γ-Sets and the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ Selection Principle

A dissertation presented to

the faculty of

the College of Arts and Sciences of Ohio University

In partial fulfillment

of the requirements for the degree

Doctor of Philosophy

Frank A. Ballone

April 2017

This dissertation titled

γ-Sets and the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ Selection Principle

by

FRANK A. BALLONE

has been approved for

the Department of Mathematics

and the College of Arts and Sciences by

Todd Eisworth

Associate Professor of Mathematics

Robert Frank

Dean, College of Arts and Sciences

ABSTRACT

BALLONE, FRANK A., Ph.D., April 2017, Mathematics

γ-Sets and the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ Selection Principle

Director of Dissertation: Todd Eisworth

The field of Selection Principles in Mathematics is in some sense the study of diagonalization processes. It has its roots in a few basic selection principles that arose from the study of problems in analysis, dimension theory, topology, and set theory. These "classical" selection principles were formally defined by M. Scheepers in 1996, but they go back to classical works of F. Rothberger, W. Hurewicz, and K. Menger. Since then, new selection principles and new types of covers have been introduced and studied in relation to the classical selection principles.

We consider the relationship between γ-sets, which are spaces satisfying a specific classical selection principle, and a newer selection principle $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ that was introduced by B. Tsaban in 2007. First, we survey known results of γ-sets due to F. Galvin and A.W. Miller and prove which results hold for the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle. Then, we establish filter characterizations for these selection principles to prove new properties and positively answer a question asked by B. Tsaban. Afterward, we prove several results about a concrete construction of a γ-set on the Cantor space due to T. Orenshtein and B. Tsaban. Lastly, we revisit our properties to formulate some open questions raised by our work.

ACKNOWLEDGEMENTS

First, I would like to thank Dr. Eisworth for all that he has done as my dissertation advisor. I greatly thank him for working with me over the past several year, spending countless hours with me in his office, and strengthening my critical thinking and research abilities.

In addition, I would like to thank Dr. Boaz Tsaban for answering any questions I had about his work and providing me with extra resources that I was unable obtain otherwise.

Finally, I thank all the members of my dissertation committee, Dr. Vladimir Uspenskiy, Dr. Sergio Lopez-Permouth, and Dr. Philip Ehrlich, for agreeing to be on my committee and taking the time to read over my dissertation.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

CHAPTER 1: INTRODUCTION TO SELECTION PRINCIPLES

The purpose of this chapter is to introduce the reader to the field of selection principles. We will first outline definitions and relations, as well as give a brief history of what are known as "classical" selection principles. Then, we will introduce two recently documented classes of selection principles, emphasizing properties and relations to the classical selection principles. We will end the chapter with brief summary of our research results.

1.1: The Spaces Considered

Many of the results mentioned in this paper apply to spaces X which are Tychonoff, perfectly normal, or Lindelöf in all powers. Unless otherwise indicated, we will consider spaces X which are sets of reals or homeomorphic to sets of reals. There will be some instances where we consider spaces with a more combinatorial structure. In particular, subsets of the Baire space ω^ω of infinite sequences of natural numbers, as these are homeomorphic to sets of reals.

Considering these types of spaces greatly narrows our scope. However, doing this will provide "good" examples of spaces we can work with, and it will filter out some pathological examples.

1.2: Covers

Definition 1.1. We say $\mathcal{U}$ is a *cover* of a space X if $X \notin \mathcal{U}$ and $X = \bigcup \mathcal{U}$.

This is also known as a non-trivial cover of X, as X cannot cover itself. Note that elements of $\mathcal{U}$ need not be open. The primary special sorts of covers that we will consider throughout this paper are as follows.

Definition 1.2. Let $\mathcal{U}$ be a cover of X.

(a) $\mathcal{U}$ is an *ω-cover* of X if each finite $F \subseteq X$ is contained in some $U \in \mathcal{U}$.

(b) $\mathcal{U}$ is a γ-*cover* of X if $\mathcal{U}$ is infinite and each $x \in X$ belongs to all but finitely many $U \in \mathcal{U}$.

Adapting B. Tsaban's notation from [29], we let the boldfaced symbols $\boldsymbol{\mathcal{O}}$, $\boldsymbol{\Omega}$, and $\boldsymbol{\Gamma}$ denote the collections of all covers, ω-covers, and γ-covers, respectively. Simple arguments establish

$$\boldsymbol{\Gamma} \subseteq \boldsymbol{\Omega} \subseteq \boldsymbol{\mathcal{O}}.$$

We also let $\mathcal{O}$, Ω, and Γ denote the corresponding collections of open covers. C_Ω will denote the collection of all clopen ω-covers.

When trying to prove results about covers, it is often more convenient to work with countable covers instead of covers of arbitrary cardinality. Notice that each infinite subset of a γ-cover is also a γ-cover and therefore each γ-cover of X contains a countable γ-subcover of X. We can also reduce an ω-cover of X to a countable ω-subcover in certain situations. J. Gerlits and Zs. Nagy [9] proved every ω-cover of a space X has a countable subset that is a ω-cover of X if and only if every finite power of X is Lindelöf. Since we are working on spaces which have this property, we can assume without loss of generality that any ω-cover or γ-cover is countable.

1.3: The $\binom{\mathscr{A}}{\mathscr{B}}$ Selection Principle

Given a cover $\mathcal{U}$ of a space X, we will first consider a property which chooses a new cover of X by selecting a sub-collection from $\mathcal{U}$ which also covers X. This is known as a *selection principle* as we are selecting a new cover of X from a given cover. We will denote this selection principle by $\binom{\mathscr{A}}{\mathscr{B}}$. Formally, let $\mathscr{A}$ and $\mathscr{B}$ be collections of covers of a space X. $\binom{\mathscr{A}}{\mathscr{B}}$ is defined as follows.

$\binom{\mathscr{A}}{\mathscr{B}}$: Every member of $\mathscr{A}$ has a subset which is a member of $\mathscr{B}$.

Many topological phenomena are captured by the general framework of $\binom{\mathscr{A}}{\mathscr{B}}$ in the situation where $\mathscr{A}$ and $\mathscr{B}$ are certain types of covers. For instance, a topological space X is compact if every open cover has a finite subcover. This can be represented by $\binom{\mathscr{A}}{\mathscr{B}}$ if we let $\mathscr{A}$ be the collection of all open covers of X and $\mathscr{B}$ be the collection of all finite elements of $\mathscr{A}$. Another example would be Lindelöf spaces, as a topological space X is Lindelöf if every open cover has a countable subcover. Letting $\mathscr{A}$ denote the collection of open covers of X and $\mathscr{B}$ the countable members of $\mathscr{A}$ will give us an $\binom{\mathscr{A}}{\mathscr{B}}$ representation for Lindelöf spaces.

Using Ω and $\mathbf{\Gamma}$ with this selection principle, we obtain what are called γ-sets.

Definition 1.3. A space X satisfying $\binom{\Omega}{\mathbf{\Gamma}}$ is called a γ*-set*. In other words, X is a γ-set if every open ω-cover contains a γ-cover.

The idea behind γ-sets was introduced by Gerlits and Nagy in [9] with the γ-property. They defined a space X to have the γ-property if given an open ω-cover $\mathcal{U}$ of X, there is family $\{V_n : n < \omega\} \subseteq \mathcal{U}$ such that V_n is a γ-cover of X. This property was used to help investigate convergence properties of $C_p(X)$, the space of all real-valued continuous functions on X with the topology of pointwise convergence. Gerlits and Nagy proved, for a space X, the space $C_p(X)$ is Fréchet Urysohn if X satisfies the γ-property.

One may also consider a "sequential" modification of the definition of γ-sets, where instead of a single open ω-cover, we look at a sequence of open ω-covers. More precisely, let $\{\mathcal{U}_n : n < \omega\}$ be a sequence of open ω-covers of X. Let us call a space X a γ'-set if there is a sequence $\{V_n : n < \omega\}$ such that for each n, $V_n \in \mathcal{U}_n$ and $\{V_n : n < \omega\}$ is a γ-cover of X.

We need the following simple observation before comparing γ-set and γ'-sets: Without loss of generality, any sequence $\{\mathcal{U}_n : n < \omega\}$ of open ω-covers of X can be replaced by a refining sequence of open ω-covers. To see this, let $\mathcal{U}_1$ and $\mathcal{U}_2$ be open ω-covers of X and consider $\mathcal{V} = \{A \cap B : A \in \mathcal{U}_1 \text{ and } B \in \mathcal{U}_2\}$. We claim $\mathcal{V}$ is an open

ω-cover of X which refines both $\mathcal{U}_1$ and $\mathcal{U}_2$. $\mathcal{V}$ refines both $\mathcal{U}_1$ and $\mathcal{U}_2$ by definition. Furthermore, $\mathcal{V}$ is open. Letting $F \subseteq X$ be finite, there is an $A \in \mathcal{U}_1$ and $B \in \mathcal{U}_2$ such that $F \subseteq A$ and $F \subseteq B$. Therefore, $F \subseteq A \cap B \in \mathcal{V}$, implying $\mathcal{V}$ is an ω-cover of X.

Now, suppose $\{\mathcal{U}_n : n < \omega\}$ is a collection of open ω-covers of X. Define

$$\begin{aligned}
\mathcal{V}_0 &= \mathcal{U}_0 \\
\mathcal{V}_1 &= \{A \cap B : A \in \mathcal{V}_0 \text{ and } B \in \mathcal{U}_1\} \\
\mathcal{V}_2 &= \{A \cap B : A \in \mathcal{V}_1 \text{ and } B \in \mathcal{U}_2\} \\
\mathcal{V}_3 &= \{A \cap B : A \in \mathcal{V}_2 \text{ and } B \in \mathcal{U}_3\} \\
&\vdots \\
\mathcal{V}_n &= \{A \cap B : A \in \mathcal{V}_{n-1} \text{ and } B \in \mathcal{U}_n\}
\end{aligned}$$

Then, $\{\mathcal{V}_n : n < \omega\}$ is a refining sequence of open ω-covers of X.

Theorem 1.1. The following are equivalent for a space X.

1. X is a γ-set.

2. X is a γ'-set.

Proof. ($2 \Rightarrow 1$). γ'-sets are clearly γ-sets, as the sequence of open ω-covers of X could be constant.

($1 \Rightarrow 2$) [9]. Let $\{\mathcal{U}_n : n < \omega\}$ be a sequence of open ω-covers of X. WLOG, we can assume $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ for each $n < \omega$. By the assumption, it is enough to show there is an infinite subsequence $\{n_k : k < \omega\}$ and a sequence $U_k \in \mathcal{U}_{n_k}$ such that $\{U_k : k < \omega\}$ is a γ-cover of X.

Now, choose a sequence $\{x_n : n < \omega\}$ of distinct points in X and let $\mathcal{V}_n = \{U - \{x_n\} : U \in \mathcal{U}_n\}$ and $\mathcal{V} = \bigcup\{\mathcal{V}_n : n < \omega\}$. $\mathcal{V}$ is an open ω-cover of X. As X is a γ-set, there is a family $\{V_k : k < \omega\} \subseteq \mathcal{V}$ that forms a γ-cover of X.

For any $k < \omega$, there is an $n_k < \omega$ and a set U_k with $V_k \subseteq U_k \in \mathcal{U}_{n_k}$. Now, if $n < \omega$ and $\{x_i : i \leq n\} \subseteq V_k$, then $n_k > n$. Thus, $\{n_k : k < \omega\}$ is infinite. ■

1.4: Other Classical Selection Principles

It turns out that γ'-sets satisfy a specific instance of the S_1 selection principle. This, along with two other classical selection principles, were formally defined by M. Scheepers in [24], although they go back to classical works of F. Rothberger, W. Hurewicz, and K. Menger. The classical selection principles are defined as follows. Let $\mathscr{A}$ and $\mathscr{B}$ be collections of covers of a space X.

$\mathsf{S}_1(\mathscr{A}, \mathscr{B})$: For each sequence $\{\mathcal{U}_n\}_{n<\omega}$ of members of $\mathscr{A}$, there exists members $U_n \in \mathcal{U}_n$, $n < \omega$, such that $\{U_n : n < \omega\} \in \mathscr{B}$.

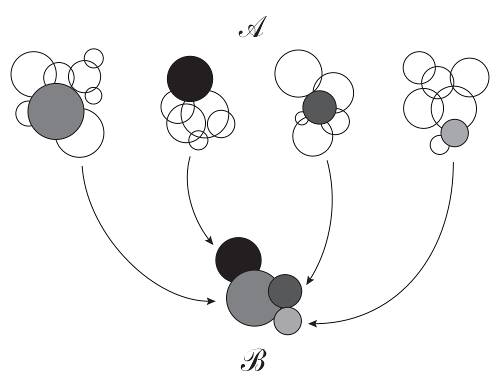

Figure 1.1: The $\mathsf{S}_1(\mathscr{A}, \mathscr{B})$ Selection Principle

This selection principle builds a new cover by choosing a single element from each $\mathcal{U}_n$. Theorem 1.1 shows X satisfies $\binom{\Omega}{\mathbf{\Gamma}}$ if and only if X satisfies $\mathsf{S}_1(\Omega, \mathbf{\Gamma})$.

The next selection principle chooses a new cover of X by piecing together unions of finite subsets.

$\mathsf{U}_{\text{fin}}(\mathscr{A},\mathscr{B})$: For each sequence $\{\mathcal{U}_n\}_{n<\omega}$ of members of $\mathscr{A}$ which do not contain a finite subcover, there exist finite (possibly empty) subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n<\omega$, such that $\{\bigcup \mathcal{F}_n : n<\omega\} \in \mathscr{B}$.

Instead of selecting a single element of $\mathcal{U}_n$, this selection principle chooses finitely many elements and takes their union. Note that since each $\mathcal{U}_n$ does not have a finite subcover, $\mathcal{F}_n$ will not cover X. This also applies to each $\bigcup \mathcal{F}_n$.

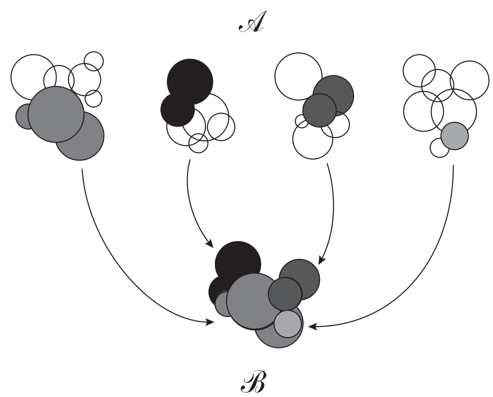

Figure 1.2: The $\mathsf{U}_{\text{fin}}(\mathscr{A},\mathscr{B})$ Selection Principle

The final classical selection principle we will consider is similar to $\mathsf{U}_{\text{fin}}(\mathscr{A},\mathscr{B})$, but it does not "glue" the finite sub-collections together.

$\mathsf{S}_{\text{fin}}(\mathscr{A},\mathscr{B})$: For each sequence $\{\mathcal{U}_n\}_{n<\omega}$ of members of $\mathscr{A}$, there exists finite (possibly empty) subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n<\omega$, such that $\displaystyle\bigcup_{n<\omega} \mathcal{F}_n \in \mathscr{B}$.

Each $\mathcal{F}_n$ consists of finitely many subsets of X. Therefore, $\bigcup_{n<\omega} \mathcal{F}_n$ is the family of subsets of X obtained when we collect those sets together. $\mathsf{S}_{\text{fin}}(\mathscr{A},\mathscr{B})$ says this family is a cover of X and it belongs to the collection $\mathscr{B}$ as well.

1.5: A Brief History

In this section, we will give a brief history of the S_1, S_{fin}, and U_{fin} selection principles.

In 1925, W. Hurewicz derived two selection principles in [10]. Hurewicz first derived $\mathsf{S}_{\text{fin}}(\mathcal{O}, \boldsymbol{\mathcal{O}})$ from a conjecture by K. Menger in [16]. In his paper, Menger introduced the following basis covering property for metric spaces.

The Menger Basis Property. For each basis $\mathcal{B}$ of a metric space (X, d), there is a sequence $\{B_n : n < \omega\}$ in $\mathcal{B}$ such that $\{B_n : n < \omega\}$ covers X and $\lim\limits_{n\to\infty} diam_d(B_n) = 0$.

Menger noticed every σ-compact metric space has this property and conjectured having this property implies that a metric space is σ-compact. Hurewicz showed that a metric space has the Menger basis property if and only if it has the selection principle $\mathsf{S}_{\text{fin}}(\mathcal{O}, \boldsymbol{\mathcal{O}})$. This leads us to the following definition.

Definition 1.4. The covering property $\mathsf{S}_{\text{fin}}(\mathcal{O}, \boldsymbol{\mathcal{O}})$ is called the *Menger property*. Any topological space with the Menger property is called a *Menger space*, that is, X is Menger if for any countable sequence of open covers $\{\mathcal{U}_n\}_{n<\omega}$ of X, there exists finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n < \omega$, such that $\bigcup_{n<\omega} \mathcal{F}_n$ is an open cover of X.

Motivated by Menger's conjecture, Hurewicz also introduced the selection principle $\mathsf{U}_{\text{fin}}(\mathcal{O}, \boldsymbol{\Gamma})$.

Definition 1.5. The covering property $\mathsf{U}_{\text{fin}}(\mathcal{O}, \boldsymbol{\Gamma})$ is called the *Hurewicz property*. Any topological space with the Hurewicz property is called a *Hurewicz space*, that is, X is Hurewicz if for any countable sequence of open covers $\{\mathcal{U}_n\}_{n<\omega}$ of X which do not contain a finite subcover, there exists finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n < \omega$, such that $\{\bigcup \mathcal{F}_n : n < \omega\}$ is a γ-cover of X.

In relation to Menger spaces, Hurewicz observed: σ-compact $\Rightarrow$ Hurewicz property $\Rightarrow$ Menger property in [11]. Hurewicz also questioned if his property was strictly stronger than Menger's, that is, $\mathsf{S}_{\text{fin}}(\mathcal{O},\boldsymbol{\mathcal{O}}) \not\Rightarrow \mathsf{U}_{\text{fin}}(\mathcal{O},\boldsymbol{\Gamma})$. In 2002, J. Chamber and R. Pol provided a positive solution to Hurewicz's question in [4] using the topological "Michael technique" and a dichotomic argument with the cases $\mathfrak{b} = \mathfrak{d}$ and $\mathfrak{b} < \mathfrak{d}$. Their solution established the existence of a set of reals X without the Hurewicz property such that all finite powers of X have the Menger property. In 2005, T. Tsaban and L. Zdomsky constructed a concrete set in [30] having the Menger property but not the Hurewicz property. Their construction gave a non-dichotomic, combinatorial proof. Finally, in 2006, A. Rinot proved the existence of a positive solution in [19] using a more direct, dichotomic proof, again distinguishing between the cases $\mathfrak{b} = \mathfrak{d}$ and $\mathfrak{b} < \mathfrak{d}$.

The S_1 selection principle was first introduced in 1938 by F. Rothberger in [20]. Rothberger was interested in strong measure zero sets in metric spaces. Strong measure zero sets were first defined in 1919 by Borel in [3].

Definition 1.6. A metric space (X, d) has *strong measure zero* if for any sequence of positive reals $\{\epsilon_n\}_{n<\omega}$, there is a cover $\{\mathcal{U}_n : n < \omega\}$ of X such that $diam_d(\mathcal{U}_n) < \epsilon_n$ for all n.

Rothberger observed that if a metric space has the property $\mathsf{S}_1(\mathcal{O},\boldsymbol{\mathcal{O}})$, then it has strong measure zero. This lead to a new type of space.

Definition 1.7. The covering property $\mathsf{S}_1(\mathcal{O},\boldsymbol{\mathcal{O}})$ is called the *Rothberger property*. Any topological space with the Rothberger property is called a *Rothberger space*, that is, X is Rothberger if for each sequence $\{\mathcal{U}_n\}_{n<\omega}$ of open covers of X, there exists $V_n \in \mathcal{U}_n$ such that $\{V_n : n < \omega\}$ is an open cover of X.

Another famous S_1 selection principle, which we previously mentioned, was introduced by J. Gerlits and Zs. Nagy in [9], namely $\mathsf{S}_1(\Omega,\boldsymbol{\Gamma})$. We will focus more on this selection principle in later sections.

1.6: Relation Properties of Selection Principles

The classical selection principles can be related in various ways. One relation is a type of monotonicity property, being antimonotonic in the first component and monotonic in the second component. This property was formally introduced by Scheepers in [24].

Proposition 1.1. Suppose Π is a selection principle and $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ are families of subsets of an infinite set.

1. If $\mathscr{A} \subseteq \mathscr{C}$, then $\Pi(\mathscr{C}, \mathscr{B}) \Rightarrow \Pi(\mathscr{A}, \mathscr{B})$.

2. If $\mathscr{B} \subseteq \mathscr{D}$, then $\Pi(\mathscr{A}, \mathscr{B}) \Rightarrow \Pi(\mathscr{A}, \mathscr{D})$.

This proposition is illustrated in Figure 1.3.

$$\begin{array}{ccc} \Pi(\mathscr{A},\mathscr{B}) & \longrightarrow & \Pi(\mathscr{A},\mathscr{D}) \\ \uparrow & & \uparrow \\ \Pi(\mathscr{C},\mathscr{B}) & \longrightarrow & \Pi(\mathscr{C},\mathscr{D}) \end{array}$$

Figure 1.3: Monotonicity Laws

We will now show how the classical selection principles are related. Let X be a space, $\mathscr{A} \in \{\Gamma, \Omega, \mathcal{O}\}$, and $\mathscr{B} \in \{\boldsymbol{\Gamma}, \boldsymbol{\Omega}, \boldsymbol{\mathcal{O}}\}$. Note that if X satisfies $\mathsf{S}_1(\mathscr{A}, \mathscr{B})$, then X satisfies $\mathsf{S}_{\text{fin}}(\mathscr{A}, \mathscr{B})$. The proof of this relation is trivial as the finite subsets considered in the $\mathsf{S}_{\text{fin}}(\mathscr{A}, \mathscr{B})$ definition can all have size 1.

Lemma 1.1. If X satisfies $\mathsf{S}_{\text{fin}}(\mathscr{A}, \mathscr{B})$, then X satisfies $\mathsf{U}_{\text{fin}}(\mathscr{A}, \mathscr{B})$.

We will prove this result shortly.

Lemma 1.2. If X satisfies $\mathsf{S}_{\text{fin}}(\mathscr{A}, \mathscr{B})$, then X satisfies $\binom{\mathscr{A}}{\mathscr{B}}$.

Proof. Let $\mathcal{U} \in \mathscr{A}$. Consider a sequence of members of $\mathscr{A}$ that is constant with a value of $\mathcal{U}$. As X satisfies $\mathsf{S}_{\text{fin}}(\mathscr{A}, \mathscr{B})$, there exists finite (possibly empty) subsets $\mathcal{F}_n \subseteq \mathcal{U}$ for each $n < \omega$ such that $\bigcup_{n<\omega} \mathcal{F}_n \in \mathscr{B}$. ■

Figure 1.4 summarizes the relations mentioned thus far.

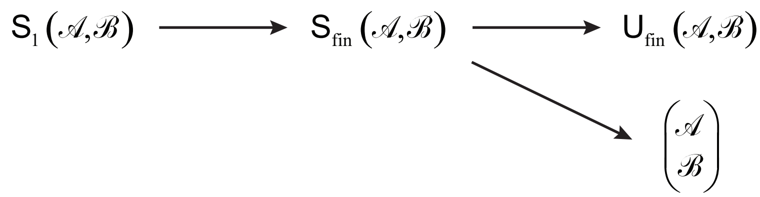

Figure 1.4: Relating Classical Selection Principles

If we let $\mathscr{A}$ range over $\{\Gamma, \Omega, \mathcal{O}\}$ and $\mathscr{B}$ over $\{\mathbf{\Gamma}, \mathbf{\Omega}, \boldsymbol{\mathcal{O}}\}$, then there are potentially nine distinct classes of spaces for each selection principle. Figure 1.5 illustrates how the nine S_1 selection principles are related using the monotonicity laws.

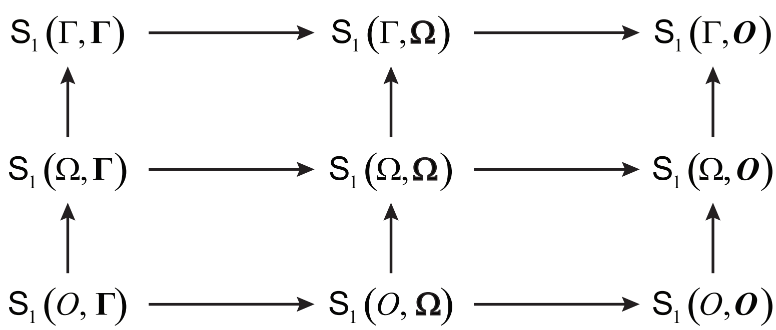

Figure 1.5: $\mathsf{S}_1(\mathscr{A}, \mathscr{B})$ Classes

$\mathsf{S}_1(\mathcal{O}, \mathbf{\Gamma})$ and $\mathsf{S}_1(\mathcal{O}, \mathbf{\Omega})$ are impossible for X, as any regular T_2 space X with at least two points has an open cover which is not an ω-cover. To see this, let $x, y \in X$ with $x \neq y$. Then, there is an open set U in X such that $x \in U$ and $y \notin U$. By regularity, there exists another open set V in X such that $x \in V$ with $\overline{V} \subseteq U$. $\mathcal{U} = U \cup (X \setminus \overline{V})$ is an open cover of X, but there is no set in $\mathcal{U}$ which contains both x and y. Therefore, X cannot satisfy $\mathsf{S}_1(\mathcal{O}, \mathbf{\Omega})$. Furthermore, X cannot satisfy $\mathsf{S}_1(\mathcal{O}, \mathbf{\Gamma})$ as $\mathsf{S}_1(\mathcal{O}, \mathbf{\Gamma}) \Rightarrow \mathsf{S}_1(\mathcal{O}, \mathbf{\Omega})$.

A similar relation diagram can be created for the S_{fin} selection principle. There is a stronger relation when considering the U_{fin} selection principle.

Lemma 1.3. $\mathsf{U}_{\text{fin}}(\mathcal{O}, \mathscr{B}) \Leftrightarrow \mathsf{U}_{\text{fin}}(\Gamma, \mathscr{B})$ for $\mathscr{B} \in \{\mathbf{\Gamma}, \mathbf{\Omega}, \boldsymbol{\mathcal{O}}\}$.

Proof. $\mathsf{U}_{\text{fin}}(\mathcal{O}, \mathscr{B}) \Rightarrow \mathsf{U}_{\text{fin}}(\Gamma, \mathscr{B})$ by the monotonicity laws. To show the reverse implication, suppose $\{\mathcal{U}_n\}_{n<\omega}$ is a sequence of open covers which do not contain a finite sub-cover. Notice that given an open cover $\{U_m : m < \omega\}$, we may replace it by $\{\bigcup_{i<m} U_i : m < \omega\}$. This set is infinite as $\{U_m : m < \omega\}$ does not contain a finite subcover. Furthermore, it is a γ-cover. Therefore, we can apply $\mathsf{U}_{\text{fin}}(\Gamma, \mathscr{B})$. ■

This lemma implies that the diagram for the nine potential classes of the U_{fin} can be reduced to any of its rows since the three selection principles in each column are equivalent. This means it is enough to prove $\mathsf{S}_{\text{fin}}(\Gamma, \mathscr{B}) \Rightarrow \mathsf{U}_{\text{fin}}(\Gamma, \mathscr{B})$ for $\mathscr{B} \in \{\mathbf{\Gamma}, \mathbf{\Omega}, \boldsymbol{\mathcal{O}}\}$ in order to show $\mathsf{S}_{\text{fin}}(\mathscr{A}, \mathscr{B}) \Rightarrow \mathsf{U}_{\text{fin}}(\mathscr{A}, \mathscr{B})$.

Proof of Lemma 1.1. We consider three cases for $\mathscr{B}$. In each case, let $\{\mathcal{U}_n\}_{n\in\omega}$ be a sequence of open γ-covers of X, each without a finite subcover.

<u>Case 1</u>: Suppose X satisfies $\mathsf{S}_{\text{fin}}(\Gamma, \boldsymbol{\mathcal{O}})$. Then, there exists finite subsets $\mathcal{F}_n$ of $\mathcal{U}_n$ such that $\bigcup_{n<\omega} \mathcal{F}_n$ covers X. Notice that,

$$X \subseteq \bigcup \bigcup_{n<\omega} \mathcal{F}_n = \bigcup_{n<\omega} \bigcup \mathcal{F}_n.$$

Therefore, X satisfies $\mathsf{U}_{\text{fin}}(\Gamma, \boldsymbol{\mathcal{O}})$.

<u>Case 2</u>: Suppose X satisfies $\mathsf{S}_{\text{fin}}(\Gamma, \mathbf{\Omega})$. Again, choose a sequence $\{\mathcal{U}_n\}_{n<\omega}$ of open γ-covers of X such that no $\mathcal{U}_n$ has a finite subcover. Applying $\mathsf{S}_{\text{fin}}(\Gamma, \mathbf{\Omega})$, there exists finite subsets $\mathcal{F}_n$ of $\mathcal{U}_n$ such that $\bigcup_{n<\omega} \mathcal{F}_n$ is an ω-cover of X. If F is any finite subset of X, then some element of $\bigcup_{n<\omega} \mathcal{F}_n$ contains F. There is an $n < \omega$ and $U \in \mathcal{F}_n$ such that $F \subseteq U$. Therefore, $F \subseteq \bigcup \mathcal{F}_n$ and the desired result follows.

<u>Case 3</u>: Suppose X satisfies $\mathsf{S}_{\text{fin}}(\Gamma, \mathbf{\Gamma})$. Choose a sequence $\{\mathcal{U}_n\}_{n<\omega}$ of open γ-covers of X such that no $\mathcal{U}_n$ has a finite subcover and apply $\mathsf{S}_{\text{fin}}(\Gamma, \mathbf{\Gamma})$ to get finite

subsets $\mathcal{F}_n$ of $\mathcal{U}_n$ such that $\bigcup_{n<\omega}\mathcal{F}_n$ is an γ-cover of X. Note that if $x \in X$, then x is contained in all but finitely many sets in $\bigcup\{\mathcal{F}_n : n < \omega\}$. Thus, x is contained in all but finitely many members of $\{\bigcup\mathcal{F}_n : n < \omega\}$. ■

We will now consider equivalent selection principles. For a space X, Theorem 1.1 stated $\mathsf{S}_1(\Omega, \mathbf{\Gamma}) \Leftrightarrow \binom{\Omega}{\mathbf{\Gamma}}$. Combining this result with Figure 1.4, it follows that $\mathsf{S}_1(\Omega, \mathbf{\Gamma}) \Leftrightarrow \mathsf{S}_{\text{fin}}(\Omega, \mathbf{\Gamma})$.

We will present a proof from [24] which shows X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}, \boldsymbol{\mathcal{O}})$ if and only if it satisfies $\mathsf{U}_{\text{fin}}(\Gamma, \boldsymbol{\mathcal{O}})$.

Proposition 1.2. For a space X, $\mathsf{S}_{\text{fin}}(\mathcal{O}, \boldsymbol{\mathcal{O}}) \Leftrightarrow \mathsf{U}_{\text{fin}}(\Gamma, \boldsymbol{\mathcal{O}})$.

Proof. Using the monotonicity laws and Figure 1.4 relations, $\mathsf{S}_{\text{fin}}(\mathcal{O}, \boldsymbol{\mathcal{O}}) \Rightarrow \mathsf{S}_{\text{fin}}(\Gamma, \boldsymbol{\mathcal{O}}) \Rightarrow \mathsf{U}_{\text{fin}}(\Gamma, \boldsymbol{\mathcal{O}})$.

To reverse the implication, suppose X satisfies $\mathsf{U}_{\text{fin}}(\Gamma, \boldsymbol{\mathcal{O}})$. Let $\{\mathcal{U}_n\}_{n<\omega}$ be a sequence of open covers of X. We may assume each $\mathcal{U}_n$ is countably infinite and no finite subset covers X, as X is not compact. For each n, enumerate each $\mathcal{U}_n$ bijectively as $\mathcal{U}_n = \{U^n_k : k = 1, 2, 3, \ldots\}$. Then, let $\mathcal{W}_n$ be the collection whose mth member is $\bigcup_{k=1}^{m} U^n_k$.

$\mathcal{W}_n$ is infinite since $\mathcal{U}_n$ has no finite subcover, and $x \in \mathcal{W}_n$ for large enough n. Therefore, $\mathcal{W}_n$ is a γ-cover of X. Now, apply the $\mathsf{U}_{\text{fin}}(\Gamma, \boldsymbol{\mathcal{O}})$ selection principle and choose a finite subset $\mathcal{F}_n$ from each $\mathcal{W}_n$ such that $\bigcup_{n=1}^{\infty}\mathcal{F}_n$ is an open cover of X.

As $\mathcal{W}_n$ is increasing, choose $\mathcal{F}_n = \{U^n_0 \cup \ldots \cup U^n_{k_n}\}$ for some k. Then, $\{U^n_i : n < \omega, i \leq k_n\}$ covers X. ■

Below are additional equivalences that have been established over the years. We give references rather than proofs and use "=" to denote equivalence when considering selection principles.

- $\mathsf{S}_1(\Gamma, \mathbf{\Gamma}) = \mathsf{S}_{\text{fin}}(\Gamma, \mathbf{\Gamma})$ [27]

- $\mathsf{S}_{\text{fin}}(\mathcal{O},\boldsymbol{\mathcal{O}}) = \mathsf{S}_{\text{fin}}(\Omega,\boldsymbol{\mathcal{O}}) = \mathsf{S}_{\text{fin}}(\Gamma,\boldsymbol{\mathcal{O}}) = \mathsf{U}_{\text{fin}}(\Gamma,\boldsymbol{\mathcal{O}})$ [24]

- $\mathsf{S}_1(\mathcal{O},\boldsymbol{\mathcal{O}}) = \mathsf{S}_1(\Omega,\boldsymbol{\mathcal{O}})$ [24]

Figure 1.6 illustrates all the selection principle relationships mentioned thus far. Most articles refer to this diagram as the Scheepers diagram.

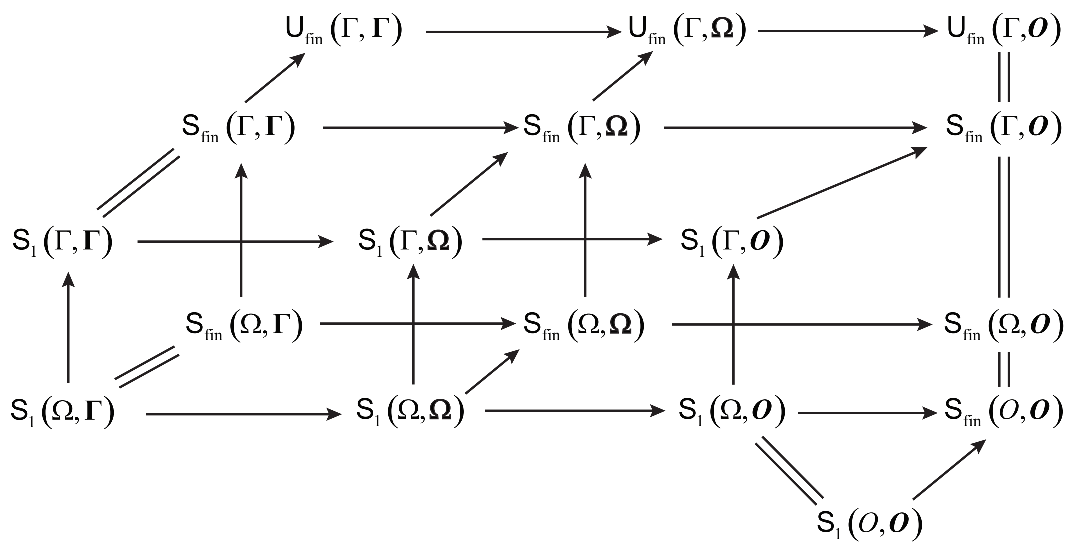

Figure 1.6: The Scheepers Diagram

1.7: Newer Selection Principles

In this section, we will survey newer selection principles which were recently introduced by Tsaban in [29]. These selection principles will play a big role throughout this paper.

Definition 1.8. For a family $\mathscr{A}$ of covers of a set X, $\mathscr{A}_\infty$ is the family of all $\mathcal{U}$ such that there exists infinite sets $\mathcal{U}_n \subseteq \mathcal{U}$, $n < \omega$, with $\{\bigcap \mathcal{U}_n : n < \omega\} \in \mathscr{A}$.

By combining this new family of covers with the $\binom{\mathscr{A}}{\mathscr{B}}$ selection principle, Tsaban obtained the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle. The motivation behind Tsaban's study of this selection principle began with a result of M. Sakai about Tychonoff spaces and the Pytkeev property in [22]. We will focus more on the Pytkeev property in chapter 5.

There is a relation between the $\binom{\mathscr{A}}{\mathscr{B}}$ and $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principles when considering surjectively derefinable covers.

Definition 1.9. A family $\mathscr{B}$ of covers of X is *surjectively derefinable* if $\{f(U) : U \in \mathcal{U}\} \in \mathscr{B}$ for each $\mathcal{U} \in \mathscr{B}$ and each $f : \mathcal{U} \to \mathcal{P}(X) \setminus \{X\}$ such that for each $U \in \mathcal{U}$, $f(U)$ is open and contains U.

Tsaban proved the following result in [29]. We will also present the proof.

Lemma 1.4. Assume that $\mathscr{B}$ is a surjectively derefinable family of covers of X. Then

$$\binom{\mathscr{A}}{\mathscr{B}_\infty} \Rightarrow \binom{\mathscr{A}}{\mathscr{B}}.$$

Proof. Let $\mathscr{A}$ and $\mathscr{B}$ be families of covers of X, with $\mathscr{B}$ surjectively derefinable, and suppose X satisfies $\binom{\mathscr{A}}{\mathscr{B}_\infty}$. Furthermore, let $\mathcal{U} \in \mathscr{A}$. Since X satisfies $\binom{\mathscr{A}}{\mathscr{B}_\infty}$, there are infinite sets $\mathcal{U}_1, \mathcal{U}_2, \ldots \subseteq \mathcal{U}$ such that $\mathcal{V} = \{\bigcap \mathcal{U}_n : n < \omega\} \in \mathscr{B}$. Now, for each n, choose $f(\bigcap \mathcal{U}_n) \in U_n$. Since $\mathscr{B}$ is surjectively derefinable, $\mathcal{W} = \{f(\bigcap \mathcal{U}_n) : n < \omega\} \in \mathscr{B}$. It follows that $\mathcal{W} \subseteq \mathcal{U}$. Therefore, $\mathcal{U}$ has a subset which is in $\mathscr{B}$. ∎

Note that the families $\boldsymbol{\mathcal{O}}$, $\boldsymbol{\Omega}$, and $\boldsymbol{\Gamma}$ are surjectively derefinable, meaning we can use Lemma 1.4 to create new relations using the $\binom{\mathscr{A}}{\mathscr{B}}$ selection principles. For instance, $\binom{\Omega}{\boldsymbol{\Gamma}_\infty} \Rightarrow \binom{\Omega}{\boldsymbol{\Gamma}}$.

If we let $\mathscr{A}$ range over $\{\Gamma, \Omega, \mathcal{O}\}$ and $\mathscr{B}$ over $\{\boldsymbol{\Gamma}, \boldsymbol{\Omega}, \boldsymbol{\mathcal{O}}\}$, then there are potentially nine distinct classes of spaces for the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle. Using the monotonicity laws, we can create a diagram similar to Figure 1.5 to see how all nine of these classes are related. Several of these classes can be ruled out. As mentioned in [29], any T_1 space with more than one element has a finite open cover. Since the spaces considered are T_1, this will rule out the $\binom{\mathcal{O}}{\mathscr{B}_\infty}$ selection principle for $\mathscr{B} \in \{\boldsymbol{\Gamma}, \boldsymbol{\Omega}, \boldsymbol{\mathcal{O}}\}$.

Tsaban proved in [29] that the $\binom{\Gamma}{\boldsymbol{\Gamma}_\infty}$ selection principle will always hold in X. We include his proof for completeness.

Proposition 1.3. Every space satisfies $\binom{\Gamma}{\mathbf{\Gamma}_\infty}$.

Proof. Assume $\mathcal{U}$ is a γ-cover of a space X. We may assume $\mathcal{U}$ is countable as any infinite subset of a γ-cover is a γ-cover. Now, enumerate $\mathcal{U} = \{U_n : n < \omega\}$ bijectively and take $\mathcal{U}_n = \{U_k : k \geq n\}$ for each n. Then, $\mathcal{V} = \{\bigcap \mathcal{U}_n : n < \omega\} \in \Gamma$ as $\mathcal{V}$ is infinite and X is not covered by finitely many members from $\mathcal{V}$. Therefore, $\mathcal{U}$ contains a subset which is a member of $\mathbf{\Gamma}_\infty$. ■

There is also an equivalence in [29] between γ-sets and sets satisfying the $\binom{\Omega}{\mathbf{\Gamma}_\infty}$ selection principle.

Corollary 1.1. $\binom{\Omega}{\mathbf{\Gamma}} = \binom{\Omega}{\mathbf{\Gamma}_\infty}$.

Proof. By Lemma 1.4,

$$\binom{\Omega}{\mathbf{\Gamma}_\infty} \Rightarrow \binom{\Omega}{\mathbf{\Gamma}}.$$

By Proposition 1.3,

$$\binom{\Omega}{\mathbf{\Gamma}} \Rightarrow \binom{\Omega}{\mathbf{\Gamma}} \cap \binom{\Gamma}{\mathbf{\Gamma}_\infty} \Rightarrow \binom{\Omega}{\mathbf{\Gamma}_\infty},$$

with the last implication following from the definition of $\binom{\mathscr{A}}{\mathscr{B}}$. ■

Considering clopen ω-covers in the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principles leads us once more to strong measure zero sets. The following definition of strong measure zero sets is slightly modified from the metric space definition mentioned earlier.

Definition 1.10. A set $X \subseteq \mathbb{R}$ has *strong measure zero* if for each sequence of positive reals $\{\epsilon_n\}_{n<\omega}$, there exists a cover $\{I_n\}_{n<\omega}$ of X such that for each n, the diameter of I_n is smaller than ϵ_n.

The following theorem relating strong measure zero sets to the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle was proved in [28] and [29].

Theorem 1.2. If $X \subseteq \mathbb{R}$ and X satisfies $\binom{C_\Omega}{\boldsymbol{\mathcal{O}}_\infty}$, then X has strong measure zero.

Gerlits and Nagy proved in [9] that every γ-set has strong measure zero. We are now able to prove this theorem using an alternate method with the results and definitions mentioned throughout this section.

Corollary 1.2. Every γ-set has strong measure zero

Proof. We have the following.

$$\binom{\Omega}{\boldsymbol{\Gamma}} = \binom{\Omega}{\boldsymbol{\Gamma}_\infty} \Rightarrow \binom{\Omega}{\boldsymbol{\Omega}_\infty} \Rightarrow \binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty} \Rightarrow \binom{C_\Omega}{\boldsymbol{\mathcal{O}}_\infty} \qquad \blacksquare$$

The classical selection principles can also be connected to the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle. For instance, we have the following relation and proof from [29].

Proposition 1.4. If X satisfies $\mathsf{S}_{\text{fin}}(\Omega, \boldsymbol{\Omega})$ and $\binom{\Omega}{\boldsymbol{\Omega}_\infty}$, then X satisfies $\mathsf{S}_1(\Omega, \boldsymbol{\Omega})$.

Proof. By definition,

$$\mathsf{S}_{\text{fin}}(\Omega, \boldsymbol{\Omega}) \cap \binom{\Omega}{\boldsymbol{\Omega}_\infty} = \mathsf{S}_{\text{fin}}(\Omega, \boldsymbol{\Omega}_\infty).$$

Therefore, it is enough to show $\mathsf{S}_{\text{fin}}(\Omega, \boldsymbol{\Omega}_\infty)$ implies $\mathsf{S}_1(\Omega, \boldsymbol{\Omega})$.

Suppose $\mathcal{U}_n$, $n < \omega$, are open ω-covers of X. By $\mathsf{S}_{\text{fin}}(\Omega, \boldsymbol{\Omega}_\infty)$, choose finite $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\mathcal{V} = \bigcup_{n<\omega} \mathcal{F}_n \in \boldsymbol{\Omega}_\infty$.

Now, choose infinite $\mathcal{V}_n \subseteq \mathcal{V}$ for each $n < \omega$ such that $\{\bigcap \mathcal{V}_n : n < \omega\}$ is an ω-cover of X. Since each $\mathcal{F}_n$ is finite, each $\mathcal{V}_n$ will contains sets from $\mathcal{U}_k$ for arbitrarily large k. Thus, we can construct an increasing sequence m_n such that $\mathcal{V}_n$ contains an element U_{m_n} of $\mathcal{U}_{m_n}$. If $k \neq m_n$ for all n, let U_k be any element of $\mathcal{U}_k$. It follows that $\{U_k : k < \omega\}$ witnesses $\mathsf{S}_1(\Omega, \boldsymbol{\Omega})$. $\blacksquare$.

We immediately have the following.

Corollary 1.3. $\mathsf{S}_{\text{fin}}(\Omega, \boldsymbol{\Omega}_\infty) \Rightarrow \mathsf{S}_{\text{fin}}(\Omega, \boldsymbol{\Omega})$.

This next relation was also proved by Tsaban in [29].

Corollary 1.4. $\mathsf{S}_1(\Omega, \boldsymbol{\Omega_\infty}) \Rightarrow \mathsf{S}_1(\Omega, \boldsymbol{\Omega})$.

Proof. By previous results:

$$\begin{aligned} \mathsf{S}_1(\Omega, \boldsymbol{\Omega_\infty}) &= \mathsf{S}_1(\Omega, \boldsymbol{\Omega}) \cap \binom{\Omega}{\boldsymbol{\Omega}_\infty}, \\ &\Rightarrow \mathsf{S}_{\text{fin}}(\Omega, \boldsymbol{\Omega}) \cap \binom{\Omega}{\boldsymbol{\Omega}_\infty}, \\ &= \mathsf{S}_1(\Omega, \boldsymbol{\Omega}). \quad \blacksquare \end{aligned}$$

Tsaban also introduced a new type of selection principle in [29] that is a selective version of the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle, similar to how $\binom{\mathscr{A}}{\mathscr{B}}$ and $\mathsf{S}_1(\mathscr{A}, \mathscr{B})$ are related. Let $\mathscr{A}$ and $\mathscr{B}$ be collections of covers of a space X.

$\bigcap_\infty(\mathscr{A}, \mathscr{B})$: For every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of members of $\mathscr{A}$, there is for each n an infinite set $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\{\bigcap V_n : n < \omega\} \in \mathscr{B}$.

A trivial argument establishes the following.

Proposition 1.5. $\bigcap_\infty(\mathscr{A}, \mathscr{B}) \Rightarrow \binom{\mathscr{A}}{\mathscr{B}_\infty}$.

Tsaban also proved in [29] how the $\bigcap_\infty(\mathscr{A}, \mathscr{B})$ selection principle is related to the $\mathsf{S}_1(\mathscr{A}, \mathscr{B})$ selection principle. We omit the proof.

Proposition 1.6. Assume $\mathscr{B}$ is a surjectively derefinable family of covers of X. Then,

$$\bigcap_\infty(\mathscr{A}, \mathscr{B}) \Rightarrow \mathsf{S}_1(\mathscr{A}, \mathscr{B}).$$

Finally, we note the following results of Tsaban from [29].

Theorem 1.3. $\bigcap_\infty(\Gamma, \boldsymbol{\Gamma}) = \mathsf{S}_1(\Gamma, \boldsymbol{\Gamma})$,

Corollary 1.5. $\bigcap_\infty(\Omega, \boldsymbol{\Gamma}) = \mathsf{S}_1(\Omega, \boldsymbol{\Gamma})$.

1.8: Research Motivation and Results

Our motivation behind the study of γ-sets and the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle originated from the following conjecture in [29].

Conjecture 1.1. The Continuum Hypothesis (CH) implies

1. there is a set of reals X satisfying $\binom{\Omega}{\mathbf{\Omega}_\infty}$ but not $\binom{\Omega}{\Gamma}$, and
2. there is a set of reals X satisfying $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$ but not $\binom{\Omega}{\mathbf{\Omega}_\infty}$.

We are interested in proving the consistency of both items in this conjecture. In other words, do we have strict implications in the following:

$$\binom{\Omega}{\mathbf{\Gamma}} = \binom{\Omega}{\mathbf{\Gamma}_\infty} \Rightarrow \binom{\Omega}{\mathbf{\Omega}_\infty} \Rightarrow \binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}. \tag{1.1}$$

Showing any of the implications cannot be reversed, assuming CH, will prove the consistency of Conjecture 1.1. It is also worth noting that a similar situation arises for the $\bigcap_\infty(\mathscr{A},\mathscr{B})$ selection principle. Using the monotonicity laws and properties previously mentioned, $\bigcap_\infty(\Omega,\mathbf{\Omega})$ and $\bigcap_\infty(\Omega,\boldsymbol{\mathcal{O}})$ lie between $\binom{\Omega}{\mathbf{\Gamma}}$ and $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$. This leads to the following relations.

$$\binom{\Omega}{\mathbf{\Gamma}} = \bigcap_\infty(\Omega,\mathbf{\Gamma}) \Rightarrow \bigcap_\infty(\Omega,\mathbf{\Omega}) \Rightarrow \bigcap_\infty(\Omega,\boldsymbol{\mathcal{O}}) \Rightarrow \binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty} \tag{1.2}$$

Question 1.1. Assuming CH, what can be said about the reverse implications in (1.2)?

Below is an itemized list of our main research results. Any undefined notation and terminology will be discussed in later chapters.

- The properties $\binom{\Omega}{\mathbf{\Omega}_\infty}$ and $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$ are F_σ-hereditary, meager-additive, and linearly σ-additive.

- It is consistent that spaces satisfying $\binom{\Omega}{\mathbf{\Omega}_\infty}$ and $\binom{\Omega}{\mathcal{O}_\infty}$ are countable while every set of reals of cardinality $\aleph_1$ has strong measure zero.

- The $\binom{\Omega}{\mathcal{O}_\infty}$ property is countably-additive.

- Assuming $\diamond_{\omega_1}$, there exists $X \subseteq \mathbb{R}$ which satisfies $\binom{\Omega}{\mathcal{O}_\infty}$ but isn't a γ-set.

- A space X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$ if and only if X^n satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$ for all $n < \omega$. This will positively answer a question asked by Tsaban in [29].

- Spaces satisfying $\binom{\Omega}{\mathbf{\Gamma}_\infty}$, $\binom{\Omega}{\mathbf{\Omega}_\infty}$, and $\binom{\Omega}{\mathcal{O}_\infty}$ can be characterized using filters on ω.

- Spaces satisfying $\binom{\Omega}{\mathbf{\Omega}_\infty}$ under finite powers can be characterized using filters and countable π-bases of order n.

- The union of two "unbounded tower spaces" of cardinality $\mathfrak{p}$ is a γ-set.

In chapter 2, we will survey known properties of γ-sets due to F. Galvin and A.W. Miller in [8] and determine which of these properties hold for the other selection principles in (1.1). Results and open questions concerning additivity, meager-additivity and countability will be discussed. In chapter 3, we will provide characterizations for the selection principles in (1.1) using filters on ω in order to prove linear σ-additivity and characterize spaces satisfying $\binom{\Omega}{\mathbf{\Omega}_\infty}$. In chapter 4, we show that the family of γ-sets constructed by T. Orenshtein and B. Tsaban from $\mathfrak{t} = \mathfrak{b}$ is closed under finite unions. In chapter 5, known results about ω-shrinkable open ω-covers and the Pytkeev property will be utilized to make new observations and open questions about the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle. In chapter 6, we will focus on filters on ω by revisiting properties from the previous chapters and formulating some open questions raised by our work.

CHAPTER 2: CONSEQUENCES OF F. GALVIN'S AND A.W. MILLER'S RESULTS

2.1: A Few Properties

We will first outline a couple properties of γ-sets due to Galvin and Miller in [8] and prove which properties hold when considering the $\binom{\Omega}{\mathbf{\Omega}_\infty}$ and $\binom{\Omega}{\mathcal{O}_\infty}$ selection principles. Galvin and Miller investigated when the property of being a γ-set is preserved. They proved γ-sets are preserved when intersected with F_σ-sets. First, we remind the reader of the relevant definition.

Definition 2.1.

(a) The subset $A \subseteq X$ is said to be an F_σ*-set* if it can be expressed as a countable union of closed sets of X, that is, if there exist closed sets $F_1, F_2, \ldots$ in X such that

$$A = \bigcup_{k=1}^{\infty} F_k.$$

(b) The subset $B \subseteq X$ is said to be a G_δ*-set* if it can be expressed as a countable intersection of open sets of X, that is, if there exist open sets $G_1, G_2, \ldots$ in X such that

$$B = \bigcap_{k=1}^{\infty} G_k.$$

Theorem 2.1. [8] Suppose X is a γ-set and Y is a F_σ subset of X. Then, $X \cap Y$ is a γ-set.

As $\binom{\Omega}{\mathbf{\Gamma}} = \binom{\Omega}{\mathbf{\Gamma}_\infty}$, we have the following.

Corollary 2.1. Suppose X satisfies $\binom{\Omega}{\mathbf{\Gamma}_\infty}$ and Y is a F_σ-set. Then, $X \cap Y$ satisfies $\binom{\Omega}{\mathbf{\Gamma}_\infty}$.

We now show spaces satisfying $\binom{\Omega}{\mathbf{\Omega}_\infty}$ or $\binom{\Omega}{\mathcal{O}_\infty}$ have this property.

Theorem 2.2. Suppose X satisfies $\binom{\Omega}{\mathcal{O}_\infty}$ and Y is an F_σ-subset of X. Then, Y satisfies $\binom{\Omega}{\mathcal{O}_\infty}$.

Proof. Let $Y = \bigcup_{n<\omega} Y_n$ with Y_n closed and $Y_n \subseteq Y_{n+1}$. Suppose $\mathcal{U}$ is an open ω-cover of Y. Consider

$$\mathcal{V} = \{D \cup (X \setminus Y_n) : D \in \mathcal{U} \text{ and } n < \omega\}.$$

$\mathcal{U}$ is an open ω-cover of X. Therefore, there exists infinite $\mathcal{V}_n \subseteq \mathcal{U}$ such that $\{\bigcap_n \mathcal{V}_n : n < \omega\}$ covers X. The proof follows as $Y \subseteq X$. ■

The proof of this result for spaces satisfying $\binom{\Omega}{\mathbf{\Omega}_\infty}$ is similar and uses the following observation. Note that if $Y \subseteq X$, where X and Y are nonempty sets, then any ω-cover of X is an ω-cover of Y. Let $\mathcal{U} = \{U_n : n < \omega\}$ be an ω-cover of X. By definition, $X \notin \mathcal{U}$, $X = \bigcup_n U_n$, and each finite subset $F \subseteq X$ is contained in some $U_n \in \mathcal{U}$ As $Y \subseteq X$, $\mathcal{U}$ covers Y. Furthermore, any finite subset of Y is in X, and hence is contained in some $U_n \in \mathcal{U}$. This leads to the following result.

Theorem 2.3. Suppose X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$ and Y is an F_σ-subset of X. Then, Y satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$.

The next property of γ-sets proved by Galvin and Miller in [8] was motivated by results from [5] and [7] and focuses on sets of first category, also known as meager sets.

Definition 2.2.

(a) A subset Y of a space X is of *first category* in X, or is *meager* in X, if it can be written as the countable union of nowhere dense subsets of X.

(b) A subset N of a space X is *nowhere dense* in X if there is no open, nonempty set U of X in which N is dense, that is, for every nonempty open $U \subseteq X$, there is an open $V \subseteq U$ such that

$$N \cap V = \emptyset.$$

It was shown in [5] and [7] under the assumption of Martin's axiom that there is a set of reals X of cardinality $\mathfrak{c}$ with the property that the set $X + Y$ has first category for every set Y of first category. Sets satisfying this property are called meager-additive.

Definition 2.3. A set $X \subseteq \mathbb{R}$ is *meager-additive* if for each meager subset $M \subseteq \mathbb{R}$, $X + M$ is meager.

Galvin and Miller proved γ-set are meager-additive in [8]. We will prove a space satisfying $\binom{\Omega}{\mathcal{O}_\infty}$ is meager-additive. To show this, we need the following lemma and its proof from [8]. $[A]^{<\omega}$ will denote the set of all finite subsets of A.

Lemma 2.1. Suppose P is a compact nowhere dense set, $F \in [\mathbb{R}]^{<\omega}$, and I_i is a bounded interval for $i < n$. Then, there exists a finite union of intervals C with $F \subseteq C$ and intervals J_i with $J_i \subseteq I_i$ for $i < n$ such that

$$\bar{J}_i \cap (\bar{C} + P) = \emptyset.$$

Proof. Let C_i for $i < \omega$ be a decreasing finite union of intervals with $F \subseteq C_i$ and $\bigcap_{i<\omega} \overline{C_i} = F$. Recall F is finite, so $F + P$ will be closed and nowhere dense. Therefore, there exists intervals $J_i \subseteq I_i$ for $i < n$ with

$$\overline{J}_i \cap (F + P) = \emptyset.$$

As $(\bigcap_{m<\omega} \overline{C}_m) + P = \bigcap_{m<\omega}(\overline{C}_m + P)$, it follows by compactness that there exists m

for every $i < n$ such that

$$\overline{J}_i \cap (\overline{C}_m + P) = \emptyset. \qquad \blacksquare$$

We come now to our result.

Theorem 2.4. If X satisfies $\binom{\Omega}{\mathcal{O}_\infty}$, then X is meager-additive.

Proof. Let X satisfy $\binom{\Omega}{\mathcal{O}_\infty}$ and Y be a meager subset of $\mathbb{R}$. We will show $X + Y$ is meager. It suffices to prove $X + P$ is meager whenever P is a compact nowhere dense set. Consider the set $\{I_n : n < \omega\}$ intervals I_n with rational endpoints. By the previous lemma, let $\mathcal{O}_n$ be a family of open sets such that for all $C \in \mathcal{O}_n$, there exists an interval $J_m \subseteq I_m$ for $m < n$ such that

$$\overline{J}_m \cap (\overline{C} + P) = \emptyset,$$

and $\mathcal{O}_n$ covers the n-element subsets of X. Next, let $\{x_n : n < \omega\}$ be distinct elements of X and let

$$\mathcal{U} = \bigcup_n \{C \setminus \{x_n\} : C \in \mathcal{O}_n\}.$$

Then, $\mathcal{U}$ is an open ω-cover of X. Since X satisfies $\binom{\Omega}{\mathcal{O}_\infty}$, there exists infinite subsets $C_1, C_2, \ldots C_i$ of $\mathcal{U}$ such that $\{\bigcap_i C_i : i < \omega\}$ covers X. We may assume $C_i \in \mathcal{O}_{k_i}$, where the k_i are distinct. By construction, $\bigcap_i \overline{C}_i + P$ is nowhere dense. Therefore, $X + P$ can be written as the union of countably many nowhere dense sets. $\blacksquare$

2.2: The γ-Borel Conjecture

It is known that every γ-set has strong measure zero. In Chapter 1, we briefly mentioned that Borel introduced the concept of strong measure zero sets of real numbers in [3]. He conjectured that only countable sets of real numbers have strong measure zero. This became known as the Borel conjecture. Laver later proved in [14] the consistency of the Borel conjecture in 1976.

Miller [17] used Hechler forcing, $\mathbb{H}$ and Laver forcing $\mathbb{L}$ to prove it is consistent that every γ-set is countable while not every strong measure zero set is countable. In fact, every set of reals of cardinality $\aleph_1$ has strong measure zero in this model.

Miller's motivation behind this idea originated from P. Szeptycki's question of whether or not it was possible to have a weak Borel cojecture, stating that every γ-set is countable while the actual Borel conjecture is false. This became known as the γ-Borel conjecture. We will now show a similar result holds for spaces satisfying $\binom{\Omega}{\mathcal{O}_\infty}$.

Theorem 2.5. If $\mathbb{H}$ is iterated ω_2 times with finite support over a model of CH, then in the resulting model, every space satisfying $\binom{\Omega}{\mathcal{O}_\infty}$ is countable and every set of reals of cardinality $\aleph_1$ has strong measure zero.

We are going to need two facts from [17] about Hechler forcing and its iteration in order to prove theorem 2.5. First, we provide a brief definition and consequence of Hechler forcing.

Definition 2.4. Let $\mathbb{H} = \{(s, f) : s \in \omega^{<\omega}, f \in \omega^\omega \text{ and } s \subseteq f\}$, ordered by $(t, g) \leq (s, f)$ if $s \subseteq t$, let g dominate f everywhere, and let $f(i) \leq t(i)$ for all $i \in |t| \setminus |s|$. $\mathbb{H}$ is called *Hechler forcing.*

Defining $d = \bigcup\{s : (s, f) \in G \text{ for some } f \in \omega^\omega\}$ yields a dominating real, where $G \subseteq \mathbb{D}$ is a generic filter. A dominating real in a generic extension is a real $y \in \omega^\omega$ such that for all $f \in V \cap \omega^\omega$, $f(n) \leq y(n)$ for all n, where V denotes the von Neumann universe.

Let A^B denote the set of all functions f from A to B. $[\omega]^\omega$ denotes all the infinite subsets of ω. For $f \in \omega^\omega$, define $\mathcal{U}_f$ to be the following family of clopen subsets of 2^ω, the collection of functions from ω to $\{0, 1\}$,

$$\mathcal{U}_f = \{C_F : \exists n \;\; F \subseteq 2^{f(n)}, |F| \leq n\} \text{ where } C_F = \{x \in 2^\omega : x \restriction f(n) \in F\}.$$

Lemma 2.2. Suppose M is a model of set theory, f is $\mathbb{H}$-generic over M, and $X \subseteq 2^\omega$ is in M. Then

$$M[f] \models \text{“}\forall \mathcal{C} \in [\mathcal{U}_f]^\omega \quad \left|\bigcap C \cap X\right| \leq \omega.\text{”}$$

Definition 2.5. $(a_\alpha \in [\omega]^\omega : \alpha < \omega_1)$ is *eventually narrow* if and only if for every $b \in [\omega]^\omega$ there exists $\alpha < \omega_1$ so that $b \setminus a_\beta$ is infinite for all $\beta > \alpha$.

Lemma 2.3. Suppose N is a model of set theory and

$$N \models (a_\alpha \in [\omega]^\omega : \alpha < \omega_1) \text{ is eventually narrow.}$$

Then for any G_{ω_2} which is $\mathbb{H}_{\omega_2}$ generic over N, we have that

$$N[G_{\omega_2}] \models (a_\alpha \in [\omega]^\omega : \alpha < \omega_1) \text{ is eventually narrow.}$$

Proof of Theorem 2.5. Since any space X satisfying $\binom{\Omega}{\mathcal{O}_\infty}$ is zero-dimensional, as any γ-set is zero-dimensional, we only have to consider uncountable $Y \subseteq 2^\omega$. Suppose $Y \subseteq 2^\omega$ is uncountable. We claim Y does not satisfy $\binom{\Omega}{\mathcal{O}_\infty}$ in $M[G_{\omega_2}]$. Let $X \subseteq Y$ be a subset of size ω_1. Next, construct $g : \omega \to \omega$ so that for every $n < \omega$

$$|\{x \restriction g(n) : x \in X\}| > n.$$

By the usual ccc finite support arguments, find $\alpha > \omega_2$ so that $X, g \in M[G_\alpha]$. Let $h = h_\alpha$ be the next Hechler real added so that $h(n) \geq g(n)$ for all n. From the results above, the set $\bigcap C \cap X$ is countable for every infinite $C \subseteq \mathcal{U}_h$, i.e.

$$N = M[G_{\alpha+1}] \models \forall C \in [\mathcal{U}_h]^\omega \bigcap C \cap X \text{ is countable.}$$

As $h(n) > g(n)$, there is no $U \in \mathcal{U}_h$ which covers X. By definition, $\mathcal{U}_h$ is an ω-cover of 2^ω and hence Y.

We claim $\mathcal{U}_h$ does not contain an $\mathcal{O}_\infty$ cover. Since X is a subset of size ω_1, let

$X = \{X_\alpha : \alpha < \omega_1\}$ and $\mathcal{U}_h = \{U_n : n < \omega\}$. In the model $N = M[G_{\alpha+1}]$ define

$$a_\alpha = \{n < \omega : x_\alpha \in U_n\}.$$

Note that a_α is the set of n's for which x_α is in U_n. We now claim $N \models (a_\alpha \in [\omega]^\omega : \alpha < \omega_1)$ is eventually narrow. By definition, $(a_\alpha \in [\omega]^\omega : \alpha < \omega_1)$ is eventually narrow if and only if for any $b \in [\omega]^\omega$, there exists $\alpha < \omega_1$ so that $b \setminus a_\beta$ is infinite for every $\beta > \alpha$. If $b \setminus a_\alpha$ is finite for uncountably many $\alpha < \omega_1$, then for some infinite subsequence $c \subseteq b$, the set

$$Z = \{x_\alpha : c \subseteq a_\alpha\}$$

is uncountable. However, this implies Z is in every U_n for $n \in c$, that is, $Z \subseteq \bigcap\{U_n : n \in c\}$. This contradicts Lemma 2.2. Therefore, $N \models (a_\alpha \in [\omega]^\omega : \alpha < \omega_1)$ is eventually narrow.

The tail of a finite iteration of $\mathbb{H}$ is itself a finite support iteration of $\mathbb{H}$. By Lemma 2.3,

$$N[G_{[\alpha+2,\omega_2)}] = M[G_{\omega_2}]$$

models that $(a_\alpha : \alpha < \omega_1)$ is eventually narrow. We claim Y does not satisfy $\binom{\Omega}{\mathcal{O}_\infty}$. Suppose $\{\bigcap U_n \in \mathcal{U} : n \in b\}$ is an open cover of $X \subseteq Y$. This means any member of X is contained in every U_n. However, this would imply for some infinite $c \subseteq b$, $X \cap \bigcap\{U_n : n \in c\}$ is uncountable. Thus, $c \subseteq a_\alpha$ as any member of c is in a_α. However, $c \subseteq a_\alpha$ contradicts the fact that a_α is eventually narrow. Therefore, Y does not satisfy $\binom{\Omega}{\mathcal{O}_\infty}$. ■

T. Eisworth noted every set reals of cardinality $\aleph_1$ is in fact meager additive in Miller's model. This leads to the following.

Corollary 2.2. It is consistent that every space satisfying $\binom{\Omega}{\mathcal{O}_\infty}$ is countable, but every set of reals of size $\aleph_1$ is meager additive.

This shows that spaces satisfying $\binom{\Omega}{\mathcal{O}_\infty}$ are theoretically a much smaller class than meager additive sets, even though it is consistent that they coincide.

2.3: Results on Additivity

Unlike many other properties considered earlier, Galvin and Miller showed in [8] it is consistent that neither the union nor the product of two γ-sets is a γ-set. They utilized the following theorem from Todorčević and an extra result about F_σ and G_δ sets to prove this result.

Theorem 2.6. Assuming $\diamond_{\omega_1}$, there exists a γ-set of X of cardinality $\omega_1 = \mathfrak{c}$ all of whose subsets are also a γ-set.

The diamond principle $\diamond_{\omega_1}$ was used in this theorem to construct an Aronzajn tree. This principle is due to Jensen in [12].

Theorem 2.7. Suppose $A \subseteq X \subseteq [0,1]$ and $(X \setminus A) \cup (A+1)$ is a γ-set. Then A is G_δ and F_σ in X.

Corollary 2.3. If X and Y are γ-sets, it is consistent assuming $\diamond_{\omega_1}$ that neither $X \cup Y$ nor $X \times Y$ is necessarily a γ-set.

We now look at a generalization of Theorem 2.7 using γ^*-spaces.

Definition 2.6. We say that X is a γ^**-set* if for any open ω-cover $\mathcal{U}$ of X, there are infinite $\mathcal{U}_n \subseteq \mathcal{U}$ for $n < \omega$ such that

1. $\{\bigcap \mathcal{U}_n : n < \omega\}$ covers X, and
2. $m \neq n$ implies $\mathcal{U}_m \cap \mathcal{U}_n \neq \emptyset$.

Clearly a γ^*-set satisfies the $\binom{\Omega}{\mathcal{O}_\infty}$ covering property by the first condition. We claim γ-sets are γ^*-sets.

Proof of Claim. Suppose X is a γ-set, let $\mathcal{U}$ be an open ω-cover of X, and let $\mathcal{V} \subseteq \mathcal{U}$ be a countable γ-subcover of $\mathcal{U}$. Enumerate $\mathcal{V}$ bijectively as $\{V_n : n < \omega\}$.

For each $n < \omega$, let $\mathcal{U}_n = \{V_m : m > n\}$. The collection $\{\mathcal{U}_n : n < \omega\}$ satisfies the conditions of definition 2.6. ■

Therefore, γ^*-set are "in-between" γ-sets and spaces satisfying $\binom{\Omega}{\mathcal{O}_\infty}$, that is, $\gamma \Rightarrow \gamma^* \Rightarrow \binom{\Omega}{\mathcal{O}_\infty}$.

Notice that if we strengthen the second condition in the definition of γ^*-sets so that the collection $\{\mathcal{U}_n : n < \omega\}$ has the finite intersection property instead of pairwise non-empty intersections, then we end up with γ-sets. Choosing $\mathcal{V}$ to be a pseudo-intersection for countably many $\mathcal{U}_n$ yields a γ-subcover of $\mathcal{U}$. It isn't clear how γ^*-sets are related to the $\binom{\Omega}{\mathbf{\Omega}_\infty}$ covering property.

We now prove Theorem 2.7 for γ^*-spaces.

Theorem 2.8. Suppose $A \subseteq X \subseteq [0,1]$ and $(X \setminus A) \cup (A+1)$ is a γ^*-set. Then A is G_δ and F_σ in X.

Proof. Let $Y = (X \setminus A) \cup (A + 1)$, $F \subseteq Y$ be finite, and choose open subsets C_F and D_F in $[0,1]$ with disjoint closures such that $F \subseteq C_F \cup (D_F + 1)$. Then, $\mathcal{U} = C_F \cup (D_F + 1)$ is an open ω-cover of Y.

Since Y is a γ^*-set, there exists finite F_n^m subsets of Y and infinite $\mathcal{U}_n = \{C_{F_n^m} \cup (D_{F_n^m} + 1) : n < \omega\} \subseteq \mathcal{U}$ for $m < \omega$ such that

1. $\left\{\bigcap(C_{F_n^m} \cup (D_{F_n^m} + 1)) : n < \omega\right\}$ covers X, and
2. $k \neq n$ implies $\mathcal{U}_k \cap \mathcal{U}_n \neq \emptyset$.

We claim $X \setminus A = \bigcup_{m<\omega} \bigcap_{n<\omega} \overline{C}_{F_n^m}$. Suppose $x \in X \setminus A$. Then choose $m > \omega$ such that $x \in \bigcap_{n<\omega}(C_{F_n^m} \cup (D_{F_n^m} + 1))$. As x is not a member of any $D_{F_n^m}$, it follows that $x \in \bigcap_{n<\omega} C_{F_n^m} \subseteq \bigcap_{n<\omega} \overline{C}_{F_n^m}$.

Conversely, suppose $x \in \bigcup_{m<\omega} \bigcap_{n<\omega} \overline{C}_{F_n^m}$. We want to show $x \in X \setminus A$. Suppose otherwise, that is, $x \in A$. Then, there exists $m^* < \omega$ such that $x+1 \in \bigcap_{n<\omega}(C_{F_n^{m^*}} \cup (D_{F_n^{m^*}}+1))$. By the second condition, there exists n and n^* such that $C_{F_n^m} \cup (D_{F_n^m} + 1) = C_{F_{n^*}^{m^*}} \cup (D_{F_{n^*}^{m^*}} + 1)$. This implies $x \in \overline{D}_{F_n^{m^*}}$ However, $x \in \overline{C}_{F_n^m}$ for all $n < \omega$. Thus, $x \notin A$. This claim shows A is G_δ in X. By a similar argument, $A = \bigcup_{m<\omega} \bigcap_{n<\omega} \overline{D}_{F_n^m}$, which implies A is F_σ in X. ■

This result leads to the existence two γ-spaces whose union is not a γ^*-space.

Corollary 2.4. If X and Y are γ-sets, it is consistent assuming $\diamond_{\omega_1}$ that neither $X \cup Y$ nor $X \times Y$ is necessarily a γ^*-set.

We will we now prove spaces satisfying $\binom{\Omega}{\mathcal{O}_\infty}$ are additive, that is, preserved under finite unions.

Theorem 2.9. Suppose X and Y both satisfy $\binom{\Omega}{\mathcal{O}_\infty}$. Then, $X \cup Y$ satisfies $\binom{\Omega}{\mathcal{O}_\infty}$.

Proof. Let $\mathcal{U} = \{U_i : i < \omega\}$ be an open ω-cover of $X \cup Y$. By definition, $\mathcal{U}$ is an ω-cover of both X and Y. Since X satisfies $\binom{\Omega}{\mathcal{O}_\infty}$, there are infinite $\mathcal{V}_1^x, \mathcal{V}_2^x, \ldots \subseteq \mathcal{U}$ such that $\{\bigcap \mathcal{V}_n^x : n < \omega\}$ covers X. Similarly, there are infinite $\mathcal{V}_1^y, \mathcal{V}_2^y, \ldots \subseteq \mathcal{U}$ such that $\{\bigcap \mathcal{V}_n^y : n < \omega\}$ covers Y. Therefore, $\{\bigcap \mathcal{V}_n^x : n < \omega\} \cup \{\bigcap \mathcal{V}_n^y : n < \omega\}$ covers $X \cup Y$. Thus, $X \cup Y$ satisfies $\binom{\Omega}{\mathcal{O}_\infty}$. ■

Note that the same proof establishes closure under countable unions. As a consequence of this theorem, we can show it is consistent that spaces satisfying $\binom{\Omega}{\mathcal{O}_\infty}$ need not be γ^* nor γ-sets assuming $\diamond_{\omega_1}$.

Corollary 2.5. $(\diamond_{\omega_1})$ There exists a space $X \subseteq \mathbb{R}$ satisfying $\binom{\Omega}{\mathcal{O}_\infty}$, but not $\binom{\Omega}{\Gamma}$.

We will end the chapter by presenting a list of open questions from the results mentioned in this section.

Question 2.1. If both spaces X and Y satisfy $\binom{\Omega}{\mathbf{\Omega}_\infty}$, does $X \cup Y$ satisfy $\binom{\Omega}{\mathbf{\Omega}_\infty}$?

Question 2.2. If X and Y are γ-sets, does $X \cup Y$ satisfy $\binom{\Omega}{\mathbf{\Omega}_\infty}$?

Question 2.3. If X and Y both satisfy the $\binom{\Omega}{\mathbf{\Omega}_\infty}$ selection principle, does $X \times Y$ satisfy the $\binom{\Omega}{\mathbf{\Omega}_\infty}$ selection principle?

We will later show that if X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$, then X^n satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$ for all $n < \omega$.

Question 2.4. If X and Y both satisfy the $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$ selection principle, does $X \times Y$ satisfy the $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$ selection principle? What can be said about X^n?

A positive answer to the first three questions, assuming CH, will prove the consistency of Conjecture 1.1.

CHAPTER 3: GENERAL FILTER CHARACTERIZATIONS

3.1: ω-covers and Footprint Filters

Basic results for filters on ω will allow us to prove new results concerning the selection principles mentioned in Conjecture 1.1.

Definition 3.1. A *filter* $\mathscr{F}$ on a nonempty set X is a nonempty collection of subsets of X such that

1. $\emptyset \notin \mathscr{F}$,
2. if $F_1, F_2 \in \mathscr{F}$, then $F_1 \cap F_2 \in \mathscr{F}$,
3. if $F \in \mathscr{F}$ and $F \subseteq A$, then $A \in \mathscr{F}$.

Recall the following basic definitions.

Definition 3.2.

(a) A *base* (*filter base*) for a filter $\mathscr{F}$ is a subfamily of $\mathscr{F}$ that contains subsets of all the sets in $\mathscr{F}$. In other words, a subfamily $\mathscr{B}$ of $\mathscr{F}$ is a base for $\mathscr{F}$ if for every set $F \in \mathscr{F}$, there is a set $B \in \mathscr{B}$ such that $B \subseteq F$.

(b) $\mathscr{B}$ is a *sub-base* for a filter $\mathscr{F}$ if $\mathscr{B} \subseteq \mathscr{F}$ and every element of $\mathscr{F}$ contains a finite intersection of sets from $\mathscr{B}$.

(c) Let $\mathscr{A}$ be a family of subsets of X. We say $\mathscr{A}$ satisfies the *finite intersection property* if every finite subfamily of $\mathscr{A}$ has a nonempty intersection.

Also recall the following properties.

Proposition 3.1.

1. If $\mathscr{B}$ is a sub-base for a filter $\mathscr{F}$, then the collection of all finite intersections of elements of $\mathscr{B}$ is a base for $\mathscr{F}$.

2. If $\mathscr{B}$ has the finite intersection property, then the collection of supersets of finite intersections from $\mathscr{B}$ is a filter with $\mathscr{B}$ as a sub-base. If $\mathscr{B}$ is closed under finite intersections, then $\mathscr{B}$ is a base for $\mathscr{F}$.

Notice that conditions 1 and 2 of Definition 3.1 imply that a filter on a set satisfies the finite intersection property. Conversely, any family of sets satisfying the finite intersection property generates a filter. Throughout this chapter, when generating a filter, we mean "as a sub-base."

There is a natural relation between filters on ω and ω-covers on a set X. Let X be an infinite set and let $\mathcal{U}$ be an ω-cover of X. By definition, $\mathcal{U}$ is countable and $X \notin \mathcal{U}$. As $\mathcal{U}$ is countable, we can enumerate it as $\langle U_n : n < \omega \rangle$. For $x \in X$, define:

$$A_x = \{n < \omega : x \in U_n\}.$$

Then, the family

$$\{A_x : x \in X\}$$

has the finite intersection property. To see this, consider a finite subfamily of $\{A_x : x \in X\}$, consisting of $A_{x_1}, A_{x_2}, \ldots, A_{x_k}$ for $x_1, x_2, \ldots x_k \in X$ and $k < n$. The intersection of this subfamily is the collection of $n < \omega$ such that every $x_1, x_2, \ldots x_k$ is in U_n. As $\mathcal{U}$ is an ω-cover of X, such an n exists. Therefore, the intersection of this subfamily is nonempty. As a result, $\mathscr{A} = \{A_x : x \in X\}$ generates a filter on ω.

$$\mathscr{F} = \left\{ Y \subseteq \omega : \bigcap_{i<n} A_{x_i} \subseteq Y \text{ for } n < \omega, A_{x_i} \in \mathscr{A}, \text{ and } x_i \in X \right\}.$$

Note that this depends on the enumeration of $\mathcal{U}$, but different enumerations will produce isomorphic filters.

Definition 3.3. Let $\mathcal{U} = \{U_n : n < \omega\}$ be a countable ω-cover of the space X. The *footprint filter* $\mathscr{F}(\mathcal{U})$ associated with $\mathcal{U}$ is the filter on ω generated by the collection $\mathscr{A} = \{A_x : x \in X\}$ (as a sub-base).

We are going to use these ideas to investigate the spaces under consideration in Conjecture 1.1. We will first need the notion of a pseudo-intersection of a family of sets.

Definition 3.4. The quasi-order $\subseteq^*$ on $\mathcal{P}(\omega)$ is defined by $A \subseteq^* B$ if $A \setminus B$ is finite . $A \subseteq \omega$ is a *pseudo-intersection* of a filter $\mathscr{F}$ if A is infinite and $A \subseteq^* F$ for each $F \in \mathscr{F}$.

γ-sets can be characterized using pseudo-intersections.

Theorem 3.1. The following are equivalent for a space X.

1. X is a γ-set.
2. for any open ω-cover $\mathcal{U}$ of X, the footprint filter $\mathscr{F}(\mathcal{U})$ has an infinite pseudo-intersection.

Proof. $(1 \Rightarrow 2)$. Suppose X is a γ-set. If $\mathcal{U}$ is a countable open ω-cover of X, then there is a subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is a γ-cover of X. Generate the footprint filter $\mathscr{F}(\mathcal{U})$ on ω from $\{A_x : x \in X\}$ where $A_x = \{n < \omega : x \in U_n\}$. We claim $\mathscr{F}(\mathcal{U})$ has a pseudo-intersection. Let $Y \in \mathscr{F}(\mathcal{U})$ and $B = \{m < \omega : U_m \in \mathcal{V}\}$. B is infinite, as $\mathcal{V}$ is infinite. For $U_n \in \mathcal{V}$, $C = \{n < \omega : x \notin U_n\}$ is finite as $\mathcal{V}$ is a γ-cover of X. Recall, $\bigcap_{i<k} A_{x_i}$ is the set of $n < \omega$ such that U_n contains every x_i for $i < k$. However, if $m \in C$, then $m \notin \bigcap_{i<k} A_{x_i}$. Therefore, $C \nsubseteq Y$, and $B \setminus Y$ is finite.

$(2 \Rightarrow 1)$. Let $\mathcal{U} = \{U_n : n < \omega\}$ be a countable open ω-cover of X. Suppose the footprint filter $\mathscr{F}(\mathcal{U})$ has an infinite pseudo-intersection $\mathcal{B}$. We claim $\{U_n : n \in \mathcal{B}\}$ is a γ-cover for X. For $x \in X$ and sufficiently large $n \in \mathcal{B}$, $n \in A_x$. Thus, $x \in U_n$ for sufficiently large n, which proves the claim. ■

The $\binom{\Omega}{\mathbf{\Omega}_\infty}$ and $\binom{\Omega}{\mathcal{O}_\infty}$ selection principles can be characterized with the notion of a π-base.

Definition 3.5. A *π-base* for $\mathcal{B} \subseteq [\omega]^\omega$ is a family $\mathscr{A} \subseteq [\omega]^\omega$ such that every set in $\mathcal{B}$ has a subset in $\mathscr{A}$.

We will be interested in case when $\mathcal{B}$ is a filter or filter base. The difference between a π-base and a pseudo-intersection for a filter $\mathscr{F}$ and a base for $\mathscr{F}$ is that a π-base need not be a subfamily of $\mathscr{F}$. In particular, a π-base for a filter need not have the finite intersection property.

Theorem 3.2. The following are equivalent for a space X.

1. X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$.

2. For any open ω-cover $\mathcal{U}$ of X, the footprint filter $\mathscr{F}(\mathcal{U})$ has a countable π-base.

Proof. ($1 \Rightarrow 2$). Suppose X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$ and $\mathcal{U} = \{U_n : n < \omega\}$ is an open ω-cover of X. By definition, there are infinite sets $\mathcal{V}_1, \mathcal{V}_2, \ldots \subseteq \mathcal{U}$ such that $\{\bigcap \mathcal{V}_n : n < \omega\}$ is an ω-cover of X. Consider the footprint filter $\mathscr{F}(\mathcal{U})$ generated by $\{A_x : x \in X\}$, where $A_x = \{n < \omega : x \in U_n\}$. We will show $\mathscr{F}(\mathcal{U})$ has a countable π-base. Let

$$B_n = \{m < \omega : U_m \in \mathcal{V}_n\}.$$

We claim there is an n such that $B_n \subseteq Y$ for any $Y \in \mathscr{F}(\mathcal{U})$.

B_n is infinite as each $\mathcal{V}_n$ is infinite. Let $Y \in \mathscr{F}(\mathcal{U})$. Then for $i < k < \omega$, there are $x_1, x_2, \ldots x_i \in X$ such that $\bigcap_{i<k} A_{x_i} \subseteq Y$ as $\{A_x : x \in X\}$ has the finite intersection property and generates $\mathscr{F}(\mathcal{U})$. It is given that $\{\bigcap \mathcal{V}_n : n < \omega\}$ is an ω-cover of X, so choose n such that $\{x_1, x_2, \ldots x_k\} \subseteq \bigcap \mathcal{V}_n$. Now, we claim $B_n \subseteq \bigcap_{i<k} A_{x_i}$. Recall, $\bigcap_{i<k} A_{x_i}$ is the set of $n < \omega$ such that U_n contains all the x_i for $i < k$. If $m \in B_n$, then $m < \omega$ such that $U_m \in \mathcal{V}_n$. However, $\{x_1, x_2, \ldots x_i\} \subseteq \bigcap \mathcal{V}_n$, so $x_i \in U_m$ for

$i < k$. Therefore, $m \in \bigcap_{i<k} A_{x_i}$, implying $B_n \subseteq \bigcap_{i<k} A_{x_i}$. Thus $B_n \subseteq Y$, and B_n is a countable π-base for $\mathscr{F}(\mathcal{U})$.

(2 $\Rightarrow$ 1). Let $\mathcal{U} = \{U_n : n < \omega\}$ be a countable open ω-cover of X and $\mathcal{B} = \{B_n : n < \omega\}$ be a countable π-base for the footprint filter $\mathscr{F}(\mathcal{U})$. Then, there exists an $n < \omega$ such that $B_n \subseteq Y$ for any $Y \in \mathscr{F}(\mathcal{U})$. Let $\{x_1, x_2, \dots x_k\}$, $k < n$, be a finite subset of X. It follows that $B_n \subseteq \bigcap_{i=1}^{k} A_{x_i}$. Thus, for $m \in B_n$, $x_i \in U_m$ for all $i \leq k$. Therefore, $\{x_1, x_2 \dots x_k\} \in \bigcap_{m \in B_n} U_m$. The collection $\langle \mathcal{U}_m : n < \omega \rangle$ where $\mathcal{U}_m = \{U_m : m \in B_n\}$ will let X satisfy $\binom{\Omega}{\mathbf{\Omega}_\infty}$. ■

Theorem 3.3. The following are equivalent for a space X.

1. X satisfies $\binom{\Omega}{\mathcal{O}_\infty}$.

2. for any open ω-cover $\mathcal{U}$ of X, the collection $\mathscr{A} = \{A_x : x \in X\}$ which generates the footprint filter $\mathscr{F}(\mathcal{U})$ has a countable π-base.

Proof. (1 $\Rightarrow$ 2). X satisfies $\binom{\Omega}{\mathcal{O}_\infty}$. Then for an open ω-cover $\mathcal{U} = \{U_n : n < \omega\}$ of X, there are infinite sets $\mathcal{V}_1, \mathcal{V}_2, \dots \subseteq \mathcal{U}$ such that $\{\bigcap \mathcal{V}_n : n < \omega\}$ is a cover of X. As before, the family $\mathscr{A} = \{A_x : x \in X\}$, where $A_x = \{n < \omega : x \in U_n\}$, generates $\mathscr{F}(\mathcal{U})$. We claim $\mathscr{A}$ has a countable π-base. Let

$$B_n = \{m < \omega : U_m \in \mathcal{V}_n\}.$$

B_n is infinite as each $\mathcal{V}_n$ is infinite. Suppose $m \in B_n$, then $m < \omega$ such that $U_m \in \mathcal{V}_n$. However, since $\{\bigcap \mathcal{V}_n : n < \omega\}$ covers X, $\{x_1, x_2, \dots x_i\} \subseteq \bigcap \mathcal{V}_n$ for $x_i \in X$ with $i < \omega$. Therefore, $x_i \in U_m$ for $i < \omega$, meaning $m \in A_{x_i}$. This implies $B_n \subseteq \mathscr{A}$, and hence $\mathscr{A}$ has a countable π-base.

(2 $\Rightarrow$ 1). Let $\mathcal{U} = \{U_n : n < \omega\}$ be a countable open ω-cover of X and let $\mathcal{B} = \{B_n : n < \omega\}$ be a countable π-base for $\mathscr{A}$. Then, for every $A \in \mathscr{A}$, there exists $n < \omega$ such that $B_n \subseteq A$. This implies for every $x \in X$, there is an $n < \omega$

such that $B_n \subseteq A_x$. Thus, $x \in U_m$ for every $m \in B_n$, that is, $x \in \bigcap_{m \in B_n} U_m$. Let $\mathcal{U}_n = \{U_m : m \in B_n\}$. The collection $\langle \mathcal{U}_n : n < \omega \rangle$ will let X satisfy $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$. ■

Note that if $X = \bigcup_i X_i$ is a countable union and each X_i has a countable π-base B_i, then $\bigcup_i B_i$ is a countable π-base for X. This leads us to the following.

Proposition 3.2. The property $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$ is countably-additive.

It is not known how the $\bigcap_\infty(\mathscr{A}, \mathscr{B})$ selection principle is associated with filters on ω.

Question 3.1. Is there a characterization of the $\bigcap_\infty(\Omega, \mathscr{B})$ selection principle for $\mathscr{B} \in \{\boldsymbol{\Gamma}, \boldsymbol{\Omega}, \boldsymbol{\mathcal{O}}\}$ using filters on ω?

3.2: Applications of Footprint Filters to $\binom{\Omega}{\boldsymbol{\Omega}_\infty}$-Spaces

We are going to use the ideas from Section 3.1 to answer some questions about spaces satisfying $\binom{\Omega}{\boldsymbol{\Omega}_\infty}$.

Definition 3.6. A property P is *linearly σ-additive* if it is preserved by countable increasing unions.

T. Orenshtein and Tsaban proved in [18] that the covering property $\Pi(\mathscr{A}, \boldsymbol{\Omega})$ is linearly σ-additive for all $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}, \mathsf{U}_{\text{fin}}\}$ and $\mathscr{A} \in \{\Gamma, \Omega, \mathcal{O}\}$. The classical selection principles $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$, $\mathsf{S}_1(\Gamma, \mathcal{O})$, $\mathsf{S}_1(\Gamma, \Gamma)$, $\mathsf{S}_{\text{fin}}(\mathcal{O}, \mathcal{O})$, and $\mathsf{U}_{\text{fin}}(\mathcal{O}, \Omega)$ are each linearly σ-additive. This was shown in [27].

The class of γ-sets were shown to be linearly σ-additive by F. Jordan [13] using results about Fréchet filters, and Orenshtein and Tsaban [18] proved the linear σ-additivity of the class of γ-sets directly from properties of selection principles. Thus, the $\binom{\Omega}{\boldsymbol{\Gamma}_\infty}$ covering property is linearly σ-additive. We already established the $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$ covering property is countably additive, implying it is linearly σ-additive. This leaves the open question of the behavior of the $\binom{\Omega}{\boldsymbol{\Omega}_\infty}$ covering property. We are able to prove this covering property is linearly σ-additive.

Theorem 3.4. The $\binom{\Omega}{\mathbf{\Omega}_\infty}$ covering property is linearly σ-additive.

Proof. Let $X = \bigcup_i X_i$, where $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ and each X_i satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$. Suppose $\mathcal{U}$ is a countable open ω-cover for X. We may assume $\mathcal{U}$ is an open ω-cover for each X_i. Then, there exists corresponding footprint filters $\mathscr{F}_i(\mathcal{U})$ on each X_i. By assumption, $\mathscr{F}_i(\mathcal{U})$ has a countable π-base $\mathcal{B}_i$. Therefore, $\mathcal{B}_i \subseteq Y_i$ for any $Y_i \in \mathscr{F}_i(\mathcal{U})$.

We claim the footprint filter $\mathscr{F}(\mathcal{U})$ on X will have a countable π-base. To see this, notice that $\mathscr{F}_i(\mathcal{U}) \subseteq \mathscr{F}_{i+1}(\mathcal{U})$ as the union is increasing. Thus, $\mathscr{F}(\mathcal{U}) = \bigcup \mathscr{F}_i(\mathcal{U})$. This is a filter as $\mathscr{F}_i(\mathcal{U}) \subseteq \mathscr{F}_{i+1}(\mathcal{U})$. We want to show there exists a countable $\mathcal{B} \subseteq [\omega]^\omega$ such that if $Y \in \mathscr{F}(\mathcal{U})$, then $B \subseteq Y$ for some $B \in \mathcal{B}$. Consider $\mathcal{B} = \bigcup_i \mathcal{B}_i$. If we assume $Y \in \mathscr{F}(\mathcal{U})$, then $Y \in \mathscr{F}_i(\mathcal{U})$ for some $i < \omega$. Therefore, there exists $\mathcal{B}_i \in \mathcal{B}$ such that $\mathcal{B}_i \subseteq Y$. ■

We will now show how the filter characterizations can be used to prove results about covering properties for the finite powers of a space. Tsaban proved the following in [29].

Proposition 3.3. If all finite powers of X satisfy $\binom{\Omega}{\mathcal{O}_\infty}$, then X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$.

It was asked by Tsaban in [29] if the converse of this implication is provable. We will provide a positive answer using filters.

Theorem 3.5. The following are equivalent for a separable metric space X.

1. X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$.
2. X^n satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$ for all $n < \omega$.
3. X^n satisfies $\binom{\Omega}{\mathcal{O}_\infty}$ for all $n < \omega$.

Proof. $(1 \Rightarrow 2)$. Assume X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$. We will show condition 2 holds for $X \times X$. Suppose $\mathcal{U}$ is a countable ω-cover of $X \times X$. We can assume $\mathcal{U} = \{U_n \times U_n : n < \omega\}$ where U_n is open in X.

We claim $\{U_n : n < \omega\}$ is an ω-cover of X. Let $\{x_i : i < k\}$, where $k < \omega$, be a finite subset of X. Consider a finite subset $\{< x_i, x_i >: i < k\} \subseteq X \times X$. As $\mathcal{U}$ is an ω-cover of $X \times X$, choose $n < \omega$ such that $\{< x_i, x_i >: i < k\} \subseteq U_n \times U_n$. Therefore, $\{x_i : i < k\} \subseteq U_n$ as required.

Let $\mathcal{U} = \{U_n : n < \omega\}$ be an ω-cover of X and consider the corresponding footprint filter $\mathscr{F}(\mathcal{U})$ on ω. As X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$, $\mathscr{F}(\mathcal{U})$ has a countable π-base. Let $\{B_n : n < \omega\}$ be a π-base for $\mathscr{F}(\mathcal{U})$. Recall, given $\{x_i : i < k\} \subseteq X$, there is an n such that $B_n \subseteq \bigcap_{i<k} A_{x_i}$. Therefore, $\{x_i : i < k\} \subseteq \bigcap_{m \in B_n} U_m$.

It follows that $\{B_n : n < \omega\}$ is a π-base for the footprint filter on $X \times X$. To see this, consider a finite subset $\{\langle x_i, y_i \rangle : i < k\} \subseteq X \times X$. Let $X_0 = \{x_i : i < k\} \cup \{y_i : i < k\} \subseteq X$. As B_n is a π-base for $\mathscr{F}(\mathcal{U})$, choose $n < \omega$ such that $X_0 \subseteq \bigcap_{m \in B_m} U_m$. Thus, $\{\langle x_i, y_i \rangle : i < k\} \subseteq X_0 \times X_0 \subseteq \bigcap_{m \in B_n} U_m \times U_m$.

Then, there is an n such that for all $m \in B_n$ and for all $i < k$ with $\langle x_i, y_i \rangle \in U_m \times U_m$, $B_n \subseteq \bigcap_{i<k} A_{\langle x_i, y_i \rangle}$. Therefore, B_n is a π-base for the footprint filter on $X \times X$. Using induction on n yields the desired result.

$(2 \Rightarrow 3)$. This holds as $\binom{\Omega}{\mathbf{\Omega}_\infty} \Rightarrow \binom{\Omega}{\mathcal{O}_\infty}$.

$(3 \Rightarrow 1)$. [29] ■

It is natural to ask for a characterization of those spaces X for which X^n satisfies $\binom{\Omega}{\mathcal{O}_\infty}$, where $n < \omega$ is fixed. To do this, we need to introduce the concept of a π-base of order n.

Definition 3.7. Let $\mathcal{A}$ be a centered subset of $[\omega]^\omega$, that is, $\mathcal{A}$ has the finite intersection property. A set $\mathcal{B} \subseteq [\omega]^\omega$ is a *π-base of order n* for $\mathcal{A}$ if for any n elements $\{A_i : i < n\}$, there is a set $B \in \mathcal{B}$ such that

$$B \subseteq \bigcap_{i<n} A_i.$$

Now we come to the promised characterization.

Theorem 3.6. The following properties are equivalent for a separable metric space X and $n < \omega$.

1. X^n satisfies $\binom{\Omega}{\mathcal{O}_\infty}$.

2. For any countable open ω-cover $\mathcal{U}$ of X, there are infinite $\mathcal{U}_m \subseteq \mathcal{U}$ for $m < \omega$ such that whenever F is an n-element subset of X, there is an m such that $F \subseteq \bigcap \mathcal{U}_m$.

3. For any countable open ω-cover $\mathcal{U}$ of X, $\{A_x : x \in X\}$ has a countable π-base of order n.

Proof. $(3 \Rightarrow 2)$. Let $\mathcal{U}$ be a countable open cover of X and suppose the collection $\{A_x : x \in X\}$, where $A_x = \{n < \omega : x \in U_n\}$, has a countable π-base of order n. This implies there is an infinite set $\mathcal{B} \subseteq [\omega]^\omega$ such that for any n elements $\{A_{x_i} : i < n\}$ of A_x, there is a $B \in \mathcal{B}$ such that

$$B \subseteq \bigcap_{i<n} A_{x_i}.$$

Recall, $\bigcap_{i<n} A_{x_i}$ is the set of $n < \omega$ such that U_n contains every x_i for $i < n$. Therefore, if $m \in B$, then $\bigcap_{m \in B} U_m$ contains the finite subset $\{x_i : i < n\}$ of X.

$(2 \Rightarrow 1)$. Let $\mathcal{U}$ be a countable ω-cover of X^n. We may assume $\mathcal{U} = \{\mathcal{U}_k \times \mathcal{U}_k \times \cdots \times \mathcal{U}_k : k < \omega\}$, taking the product n times, where each $\mathcal{U}_k$ is open in X. By condition 2, there are infinite $\mathcal{U}_m \subseteq \mathcal{U}_k$ for $m < \omega$ such that whenever F is an j-element subset of X with $j < \omega$, there is an m such that $F \subseteq \bigcap \mathcal{U}_m$. Therefore, $\bigcap \mathcal{U}_m$ is an ω-cover of X, implying each X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$. By the previous lemma, this is equivalent to X^n satisfying $\binom{\Omega}{\mathcal{O}_\infty}$.

$(1 \Rightarrow 3)$. Suppose X^n satisfies $\binom{\Omega}{\mathcal{O}_\infty}$. Let $\mathcal{U} = \{U_i : i < \omega\}$ be a countable ω-cover of X. The collection $\{U_i^n : i < \omega\}$ is an ω-cover of X^n. To see this, let $\{\overline{a}_j : j < k\}$ be a finite subset of X^n, where $\overline{a}_j = \{a_t^j : t < n\}$ and $k < \omega$. Since $\mathcal{U}$ is an ω-cover of

X, there is an $i < \omega$ such that

$$\{a_t^j : t < n, j < k\} \subseteq U_i.$$

Note that, U_i^n contains each $\overline{a}_j$.

The collection of sets $\{A_{\overline{x}} : \overline{x} \in X\}$ has a countable π-base $\{B_n : n < \omega\}$ by the initial assumption on X^n using filters. Given finite subset $\{x_t : t < n\}$ of X, we need to show there is an i such that

$$B_i \subseteq \bigcap_{i<n} A_{x_t}.$$

Let $\overline{x} = \langle x_t : t < n \rangle \in X^n$. As X^n has a π-base, choose i such that

$$B_i \subseteq A_{\overline{x}}.$$

Suppose $m \in B_i$. We need to show $m \in \bigcap_{i<n} A_{x_t}$, meaning every x_t is in U_m. This automatically follows as $\overline{x} \in U_m^n = U_m \times \cdots \times U_m$. ∎

CHAPTER 4: CONSTRUCTING γ-SETS

We saw in Chapter 2 that there is an example of a space satisfying $\binom{\Omega}{\mathcal{O}_\infty}$ that isn't a γ-space if we assume $\diamond_{\omega_1}$. It is natural to ask if we can weaken this assumption somewhat. We know that a "real" example of this phenomenon does not exist. The next step is to try to construct a counterexample from assumptions about cardinal characteristics of the continuum.

4.1: Cardinal Characteristics

We first recall a few cardinal characteristics of the continuum. In particular, $\mathfrak{p}$, $\mathfrak{b}$, and $\mathfrak{t}$.

Definition 4.1. A family of countable sets had the *strong finite intersection property* if every nonempty finite subfamily has an infinite intersection. The cardinal characteristic $\mathfrak{p}$ is the smallest cardinality of any family $\mathscr{F} \subseteq [\omega]^\omega$ which has the strong finite intersection property, but does not have a pseudo-intersection.

Notice that this means whenever $X \subseteq [\omega]^\omega$ has the strong finite intersection property and $|X| < \mathfrak{p}$, then X has a pseudo-intersection.

Definition 4.2.

(a) $\leq^*$ is the quasi-ordering defined on ω^ω by $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n < \omega$.

(b) $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if the set of all increasing enumerations of elements of $\mathcal{B}$ is unbounded in ω^ω with respect to $\leq^*$.

(c) $\mathfrak{b}$ is the minimal cardinality of a $\leq^*$-unbounded subset of ω^ω.

Definition 4.3. A *tower* of cardinality κ is a set $T \subseteq [\omega]^\omega$ which can be enumerated bijectively as $\{x_a : \alpha < \kappa\}$, such that for all $\alpha < \beta < \kappa$, $x_\beta \subseteq^* x_\alpha$. The cardinal characteristic $\mathfrak{t}$ is the smallest cardinality of a tower which has no pseudo-intersection.

It is known that $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{c}$. In addition, M. Malliaris and S. Shelah proved $\mathfrak{p} = \mathfrak{t}$ in [15].

4.2: γ-Sets and Unbounded Towers

Let P be a nontrivial property of the set of reals. The *critical cardinality* of P, denoted $\mathrm{non}(P)$, is the minimal cardinality of a set of reals not satisfying P. It is naturally asked whether or not there is a set of reals of cardinality at least $\mathrm{non}(P)$ which satisfies P, that is, a nontrivial example. It is known that $\mathrm{non}\left(\binom{\Omega}{\mathbf{\Gamma}}\right) = \mathfrak{p}$ [8]. In other words, if $|X| < \mathfrak{p}$, then X is a γ-set.

Observe that if we want to construct a space satisfying $\binom{\Omega}{\mathcal{O}_\infty}$ that isn't a γ-set, then we need to look at spaces of cardinality of at least $\mathfrak{p}$. We will turn our attention to γ-sets.

Weakening the relations of the cardinal characteristics of the continuum has lead to the existence of γ-sets. Galvin and Miller proved in [8] that $\mathfrak{p} = \mathfrak{c}$ implies the existence of a γ-set of cardinality $\mathfrak{p}$. We saw in the Hechler model from Miller, see [17], that there are no uncountable γ-sets. In this model, $\aleph_1 = \mathfrak{p} = \mathfrak{t} < \mathfrak{b}$. Thus, $\mathfrak{p} > \aleph_1$ implies the existence of uncountable γ-sets. Orenshtein and Tsaban constructed a γ-set assuming $\mathfrak{p} = \mathfrak{t}$ in [18]. We will introduce their construction, prove slight modifications, and present new results.

Orenshtein and Tsaban considered the Cantor space equipped with the product topology. The Cantor space $\{0,1\}^\omega$ is the space of all infinite sequences of 0's and 1's. The Cantor space can also be identified with $\mathcal{P}(\omega)$ using characteristic functions. This defines the topology of $\mathcal{P}(\omega)$. The partition $\mathcal{P}(\omega) = [\omega]^\omega \cup [\omega]^{<\omega}$, into the infinite and the finite sets, respectively, will be considered. These spaces are homeomorphic to sets of reals.

The γ-set in question was constructed from an unbounded tower of cardinality $\mathfrak{p}$.

Definition 4.4. An *unbounded tower* of cardinality κ is an unbounded set $T \subseteq [\omega]^\omega$

which is a tower of cardinality κ.

The following results were used by Orenshtein and Tsaban in [18] to construct a set of reals that is a γ-set. We present the proofs for the first two lemmas.

Lemma 4.1. (Folklore). If $B \subseteq [\omega]^\omega$ is unbounded, then for each increasing $f \in \omega^\omega$, there is an $x \in B$ such that $x \cap (f(n), f(n+1)) = \emptyset$ for infinitely many n.

Proof. Assume otherwise, that is, there is an increasing $f \in \omega^\omega$ such that for every $x \in B$, $x \cap (f(n), f(n+1)) \neq \emptyset$. Let g dominate all the functions $f_m(n) = f(n+m)$ for $m < \omega$. This means $f(n+m) \leq g(n)$ for all but finitely many n. Then, for each $x \in B$, $x \leq^* g$. To see why this is true, let m be such that for all $n \geq m$,

$$x \cap (f(n), f(n+1)) \neq \emptyset.$$

We can choose such an m by assumption. For each n, the nth element of x is smaller than $f_{m+1}(n) = f(n+m+1) = f((n+1)+m)$, as $f(x) \leq x \leq f(n+1)$. Thus, x is smaller than $f_{m+1}(n)$ which is dominated by g. This implies B is not unbounded, leading to a contradiction. ■.

Lemma 4.2. $\mathfrak{t} = \mathfrak{b}$ if and only if there is an unbounded tower of cardinality $\mathfrak{t}$.

Proof. ($\Rightarrow$). Let $\mathfrak{t} = \mathfrak{b}$. We will construct an unbounded tower x_α by induction on α and show it has cardinality $\mathfrak{t}$.

Let $\{b_\alpha : \alpha < \mathfrak{b}\} \subseteq [\omega]^\omega$ be an unbounded tower of cardinality $\mathfrak{b}$ by the inductive hypothesis. At step α, let a be a pseudo-intersection of $\{x_\beta : \beta < \alpha\}$. Choose the pseudo-intersection of this set as $\mathfrak{b} = \mathfrak{t}$ and anything smaller than α has a pseudo-intersection. Furthermore, take $x_\alpha \subseteq a$, which is in the pseudo-intersection of x_β, such that the increasing enumeration of x_α dominates b_α.

($\Leftarrow$). Suppose there is an unbounded tower T of cardinality $\mathfrak{t}$. It is known that $\mathfrak{t} \leq \mathfrak{b}$. Since T is unbounded, $|T| \geq \mathfrak{b}$. Furthermore, by assumption, $|T| = \mathfrak{t}$. Therefore $\mathfrak{t} \leq \mathfrak{b} \leq |T| = \mathfrak{t}$, implying $\mathfrak{t} = \mathfrak{b}$.

Lemma 4.3. Suppose $[\omega]^{<\omega} \subseteq X \subseteq \mathcal{P}(\omega)$ and $\mathcal{U}$ is an ω-cover of X. Then, there are $m_1 < m_2 < \ldots$ and distinct $U_1, U_2, \ldots \in \mathcal{U}$ such that $\{U_n : n < \omega\}$ is a γ-cover of X and for each $x \subseteq \omega$, if $x \cap (m_n, m_{n+1}) = \emptyset$ then $x \in U_n$.

Corollary 4.1. Suppose $[\omega]^{<\omega} \subseteq X \subseteq \mathcal{P}(\omega)$ and X satisfies $\binom{\Omega}{\mathbf{\Gamma}}$. Then for each ω-cover $\mathcal{U}$ of X, there are $m_1 < m_2 < \ldots$ and distinct $U_1, U_2, \ldots \in \mathcal{U}$ such that $\{U_n : n < \omega\}$ is a γ-cover of X and for each $x \subseteq \omega$, if $x \cap (m_n, m_{n+1}) = \emptyset$ then $x \in U_n$.

Orenshtein's and Tsaban's main result is as follows. Recall, $\mathsf{S}_1(\Omega, \mathbf{\Gamma}) = \binom{\Omega}{\mathbf{\Gamma}}$.

Theorem 4.1. If $T \subseteq [\omega]^{\omega}$ is an unbounded tower of cardinality $\mathfrak{p}$, then $T \cup [\omega]^{<\omega}$ satisfies $\mathsf{S}_1(\Omega, \mathbf{\Gamma})$.

We will construct a γ-set by modifying the initial assumption of Theorem 4.1 and modifying Lemma 4.1.

Lemma 4.4. Suppose $T = \{x_\alpha : \alpha < \kappa\} \subseteq [\omega]^{\omega}$ is an unbounded tower. Then for every increasing $f : \omega \to \omega$ and for all sufficiently large $\alpha < \kappa$, it follows that $x_\alpha \cap (f(n), f(n+1)) = \emptyset$ for infinitely many n.

Theorem 4.2. Let $T = \{x_\alpha : \alpha < \kappa\} \subseteq [\omega]^{\omega}$, where κ is regular, T is an unbounded tower, and $\{x_\beta : \beta < \alpha\}$ is a γ-set for each $\alpha < \kappa$. Then, $T \cup [\omega]^{<\omega}$ is a γ-set.

Proof. Let $T = \{x_\alpha : \alpha < \kappa\}$ such that κ is regular and $\{x_\beta : \beta < \alpha\}$ is a γ-set for each $\alpha < \kappa$. For each α, let $X_\alpha = \{x_\beta : \beta < \alpha\} \cup [\omega]^{<\omega}$. Let $\mathcal{U}$ be a ω-cover of $T \cup [\omega]^{<\omega}$. We will show $\mathcal{U}$ contains a γ-subcover of $T \cup [\omega]^{<\omega}$.

We may assume $\mathcal{U}$ is countable. As κ is regular, there is an $\alpha_1 < \kappa$ such that X_{α_1} is not contained in any member of $\mathcal{U}$.

By assumption, X_{α_1} is a γ-set. By Lemma 4.3, since $\mathcal{U}$ is an γ-cover of X_{α_1}, there are $m_1^1 < m_2^1 < \ldots$ and distinct $U_1^1, U_2^1, \ldots \in \mathcal{U}$ such that $\{U_n^1 : n < \omega\}$ is a γ-cover of X_{α_1}, and for each $x \in \mathcal{P}(\omega)$, if $x \cap (m_n^1, m_{n+1}^1) = \emptyset$, then $x \in U_n^1$. Let $D_1 = \omega$.

The set $\{x_\alpha : \alpha_1 < \alpha < \kappa\}$ is unbounded. By Lemma 4.1, there is $\alpha_2 > \alpha_1$ such that $D_2 = \{n : x_{\alpha_2} \cap (m^1_n, m^1_{n+1}) = \emptyset\}$ is infinite. By our initial assumption, X_{α_2} is a γ-set. As $\mathcal{U}$ is an ω-cover of X_{α_2}, Corollary 4.1 implies there are $m^2_1 < m^2_2 < \dots$ and distinct $U^2_1, U^2_2, \dots \in \mathcal{U}$ such that $\{U^2_n : n < \omega\}$ is a γ-cover of X_{α_2}, and for each $x \in \mathcal{P}(\omega)$, if $x \cap (m^2_n, m^2_{n+1}) = \emptyset$, then $x \in U^2_n$. Furthermore, $\{U^2_n : n \in D_2\}$ is a γ-cover of X_{α_2} since D_2 is infinite.

Continuing in the same manner, define for each $k > 1$, objects α_k, D_k, and $\{U^k_i : i < \omega\}$ such that:

(1) $\alpha_k > \alpha_{k-1}$;

(2) $D_k = \{n : x_{\alpha_k} \cap (m^{k-1}_n, m^{k-1}_{n+1}) = \emptyset\}$ is infinite;

(3) $m^k_1 < m^k_2 < \dots$;

(4) $U^k_1, U^l_2, \dots \in \mathcal{U}$ are distinct;

(5) $\{U^k_n : n \in D_k\}$ is a γ-cover of X_{α_k};

(6) for each $x \in \mathcal{P}(\omega)$, if $x \cap (m^k_n, m^k_{n+1}) = \emptyset$, then $x \in U^k_n$.

Let $\alpha = \sup_k \alpha_k$. Note that, $X_\alpha = \bigcup_k X_{\alpha_k}$ is a countable increasing union. By our initial assumption, since X_α is a γ-set, it also satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. Therefore, by Lemma 2.9 in [18], there are infinite $I_1 \subseteq D_1$, $I_2 \subseteq D_2$, $\dots$ such that each $x \in X_\alpha$ belongs to $\bigcap_{n \in I_k} U^k_n$ for all but finitely many $k < \omega$.

Take $n_1 \in I_2$. For $k > 1$ take $n_k \in I_{k+1}$ such that

- $m^k_{n_k} > m^{k-1}_{n_{k-1}+1}$,
- $x_\alpha \cap (m^k_{n_k}, m^k_{n_k+1}) \subseteq x_{\alpha_{k+1}} \cap (m^k_{n_k}, m^k_{n_k+1})$, and
- $U^k_{n_k} \notin \{U^1_n, \dots U^{k-1}_{n_{k-1}}\}$.

We claim $\{U^k_{n_k} : k < \omega\}$ is a γ-cover of $T \cup [\omega]^{<\omega}$. As $\{U^k_{n_k} : k < \omega\}$ is already a γ-cover of X_α, it remains to show that for each $x \subseteq^* x_\alpha$, $x \in U^k_{n_k}$ for all but finitely many k. For each large enough k, $m^k_{n_k}$ is large enough so that

$$\begin{aligned} x_\alpha \cap (m^k_{n_k}, m^k_{n_k+1}) &\subseteq x_\alpha \cap (m^k_{n_k}, m^k_{n_k+1}) \\ &\subseteq x_{\alpha_{k+1}} \cap (m^k_{n_k}, m^k_{n_k+1}) \\ &= \emptyset. \end{aligned}$$

since $n_k \in D_{k+1}$. Thus, $x \in U^k_{n_k}$. ■

It is not known what will happen if the initial segments satisfy $\binom{\Omega}{\mathcal{O}_\infty}$.

Question 4.1. Let $T = \{x_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$, where κ is regular, T is an unbounded tower, and $\{x_\beta : \beta < \alpha\}$ satisfies $\binom{\Omega}{\mathcal{O}_\infty}$ for each $\alpha < \kappa$. What can be said about $T \cup [\omega]^{<\omega}$?

Question 4.2. Can unbounded towers be used to construct a set X satisfying $\binom{\Omega}{\mathbf{\Omega}_\infty}$ or $\binom{\Omega}{\mathcal{O}_\infty}$?

We have noted earlier that $\diamond_{\omega_1}$ implies the existence of two γ-sets whose union is not a γ-set. $\mathfrak{p} = \mathfrak{b}$ is currently the weakest assumption known to produce γ-sets. We will now show that the union of two unbounded towers also produces a γ-set.

Proposition 4.1. Let T_0 and T_1 be unbounded towers of cardinality $\mathfrak{p}$. Then, $T_0 \cup T_1 \cup [\omega]^{<\omega}$ is a γ-set.

Proof. Let T_i for $i \in \{0, 1\}$ be an unbounded tower of cardinality $\mathfrak{p}$. This is equivalent to $\mathfrak{p} = \mathfrak{b}$. Let

$$T_i = \{x^i_\alpha : \alpha < \mathfrak{b}\}$$

be an unbounded tower for $i \in \{0, 1\}$. Furthermore, for each α, let

$$X_\alpha = \{x^0_\beta : \beta < \alpha\} \cup \{x^1_\beta : \beta < \alpha\} \cup [\omega]^{<\omega}.$$

As $|X_\alpha| < \mathfrak{p}$, X_α is a γ-set for each α. Let $\mathcal{U}$ be a countable ω-cover of $T_0 \cup T_1 \cup [\omega]^{<\omega}$. We need to show $\mathcal{U}$ has a γ-subcover.

Choose α_1 such that X_{α_1} is not is not contained in any member of $\mathcal{U}$. This will guarantee that $\mathcal{U}$ is a ω-cover for each $\alpha \geq \alpha_1$ as $\mathcal{U}$ contains only finitely many subsets of X_α.

Now, there are $m_1^1 < m_2^1 < \cdots$ and distinct $U_1^1, U_2^1, \ldots \in \mathcal{U}$ such that $\{U_n^1 : n < \omega\}$ is a γ-cover of X_{α_1} and $x \cap (m_n^1, m_{n+1}^1) = \emptyset$ implies $x \in U_n^1$. We will show there is $\alpha_2 > \alpha_1$ such that

$$\{n < \omega : x_{\alpha_2}^0 \cap (m_n^1, m_{n+1}^1) = \emptyset \text{ and } x_{\alpha_2}^1 \cap (m_n^1, m_{n+1}^1) = \emptyset\}$$

is infinite.

Since T_0 is unbounded, choose $\beta > \alpha_1$ such that

$$\{n : x_\beta^0 \cap (m_n^1, m_{n+1}^1) = \emptyset\}$$

is infinite. Next, define g such that for each $m < \omega$, there is an n such that $g(m) < m_n^1 < m_{n+1}^1 < g(m+1)$ and

$$x_\beta^0 \cap (m_n^1, m_{n+1}^1) = \emptyset.$$

Since g is increasing and T_1 is unbounded, we can choose $\alpha_2 > \beta$ such that

$$x_{\alpha_2}^1 \cap (g(m), g(m+1)) = \emptyset$$

for infinitely many m. Suppose $x_{\alpha_2}^1 \cap (g(m), g(m+1)) = \emptyset$ and $x_{\alpha_2}^0 \setminus x_\beta^0 \subseteq g(m)$. Furthermore, choose n such that $(m_n^1, m_{n+1}^1) \subseteq (g(m), g(m+1))$ and $x_\beta^0 \cap (m_n^1, m_{n+1}^1) = \emptyset$. It follows there are infinitely many n such that both

$$x_{\alpha_2}^0 \cap (m_n^1, m_{n+1}^1) = \emptyset \text{ and } x_{\alpha_2}^1 \cap (m_n^1, m_{n+1}^1) = \emptyset.$$

The remainder of the proof is similar to the proof of Theorem 4.2. Letting $D_1 = \omega$

and $D_2 = \{n : x^0_{\alpha_2} \cap (m^1_n, m^1_{n+1}) = \emptyset$ and $x^1_{\alpha_2} \cap (m^1_n, m^1_{n+1}) = \emptyset\}$, which is infinite, continue in the same manner and define elements with the following properties for each $k > 1$.

(1) $\alpha_k > \alpha_{k-1}$;

(2) $D_k = \{n : x^0_{\alpha_k} \cap (m^{k-1}_n, m^{k-1}_{n+1}) = \emptyset$ and $x^1_{\alpha_k} \cap (m^{k-1}_n, m^{k-1}_{n+1}) = \emptyset\}$ is infinite;

(3) $m^k_1 < m^k_2 < \dots$;

(4) $U^k_1, U^k_2, \dots \in \mathcal{U}$ are distinct;

(5) $\{U^k_n : n \in D_k\}$ is a γ-cover of X_{α_k};

(6) for each $x \in \mathcal{P}(\omega)$, if $x \cap (m^k_n, m^k_{n+1}) = \emptyset$, then $x \in U^k_n$.

Let $\alpha = \sup_k \alpha_k$. Since $\mathrm{cf}(\mathfrak{b}) > \omega$, $\alpha < \mathfrak{b}$. Furthermore, $X_\alpha = \bigcup_k X_{\alpha_k}$ is a countable increasing union. By the initial assumption, since X_{α_k} is a γ-set, it also satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. Therefore, there are infinite $I_1 \subseteq D_1$, $I_2, \subseteq D_2$, $\dots$ such that each $x \in X_\alpha$ belongs to $\bigcap_{n \in I_k} U^k_n$ for all but finitely many $k < \omega$.

Take $n_1 \in I_2$. For $k > 1$ take $n_k \in I_{k+1}$ such that

- $m^k_{n_k} > m^{k-1}_{n_{k-1}+1}$,
- $x^0_\alpha \cap (m^k_{n_k}, m^k_{n_k+1}) \subseteq x^0_{\alpha_{k+1}} \cap (m^k_{n_k}, m^k_{n_k+1})$,
- $x^1_\alpha \cap (m^k_{n_k}, m^k_{n_k+1}) \subseteq x^1_{\alpha_{k+1}} \cap (m^k_{n_k}, m^k_{n_k+1})$, and
- $U^k_{n_k} \notin \{U^1_n, \dots U^{k-1}_{n_{k-1}}\}$.

We will show $\{U^k_{n_k} : k < \omega\}$ is a γ-cover of $T \cup [\omega]^{<\omega}$. Since $\{U^k_{n_k} : k < \omega\}$ is already a γ-cover of X_α, it remains to show that for each $x \subseteq^* x_\alpha$, $x \in U^k_{n_k}$ for all but finitely many k. For each large enough k, $m^k_{n_k}$ is large enough so that

$$\begin{aligned} x_\alpha \cap (m^k_{n_k}, m^k_{n_k+1}) &\subseteq (x^0_\alpha \cup x^1_\alpha) \cap (m^k_{n_k}, m^k_{n_k+1}) \\ &\subseteq (x^0_{\alpha_{k+1}} \cup x^1_{\alpha_{k+1}}) \cap (m^k_{n_k}, m^k_{n_k+1}) \\ &= \emptyset \end{aligned}$$

as $n_k \in D_{k+1}$. Thus, $x \in U^k_{n_k}$. ■

CHAPTER 5: ω-SHRINKABLE ω-COVERS

In this chapter, we will focus on the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle using a different class of covers and show how it is related to selection principles mentioned in Conjecture 1.1. We will also present new results related to the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle and the Pytkeev property.

5.1: Consequences of the Pytkeev Property

Tsaban's motivation behind studying the $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle originated from M. Sakai's results in [21] about the Pytkeev property in the function space $C_p(X)$. Recall, for a space X, $C_p(X)$ denotes the space of all real-valued continuous functions on X with the topology of pointwise convergence. Basic open sets of $C_p(X)$ are of the form

$$[x_1, x_2, \ldots, x_k; U_1, U_2, \ldots, U_k] = \{f \in C_p(X) : f(x_i) \in U_i, i = 1, 2, \ldots, k\},$$

where $x_i \in X$ and each U_i is an open subset of the real line.

It is well-known that there are relationships between properties of X and $C_p(X)$, see [1]. In [21], Sakai characterized the Pytkeev property of $C_p(X)$ in terms of X.

Definition 5.1.

(a) For a space X and $x \in X$, a family $\mathcal{N}$ of subsets of X is called a π-*network* at x if every neighborhood of x contains some element of $\mathcal{N}$.

(b) A space X is called a *Pytkeev space* if $x \in \overline{A} \setminus A$ and $A \subseteq X$ imply the existence of a countable π-network at x of infinite subsets of A

Sakai proved for a Tychonoff space X, $C_p(X)$ has the Pytkeev property if and only if X satisfies a particular $\binom{\mathscr{A}}{\mathscr{B}_\infty}$ selection principle. This selection principle uses ω-shrinkable ω-covers.

Definition 5.2. An open ω-cover $\mathcal{U}$ of X is *ω-shrinkable* if there exists a closed ω-cover $\{C(U) : U \in \mathcal{U}\}$ with $C(U) \subseteq U$ for every $U \in \mathcal{U}$.

Note some of the sets $C(U)$ may be empty. We will use $\tilde{\Omega}$ to denote the family of all open ω-shrinkable ω covers, and we can now state Sakai's main result.

Theorem 5.1. For a Tychonof space X, $C_p(X)$ is a Pytkeev space if and only X satisfies $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$.

It is unknown whether or not the ω-shrinkable condition can be removed.

Question 5.1. For a Tychonof space X, is $C_p(X)$ a Pytkeev space if and only X satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$?

Sakai also proved the $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$ covering property is linearly σ-additive in [21]. We will present the proof.

Theorem 5.2. $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$ is linearly σ-additive.

Proof. Let $X = \bigcup_i X_i$, where $X_1 \subseteq X_2 \subseteq X_3 \cdots$ and each X_n satisfies $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$. Furthermore, let $\mathcal{U}$ be an ω-shrinkable open ω-cover of X.

Consider the set $\mathcal{V}_n = \{U \cap X_n : U \in \mathcal{C}\}$. By definition, $\mathcal{V}_n$ is a ω-shrinkable open ω-cover of X_n. Ruling out the trivial case, assume the set $\{n < \omega : X_n \in \mathcal{V}_n\}$ is infinite. Then, there are sequences $n_0 < n_1 < \cdots$ and $U_0, U_1, \cdots \in \mathcal{U}$ such that $X_{n_i} \subseteq U_i$, $i < \omega$. Since $\mathcal{U}$ is non-trivial, $\{U_i : i < \omega\}$ is infinite. Let $\mathcal{U}_i = \{U_n : n \geq i\}$ for each $i < \omega$. Then, $\mathcal{U}_i$ is infinite and $\{\bigcap \mathcal{U}_i\}_{i<\omega}$ is a ω-cover of X. Thus, arriving at the desired result. Therefore, without loss of generality, we may assume that each $\mathcal{V}_n$ is non-trivial.

As each X_n satisfies $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$ for $n < \omega$, take a sequence $\{\mathcal{V}_{nm}\}_{m<\omega}$ of subfamiles of $\mathcal{V}_n$ such that V_{nm} is infinite and $\{\bigcap \mathcal{V}_{nm}\}_{m<\omega}$ is an ω-cover of X_n.

Next, take a subfamily $\mathcal{U}_{nm} \subseteq \mathcal{U}$ such that $\mathcal{V}_{nm} = \{U_{nm} \cap X_n : U_{nm} \in \mathcal{U}_{nm}\}$. The collection $\{\mathcal{U}_{nm} : n, m < \omega\}$ is infinite since $\mathcal{U}$ is non-trivial, and an ω-cover of X since $\{\bigcap \mathcal{V}_{nm}\}_{m<\omega}$ is an ω-cover of X_n. ■

The $\binom{\tilde{\Omega}}{\Pi_\infty}$ selection principle for $\Pi \in \{\mathbf{\Gamma}, \mathbf{\Omega}, \boldsymbol{\mathcal{O}}\}$ relates nicely to the selection principles in (1.1). The monotonicity laws and the fact that $\tilde{\Omega} \subseteq \Omega$ leads to the following.

Corollary 5.1. For $\Pi \in \{\mathbf{\Gamma}, \mathbf{\Omega}, \boldsymbol{\mathcal{O}}\}$,

$$\binom{\Omega}{\Pi_\infty} \Rightarrow \binom{\tilde{\Omega}}{\Pi_\infty}.$$

It is unknown whether or not this implication can be reversed.

Question 5.2. Does $\binom{\Omega}{\Pi_\infty} \Leftarrow \binom{\tilde{\Omega}}{\Pi_\infty}$ for $\Pi \in \{\mathbf{\Gamma}, \mathbf{\Omega}, \boldsymbol{\mathcal{O}}\}$?

There is also a similar result and question when considering γ-sets.

Corollary 5.2. $\binom{\Omega}{\mathbf{\Gamma}} \Rightarrow \binom{\tilde{\Omega}}{\mathbf{\Gamma}}$.

Question 5.3. Does $\binom{\Omega}{\mathbf{\Gamma}} \Leftarrow \binom{\tilde{\Omega}}{\mathbf{\Gamma}}$?

It also follows that a γ-set satisfies the $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$ selection principle. Let $\mathcal{U}$ be a ω-shrinkable non-trivial open ω-cover of a γ-set X. Shrinkable implies for each $U \in \mathcal{U}$, there is a closed set $C(U)$ of X such that $C(U) \subseteq U$ and $\mathcal{C} = \{C(U) : U \in \mathcal{U}\}$ is an ω-cover of X. As X is a γ-set, $\mathcal{C}$ has a γ-subcover. Therefore, there exists a subcover $\{C(U_j)\}_{j\in\omega}$ with $X = \bigcup_{j\in\omega}\{\bigcap_{m\geq j} C(U_j)\}$. As $\mathcal{U}$ is non-trivial, this yields a $\mathbf{\Omega}_\infty$-subcover of $\mathcal{U}$.

This result can also be shown using the relations in (1.1) and the previous corollary as $\binom{\Omega}{\mathbf{\Gamma}} = \binom{\Omega}{\mathbf{\Gamma}_\infty} \Rightarrow \binom{\Omega}{\mathbf{\Omega}_\infty} \Rightarrow \binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$.

It is unknown whether or not this implication can be reversed.

Question 5.4. Does $\binom{\Omega}{\mathbf{\Gamma}} \Leftarrow \binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$?

One way to partially answer the questions mentioned in this chapter is to determine which conditions guarantee that an open ω-cover is ω-shrinkable. Sakai proved in [21] that the Menger property, $\mathsf{S}_{\text{fin}}(\mathcal{O}, \boldsymbol{\mathcal{O}})$, is associated with ω-shrinkable covers.

Theorem 5.3. Let X be a space such that each finite power of X has the Menger property. Then, every open ω-cover of X is ω-shrinkable.

Furthermore, there is an interesting equivalence in [27].

Theorem 5.4. For a space X, the following are equivalent:

1. Every finite power of X has the Menger property.
2. X satisfies $\mathsf{S}_{\text{fin}}(\Omega, \boldsymbol{\Omega})$.

Therefore, if it is assumed that X satisfies $\mathsf{S}_{\text{fin}}(\Omega, \boldsymbol{\Omega})$, then every open ω-cover of X is ω-shrinkable. Thus, questions 5.2 and 5.3 are positively answered under this assumption.

In chapter 2, we saw it was consistent that any set X satisfying $\binom{\Omega}{\boldsymbol{\Gamma}_\infty}$, $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$, or $\binom{\Omega}{\boldsymbol{\Omega}_\infty}$ is countable. In [28] Miller proved if $X \subseteq \mathbb{R}$ and $C_p(X)$ has the Pytkeev property, then X has strong measure zero. Furthermore, it is consistent that there are no uncountable strong measure zero sets. Therefore, we have the following consistency result for the $\binom{\tilde{\Omega}}{\boldsymbol{\Omega}_\infty}$ selection principle.

Corollary 5.3. It is consistent that every set X satisfying $\binom{\tilde{\Omega}}{\boldsymbol{\Omega}_\infty}$ is countable.

Not much is known about the finite union nor product of sets satisfying $\binom{\tilde{\Omega}}{\boldsymbol{\Omega}_\infty}$.

Question 5.5. If X and Y both satisfy$\binom{\tilde{\Omega}}{\boldsymbol{\Omega}_\infty}$, does $X \cup Y$ satisfy $\binom{\tilde{\Omega}}{\boldsymbol{\Omega}_\infty}$?

Question 5.6. If X and Y both satisfy$\binom{\tilde{\Omega}}{\boldsymbol{\Omega}_\infty}$, does $X \times Y$ satisfy $\binom{\tilde{\Omega}}{\boldsymbol{\Omega}_\infty}$?

The $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$ selection principle can also be used to reverse some of the implications in Figure 5.1. This diagram is slightly modified from our previous Scheepers diagram. As mentioned Lemma 1.3, we can replace $\mathsf{U}_{\text{fin}}(\Gamma, \mathbf{\Gamma})$ with $\mathsf{U}_{\text{fin}}(\mathcal{O}, \mathbf{\Gamma})$ and $\mathsf{U}_{\text{fin}}(\Gamma, \mathbf{\Omega})$ with $\mathsf{U}_{\text{fin}}(\mathcal{O}, \mathbf{\Omega})$.

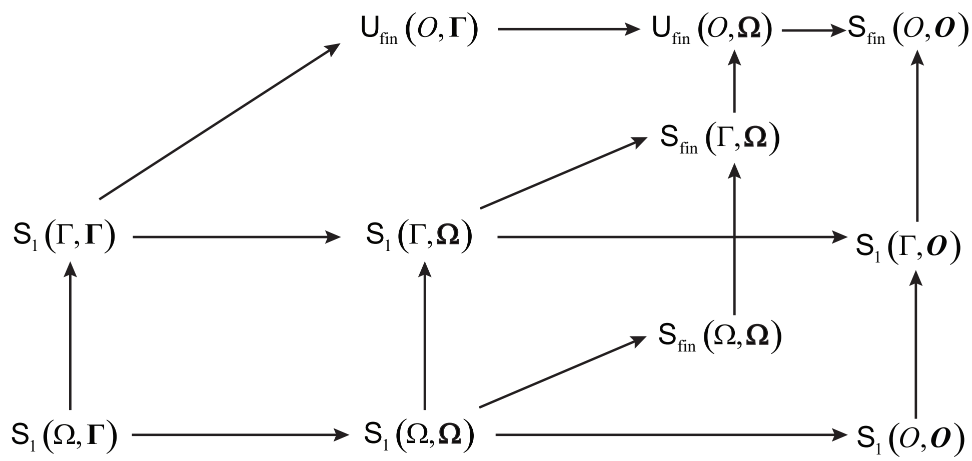

Figure 5.1: Modified Scheepers Diagram

Tsaban and Zdomskyy utilized several combinatorial results to prove the following theorem in [31].

Theorem 5.5. If $C_p(X)$ has the Pytkeev property and X satisfies $\mathsf{U}_{\text{fin}}(\mathcal{O}, \mathbf{\Omega})$, then X satisfies $\mathsf{U}_{\text{fin}}(\mathcal{O}, \mathbf{\Gamma})$ as well as $\mathsf{S}_1(\mathcal{O}, \boldsymbol{\mathcal{O}})$

The following is an immediate consequence of Theorems 5.1 and 5.5.

Corollary 5.4. Suppose X satisfies the $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$, then

1. $\mathsf{U}_{\text{fin}}(\mathcal{O}, \mathbf{\Gamma}) \Leftarrow \mathsf{U}_{\text{fin}}(\mathcal{O}, \mathbf{\Omega})$, and

2. $\mathsf{S}_1(\mathcal{O}, \boldsymbol{\mathcal{O}}) \Leftarrow \mathsf{U}_{\text{fin}}(\mathcal{O}, \mathbf{\Omega})$.

Question 5.7. Can other implications in Figure 5.1 be reversed using the $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$ selection principle?

One can also ask if assuming $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$ will reverse the implications for the newer selection principles in (1.1).

Question 5.8. Can any of the implications in (1.1) be reversed assuming $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$?

It is unknown whether or not we can prove the consistency of Conjecture 1.1 assuming CH. A positive answer to Question 5.8 will at least allow us to prove there exists such spaces mentioned in Conjecture 1.1 if we assume $\binom{\tilde{\Omega}}{\mathbf{\Omega}_\infty}$ instead of CH.

CHAPTER 6: RESULTS CONCERNING FILTERS ON ω

We were unable to prove the consistency of both items in Conjecture 1.1, but we were able to prove new results about the selections principles considered in the conjecture. We will revisit these properties and the open questions asked in the previous chapters to develop new results and open questions concerning filters on ω.

6.1: Properties and Open Questions

It is consistent that the union of two γ-sets need not be a γ-set. Using the filter characterizations, this means if we have two filters $\mathscr{F}_1$ and $\mathscr{F}_2$ on ω, each having a pseudo-intersection, then the filter generated by $\mathscr{F}_1 \cup \mathscr{F}_2$ is not guaranteed to have a pseudo-intersection. We will given an example illustrating why the union is not guaranteed to have a pseudo-intersection. First, recall the following definitions.

Definition 6.1.

(a) A filter $\mathscr{F}$ is an *ultrafilter* on X if for any $A \subseteq X$, either $A \in \mathscr{F}$ or $X \setminus A \in \mathscr{F}$.

(b) An ultrafilter on X is *non-principle* if it contains no finite subsets of X. In other words, it contains only infinite subsets of X

Unlike filters on ω, it is impossible for a non-principle ultrafilter on ω to have an infinite pseudo-intersection.

Lemma 6.1. No non-principle ultrafilter on ω has a pseudo-intersection.

Proof. Let $\mathscr{F}$ be a non-principle ultrafilter on ω. Assume $\mathscr{F}$ has a pseudo-intersection $P \in [\omega]^\omega$. Then, $P \setminus F$ is finite for every $F \in \mathscr{F}$. Since $\mathscr{F}$ is an ultrafilter on ω, either $P \in \mathscr{F}$ or $\omega \setminus P \in \mathscr{F}$. However, $P \setminus (\omega \setminus P) = P$ is infinite, so $\omega \setminus P \notin \mathscr{F}$. Therefore, $P \in \mathscr{F}$.

Consider a set G such that both G and $\omega \setminus G$ contain infinitely many elements of P. Then, either $G \in \mathscr{F}$ or $\omega \setminus G \in \mathscr{F}$. If $G \in \mathscr{F}$, then $P \setminus G$ is finite, which is a contradiction. If $\omega \setminus G \in \mathscr{F}$, then $P \setminus (\omega \setminus G) = P \cap G$ is finite, leading to another contradiction. Thus $\mathscr{F}$ cannot have a pseudo-intersection. ■

We now present an example of two filters having a pseudo-intersection such that the filter generated by the their union does not have a pseudo-intersection.

Define the filters $\mathscr{F}_1$ and $\mathscr{F}_2$ as follows. Let $X \in \mathscr{F}_1$ if and only if X contains cofinitely many even numbers and $\mathscr{U}$-many odd numbers, where $\mathscr{U}$ is a non-principle ultrafilter. Let $\mathscr{F}_2$ be the cofinite filter on the evens.

$\mathscr{F}_1$ has a pseudo-intersection, namely the evens. $\mathscr{F}_2$ has the same pseudo-intersection. Furthermore, the filter generated by $\mathscr{F}_1 \cup \mathscr{F}_2$ has the finite intersection property, as it generates $\mathscr{F}_2$.

Next, define $\mathscr{F}_3$ to be the cofinite filter on the odds. $\mathscr{F}_3$ has a pseudo-intersection, namely the odds. However, the filter generated by $\mathscr{F}_1 \cup \mathscr{F}_3$ generates the ultrafilter $\mathscr{U}$, which does not have a pseudo-intersection.

To guarantee the filter generated by the union does has a pseudo-intersection, we need the notion of "ez" filters.

Definition 6.2.

(a) Given a collection $\mathcal{B}$ of subsets of ω, we denote by $\mathcal{B}^+$ the collection of sets in ω which have nonempty intersection with each member of $\mathscr{B}$. That is,

$$\mathcal{B}^+ = \{S \subseteq \omega : S \cap T = \emptyset, \forall T \in \mathcal{B}\}.$$

(b) We say a filter $\mathscr{F}$ on ω is *ez* if every $\mathscr{F}^+$ set has a pseudo-intersection.

Lemma 6.2. Let X be a γ-set, $\mathcal{U} = \{U_n : n < \omega\}$ be an ω-cover of X, and $\mathscr{F}$ be the footprint filter associated with $\mathcal{U}$ on ω. $B \subseteq \omega$ is in $\mathscr{F}^+$ if and only if $\{U_n : n \in B\}$ is an ω-cover.

Proof. Suppose $B \subset \omega$ is in $\mathscr{F}^+$. Then B has a nonempty intersection with every $F \in \mathscr{F}$. Recall,

$$\mathscr{F} = \left\{ Y \subseteq \omega : \bigcap_{i<n} A_{x_i} \subseteq Y \text{ for } n < \omega, A_{x_i} \in \mathscr{A}, \text{ and } x_i \in X \right\},$$

where $\mathscr{A} = \{A_x : x \in X\}$ and $A_x = \{n < \omega : x \in U_n\}$. This means for any $F \in \mathscr{F}$, $\bigcap_{i<n} A_{x_i} \subseteq F$. Note that $\bigcap_{i<n} A_{x_i}$ is the set of $n < \omega$ such that $\{x_1, x_2, \ldots x_i\} \subseteq U_n$. Since $B \cap F \neq \emptyset$, there exists an $n < \omega$ such that $\{U_n : n \in B\}$ is an ω-cover of X.

To prove the reverse implication, let $\{U_n : n \in B\}$ be an ω-cover of X. Then, $\{x_i : i < k\}$ is contained in some U_n for $n \in B$. This implies $\bigcap_{i<n} A_{x_i} \subseteq B$. As $\mathscr{F} = \left\{Y \subseteq \omega : \bigcap_{i<n} A_{x_i} \subseteq Y \text{ for } n < \omega\right\}$, B and any $F \in \mathscr{F}$ would have a nonempty intersection. Therefore, $B \in \mathscr{F}^+$. ■

Given a γ-set X and a countable ω-cover $\mathcal{U}$ of X, the footprint filter associated with $\mathcal{U}$ on ω is always ez. Recall, being a γ-set means the footprint filter $\mathscr{F}$ has a pseudo-intersection. By the previous lemma, a set B is in $\mathscr{F}^+$ if and only if $\{U_n : n \in B\}$ is an ω-cover of X. As X is a γ-set, there is a γ-subcover of $\{U_n : n \in B\}$. Representing γ-sets using filters, it follows $\mathscr{F}^+$ would have a pseudo-intersection. This leads us to the following.

Corollary 6.1. Any filter on ω generated by fewer than $\mathfrak{p}$ sets is ez.

Recall, $\mathscr{F}_1$ and $\mathscr{F}_3$ each had a pseudo-intersection, but the filter generated by $\mathscr{F}_1 \cup \mathscr{F}_3$ did not. Furthermore, these filters were not ez. Adding the condition that two filters are ez will guarantee the union generated by both filters will have a pseudo-intersection. As ez filters are relevant to γ-sets, this emphasizes why this kind of pathology does not happen with γ-sets.

It is also worth noting that ez filters share a connection with certain topological spaces.

Definition 6.3. A space X is called *Fréchet-Urysohn* space (FU-space) if for each $x \in X$ such that x is in the closure of A, there is a sequence in A converging to x.

Given a filter $\mathscr{F}$ on ω, we obtain a natural topological space $X_{\mathscr{F}}$ defined as follows:

1. The underlying set of $X_{\mathscr{F}}$ is $\omega \cup \{\infty\}$.

2. Points in ω are isolated.

3. The neighborhoods of the point at infinity are given by elements of the filter, that is, a basic open neighborhood of ∞ is given by $U \cup \{\infty\}$ where $U \subseteq \omega$ is in $\mathscr{F}$.

It follows that a subset A of ω will "pick up" ∞ in its closure if and only if A is positive with respect to $\mathscr{F}$. Furthermore, $P \subseteq \omega$ is a pseudo-intersection for $\mathscr{F}$ if and only if P converges to $\{\infty\}$ in the usual sense. Therefore, $\mathscr{F}$ is ez if and only if the topological space $X_{\mathscr{F}}$ is a FU-space.

Recall, it is consistent that the product of two γ-sets need not be a γ-set. It is not known which conditions need to be placed on the footprint filters in order for the filter generated by the product to have a pseudo-intersection.

Question 6.1. Suppose $\mathscr{F}_1$ and $\mathscr{F}_2$ are filters on ω, each having a pseudo-intersection. Under what conditions would the filter generated by $\mathscr{F}_1 \times \mathscr{F}_2$ have a pseudo-intersection?

We will now revisit open questions from the previous chapters using filters on ω. It is not known whether or not the union or product of two spaces satisfying $\binom{\Omega}{\mathbf{\Omega}_\infty}$ also satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$. Furthermore, it is not known if the $\binom{\Omega}{\boldsymbol{\mathcal{O}}_\infty}$ property is preserved under finite products. This leads to the following open questions for filters on ω.

Question 6.2. Suppose $\mathscr{F}_1$ and $\mathscr{F}_2$ are filters on ω such that each filter has a countable π-base. Does the filter generated by $\mathscr{F}_1 \cup \mathscr{F}_2$ or $\mathscr{F}_1 \times \mathscr{F}_2$ have a countable

π-base? If the answer is negative, then which conditions would provide a positive solution?

Question 6.3. Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ generate the filters $\mathscr{F}_1$ and $\mathscr{F}_2$, respectively, on ω. Furthermore, suppose both $\mathscr{A}_1$ and $\mathscr{A}_2$ have a countable π-base. Does $\mathscr{A}_1 \times \mathscr{A}_2$ have a countable π-base? Which conditions will provide a positive solution?

We do know however that the union of two spaces satisfying $\binom{\Omega}{\mathcal{O}_\infty}$ also satisfies $\binom{\Omega}{\mathcal{O}_\infty}$. This means if $\mathscr{F}_1$ and $\mathscr{F}_2$ are filters generated by $\mathscr{A}_1$ and $\mathscr{A}_2$ on ω, then $\mathscr{A}_1 \cup \mathscr{A}_2$ has a countable π-base.

It is also unclear whether or not the union of two γ-sets satisfies $\binom{\Omega}{\mathbf{\Omega}_\infty}$.

Question 6.4. Suppose $\mathscr{F}_1$ and $\mathscr{F}_2$ are filters on ω, each having a pseudo-intersection. Does the filter generated by $\mathscr{F}_1 \cup \mathscr{F}_2$ have a countable π-base? If not, then under which conditions does it have a countable π-base?

We will now outline our results on linear σ-additivity using filters. Consider the filter on ω generated by $\mathscr{F} = \bigcup_n \mathscr{F}_n$ with filters $\mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \mathscr{F}_3 \dots$ on ω. Furthermore, let $\mathscr{A} = \bigcup_n \mathscr{A}_n$ with $\mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \mathscr{A}_3 \dots$, where $\mathscr{A}_n$ generates $\mathscr{F}_n$ for each n. We have the following.

1. If each filter $\mathscr{F}_n$ has a pseudo-intersection, then the filter generated by $\mathscr{F}$ has a pseudo-intersection.

2. If each filter $\mathscr{F}_n$ has a countable π-base, then the filter generated by $\mathscr{F}$ has a countable π-base.

3. If each generating set $\mathscr{A}_n$ has a countable π-base, then $\mathscr{A}$ has a countable π-base.

We also proved the covering property $\binom{\Omega}{\mathbf{\Omega}_\infty}$ is preserved under finite powers. It was shown that the $\binom{\Omega}{\mathbf{\Gamma}}$ selection principle is preserved under finite powers by Scheepers, Just, Miller, and Szeptycki in [27]. It is unknown if the covering property $\binom{\Omega}{\mathcal{O}_\infty}$ is preserved under finite powers.

Below is a table summarizing the known results and open questions for filters on ω.

Table 6.1: Closure Properties for Filters on ω

	Fin. Unions	Inc. Unions	Fin. Products	Fin. Powers
Filter has a pseudo-intersection	No	Yes	No	Yes
Filter has a countable π-base	?	Yes	?	Yes
Generating set has a countable π-base	Yes	Yes	?	?

REFERENCES

[1] A.V. Arkhangel'skii. *Topological Function Spaces*. Kluwer Academic Publishers, 1991.

[2] A. Blass. Combinatorial Cardinal Characteristics of the Continuum. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*, volume 1, pages 395–490. Springer, 2010.

[3] E. Borel. Sur la Classification des Ensembles de Mesure Nulle. *Bulletin de la Societe Mathematique de France*, 47:97–125, 1919.

[4] J. Chamber and R. Pol. A Remark on Fremlin-Miller Theorem Concerning the Menger Property and Michael Concentrated Sets. *Unpublished*, 2002.

[5] P. Erdos, K. Kunen, and R.D. Mauldin. Some Additive Properties of Sets of Real Numbers. *Fundamenta Mathematicae*, 93:187–199, 1981.

[6] D.D. Fremlin. *Consequences of Martin's Axiom*. Cambridge University Press, 1984.

[7] H. Friedman and M. Talagrand. Un Ensemble Singulier. *Bulletin des Sciences Mathématique*, 104:337–340, 1980.

[8] F. Galvin and A.W. Miller. γ-Sets and Other Singular Sets of Real Numbers. *Topology and its Applications*, 17:145–155, 1984.

[9] J. Gerlits and Zs. Nagy. Some Properties of $C(X)$ I. *Topology and its Applications*, 14:151–161, 1982.

[10] W. Hurewicz. Über eine Verallgemeinerung des Borelschen Theorems. *Mathematische Zeitschrifte*, 24:401–425, 1925.

[11] W. Hurewicz. Über Folgen stetiger Funktionen. *Fundamenta Mathematicae*, 9:193–204, 1927.

[12] R.B. Jensen. The Fine Structure of the Constructible Hierarchy. *Annals of Mathematical Logic*, 4(3):229–308, 1972.

[13] F. Jordan. There are no Hereditary Productive γ-spaces. *Topology and its Applications*, 155:1786–1791, 1999.

[14] R. Laver. On the Consistency of Borels Conjecture. *Acta Mathematica*, 137(1):151–169, 1976.

[15] M. Malliaris and S. Shelah. Cofinality Spectrum Theorems in Model Theory, Set Theory, and General Topology. *Journal of the American Mathematical Society*, 29:237–297, 2016.

[16] M.K. Menger. Überdeckungssätze der Punktmengenlehre. *Sitzungsberichte der Wiener Akademie*, 133:421–444, 1924.

[17] A.W. Miller. The γ-Borel Conjecture. *Archive for Mathelatical Logic*, 44:425–434, 2005.

[18] T. Orenshtein and B. Tsaban. Linear σ-Additivity and Some Applications. *Transactions of the American Mathematical Society*, 633:3621–3637, 2011.

[19] A Rinot. Covering Properties: Separating Between Menger and Hurewicz. `http://papers.assafrinot.com/Menger_Hurewicz.pdf`. Published Online.

[20] F. Rothberger. Eine Verschärfung der Eigenschaft. *Fundamenta Mathematicae*, 30:50–55, 1938.

[21] M. Sakai. The Pytkeev Property and the Reznichenko Property in Function Spaces. *Note di Matematica*, 2:43–52, 2003.

[22] M. Sakai. The Pytkeev Property and the Reznichenko Property in Funtion Spaces. *Note di Matematica*, 22:43–52, 2003.

[23] M. Sakai and M. Scheepers. The Combinatorics of Open Covers. *Recent Progress in General Topology*, 3:751–799, 2014.

[24] M Scheepers. Combinatorics of Open Covers I: Ramsey Theory. *Topology and its Applications*, 69:31–62, 1996.

[25] M. Scheepers. Sequential Convergence in $C_p(X)$ and a Covering Property. *East-West Journal of Mathematics*, 1:207–214, 1999.

[26] M. Scheepers. Selection Principles and Covering Properties in Topology. *Note di Matematica*, 22(2):3–41, 2003.

[27] M. Scheepers, W. Just, A. Miller, and P. Szeptycki. The Combinatorics of Open Covers II. *Topology and its Applications*, 73:241–266, 1996.

[28] P. Simon and B. Tsaban. On the Pytkeev Property in Spaces of Continuous Functions. *Proceedings of the American Mathematical Society*, 136(2):1125–1135, 2007.

[29] B. Tsaban. A New Selection Principle. *Topology Proceedings*, 31:319–329, 2007.

[30] B. Tsaban and L Zdomsky. Scales, Fields, and a Problem of Hurewicz. *Journal of the European Mathematical Society*, 10:837–866, 2008.

[31] B. Tsaban and L. Zdomskyy. On the Pytkeev Property in Spaces of Continuous Functions II. *Houston Journal of Mathematics*, 35(2):563–571, 2009.