The Interface Dynamics in the Hele-Shaw Cell

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Abstract

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Hele-Shaw free boundary problems have been attracting attention of engineers, physicists and mathematicians for many decades because of the fact that the number of moving boundary processes such as solidification, electro-deposition, flows in porous media can be reduced to Hele-Shaw problems. There are two classical formulation of Hele-Shaw problems which are the one-phase and two-phase Hele-Shaw problems. The one-phase Hele-Shaw problem has been studied extensively by many researchers and many explicit solutions are known. Regarding the two-phase Hele-Shaw problem (also know as the "Muskat problem"), much less progress has been made.

In this dessertation, we study the evolution of a two-phase Hele-Shaw problem under assumption of a negligible surface tension. Our models involve the sinks and sources to be line distributions with disjoint supports located in the exterior and the interior domains as well as time dependent change in the gap width. We use the tools of complex analysis such as the Schwarz function and the complex potential. We give examples of exact solutions when the interface belongs to a certain family of algebraic curves, defined by the initial shape of the boundary.

To my parents: Kolawole and Rasheedat.

To my family: my wife Shakirat and my daughters Maryam and Khadijat.

To all my brothers and sisters.

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Not forgetting my esteemed department and all other departmental staff that contributed in one form or the other to the successful completion of my dissertation. I pray that you all continue to grow from strength to strength and in health. Thank you. God bless.

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1 INTRODUCTION

1.1 Brief description of Hele Shaw problem

In the 19th century, one of the famous works in Fluid Dynamic was a succession of papers written by Henry Selby Hele-Shaw (1854 – 1941). The Hele-Shaw cell is a device used to study two-dimensional flows of viscous fluids. It consists of two parallel plates and a viscous fluid, sandwiched in a narrow gap between the plates, see Figure 1.1.

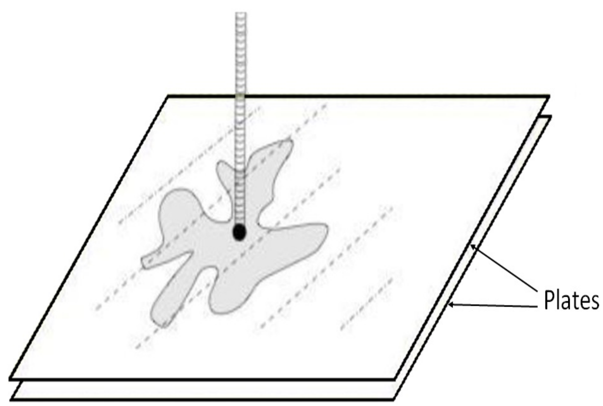


Figure 1.1: Hele-Shaw cell [89].

The Hele-Shaw cell is extensively used as a model in various fields of engineering and natural sciences, in particularly, fluid mechanics, materials science, and crystal growth [86]. One of the important characteristics of the flows in a Hele-Shaw problem is that the Navier-Stokes equations averaged over the gap reduce to a much simpler relation as Darcy's law and then to the Laplace equation for pressure [29]. Hele-Shaw free boundary problems have been extensively studied over the last century (see [29], [86] and references therein).

There are two classical formulations of the Hele-Shaw problems: the one-phase problem and the two-phase problem (also known as the “Muskat problem”). In one-phase problem one of the fluids is assumed to be viscous while the other is effectively inviscid (that is, it has a constant pressure). There are two statements of one-phase problem,

- The interior problem: where oil is surrounded by air in a Hele-Shaw cell.
- The exterior problem: where an air bubble is surrounded by oil in a Hele-Shaw cell.

The two-phase Hele-Shaw problem is a free boundary problem related to the theory of flows in porous media [54]. The problem describes an evolution of an interface between two immiscible fluids, oil and water, in a Hele-Shaw cell or in a porous medium. While the one-phase problem has been studied for many decades and many explicit solutions are known, much less progress has been made for the two-phase problem. Concerning the two-phase Hele-Shaw problem, the global existence of solutions to some specific two-phase Hele-Shaw problems was considered in [24], [80] and [91]. Howison [38], has obtained several simple solutions including travelingwave solutions and stagnation point flow. In [38], an idea of a method for solving some two-phase problems was proposed and used to reappraise the Jacquard and Séguier solution [44]. Crowdy [14], presented an exact solution to the Muskat problem for the elliptical initial interface between two fluids of different viscosity. In [14], it was shown that an elliptical inclusion of one fluid remains elliptical when placed in a linear ambient flow of another fluid. Bazaliy and Vasylyeva [8], proved that the problem has a unique solution in the weighted Holder classes locally in time and specify the sufficient conditions for the existence of the “waiting time” phenomenon.

The main difficulty of the two-phase problems is the fact that the pressure on the interface is unknown. That is why there are not too many solutions to the two-phase problem. However, due to the mathematical analogy with free boundary problems such as solidification, electro-deposition, ice crystal growth, flows in porous media it is important to have better understanding of solutions to the two-phase problem.

The dissertation is devoted to the two-phase Hele-Shaw problem with a negligible surface tension. The material is taken from two published papers. One of them, [3], is related to the two-phase problem in the presence of the line distributions of sinks and sources, while the other problem, in addition to line distributions of sinks and sources, involves a time dependent change in the gap width [76]. Next, we give a summary of topics included throughout the course of this research.

1.2 Organization of thesis

In chapter 2, we discuss preliminaries and definitions which includes the mathematical description and a brief explanation of fluid mechanics. In section 2.1, we derive the Navier-Stokes equations equations for incompressible fluids which constitute the continuity equations and basic conservation applied to properties of fluids. The equations are as follows

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{\mathbf{1}}{\rho}(-\nabla \mathbf{p} + \nu \boldsymbol{\Delta} \mathbf{v}), \tag{1.2.1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.2.2}$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ is the del-operator, $\Delta = \nabla \cdot \nabla = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$, $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$ is the velocity of the flows, p is the pressure of the fluid, ν is the viscosity coefficient and ρ is the density of the fluid. Equation 1.2.1 is called the momentum equation while equation 1.2.2 is called the continuity equation. In section 2.2, we define the complex and velocity potentials. In section 2.3, we derive the one-phase Hele-Shaw problem by considering the Navier-Stokes equations for incompressible fluids and neglecting the gravity. In section 2.4, we define the Schwarz function, give some examples of Schwarz function, and describe the properties of the Schwarz function.

Chapter 3 consists of a paper [3], which is joint work with Drs Tatiana Savina and Alexander Nepomnyashchy. In this chapter we obtain the exact solutions to Muskat problem with line distributions of sinks and sources. Section 3.1 is the introduction extracted from [3]. In section 3.2, we discuss the formulation of the problem which is as follows:

Let $\Omega_2(t) \subset \mathbb{R}^2$ with a boundary $\Gamma(t)$ at time t be a simply-connected bounded domain occupied by a fluid with a constant viscosity ν_2, and let $\Omega_1(t)$ be the region $\mathbb{R}^2 \setminus \bar{\Omega}_2(t)$ occupied by a different fluid of viscosity ν_1. Consider the two-phase Hele-Shaw problem forced by sinks and sources:

$$\mathbf{v}_j = -k_j \nabla p_j, \qquad j = 1, 2, \tag{1.2.3}$$

where the pressure p_j is a harmonic function almost everywhere in the region $\Omega_j(t)$, satisfying boundary conditions

$$p_1(x, y, t) = p_2(x, y, t) \quad \text{on} \quad \Gamma(t) \tag{1.2.4}$$

and

$$-k_1 \frac{\partial p_1}{\partial n} = -k_2 \frac{\partial p_2}{\partial n} = v_n \quad \text{on} \quad \Gamma(t). \tag{1.2.5}$$

Equation (1.2.4) states the continuity of the pressure under the assumption of negligible surface tension. Equation (1.2.5) means that the normal velocity of the boundary itself coincides with the normal velocity of the fluid at the boundary. Here $\mathbf{v}_j$ is a velocity vector of fluid j, $k_j = h^2/12\nu_j$, and h is the gap width of the Hele-Shaw cell. In section 3.3 we reformulate the two phase Hele-Shaw problem in terms of the Schwarz function and we point out the steps of the suggested method of finding the exact solutions. Section 3.4 is devoted to the two-phase mother body in the context of the Muskat problem. In Section 3.5, we consider some algebraic curves as initial position of the interface, and we compute the corresponding exact solutions, and finally, in section 3.6 we give the conclusions.

Chapter 4 consists of a paper in [76], which is joint work with Dr.Tatiana Savina and Avital Savin. In this chapter, we reformulate the Muskat problem with the time dependent gap in terms of the Schwarz function equation. We describe a method of constructing exact solutions, and using this method we consider examples in the presence and in the absence of additional sinks and sources. The structure of the chapter is as follows. Section 4.1 is the introduction extracted from [76]. In section 4.2, we discuss the formulation of the problem which is as follows:

Let $\Omega_2(t) \subset \mathbb{R}^2$ with a boundary $\Gamma(t)$ at time t be a simply-connected bounded domain occupied by a fluid with a constant viscosity ν_2, and let $\Omega_1(t)$ be the region $\mathbb{R}^2 \setminus \bar{\Omega}_2(t)$ occupied by a different fluid of viscosity ν_1. To consider a two-phase Hele-Shaw flow forced by a time-dependent gap, we start with the Darcy's law

$$\mathbf{v}_j = -k_j \nabla p_j \quad \text{in} \quad \Omega_j(t), \qquad j = 1, 2, \tag{1.2.6}$$

where $\mathbf{v}_j$ and p_j are a two-dimensional gap-averaged velocity vector and a pressure of fluid j respectively, $k_j = \frac{h^2(t)}{12\nu_j}$, and $h(t)$ is the gap width of the Hele-Shaw cell. Equation (1.2.6) is complemented by the volume conservation,

$$A(t)h(t) = A(0)h(0) \tag{1.2.7}$$

for any time t, where $A(t)$ and $A(0)$ are the areas of $\Omega_2(t)$ and $\Omega_2(0)$ respectively. The conservation of volume for a time-dependent gap may be written as a modification of the usual incompressibility condition

$$\nabla \cdot \mathbf{V_2} = 0,$$

where $\mathbf{V_2} = (u, v, w)$ is a three-dimensional velocity vector of the fluid occupying the domain $\Omega_2(t)$. Indeed, the averaging of the three-dimensional incompressibility condition across the gap gives [79]:

$$
\begin{aligned}
0 &= \frac{1}{h(t)} \int_0^{h(t)} (u_x + v_y + w_z)\, dz \\
&= \frac{1}{h(t)} \int_0^{h(t)} u_x\, dz \;+\; \frac{1}{h(t)} \int_0^{h(t)} v_y\, dz \;+\; \frac{1}{h(t)} \int_0^{h(t)} w_z\, dz \\
&= u_x^{av} + v_y^{av} + \frac{(w(h(t)) - w(0))}{h(t)} \\
&= u_x^{av} + v_y^{av} + \frac{\dot{h}(t)}{h(t)}.
\end{aligned}
\tag{1.2.8}
$$

Here $z = 0$ corresponds to the lower plate and $z = h(t)$ corresponds to the upper plate, and and $h(t)$ and $\dot{h}(t)$ are assumed to be small enough to avoid any inertial effects as well as to keep the large aspect ratio. The latter implies [79]

$$
\nabla \cdot \mathbf{v_2} = -\frac{\dot{h}(t)}{h(t)} \quad \text{in} \quad \Omega(t),
\tag{1.2.9}
$$

where $\mathbf{v_2} = (\mathbf{u^{av}}, \mathbf{v^{av}})$. We also note that similar consideration may be applied to any finite part of the region $\Omega_1(t)$. Thus, from equation 1.2.6,

$$
\nabla \cdot \mathbf{v_2} = \nabla \cdot (-k_j \nabla p_j).
$$

Hence, from equation 1.2.9 we obtain

$$
\nabla \cdot \mathbf{v_2} = -k_j\, (\nabla \cdot \nabla p_j) = -\frac{\dot{h}(t)}{h(t)}.
$$

Since $\Delta = \nabla \cdot \nabla$, the problem in terms of the pressure p_j as a solution to Poisson's equation is given as follows,

$$
\Delta p_j = \frac{1}{k_j} \frac{\dot{h}(t)}{h(t)},
\tag{1.2.10}
$$

almost everywhere in the region $\Omega_j(t)$, satisfying boundary conditions

$$
p_1(x, y, t) = p_2(x, y, t) \quad \text{on} \quad \Gamma(t)
\tag{1.2.11}
$$

and

$$-k_1 \frac{\partial p_1}{\partial n} = -k_2 \frac{\partial p_2}{\partial n} = v_n \quad \text{on} \quad \Gamma(t). \tag{1.2.12}$$

Equation (1.2.11) states the continuity of the pressure under the assumption of negligible surface tension. Equation (1.2.12) means that the normal velocity of the boundary itself coincides with the normal velocity of the fluid at the boundary. In Section 4.3 we describe the method of finding exact solutions for a Muskat problem with a time-dependent gap. In Section 4.4, we gives examples of the exact solutions, and the concluding remarks are given in Section 4.5.

In chapter 5, we list of some future proposed problems.

2 Preliminaries and Definitions

2.1 Newtonian fluid

It is useful to begin with the definition of a fluid. A fluid is a substance which deforms continuously whenever there is a presence of a small shear stress. A fluid, in which there is a linear relationship between viscosity and shear stress, is called a Newtonian fluid, which was named after Sir Issac Newton (1642-1726). Many common fluids such as water and petroleum products are Newtonian fluid. Newtonian fluids obey the Newton's law of viscosity,

$$\sigma = \nu \frac{\partial \mathbf{v}}{\partial y}. \tag{2.1.1}$$

Here the coefficient of proportionality ν is called the coefficient of viscosity or simply dynamic viscosity, σ is the shear stress, which is the ratio between the force and the area,

$$\sigma = \frac{F}{A},$$

and $\frac{\partial \mathbf{v}}{\partial y}$ is the rate of change of velocity with respect y or the rate of shear deformation. In words, we say that the shear stress in the direction of x of the flow is proportional to the velocity gradient with respect y with the constant of proportionality ν.

2.1.1 Navier-Stokes equations

To derive the Navier-Stokes equations, we use the continuity equation and the momentum equation. The conservation of mass is used in deriving the continuity equation while the Newton's second law results eventually in the momentum equation. For derivation of the continuity equation we use the Reynolds' Transportation Theorem, which was named after Osborne Reynolds (1842–1912). Let us consider a fluid that occupies a control volume CV, and it is bounded by a control surface CS at time t. The change in time $t + dt$, denotes that the system has begun to move out of the control volume (see Figure 2.1).

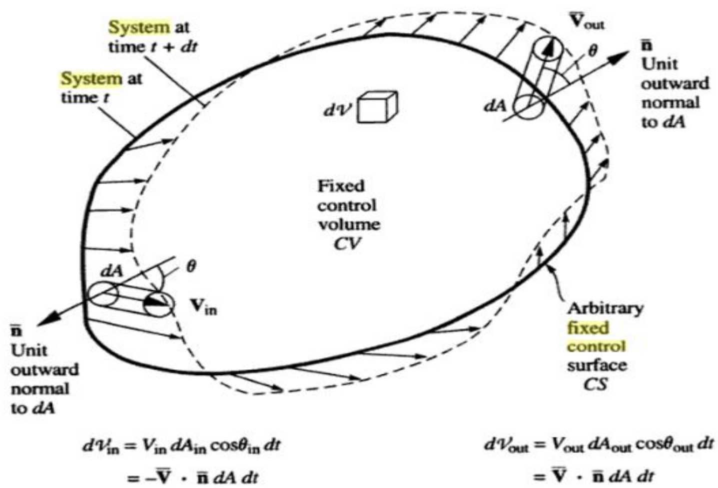


Figure 2.1: The system and the fixed control volume, adopted from [77].

Theorem 2.1.1. *([30], [77]) (**Reynolds' Transport Theorem**). The rate of change of an extensive property* Φ *of the system is equal to the rate of change the extensive property* Φ *inside the control volume CV plus the rate of the flux of N across the control surface CS at time t:*

$$\left.\frac{d\Phi}{dt}\right|_{sys} = \int_{CV} \frac{\partial}{\partial t}(\eta\rho)\, dV \; + \; \oint_{CS} \eta\rho\mathbf{v}\cdot \boldsymbol{n}\, dS, \tag{2.1.2}$$

where η *is intensive property which is equal to the extensive property per unit mass, that is,* $\eta = d\Phi/dm$, ρ *is the fluid density, the velocity vector* $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$, *and* $\boldsymbol{n}$ *is the unit normal vector in the outward direction.*

Theorem 2.1.2. *([81]) (**Divergence Theorem**). Let F be a vector field, whose component functions have continuous partial derivative on an open region that contains R. Where R is the region inside of* S, *and* S *is a closed surface. Then*

$$\iint_S F \cdot dA = \iiint_R div\,(F)\, dV,$$

where $div\, F = \nabla \cdot F$ *and* $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$.

Using the Divergence theorem (2.1.2) (also know as Guass theorem), equation (2.1.2) implies

$$\frac{d\Phi}{dt}\bigg|_{sys} = \int_{CV}\left[\frac{\partial}{\partial t}(\eta\rho) + \nabla\cdot(\eta\rho\mathbf{v})\right]dV. \tag{2.1.3}$$

For simplicity, we introduce an Eulerian derivative $\frac{D}{Dt}$, which is defined as follows

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla,$$

which can also be written in coordinate system as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathrm{v}_1\frac{\partial}{\partial x_1} + \mathrm{v}_2\frac{\partial}{\partial x_2} + \mathrm{v}_3\frac{\partial}{\partial x_3}.$$

Then from equation (2.1.3), we have

$$\frac{d\Phi}{dt}\bigg|_{sys} = \int_{CV}\left(\frac{D(\eta\rho)}{Dt} - \mathbf{v}\cdot\nabla(\eta\rho) + \nabla\cdot(\eta\rho\mathbf{v})\right)dV.$$

Since,

$$\nabla\cdot(\eta\rho\mathbf{v}) = \nabla(\eta\rho)\cdot\mathbf{v} + \eta\rho(\nabla\cdot\mathbf{v})$$

Hence, we obtain

$$\frac{d\Phi}{dt}\bigg|_{sys} = \int_{CV}\left(\frac{D(\eta\rho)}{Dt} + \eta\rho(\nabla\cdot\mathbf{v})\right)dV. \tag{2.1.4}$$

2.1.2 The continuity equation

Since the mass neither created nor destroyed, the law of conversation of mass states that the rate of change of mass with time in a system is zero. That is,

$$\frac{dm}{dt}\bigg|_{sys} = 0. \tag{2.1.5}$$

If we now consider a control volume CV, occupying fluid mass m, then, equation (2.1.4) with $\Phi \equiv m$ and $\eta \equiv 1$ reads as

$$\frac{dm}{dt}\bigg|_{sys} = \int_{CV}\left(\frac{D\rho}{Dt} + \rho(\nabla\cdot\mathbf{v})\right)dV.$$

According to equation (2.1.5) the above equation implies

$$\int_{CV} \left(\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) \right) dV = 0.$$

Since this equation holds for any control volume, we have

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0.$$

In the case of an incompressible fluid, we considered ρ to be constant and the above equation reduces to

$$\nabla \cdot \mathbf{v} = 0, \tag{2.1.6}$$

since in coordinate system

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathrm{v}_1 \frac{\partial \rho}{\partial x_1} + \mathrm{v}_2 \frac{\partial \rho}{\partial x_2} + \mathrm{v}_3 \frac{\partial \rho}{\partial x_3} = 0.$$

Equation 2.1.6 is known as the *continuity equation* [66], [77].

2.1.3 The conservation of momentum

Let us consider a fluid that occupies by a control volume CV and bounded by a control surface CS at time t. Its momentum is given by

$$\int_{CV} \rho \mathbf{v} \, dV.$$

Hence, the rate of change of momentum is

$$\frac{d}{dt} \int_{CV} \rho \mathbf{v} \, dV = \int_{CV} \rho \frac{D\mathbf{v}}{Dt} \, dV. \tag{2.1.7}$$

By Newtons second law, the rate of change of momentum is equal to the sum of all external forces. The total force $\mathbf{F_p}$ is the sum of the body force due to gravity and surface force due to viscous shear and normal stresses. That is,

$$\mathbf{F_p} = \int_{\mathbf{CV}} \rho \mathbf{g} \, \mathbf{dV} + \int_{\mathbf{CS}} [\sigma] \cdot \mathbf{dS},$$

where g is the gravity constant, $[\sigma]$ is the stress tensor. Using the Divergence theorem (2.1.2) to the above equation we obtain

$$\mathbf{F_p} = \int_{\mathbf{CV}} (\rho\mathbf{g} + \nabla \cdot [\sigma])\,\mathbf{dV}. \tag{2.1.8}$$

For any control volume CV, equation (2.1.7) and (2.1.8) must be equal according to Newtons second law, then we obtain

$$\int_{CV} \rho \frac{D\mathbf{v}}{Dt}\, dV = \int_{CV} (\rho g + \nabla \cdot [\sigma])\, dV.$$

Hence,

$$\rho \frac{D\mathbf{v}}{Dt} = \rho g + \nabla \cdot [\sigma]\,. \tag{2.1.9}$$

Equation (2.1.9) is known as the *Cauchy equation* [66], [77] . The physical interpretation of this Cauchy equation is seen clearly in Cartesian coordinates in two dimensions (see Figure 2.2) as follows. Momentum equation in x axis:

$$\rho \frac{D\mathrm{v}_1}{Dt} = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z},$$

momentum equation in y axis:

$$\rho \frac{D\mathrm{v}_2}{Dt} = \rho g_y + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z},$$

and momentum equation in z axis:

$$\rho \frac{D\mathrm{v}_3}{Dt} = \rho g_z + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}.$$

The components σ_{ij} can be expressed as the ratio of the force to the unit area in the i direction across a plane with normal in the j direction. The gradients in the stress tensor must be there to serve as a net force on any element. In particular, the stress tensor is symmetric, so $\sigma_{xy} = \sigma_{yx}$ [5].

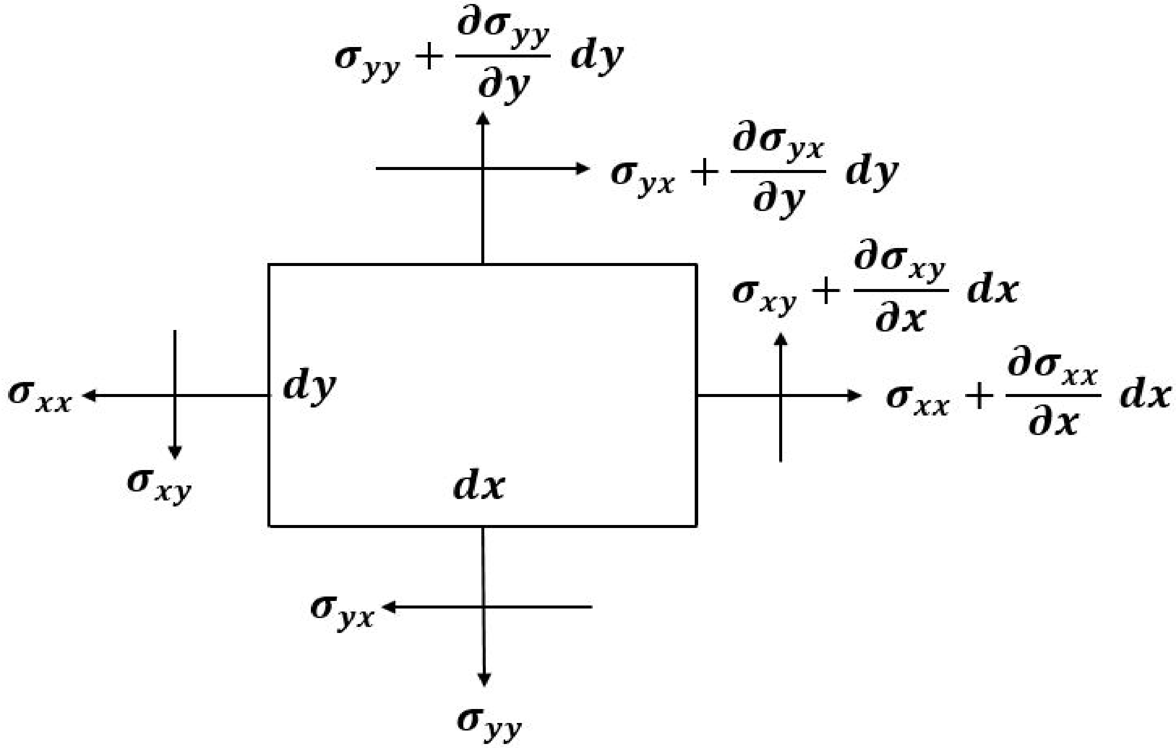


Figure 2.2: The surface stresses on a fluid element in two dimensions [53], [77].

The Stokes' viscosity law for incompressible fluid states that the stress tensor $[\sigma]$ is given by [30],

$$\sigma_{ij} = -P\delta_{ij} + \nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{2.1.10}$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j. \end{cases}$$

Substituting equation (2.1.10) into (2.1.9) we obtain the momentum in the x, y and z axis as follows. Momentum equation in x axis:

$$\rho\frac{D\mathrm{v}_1}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 \mathrm{v}_1}{\partial x^2} + \frac{\partial^2 \mathrm{v}_1}{\partial y^2} + \frac{\partial^2 \mathrm{v}_1}{\partial z^2}\right),$$

momentum equation in y axis:

$$\rho\frac{D\mathrm{v}_2}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 \mathrm{v}_2}{\partial x^2} + \frac{\partial^2 \mathrm{v}_2}{\partial y^2} + \frac{\partial^2 \mathrm{v}_2}{\partial z^2}\right),$$

momentum equation in z axis:

$$\rho\frac{D\mathrm{v}_3}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 \mathrm{v}_3}{\partial x^2} + \frac{\partial^2 \mathrm{v}_3}{\partial y^2} + \frac{\partial^2 \mathrm{v}_3}{\partial z^2}\right).$$

In a vector form, we can rewrite the momentum equation as

$$\rho\frac{D\mathbf{v}}{Dt} = \rho g - \nabla p + \nu\Delta\mathbf{v}. \tag{2.1.11}$$

If we neglect the body force due to gravity, then we have $\int_{CV}\rho g\, dV = 0$ and equation (2.1.11) reduce to

$$\frac{D\mathbf{v}}{Dt} = \frac{1}{\rho}(-\nabla p + \nu\Delta\mathbf{v}), \tag{2.1.12}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla.$$

Hence the equations (2.1.6) and (2.1.12) are called the Navier-stokes equations for incompressible fluids.

2.2 Complex potential

Definition 2.2.1. *Consider a complex variable $z = x + iy$. Let φ be a twice differentiable function which satisfy a Laplace's equation and therefore, harmonic. Then there exist a harmonic conjugate, denoted by ψ, such that*

$$W(z) = \varphi(x, y) + i\psi(x, y)$$

which is analytic. The function ψ is called the stream function and the analytic function W is called the complex potential [67]. The complex velocity is obtained by computing the derivative of W with respect to z.

The component functions φ and ψ of the complex potential W is harmonic, satisfying Laplaces equation, that is,

$$\begin{cases} \nabla^2\varphi = 0 \\ \nabla^2\psi = 0. \end{cases} \tag{2.2.1}$$

Since the complex potential W is analytic, its component functions φ and ψ satisfies the Cauchy-Riemann equations, that is,

$$\begin{cases} \dfrac{\partial\varphi}{\partial x} = \dfrac{\partial\psi}{\partial y} \\ \dfrac{\partial\varphi}{\partial y} = -\dfrac{\partial\psi}{\partial x}. \end{cases} \tag{2.2.2}$$

2.3 The derivation of the one-phase Hele-Shaw model

To derive the Hele-Shaw equations we first consider the Navier-Stokes equations neglecting the gravity (2.1.6) and (2.1.11),

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{\mathbf{1}}{\rho}(-\nabla\mathbf{p} + \nu\boldsymbol{\Delta}\mathbf{v}), \tag{2.3.1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2.3.2}$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ is the del-operator, $\Delta = \nabla\cdot\nabla = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}$, $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$ is the velocity of the flows, p is the pressure of the fluid, ν is the viscosity coefficient and ρ is the density of the fluid. Assume that the injection of the fluid is slow enough for the flow to be approximately stationary and that the flow is entirely horizontal. This means that the term $\frac{\partial\mathbf{v}}{\partial t}$ can be neglected in equation (2.3.1) and $\mathrm{v}_3 = 0$. Then equations (2.3.1) and (2.3.2) reduce to [30], [31],

$$\left(\mathrm{v}_1\frac{\partial}{\partial x_1}+\mathrm{v}_2\frac{\partial}{\partial x_2}\right)\mathrm{v}_1=-\frac{1}{\rho}\left(\frac{\partial p}{\partial x_1}\right)+\frac{\nu}{\rho}\Delta\mathrm{v}_1, \tag{2.3.3}$$

$$\left(\mathrm{v}_1\frac{\partial}{\partial x_1}+\mathrm{v}_2\frac{\partial}{\partial x_2}\right)\mathrm{v}_2=-\frac{1}{\rho}\left(\frac{\partial p}{\partial x_2}\right)+\frac{\nu}{\rho}\Delta\mathrm{v}_2, \tag{2.3.4}$$

$$0=-\frac{1}{\rho}\left(\frac{\partial p}{\partial x_3}\right). \tag{2.3.5}$$

Consider the boundary conditions $\mathrm{v}_1=\mathrm{v}_2=0$ whenever $x_3=0$ or $x_3=h$. By putting

$$\frac{\partial \mathrm{v}_1}{\partial x_j}=\frac{\partial \mathrm{v}_2}{\partial x_j}=\frac{\partial^2 \mathrm{v}_1}{\partial x_j^2}=\frac{\partial^2 \mathrm{v}_2}{\partial x_j^2}=0$$

for $j=1,2$, then equations (2.3.3), (2.3.4) and (2.3.5) reduce to

$$\frac{1}{\nu}\left(\frac{\partial p}{\partial x_1}\right)=\frac{\partial^2 \mathrm{v}_1}{\partial x_3^2}, \qquad \frac{1}{\nu}\left(\frac{\partial p}{\partial x_2}\right)=\frac{\partial^2 \mathrm{v}_2}{\partial x_3^2}. \tag{2.3.6}$$

Integrating both side of equations (2.3.6) with respect to x_3 and applying the boundary conditions $\mathrm{v}_1=\mathrm{v}_2=0$ whenever $x_3=0$ or $x_3=h$ and finally solving for v_1 and $\mathrm{v}_2=0$ we obtain

$$\mathrm{v}_1=\frac{1}{2}\frac{\partial p}{\partial x_1}\left(\frac{x_3^2}{\nu}-\frac{hx_3}{\nu}\right), \tag{2.3.7}$$

$$\mathrm{v}_2=\frac{1}{2}\frac{\partial p}{\partial x_2}\left(\frac{x_3^2}{\nu}-\frac{hx_3}{\nu}\right). \tag{2.3.8}$$

The average velocity v_j^{ave} over the gap is given by

$$v_j^{ave}=\frac{1}{h}\int_0^h V_j\,dx_3, \tag{2.3.9}$$

where $j = 1, 2$. Evaluating this average velocity for $j = 1$, we obtain

$$\begin{aligned}
v_1^{ave} &= \frac{1}{h}\int_0^h V_1 dx_3, \\
&= \frac{1}{h}\int_0^h \frac{1}{2}\frac{\partial p}{\partial x_1}\left(\frac{x_3^2}{\nu} - \frac{hx_3}{\nu}\right)dx_3, \\
&= \frac{1}{2h}\frac{\partial p}{\partial x_1}\left(\frac{x_3^3}{3\nu} - \frac{hx_3^2}{2\nu}\right)\Bigg|_{x_3=0}^{x_3=h}, \\
&= \frac{1}{2h\nu}\frac{\partial p}{\partial x_1}\left(\frac{h^3}{3} - \frac{h^3}{2}\right), \\
&= -\frac{h^2}{12\nu}\frac{\partial p}{\partial x_1}.
\end{aligned} \tag{2.3.10}$$

Similarly by repeating the same procedure for $j = 2$ we derive v_2 to be

$$\begin{aligned}
v_2^{ave} &= \frac{1}{h}\int_0^h V_2 dx_3, \\
&= \frac{1}{h}\int_0^h \frac{1}{2}\frac{\partial p}{\partial x_2}\left(\frac{x_3^2}{\nu} - \frac{hx_3}{\nu}\right)dx_3, \\
&= \frac{1}{2h}\frac{\partial p}{\partial x_2}\left(\frac{x_3^3}{3\nu} - \frac{hx_3^2}{2\nu}\right)\Bigg|_{x_3=0}^{x_3=h}, \\
&= \frac{1}{2h\nu}\frac{\partial p}{\partial x_2}\left(\frac{h^3}{3} - \frac{h^3}{2}\right), \\
&= -\frac{h^2}{12\nu}\frac{\partial p}{\partial x_2}.
\end{aligned} \tag{2.3.11}$$

Hence,

$$\mathbf{V} = -\frac{h^2}{12\nu}\nabla p, \tag{2.3.12}$$

where $\mathbf{V} = (v_1^{ave}, v_2^{ave}, v_3^{ave})$ and $v_3^{ave} = 0$. Where $\mathbf{V}$ and p depend only on x_1 and x_2. Therefore the equation (2.3.12) describes a two dimensional potential flow, where the potential function is proportional to the pressure. Due to compressibility of the flow the pressure is a harmonic function. Equation (2.3.12) is called the **Hele-Shaw equation**. It is of the same form as Darcy's law, which governs flow in porous media [30].

2.4 The Schwarz function for an analytic curve

The Schwarz function has played a significant role in reformulating, understanding and generating new ideas and concepts on how to find an exact solutions to the Hele-Shaw and Laplacian growth problems in the plane. According to Davis [17], the Schwarz function for an analytic curve is defined as follows. Let $\Gamma(t)$ be an analytic curve for a fixed t given by the equation $F(t,x,y)=0$, where $F(t,x,y)$ is a polynomial with respect to x and y with real coefficients and its partial derivatives $F_x(t,x,y)$ and $F_y(t,x,y)$ do not vanish simultaneously along an analytic curve Γ. Using the change of variables $z=x+iy$ and $\bar{z}=x-iy$, then, this function for a real-analytic curve $\Gamma := \{F(t,x,y)=0\}$ is defined as a solution to the equation $F\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}, t\right)=0$ with respect to $\bar{z}$. Such a solution exists in some neighborhood Ω_Γ of the curve Γ and a uniquely determined analytic function $S(t,z)$, for $z\in\Omega_\Gamma$, such that $\bar{z}=S(t,z)$ for $z\in\Gamma$. This function $S(t,z)$ is called the *Schwarz function* and is defined and analytic in a neighborhood $\Omega_\Gamma(t)$ of $\Gamma(t)$. Let us consider a few examples of some familiar curves.

2.4.1 Examples of Schwarz function

Example 1. *The equation of a circle centered at the point* (x_0, y_0) *of radius* $r(t)$ *for fixed* t *in Cartesian coordinates is given by*

$$(x-x_0)^2+(y-y_0)^2=r^2(t).$$

By performing the change of variables $x=\frac{z+\bar{z}}{2}$, $y=\frac{z-\bar{z}}{2i}$, *and solving for* $\bar{z}$ *in terms of* z, *we obtain the corresponding Schwarz function*

$$\bar{z}=S(t,z)=\frac{r^2(t)}{z-\bar{z}_0}+\bar{z}_0,$$

which has a simple pole at the point $z=\bar{z}_0$ *[71].*

Example 2. *The equation of an ellipse centered at the origin for fixed t in rectangular form is given by*

$$\frac{x^2}{a^2(t)} + \frac{y^2}{b^2(t)} = 1.$$

By performing the change of variables $x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$, *and solving for* $\bar{z}$, *we obtain the corresponding Schwarz function as follows*

$$\bar{z} = S(t, z) = \frac{a^2(t) + b^2(t)}{a^2(t) - b^2(t)} z + \frac{2a(t)b(t)}{b^2(t) - a^2(t)} \sqrt{z^2 + b^2(t) - a^2(t)},$$

which has foci at the points $z = \pm\sqrt{a^2(t) - b^2(t)}$ *for fixed t [71].*

Example 3. *A curve of the fourth order (called the ovals of Cassini) in rectangular form is given by the equation*

$$(x^2 + y^2)^2 - 2b^2(x^2 - y^2) = a^4 - b^4,$$

where a and b are unknown positive function of t. This equation describes a simple closed curve if a > b. and two closed curves otherwise. Using the change of variables $x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$, *and solving for* $\bar{z}$ *in terms of z, we obtain its Schwarz function as*

$$S(t, z) = \frac{\sqrt{b^2 z^2 + a^4 - b^4}}{\sqrt{z^2 - b^2}}.$$

Which has two interior singularities at the points $z = \pm b$ *and two exterior singularities at the points* $z = \pm\sqrt{\frac{b^4 - a^4}{b^2}}$ *[71].*

Example 4. *Consider a Neumann's oval [78] whose boundary* $\Gamma(t)$ *is given by the eaquation*

$$\Gamma(t) = \left\{(x^2 + y^2)^2 - a^2(t)x^2 - b^2(t)y^2 = 0\right\},$$

After performing change of variables $x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$, *and solving for* $\bar{z}$ *in terms of z, we obtain its Schwarz function as*

$$S(z, t) = \frac{z(a^2(t) + b^2(t)) + 2z\sqrt{z^2 d^2(t) + a(t)^2 b^2(t)}}{4z^2 - d^2(t)}.$$

Which has two interior simple poles at $z = \pm\frac{d}{2}$*, and two exterior branch points at* $z = \pm\frac{iab}{d}$*, where* $d^2(t) = a^2(t) - b^2(t)$*.* [71]

2.4.2 Properties of the Schwarz function

Let $\Gamma(t)$ be an analytic curve for fixed t with the Schwarz function $\bar{z} = S(t,z)$ where $z \in \Gamma(t)$, then the following properties hold along $\Gamma(t)$ [17]:

1. The derivative of the Schwarz function does not vanish at any point on the curve, that is, $\partial_z S \neq 0$ and $|\partial_z S| = 1$.

   Indeed, along the analytic curve $\Gamma(t)$, differentiating $\bar{z}$ w.r.t z, we have

   $$\frac{d\bar{z}}{dz} = \frac{dx - idy}{dx + idy}. \tag{2.4.1}$$

   Since the derivative of an analytic function is independent of the direction, in which increments are taken, and $\bar{z} = S(t,z)$ where $z \in \Gamma(t)$ we have

   $$\partial_z S = \frac{d\bar{z}}{dz} = \frac{dx - idy}{dx + idy} \tag{2.4.2}$$

   and

   $$|\partial_z S| = \left|\frac{d\bar{z}}{dz}\right| = \left|\frac{dx - idy}{dx + idy}\right|. \tag{2.4.3}$$

   It follows that $|\partial_z S| = 1$ and $\partial_z S \neq 0$.

2. The unit tangent vector and its conjugate along the curve can be reformulated in terms of the Schwarz function as

   $$\frac{dz}{ds} = \frac{1}{\sqrt{\partial_z S}}$$

and

$$\frac{d\bar{z}}{ds} = \sqrt{\partial_z S},$$

respectively. Where s is the arc length parameter on $\Gamma(t)$ at any time t.

Indeed, since s is the arc length parameter on the analytic curve, we have

$$ds^2 = dx^2 + dy^2 = (dx + idy)(dx - idy) = dzd\bar{z}. \tag{2.4.4}$$

Using the fact that

$$d\bar{z} = \partial_z S(t, z)dz$$

Thus, from equation (2.4.4), we obtain

$$ds = \sqrt{dz\partial_z S\, dz} = \sqrt{\partial_z S\,(dz)^2}. \tag{2.4.5}$$

Hence,

$$\frac{dz}{ds} = \frac{1}{\sqrt{\partial_z S}}. \tag{2.4.6}$$

In addition, since $\bar{z} = S(z, t)$, computing the derivative with respect to the arc length s we obtain

$$\frac{d\bar{z}}{ds} = \frac{d\bar{z}}{dz}\frac{dz}{ds} = \partial_z S\,\frac{dz}{ds}. \tag{2.4.7}$$

Substituting equation (2.4.6) into equation (2.4.7), we obtain

$$\frac{d\bar{z}}{ds} = \partial_z S\,\frac{1}{\sqrt{\partial_z S}} = \sqrt{\partial_z S}. \tag{2.4.8}$$

Hence, $\frac{dz}{ds} = \frac{1}{\sqrt{\partial_z S}}$ and $\frac{d\bar{z}}{ds} = \sqrt{\partial_z S}$ are respectively the tangent vector and its complex conjugate in terms of the Schwarz function.

3. The directional derivatives of a function $F(z, \bar{z})$ along and normal to the curve $\Gamma(t)$ can also be written in term of the Schwarz function as follows

$$\frac{dF}{ds} = F_z\frac{1}{\sqrt{\partial_z S}} + F_{\bar{z}}\sqrt{\partial_z S}, \tag{2.4.9}$$

and

$$\frac{dF}{dn} = -i\left(F_z \frac{1}{\sqrt{\partial_z S}} - F_{\bar{z}} \sqrt{\partial_z S}\right) \tag{2.4.10}$$

respectively.

Indeed, according to Davis [17], we can write $\frac{dF}{ds}$ and $\frac{dF}{dn}$ with the tangent angle ω as follows

$$\frac{dF}{ds} = F_z e^{i\omega} + F_{\bar{z}} e^{-i\omega} \tag{2.4.11}$$

and

$$\frac{dF}{dn} = -i\left(F_z e^{i\omega} - F_{\bar{z}} e^{-i\omega}\right). \tag{2.4.12}$$

Letting $F(z, \bar{z}) = z$ and $F(z, \bar{z}) = \bar{z}$ in equation (2.4.11) and (2.4.12) respectively, we obtain

$$\frac{dz}{ds} = e^{i\omega} = i\frac{dz}{dn}, \tag{2.4.13}$$

and

$$\frac{d\bar{z}}{ds} = e^{-i\omega} = -i\frac{d\bar{z}}{dn}. \tag{2.4.14}$$

Hence, equation (2.4.11) and (2.4.12) combined with (2.4.13) and (2.4.14) yields

$$\frac{dF}{ds} = F_z \frac{dz}{ds} + F_{\bar{z}} \frac{d\bar{z}}{ds}, \tag{2.4.15}$$

and

$$\frac{dF}{dn} = F_z \frac{dz}{dn} + F_{\bar{z}} \frac{d\bar{z}}{dn}. \tag{2.4.16}$$

Substituting equation (2.4.6) and (2.4.8) into (2.4.13) and (2.4.14), we obtain, in addition, the normal vector and its complex conjugate in term of the Schwarz function as

$$\frac{dz}{dn} = -\frac{i}{\sqrt{\partial_z S}}, \tag{2.4.17}$$

and

$$\frac{d\bar{z}}{dn} = i\sqrt{\partial_z S}. \tag{2.4.18}$$

Finally, substituting equations (2.4.6), (2.4.8), (2.4.17) and (2.4.18) into equations (2.4.15) and (2.4.16), yields the relations (2.4.9) and (2.4.10)

4. If we denote the curvature of $\Gamma(t)$ by κ, according to Davis [17] then κ can be expressed in terms of the Schwarz function as follows

$$\kappa = \frac{i}{2}\frac{\partial_{zz} S}{(\partial_z S)^{\frac{3}{2}}}, \tag{2.4.19}$$

where the $\partial_z S = \frac{\partial S}{\partial z}$ and $\partial_{zz} S = \frac{\partial^2 S}{\partial z^2}$. Also the derivative of the curvature along the boundary is given by [17]:

$$\partial_s \kappa = \frac{i}{2}\left(\frac{\partial_z S\,\partial_{zzz} S - 3/2(\partial_{zz} S)^2}{(\partial_z S)^3}\right). \tag{2.4.20}$$

Let ω be the angle between the tangent to the $\Gamma(t)$ and the real axis. Then,

$$\tan\omega = y' = \frac{dy}{dx},$$

and

$$\omega = \tan^{-1} y'.$$

The curvature is defined as

$$\kappa = \frac{d\omega}{ds} = \frac{d(\tan^{-1} y')}{dx}\,\frac{dx}{ds}. \tag{2.4.21}$$

Since $\frac{dx}{ds} = \frac{dx}{dz}\frac{dz}{ds}$, equation (2.4.21) yields

$$\kappa = \frac{d(\tan^{-1} y')}{dx}\frac{dx}{dz}\frac{dz}{ds}. \tag{2.4.22}$$

Hence,

$$\kappa = \frac{y''}{1 + y'^2}\frac{dx}{dz}\frac{dz}{ds}. \tag{2.4.23}$$

Where y', y'', $\frac{dz}{dx}$ and $\frac{dz}{ds}$ in term of the Schwarz function are as follows

$$y' = -i\frac{1 - \partial_z S}{1 + \partial_z S}, \quad \frac{dz}{dx} = \frac{2}{1 + \partial_z S}, \quad \frac{dz}{ds} = \frac{1}{\sqrt{\partial_z S}} \tag{2.4.24}$$

and

$$y'' = \frac{dy'}{dz}\frac{dz}{dx} = \frac{4i\partial_{zz}S}{(1 + \partial_z S)^3}. \tag{2.4.25}$$

Substituting equations (2.4.24) and (2.4.25) into (2.4.23), we obtain the curvature κ in term of the Schwarz function

$$\kappa = \frac{i}{2}\frac{\partial_{zz}S}{(\partial_z S)^{\frac{3}{2}}}.$$

Differentiating the curvature κ with respect to s we obtain

$$\partial_s\kappa = \frac{\partial\kappa}{\partial s} = \frac{\partial\kappa}{\partial z}\frac{\partial z}{\partial s} = \frac{i}{2}\left(\frac{\partial_z S\,\partial_{zzz}S - 3/2(\partial_{zz}S)^2}{(\partial_z S)^3}\right).$$

Hence,

$$\partial_s\kappa = \frac{i}{2}\{S, z\}, \tag{2.4.26}$$

where $\{S, z\} = \left(\frac{\partial_z S\,\partial_{zzz}S - 3/2(\partial_{zz}S)^2}{(\partial_z S)^3}\right)$.

Example 5. *A circle centered at the origin has a constant curvature.*

From example 1, the Schwarz function $\bar{z} = S(t,z) = \frac{a^2}{z}$*, which has a simple pole at the origin and radius a. Then,*

$$\partial_z S = \frac{-a^2}{z^2}, \quad \partial_{zz} S = \frac{2a^2}{z^3}, \quad \partial_{zzz} S = \frac{-6a^2}{z^4}. \tag{2.4.27}$$

Using equation (2.4.19) *we obtain* $\kappa = -\frac{1}{a}$*. Hence,*

$$\{S, z\} = 0, \quad \partial_s \kappa = \frac{i}{2}\{S, z\} = 0. \tag{2.4.28}$$

Proposition 2.4.1. *([37], [48]) The normal velocity* v_n *in* $\Omega_\Gamma(t)$ *of* $\Gamma(t)$ *can be written in terms of the Schwarz function as follows*

$$v_n = \frac{-i\partial_t S(t,z)}{2\sqrt{\partial_z S(t,z)}} = \frac{\partial_t S(t,z)}{2i\sqrt{\partial_z S(t,z)}}. \tag{2.4.29}$$

Proof. The normal velocity of the boundary, $v_n = \mathbf{v}\cdot\mathbf{n}$, where $\mathbf{v}$ is the velocity vector which after change of variables $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ can be written in a complex plane as

$$\mathbf{v} = \left(\frac{z_t + \bar{z}_t}{2}, \frac{z_t - \bar{z}_t}{2i}\right)$$

and $\mathbf{n}$ is the outward normal,

$$\mathbf{n} = \left(\frac{z_n + \bar{z}_n}{2}, \frac{z_n - \bar{z}_n}{2i}\right).$$

Thus,

$$\mathbf{v}\cdot\mathbf{n} = \left(\frac{z_t + \bar{z}_t}{2}, \frac{z_t - \bar{z}_t}{2i}\right) \cdot \left(\frac{z_n + \bar{z}_n}{2}, \frac{z_n - \bar{z}_n}{2i}\right),$$

and we obtain

$$\mathbf{v}\cdot\mathbf{n} = \left(\frac{z_t + \bar{z}_t}{2}\right) \cdot \left(\frac{z_n + \bar{z}_n}{2}\right) + \left(\frac{z_t - \bar{z}_t}{2i}\right) \cdot \left(\frac{z_n - \bar{z}_n}{2i}\right).$$

Hence,

$$\mathbf{v}\cdot\mathbf{n} = \frac{z_t\,\bar{z}_n + \bar{z}_t\,z_n}{2} \tag{2.4.30}$$

Using equations (2.4.17) and (2.4.18) in the third property of Schwarz function, equation (2.4.30) yields

$$v_n = \mathbf{v}\cdot\mathbf{n} = \frac{i}{2}\left(z_t\sqrt{\partial_z S} - \bar{z}_t\frac{1}{\sqrt{\partial_z S}}\right). \tag{2.4.31}$$

Let us differentiate the equation describing the boundary, $\bar{z} = S(t, z)$, with respect to time t, we obtain,

$$\bar{z}_t = \partial_t S + \partial_z S\, z_t. \tag{2.4.32}$$

Substitution (2.4.32) into (2.4.31) we obtain the normal velocity of the boundary,

$$v_n = \mathbf{v}\cdot\mathbf{n} = \frac{i}{2}\left(z_t\sqrt{\partial_z S} - (\partial_t S + \partial_z S\, z_t)\frac{1}{\sqrt{\partial_z S}}\right). \tag{2.4.33}$$

Hence, we have derived our desired formula (2.4.29). The Schwarz function was used by many authors in the context of the Hele-Shaw problem in order to reformulate the problem in terms of so called complex potential. □

3 Exact Solutions to Muskat Problem With Line Distribution of Sinks and Sources

3.1 Introduction

This chapter was published in the AMS journal "Contemporary Mathematics" (see the reference [3]). It concerns exact solutions to the two-phase Hele-Shaw problem with the line of distribution of sinks and sources [3], extending the results obtained by Crowdy [14]. The main difficulty of the two-phase problems is the fact that the pressure on the interface, that is separating the two fluids, is unknown. However, if we assume that the free boundary remains within the family of curves, specified by the initial shape of the interface separating the fluids (which is feasible if the surface tension is negligible) then the problem is drastically simplified. Our study is devoted to the situations when the evolution of the interface is controlled by a special choice of sinks and sources. The suggested method allows to obtain exact solutions for a certain class of curves for which the Schwarz function can be computed.

3.2 Mathematical formulation of the Muskat problem

The mathematical formulation of the problem is as follows. Let $\Omega_2(t) \subset \mathbb{R}^2$ with a boundary $\Gamma(t)$ at time t be a simply-connected bounded domain occupied by a fluid with a constant viscosity ν_2, and let $\Omega_1(t)$ be the region $\mathbb{R}^2 \setminus \bar{\Omega}_2(t)$ occupied by a different fluid of viscosity ν_1. To consider the two-phase Hele-Shaw problem forced by sinks and sources, we start with the Darcy's law, which stated that the velocities of fluids are proportional to the pressure gradients [16].

$$\mathbf{v}_j = -k_j \nabla p_j, \qquad j = 1, 2, \tag{3.2.1}$$

where the pressure p_j is a harmonic function almost everywhere in the region $\Omega_j(t)$, that is,

$$\Delta p = 0 \quad \text{in} \quad \Omega(t), \tag{3.2.2}$$

satisfying boundary conditions

$$p_1(x, y, t) = p_2(x, y, t) \quad \text{on} \quad \Gamma(t), \tag{3.2.3}$$

$$-k_1 \frac{\partial p_1}{\partial n} = -k_2 \frac{\partial p_2}{\partial n} = v_n \quad \text{on} \quad \Gamma(t). \tag{3.2.4}$$

Here $\mathbf{v}_j$ is a velocity vector of fluid j, $k_j = h^2/12\nu_j$, and h is the gap width of the Hele-Shaw cell. Equation (3.2.3) states the continuity of the pressure under the assumption of negligible surface tension. Equation (3.2.4) means that the normal velocity of the boundary itself coincides with the normal velocity of the fluid at the boundary. The free boundary $\Gamma(t)$ moves due to the sources and sinks located in both regions. Therefore, we adopt a natural physical assumption that the fluid flux generated by the system of sources and sinks is finite. That allows no more than the logarithmic growth of the fluid pressure near a point source/sink or at infinity,

$$|p_j(x, y, t)| \leq \left| -\frac{Q_a(t)}{2\pi k_j} \log \sqrt{(x - x_a)^2 + (y - y_a)^2} \right|, \tag{3.2.5}$$

where $Q_a(t)$ is the strength of the source/sink. We would like to stress that this physical assumption is not a restriction of the suggested method. In Section 3.5, for the sake of curiosity, we obtain a solution, whose far field flow is linear. The latter type of flow was obtained by Crowdy [14].

In this study, we allow the supports of sinks and sources to not only be points, but lines/curves as well, which could essentially change the dynamics of the evolution of the interface. A similar approach was used for the one-phase problems [71], [75]. In the case of the interior problem, this approach was motivated by the fact that during extraction through a point sink located within a viscous fluid, the free boundary is unstable, and the solution breaks down before all the fluid is extracted due to the formation of cusps, except for the situation of a circular boundary with a sink in the center. In [71] it was shown that a choice of sinks with line distributions linked to the initial shape occupied by the viscous fluid allows to enlarge the class of domains, from which the viscous fluid can be completely extracted without a cusp formation. Analogously, for the exterior problem with uniform extraction at infinity Howison [41] has proven that the elliptical bubbles are the only finite bubbles which exist for all times and whose boundary crosses all points initially outside the bubble. In all other cases, the solution either fails to exist in a finite time or the solution has some points on the interface that have a finite limit as time approaches infinity, so some fluid is "left behind" [11]. In the recent work [75], it was shown that if a point sink at infinity is replaced with a specific line distribution of sinks in the exterior region, then the evolution changes and it is possible to find other than elliptical shapes for which in the course of growing, the boundary of the air bubble crosses all points outside the bubble.

The notion of the two-phase mother body, which generalizes the dynamical (one-phase) mother body was used in [71] and [75]. We would like to acknowledge the results obtained by Karp related to the unbounded quadrature domains, including the asymptotic behavior of the boundary in $\mathbb{R}^2$ [45] and the connections between the generalized Newtonian potential and the unbounded quadrature domains in $\mathbb{R}^n$, $n \geq 3$ [46]. It is also worth mentioning the recent development in the two-phase quadrature domain theory [18], [26]. In the spirit of the latter theory the problem in question could be reformulated as follows. Let $u(x, y)$ be a continuous across $\Gamma(t)$ function, such that $u(x, y)\chi_{[\Omega_1]} = p_1$ and $u(x, y)\chi_{[\Omega_2]} = p_2$, where

$$\chi_{[\Omega_j]} = \begin{cases} 1, & \text{if} \quad (x, y) \in \Omega_j \cup \Gamma \\ 0, & \text{if} \quad (x, y) \notin \Omega_j \cup \Gamma. \end{cases}$$

Find a solution to the problem

$$\Delta u = \nu_1 + \nu_2 \quad \text{in} \quad \mathbb{R}^2, \tag{3.2.6}$$

$$-k_j \frac{\partial u}{\partial n} \chi_{[\Omega_j]} \to v_n \quad \text{as} \quad (x, y) \to \Gamma(t), \tag{3.2.7}$$

where $\nu_j(t)$ are time dependent distributions with supp $\nu_j(t) \subset \Omega_j$, supp $\nu_1 \cap$ supp $\nu_2 = \varnothing$.

3.3 The Schwarz function method of finding exact solutions for the Muskat problem

As mentioned above, the evolution of the interface separating the fluids is determined by the distributions of sinks and sources, which in the absence of the surface tension, could be chosen in such a way that keeps $\Gamma(t)$ within a family of curves defined by $\Gamma(0)$. For what follows, it is convenient to reformulate problem (3.2.1)–(3.2.5) in terms of the Schwarz function $S(z, t)$ of the curve $\Gamma(t)$ [17], [78]. This function for a real-analytic curve $\Gamma := \{g(x, y, t) = 0\}$ is defined as a solution to the equation

$$g\left(\frac{(z + \bar{z})}{2}, \frac{(z - \bar{z})}{2i}, t\right) = 0,$$

with respect to $\bar{z}$. Such (regular) solution exists in some neighborhood U_Γ of the curve Γ, if the assumptions of the implicit function theorem are satisfied [17]. Note that if g is a polynomial, then the Schwarz function is continuable into Ω_j, generally as a multiple-valued analytic function with a finite number of algebraic singularities (and poles). In U_Γ, the normal velocity, v_n, of $\Gamma(t)$ can be written in terms of the Schwarz function [37] as,

$$v_n = \frac{-i\partial_t S(z, t)}{2\sqrt{\partial_z S(z, t)}}$$

see proposition 2.4.1 for the proof.

To complete the reformulation of this two phase problem in terms of the Schwarz function, we introduce the complex potential,

$$W_j = p_j - i\psi_j,$$

which is an analytic multiple-valued function defined on $\Gamma(t)$ and in $\Omega_j(t) \cap U_\Gamma$, for $j = 1, 2$. Here ψ_j be a stream function. Let τ be an arclength along $\Gamma(t)$. Following [15], [48] and [49]. For the derivative of $W_j(z, t)$ with respect to z on $\Gamma(t)$ we have

$$\begin{aligned} \partial_z W_j = \frac{\partial_\tau W_j}{\partial_\tau z} &= \frac{\partial_\tau (p_j - i\psi_j)}{\partial_\tau z} \\ &= \frac{\partial_\tau p_j - i\partial_\tau \psi_j}{\partial_\tau z}, \end{aligned} \tag{3.3.1}$$

Taking into account the Cauchy-Riemann equations in the (n, τ) coordinates,

$$\frac{\partial p_j}{\partial n} = -\frac{\partial \psi_j}{\partial \tau}, \quad \frac{\partial \psi_j}{\partial n} = \frac{\partial p_j}{\partial \tau}, \tag{3.3.2}$$

we have

$$\begin{aligned} \partial_z W_j =& \frac{\partial_\tau p_j - i\partial_\tau \psi_j}{\partial_\tau z} \\ &= \frac{\partial_\tau p_j + i\partial_n p_j}{\partial_\tau z} \\ &= \frac{\partial_\tau p_j - i\frac{v_n}{k_j}}{\partial_\tau z}. \end{aligned}$$

Expressing v_n and $\partial_\tau z$ in terms of the Schwarz function, $v_n = -\frac{i\partial_t S}{2\sqrt{\partial_z S}}$ and $\partial_\tau z = \frac{1}{\sqrt{\partial_z S}}$, we obtain

$$\partial_z W_j = \partial_\tau p_j \sqrt{\partial_z S} - \frac{\partial_t S}{2k_j}. \tag{3.3.3}$$

Here $\partial_z W_j \equiv \frac{\partial W_j}{\partial z}$, $\partial_\tau z \equiv \frac{\partial z}{\partial \tau}$ etc. Since $p_1 = p_2$ on $\Gamma(t)$, equation (3.3.3) implies

$$\partial_z W_1 + \frac{\partial_t S}{2k_1} = \partial_z W_2 + \frac{\partial_t S}{2k_2} = \partial_\tau p_j \sqrt{\partial_z S}. \tag{3.3.4}$$

To keep $\Gamma(t)$ in a certain family of curves defined by $\Gamma(0)$, for example, in a family of ellipses, we assume that p_j on $\Gamma(t)$ is a function of time only. In that case the problem is simplified drastically, and on $\Gamma(t)$ we have

$$\partial_z W_j = -\frac{\partial_t S}{2k_j} \qquad j = 1, 2. \tag{3.3.5}$$

The reformulation of the two-phase problem in terms of the Schwarz function of the interface can be summarized in the following theorem.

Theorem 3.3.1. *Let* $\Gamma(t)$ *be an analytic curve for fixed t. Then there exist multiple-valued analytic complex potential functions* $W_j(z,t)$*, for* $j = 1, 2$*, defined on* $\Gamma(t)$ *and in* $\Omega_j \cap U_\Gamma(t)$ *that satisfy the equations*

$$\partial_z W_j = \frac{\partial W_j}{\partial z} = -\frac{\partial_t S}{2k_j},$$

whose real parts $\Re[W_j] = p_j$ *solves the free boundary problem* (3.2.1)-(3.2.4).

Taking into account that $\partial_t S$ can be continued off of $\Gamma(t)$, each equation (3.3.5) can be continued off of Γ into the corresponding Ω_j, where W_j is a multiple-valued analytic function. Note that equations (3.3.5) indicate that the singularities of W_1, W_2 and the Schwarz function are linked. Therefore, as we show below, those singularities in some cases can be used to control the interface between the fluids. Thus, the problem reduces to finding the distributions $\mu_1(t)$ and $\mu_2(t)$, that keep $\Gamma(t)$ in a family of curves generated by $\Gamma(0)$. The latter problem can be viewed as a generalization of a classical problem of electrostatics: find a two-charge system that yields the (desired) zero potential on a conducting plate.

To find the exact solutions, suppose that at $t = 0$ the interface is an algebraic curve,

$$\sum_{k=0}^{n} a_k(0) x^{k-n} y^n = 0,$$

with the Schwarz function $S(z, a_k^0)$. Assume that during the course of evolution the Schwarz function of the interface $S(z, a_k(t)) \equiv S(z,t)$ is such that $S(z, a_k(0)) = S(z, a_k^0)$. The steps of the method are

1. Compute the Schwarz function $S(z,t)$, locate its singularities, and define their type.

2. Using equations (3.3.5) find preliminary expressions for $\partial_z W_j$ and, by putting restrictions on the coefficients $a_k(t)$, eliminate their terms that involve non-integrable singularities.

3. Find the quantities W_j by integrating (3.3.5) with respect to z.

4. Compute the quantities p_j by taking the real parts of W_j.

5. Evaluate the quantities p_j on the interface to determine the independent of z function of integration from the step 3).

6. Compute the two-phase mother body.

   We comment that the steps 1-5 are straight-forward, and the step 6 is discussed below.

3.4 A two-phase mother body

Generally, the complex potentials W_j are multiple-valued functions in Ω_j. To choose a branch for each of these functions, one has to introduce the cuts, that serve as supports for the distributions of sinks and sources. The union of these distributions $\mu_1(t)$, $\mu_2(t)$ with disjoint supports (see formula (3.2.6)) and integrable densities, which allows a smooth evolution of the interface, is called below *a two-phase mother body*. The notion of a mother body comes from the potential theory [18], [26], [32]-[70]. The supports of these distributions consist of sets of arcs and/or points and do not bound any two-dimensional subdomains in $\Omega_j(t)$, $j = 1, 2$. Each cut included in the support of $\mu_j(t)$ is contained in the domain $\Omega_j(t)$, and the limiting values of the pressure on each side of the cut are equal. The value of the source/sink density on the cut is equal to the jump of the normal derivative $\partial_n p_j$ of the pressure p_j. To ensure that the total flux through the sources/sinks is finite, all of the singularities of the function W_j must have no more than logarithmic growth.

If $\Gamma(t)$ is an algebraic curve, then the singularities of W_j are either poles or algebraic singularities. Thus, each cut originates from an algebraic singularity $z_a(t)$ of the potential W_j. To ensure that the limiting values of p_j on both sides of each cut are equal, using the terminology of algebraic topology ([35], p. 21), we proceed as follows. We fix a point

$z_b \in \Gamma(t)$, the base point, and consider a fundamental group of loops $l \subset \Omega_j(t)$, having z_b as their starting and the terminal point, and surrounding the singular point $z_a(t)$, the group $\pi_1(\Omega_j \backslash \mathrm{sing}\,(W_j), z_a)$. A multiple-valued function W_j varies along l. We denote its variation by $\mathrm{var}_l\, W_j$, and the real part of its variation by $\mathrm{var}_l\, p_j$. Then, the zero level sets $\mathrm{var}_l\, p_j = 0$ describe the location of the desired cuts. Typically, the algebraic singularities are not stationary, that is $\dot{z}_a \neq 0$. The location of $z_a(t)$ is determined by the Schwarz function. The theorem below states the uniqueness of the direction of the cut at a non-stationary singularity $z_a(t)$ in general position. The latter means that the singularity $z_a(t)$ appears from a finite regular characteristic point of the complexification of the boundary $\Gamma(t)$, and the tangency between this singular point and the corresponding characteristic ray is quadratic. Under such requirements the function $S\,(z,t)$ at $z_a(t)$ has the square root type singularity:

$$S\,(z,t) = \Phi\,(z,t)\;\sqrt{z - z_a(t)} + \Psi\,(z,t)\,. \tag{3.4.1}$$

Here $\Phi\,(z,t)$ and $\Psi\,(z,t)$ are regular functions of z in a neighborhood of the point $z_a(t)$, and $\Phi\,(z_a(t),t) \neq 0$. The following theorem describes restrictions on the branch cuts in terms of their admissible slopes in the neighborhood of $z_a(t)$.

Theorem 3.4.1. *Let z_a be a singular point of the complex potential W_j located in $\Omega_j(t)$, $j = 1, 2$, such that $\dot{z}_a \neq 0$. Then, under the assumption of general position* (3.4.1)*, the direction of the cut, on which* $\mathrm{var}_l\, p_j = 0$ *near this point, is uniquely defined by the formula*

$$\varphi = \pi - 2(\arg[\Phi\,(z_a(t),t)] + \arg[\dot{z}_a]) + 2\pi k, \quad k = 0, \pm 1, \pm 2.... \tag{3.4.2}$$

Proof. We start with representation (3.4.1) dropping the regular part, $\Psi\,(z,t)$, in it and expanding the function $\Phi\,(z,t)$ into the Taylor series with respect to z at the point $(z_a(t),t)$,

$$S\,(z,t) = \sqrt{z - z_a(t)}\;\sum_{m=0}^{\infty} c_m(t)\,(z - z_a(t))^m\,. \tag{3.4.3}$$

The time derivative of the Schwarz function (3.4.3) has the form:

$$\dot{S}\,(z,t) = -\frac{\dot{z}_a}{2}\sum_{m=0}^{\infty} c_m\,(z-z_a)^{m-1/2} + \sum_{m=0}^{\infty}\Big(\dot{c}_m - \dot{z}_a(m+1)c_{m+1}\Big)\,(z-z_a)^{m+1/2}\,, \qquad (3.4.4)$$

therefore, formula (3.3.5) implies:

$$W_j\,(z,t) = \dot{z}_a c_0 (z-z_a)^{1/2}\left(\frac{1}{2k_j} + \xi_1(z,t)\right) - \dot{c}_a (z-z_a)^{3/2}\left(\frac{1}{3k_j} + \xi_2(z,t)\right),$$

where ξ_1 and ξ_2 are regular functions near z_a vanishing at this point, and $c_0 = \Phi\,(z_a(t),t)$. Hence, the variation of the pressure along the loop l is

$$\mathrm{var}_l\, p_j = \Re\left\{\dot{z}_a c_0 (z-z_a)^{1/2}\Big(\frac{1}{k_j} + 2\xi_1(z,t)\Big) - \dot{c}_0 (z-z_a)^{3/2}\left(\frac{2}{3k_j} + 2\xi_2(z,t)\right)\right\}. \qquad (3.4.5)$$

Consider a small neighborhood of the point z_a, and set there

$$z = z_a + \rho e^{i\varphi},$$

assuming that ρ is small. Since $\dot{z}_a \neq 0$, the second term in (3.4.5) is small with respect to the first term and therefore should be dropped. Setting the principal part of $\mathrm{var}_l\, p$ to zero, we have

$$\mathrm{var}_l\, p_j = \frac{1}{k_j}|\dot{z}_a||c_0|\,\sqrt{\rho}\,\Re\,\left\{\exp i\left(\frac{\varphi}{2} + \pi n + \arg\dot{z}_a + \arg c_0\right)\right\} = 0, \qquad (3.4.6)$$

where $n = 0, \pm 1, \pm 2, \ldots$. Equation (3.4.6) implies formula (3.4.2), which finishes the proof.

Remark that if $\dot{z}_a = 0$ that is, z_0 is a stationary singular point, and $\dot{\Phi}\,(z_a(t),t) \neq 0$, from formula (3.4.5) follows

$$\mathrm{var}_l\, p_j = -\frac{2}{3k_j}\rho^{\frac{3}{2}}\,\Re\,\left[e^{\left(\frac{3i\varphi}{2}+i\theta_0\right)}\{\dot{R}_0 + iR_0\dot{\theta}_0\}\left(1+\xi_1\left(z_a+\rho e^{i\varphi}\right)\right)\right] = 0,$$

which results in three direction of admissible cuts

$$\varphi = \varphi_k = \frac{2}{3}(\pi k - \theta_0 + \nu_0),$$

and $k = 0, \pm 1, \pm 2, ...$, where

$$\nu_0 = \arcsin(\dot{R}_0 / \sqrt{\dot{R}_0^2 + R_0^2 \dot{\theta}_0^2}),$$

R_0 and θ_0 are the modulus and the argument of $\Phi(z_a(t), t)$ respectively. In the case when $\dot{z}_a = 0$ and first $(j-1)$ time derivatives of $\Phi(z, t)$ at $z_a(t)$ equal to zero, but the j-th derivative is not, the number of directions is $(2j + 3)$.

The constructed support of the distribution μ_j must satisfy the following conditions: each cut emanates from a singular point of W_j, located in the domain $\Omega_j(t)$, and $\text{var}_l\, p_j$ vanishes on each cut. To obtain a two-phase mother body, one has to calculate the corresponding density along each cut. In the next section we show examples when a two-phase mother body exists and is unique, and we use the constructed mother bodies to derive the exact solutions to the Muskat problems.

3.5 Examples of specific $\Gamma(0)$

3.5.1 Circle

To illustrate the method, we start with the simplest example for which solution is known. Suppose that the initial shape of the interface is a circle with equation

$$x^2 + y^2 = a^2(0),$$

and during the evolution the boundary remains circular,

$$x^2 + y^2 = a^2(t).$$

The corresponding Schwarz function see example (1) is

$$S = \frac{a^2(t)}{z},$$

and the derivative with respect to t is given as

$$\dot{S} = \frac{2a\dot{a}}{z}.$$

Equation (3.3.4) in this case reads as

$$\partial_z(W_2 - W_1) = \Big(\frac{1}{k_1} - \frac{1}{k_2}\Big)\frac{a\dot{a}}{z}.$$

Integrating this equation with respect to z, we have

$$W_2 - W_1 = \Big(\frac{1}{k_1} - \frac{1}{k_2}\Big)a\dot{a}\log z \; + \; C(t).$$

The complex potential W_2 has a singularity at zero, while W_1 has a singularity at infinity. Thus, the two-phase mother body has support at these two points, one of which serves as a sink and the other as a source. Taking the real parts of both sides of the previous equation, we obtain

$$p_2 - p_1 = \Big(\frac{1}{k_1} - \frac{1}{k_2}\Big)a\dot{a}\ln\sqrt{x^2 + y^2} + \Re[C(t)],$$

which is satisfied if

$$p_j = -\frac{a\dot{a}}{2k_j}\ln(x^2 + y^2) + C_j(t),$$

for $j = 1, 2$ with C_j chosen from the condition (3.2.3). This choice is, obviously, not unique. To specify, for instance, $C_1(t)$ one could use the condition at infinity. If the condition reads as

$$p_1(x, y, t) = -\frac{Q(t)}{2\pi k_1}\ln\sqrt{x^2 + y^2} \quad \text{as} \quad \sqrt{x^2 + y^2} \to \infty$$

with a defined sink/source strength $Q(t)$ at infinity, then $C_1 = 0$ and

$$p_1 = -\frac{a\dot{a}}{2k_1}\ln(x^2 + y^2).$$

The evolution of the boundary is defined by the equation $Q(t) = \dot{A} = 2\pi a\dot{a}$ via $\dot{a}$. Here $\dot{A}(t)$ is the rate of change of the area of the interior domain $\Omega_2(t)$. The interior pressure is

$$p_2 = -\frac{a\dot{a}}{2k_2}\ln(x^2 + y^2) + \frac{a\dot{a}}{2k_2}\ln a^2 - \frac{a\dot{a}}{2k_1}\ln a^2,$$

with a source/sink at the origin of the strength $|Q(t)|$. Alternatively, we could choose C_j such that p_j vanishes on Γ. Then,

$$p_1 = -\frac{a\dot{a}}{2k_1}\ln(x^2+y^2) + \frac{a\dot{a}}{2k_1}\ln a^2, \tag{3.5.1}$$

and

$$p_2 = -\frac{a\dot{a}}{2k_2}\ln(x^2+y^2) + \frac{a\dot{a}}{2k_2}\ln a^2. \tag{3.5.2}$$

We remark, that in the case of the circular initial interface, condition (3.2.3) could be replaced with

$$p_1 - p_2 = \gamma\kappa,$$

while keeping the boundary in the family of concentric circles during the course of the interface evolution. Here γ is a constant surface tension coefficient and κ is a free boundary curvature. In that case the equation in (3.5.1) is replaced with

$$p_1 = \frac{a\dot{a}}{2k_1}\ln(a^2/(x^2+y^2)) + \frac{\gamma}{a}.$$

3.5.2 Ellipse

Consider a two-phase problem with an elliptical interface,

$$\Gamma(0) = \left\{\frac{x^2}{a^2(0)} + \frac{y^2}{b^2(0)} = 1\right\},$$

where $a(0)$ and $b(0)$ are given and $a(0) > b(0)$. The Schwarz function see example (2) of an elliptical interface with semi-axes $a(t)$ and $b(t)$ is

$$S(z,t) = \frac{a^2(t)+b^2(t)}{d^2(t)}z - \frac{2a(t)b(t)}{d^2(t)}\sqrt{z^2-d^2(t)},$$

where $d(t) = \sqrt{a^2(t)-b^2(t)}$ is the half of the inter-focal distance. Assuming that the interface remains elliptical during the course of the evolution, from equation (3.3.4) we have

$$W_2 - W_1 = \Big(\frac{1}{k_2} - \frac{1}{k_1}\Big)\Big\{-\frac{z^2}{4}\frac{\partial}{\partial t}\Big(\frac{a^2+b^2}{d^2}\Big)$$
$$+\frac{z}{2}\sqrt{z^2-d^2}\,\frac{\partial}{\partial t}\Big(\frac{ab}{d^2}\Big) - \frac{1}{2}\log(z+\sqrt{z^2-d^2})\,\frac{\partial(ab)}{\partial t}\Big\} + C(t). \tag{3.5.3}$$

The first two terms in the right hand side of (3.5.3) have poles of order two at infinity. Those terms are eliminated, if the eccentricity of the ellipse does not change with time, the latter implies that the ratio $a(t)/b(t) = const$. This ensures the existence of no more than logarithmic singularity at infinity and agrees with the solution to the exterior one-phase problem reported in [41]. Thus, the expression for the complex potentials reduces to

$$W_2 - W_1 = -\frac{1}{2}\frac{\partial(ab)}{\partial t}\left(\frac{1}{k_2} - \frac{1}{k_1}\right)\log(z+\sqrt{z^2-d^2}) + C(t). \tag{3.5.4}$$

Taking the real parts of both sides, we have

$$p_2 - p_1 = -\frac{1}{2}\frac{\partial(ab)}{\partial t}\left(\frac{1}{k_2} - \frac{1}{k_1}\right)\ln|z+\sqrt{z^2-d^2}| + C_2(t) - C_1(t),$$

which can be written as

$$p_j = -\frac{1}{2k_j}\frac{\partial(ab)}{\partial t}\ln|z+\sqrt{z^2-d^2}| + C_j(t),$$

where $C_j(t)$ may be chosen from the condition $p_j = 0$ on $\Gamma(t)$, which leads to

$$p_j = -\frac{1}{2k_j}\frac{\partial(ab)}{\partial t}\Big(\ln|z+\sqrt{z^2-d^2}| - \ln(a+b)\Big), \tag{3.5.5}$$

or

$$p_j = -\frac{1}{2k_j}\frac{\partial(ab)}{\partial t}\Big(\ln\sqrt{(x+\alpha)^2(1+y^2/\alpha^2)} - \ln(a+b)\Big),$$

where

$$\Re[C_j(t)] = \frac{1}{2k_j}\frac{\partial(ab)}{\partial t}\ln(a+b),$$

and

$$\alpha^2 = \Big(x^2 - y^2 - d^2 + \sqrt{(x^2-y^2-d^2)^2 + 4x^2y^2}\Big)/2.$$

Note that the inter-focal distance, $d(t) = 2b(t)\sqrt{a^2(0)/b^2(0) - 1}$, of such an ellipse changes, while the eccentricity is constant. The support of the two-phase mother body consists of a point sink/source at infinity and a source/sink distribution with density

$$\mu_2(x,t) = \frac{2ab\partial_t(\sqrt{d^2 - x^2})}{d^2 k_2}$$

along the inter-focal segment. The support of this distribution is defined using formula (3.4.2). Indeed, the singular points of W_2 are $z = \pm d$ with

$$\Phi(z,t) = \frac{-2ab\sqrt{z \pm d}}{d^2},$$

respectively. Formula (3.4.2) implies that the direction of the cut at $z = d$ is defined by the angle $\varphi = \pi + 2\pi k$, and at $z = -d$ by the angle $\varphi = 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$. Thus, the two-phase mother body, described above, allows the interface between two fluids remain elliptical for an infinite time if the domain Ω_2 grows (sources are located along the interfocal segment and a sink is located at the point of infinity). The opposite sink/source choice allows the complete removal of the fluid initially occupied domain Ω_2.

We remark that from (3.5.5) follows that the pressure at infinity grows as

$$p_1 \sim -(2k_1)^{-1} \ln|z|\partial_t(ab) = -\frac{\dot{A}}{2k_1} \ln\sqrt{x^2 + y^2},$$

which agrees with formula (3.2.5). Moreover, the strength of the sink/source at infinity is in agreement with the total strength of the source/sink distribution in Ω_2 since,

$$\int_{-d}^{d} k_2 \nu_2(x,t)\, dx = \pi\partial_t(ab) = \dot{A}.$$

Note that Crowdy [14] obtained an exact solution with a different type of growth at infinity; the solution, reported in [14], has a linear far field flow and a constant area of ellipse. We observe that under the assumption that the area of the elliptical inclusion does not change in time, that is, $a(t)b(t) = const$, equation (3.5.3) implies

$$W_j = \frac{1}{k_j}\Big\{-\frac{z^2}{4}\frac{\partial}{\partial t}\Big(\frac{a^2 + b^2}{d^2}\Big) + \frac{z}{2}\sqrt{z^2 - d^2}\,\frac{\partial}{\partial t}\Big(\frac{ab}{d^2}\Big)\Big\} + C_j(t), \quad j = 1, 2,$$

therefore, the pressure p_1 is defined by

$$p_1 = \frac{1}{k_1}\Re\Big\{-\frac{z^2}{4}\frac{\partial}{\partial t}\Big(\frac{a^2+b^2}{d^2}\Big) + \frac{ab\,z}{2}\sqrt{z^2-d^2}\,\frac{\partial}{\partial t}\Big(\frac{1}{d^2}\Big)\Big\} - \frac{b^2 a\dot{a}}{k_1 d^2},$$

which retries a linear far field flow. However, the obtained solution is different from Crowdy's, since the interior flow reported in [14] is a simple linear flow, while the solution in question, in addition to the linear flow, has another term

$$p_2 = \frac{1}{2k_2}\Big\{\frac{(y^2-x^2)}{2}\frac{\partial}{\partial t}\Big(\frac{a^2+b^2}{d^2}\Big) + \frac{ab\,x(\alpha^2-y^2)}{\alpha}\,\frac{\partial}{\partial t}\Big(\frac{1}{d^2}\Big)\Big\} - \frac{b^2 a\dot{a}}{k_2 d^2}.$$

The interior flow is generated by the density

$$\mu = \frac{ab\,\partial_t(d^2)}{k_2 d^4}\,\frac{(2x^2-d^2)}{\sqrt{d^2-x^2}},$$

supported on the inter-focal segment. Such a density changes sign along the inter-focal segment, so the area of the ellipse does not change in time: if $a(t)$ increases with time, the ellipse becomes "thiner".

3.5.3 The Neumann's oval

Let the initial free boundary have a shape of the Neumann's oval [78] given by the equation

$$\Gamma(0) = \Big\{(x^2+y^2)^2 - a(0)^2x^2 - b(0)^2y^2 = 0\Big\},$$

(see Fig. 3.1). Its Schwarz function at time $t = 0$ is given by

$$S(z,0) = \frac{z(a^2(0)+b^2(0)) + 2z\sqrt{z^2d^2(0)+a(0)^2b^2(0)}}{4z^2-d^2(0)},$$

where $d^2(0) = a^2(0) - b^2(0) > 0$ with given $a(0)$, $b(0)$. Assume that during the evolution the domain retains the Neumann's oval shape,

$$\Gamma(t) = \Big\{(x^2+y^2)^2 - a^2(t)x^2 - b^2(t)y^2 = 0\Big\},$$

with unknown $a(t)$, $b(t)$ for $t > 0$. The singularities of the Schwarz function,

$$S(z,t) = \frac{z(a^2(t)+b^2(t)) + 2z\sqrt{z^2d^2(t)+a(t)^2b^2(t)}}{4z^2-d^2(t)},$$

located in the interior domain $\Omega_2(t)$ are simple poles at $z = \pm\frac{d}{2}$, while the singularities located in the exterior domain Ω_1 are the branch points at $z = \pm\frac{ia(t)b(t)}{d(t)}$.

To ensure at most logarithmic growth of the pressure at the singular points, d must be constant. In that case

$$\dot{S} = \frac{z\partial_t(a^2+b^2)}{4z^2-d^2} + \frac{z\partial_t(a^2b^2)}{(4z^2-d^2)\sqrt{z^2d^2+a^2b^2}}. \tag{3.5.6}$$

Then equation (3.3.4) implies

$$\begin{aligned} p_2 - p_1 = & -\frac{1}{2}\Big(\frac{1}{k_2}-\frac{1}{k_1}\Big)\Big\{\frac{\partial_t(a^2+b^2)}{8}\Re[\log(4z^2-d^2)] \\ & -\frac{\partial_t(a^2b^2)}{2(a^2+b^2)}\Re\big[\tanh^{-1}\frac{2\sqrt{a^2b^2+d^2z^2}}{a^2+b^2}\big]\Big\} + C(t). \end{aligned} \tag{3.5.7}$$

Taking into account that when d is constant, $\partial_t(a^2+b^2) = 4a\dot{a}$ and $\partial_t(a^2b^2) = 2a\dot{a}(a^2+b^2)$, we have

$$\begin{aligned} p_2 - p_1 = & -\frac{a\dot{a}}{2}\Big(\frac{1}{k_2}-\frac{1}{k_1}\Big)\Re\Big\{\log(4z^2-d^2) \\ & -\log(a^2+b^2+2\sqrt{a^2b^2+d^2z^2})\Big\} + \ln(d) + C(t). \end{aligned} \tag{3.5.8}$$

Note that on the interface

$$\log\left(\frac{4z^2-d^2}{a^2+b^2+2\sqrt{a^2b^2+d^2z^2}}\right) = \log\left(\frac{z}{\bar{z}}\right),$$

whose real part is zero on $\Gamma(t)$. Therefore, to satisfy the condition (3.2.3),

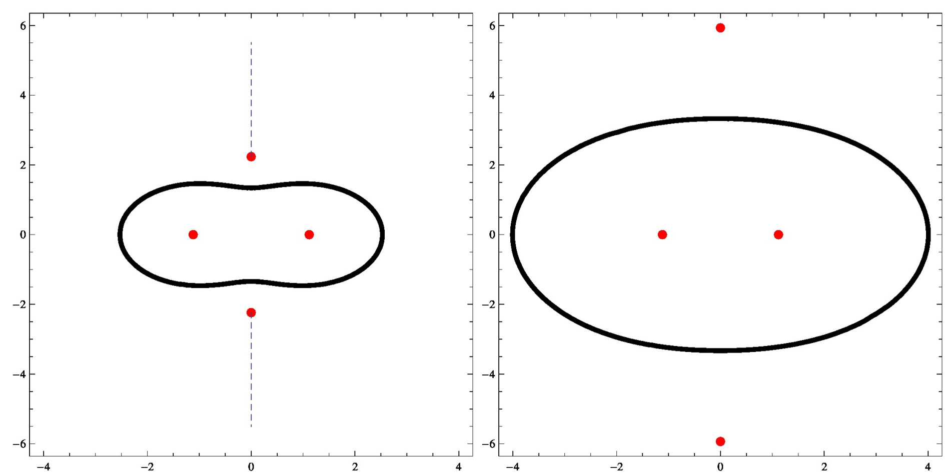
$$C(t) = -\frac{a\dot{a}\ln d}{2}\Big(\frac{1}{k_2}-\frac{1}{k_1}\Big).$$

Thus, we have

$$p_j = \frac{a\dot{a}}{2k_j}\Re\left[\log\frac{a^2+b^2+2\sqrt{a^2b^2+d^2z^2}}{4z^2-d^2}\right], \qquad j = 1, 2, \tag{3.5.9}$$

and the interior part of mother body μ_2 consists of either two point sinks or two point sources.

To find the directions of the cuts in Ω_1, we use formula (3.4.2) (it is general position like an ellipse). In the neighborhood of the branch point $z_0 = \frac{iab}{d} \in \Omega_1$, $\arg[\Phi(z_0(t), t)] = -\pi/4 + \pi k$, $\arg[\dot{z}_0] = \pm\pi/2$, where the plus corresponds to the growth of the interior domain Ω_2. The direction of the cut near this point is $\varphi = \pi/2 + 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$. Similarly, at the $z_0 = -iab/d$, $\arg[\Phi(z_0(t), t)] = \pi/4 + \pi k$, $\arg[\dot{z}_0] = \pm\pi/2$, where the plus corresponds to the decrease of the interior domain Ω_2. The direction of the cut near this point is $\varphi = -\pi/2 + 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$.



(a) (b)

Figure 3.1: The Neumann's ovals (solid lines), singularities of the complex potential (dots), and the cuts (dashed lines) for $d = \sqrt{5}$: (a) $a = 2.5$, $b = \sqrt{5}/2$; (b) $a = 4$, $b = \sqrt{11}$ [3].

The support of the two-phase mother body for the Neumann's oval is shown in Fig. 3.1. The cuts are the dashed lines that go along the imaginary axis starting at each branch point (the dots in the exterior domain in Fig. 3.1) to infinity. The dots in the interior domain correspond to the simple poles.

To obtain the sink/source density along the cut located above the x-axis, we first compute the variation of $S(z,t)$

$$\mathrm{var}_l\, S(z) = 4z(4z^2 - d^2)^{-1}\sqrt{z^2d^2 + a^2b^2}\Big|_{z=iy, y>ab/d} = \frac{4y\sqrt{y^2d^2 - a^2b^2}}{4y^2 + d^2},$$

then the jump of $\partial_z W_1$, that is,

$$\mathrm{var}_l\, \partial_z W_1 = -\frac{1}{2k_1}\partial_t(\mathrm{var}_l\, S(z)) = -\frac{2y}{k_1}\partial_t\Big(\frac{\sqrt{y^2d^2 - a^2b^2}}{4y^2 + d^2}\Big),$$

finally, the sink distribution on both cuts equals

$$\mu_1(y,t) = \frac{1}{k_1}\Big|\frac{\partial_t(a^2b^2)y}{(4y^2 + d^2)\sqrt{y^2d^2 - a^2b^2}}\Big|.$$

To compute the rate, $Q_1(t)$, through the cuts located in the exterior domain, one has to integrate $k_1\mu_1(y,t)$ along the cuts, which implies

$$Q_1(t) = \frac{\pi}{2}\frac{\partial_t(a^2b^2)}{(a^2 + b^2)} = \frac{\pi}{2}\partial_t(a^2). \tag{3.5.10}$$

From formula (3.5.9) follows that the rate at infinity is

$$Q(t) = \frac{\pi}{4}\partial_t(a^2 + b^2) = \frac{\pi}{2}\partial_t(a^2). \tag{3.5.11}$$

Those rates are linked to the change of the area

$$Q(t) + Q_1(t) = \pi\partial_t(a^2) = \dot{A}, \tag{3.5.12}$$

where $A(t)$ is the area of the interior domain. Indeed, the area of the Neumann's oval is $A = \pi(a^2 + b^2)/2$ (see [78], p. 20), which can be rewritten as $A = \pi(a^2 - d^2/2)$. Since in the case in question d is constant, $\dot{A} = \pi\partial_t(a^2)$. Note also that the rate through each of the two point sources/sinks located in the domain Ω_2 equals $\dot{A}/2$.

We remark that since $d^2(t) = a^2(t) - b^2(t) = const$, in the case of decreasing area of Ω_2, the obtained solution is valid up to the limiting case when b approaches zero and Γ splits into two circles $(x + \frac{d}{2})^2 + y^2 = \frac{d^2}{4}$ and $(x - \frac{d}{2})^2 + y^2 = \frac{d^2}{4}$.

3.5.4 The Cassini's oval

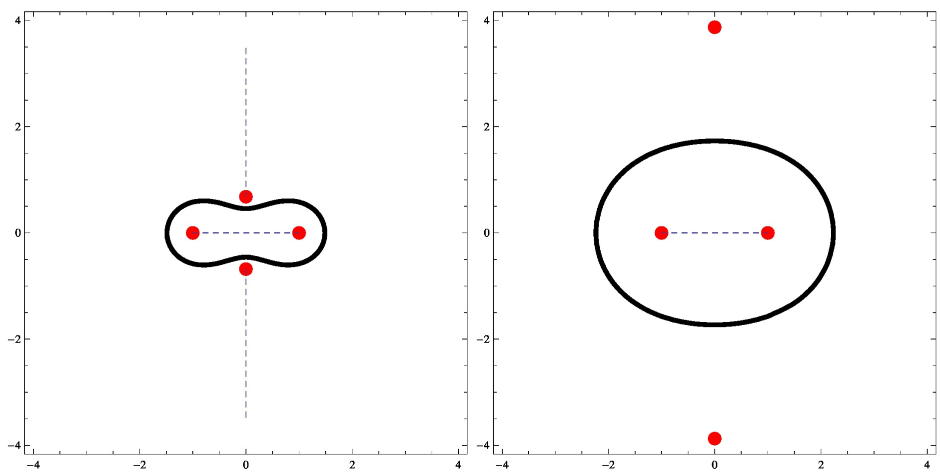
Similar to the previous examples, assume that $\Gamma(t)$ remains in the specific family of curves, the Cassini's ovals, given by the equation

$$\left(x^2+y^2\right)^2-2b(t)^2\left(x^2-y^2\right)=a(t)^4-b(t)^4,$$

or

$$\left[(x-b)^2+y^2\right]\left[(x+b)^2+y^2\right]=a^4,$$

where $a(t)$ and $b(t)$ are unknown positive functions of time. This curve consists of one closed curve if $a(t)>b(t)$ (see Fig. 3.2), and two closed curves otherwise. Assume that at $t=0$ $a(0)>b(0)$.



(a) (b)

Figure 3.2: The Cassini's ovals (solid lines), singular points of the complex potential (dots), and the cuts (dashed lines) for $b=1$: (a) $a=1.1$, (b) $a=2$ [3].

The Schwarz function of the Cassini's oval see example (2.4.1),

$$S\,(z,t)=\frac{\sqrt{b^2z^2+a^4-b^4}}{\sqrt{z^2-b^2}},$$

has two singularities in $\Omega_1(t)$, $z = \pm i\sqrt{(a^4 - b^4)/b^2}$, and two singularities, $z = \pm b$, in $\Omega_2(t)$. Differentiating the Schwarz function with respect to t, we have

$$\dot{S}(z,t) = \frac{b\dot{b}z^2 + 2a^3\dot{a} - 2b^3\dot{b}}{\sqrt{b^2z^2 + a^4 - b^4}\sqrt{z^2 - b^2}} + \frac{b\dot{b}\sqrt{b^2z^2 + a^4 - b^4}}{\sqrt{(z^2 - b^2)^3}}.$$

To ensure that the singularities of the complex potential have no more than the logarithmic type, $\dot{b}$ must be zero. Thus, we have

$$\dot{S}(z,t) = \frac{2a^3\dot{a}}{\sqrt{b^2z^2 + a^4 - b^4}\sqrt{z^2 - b^2}}.$$

Then equation (3.3.4) implies

$$W_2 - W_1 = -a\dot{a}\Big(\frac{1}{k_2} - \frac{1}{k_1}\Big)F\Big(\cos^{-1}(\frac{b}{z}),\ \frac{\sqrt{a^4 - b^4}}{a^2}\Big) + C(t), \tag{3.5.13}$$

where $F(\alpha,\beta)$ is the incomplete elliptic integral of the first kind,

$$\begin{aligned} F\left(\cos^{-1}(\frac{b}{z}),\ \frac{\sqrt{a^4 - b^4}}{a^2}\right) &= \int_0^{\sqrt{1-\frac{b^2}{z^2}}} \frac{dt}{\sqrt{1 - \frac{a^4-b^4}{a^4}t^2}\ \sqrt{1 - t^2}} \\ &= \int_0^{\cos^{-1}(\frac{b}{z})} \frac{dt}{\sqrt{1 - \frac{a^4-b^4}{a^4}\sin^2 t}}. \end{aligned} \tag{3.5.14}$$

Then

$$p_2 - p_1 = -\frac{a\dot{a}}{2}\Big(\frac{1}{k_2} - \frac{1}{k_1}\Big)\Big[F\Big(\xi,\ \frac{\sqrt{a^4 - b^4}}{a^2}\Big) + \overline{F\Big(\xi,\ \frac{\sqrt{a^4 - b^4}}{a^2}\Big)}\Big] + C(t), \tag{3.5.15}$$

where $\xi = \cos^{-1}(\frac{b}{z})$. Using the property $\overline{F(\alpha,\beta)} = F(\overline{\alpha},\beta)$ and the summation formula for the elliptic integrals [6], we have

$$p_2 - p_1 = -\frac{a\dot{a}}{2}\left(\frac{1}{k_2} - \frac{1}{k_1}\right)F\left(\alpha,\ \frac{\sqrt{a^4 - b^4}}{a^2}\right) + C(t), \tag{3.5.16}$$

where

$$\alpha = \sin^{-1}\left(\frac{\cos\overline{\xi}\sin\xi\sqrt{1 - \frac{a^4-b^4}{a^4}\sin^2\overline{\xi}} + \cos\xi\sin\overline{\xi}\sqrt{1 - \frac{a^4-b^4}{a^4}\sin^2\xi}}{1 - \frac{a^4-b^4}{a^4}\sin^2\xi\sin^2\overline{\xi}}\right). \tag{3.5.17}$$

Since, $\xi = \cos^{-1}(\frac{b}{z})$, then,

$$\cos\xi = (\frac{b}{z}), \qquad \cos\bar{\xi} = (\frac{b}{\bar{z}}),$$

and

$$\sin\xi = \sqrt{1 - \frac{b^2}{z^2}}, \qquad \sin\bar{\xi} = \sqrt{1 - \frac{b^2}{\bar{z}^2}}.$$

Rewriting (3.5.17) in terms of z and $\bar{z}$, we obtain the following expression

$$\alpha = \sin^{-1} \frac{\frac{b}{\bar{z}}\sqrt{1-\frac{b^2}{z^2}}\sqrt{1-\frac{a^4-b^4}{a^4}\frac{(\bar{z}^2-b^2)}{\bar{z}^2}} + \frac{b}{z}\sqrt{1-\frac{b^2}{\bar{z}^2}}\sqrt{1-\frac{(a^4-b^4)}{a^4}\frac{(z^2-b^2)}{z^2}}}{1-\frac{(a^4-b^4)}{a^4}\frac{(z^2-b^2)}{z^2}\frac{(\bar{z}^2-b^2)}{\bar{z}^2}}, \tag{3.5.18}$$

and finally we obtain

$$\alpha = \sin^{-1}\left(\frac{a^2 z\sqrt{z^2-b^2}\sqrt{b^2\bar{z}^2+a^4-b^4} + a^2\bar{z}\sqrt{\bar{z}^2-b^2}\sqrt{b^2z^2+a^4-b^4}}{b^2z^2\bar{z}^2+(a^4-b^4)(z^2+\bar{z}^2-b^2)}\right). \tag{3.5.19}$$

The pressures satisfying (3.5.16) are

$$p_j = -\frac{a\dot{a}}{2k_j}F\Big(\alpha, \frac{\sqrt{a^4-b^4}}{a^2}\Big) + C_j(t), \tag{3.5.20}$$

where the terms $C_j(t)$ are computed from the values of p_j on the interface, on which

$$p_j = -\frac{a\dot{a}}{2k_j}F\Big(\frac{\pi}{2}, \frac{\sqrt{a^4-b^4}}{a^2}\Big) + C_j(t).$$

Finally, for the pressure we have

$$p_j = -\frac{a\dot{a}}{2k_j}F\Big(\alpha, \frac{\sqrt{a^4-b^4}}{a^2}\Big) + \frac{a\dot{a}}{2k_j}F\Big(\frac{\pi}{2}, \frac{\sqrt{a^4-b^4}}{a^2}\Big). \tag{3.5.21}$$

Let us construct the two-phase mother body starting with its part located in the domain Ω_1. This is a generic situation, so we can use formula (3.4.2). In the neighborhood of the point $z_0 = i\sqrt{(a^4-b^4)}/b$, $\arg[\dot{z}_0] = \pi/2 + 2\pi k$, $\arg[\Phi(z_0(t), t)] = \pm\pi/2 + \pi/4 + \pi k$. Thus, according to (3.4.2) the direction of the cut is $\varphi = \pi/2 + 2\pi k$, $k = 0, \pm1, \pm2, \ldots$. Similarly, at the point $z_0 = -i\sqrt{(a^4-b^4)}/b$, $\arg[\dot{z}_0] = -\pi/2 + 2\pi k$, $\arg[\Phi(z_0(t), t)] = \pm\pi/2 - \pi/4 + \pi k$. Therefore, the direction of the cut is $\varphi = -\pi/2 + 2\pi k$.

Taking into account symmetry with respect to x-axis, we conclude that the support of μ_1 consists of two rays starting at the branch points and going to infinity. The corresponding density is

$$\mu_1(y,t) = -\frac{2a^3\dot{a}}{k_1\sqrt{b^2y^2 - a^4 + b^4}\sqrt{y^2 + b^2}}.$$

The singularities of the Schwarz function in the interior domain Ω_2 have the inverse square root type (which is not the generic case),

$$S(z,t) = \frac{\Phi(z,t)}{\sqrt{z - z_0}}, \tag{3.5.22}$$

where,

$$\Phi(z,t) = \sqrt{b^2z^2 + a^4 - b^4},$$

is a regular functions of z in the neighborhood of the point $z_0 = \pm b$ with $\Phi(z_0,t) \neq 0$. Expanding function $\Phi(z,t)$ into the Taylor series with respect to z at the point (z_0,t),

$$S(z,t) = \sum_{m=0}^{\infty} c_m(t)\,(z - z_0)^{m-1/2}. \tag{3.5.23}$$

Differentiating (3.5.23) with respect to t, taking into account that z_0 is a stationary singularity since $\dot{b} = 0$, we have:

$$\dot{S}(z,t) = \sum_{m=0}^{\infty} \dot{c}_m\,(z - z_0)^{m-1/2} \tag{3.5.24}$$

Integration of the latter formula with respect to z using (3.3.5) implies:

$$W_2(z,t) = -\frac{\dot{c}_0}{2k_2}\sqrt{z - z_0}\Big(1 + \xi(z,t)\Big),$$

where ξ is a regular function near z_0 vanishing at this point, and $c_0 = \Phi(z_0,t)$.

The variation of the pressure along the loop l is

$$\mathrm{var}_l\, p_2 = \Re\left\{-\frac{\dot{c}_0}{k_2}\sqrt{z - z_0}\Big(1 + \xi(z,t)\Big)\right\}. \tag{3.5.25}$$

Consider a small neighborhood of the point z_0, where,

$$z = z_0 + \rho e^{i\varphi},$$

with a small ρ. Setting the principal part of $\text{var}_l\, p_2$ to zero, we have

$$\text{var}_l\, p_2 = -\frac{|\dot{c}_0|}{k_2} \sqrt{\rho}\, \Re \left\{ \exp i \left(\frac{\varphi}{2} + \pi n + \arg[\dot{c}_0] \right) \right\} = 0, \qquad (3.5.26)$$

where $n = 0, \pm 1, \pm 2, \ldots$.

Thus,

$$\varphi = \pi - 2 \arg[\dot{\Phi}\,(z_0, t)] + 2\pi k,$$

which implies that the support of μ_2 is the segment $[-b,\, b]$ with the density

$$\mu_2 = \frac{2a^3 \dot{a}}{k_2 \sqrt{b^2 x^2 + a^4 - b^4}\, \sqrt{b^2 - x^2}}.$$

Integrating $k_j \mu_j$ along the corresponding cuts, one obtains the rate of change of the area of Ω_2, which is given by the formula

$$\dot{A} = \partial_t (a^2)\, F(\pi, \frac{b^2}{a^2}) = \pi\, \partial_t (a^2)\, {}_2F_1\, (\frac{1}{2}, \frac{1}{2}; 1; \frac{b^4}{a^4}),$$

where ${}_2F_1$ is the hypergeometric series [88].

3.6 Conclusions

We have studied a Two-Phase problem with a negligible surface tension and suggested a method of finding exact solutions. The idea of the method was to keep the interface within a certain family of curves defined by its initial shape by constructing two distributions with disjoint supports located on the different sides of the moving interface.

This study extended the results reported in [14] and [38]. We gave new examples of exact solutions including the circle, an ellipse, and two ovals: Neumann's and Cassini's. In those examples we assumed that the flux generated by the sinks/sources is finite, that is, the pressure may have at most a logarithmic growth. To demonstrate that this physical assumption does not restrict our method, we have presented an example of an exact solution with a linear far field flow. Our study showed the possibility for the control of the interface via the two-phase mother body for the two-phase Hele-Shaw problem.

4 On a Two-Phase Hele-Shaw Problem With a Time-Dependent Gap and Distributions of Sinks and Sources

4.1 Introduction

This chapter was published in the Journal of Physics A (see the reference [76]). Free boundary problems have been a significant part of modern mathematics for more than a century, since the celebrated Stefan problem, which describes solidification, that is, an evolution of the moving front between liquid and solid phases. Free boundary problems also appear in fluid dynamics, geometry, finance, and many other applications (see [10] for a detailed discussion). Recently, they started to play an important role in modeling of biological processes involving moving fronts of populations or tumors [21]. These processes include cancer, biofilms, wound healing, granulomas, and atherosclerosis [21]. Biofilms are defined as communities of microorganisms, typically bacteria, that are attached to a surface. The biofilms motivated Friedman et al [22] to consider a two-phase free boundary problem, where one phase is an incompressible viscous fluid, and the other phase is a mixture of two incompressible fluids, which represent the viscous fluid and the polymeric network (with bacteria attached to it) associated with a biofilm. Free boundary problems are also used in modeling of a tumor growth with one phase to be the tumor region, and the other phase to be the normal tissue surrounding the tumor [23].

A Muskat problem is a free boundary problem related to the theory of flows in porous media [54]. It describes an evolution of an interface between two immiscible fluids, ‘oil’ and ‘water’, in a Hele-Shaw cell or in a porous medium. Here we study a two-phase Hele-Shaw flow assuming that the upper plate uniformly moves up or down changing the gap width of a Hele-Shaw cell. Hele-Shaw free boundary problems have been extensively studied over the last century (see [29], [86] and references therein). There are two classical formulations of the Hele-Shaw problems: the one-phase problem, when one of the fluids is assumed to be viscous while the other is effectively inviscid (the pressure there is constant), and the two-phase (or Muskat) problem. A statement of the problem with a time-dependent gap between the plates was mentioned in [19] among other generalized Hele-Shaw flows. The one-phase (interior) version of this problem was considered in [79], where conditions of existence, uniqueness, and regularity of solutions were established under assumption that surface tension effects on the free boundary are negligible; some exact solutions were constructed as well. An interior problem with a time-dependent gap and a non-zero surface tension was considered in [74], where asymptotic solutions were obtained for the case when initial shape of the droplet is a weakly distorted circle. Note also that the mathematical formulation of the interior problem with a time-dependent gap is similar to the problem of evaporation of a thin film [2]. When the surface tension is negligible, the pressure in both formulations can be obtained as a solution to the Poissons equation in a bounded domain with homogeneous Dirichlet data on the free boundary.

Much less progress has been made for the Muskat problem. Regarding the problem with a constant gap width, we should mention works [3], [14], [24], [44], [38], [80] and [91]. Specifically, Howison [38] has obtained several simple solutions including the traveling-wave solutions and the stagnation point flow. In [38], an idea of a method for solving some two-phase problems was proposed and used to reappraise the Jacquard-Séguier solution [44]. Global existence of solutions to some specific two-phase problems was considered in [24], [80] and [91]. Crowdy [14] presented an exact solution to the Muskat problem for the elliptical initial interface between two fluids of different viscosity. In [14], it was shown that an elliptical inclusion of one fluid remains elliptical when placed in a linear ambient flow of another fluid. In [3], new exact solutions to the Muskat problem were constructed, extending the results obtained in [14], to other types of inclusions. This chapter is concerned with a two-phase Hele-Shaw problem with a variable gap width in the presence of sinks and sources.

4.2 The mathematical formulation of the two phase problem

Let $\Omega_2(t) \subset \mathbb{R}^2$ with a boundary $\Gamma(t)$ at time t be a simply-connected bounded domain occupied by a fluid with a constant viscosity ν_2, and let $\Omega_1(t)$ be the region $\mathbb{R}^2 \setminus \bar{\Omega}_2(t)$ occupied by a different fluid of viscosity ν_1. To consider a two-phase Hele-Shaw flow forced by a time-dependent gap, we start with the Darcy's law

$$\mathbf{v}_j = -k_j \nabla p_j \quad \text{in} \quad \Omega_j(t), \qquad j = 1, 2, \tag{4.2.1}$$

where $\mathbf{v}_j$ and p_j are a two-dimensional gap-averaged velocity vector and a pressure of fluid j respectively, $k_j = \frac{h^2(t)}{12\nu_j}$, and $h(t)$ is the gap width of the Hele-Shaw cell. Equation (4.2.1) is complemented by the volume conservation,

$$A(t)h(t) = A(0)h(0) \tag{4.2.2}$$

for any time t, where $A(t)$ and $A(0)$ are the areas of $\Omega_2(t)$ and $\Omega_2(0)$ respectively. The conservation of volume for a time-dependent gap may be written as a modification of the usual incompressibility condition

$$\nabla \cdot \mathbf{V_3} = 0,$$

where $\mathbf{V_3} = (u, v, w)$ is a three-dimensional velocity vector of the fluid occupying the domain $\Omega_2(t)$. Indeed, the averaging of the three-dimensional incompressibility condition across the gap gives [79]:

$$\begin{aligned}
0 &= \frac{1}{h(t)} \int_0^{h(t)} (u_x + v_y + w_z)\, dz \\
&= \frac{1}{h(t)} \int_0^{h(t)} u_x\, dz + \frac{1}{h(t)} \int_0^{h(t)} v_y\, dz + \frac{1}{h(t)} \int_0^{h(t)} w_z\, dz \\
&= u_x^{av} + v_y^{av} + \frac{(w(h(t)) - w(0))}{h(t)} \\
&= u_x^{av} + v_y^{av} + \frac{\dot{h}(t)}{h(t)}.
\end{aligned} \tag{4.2.3}$$

Here $z = 0$ corresponds to the lower plate and $z = h(t)$ corresponds to the upper plate, and and $h(t)$ and $\dot{h}(t)$ are assumed to be small enough to avoid any inertial effects as well as to keep the large aspect ratio. The latter implies [79]

$$\nabla \cdot \mathbf{v_2} = -\frac{\dot{h}(t)}{h(t)} \quad \text{in} \quad \Omega(t). \tag{4.2.4}$$

Note that similar consideration may be applied to any finite part of the region $\Omega_1(t)$. Thus, equations (4.2.1) and (4.2.4) suggest to formulated the problem in terms of the pressure p_j as a solution to Poisson's equation,

$$\Delta p_j = \frac{1}{k_j}\frac{\dot{h}(t)}{h(t)}, \tag{4.2.5}$$

almost everywhere in the region $\Omega_j(t)$, satisfying boundary conditions

$$p_1(x, y, t) = p_2(x, y, t) \quad \text{on} \quad \Gamma(t), \tag{4.2.6}$$

$$-k_1\frac{\partial p_1}{\partial n} = -k_2\frac{\partial p_2}{\partial n} = v_n \quad \text{on} \quad \Gamma(t). \tag{4.2.7}$$

We remark that when sinks and sources are present in $\Omega_j(t)$, equation (4.2.5) has an additional term,

$$\Delta p_j = \frac{1}{k_j}\frac{\dot{h}(t)}{h(t)} + \mu_j,$$

describing the corresponding distribution. Equation (4.2.6) states the continuity of the pressure under the assumption of negligible surface tension. Equation (4.2.7) means that the normal velocity of the boundary itself coincides with the normal velocity of the fluid at the boundary.

The free boundary $\Gamma(t)$ moves due to a change of the gap width as well as the presence of sinks and sources located in both regions. The supports of the sinks and sources, specified in section 4.3, are either points or lines/curves. The presence of sinks and sources obviously changes the dynamics of the evolution of the interface between the fluids, which is shown for an elliptical interface in section 4.4.

For what follows, it is convenient to reformulate the problem in terms of harmonic functions $\tilde{p}_j$, where

$$p_j(x, y, t) = \tilde{p}_j(x, y, t) + \frac{1}{4k_j}\frac{\dot{h}(t)}{h(t)}(x^2 + y^2). \qquad (4.2.8)$$

Then the problem (4.2.5)-(4.2.6) reduces to

$$\Delta \tilde{p}_j = \chi_j \mu_j \quad \text{in} \quad \Omega_j(t), \qquad (4.2.9)$$

where $\chi_j = 0$ or $\chi_j = 1$ in the absence or presence of sinks and sources in $\Omega_j(t)$ respectively,

$$\tilde{p}_1(x, y, t) = \tilde{p}_2(x, y, t) + \frac{k_1 - k_2}{4k_1k_2}\frac{\dot{h}(t)}{h(t)}(x^2 + y^2) \quad \text{on} \quad \Gamma(t), \qquad (4.2.10)$$

$$-k_1\frac{\partial \tilde{p}_1}{\partial n} = -k_2\frac{\partial \tilde{p}_2}{\partial n} = v_n + \frac{1}{4}\frac{\dot{h}(t)}{h(t)}\frac{\partial}{\partial n}(x^2 + y^2) \quad \text{on} \quad \Gamma(t). \qquad (4.2.11)$$

The main difficulty of the two-phase problems is the fact that the pressure on the interface is unknown. However, if we assume that the free boundary remains within the family of curves, specified by the initial shape of the interface separating the fluids (which is feasible if the surface tension is negligible), the problem is drastically simplified. Using reformulation of the Muskat problem with the time-dependent gap in terms of the Schwarz function equation, we describe a method of constructing exact solutions, and using this method we consider examples in the presence and in the absence of additional sinks and sources. Next, we will describe the method of finding exact solutions, gives examples of the exact solutions and finally we conclude.

4.3 The method of finding exact solutions for a Muskat problem with a time-dependent gap

Consider a problem

$$\Delta \tilde{p}_j = \chi_j \mu_j \quad \text{in} \quad \Omega_j(t), \qquad (4.3.1)$$

$$\tilde{p}_1(x,y,t) + \Psi_1(x,y,t) = \tilde{p}_2(x,y,t) + \Psi_2(x,y,t) \quad \text{on} \quad \Gamma(t), \tag{4.3.2}$$

$$-k_1\frac{\partial \tilde{p}_1}{\partial n} = -k_2\frac{\partial \tilde{p}_2}{\partial n} = v_n + \Phi(x,y,t) \quad \text{on} \quad \Gamma(t). \tag{4.3.3}$$

In the case when

$$\Psi_j = \frac{1}{4k_j}\frac{\dot{h}(t)}{h(t)}(x^2+y^2), \qquad j = 1,2, \tag{4.3.4}$$

$$\Phi = \frac{1}{4}\frac{\dot{h}(t)}{h(t)}\frac{\partial}{\partial n}(x^2+y^2), \qquad j = 1,2, \tag{4.3.5}$$

the problem (4.3.1)-(4.3.3) coincides with (4.2.9)-(4.2.11).

As stated before, the evolution of the interface separating the fluids is forced by the change in the gap width and the presence of sinks and sources. In the absence of the surface tension, there is a possibility to control the interface by keeping $\Gamma(t)$ within a family of curves defined by $\Gamma(0)$. For what follows, it is convenient to reformulate problem (4.3.1)–(4.3.3) in terms of the Schwarz function $S(z,t)$ of the curve $\Gamma(t)$ [17], [47], [72], and [78].

This function for a real-analytic curve $\Gamma := \{g(x,\ y,\ t) = 0\}$ is defined as a solution to the equation

$$g\left(\frac{z+\bar{z}}{2},\ \frac{z-\bar{z}}{2i},\ t\right) = 0$$

with respect to $\bar{z}$. This (regular) solution exists in some neighborhood U_Γ of the curve Γ, if the assumptions of the implicit function theorem are satisfied [17]. Note that if g is a polynomial, then the Schwarz function is continuable into Ω_j, generally as a multiple-valued analytic function with a finite number of algebraic singularities (and poles). In U_Γ, the normal velocity, v_n, of $\Gamma(t)$ can be written in terms of the Schwarz function [37],

$$v_n = -\frac{i\partial_z S(z,t)}{2\sqrt{\partial_z S(z,t)}},$$

see proposition 2.4.1 for the proof.

To complete the reformulation of problem in terms of the Schwarz function, we introduce the complex potential

$$W_j = p_j - i\psi_j,$$

which is an analytic multiple-valued function defined on $\Gamma(t)$ and in $\Omega_j(t) \cap U_\Gamma$, $j = 1,\ 2$. Here, ψ_j be a stream function. Following [15], [48], [49], and [51] and also taking into account the Cauchy-Riemann conditions in the (n, τ) coordinates,

$$\frac{\partial p_j}{\partial n} = -\frac{\partial \psi_j}{\partial \tau}, \quad \frac{\partial \psi_j}{\partial n} = \frac{\partial p_j}{\partial \tau}. \tag{4.3.6}$$

For the derivative of $W_j(z, t)$ with respect to z on $\Gamma(t)$ we have

$$\begin{aligned} \partial_z W_j = \frac{\partial_\tau W_j}{\partial_\tau z} &= \frac{\partial_\tau(\tilde{p}_j - i\psi_j)}{\partial_\tau z} \\ &= \frac{\partial_\tau \tilde{p}_j + i\partial_n \tilde{p}_j}{\partial_\tau z} \\ &= \frac{\partial_\tau \tilde{p}_j - i(v_n + \Phi)/k_j}{\partial_\tau z}, \end{aligned} \tag{4.3.7}$$

where τ is an arclength along $\Gamma(t)$. Expressing v_n and $\partial_\tau z$ in terms of the Schwarz function, $v_n = -\frac{i\partial_z S}{2\sqrt{\partial_z S}}$ and $\partial_\tau z = \frac{1}{\partial_z S}$, we obtain

$$\partial_z W_j = \partial_\tau \tilde{p}_j \sqrt{\partial_z S} - \frac{\partial_t S}{2k_j} - \frac{i\Phi}{k_j}\sqrt{\partial_z S}. \tag{4.3.8}$$

Here $\partial_z W_j \equiv \frac{\partial W_j}{\partial z}$, $\partial_\tau \equiv \frac{\partial}{\partial \tau}$. Equation (4.3.2) implies that

$$\tilde{p}_1 + \Psi_1 = \tilde{p}_2 + \Psi_2 = f$$

on $\Gamma(t)$, where f is an unknown function. To keep $\Gamma(t)$ in a certain family of curves defined by $\Gamma(0)$, for example, in a family of ellipses, we assume that f on $\Gamma(t)$ is a function of time only. This possibility is shown in Section 4.4, where specific examples are discussed. In that case the problem is simplified drastically, and on $\Gamma(t)$ we have

$$\partial_z W_j = -\frac{\partial_t S}{2k_j} - \partial_z(\Psi_j(z, S(z,t)) - \frac{i\Phi}{k_j}\sqrt{\partial_z S} \qquad j = 1, 2. \tag{4.3.9}$$

For the special case when Ψ_j and Φ are given by (4.3.4), (4.3.5), the last equation reduces to

$$\partial_z W_j = -\frac{1}{2k_j}(\dot{S} + \frac{\dot{h}}{h}S) \qquad j = 1, 2. \tag{4.3.10}$$

The reformulation of the two-phase problem with a time-dependent gap in terms of the Schwarz function of the interface is summarized in the following theorem.

Theorem 4.3.1. *Let* $\Gamma(t)$ *be an analytic curve for fixed t. Then there exist multiple-valued analytic complex potential functions* $W_j(z, t)$, *for* $j = 1, 2$, *defined on* $\Gamma(t)$ *and in* $\Omega_j \cap U_\Gamma(t)$ *that satisfy the equations*

$$\partial_z W_j = -\frac{1}{2k_j}\left(\partial_t S + \frac{\dot{h}}{h}S\right), \tag{4.3.11}$$

whose real parts $\Re[W_j] = \tilde{p}_j$ *solves the free boundary problem* (4.2.9)-(4.2.11).

Corollary 4.3.2. *The solution to the problem (4.2.5)-(4.2.7) has the form*

$$p_j = \Re[W_j] + \frac{\dot{h}(t)}{4k_j h(t)}(x^2 + y^2), \tag{4.3.12}$$

where W_j *satisfies the equation (4.3.11) for* $j = 1, 2$.

Remark that each equation (4.3.10) can be continued off of Γ into the corresponding Ω_j, where W_j is a multiple-valued analytic function. The equations (4.3.9) and (4.3.10) imply that the singularities of W_1, W_2, and the singularities of the Schwarz function are linked. As such, the singularities of the Schwarz function play the crucial role in the construction of solutions in question.

To find the exact solutions, suppose that at $t = 0$ the interface is an algebraic curve,

$$\sum_{k=0}^{n} a_k(0)x^{k-n}y^n = 0,$$

with the Schwarz function $S(z, a_k^0)$. Assume that during the course of evolution the Schwarz function of the interface $S(z, a_k(t)) \equiv S(z, t)$ is such that $S(z, a_k(0)) = S(z, a_k^0)$, which leads us to the following six steps method:

1. Compute the Schwarz function $S(z,t)$, locate its singularities, and define their type.

2. Using equations (4.3.10) find preliminary expressions for $\partial_z W_j$.

3. By putting restrictions on the coefficients $a_k(t)$ in the preliminary expressions for $\partial_z W_j$ eliminate the terms involving undesirable singularities (if possible).

4. Integrate (4.3.10) with respect to z in order to find W_j up to an arbitrary function of time.

5. Take the real part of W_j in order to obtain p_j up to an arbitrary function of time.

6. Evaluate the quantities p_j on the interface to determine the independent of z function of integration from the steps 3 and 4.

7. Locate the supports and compute the distributions of sinks and sources.

Before describing how to locate the supports, we remark that the distributions in step 7 are related to the two-phase mother body [3]. The notion of a mother body arises from the potential theory [18], [26], [32], [33], and [70] and was adopted to the one-phase Hele-Shaw problem in [75].

As mentioned above, generally, the complex potentials W_j are multiple-valued functions in Ω_j. For instance, if $\Gamma(t)$ is an algebraic curve, then the singularities of W_j are either poles or algebraic singularities. To choose a branch of W_j, one has to introduce the cuts, $\gamma_j(t)$, that serve as supports for the distributions of sinks and sources, $\mu_j(t)$, $j = 1, 2$. Thus, each cut originates from an algebraic singularity $z_a(t)$ of the potential W_j. The supports consist of those cuts and/or points and do not bound any two-dimensional subdomains in $\Omega_j(t)$, $j = 1, 2$. Each cut included in the support of $\mu_j(t)$ is contained in the domain $\Omega_j(t)$, and the limiting values of the pressure on each side of the cut are equal. The value of the density of sinks and sources located on the cut is equal to the jump of the normal derivative $\partial_n p_j$ of the pressure p_j. In order for the total flux through the sinks and sources to be finite, all of the singularities of the function W_j must have no more than the logarithmic growth.

The location of $z_a(t)$, as well as the directions of the cuts emanating from $z_a(t)$, are determined by the Schwarz function via (4.3.10). In the examples considered below, the Schwarz function has the following two representations near its singular points. The first representation being the square root (general position)

$$S^g(z,t) = \xi^g(z,t)\ \sqrt{z - z_a(t)} + \zeta^g(z,t)\,, \tag{4.3.13}$$

where $z_a(t)$ is a non-stationary singularity, that is $\dot{z}_a \neq 0$. The second being the reciprocal square root

$$S^r(z,t) = \frac{\xi^r(z,t)}{\sqrt{z - z_a(0)}} + \zeta^r(z,t)\,, \tag{4.3.14}$$

where $z_a(0)$ is a stationary singularity, that is $\dot{z}_a = 0$. Here $\xi^{g,r}(z,t)$ and $\zeta^{g,r}(z,t)$ are regular functions of z in a neighborhood of the point $z_a(t)$, and $\xi^{g,r}(z_a(t),t) \neq 0$.

By plugging (4.3.13) and (4.3.14) into (4.3.10), in a small neighborhood of $z_a(t)$ we have

$$W_j^g(z,t) = \frac{1}{2k_j}\dot{z}_a\xi^g(z_a(t),t)\ \sqrt{z - z_a(t)} + \ldots, \tag{4.3.15}$$

$$W_j^r(z,t) = \frac{1}{k_j}C_0(t)\ \sqrt{z - z_a(0)} + \ldots, \tag{4.3.16}$$

where the dots correspond to the smaller and regular terms that do not affect the computation of the directions of the cuts. The quantity $C_0(t)$ is defined by

$$C_0(t) = \dot{\xi}^r(z_a(0),t) + \frac{\dot{h}(t)}{h(t)}\xi^r(z_a(0),t)\,.$$

Formulas (4.3.15) and (4.3.16) along with the substitutions

$$z = z_a + \rho\exp(i\varphi^{g,r}),$$

with small ρ, imply that

$$p_j^g(z,t) = \frac{\sqrt{\rho}}{2k_j}\Re[\dot{z}_a\xi^g(z_a(t),t)\exp(\frac{i\varphi^g}{2})] + \ldots, \tag{4.3.17}$$

$$p_j^r(z,t) = -\frac{\sqrt{\rho}}{k_j}\Re[C_0(t)\exp(\frac{i\varphi^r}{2})] + \ldots. \tag{4.3.18}$$

Computing the zero level of a variation of p_j along a small loop surrounding the singular point, we finally obtain the following directions of the cuts: for the general position

$$\varphi^g = \pi - 2(\arg[\xi^g(z_a(t),t)] + \arg[\dot{z}_a]) + 2\pi k, \quad k = 0, \pm 1, \pm 2.... \tag{4.3.19}$$

and for the reciprocal square root

$$\varphi^r = \pi - 2\arg[C_0(t)] + 2\pi k, \quad k = 0, \pm 1, \pm 2.... \tag{4.3.20}$$

In the next section, we use the described method to construct exact solutions to the Muskat problem. In the considered examples, the evolution of the interface is driven by the change in the gap width of the Hele-Shaw cell. The examples include the elliptical shape with and without sinks and sources in the finite domain as well as the Cassini's oval in the presence of sinks and sources.

4.4 Examples of specific initial interfaces

4.4.1 Circle

To illustrate the method, we start with the simplest example for which the solution is known. Suppose that the initial shape of the interface is a circle with the equation

$$x^2 + y^2 = a^2(0),$$

and during the evolution the boundary remains circular,

$$x^2 + y^2 = a^2(t).$$

The corresponding Schwarz function is

$$S = \frac{a^2(t)}{z},$$

and the derivative with respect to t is given as

$$\dot{S} = \frac{2a\dot{a}}{z}.$$

Due to the volume conservation, functions $a(t)$ must satisfy the equation

$$h(t) = \frac{a_0^2 h_0}{a^2(t)}, \qquad \frac{\dot{h}(t)}{h(t)} = -\frac{2\dot{a}(t)}{a(t)}, \tag{4.4.1}$$

h(t) is the width gap. Taking into account equation 4.4.1 then, equation (4.3.10) in this case reads as

$$\partial_z W_j = -\frac{1}{2k_j}\left(\frac{2a\dot{a}}{z} + \frac{-2\dot{a}}{a}\left[\frac{a^2}{z}\right]\right) = 0. \tag{4.4.2}$$

Hence,

$$\partial_z W_j = 0,$$

which implies that $\tilde{p}_j$ is a function depending on t only,

$$\tilde{p}_j = -\frac{a_0^2 h_0 \dot{h}}{4k_j h^2} + f(t), \tag{4.4.3}$$

therefore,

$$p_j(x, y, t) = \frac{1}{4k_j}\frac{\dot{h}(t)}{h(t)}\left(x^2 + y^2 - \frac{a_0^2 h_0}{h(t)}\right) + f(t) \tag{4.4.4}$$

and $a(t) = a_0\sqrt{\frac{h_0}{h(t)}}$.

4.4.2 Ellipse

Consider a two-phase problem with an elliptical interface,

$$\Gamma(0) = \left\{\frac{x^2}{a(0)^2} + \frac{y^2}{b(0)^2} = 1\right\},$$

where $a(0)$ and $b(0)$ are given and $a(0) > b(0)$. The Schwarz function of an elliptical interface with semi-axes $a(t)$ and $b(t)$ is

$$S\,(z,t) = \frac{a(t)^2 + b(t)^2}{d^2(t)} z - \frac{2a(t)b(t)}{d^2(t)} \sqrt{z^2 - d(t)^2}, \tag{4.4.5}$$

where $d(t) = \sqrt{a(t)^2 - b(t)^2}$ is the half of the inter-focal distance. Differentiating the Schwarz function (4.4.5) with respect to time t, we have

$$\partial_t S = \frac{\partial S}{\partial t} = z \frac{\partial}{\partial t}\left(\frac{a^2 + b^2}{d^2}\right) - \sqrt{z^2 - d^2}\frac{\partial}{\partial t}\left(\frac{2ab}{d^2}\right) + \frac{ab}{d^2\sqrt{z^2 - d^2}}\frac{\partial}{\partial t}(d^2). \tag{4.4.6}$$

Assuming that the interface remains elliptical during the course of the evolution, we use equation (4.3.10)

$$\partial_z W_j = -\frac{1}{2k_j}\left(\partial_t S + \frac{\dot{h}}{h} S\right).$$

Due to the volume conservation of the fluid occupying $\Omega_2(t)$, the product of functions $a(t)$ and $b(t)$ is linked to the gap width, $h(t)$, via the equation

$$h(t) = \frac{a_0 b_0 h_0}{a(t)b(t)},$$

where $a_0 = a(0)$, $b_0 = b(0)$, and $h_0 = h(0)$. Therefore,

$$\frac{\dot{h}(t)}{h(t)} = -\frac{\partial_t(ab)}{ab},$$

and the equation (4.3.10) could be rewritten as

$$\partial_z W_j = -\frac{1}{2k_j}\left(\partial_t S - \frac{\partial_t(ab)}{ab} S\right), \tag{4.4.7}$$

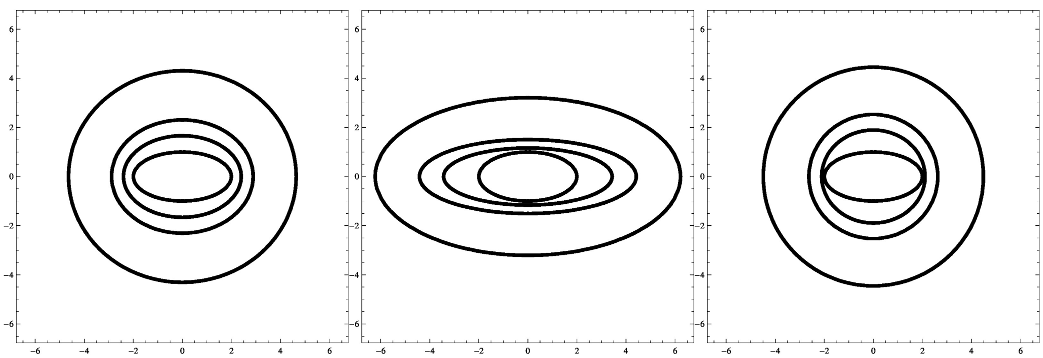
which results in

$$\begin{aligned}\partial_z W_j = -\frac{z}{2k_j}\Big\{\frac{\partial}{\partial t}\Big(\frac{a^2 + b^2}{d^2}\Big) - \frac{(a^2 + b^2)}{a\,b\,d^2}\frac{\partial}{\partial t}(ab)\Big\} \\ - \frac{(2z^2 - d^2)}{\sqrt{z^2 - d^2}}\,\frac{ab}{2k_j d^4}\frac{\partial}{\partial t}\Big(d^2\Big)\end{aligned} \tag{4.4.8}$$

and

$$W_j = -\frac{z^2}{4k_j}\Big\{\frac{\partial}{\partial t}\Big(\frac{a^2+b^2}{d^2}\Big) - \frac{(a^2+b^2)}{a\,b\,d^2}\frac{\partial}{\partial t}(ab)\Big\}$$
$$-\frac{a\,b\,z}{2k_j d^4}\sqrt{z^2-d^2}\,\frac{\partial}{\partial t}(d^2) + C_j(t), \qquad (4.4.9)$$

where $C_j(t)$ is an arbitrary function of time.



(a) (b) (c)

Figure 4.1: Squeezing of an ellipse: $a_0 = 2$, $b_0 = 1$, $h_0 = 0.1$, $h(t) = h_0 - t$; $t = 0$, $t = 0.05$, $t = 0.07$, $t = 0.09$: (a) $d^2 = const$, (b) $d^2(t) = d_0^2 \exp(25t)$, (c) $d^2(t) = d_0^2 \exp(-25t)$ [76].

(a) Evolution with constant inter-focal distance.

To obtain an exact solution in the absence of sinks and sources in the finite part of the plane, we set $d(t) = d(0)$. Then, the second term in the formula (4.4.9) vanishes, which implies the following expression for the pressure

$$\tilde{p}_j = \Re[W_j] = \frac{1}{4k_j}\Big((x^2-y^2)\frac{\dot{a}\,d_0^2}{a(a^2-d_0^2)} + 2\dot{a}a\Big) + f(t), \qquad (4.4.10)$$

therefore,

$$p_j = \frac{\dot{a}}{2k_j a(a^2-d_0^2)}\Big(d_0^2x^2 - a^2(x^2+y^2) + a^2(a^2-d_0^2)\Big) + f(t) \qquad (4.4.11)$$

is the solution to the problem (4.2.5)-(4.2.7). Note that when $d_0 = 0$, this formula coincides with formula (4.4.4) related to the circular interface.

Hence, $\Gamma(t)$ is a family of co-focal ellipses,

$$\frac{x^2}{a^2(t)} + \frac{y^2}{b^2(t)} = 1,$$

controlled by one of the functions $a(t)$, $b(t)$ or $h(t)$. If $h(t)$ is given, then by volume conservation, $A(t)h(t) = A(0)h(0)$ for any time t. Then, functions $a(t)$ and $b(t)$ must satisfy the equation

$$a(t) = \frac{a_0 b_0 h_0}{b(t)h(t)}, \qquad b(t) = \frac{a_0 b_0 h_0}{a(t)h(t)}.$$

Let $e(t) = a_0 b_0 h_0 / h(t)$, then $a(t) = e(t)/b(t)$ and $b(t) = e(t)/a(t)$. Using the fact that $d^2(t) = a^2(t) - b^2(t)$ we obtain the following equations

$$b^4(t) + d^2(t)b^2(t) - e^2(t) = 0, \qquad a^4(t) - d^2(t)a^2(t) - e^2(t) = 0. \tag{4.4.12}$$

Applying Quadratic formula to equations 4.4.12, we obtain

$$a^2(t) = \tfrac{1}{2}\Big(a_0^2 - b_0^2 + \sqrt{(a_0^2 - b_0^2)^2 + 4e^2(t)}\Big), \tag{4.4.13}$$

$$b^2(t) = \tfrac{1}{2}\Big(b_0^2 - a_0^2 + \sqrt{(a_0^2 - b_0^2)^2 + 4e^2(t)}\Big), \tag{4.4.14}$$

where $e(t) = a_0 b_0 h_0 / h(t)$. An example of such an evolution with a linear function $h(t)$ is shown in Fig. 4.1 (a).

(b) Evolution with variable inter-focal distance.

If we admit solutions with variable inter-focal distance by keeping all terms in (4.4.9), we must allow, in addition to the gap change, some sinks/sources located in Ω_2. In that case, the pressure is

$$\tilde{p}_j = -\frac{(x^2 - y^2)}{4k_j}\Big\{\frac{\partial}{\partial t}\Big(\frac{a^2 + b^2}{d^2}\Big) - \frac{(a^2 + b^2)}{a\,b\,d^2}\frac{\partial}{\partial t}(ab)\Big\}$$
$$-\frac{a\,b}{2k_j d^4}\frac{\partial}{\partial t}(d^2)\frac{x\,(\alpha^2 - y^2)}{\alpha} - \frac{ab\,(\dot{a}b - a\dot{b})}{2k_j d^2} + f(t), \tag{4.4.15}$$

where

$$\alpha^2 = \Big(x^2 - y^2 - d^2 + \sqrt{(x^2 - y^2 - d^2)^2 + 4x^2y^2}\Big)/2,$$

therefore, making

$$p_j = -\frac{(x^2 - y^2)}{4k_j}\Big\{\frac{\partial}{\partial t}\Big(\frac{a^2 + b^2}{d^2}\Big) - \frac{(a^2 + b^2)}{a\,b\,d^2}\frac{\partial}{\partial t}(ab)\Big\} - \frac{ab\,(\dot{a}b - a\dot{b})}{2k_j d^2}$$
$$- \frac{a\,b}{2k_j d^4}\,\frac{\partial}{\partial t}(d^2)\frac{x\,(\alpha^2 - y^2)}{\alpha} - \frac{\partial_t(ab)}{4k_j ab}(x^2 + y^2) + f(t). \tag{4.4.16}$$

Equation (4.4.9) implies that there are two singular points in the interior domain Ω_2, $z = \pm d$. The Schwarz function near those points has the square root representation (4.2.7) with

$$\xi^g = -\frac{2ab}{d^2}\,\sqrt{z \pm d}.$$

The direction of the cut at each point is defined by formula (4.3.19), which implies that at the point $z_a = d$, the angle is $\varphi^g = \pi + 2\pi k$ and at the point $z_a = -d$, the angle is $\varphi^g = 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$. Thus, the cut $\gamma_2(t)$ is located along the inter-focal segment $[-d, d]$. The density of the distribution of sinks and sources along that segment is given by the formula

$$\mu_2 = \frac{ab\,\partial_t(d^2)}{k_2 d^4}\;\frac{(2x^2 - d^2)}{\sqrt{d^2 - x^2}}.$$

Such a density changes its sign along the inter-focal segment, so its presence does not affect the area of the ellipse,

$$\dot{A} = \int_{-d}^{d} k_2\mu_2(x, t)\,dx = 0.$$

Fig. 4.1 shows how the sinks and sources change the evolution of the interface with increasing (see Fig. 4.1 (b)) and decreasing (see Fig. 4.1 (c)) inter-focal distances.

4.4.3 The Cassini's oval

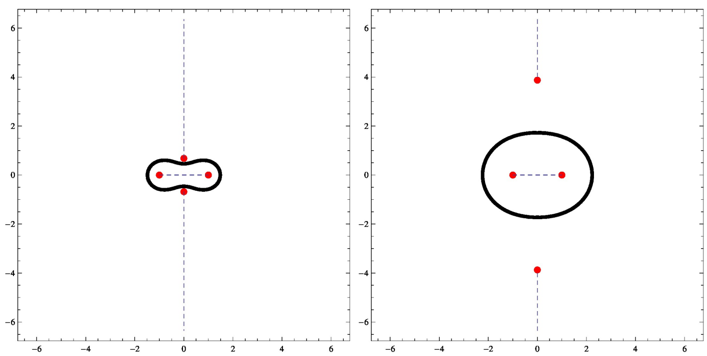
Similar to the previous examples, assume that $\Gamma(t)$ remains in the specific family of curves, the Cassini's ovals, given by the equation

$$\left(x^2 + y^2\right)^2 - 2b^2(t)\left(x^2 - y^2\right) = a^4(t) - b^4(t),$$

or

$$\left[(x-b)^2+y^2\right]\left[(x+b)^2+y^2\right]=a^4,$$

where $a(t)$ and $b(t)$ are unknown positive functions of time. This curve consists of one closed curve, if $a(t) > b(t)$ (see Fig. 4.2), and two closed curves otherwise. Assume that at $t = 0$, $a(0) > b(0)$.



(a) (b)

Figure 4.2: Squeezing of the Cassini's ovals for $b(t) = b_0 = 1$, $a_0 = 1.1$, $h_0 = 0.1$, $h(t) = h_0 - t$: (a) $t = 0$, (b) $t = 0.05$ [76].

The Schwarz function of Cassini's oval,

$$S\,(z,t)=\frac{\sqrt{b^2z^2+a^4-b^4}}{\sqrt{z^2-b^2}},$$

has two singularities in $\Omega_1(t)$, $z = \pm i\sqrt{(a^4-b^4)/b^2}$, and two singularities in $\Omega_2(t)$, $z = \pm b$. Differentiating the Schwarz function with respect to t, we have

$$\partial_t S \;=\; \frac{b\dot{b}z^2+2a^3\dot{a}-2b^3\dot{b}}{\sqrt{b^2z^2+a^4-b^4}\,\sqrt{z^2-b^2}}+\frac{b\dot{b}\sqrt{b^2z^2+a^4-b^4}}{\sqrt{(z^2-b^2)^3}}. \tag{4.4.17}$$

The corresponding complex velocities using equation (4.3.10) have singularities at the same points and is given by

$$\partial_z W_j = -\frac{1}{2k_j}\left(\frac{B_1z^2+B_2}{\sqrt{(b^2z^2+a^4-b^4)(z^2-b^2)}}+\frac{b\dot{b}\sqrt{b^2z^2+a^4-b^4}}{\sqrt{(z^2-b^2)^3}}\right). \tag{4.4.18}$$

Here

$$B_1 = b\dot{b} + b^2\dot{h}/h, \qquad B_2 = 2a^3\dot{a} - 2b^3\dot{b} + (a^4 - b^4)\dot{h}/h,$$

and

$$\frac{\dot{h}}{h} = -\frac{\dot{A}}{A},$$

due to volume conservation. The area of Cassini's oval can be computed in polar coordinates,

$$A = a^2 E\left(\pi, \frac{b^2}{a^2}\right) = 2a^2 E\left(\frac{b^2}{a^2}\right),$$

where

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 t}\, dt,$$

and $E(k) = E(\pi/2, k)$, resulting in

$$\frac{\dot{A}}{A} = \frac{2\dot{a}}{a} + \frac{\partial_t E\left(\pi, \frac{b^2}{a^2}\right)}{E\left(\pi, \frac{b^2}{a^2}\right)}. \tag{4.4.19}$$

Taking into account ([58], p. 772),

$$\frac{\partial E(\phi, k)}{\partial k} = \frac{1}{k}\Big(E(\phi, k) - F(\phi, k)\Big),$$

where

$$F(\phi, k) = \int_0^{\phi} \frac{1}{\sqrt{1 - k^2 \sin^2 t}}\, dt, \tag{4.4.20}$$

and $F(\pi/2, k) = K(k)$. Then,

$$\frac{\partial}{\partial t}\left(E(\pi, \frac{b^2}{a^2})\right) = \left(E(\pi, \frac{b^2}{a^2}) - F(\pi, \frac{b^2}{a^2})\right)\frac{2a\dot{b} - 2b\dot{a}}{ab},$$

we have

$$B_1(t) = \frac{b}{aE(\pi, \frac{b^2}{a^2})}\Big(-a\dot{b}E(\pi, \frac{b^2}{a^2}) + 2(a\dot{b} - \dot{a}b)F(\pi, \frac{b^2}{a^2})\Big), \tag{4.4.21}$$

$$B_2(t) = \frac{2(\dot{a}b - a\dot{b})}{abE(\pi, \frac{b^2}{a^2})}\Big(a^4 E(\pi, \frac{b^2}{a^2}) - (a^4 - b^4)F(\pi, \frac{b^2}{a^2})\Big), \tag{4.4.22}$$

and

$$W_j = -\frac{1}{2k_j}\Big(B_1 I_1 + B_2 I_2 + b\dot{b} I_3\Big). \tag{4.4.23}$$

Here

$$I_1 = \frac{b^2}{a^2}F\Big(\cos^{-1}(\frac{b}{z}),\ \frac{\sqrt{a^4-b^4}}{a^2}\Big) - \frac{a^2}{b^2}E\Big(\cos^{-1}(\frac{b}{z}),\ \frac{\sqrt{a^4-b^4}}{a^2}\Big) + \frac{\sqrt{(z^2b^2+a^4-b^4)(z^2-b^2)}}{zb^2},$$

$$I_2 = \frac{1}{a^2}F\Big(\cos^{-1}(\frac{b}{z}),\ \frac{\sqrt{a^4-b^4}}{a^2}\Big) = \frac{1}{a^2}\int_0^{\sqrt{1-b^2/z^2}} \frac{dt}{\sqrt{1-\frac{a^4-b^4}{a^4}t^2}\ \sqrt{1-t^2}} = \frac{1}{a^2}\int_0^{\cos^{-1}(b/z)} \frac{dt}{\sqrt{1-\frac{a^4-b^4}{a^4}\sin^2 t}},$$

and the integral I_3 corresponds to the last term in (4.4.18).

To ensure that the singularities of the complex potential have no more than the logarithmic type, we eliminate this term by setting $\dot{b}$ to zero. Thus, we have

$$\dot{S}\,(z) = \frac{2a^3\dot{a}}{\sqrt{b^2z^2+a^4-b^4}\,\sqrt{z^2-b^2}},$$

and the equation (4.3.10) implies

$$W_j = -\frac{a\dot{a}}{k_j E(\frac{b^2}{a^2})}\Big[\Big(E(\tfrac{b^2}{a^2}) - K(\tfrac{b^2}{a^2})\Big)F\Big(\xi,\ \tfrac{\sqrt{a^4-b^4}}{a^2}\Big) \tag{4.4.24}$$
$$+ K(\tfrac{b^2}{a^2})\,E\Big(\xi,\ \tfrac{\sqrt{a^4-b^4}}{a^2}\Big) - \frac{K(\frac{b^2}{a^2})\ \sqrt{(z^2b^2+a^4-b^4)(z^2-b^2)}}{a^2z}\Big] + C(t),$$

where $\xi = \cos^{-1}(\frac{b}{z})$ and $F(\alpha\,,\beta)$ is the incomplete elliptic integral of the first kind (4.4.20),

$$F\Big(\cos^{-1}(\frac{b}{z}),\ \frac{\sqrt{a^4-b^4}}{a^2}\Big) = \int_0^{\sqrt{1-b^2/z^2}} \frac{dt}{\sqrt{1-\frac{a^4-b^4}{a^4}t^2}\ \sqrt{1-t^2}}$$
$$= \int_0^{\cos^{-1}(b/z)} \frac{dt}{\sqrt{1-\frac{a^4-b^4}{a^4}\sin^2 t}}.$$

Since $p_j = \Re\,[W_j]$, we need to compute the real parts for each term in (4.4.24). Using the property $\overline{F(\alpha\,,\beta)} = F(\overline{\alpha}\,,\beta)$ and the summation formula for the elliptic integrals of the first kind [6], we have

$$\frac{1}{2}\Big[F\Big(\xi,\ \frac{\sqrt{a^4-b^4}}{a^2}\Big) + \overline{F\Big(\xi,\ \frac{\sqrt{a^4-b^4}}{a^2}\Big)}\Big] = \frac{1}{2}F\Big(\alpha,\ \frac{\sqrt{a^4-b^4}}{a^2}\Big),$$

where

$$\alpha = \sin^{-1} \frac{\cos\overline{\xi}\sin\xi\sqrt{1-\frac{a^4-b^4}{a^4}\sin^2\overline{\xi}}+\cos\xi\sin\overline{\xi}\sqrt{1-\frac{a^4-b^4}{a^4}\sin^2\xi}}{1-\frac{a^4-b^4}{a^4}\sin^2\xi\sin^2\overline{\xi}} \tag{4.4.25}$$

or

$$\alpha = \sin^{-1} \frac{a^2 z\sqrt{z^2-b^2}\sqrt{b^2\overline{z}^2+a^4-b^4}+a^2\overline{z}\sqrt{\overline{z}^2-b^2}\sqrt{b^2z^2+a^4-b^4}}{b^2z^2\overline{z}^2+(a^4-b^4)(z^2+\overline{z}^2-b^2)}. \tag{4.4.26}$$

Similarly, using the property $\overline{E(\alpha\,,\beta)} = E(\overline{\alpha}\,,\beta)$ and the summation formula for the elliptic integrals of the second kind [6], we have

$$\frac{1}{2}\Big[E\Big(\xi,\ \frac{\sqrt{a^4-b^4}}{a^2}\Big)+\overline{E\Big(\xi,\ \frac{\sqrt{a^4-b^4}}{a^2}\Big)}\Big] = \frac{1}{2}E\Big(\alpha,\ \frac{\sqrt{a^4-b^4}}{a^2}\Big)+\frac{(a^4-b^4)\sqrt{(z^2-b^2)(\overline{z}^2-b^2)}}{2a^4 z\overline{z}}\sin\alpha.$$

Consequently, the pressure is determined by

$$\begin{aligned}\tilde{p}_j = \quad & -\tfrac{a\dot{a}}{2k_jE(\frac{b^2}{a^2})}\Big[\Big(E(\tfrac{b^2}{a^2})-K(\tfrac{b^2}{a^2})\Big)F\Big(\alpha,\ \tfrac{\sqrt{a^4-b^4}}{a^2}\Big)+K(\tfrac{b^2}{a^2})\,E\Big(\alpha,\ \tfrac{\sqrt{a^4-b^4}}{a^2}\Big) \\ & +K(\tfrac{b^2}{a^2})\,\tfrac{(a^4-b^4)\sqrt{(z^2-b^2)(\overline{z}^2-b^2)}}{a^4z\overline{z}}-\tfrac{2K(\frac{b^2}{a^2})}{a^2}\,\Re\{\tfrac{\sqrt{(z^2b^2+a^4-b^4)(z^2-b^2)}}{z}\}\Big]+C_j(t).\end{aligned} \tag{4.4.27}$$

Here

$$\Re\left\{\frac{\sqrt{(z^2b^2+a^4-b^4)(z^2-b^2)}}{z}\right\} = \frac{x(\alpha_1^2\alpha_2^2-x^2y^2b^2+y^2(\alpha_1^2b^2+\alpha_2^2))}{(x^2+y^2)\,\alpha_1\alpha_2},$$

where

$$\alpha_1^2 = (x^2-y^2-b^2+\sqrt{(x^2-y^2-b^2)^2+4x^2y^2})/2$$

and

$$\alpha_2^2 = ((x^2-y^2)b^2+a^4-b^4+\sqrt{((x^2-y^2)b^2+a^4-b^4)^2+4x^2y^2b^2})/2.$$

Taking into account the boundary condition to determine $C_j(t)$, we have

$$\begin{aligned}\tilde{p}_j = & -\tfrac{a\dot{a}}{2k_jE(\frac{b^2}{a^2})}\Big[\Big(E(\tfrac{b^2}{a^2})-K(\tfrac{b^2}{a^2})\Big)F\Big(\alpha,\ \tfrac{\sqrt{a^4-b^4}}{a^2}\Big)+K(\tfrac{b^2}{a^2})\,E\Big(\alpha,\ \tfrac{\sqrt{a^4-b^4}}{a^2}\Big) \\ & +K(\tfrac{b^2}{a^2})\,\tfrac{(a^4-b^4)\sqrt{(z^2-b^2)(\overline{z}^2-b^2)}}{a^4z\overline{z}}-\tfrac{2K(\frac{b^2}{a^2})}{a^2}\,\Re\{\tfrac{\sqrt{(z^2b^2+a^4-b^4)(z^2-b^2)}}{z}\} \\ & -\Big(E(\tfrac{b^2}{a^2})-K(\tfrac{b^2}{a^2})\Big)K\Big(\tfrac{\sqrt{a^4-b^4}}{a^2}\Big)-K(\tfrac{b^2}{a^2})\,E\Big(\tfrac{\sqrt{a^4-b^4}}{a^2}\Big)\Big]+f(t)\end{aligned} \tag{4.4.28}$$

or

$$\tilde{p}_j = -\frac{a\dot{a}}{2k_j E(\frac{b^2}{a^2})}\Big[\Big(E(\tfrac{b^2}{a^2}) - K(\tfrac{b^2}{a^2})\Big)\, F\Big(\alpha,\ \tfrac{\sqrt{a^4-b^4}}{a^2}\Big) + K(\tfrac{b^2}{a^2})\, E\Big(\alpha,\ \tfrac{\sqrt{a^4-b^4}}{a^2}\Big)$$
$$+K(\tfrac{b^2}{a^2})\, \frac{(a^4-b^4)\sqrt{(x^2+y^2)^2-2b^2(x^2-y^2)+b^4}}{a^4(x^2+y^2)} \tag{4.4.29}$$
$$-\frac{2K(\frac{b^2}{a^2})}{a^2}\, \frac{x(\alpha_1^2\alpha_2^2 - x^2y^2b^2 + y^2(\alpha_1^2 b^2 + \alpha_2^2))}{(x^2+y^2)\,\alpha_1\alpha_2} - \frac{\pi}{2}\Big] + f(t).$$

Thereby,

$$p_j = \tilde{p}_j - \frac{\dot{a}K(\frac{b^2}{a^2})}{2k_j a E(\frac{b^2}{a^2})}(x^2 + y^2).$$

To find the location of sinks and sources in the interior domain Ω_2, note that the Schwarz function near its singular points $z = \pm b$ has the reciprocal square root representation (4.3.14) with

$$\xi^r(z,t) = \frac{\sqrt{b^2z^2 + a^4 - b^4}}{\sqrt{z \pm b}}.$$

Formula (4.3.20) implies that $\varphi^r(b) = \pi$ and $\varphi^r(-b) = 0$. This results (taking into account the symmetry of the problem) in the segment $x \in [-b, b]$ as a location of sinks and sources. The corresponding density is

$$\mu_2 = \frac{B_1x^2 + B_2}{k_2\sqrt{(b^2x^2 + a^4 - b^4)(b^2 - x^2)}}.$$

Note that

$$\int_{-b}^{b} \mu_2(x)\,dx = 0,$$

which is consistent with the volume conservation.

To determine the location of the sinks and sources in domain Ω_1, we start with singular points $z_a(t) = \pm i\sqrt{(a^4 - b^4)}\,/b$. The Schwarz function near these points has the square root representation (4.2.7), and the directions of the cuts are defined by formula (4.3.19).

In the neighborhood of the point $z_a(t) = i\sqrt{(a^4 - b^4)}/b$, we have $\arg[\dot{z}_a] = \pi/2 + 2\pi k$ and $\arg[\xi^g(z_a(t), t)] = -\pi/4 + \pi k$. Thus, according to (4.3.19) the direction of the cut is $\varphi^g = \pi/2 + 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$.

Similarly, at the point $z_a(t) = -i\sqrt{(a^4 - b^4)}/b$, $\arg[\dot{z}_a] = -\pi/2 + 2\pi k$, $\arg[\xi^g(z_a(t), t)] = -3\pi/4 + \pi k$. Therefore, the direction of the cut is $\varphi^g = -\pi/2 + 2\pi k$.

Taking into consideration symmetry with respect to the x-axis, we conclude that the support of μ_1 consists of two rays starting at the branch points and going to infinity (see the dashed lines in Fig. 4.2). The density of sinks and sources is defined by

$$\mu_1 = \frac{B_1 y^2 - B_2}{k_1\sqrt{(b^2y^2 - a^4 + b^4)(b^2 + y^2)}}.$$

The evolution of the oval is controlled by a single function $h(t)$, where b is constant and the parameter $a(t)$ is defined by the equation:

$$\frac{\dot{h}}{h} = -\frac{\dot{a}}{a}\frac{K(b^2/a^2)}{E(b^2/a^2)}.$$

Fig. 4.2 shows the evolution of the Cassini's oval under squeezing with $h(t) = h_0 - t$ at $t = 0$ (see Fig. 4.2 (a)) and $t = 0.05$ (see Fig. 4.2 (b)). The dots correspond to the singular points z_a, the dashed lines correspond to the cuts.

4.5 Conclusions

We studied a Muskat problem with a negligible surface tension and a gap width dependent on time. This study extended the results reported in [3], [74], and [79]. We have suggested a method of finding exact solutions and applied it to find new exact solutions for initial elliptical shape and Cassinis oval. The idea of the method was to keep the interface within a certain family of curves defined by its initial shape.

For the elliptical shape, we found two types of solutions: without sinks and sources in the interior domain, and with the presence of a special distribution of sinks and sources along the inter-focal distance. In the former solution, the inter-focal distance remains constant, while in the latter, it changes. In the case when the inter-focal distance decreases, the presence of the sink-source distribution since it does not change the area of the interior domain could be possibly used to simulate the effect of surface tension. It will be studied elsewhere. For the Cassinis oval, we found a solution to the problem when both a gap change and special distributions of sinks and sources in both the interior and exterior domains are present.

5 Future Work

In this section, we discuss the list of some future proposed problems.

1. To obtain some exact solutions of the two-phase Hele-Shaw problem with the presence surface tension on the boundary and allow the sinks and sources to be line distributions with disjoint supports located in the exterior and the interior domains, a two-phase mother body. The mathematical formulation of the proposed future problem is as follows:

   Let $\Omega_2(t) \subset \mathbb{R}^2$ with a boundary $\Gamma(t)$ at time t be a simply-connected bounded domain occupied by a fluid with a constant viscosity ν_2, and let $\Omega_1(t)$ be the region $\mathbb{R}^2 \setminus \bar{\Omega}_2(t)$ occupied by a different fluid of viscosity ν_1. Consider the two-phase Hele-Shaw problem forced by sinks and sources in the presence of the surface tension:

$$\mathbf{v}_j = -k_j \nabla p_j, \qquad j = 1, 2, \tag{5.0.1}$$

   where the pressure p_j is a harmonic function almost everywhere in the region $\Omega_j(t)$, satisfying boundary conditions

$$p_1(x, y, t) - p_2(x, y, t) = \kappa\gamma \quad \text{on} \quad \Gamma(t), \tag{5.0.2}$$

$$v_n = -k_1 \frac{\partial p_1}{\partial n} = -k_2 \frac{\partial p_2}{\partial n} \quad \text{on} \quad \Gamma(t). \tag{5.0.3}$$

   Here $\mathbf{v}_j$ is a velocity vector of fluid j, $k_j = h^2/12\nu_j$, κ is the curvature, γ is the surface tension and h is the gap width of the Hele-Shaw cell. Equation (5.0.2) states the continuity of the pressure in the presence of the surface tension. Equation (5.0.3) means that the normal velocity of the boundary itself coincides with the normal velocity of the fluid at the boundary. We will seek a solution with a constant gap and with a variable gap (time-dependent gap).

2. To obtain asymptotic solutions with surface tension and in the presence of other external force.

3. To study the dynamics of the singularities in one-phase and two-phase Hele-Shaw problems.

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