{c|ccc}
\mathbf{8327} & 2 & 1 & 0 \\ \hline
2 & 1 & 0 & 2 \\
1 & 1 & 0 & 2 \\
0 & 1 & 0 & 2
\end{array}
\quad
\begin{array}{c|ccc}
\mathbf{11355} & 2 & 1 & 0 \\ \hline
2 & 1 & 2 & 0 \\
1 & 1 & 2 & 0 \\
0 & 1 & 2 & 0
\end{array},
\quad
\begin{array}{c|ccc}
\mathbf{15897} & 2 & 1 & 0 \\ \hline
2 & 2 & 1 & 0 \\
1 & 2 & 1 & 0 \\
0 & 2 & 1 & 0
\end{array}
\quad
\begin{array}{c|ccc}
\mathbf{14383} & 2 & 1 & 0 \\ \hline
2 & 2 & 0 & 1 \\
1 & 2 & 0 & 1 \\
0 & 2 & 0 & 1
\end{array}
\tag{A.0.1}
$$

Appendix B: Operations in $\mathcal{M}(S)$ for $n = 2$

$$
\begin{array}{llll}
0 := \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array}, & 1 := \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 1 \\ \hline \end{array} & 2 := \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array} & 3 := \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 1 \\ \hline \end{array} \\
\\
4 := \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 0 \\ \hline \end{array}, & 5 := \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 1 \\ \hline \end{array}, & 6 := \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & 7 := \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \\
\\
8 := \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 0 \\ \hline \end{array}, & 9 := \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}, & 10 := \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & 0 \\ \hline \end{array} & 11 := \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & 1 \\ \hline \end{array} \\
\\
12 := \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 0 \\ \hline \end{array}, & 13 := \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}, & 14 := \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & 15 := \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}
\end{array}
\tag{B.0.1}
$$

Appendix C: 2-Sided Distributive Table for $n = 2$

Reading the table: $(N_{row} \downarrow N_{col})$ means N_{col} distributes on N_{row}

Table C.1: Distributive Table

$\downarrow$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
0	$\checkmark$								$\checkmark$							
1																
2	$\checkmark$								$\checkmark$							
3							$\checkmark$			$\checkmark$						
4	$\checkmark$								$\checkmark$							
5							$\checkmark$			$\checkmark$						
6	$\checkmark$								$\checkmark$							
7																
8	$\checkmark$								$\checkmark$		$\checkmark$		$\checkmark$		$\checkmark$	$\checkmark$
9															$\checkmark$	$\checkmark$
10	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
11															$\checkmark$	$\checkmark$
12	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
13															$\checkmark$	$\checkmark$
14	$\checkmark$								$\checkmark$		$\checkmark$		$\checkmark$		$\checkmark$	$\checkmark$
15															$\checkmark$	$\checkmark$
Outset	8	2	2	2	2	2	4	2	8	4	4	2	4	2	8	8

Appendix D: $\triangleleft$ Calculations

Reading the table: $(N_{row} \triangleleft N_{col})(a, b) = N_{col}(N_{row}(a, b), N_{row}(b, a))$

Table D.1: $\triangleleft$— Calculations

$\triangleleft$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
0	0	15	0	15	0	15	0	15	0	15	0	15	0	15	0	15
1	0	14	0	14	0	14	0	14	1	15	1	15	1	15	1	15
2	0	9	4	13	2	11	6	15	0	9	4	13	2	11	6	15
3	0	8	4	12	2	10	6	14	1	9	5	13	3	11	7	15
4	0	9	2	11	4	13	6	15	0	9	2	11	4	13	6	15
5	0	8	2	10	4	12	6	14	1	9	3	11	5	13	7	15
6	0	9	0	9	0	9	0	9	6	15	6	15	6	15	6	15
7	0	8	0	8	0	8	0	8	7	15	7	15	7	15	7	15
8	0	7	0	7	0	7	0	7	8	15	8	15	8	15	8	15
9	0	6	0	6	0	6	0	6	9	15	9	15	9	15	9	15
10	0	1	4	5	2	3	6	7	8	9	12	13	10	11	14	15
11	0	0	4	4	2	2	6	6	9	9	13	13	11	11	15	15
12	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
13	0	0	2	2	4	4	6	6	9	9	11	11	13	13	15	15
14	0	1	0	1	0	1	0	1	14	15	14	15	14	15	14	15
15	0	0	0	0	0	0	0	0	15	15	15	15	15	15	15	15

The following Magma code was used to calculate the numbers in the table, the code can also check for left distributivity.

$O := [< a, \text{Matrix}(\text{IntegerRing}(), 2, 2, [0, 0, 0, 0]) >: a \text{ in } [0..15]];$

$M := [\text{Matrix}(\text{IntegerRing}(), 2, 2, [a, b, c, d]) : a, b, c, d \text{ in } [0, 1]];$

$$\text{numb} := \text{function}(m)$$

$$n := 0;$$

$$n := m[1, 1] * 2^3 + m[1, 2] * 2^2 + m[2, 1] * 2 + m[2, 2];$$

$$\text{return } n;$$

$$\text{end function;}$$

$$\text{for m in M do}$$

$$n := numb(m);$$

$$O[n + 1][2] := m;$$

$$\text{end for;}$$

The next codes can check if one operations distributes on each other on both sides for $n = 2$

Triple:=$\{[a, b, c] : a, b, c$ in $[0, 1]\}$;

Distribute:=function(n, m)

$O1 := O[n + 1][2];$

$O2 := O[m + 1][2];$

for t in Triple do

$a := 2 - t[1];$

$b := 2 - t[2];$

$c := 2 - t[3];$

$l := O1[a, 2 - O2[b, c]];$

$r := O2[2 - O1[a, b], 2 - O1[a, c]];$

```
if l ne r then
return false;
end if;
end for;
return true;
end function;
```

The following codes can calculate ◁ for $n = 2$

```
Double:= {[a, b] : a, b in [0, 1]};
Triangle:=function(i, j)
z := ZeroMatrix(IntegerRing(), 2, 2);
O1 := O[i + 1][2];
O2 := O[j + 1][2];
for t in Double do
a := 2 − t[1];
b := 2 − t[2];
z[a, b] := O2[2 − O1[a, b], 2 − O1[b, a]];
end for;
return z;
end function;
```

Appendix E: $Aut(\mathcal{M}(S), \triangleleft)$ for $|S| = 2$

Table E.1: $Aut(\mathcal{M}(S), \triangleleft)$ for $|S| = 2$ Calculations

$\frac{*}{\sigma}$	Id	(10)	Set
12	12	12	Units
10	10	10	Units
3	3	3	Units
5	5	5	Unit
0	0	0	Constant Operation
15	15	0	Constant Operation
6	6	9	Square Root of Constant Operation
9	9	6	Square Root of Constant Operation
4	4	13	Idempotent Operation
8	8	14	Idempotent Operation
13	13	4	Idempotent Operation
14	14	8	Idempotent Operation
1	1	7	Square Root of Idempotent Operation
2	2	11	Square Root of Idempotent Operation
7	7	1	Square Root of Idempotent Operation
11	11	2	Square Root of Idempotent Operation

APPENDIX F: CODES FOR CALCULATING $\triangleleft, n = 3, 4$ FOR EACH n

The following codes can calculate $\triangleleft$ for $n = 3$ for *DCC* and *DCR*

D :=[Matrix(IntegerRing(),3,3,[a,a,a,b,b,b,c,c,c]) : a,b,c in [0,1,2]— a ne b and b ne c and a ne c];

C:=[Matrix(IntegerRing(),3,3,[a,b,c,a,b,c,a,b,c]) : a,b,c in [0,1,2]— a ne b and b ne c and a ne c];

O:=[<a,ZeroMatrix(IntegerRing(),3,3)>: a in [0..($3^9 - 1$)]];

M:=[Matrix(IntegerRing(),3,3,[a,b,c,d,e,f,g,h,i]) : a,b,c,d,e,f,g,h,i in [0,1,2]];

numb:=function(m)

$n := 0;$

$$n := m[1,1] * 3^8 + m[1,2] * 3^7 + m[1,3] * 3^6 + m[2,1] * 3^5 + m[2,2] * 3^4 + m[2,3] * 3^3 + m[3,1] * 3^2 + m[3,2] * 3 + m[3,3];$$

return n;

end function;

for m in M do

$n := numb(m);$

$O[n+1][2] := m;$

end for;

Operation:=function(n)

return $O[n+1]$;

end function;

Double:=[a,b] : a,b in [0,1,2];

Triangle:=function(i,j)

z:=ZeroMatrix(IntegerRing(), 3,3);

O1:=O[$i+1$][2];

```
O2:=O[j + 1][2];
for t in Double do
a := 3 − t[1];
b := 3 − t[2];
z[a, b] := O2[3 − O1[a, b], 3 − O1[b, a]];
end for;
return z;
end function;
```

The following codes can calculate ◁ for $n = 3$

```
Double:= [a, b] : a, bin[0, 1, 2];
Triangle3:=function(A,B)
z := ZeroMatrix(IntegerRing(), 3, 3);
for t in Double do
x := 3 − t[1];
y := 3 − t[2];
z[x, y] := B[3 − A[x, y], 3 − A[y, x]];
end for;
return z;
end function;
```

The following codes can calculate ◁ for $n = 4$

```
Double:= [a, b] : a, bin[0, 1, 2, 3];
Triangle4:=function(A,B)
z:=ZeroMatrix(IntegerRing(), 4,4);
for t in Double do
a := 4 − t[1];
b := 4 − t[2];
```

$z[a, b] := B[4 - A[a, b], 4 - A[b, a]];$

end for;

return z;

end function;

The following codes can calculate ◁ for each n

Trianglen:=function(A,B,n)

z:=ZeroMatrix(IntegerRing(), n, n);

Double:=[a,b] : a,b in [0..(n-1)];

for t in Double do

$a := n - t[1];$

$b := n - t[2];$

$z[a, b] := B[n - A[a, b], n - A[b, a]];$

end for;

return z;

end function;

Appendix G: Codes for generating and calculating inverse of an operation *

```
Inverse3:= function(A)
z:=ZeroMatrix(IntegerRing(), 3, 3);
Double:=[a,b] : a,b in [0,1,2];
for t in Double do
a:=3-t[1];
b:=3-t[2];
e:= A[a,b];
f:= A[b,a];
z[3-e,3-f]:=t[1];
z[3-f,3-e]:=t[2];
end for;
return z;
end function;
Inversen:= function(A,n)
z:=ZeroMatrix(IntegerRing(), n, n);
Double:=[a,b] : a,b in [0..(n-1)];
for t in Double do
a:=n-t[1];
b:=n-t[2];
e:= A[a,b];
f:= A[b,a];
z[n-e,n-f]:=t[1];
z[n-f,n-e]:=t[2];
```

```
end for;
return z;
end function;
```

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