To Prove or Disprove: The Use of Intuition and Analysis by Undergraduate Students to Decide on the Truth Value of Mathematical Statements and Construct Proofs and Counterexamples

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Abstract

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To Prove or Disprove: The Use of Intuition and Analysis by Undergraduate Students to Decide on the Truth Value of Mathematical Statements and Construct Proofs and Counterexamples

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Deciding on the truth value of mathematical statements is an essential aspect of mathematical practice in which students are rarely engaged. This study explored undergraduate students' approaches to mathematical statements with unknown truth values. The research questions were

1. In what ways and to what extent do students use intuition and analysis to decide on the truth value of mathematical statements?
2. What are the connections between students' process of deciding on the truth value of mathematical statements and their ability to construct associated proofs and counterexamples?
3. What types of systematic intuitive, mathematical, and logical errors do students make during the proving process, and what is the impact of these errors on the proving process?

Clinical task-based interviews utilizing the think-aloud method revealed students' reasoning processes in depth. Twelve undergraduate students each completed four mathematical tasks requiring them to decide on the truth value of a statement and prove or disprove it accordingly. Through analysis of the data, I developed a framework for

distinguishing among types of reasoning based on their cognitive and mathematical properties. The framework identifies four distinct categories of reasoning – intuitive, semantic-empirical, semantic-deductive, and syntactic – each with subcategories.

The students in this study used all four types of reasoning for deciding on the truth value of the statements in the tasks. Their use of semantic-deductive and syntactic reasoning mirrored mathematicians' use of these reasoning types for decision-making. With the exception of one task, the students' decision-making and construction processes were generally connected. Connections in which the construction process was based on decision-making process mostly facilitated proving. However, simultaneous decision-making and construction processes often led to overturned decisions. Regarding intuitive decision-making, only property-based intuitive decisions were connected to the corresponding construction process.

The students in this study made numerous systematic mathematical and logical errors, but systematic intuitive errors were limited and occurred on only one task. The systematic conceptual misunderstandings surrounding the concept of function are troubling due to the centrality of this concept to mathematics. Few errors were overcome, but a certain level of uncertainty may aid students in overcoming logical errors.

Dedication

To my family

Your unconditional love and support makes all things possible.

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Chapter 1: Introduction

Mathematicians understand that reasoning and proof are essential components of doing mathematics. According to Schoenfeld, proof is the "soul" of mathematics (2009, p. xii). As a college mathematics instructor, Hersh (2009) wished his students understood that "proof, in the broadest sense, of careful reasoning leading to definite, reliable conclusions, is what mathematics is all about" (pp. 19–20). The American Mathematical Association of Two-Year Colleges (AMATYC, 1995) and the Committee on the Undergraduate Program in Mathematics (CUPM, 2004) call for focusing the undergraduate mathematics curriculum on reasoning and proof. However, many undergraduate students do not grasp the importance of proof, struggle with numerous aspects of mathematical reasoning, and have limited facility in constructing mathematical proofs and counterexamples (Alcock, 2010; Dreyfus, 1999; Harel & Sowder, 1998, 2009; Moore, 1994; Selden & Selden, 1987, 2003, 2007; Solomon, 2006; Weber, 2001).

Intuitive and analytical reasoning are fundamental components of proof-based mathematics. *Intuition* constructs an automatic mental representation of a task, taking into consideration task cues, prior knowledge, and experience, and operates independently of working memory (Evans, 2009. 2010, 2012b; Fischbein, 1987; Wilder, 1967). On the other hand, *analysis* is a deliberate process of reasoning that can be explained and decomposed into its constituent parts, and requires the use of working memory (Evans, 2008, 2012a, 2012b; Fischbein, 1987). Although often seen as competing modes of thought, intuition and analysis can be recognized as complementary, each playing key roles in evaluating mathematical statements and producing proofs and counterexamples (Fischbein, 1987; Hersh, 1997; Tall, 1991; Wilder, 1967). Dual-process theories of reasoning and decision-making assert that intuition and analysis correspond to distinct types of cognitive processing, each with specified characteristics, roles, and difficulties (Evans, 2006, 2008, 2010; Kahneman, 2002).

Analytical reasoning includes both informal and formal reasoning, an important distinction in mathematics. *Formal* reasoning is based on logic and deduction, and *informal* reasoning includes reasoning strategies such as visual, example-based, or pattern-based reasoning. A *mathematical proof* is a justification of an assertion consisting entirely of formal reasoning. However, the *proving process* is complex and encompasses a multitude of activities including exploring and identifying patterns and relationships, generating conjectures and generalizations, and testing, refining, and proving conjectures (AMATYC, 1995; CUPM, 2004; de Villiers, 2010; Durand-Guerrier et al., 2012). Thus, although intuitive and informal reasoning have no place in a mathematical proof—the finished product – they are crucial aspects of the proving process.

Due to the important distinction between informal and formal reasoning, research in mathematics education typically groups together intuitive and informal reasoning in contrast to formal reasoning. In a series of articles (Alcock & Weber, 2010; Weber & Alcock, 2004, 2009), Alcock and Weber describe two distinct reasoning styles and approaches to proof production that they call *semantic* (or *referential*) and *syntactic*. Semantic reasoners produce proofs through a focus on general understanding guided by intuition, examples, diagrams, or informal explanations, and syntactic reasoners produce proofs mainly through formal reasoning based on logic and structure (Weber & Alcock, 2004). However, this division conceals the cognitive differences between intuitive and analytical reasoning as well as the cognitive similarities between informal and formal reasoning (both being analytical). It is important in mathematics education to distinguish between intuitive and analytical reasoning as well as informal and formal reasoning. In this study, I will reconcile dual-process theory and Alcock and Weber's theory of semantic and syntactic reasoning by separating intuition from semantic reasoning and using "semantic" and "informal" synonymously as well as "syntactic" and "formal" (see Figure 1). With this framework, I can study intuition separate from analysis, and within my study of analysis, I can distinguish semantic and syntactic reasoning.

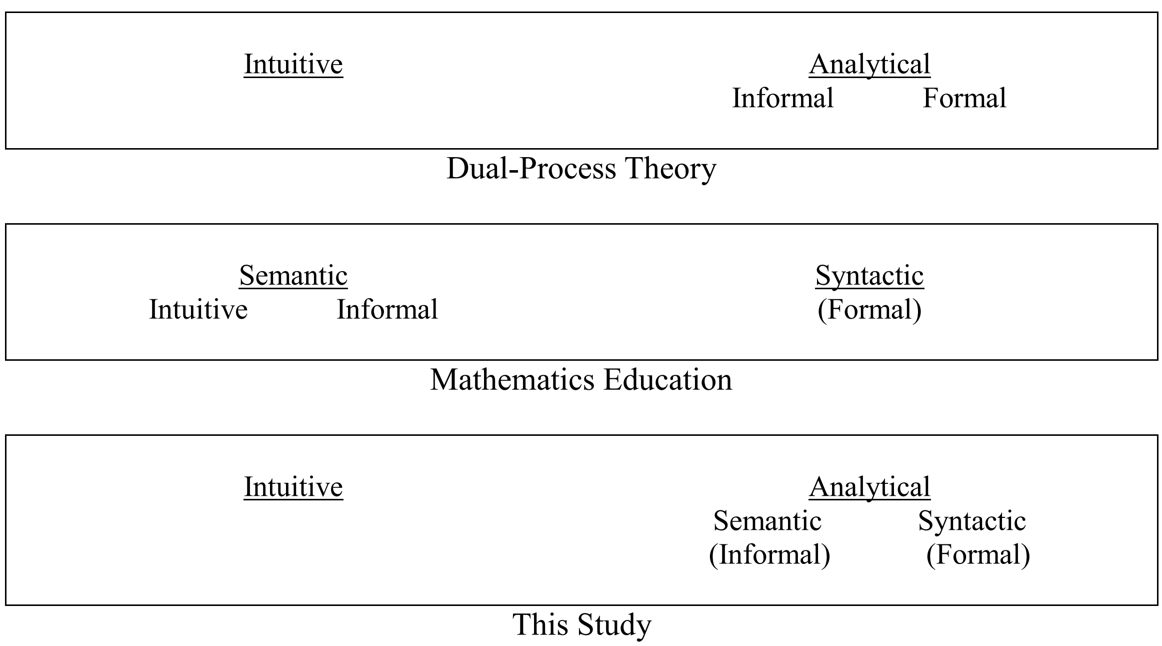


Figure 1. Distinction of reasoning types in dual-process theory, mathematics education, and this study.

The use of dual-process theory in addition to theories of mathematical reasoning can support interpretations of students' reasoning in the proving process that distinguish among intuitive, semantic, and syntactic reasoning. Some mathematics educators encourage the use of dual-process theory because of its focus on "general cognitive considerations that are not limited to mathematics," as well as its "tightening, refining, and operationalizing" of the distinction between intuition and analysis (Leron & Hazzan, 2006, pp. 115, 122). In particular, it has been suggested that the use of dual-process theory in mathematics education research can provide fresh viewpoints on students' (a) reasoning and systematic errors on mathematical tasks (Leron & Hazzan, 2006, 2009), and (b) strategies for evaluating the truth value of mathematical assertions (Buchbinder & Zaslavsky, 2007).

Problem Statement and Research Questions

Although there is extensive research on proof and proving, such research mainly addresses students' understanding of what constitutes a proof and students' difficulties in constructing proofs for statements that are given as true. There is little research on intuition as a way of reasoning separate from semantic reasoning and little research on the reasoning that students use to decide on the truth value of mathematical statements. However, the research on dual-process theory from cognitive psychology can shed new light on students' reasoning on mathematical proof tasks. This study uses dual-process theory in conjunction with theories of mathematical reasoning as a lens through which to examine students' use of intuitive and analytical reasoning while deciding on the truth value of mathematical statements and constructing proofs and counterexamples. This study explores (a) the ways intuition and analysis interact in the decision-making process, (b) the ways this decision-making process influences students' constructions of associated proofs or counterexamples for the statements, and (c) the impact of students' systematic errors on the proving process. The overarching research questions are:

1. In what ways and to what extent do students use intuition and analysis to decide on the truth value of mathematical statements?
2. What are the connections between students' process of deciding on the truth value of mathematical statements and their ability to construct associated proofs and counterexamples?
3. What types of systematic intuitive, mathematical, and logical errors do students make during the proving process, and what is the impact of these errors on the proving process?

Significance

Due to the way undergraduate mathematics is taught, many students lack an understanding of the true nature of doing mathematics and have difficulties constructing proofs and counterexamples. Mathematicians use intuition, informal reasoning, conjecture, creativity, and rigor in constructing proofs, however, instruction in proof involves "definition, theorem, proof," and repeat (Davis & Hersh, 1981, p. 151). "The definition-theorem-proof approach to mathematics has become almost the sole paradigm of mathematical exposition and advanced instruction. Of course, this not the way mathematics is created, propagated, or even understood" (Davis & Hersh, 1981, p. 306). Although this type of instruction cuts off students from the intuitive and creative aspects of proof and does not reflect the way mathematicians do mathematics, it is still the standard form of instruction of undergraduate proof-based mathematics in the United States (Davis & Hersh, 1981; Tall, 1991; Weber, 2004; Wilder, 1967).

As a consequence of standard mathematical instruction, students rarely are engaged in aspects of the proving process involving uncertainty, especially determining the truth value of mathematical statements (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012). Due to students' limited engagement in such activities, little is known about how they evaluate conjectures and what types of reasoning they use to do so. In particular, there is a lack of research on students' ways of deciding on the truth value of general mathematical statements involving general mathematical objects in the context of proof-based mathematics.

However, determining the truth value of mathematical statements is an important component of the proving process. Generating and evaluating mathematical conjectures is an essential aspect of mathematical practice. Additionally, important knowledge can be gained through the exploration of both true and false mathematical statements. When dealing with uncertainty, mathematicians often try to decide on a statement's truth value with some degree of confidence before investing a significant amount of time attempting to prove or refute it (de Villiers, 1990, 2010; Inglis, Mejia-Ramos, & Simpson, 2007). Intuitive reasoning can be helpful in this process because it can suggest what is plausible in the absence of a proof (Burton, 2004; Davis & Hersh, 1981; Fischbein, 1994). Study of this decision process is essential for determining the ways of reasoning that can lead to successful decisions about the truth or falsity of mathematical statements. Furthermore, studying successful students engaging in this process, rather than mathematicians, is more likely to yield results of pedagogical value and "suggest learning trajectories that might be applicable for many other students as well" (Weber, 2009, p. 201). Such trajectories may help students understand that although the proving process is a complex combination of creativity and rigor, it involves a variety of accessible reasoning types that they can learn to use to build ideas that connect deciding on the truth value of mathematical statements to the construction of associated proofs and counterexamples.

Definition of Terms

Analysis (analytical reasoning) is a deliberate process of reasoning that can be decomposed into its constituent parts and requires the use of working memory (Evans, 2008, 2012a, 2012b; Fischbein, 1987). Analysis includes both semantic and syntactic reasoning and is used in this study in contrast to intuition.

An **argument** is an intuitive or semantic justification for the truth or falsity of a mathematical statement. Argument is used in this study in contrast to proof.

A **clinical task-based interview** is an interview conducted in a laboratory or clinical setting that involves participants completing tasks that are determined in advance by the researcher and answering questions about their work on the task that will elicit their thought processes on how or why they took particular actions.

A **counterexample** is an example that establishes the falsity of a mathematical statement. A single counterexample is sufficient to refute a statement with absolute certainty.

Dual-process theory is a theory of reasoning and decision-making that distinguishes intuition and analysis as distinct types of cognitive processing, each with specified characteristics, roles, and difficulties (Evans, 2006, 2008, 2010; Kahneman, 2002).

First-level member check is a process in which participants confirm the accuracy of their responses to the interview questions (Bratlinger et al., 2005).

Intuition (intuitive reasoning) constructs an automatic mental representation of a task, taking into consideration task cues, prior knowledge, and experience, and operates independently of working memory (Evans, 2010, 2012b; Fischbein, 1987; Wilder, 1967). Intuition is used in this study in contrast to analysis (including semantic and syntactic reasoning).

Mathematicians in this study are individuals who are working on or have a graduate degree in mathematics. Mathematicians may or may not teach mathematics at a postsecondary institution, but engage in work that uses mathematics significantly.

A **(mathematical) proof** is a justification consisting entirely of syntactic reasoning for the truth or falsity of a mathematical statement. Proof is used in this study in contrast to argument.

The **proving process** is the process of constructing a proof or counterexample for a mathematical statement. This process encompasses a multitude of activities including exploring and identifying patterns and relationships, generating conjectures and generalizations, and testing, refining, and proving conjectures.

Reasoning refers to any kind of cognition, including intuitive, semantic, and syntactic.

Semantic (informal) reasoning includes a variety of reasoning strategies such as visuo-spatial, example-based, graphical, diagrammatic, physical, kinaesthetic, analogical, inductive, and pattern-based. Semantic reasoning is a form of analytical reasoning and is used in this study in contrast to syntactic and intuitive reasoning.

Syntactic (formal) reasoning is reasoning from definitions, axioms, assumptions, and theorems based solely on logic and deduction that conforms to specified rules regarding language, symbols, and frameworks for argumentation. Syntactic reasoning is a form of analytical reasoning and is used in this study in contrast to intuitive and semantic reasoning.

Students in this study refers to undergraduate students unless otherwise noted.

The **think-aloud method** is a clinical task-based interview method that involves participants speaking aloud everything they are thinking while they work on a task, thus reporting their thoughts concurrently with their work on the task (van Someren, Barnard, & Sandberg, 1994).

Working memory is a cognitive system "of limited capacity closely linked with executive and attentional functions . . . which seems to be the only part of the mind that is consciously accessible, at least in a cognitive sense" (Evans, 2010, p.314).

Organization of the Chapters

This dissertation is organized in 5 chapters as follows:

Chapter 1 introduces the topic of study, its significance, and my research questions. This chapter concludes with definitions of terms and organization of the chapters.

Chapter 2 reviews the related literature in order to frame this study in research from cognitive psychology and mathematics education. The cognitive psychology literature on dual-process theory provides a general framework for the study of intuition, analysis and decision-making. The mathematics education literature on proof and proving situates intuition and analysis as essential components of the proving process and further divides analysis into semantic and syntactic reasoning.

Chapter 3 describes the research methods used in this study for data collection and analysis. In order to explore students' reasoning in depth, qualitative interview methods are used. In particular, this study employs clinical task-based interviews and the think-aloud method. Benefits and limitations of these methods are discussed. Additionally, this chapter describes sampling, interview, and data analysis procedures.

Chapter 4 represents the results of the interview data. Each of the four tasks is presented separately, and within each task, the students' decision-making and construction processes are described so as to provide insight into the answers to my research questions. This chapter concludes with a discussion of themes that arose across tasks with respect to my research questions.

Chapter 5 concludes this dissertation with a summary of chapters 1-4, a discussion of the conclusions with respect to my research questions as well as other conclusions, implications of the study, and suggestions for future research.

Chapter 2: Literature Review

Intuition and analysis are fundamental components of mathematics that are crucial to the evaluation of mathematical conjectures and construction of proofs and counterexamples. This chapter presents a review of literature on (a) intuitive and analytical reasoning in dual-process theory and mathematical proof and proving, (b) the important distinction within analytical reasoning between semantic and syntactic reasoning in the proving process, (c) the process of deciding on the truth value of a mathematical statement and the uses of intuitive and analytical reasoning in this process, and (d) connections between the processes of deciding on the truth value of a mathematical statement and constructing an associated proof or counterexample.

Intuition

Intuition is an essential feature of mathematics (Fischbein, 1987; Hersh, 1997; Wilder, 1967). "If one is . . . trying to look at people who are doing mathematics and to understand what they are doing, then the problem of intuition becomes central and unavoidable" (Davis & Hersh, 1981, p. 393). However, the study of intuition in mathematics has been limited, especially as a concept separate from informal reasoning (Burton, 2004; Fischbein, 1987).

> A consideration of intuition as it is actually experienced leads to a notion which is difficult and complex, but it is not inexplicable or unanalyzable. A realistic analysis of mathematical intuition is a reasonable goal, and should become one of the central features of an adequate philosophy of mathematics. (Davis & Hersh, 1981, p. 393)

Leron and Hazzan (2006) note that dual-process theory can contribute to this goal due to its refined and operationalized distinction between intuition and analysis. In this section, the concept of intuition will be explained by (a) defining intuition; (b) describing how intuition develops; (c) indicating reliability issues associated with intuition, including systematic intuitive errors; and (d) discussing intuition in mathematical proving.

Definition of intuition. Many definitions of intuition have been proposed by researchers in a variety of disciplines, including psychology, philosophy, and mathematics. Common characteristics of intuition are that it is relatively quick, automatic, and requires little cognitive effort (Evans, 2008; Hammond, 1980). Intuition is a process of organizing and integrating information from both environment and memory into a judgment, decision, or interpretation and operates at least partially without awareness of the process by which these are formed (Evans, 2010; Fischbein, 1982; Glockner & Witteman, 2010; Hammond, 2007; Noddings & Shore, 1984; Wilder, 1967). This partial lack of awareness means that intuitive responses cannot be fully explained or decomposed into their constituent elements, unlike analytical responses (Evans, 2010; Fischbein, 1982). Following Evans (2008, 2009, 2012b), the defining feature of intuition is its independent operation from working memory. Working memory needs content with which to work, and intuition supplies working memory with mental representations (see below) (Evans, 2009).

Although the concept of intuition can encompass a variety of underlying cognitive processes, this study will focus on constructive intuition because mathematical intuition is viewed as being constructed (Davis & Hersh, 1981; Fischbein, 1987; Noddings &

Shore, 1984; Smith & Hungwe, 1998; Tall, 1991). *Constructive intuition* is based on a mental representation of a task that is constructed from given cues in the task as well as information retrieved from memory (Glockner & Witteman, 2010). These intuitive representations are constructed automatically and provide a default response to a task (Evans, 2009, 2010, 2012a; Wilder, 1967). This “natural” production of representations is what makes intuition especially important in decision-making (Fischbein, 1987, p. 14). Despite the quickness of this construction, intuition takes into account prior knowledge, experiences, beliefs, task features, and the current goal of the reasoning to create the task representation (Evans, 2006, 2010). Furthermore, “intuition is able to organize information, to synthesize previously acquired experiences . . . to guess, by extrapolation, beyond the facts at hand” (Fischbein, 1982, p. 12).

Development of intuition. Intuition is developed and modified through active experience and knowledge construction (Burton, 2004; Davis & Hersh, 1981; de Villiers, 2010; Evans, 2008, 2010; Fischbein, 1982, 1987; Noddings & Shore, 1984; Wilder, 1967; Wittmann, 1981). Experience shapes intuition by creating expectations of truth or falsity based on similar situations or related tasks (Davis & Hersh, 1981; Fischbein, 1987). Wilder (1967) notes that nonmathematicians do not have mathematical intuition, thus it must be developed through mathematical experience and knowledge. In mathematics, the development of intuition is usually linked to active knowledge construction and engagement in experimentation (Davis & Hersh, 1981; de Villiers, 2010; Fischbein, 1982, 1987; Noddings & Shore, 1984; Tall, 1991; Wilder, 1967; Wittmann, 1981).

Limited experiences with the exploration of mathematical objects and theories will inhibit the development of intuition. Thus, in order to develop students' mathematical intuition, it is imperative to adopt an instructional approach to proof and proving that focuses on the entire process of proving rather than only the product (Hadamard, 1954; Tall, 1991). Such an approach would include experimentation and argumentation in addition to deduction and proof (Burton, 1999b, 2004). Unfortunately, few students learn this way in today's mathematics classrooms, especially in the United States (Burton, 1999a; de Villiers, 2010; Fischbein, 1987; Weber, 2004).

Reliability of intuition. Due to inconsistencies or inaccuracies in prior learning experiences, a person's intuitive representation may not faithfully represent the situation at hand (Evans, 2008; Fischbein, 1987; Tall, 1991; Wilder, 1967). The reliability of intuitive representations often depends on how deeply intuition is developed through relevant experience (Burton, 2004; Evans, 2008, 2010; Noddings & Shore, 1984; Wilder, 1967). If the constructed intuitive representation correctly represents the underlying task structure, then the interpretation should lead to high achievement on the task (Evans, 2012a; Glockner & Witteman, 2010). On the other hand, intuitive representations may be distorted or deficient due to systematic errors (see below), leading to poor achievement (Evans, 2010; Glockner & Witteman, 2010; Kahneman, 2002).

Systematic intuitive errors. Systematic intuitive errors are errors of intuitive reasoning that cause misrepresentations of situations and persist across situations and people. Many systematic intuitive errors can be classified as accessibility errors (Glockner & Witteman, 2010; Kahneman, 2002). *Accessibility* is the ease with which certain knowledge is evoked or certain task features are perceived and is a crucial component of intuitive reasoning and decision-making (Kahneman, 2002). Two key accessibility errors involve (1) attribute substitution, and (2) knowledge and task feature relevance.

Attribute substitution errors occur when a more readily accessible attribute is substituted in a task for a less readily accessible attribute (Kahneman, 2002). For example, similarity is an attribute that is always accessible because it is processed intuitively (Kahneman & Frederick, 2002). Participants may intuitively notice similarities between a given task and previously encountered tasks and substitute more accessible attributes for less accessible ones based on these similarities (Kahneman & Frederick, 2002). Thus, participants often unknowingly transform the given task into a similar more accessible task. For example, consider a professor who has just listened to a talk given by a candidate for job opening and is asked the question "How likely is it that this candidate could be tenured in our department?" This professor may instead unknowingly answer the easier question "How impressive was the talk?" (Kahneman & Frederick, 2002, p. 52).

Relevance errors occur when knowledge and task features are deemed irrelevant because they are not readily accessible (Evans, 2008, 2010; Fischbein, 1987; Kahneman, 2002). When forming intuitive task representations, (a) less accessible relevant knowledge is often not brought to bear on the task (Weber, 2001), (b) less accessible relevant task features are often overlooked, and (c) more accessible irrelevant task features are often overemphasized (Evans, 2008). Leron and Hazzan (2006) provide an example of a relevance error based on the wording in a task. Students who were asked to construct an equation to represent the sentence “There are six times as many students as professors at this university” (as found in Clement et al., 1981) intuitively deemed the order of the wording to be relevant and wrote the incorrect equation 6S = P.

The influence of accessibility on reasoning and decision-making is strong, yet decreases with increased experience and knowledge. Due to the automatic operation of intuition, people are unaware of the influence of accessibility and the predominance of accessibility errors. However, the reliability of intuition improves as experience and knowledge increase, and the negative influence of accessibility often decreases (Evans, 2010; Kahneman, 2002, 2011). Thus, experience “increases the accessibility of useful responses and of productive ways to organize information” (Kahneman, 2002, p. 453). Although novices with some relevant knowledge (rather than naive novices) are most susceptible to accessibility errors in intuitive reasoning and decision-making, even experts still succumb to such errors (Kahneman, 2002, 2011).

Intuition in mathematical proving. Intuition organizes information into a meaningful representation in order to provide an initial understanding of a mathematical task (Burton, 2004; Fischbein, 1982; Noddings & Shore, 1984). This understanding can provide a starting point, suggest a direction to pursue, or guide action on the task (Burton, 2004; Fischbein, 1982, 1987; Smith & Hungwe, 1998; Wilder, 1967). This is important because many students struggle to begin a mathematical proof task (Moore, 1994). Additionally, intuition can help connect the current task to prior knowledge and experiences by helping students recognize similarities and see a "common global situation" (Burton, 2004; Fischbein, 1987, p. 53).

Intuitive representations may take on many forms in mathematics. Intuitive representations may take the form of visual images or perceptual representations of concepts or objects (Davis & Hersh, 1981; Hadamard, 1954; Hammond, 1980; Tall, 2008). Hadamard (1954) stresses the importance of intuitive representations as *vague images* that help mathematicians make sense of situations, bring to mind certain ideas, and coordinate the subparts of solutions. On the other hand, intuition can be logical or deductive (Hadamard, 1954; Tall, 1991). Hadamard (1954) and Tall (1991) suggest that deductive thinking is intuitively natural to mathematicians due to their extensive experience with logical thinking. "Thus, aspects of logic too can be honed to become more 'intuitive' to the mathematical mind" (Tall, 1991, p. 14).

Difficulties with intuitive reasoning. Despite the importance of intuition to the proving process, students have a variety of difficulties with intuitive reasoning, including falling victim to systematic intuitive errors. Students often lack intuitive understandings altogether or have narrow intuitions based on examples and visualizations that may prohibit effective intuitive reasoning during proof construction (Moore, 1994; Raman, 2003). Many students have limited logical intuition (Tall, 1991). Such limitations in intuition make it more likely that students' intuitive representations will be distorted or inaccurate and lead to incorrect conclusions (Burton, 2004; de Villiers, 2010; Fischbein, 1987; Hadamard, 1954; Noddings & Shore, 1984; Wilder, 1967). Thus, it is essential that intuitive representations are examined critically and followed up with a mathematical proof (Burton, 2004; de Villiers, 2010; Hadamard, 1954; Smith & Hungwe, 1998). However, students' intuitive understandings may not lead directly to a proof or counterexample, or students may not recognize the relationship between their intuition and a syntactic proof or counterexample (Raman, 2003; Tall, 1991).

Limited research has indicated that students are subject to attribute substitution and relevance errors when working on mathematical proof tasks. In analyzing high-school students' reasoning while deciding on the truth value of number theory conjectures, Buchbinder and Zaslavsky (2007) found that in certain situations, students' intuitive responses ignored "relevant cues or relevant content knowledge" (p. 569). Bubp (2013) reported on a mathematical task given to four undergraduate mathematics majors and one graduate student in mathematics on which all five students committed a systematic intuitive error. The task was presented to the students as follows:

> Definitions: A function $f: \mathbb{R} \to \mathbb{R}$ is said to be **increasing** if and only if for all $x_1, x_2 \in \mathbb{R}$, ($x_1 < x_2$ implies $f(x_1) < f(x_2)$. Similarly, a function $f: \mathbb{R} \to \mathbb{R}$ is said to be **decreasing** if and only if for all $x_1, x_2 \in \mathbb{R}$, ($x_1 < x_2$ implies $f(x_1) > f(x_2)$). Prove or disprove: If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are decreasing on an interval *I*, then the composite function $f \circ g$ is increasing on *I*. (p. 2-446)

Two undergraduate students committed attribute substitution errors. One student "substituted the similar concept of *negative times negative equals positive* for the task concept of *decreasing composed with decreasing equals increasing*" (p. 2-447). The other student "substituted the incorrect concept *odd times odd equals even* in place of *decreasing composed with decreasing equals increasing*" (p. 2-447). The three other students committed the same relevance error. "They each ignored the interval restriction in the task, responding as if the task was to prove or disprove the following: If $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ are decreasing, then the composite function $f \circ g$ is increasing" (p. 2-447). Each student constructed a distorted intuitive representation of the task based on a systematic intuitive error that led them to decide that the false statement was true.

Leron and Hazzan (2006) reported on an abstract algebra task on which students made attribute substitution and relevance errors. The task required students to determine the truth value of the statement "$\mathbb{Z}_3$ is a subgroup of $\mathbb{Z}_6$" and to explain their reasoning. The participants were 113 computer science majors taking an abstract algebra course. Twenty of these students invoked Lagrange's theorem to claim that this statement was true. However, Lagrange's theorem is not applicable to this statement because it requires a subgroup as an assumption and cannot be used to conclude a subset is a subgroup. The students actually used an incorrect version of the converse of Lagrange's theorem. Thus, the students committed an attribute substitution error by substituting an easier and more accessible situation represented by their distorted version of Lagrange's theorem for the more difficult and less accessible situation in the given task. Furthermore, students committed compound relevance errors. Their intuition invoked irrelevant knowledge (Lagrange's theorem) based on the irrelevant task cues of the word *subgroup* and the fact that three divides six.

Analysis

Analysis, including both semantic and syntactic reasoning, is fundamental to mathematics. The American Mathematical Association of Two-Year Colleges

(AMATYC, 1995) and the Committee on the Undergraduate Program in Mathematics (CUPM, 2004) call for focusing the undergraduate mathematics curriculum on semantic reasoning and syntactic proof. Although a proof must be a syntactic product, the process of proving can involve both semantic and syntactic reasoning. Types of semantic reasoning include: visuo-spatial, example-based, graphical, diagrammatic, physical, kinaesthetic, analogical, inductive, and pattern-based. Syntactic reasoning includes logical and deductive reasoning based on axioms, definitions, theorems, and standard proof frameworks. Both types of reasoning are important to proving, and successful mathematical provers employ semantic and syntactic reasoning strategies flexibly "in response to changing demands during a proof attempt" (Alcock, 2010, p. 84). In this section, the concept of analysis will be explained by (a) defining analysis, and (b) discussing analysis in mathematical proving, including distinguishing between semantic and syntactic analysis and specifying students' difficulties with each.

Definition of analysis. Analysis is a deliberate and systematic process of reasoning. Common characteristics of analytical reasoning are that it is relatively slow, controlled, and requires much cognitive effort (Evans, 2008, 2010). It is a decomposable process that can be separated into its constituent parts and explained or justified (Evans, 2010; Fischbein, 1987). Thus, unlike intuition, people are aware of both the process and the product of analysis (Evans, 2010; Hammond, 1980). Like intuition, there are multiple viewpoints for the defining feature of analysis, and this study will employ Evans' (2009, 2012a, 2012b) definition that analysis "requires access to a single central working memory system *among other resources*" such as "systems for attention, language processing, [or] memory retrieval" (2009, p. 38). Thus, once an intuitive representation of a task is thrust into working memory, analysis accesses that working memory content in order to mentally manipulate the now explicit representations (Evans, 2009).

Once analytical reasoning is brought into action in response to an automatically generated intuitive representation of a task situation, it may be overpowered by intuition and limited in its functionality (Evans, 2010; Kahneman, 2002; Noddings & Shore, 1984; Smith & Hungwe, 1998). First, analysis may be bound by intuition, thus depending on the same biased or incomplete task representations on which the intuition was based (Fischbein, 1994; Thompson, 2009). Second, analytical cognition may be invoked solely to justify an intuitive representation, thus failing to consider alternative representations of a task (Thompson, 2009). Lastly, analysis may be blinded by overconfidence in an intuitive response, ignoring discovered alternative representations of a task (Burton, 1999b; Thompson, 2009). The power of intuition may lead to analysis being employed erroneously to support an intuitive response to a task. Thus, it may be the case that errors that seem to be the result of poor analytical cognition are actually caused by faulty intuitive representations.

Alternatively, analysis may be used to overpower an intuitive task representation. First, analysis may override representations based on biased or incomplete intuitions by recognizing (a) relevant information that was ignored in the intuitive representation, or (b) a logical rule that contradicts the intuitive representation (Kahneman, 2002). Second, analytical cognition may be invoked with the goal of finding alternative representations of the task rather than supporting the intuitive representation (Thompson, 2009). Lastly, analysis may consider alternative representations due to a lack of confidence in an intuitive representation (Thompson, 2009). In each of these cases, analysis operates according to the structure of the task rather than relying on intuitive representations.

Analysis in mathematical proving. In a series of articles (Alcock & Weber, 2010; Weber & Alcock, 2004, 2009), Alcock and Weber describe two distinct reasoning styles and approaches to proof production that they call *semantic* (or *referential*) and *syntactic*. In this study, "semantic" and "informal" will be used synonymously as well as "syntactic" and "formal." Semantic reasoners produce proofs through a focus on general understanding guided by examples, diagrams, or other informal explanations, and syntactic reasoners produce proofs mainly through formal reasoning based on logic and structure (Weber & Alcock, 2004). Both styles of reasoning present students with opportunities and difficulties in proof production, and no correlation has been found between reasoning style and performance in constructing correct mathematical proofs (Weber & Alcock, 2009). Thus, Alcock and Weber (2010) have concluded that "neither of these approaches should be used exclusively by students and both syntactic and referential approaches to proving are necessary for proving competence" (p. 96).

Semantic reasoning. Semantic reasoning plays essential roles in mathematical proving (Alcock & Weber, 2010). The proving process can encompass experimentation and argumentation supporting conjecturing and generalization as well as the construction of counterexamples and formal proofs. Weber & Alcock (2004) found that knowledge of "multiple informal representations of specific and generic examples" supported competence in proving (Weber & Alcock, 2009, p. 332). This section will elaborate on how semantic reasoning can support the following activities during proving: (a) developing understanding of the components of the proof task, (b) formulating and testing conjectures and generalizations, and (c) fostering the construction of counterexamples and formal proofs.

Semantic reasoning helps students develop understanding of mathematical proof tasks (AMATYC, 1995; CUPM, 2004; Smith, 2006). Generating examples and nonexamples of relevant definitions can enhance students' understanding of the definitions and related task concepts (de Villiers, 2010; Harel & Sowder, 1998; Weber & Alcock, 2004, 2009). In particular, analyzing special or limiting cases can assist with determining the boundaries of a definition and can create a more complete understanding of a definition (de Villiers, 2010). Additionally, students' informal exploration of relationships and connections among multiple representations can assist with sense making (Borwein, 2005; Zbiek & Heid, 2011). For example, by comparing graphical, numerical, and symbolic representations of a task situation, students can inform, question, or confirm their understanding of mathematical concepts (Heid, Hollebrands & Iseri, 2002).

Semantic reasoning plays an important role in developing and testing conjectures and generalizations, but this aspect of the proving process is often overlooked in the classroom (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier et al., 2012). Inductive reasoning from a series of examples can guide the production and evaluation of conjectures and generalizations (de Villiers, 2010; Durand-Guerrier et al., 2012; Heid & Blume, 2008; Kasube & McCallum, 2001). Additionally, through various informal explorations, students can discover, generalize, and justify mathematical patterns and relationships (Borwein, 2005; National Council of Teachers of Mathematics (NCTM), 2000). Because the use of semantic reasoning in testing conjectures is a key component in this study, it will be developed more fully in a subsequent section.

Semantic reasoning can provide a basis for and support the development of a syntactic proof or counterexample (Alcock, 2010; de Villiers, 2010). Specifically, semantic reasoning can be essential in bridging the gap between intuition and syntactic proof (Moore, 1994; Raman, 2003; Weber & Alcock, 2004; Wittmann, 1981). First, by exploring examples, graphs, analogies, or patterns, students can draw empirical inferences that can support proof construction (de Villiers, 2010; Weber & Alcock, 2009). Second, semantic reasoning can reveal a "hidden clue or underlying structure of a problem, leading eventually to the construction or invention of a proof" (de Villiers, 2010, p. 215). In particular, Borwein (2005) and Zbiek and Heid (2011) suggest the use of graphing to expose mathematical principles, such as relationships between symbolic and graphical representations. Finally, inductive reasoning from a series of geometric constructions, examples, or patterns can guide the development of formal symbolic representations, counterexamples, and proofs (de Villiers, 2010; Durand-Guerrier et al., 2012; Kasube & McCallum, 2001).

Difficulties with semantic reasoning. Despite the importance of semantic reasoning, students have difficulty understanding its role in the proving process. This section provides an overview of the following student difficulties with semantic reasoning: (a) non-use of semantic reasoning to assist with proof production, (b) use of semantic arguments as a substitute for a syntactic proof, or (c) non-use of semantic reasoning to connect intuition to a syntactic proof.

There is evidence that students and mathematicians alike have a preferred reasoning style and some never use semantic reasoning during proof construction (Alcock & Weber, 2010; Weber, 2009; Weber & Alcock, 2004). Often, students do not generate examples for themselves to support understanding in proof construction, even when they can do so at the request of someone else (Alcock & Inglis, 2008; Weber & Alcock, 2004), whereas other students are simply unable to produce examples (Moore, 1994; Smith, 2006). Pinto and Tall (1999) found that some students, despite their attempts, could not produce adequate diagrams to help them informally understand definitions in a real analysis course.

Although semantic reasoning is important for supporting proving, it cannot be a substitute for syntactic proof. Students often fail to recognize this distinction (Harel & Sowder, 1998, 2007), perhaps partially because instructors and textbooks seldom distinguish between explanation, argument, and proof (Dreyfus, 1999). Many students tend to accept inductive arguments based on examples as proofs (Dreyfus, 1999; Harel & Sowder, 1998; Inglis et al., 2007). Additionally, perceptual arguments based on images, geometric figures, or graphs may be seen as an appropriate alternative to a syntactic proof (Dreyfus, 1999; Hadamard, 1954). However, "an example or image may incorporate properties that are not universally true, and therefore mislead the reasoner into trying to prove untrue general claims" (Weber & Alcock, 2009, p. 334).

Semantic reasoning may be the key to building bridges from intuition to mathematical proof, but students often struggle to make these connections (Hadamard, 1954). Raman (2003) notes that students seldom link their intuitive understandings to a syntactic proof. This can occur because students (a) have limited or distorted intuitions that are difficult to link to syntactic proofs (Moore, 1994); (b) have inaccurate or incomplete informal representations of concepts (Tall & Vinner, 1981); or (c) fail to recognize the relationship between their intuitive understandings and a syntactic proof (Raman, 2003). However, the key to a proof for some mathematicians is a semantic understanding of the main concept in the proof that connects intuition and formality (Raman, 2001, 2003; Weber & Alcock, 2004).

Syntactic reasoning. Syntactic reasoning constitutes the use of logical and deductive reasoning based on axioms, definitions, and theorems. Syntactic reasoning is required to produce a proof that would be acceptable to the community of mathematicians (NCTM, 2000; Weber & Alcock, 2009). The fact that mathematical results must be proved through syntactic reasoning is often seen as a defining feature of the field of mathematics and the key to its separation from the empirical sciences. Thus, the development of formal mathematical reasoning and the ability to understand and construct syntactic proofs is one of the key goals of undergraduate mathematics education (AMATYC, 1995, 2006; CUPM, 1992, 2004; Kasube & McCallum, 2001).

A variety of unique features classify syntactic reasoning and proof. First, syntactic reasoning requires the careful statement of problems and the precise use of language, notation, symbols, and definitions (CUPM, 2004; Kasube & McCallum, 2001;

Weber & Alcock, 2009). Second, a syntactic proof must be clear and coherent so that there is no ambiguity and the proof framework is apparent (CUPM, 2004; Weber & Alcock, 2009). Third, acceptable proof frameworks, such as direct proof, indirect proof, or proof by mathematical induction provide the structure for the proof and specify admissible assumptions and proper conclusions (Weber & Alcock, 2009). Fourth, syntactic proofs contain only mathematical statements that use some combination of the precise use of the English language and first-order logic. Finally, all reasoning in a syntactic proof must be based on definitions, assumptions, theorems, and the use of logical deduction. Informal representations such as graphs or examples, as well as reasoning based on informal representations are not permitted as a basis for conclusions (Weber & Alcock, 2009).

Formal reasoning can contribute to students' understanding of the relevant concepts and definitions involved in a proof construction as well as the construction process itself (AMATYC, 1995). Applying proof frameworks to organize a proof and invoking relevant definitions is often the first step in constructing a proof (Selden & Selden, 2009; Weber & Alcock, 2009). Through this use of syntactic reasoning, students can reflect on what they have done in order to gain understanding of the formal reasoning process (Weber & Alcock, 2009). Additionally, syntactic reasoning may reveal the key idea in a proof and suggest exactly what is needed to complete the proof (Selden & Selden, 2009; Solomon, 2006). Finally, syntactic reasoning can be used to develop semantic and intuitive understandings of the concepts and definitions in a proof task (NCTM, 2000; Harel & Sowder, 1998; Selden & Selden, 2009; Weber & Alcock, 2009).

Difficulties with syntactic reasoning. Although syntactic reasoning is essential to the proving process, students often struggle to understand the need for precision and conformity to structure and logical rules in mathematical proof. This section will elaborate on the following difficulties that students encounter with syntactic reasoning: (a) lack of understanding of basic logical principles and proof frameworks, and (b) acceptance of semantic knowledge of definitions and lack of understanding about how to use definitions and theorems in proof productions.

Students often lack an understanding of logical principles as well as the use of logical language and notation in a proof. Students have limited logical inferencing abilities, including difficulties with quantifiers, negations, contrapositive statements, and converse statements (Connor, Moss, & Grover, 2007; Harel & Sowder, 2007; Moore, 1994; Selden & Selden, 1987). Moore (1994) found that undergraduate students have a "lack of knowledge of logic and methods of proof" (p. 263) and trouble determining appropriate proof frameworks. In interviews with Alcock (2010), mathematicians who teach transition-to-proof courses noted that students' main difficulties with proof related to structural and logical reasoning. Additionally, in Selden and Selden's (1987) classification of reasoning errors in undergraduate students' proof productions, many were logical errors, such as beginning with the conclusion of the statement to be proved, using circular reasoning, and ignoring or extending symbols and quantifiers. Likewise, Moore (1994) observed that students' lack of understanding of appropriate uses of logical language and notation formed a key barrier to their proof production.

Students' difficulties with definitions and theorems usually relate to a lack of understanding of the need for precision in proof. Moore (1994) found that undergraduate students often fail to state definitions precisely or use them appropriately to structure a proof. Furthermore, mathematics professors in one study believed these difficulties correspond to students' lack of understanding of the merit of definitions in proof, specifically that definitions are valued for their precision, and exact statements of definitions, rather than general understandings, are necessary (Harel & Sowder, 2009). Additionally, Alcock & Simpson (2002) found evidence to support Harel and Sowder's viewpoint in a study with undergraduate students taking a real analysis course. Such difficulties with definitions may prohibit students from developing a syntactic conception of proof in mathematics (Moore, 1994). Similar misunderstandings occur with theorems involved in proof productions, both with the theorem to be proved and in using other theorems in a proof. Students' lack of understanding of precision and formality often leads them to neglect or misinterpret either the hypotheses or the conclusion in theorems and use theorems when they are not applicable (Selden & Selden, 1987).

Deciding on the Truth Value of a Mathematical Statement

The process of proving mathematical statements often starts with formulating and testing mathematical conjectures—activities that involve uncertainty. Unfortunately, students are rarely engaged in uncertain aspects of the proving process, including determining the truth value of mathematical statements (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier et al., 2012). Due to these limited opportunities, students experience a variety of difficulties studying conjectures, including (a) knowing how to begin an exploration, (b) formulating ideas and opinions about the truth of a conjecture, and (c) connecting ideas and opinions to proofs or counterexamples (Alibert, 1988). However, it is important for students to engage in such activity in order to experience these aspects of proving, engage in intuitive and semantic reasoning, and develop "an attitude of reasonable skepticism" (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier et al., 2012, p. 357). Furthermore, Alibert (1988) found that students were more involved, interested, and curious about mathematical tasks that involved uncertainty.

Despite the centrality of proof to mathematics, many students possess an understanding of mathematics for which proof is unnecessary (Harel & Sowder, 1998; Solomon, 2006). This can be attributed to an overemphasis on syntactic reasoning in proof and the over-use of tasks requiring students to prove a statement that is presented as true (Alibert & Thomas, 1991; Durand-Guerrier et al., 2012). Thus, it is imperative to find ways to motivate students' need for proof. Many mathematics educators believe that engaging students in exploring the truth value of mathematical statements can prompt a need for proof. "The necessity, the functionality, of proof can only surface in situations in which the students meet uncertainty about the truth of mathematical propositions" (Alibert, 1988, p. 32). "A main challenge in teaching argumentation and proof is to motivate students to examine whether and why statements are true or false. . . . Thus, many mathematics educators now promote the development, at every level of the curriculum, of problems where the truth-goal is at stake" (Durand-Guerrier et al., 2012, p. 362). As students' opportunities with uncertainty in the proving process increase, "previous experience of doubt of the truth or falsity of mathematical statements can lead students to see the need for validation as meaningful in terms of their own experiences, conjectures, and mathematical backgrounds" (Durand-Guerrier et al., 2012, p. 358).

The remainder of this section includes a (a) description of a general framework of decision-making pathways that students may follow that highlights the use and interaction of intuition and analysis during decision-making, (b) review of the literature on the use of intuitive and analytical reasoning by mathematicians and students during the process of deciding on the truth value of a mathematical statement, and (c) discussion of the distinctions and connections between the decision-making and construction processes in the proving process.

Decision-making pathways. Intuitive and analytical reasoning can be used together in numerous ways to make decisions. If an intuitive decision is not made, then a decision may be made through analysis (Kahneman, 2002) or not at all. If an intuitive decision is made, it may be accepted without analytical reasoning, or it may be intervened upon by analysis (Evans, 2010, 2012a). When intervention occurs, the intuitive decision may be supported or overridden. This suggests a framework for dual-process theory decision-making as illustrated in Figure 2 (Evans, 2010, 2012a; Kahneman, 2002).

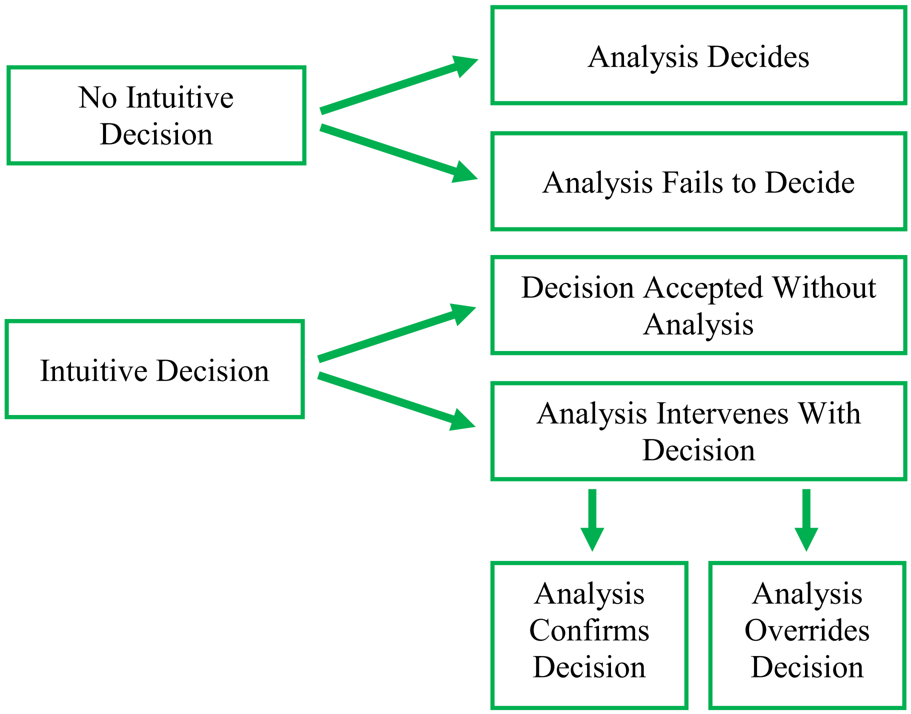


Figure 2. Dual-process decision-making pathways.

Analysis is often brought into action in response to an intuitive representation of a task (Evans, 2010; Kahneman, 2002).[1] When an intuitive decision is not accessible, decision-making becomes the task of analytical reasoning, but still may be influenced by accessibility errors in the intuitive representation.

Analytical reasoning may or may not be invoked in response to an intuitive decision on a task (Evans, 2010; Kahneman, 2002). Due to the high effort involved in analysis, intuitive decisions often are accepted with little or no analytical intervention (Evans, 2010; Kahneman, 2002). However, the need for intervention through analysis can be provoked by explicit directions to reason analytically (Evans, 2010) or by a lack of

[1]Such a representation need not include an intuitive decision.

confidence in an intuitive decision (Thompson, 2009). Upon an intervention, analytical reasoning may be used to either confirm or override an intuitive decision.

Limited research in mathematics education indicates that students use a variety of decision-making pathways to decide on the truth value of mathematical statements. Both Bubp's (under review) study of undergraduate students in transition-to-proof courses and Buchbinder and Zaslavsky's (2007) study of high school students found that a majority of students' decision-making pathways involved an intuitive decision that either was supported or overridden by analysis. Each study reported on a case in which students used sound analytical reasoning to overturn an incorrect intuitive decision or to support a correct intuitive decision (Bubp, under review; Buchbinder & Zaslavsky, 2007). Additionally, Buchbinder & Zaslavsky (2007) described two cases in which students' analytical reasoning was blinded by their overconfidence in their intuitive decisions, (a) prohibiting them from recognizing a counterexample for a statement they were convinced was true, and (b) leading them to create a nonexistent counterexample for a statement they were convinced was false. On the other hand, Bubp (under review) found that students' systematic intuitive errors were the key influence on their analytical reasoning being unable to overturn incorrect intuitive decisions.

Bubp (under review) and Buchbinder and Zaslavsky (2007) further report cases in which students did not make an intuitive decision. In one situation, a student failed to make either an intuitive or analytical decision, and indicated that she would attempt a proof, and if that failed, then she would conclude the statement was false (Bubp, under review). However, she did not know how to begin a proof and discontinued work on the task. Despite this case, most of the students used a variety of forms of analytical reasoning to evaluate the conjectures, including looking at specific examples (Buchbinder & Zaslavsky, 2007), searching for an appropriate theorem or rule to use (Buchbinder & Zaslavsky, 2007), and performing algebraic or symbolic manipulations (Bubp, under review; Buchbinder & Zaslavsky, 2007).

Types of reasoning. Most research on the types of reasoning engaged in while deciding on the truth value of a mathematical statement comes from the study of mathematicians' reasoning rather than students' reasoning. Furthermore, little research has been conducted with undergraduate students evaluating conjectures that involve general objects and their properties and for which a proof or counterexample is the expected end product. Although the types of reasoning reported on here may shed some light in the ways mathematicians and students decide on the truth value of a mathematical statement, more research is needed on undergraduate students' ways of reasoning.

Intuitive reasoning. Intuition is especially important for deciding on the truth value of a mathematical statement because it can suggest what is plausible in the absence of a proof (Burton, 2004; Davis & Hersh, 1981; Fischbein, 1994) and "provides a justification for, but is prior to, the search for convincing argument and, ultimately, proof" (Burton, 1999b, p. 32). In the limited research on intuition in mathematics education, researchers have found a variety of types of intuitive reasoning used by students and mathematicians to evaluate mathematical conjectures. Additionally, there were situations in which mathematicians' and students' intuitions were inaccurate and led them to incorrect decisions about the truth value of mathematical conjectures (Bubp, 2012, 2013; Buchbinder & Zaslavsky, 2007; Inglis et al., 2007; Leron & Hazzan, 2006).

Mathematicians' intuitive support for the truth or falsity of a mathematical statement was based on either suspected properties about mathematical objects or known relationships between mathematical concepts (Inglis et al., 2007). For various conjectures, mathematicians used expected properties of certain types of numbers as a basis for reducing their uncertainty, such as intuitive arguments about the divisors of even and odd numbers (Inglis et al., 2007). On another conjecture, the mathematicians' intuitive arguments were based on their understanding of the relationships between divisors and addition or multiplication—in particular that divisors are not preserved under addition, but are preserved under multiplication (Inglis et al., 2007).

Students' intuitive decisions on the truth value of mathematical statements were based on (a) cues in the statement (Buchbinder & Zaslavsky, 2007; Leron & Hazzan, 2006); (b) definitions and mental images (Bubp, 2012); or (c) expected relationships between mathematical objects with certain properties (Bubp, 2013). First, high school students examining the following statement, "If two triangles have 2 sides and 3 angles that are equal, then the triangles are congruent" intuitively responded to the lack of the word *respectively* and concluded that the statement was likely false (Buchbinder & Zaslavsky, 2007, p. 565). Furthermore, undergraduate students asked to determine the truth value of the statement "$\mathbb{Z}_3$ is a subgroup of $\mathbb{Z}_6$" were prompted to use Lagrange's Theorem because of the word *subgroup* and the fact that three divides six. Second, Bubp (2012) described a student whose intuition consisted of a combination of the definition of one-to-one and a vague image of a function that can "double back on itself" (p. 196), and another student who intuited a mental image of the relationship between the definitions of function and one-to-one, and the vertical and horizontal line tests. Finally, two students used their intuitive understandings of suspected relationships between increasing and decreasing functions and either positive and negative numbers, or even and odd numbers, respectively, to decide on the truth value of a mathematical statement (Bubp, 2013).

Semantic reasoning. Mathematicians have long engaged in experimentation and informal reasoning to evaluate conjectures (de Villiers, 1990, 2010; Hanna, 2007). Mathematicians use a variety of semantic reasoning strategies to decide on the truth value of a mathematical statement, including: drawing geometric figures or diagrams (de Villiers, 2010), examining special or limiting cases (Alcock & Inglis, 2008; de Villiers, 2010), reasoning by analogy (Alcock & Inglis, 2008; de Villiers, 2010), exploring patterns (Alcock & Inglis, 2008; de Villiers, 2010), studying specific or generic examples (Alcock & Inglis, 2008; Inglis et al., 2007), searching for counterexamples (Inglis et al., 2007), and engaging in informal plausibility argumentation about properties of relevant mathematical objects (Alcock & Inglis, 2008; Inglis et al., 2007). Although much less is known about how undergraduate students reason informally about the truth value of a mathematical conjecture, evidence has been found that they (a) study specific examples (Buchbinder & Zaslavsky, 2007; Connor, Moss, & Grover, 2007; Durand-Guerrier et al., 2012; Weber & Mejia-Ramos, 2009); (b) search for counterexamples (Durand-Guerrier et al., 2012); (c) draw diagrams, especially Venn diagrams (Weber, Brophy, & Lin, 2008); and (d) construct semantic arguments about mathematical properties of examples (Weber

& Mejia-Ramos, 2009). Despite this overwhelming use of semantic reasoning for evaluating conjectures, it is important to remember that such reasoning can be misleading and should be followed up with syntactic proof (Hadamard, 1954).

Syntactic reasoning. Mathematicians and students both employ various syntactic reasoning strategies to decide on the truth value of a mathematical statement. Inglis et al. (2007) found that mathematicians engaged in reasoning from definitions, algebraic reasoning, and counterexamples to determine the truth value of a mathematical conjecture. Weber (2009) provides an account of a successful undergraduate student who used only syntactic reasoning to evaluate conjectures. This student would reformulate a conjecture by using logically equivalent statements or alternate definitions, determine logical inferences that could be made from the assumptions, or attempt a proof. If he got stuck on a proof attempt, he would examine why he got stuck in order to try to determine logically why the statement may be false instead. Despite this student's success in simply attempting a proof in order to evaluate a conjecture, Weber et al. (2008) indicate that most of the less successful undergraduate students in their study used this technique. Additionally, students may perform algebraic or symbolic manipulations (Bubp, under review; Buchbinder & Zaslavsky, 2007) or consider possibly relevant theorems, rules, or definitions when determining the truth value of a mathematical statement (Buchbinder & Zaslavsky, 2007; Durand-Guerrier et al., 2012).

Connecting decision-making to proof or counterexample construction. When approaching the uncertainty of a conjecture, mathematicians and students usually decide on the conjecture's truth or falsity with some degree of confidence before investing a significant amount of time attempting to prove or refute it (Bubp, 2012; Buchbinder & Zaslavsky, 2007; de Villiers, 1990, 2010; Inglis et al., 2007).

> Conjectures vary in the degrees of faith a person has in their potential truth. The amount of effort a person is willing to make in seeking evidence that would render the conjecture a fact (or refute it) is in some proportion to her or his faith in the truth (or falsity) of the conjecture. (Harel & Sowder, 1998, p. 242)

Thus, there are often distinct decision-making and construction phases during the proving process. The decision-making process entails reasoning that is used to reduce uncertainty about the truth value of a conjecture to the point that a decision is made as to whether a proof or counterexample should be pursued (Harel & Sowder, 1998; Inglis et al., 2007). The construction process entails the construction of a proof or counterexample to support the decision and remove all uncertainty regarding the statement's truth value. Because mathematical proofs and counterexamples are syntactic, intuitive and semantic reasoning are only appropriate to reduce uncertainty about the truth value of a mathematical statement, not remove it (Inglis et al., 2007).

Inglis et al. (2007) suggest the beginnings of a framework for categorizing reasoning used to reduce uncertainty regarding a statement's truth value during decision-making. Such reasoning categories are called *warrant-types* and grounded in empirical evidence. They offer this categorization as a companion to Harel and Sowder's (1998) categorization of reasoning used to remove uncertainty during the construction phase of the proving process. Harel and Sowder (1998) call their reasoning categories *proof schemes*. The key difference is that warrant-types are used to reduce uncertainty during decision-making whereas proof schemes are used to remove uncertainty during the construction process. These two frameworks form the basis for deeper classification of reasoning types beyond the broad categories of intuitive, semantic, and syntactic reasoning. Furthermore, these frameworks suggest different forms of reasoning that are appropriate for different phases of the proving process.

Harel and Sowder (1998) describe three main classes of proof schemes, some with subcategories that students may use to remove uncertainty about the truth or falsity of a conjecture. Harel (2007) updated these proof schemes, and I will follow the updated version here. The three classes of proof schemes are: (a) the *external conviction* proof scheme class, (b) the *empirical* proof scheme class, and (c) the *deductive* proof scheme class. I will not discuss the external conviction proof schemes because they are not based on intuitive, semantic, or syntactic reasoning. The empirical proof scheme class includes (a) the *inductive* proof scheme in which conjectures are evaluated quantitatively in one or more cases, and (b) the *perceptual* proof scheme in which conjectures are evaluated based on static mental images that cannot be transformed.

The deductive proof scheme class contains the subcategories of *transformational* and *modern axiomatic* proof schemes. Transformational proof schemes utilize logical inference, but there are no strict rules for such inferences, so semantic elements are often included. Additionally, transformational proof schemes include a restriction related to at least one of the following: (a) context of the argument, (b) generality of the argument's justification, or (c) mode of the justification (Harel, 2007, p. 68). The restrictions reflect students' difficulties with context-dependent reasoning, justification in general terms, and understanding of causality, respectively. On the other hand, modern axiomatic proof schemes include strict rules for logical inference, resulting in only syntactic reasoning, and is free from the three restrictions described above (Harel, 2007).

Inglis et al. (2007) discuss three categories of reasoning for reducing uncertainty during decision-making: (a) the *inductive* warrant-type, (b) the *structural-intuitive* warrant-type, and (c) the *deductive* warrant-type. The inductive warrant-type corresponds to Harel and Sowder's (1998) inductive proof scheme in which conjectures are evaluated quantitatively in one or more cases. The structural-intuitive warrant-type is a decision based on "observations about, or experiments with, some kind of mental structure, be it visual or otherwise" (Inglis et al., 2007, p. 12). This warrant-type often corresponds to intuitive reasoning. Finally, deductive warrant-types correspond to the modern axiomatic proof scheme and include only syntactic reasoning. Deductive warrant-types include mathematical proofs, algebraic manipulations, and counterexamples (Inglis et al., 2007, p. 15).

Mathematicians are aware that any intuitive or semantic reasoning used during the decision-making phase must be confirmed with a syntactic proof, but students often lack this awareness (Alcock & Simpson, 2002; Dreyfus, 1999; Harel & Sowder, 1998; Inglis et al., 2007). Additionally, students often have difficulty connecting the decision-making and construction phases because of their difficulty in connecting intuitive and semantic reasoning to syntactic reasoning (Moore, 1994; Raman, 2003; Tall, 1991; Weber & Alcock, 2004). However, researchers suggest that connections between the decision-making and construction processes facilitate proving whereas disconnections hinder it

(Garuti, Boero, & Lemut, 1998). In particular, Garuti, Boero, and Lemut (1998) hypothesize that the greater the gap between the arguments for the truth value of the conjecture and the arguments that can be translated into mathematical proofs, the greater the difficulty in constructing a mathematical proof.

Summary

This chapter reviewed literature on intuition, analysis, and the process of deciding on the truth value of a mathematical statement. Intuition and analysis were defined and situated in dual-process theory as a general cognitive theory of reasoning and decision-making. Furthermore, the distinction between semantic and syntactic analytical reasoning was developed. Studies have shown that there are benefits and difficulties involved when using both intuitive and analytical reasoning in mathematical proof and proving. In the context of deciding on the truth value of a mathematical statement, decision-making pathways were described, and the use of intuitive, semantic, and syntactic reasoning was discussed. Finally, connections between the processes of deciding on the truth value of a mathematical statement and constructing an associated proof or counterexample were considered.

Chapter 3: Research Methods

In order to understand students' ways of reasoning while deciding on the truth value of mathematical statements and constructing proofs or counterexamples, I used qualitative interview methods. In particular, clinical task-based interviews and the think-aloud method provided me with the opportunity to explore students' reasoning processes in depth.

Clinical task-based interviews and the think-aloud method are prominent interviewing methods in research in mathematics education (Ginsburg, 1981, 1997; Goldin, 1998; Hunting, 1997; Koichu & Harel, 2007; Teppo, 1998; van Someren, Barnard, & Sandberg, 1994). "There is overwhelming evidence that clinical task-based interviews open a window into the subject's knowledge, problem-solving behaviors, and reasoning" (Koichu & Harel, 2007, p. 349). Furthermore, studying problem solving and reasoning processes can shed light on "the mechanisms underlying human reasoning, the cause of errors, [and] the character and origin of differences in performance between people" (van Someren, Barnard, & Sandberg, 1994, p. 14). Both clinical task-based interviews and the think-aloud method can be used either inductively to create theories of reasoning processes or deductively to test the validity of an existing theory (Ginsburg, 1981, 1997; van Someren, Barnard, & Sandberg, 1994).

In this chapter, I describe (a) the methods, benefits, and limitations associated with clinical task-based interviews and the think-aloud method; (b) my personal perspective and how it influenced my decisions in this study; (c) my sampling, data collection, interview, and data analysis procedures.

Clinical Task-Based Interviews

Clinical task-based interview methods are a class of methods in which interviews are conducted in a laboratory or clinical setting, and participants complete tasks that are determined in advance by the researcher (Ginsburg, 1981, 1997; Goldin, 1998; Hunting, 1997; Koichu & Harel, 2007; Opper, 1977). The researcher asks the participants various predetermined questions about their work on the task, specifically including questions that will elicit participants' thought processes on how or why they took particular actions. Additionally, the researcher asks probing and exploratory questions and may have participants complete further tasks to investigate emerging hypotheses based on the participants' solutions or explanations to the predetermined tasks (Ginsburg, 1997). "The essence of the clinical interview is deliberate nonstandardization and flexibility" (Ginsburg, 1997, p. 70). A variety of tasks can be used in clinical task-based interviews to explore participants' thinking processes (Ginsburg, 1997). Such tasks include analyzing a video or a solution to a problem written by another person, constructing a solution to a given problem, or analyzing one's own work on a problem completed at a prior time.

Piaget developed the clinical task-based interview in the 1920s as a method to explore deeply participants' thinking processes (Ginsburg, 1997; Opper, 1977). Piaget thought that the standard research methods of the time—standardized tests and natural observation—were unable to reveal thinking processes. Thus, he created his own method, modeling it on the method psychiatrists used in diagnoses. Piaget's original version of the method was open-ended and allowed the researcher the freedom to use any means necessary to explore the concept at hand and tailor the interview process to each participant (Opper, 1977). As it is used today, the clinical task-based interview is semi-structured rather than open-ended (Ginsburg, 1997; Hunting, 1997; Opper, 1977). Piaget's goals for his method were to (a) elicit participants' spontaneous thought through exploration, (b) identify participants' thought processes through on-the-spot hypothesis development and testing, and (c) consider participants' entire mental context through examining non-cognitive aspects such as attention and motivation (Ginsburg, 1997).

The Think-Aloud Method

The think-aloud method is another task-based interview method that occurs in a clinical setting and elicits participants' thought processes (Patton, 2002; van Someren, Barnard, & Sandberg, 1994). In this method, participants speak aloud everything they are thinking while they work on a task. If participants stop speaking, the interviewer will prompt them to continue to think aloud throughout the process. The key to the think-aloud method is that participants report their thoughts concurrently with their work on the task rather than retrospectively after they have completed the task (van Someren, Barnard, & Sandberg, 1994). A variety of types of tasks can use the think-aloud method, such as problem solving, painting, or medical diagnoses.

The think-aloud method originated in psychological research in the 1940s as a reaction to issues with the standard method of introspection (van Someren, Barnard, & Sandberg, 1994). The introspection method involves participants choosing intermediate points within the process of completing a task to stop and report on their thinking. Participants are expected to provide "an accurate, complete, and coherent report on a cognitive process" (van Someren, Barnard, & Sandberg, 1994, p. 22). However, this can be difficult for participants because a report of the process often requires participants to interpret their actions, may necessitate the use of psychological terminology, and is separated from the process itself (van Someren, Barnard, & Sandberg, 1994). Furthermore, the method introspection does not treat the verbal reports as data. Instead, the data consists of the actual events that are being analyzed and explained by the participant. "However, these data are fundamentally accessible only to a single observer, who also performs the thought process. This makes it impossible to replicate empirical studies and thereby to settle scientific discussions about thought processes" (van Someren, Barnard, & Sandberg, 1994, p. 30). These issues turned the psychological community away from cognitive research methods and contributed to the rise of behaviorism in the 1930s (van Someren, Barnard, & Sandberg, 1994). Then, in the 1940s, the think-aloud method emerged as a cognitive method that avoids the issues of introspection because it uses a verbalization process rather than an interpretation process and verbal protocols that are accessible to everyone as data.

The techniques of clinical task-based interviews and the think-aloud method are quite similar. Both methods involve (a) participant solutions to tasks requiring decisions, (b) researcher questioning in response to participants' work on the tasks, (c) limited researcher interference in participants' thought processes, (d) researcher neutrality toward correctness of solutions, and (e) researcher emphasis on how participants arrived at their solutions (Ginsburg, 1981, 1997; Hunting, 1997; Koichu & Harel, 2007; van Someren, Barnard, & Sandberg, 1994). The key difference is that the think-aloud method requires concurrent verbalization and completion of tasks whereas clinical task-based interviews depend mostly on verbal reflections after completing a task (Ginsburg, 1981, 1997; van Someren, Barnard, & Sandberg, 1994). However, many researchers consider the think-aloud method to be an integral component of clinical task-based interviewing (Hunting, 1997; Koichu & Harel, 2007).

There are several key differences between more common interviewing methods and the clinical task-based and think-aloud interview methods. Clinical task-based interviews must be semi-structured and occur in a clinical setting. Thus, these techniques cannot be used in standardized or open-ended interviews, cannot take place in a naturalistic setting, and will not work in the absence of a task. The key difference however is the goal of the interview. The purpose of most interviews is to learn about participants' experiences, life stories, or thoughts on a particular concept. The goal of clinical task-based interviews and the think-aloud method is to learn about participants' underlying cognitive processes that are engaged during completion of a particular task. These interview methods seek to uncover cognitive processes and reveal what and how participants are thinking.

Limitations of Interview Methods

Three limitations of clinical task-based interviews and the think-aloud method should be considered when using these methods to elicit participants' reasoning processes: (a) participant verbalization, (b) participant introspection, and (c) researcher influence.

Participant verbalization. Researchers conducting clinical task-based interviews and think-aloud methods need to consider limitations of these methods related to participants' verbalization of their thinking processes. Participants often have difficulty verbalizing their thinking (Ginsburg, 1981; Hunting, 1997; van Someren, Barnard, & Sandberg, 1994). Participant's use of language may make it difficult for researchers to make strong, trustworthy interpretations, and these interviewing techniques rely heavily on researcher interpretation (Ginsburg, 1981; Hunting, 1997; van Someren, Barnard, & Sandberg, 1994). Finally, the data are inevitably incomplete because participants can only verbalize what is in their working memory, so it is impossible for them to verbalize their entire thought process (van Someren, Barnard, & Sandberg, 1994).

Participant introspection. Participants may be inclined to attempt to interpret their thinking during think-aloud tasks rather than simply report it. Although the think-aloud method was designed to encourage only verbal reports, it is common for participants to interpret their thinking instead (van Someren, Barnard, & Sandberg, 1994). Such introspection can distort participants' actual thoughts, and may affect the quality of the data (van Someren, Barnard, & Sandberg, 1994). However, it is common for researchers conducting clinical task-based interviews to ask participants to reflect on their thinking after completion of the tasks. The difference is that participants are reflecting on their thinking after the task rather than interpreting their thinking during the task. In order to ensure reliability of the data, interviewers must encourage participants to only report their thinking, and not interpret it, while working on the tasks (Ginsburg, 1981; Hunting, 1997; van Someren, Barnard, & Sandberg, 1994).

Researcher influence. Researchers' must acknowledge that their interactions with participants influence the interviews. Regardless of the level of interaction, the researcher is still an active member of the interview process (Goldin, 1998; Hunting, 1997; Koichu & Harel, 2007; Rubin & Rubin, 2012). Thus, researchers must consider the effects of this social interaction on participants' responses (Goldin, 1998; Koichu & Harel, 2007). Participants may say what they think the researcher wants to hear or worry about the "right" answer as opposed to simply describing their thoughts or experiences (Goldin, 1998). Seidman (1998) warns that too much interaction can cloud the relationship between the researcher and the participant so that "the question of whose experience is being related and whose meaning is being made is critically confounded" (p. 80).

Interactions are of particular interest in clinical task-based interviews and the think-aloud method because a trade-off is often required with respect to the amount of prompting the researcher does. Prompts to think aloud or clarification questions can lead to completeness, but these can also disturb a participant by interfering with their thought processes (Goldin, 1998; Koichu & Harel, 2007; van Someren, Barnard, & Sandberg, 1994). Furthermore, interactions between the researcher, participant, and knowledge being constructed create a unique situation in each interview (Koichu & Harel, 2007; Patton, 2002). This makes replication of interview procedures difficult and may result in concerns for reliability (Goldin, 1998; Koichu & Harel, 2007).

Research Questions

In this study, I was interested in the ways that undergraduate students decide on the truth value of a mathematical statement. This includes their reasoning during the decision-making process as well as how that reasoning influences the subsequent construction of a corresponding proof or counterexample. Furthermore, I was interested in the types of systematic errors that students make during the proving process. Through clinical task-based interviews and the think-aloud method, I gained insight into students' reasoning during decision-making and constructing proofs and counterexamples. My overarching research questions were:

1. In what ways and to what extent do students use intuition and analysis to decide on the truth value of mathematical statements?
2. What are the connections between students' process of deciding on the truth value of mathematical statements and their ability to construct associated proofs and counterexamples?
3. What types of systematic intuitive, mathematical, and logical errors do students make during the proving process, and what is the impact of these errors on the proving process?

Reflexivity: Personal Perspective

The personal perspective of researchers, including their opinions and experiences, affects what questions they choose to ask as well as how they understand the answers to those questions (Goldin, 1998; Patton, 2002; Rubin & Rubin, 2012). Researchers need to acknowledge their impact on the interview process as well as their role in interpreting, analyzing, and describing the situation under study (Ernest, 1998; Goldin, 1998; Hunting, 1997; Koichu & Harel, 2007; Kvale & Brinkmann, 2009; Patton, 2002; Rubin & Rubin, 2012; Seidman, 1998). Consequently, researchers must reflect about how their perspective influences the interview process. My perspective on proof and proving evolved through three stages of experience: (a) my undergraduate study of mathematics, (b) my graduate study of mathematics, and (c) my study of mathematics education. This perspective influenced my choice of research questions and research design for this study.

My undergraduate study of mathematics. As an undergraduate student, proof and proving were about logic and deductive reasoning. My love of logic and deduction blossomed through completion of a variety of courses on formal logic, computer programming, and proof-based mathematics. As a mathematics major, I took a transition-to-proof course my sophomore year, and all of my upper-division undergraduate mathematics courses were proof-based. I had little difficulty unpacking the logic of definitions or theorems, understanding quantifiers, or proof methods. I could easily set up the logical structure of a proof and determine relevant inferences from definitions and the assumptions.

I had no reason to believe that proof was about anything other than logical reasoning. My mathematics instructors presented proofs in their polished form as if the entire thought process proceeded in a linear and deductive fashion. An acceptable proof on homework or exams was a deductive argument using definitions and previously proved theorems that led from the assumptions to the conclusion. All inferences were based on logical reasoning. It never occurred to me that there could be a difference between *proving* and *proof*. The process of proving was to construct a deductive proof, and it seemed that logical reasoning was the clear choice to lead to a deductive proof.

Looking back on my undergraduate career, I realize now that I could construct straightforward deductive proofs in which the proof essentially fell out from the definitions, but not much else. Additionally, I had difficulty understanding the abstract content of some of my upper-division courses, especially abstract algebra. Even though I could construct most of the proofs for the class, I did not understand how abstract algebra related to the algebra I had been doing since eighth grade. Thus, the proofs were not helping me make sense of the material. There was a disconnect between my ability to construct proofs and my ability to understand the material.

My graduate study of mathematics. As I completed my Master's degree in mathematics, my understanding of proof progressed and my success in constructing proofs declined. Abstract algebra made perfect sense to me the second time around. This understanding, along with algebra being congenial to a logic-based proof style, led to success in this area of mathematics. Analysis, however, was a different story. My approach did not work well on most analysis proofs because they seemed to depend on some insight or "trick" into how to construct an object with desired properties. Although I could not come up with the "tricks" on my own, I could read and understand analysis proofs, and could follow the description of the "trick" and its placement within the rest of the logical structure of the proof. Thus, I had no difficulty understanding the end result, the deductive proof, but I could not construct such proofs. Although proofs were still deductive arguments, the proving process now seemed to require something other than logical reasoning, something that I called “creativity.”

What was this creativity in mathematics that allowed for insight into “tricks” in proofs? Until graduate school, I had never considered it. Then, it became this magical quality that I was lacking. I did not think that I could learn to be creative in mathematics. I thought of creativity as some innate ability that I lacked, and there was nothing I could do about it. I had classmates who were creative, in my determination, and I would study with them, but I was unable to figure out the source of their creativity. Thus, the idea of creativity remained a mystery and an unattainable goal.

My study of mathematics education. My experiences with proof and my love of logic and mathematics led me to investigate proof and proving as a doctoral student in mathematics education. I wondered whether there was something more to mathematical creativity and whether it could be taught. I had heard that students did not need to learn logic in order to do mathematical proofs. How could that be possible when all of my proofs were built on logic?

Through my readings on semantic and syntactic reasoning by Alcock and Weber (Alcock & Weber, 2010; Weber & Alcock, 2004, 2009), I realized two important things: (a) I was a syntactic reasoner, and (b) what I called “creativity” closely resembled what they call “semantic reasoning.” One of the key difficulties syntactic provers face is that they can often produce a correct proof of a statement without understanding why it is true (Weber & Alcock, 2009). As an undergraduate, this was often the case for me. I did not use examples, graphs, visualizations, or other semantic aspects of mathematics to assist me with constructing proofs. I did not use these to help me obtain an informal understanding of a statement before attempting a proof construction. Maybe my difficulty with proving was not due to a lack of creativity in mathematics, maybe I lacked semantic reasoning skills. Even better, maybe I could learn such skills. Unfortunately, my instructors did not indicate that there was a place for semantic reasoning in proof productions. They did not demonstrate such skills while presenting polished proofs in class. They did not assign exercises geared toward developing semantic reasoning skills or using these in the proving process. This realization made me feel as though my mathematical education had been incomplete in process preparation, even if it was complete in product preparation. Additional readings, such as Selden and Selden (2009) and Tall (1991), reiterated the idea that proving is a process that entails semantic reasoning in addition to logical reasoning. Furthermore, they indicated that traditional mathematics instruction does not address the full range of reasoning competencies needed to construct mathematical proofs successfully.

In addition to semantic reasoning strategies, intuition is another idea that I had not considered as a part of proof production until I encountered it in my exploratory research study. The idea of intuition as an important component of proving led me to the literature on dual-process theory. This theory has changed my view of proving to more of a cognitive psychological rather than a mathematical perspective and has shaped my plan for data analysis in this study.

My current understanding of proof and proving is that proving is a complex process that involves intuition, semantic reasoning, creativity, logical, and deductive reasoning. Although the end result, the proof itself, is a syntactic argument, much more than logical reasoning goes into the process of constructing a proof. Additionally, the struggles that I encountered as a student are shared by others. Furthermore, other students have other types of difficulties, such as with logical reasoning skills. I am excited about the idea of being able to contribute to research on the teaching and learning of proof and to help students realize that the proving process involves an accessible variety of reasoning strategies that they can learn and learn to use successfully as creative and rigorous mathematics students.

My evolving understanding of proof and proving influenced the research questions and design of this study. I spent most of my mathematical career not knowing that there were intuitive, semantic, and syntactic reasoning strategies and not understanding why I was unsuccessful at intuitive and semantic aspects of the proving process although I could comprehend the proofs I read and thought that I had a strong understanding of the material. I want future mathematics majors to have better experiences with proving and more opportunities to develop intuitive and semantic reasoning skills and use these in proof constructions. Thus, the ultimate goal of this study is to improve instruction in proof and proving. However, as a stepping stone, I concentrated on students' *learning* rather than *instruction*. This study focused on students' use of intuitive, semantic, and syntactic reasoning in the proving process, especially when dealing with the uncertainty of deciding on the truth value of a mathematical statement. In the remainder of this chapter, I describe my sampling, data collection, interview, and data analysis procedures.

Sampling Procedures

The participants in this study were chosen according to purposeful sampling (Patton, 2002; Seidman, 1998), whereby I selected information-rich cases that were relevant to the purpose of the study. Seidman (1998) suggests that purposeful sampling is the best sampling strategy when random sampling is not an option and that maximum variation purposeful sampling is "the most effective basic strategy for selecting participants for an interview study" (p. 45). This sampling technique allowed for a range of participants that fit certain criteria and were "fair to the larger population" (p. 45). Patton (2002) notes that this sampling strategy yields two kinds of information: "(1) high-quality, detailed descriptions of each case that are useful for documenting uniquenesses, and (2) important shared patterns that cut across cases and derive their significance from having emerged out of heterogeneity" (p. 235). Being able to gather data on shared patterns helped me determine the extent to which my findings may be applicable in other situations (Rubin & Rubin, 2012). Additionally, this strategy allowed me to "explore the richness of a particular that may serve as an exemplar of something more general" (Ernest, 1998, p. 34). This section is organized into three parts: (a) participant selection, (b) sample size, and (c) description of participants. Participant recruitment and data collection took place between September and December of 2013 and occurred after the Ohio University Institutional Review Board (IRB) approved the proposed study.

Participant selection. Purposeful sampling was used to select participants from the main campus of a public university in the Midwest United States. I recruited participants from the students enrolled in the following upper-division undergraduate mathematics courses during fall semester 2013: College Geometry, Secondary Mathematics Methods, Secondary Mathematics Curriculum, Abstract Algebra I, Advanced Calculus I, or Undergraduate Mathematics Seminar. This selection strategy provided variation in students' experiences with proof as well as students' majors.

I attended a class meeting of each of the courses, described the study and what would be required of the participants, and asked the students to complete a short questionnaire (see Appendix A). The questionnaire included questions that determined if the student was interested in participating in a research study on mathematical proof, and if they met the selection criteria of (a) being an undergraduate student at the university, and (b) having passed at least one proof-based mathematics course with a grade of B or better. If a student was interested and met the selection criteria, they were asked to provide me with their email address so that I could contact them with additional information and set up an interview. However, this strategy did not yield a sufficient sample size, so I had an email sent to all mathematics majors at the university, including the questionnaire and asking the students to email me if they were interested in participating in the study. The use of this additional recruitment technique allowed me to achieve my desired sample size (n = 12).

By restricting my participants to those who had earned a B or better in a proof-based mathematics course, I intended to choose participants who could be successful on the interview tasks and had demonstrated successful training in constructing proofs and counterexamples. Researchers in mathematics education have indicated that in addition to understanding students' difficulties with proof, we need to understand how successful students reason during proof tasks (Harel & Sowder, 2007; Weber, 2009). Furthermore, Evans (2010) indicates that researchers applying dual process theories should move away from using participants that have no formal training in logical reasoning to studying participants with formal training in reasoning techniques. This may provide a different perspective on how intuition and analysis interact during reasoning and decision-making and can be used to test the scope of systematic intuitive errors.

Sample size. I selected a sample of 12 participants based on the recommendations of Guest, Bunce, and Johnson (2006) and a review of the sample sizes used in quality, peer-reviewed research on proof and proving utilizing clinical task-based interviews and the think-aloud method, as suggested by Onwuegbuzie and Leech (2007). My review of 11 articles yielded the following information: (a) there were 16 sample sizes reported as some articles reported on multiple studies; (b) the sample sizes ranged from two through 18 participants; (c) of the two studies that had 18 participants, one of them indicated that a sample of six was representative of the entire sample; (d) the mean sample size was approximately eight participants; (e) the median sample size was six; (f) the mode of the sample sizes was four. Thus, I chose to use 12 participants. This was reasonable based on this review, corresponded to the recommendation of Guest, Bunce, and Johnson (2006), and seemed sufficient to generate the variability described by Seidman (1998) and necessary for this study.

Description of participants. The participants in this study had one of the following majors: (a) mathematics, (b) Adolescent-to-Young Adult (AYA) integrated mathematics education (secondary mathematics teacher certification), (c) dual majors in mathematics and AYA integrated mathematics education, or (d) economics. Most students were in their fourth year of undergraduate study, but there was one student in his second year, and one student in his third year. The minimum number of college level mathematics course that a participant had taken or was enrolled in during data collection was six, and the maximum number was 17. However, this number is not necessarily an accurate reflection of the participants' mathematical background because (a) the university switched from quarters to semesters during the participants' undergraduate studies so that three courses on the quarter system would be equivalent to two courses on the semester system, and (b) the students' lowest level collegiate mathematics course ranged from college algebra to calculus III. The minimum number of proof-based mathematics courses completed by a participant was one and the maximum was seven, and some participants were enrolled in proof-based mathematics courses during data collection. However, different proof-based courses represent different levels of rigor so that an equivalent number of proof-based courses does not necessarily correspond to equivalent experiences with proof. Table 1 provides an overview of these demographics for each participant.

Table 1

Participant Demographics

Name (Pseudonym)	Major	Year in School	# college level math courses	# proof based courses completed	# proof based courses enrolled in Fall 2013
Aurelia	AYA math education	4	15	3	0
Edward	Economics	4	6	1	1
Elliot	Math	3	16	6	2
Emily	Math	4	15	4	1
Evan	Math	2	8	3	1
Inigo	Math/AYA math education	4	16	4	2
Jalynn	AYA math education	4	17	3	0
Jay	Math	4	11	3	1
Julie	AYA math education	4	14	4	0
Louis	AYA math education	4	10^	3	0
Michael	Math	4	17	7	1
Tina	Math/AYA math education	4	10	2	1

Note: ^ data incomplete

Despite differences in the participants' number of mathematics courses and proof-based mathematics courses taken, there were similarities in the mathematics courses they had taken at the university level. Every student passed either third semester or fourth quarter calculus and passed or was enrolled in some form of linear algebra (eight proof-based and four non-proof-based). Eleven of the 12 participants passed a non-proof-based discrete mathematics course and 10 of the 12 students passed a proof-based number theory course. Eight of the twelve students passed a college geometry course and passed or were enrolled in some form of abstract algebra (either introductory abstract algebra or abstract algebra). Finally, seven of the twelve students passed or were enrolled in an advanced calculus course. See Appendix F for a detailed list of the proof-based courses taken by each student.

Data Collection

I conducted two interviews with each participant and each lasted approximately one hour. I separated each interview with the same participant by roughly one week, but was flexible to accommodate participants' schedules. During the first interview, participants completed a questionnaire on their mathematical history, solved three tasks, and answered follow-up questions regarding the tasks (see Appendix C). There was one exception as Julie only completed two tasks due to time constraints (she completed three tasks during the second interview). In the second interview, participants completed two tasks and answered follow up questions to the tasks and general questions on mathematical proof and intuition (see Appendix D). The interviews were audio recorded for transcription by a paid transcriber. Each task used in the interviews provided a different piece of information related to my research questions, thus, this strategy allowed me to see a complete picture of this information from all participants. Additionally, I was able to analyze the data both across participants and across tasks, providing multiple layers of interpretation.

Data sources. The data for this dissertation came from (a) a questionnaire on participants' mathematical backgrounds, (b) transcripts from the participants' clinical task-based interviews using the think-aloud method, (c) participants' written work on the tasks in the interviews, and (d) my field notes from the interviews. The mathematical background questionnaire (see Appendix B) provided information on students' major, undergraduate experience with mathematics, which proof-based courses they had taken, and their grades in their mathematics courses (self-reported). This information offered insight into the population(s) from which my sample was drawn.

The two task-based interviews using the think-aloud method produced two data sources: verbal protocols (transcribed from the interview) and written documents. The protocol for the first interview included (a) three prove-or-disprove tasks, and (b) follow-up questions regarding the interview tasks (see Appendix C). The protocol for the second interview included (a) two prove-or-disprove tasks, (b) follow-up questions regarding the interview tasks, and (c) general questions regarding proof and intuition (see Appendix D). A random selection from the tasks was used to determine the order and choice of tasks on each interview. Furthermore, there were two different sets of directions for the tasks: (a) prove or disprove, and (b) determine, with proof or refutation, whether the following statement is true or false. For each participant, one set of directions was randomly chosen for the tasks in the first interview, and then the other set of directions was used for the tasks in the second interview. Thus, each participant saw both sets of directions, but the directions were consistent within each interview.

During completion of the tasks, participants were provided with paper on which to write any notes or scratch work as well as their completed proofs and counterexamples for the tasks. This paper was collected as further documentation of the students' thinking. Additionally, I provided participants with a LiveScribe Pen to use to complete the tasks. This Pen, when used on special paper, recorded audio and writing synchronously. This synchronization allowed me to listen to what participants said at each moment that they wrote.

I asked follow-up questions to the interview tasks in order to gather data retrospectively regarding participants' work on the tasks. These data were used to check against the participants' work during the task. Additionally, it provided reflective insight into students' thinking that was not expressed during their work on the task.

The general questions on proof and intuition provided information on the participants' typical approaches to prove-or-disprove tasks to indicate whether their approaches on the interview tasks were consistent with what they think they usually do. Additionally, the questions provided information on the participants' conceptions of proof and intuition.

Confidentiality. In order to minimize the possibility of identification, all participants were asked to give themselves a pseudonym under which their data were recorded and stored. However, not all participants chose to use a pseudonym. Participants were asked to write their chosen name on their mathematical history questionnaire and that name was used to identify the subsequent interviews, written work, and interview transcriptions. There is no written record linking the pseudonyms to participants' real names. The audio recordings and written work from the LiveScribe Pen and paper were uploaded to a password protected PC. The audio recordings were erased from the Pen itself between each interview. Additionally, the interviews were audio recorded on a second recorder as a backup. The interviews were deleted from this recorder after the transcription was complete. Finally, the participants' written work is currently stored in a secure cabinet accessible only to me.

Interview Procedures

Building rapport. Building rapport with participants was essential for making them feel comfortable enough to speak freely. Researchers can build rapport by being "open, honest, fair, and accepting" (Rubin & Rubin, 2012, p.79). Additionally, showing empathy and understanding helps build trust with participants (Kvale & Brinkmann, 2009; Patton, 2002; Rubin & Rubin, 2012). Participants feel more comfortable with researchers when they "feel some personal connection" or "share a common background" (Patton, 2002; Rubin & Rubin, 2012, p.79). I helped participants feel this way by sharing my experiences with mathematics and proof (see Appendix C). During the interview, I was careful not to press participants too hard for information and put them on the defensive (Rubin & Rubin, 2012; Seidman 1998). I remained neutral to participants' responses and did not convey judgment on their reasoning or the correctness of their answers through either verbal or non-verbal reactions (Hunting, 1997; Patton, 2002; Rubin & Rubin, 2012; Seidman, 1998). During the interviews, I let the participants know that I was not interested in the correctness of their answers, only their thinking processes. However, I showed empathy toward the participants, providing emotional scaffolding as needed to encourage them that they were doing fine, were being helpful, and to continue in the face of frustration.

Interview tasks. In order to design a study that would provide information on students' uses of intuition and analysis in deciding on the truth value of mathematical assertions and constructing proofs and counterexamples, the first step was to determine which types of mathematical tasks addressed these aspects of the proving process. I decided on prove-or-disprove tasks, inspired by the textbook *Extending the Frontiers of Mathematics: Inquiries Into Proof and Argumentation* by Burger (2007). Such tasks have two parts. First, students must decide if they think the statement in the task is true or false. This decision could use a variety of reasoning strategies, both intuitive and analytical. Second, students must construct either a proof or counterexample that supports their decision and may additionally use intuitive, semantic, and syntactic reasoning. This structure allowed me to investigate the ways that students reasoned during both the decision and construction processes, including whether students' reasoning in the decision process informed their proof or counterexample construction.

Participants were provided with five prove-or-disprove tasks, one at a time on separate sheets of paper, over the course of two interviews, each approximately an hour long. Two sets of directions were used for the tasks, one set per interview so that each participant received both over the course of their two interviews. The directions read either "Prove or disprove" or "Determine, with proof or refutation, whether the following statement is true or false." The differing directions were intended to promote deliberation on the truth value of the statement prior to attempting a proof. Participants were not (a) given any instructions other than to think aloud during the process, (b) provided with any information other than a list of definitions of terms in the tasks (see Appendix E), or (c) given the use of any materials. I indicated to the participants in the beginning of the interview (and reminded them as necessary) that I was not concerned with the correctness of their work, and that I was only interested in their reasoning processes. Although I wanted them to be successful, valuable data were collected from students' unsuccessful attempts. During the interviews, I asked participants to clarify or expand on their thinking if something was unclear or particularly interesting to me. Otherwise, I interfered with the thinking process as little as possible, except to remind them to continue speaking aloud when necessary.

The following tasks were used in this study.

Task A: *Injective Function Task*: Let $f: A \to B$ be a function and suppose that $a_0 \in A$ and $b_0 \in B$ satisfy $f(a_0) = b_0$. Prove or disprove: If $f(a) = b$ and $a \neq a_0$, then $b \neq b_0$.

Task B: *Monotonicity Task*: Prove or disprove: If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are decreasing on an interval *I*, then the composite function $f \circ g$ is increasing on *I*.

Task C: *Equivalence Relation Task*: Let *D* be a set. Define a relation ≈ on functions with domain *D* as follows: $f \approx g$ if and only if there exists $x \in D$ such that $f(x) = g(x)$. Prove or disprove: The relation ≈ is an equivalence relation.

Task D: *Global Maximum Task*: Prove or disprove: If f is an increasing function, then there is no real number c that is a global maximum for f.

Task E: *Composite Function Task*: Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions. Prove or disprove: If the composite function $f \circ g$ is one-to-one, then g is one-to-one.

The tasks cover basic information about functions and relations with which students who have taken at least one proof-based mathematics course should be familiar because all such students will have taken at least Calculus II. The tasks were chosen to (a) be accessible to the participants, (b) cover the same general topic of functions and relations, (c) be approachable using intuitive, semantic, or syntactic reasoning methods, and (d) provide opportunities to construct both proofs and counterexamples (see Appendix F for an elaboration on the origin of the tasks). Although the tasks were chosen to be accessible and within the same general topic, it became apparent during data analysis that the equivalence relation task was not as accessible as I had intended and did not seem to be in line with the other tasks that focused on functions. Subsequently, the Equivalence Relation Task was not included in the data analysis for this study.

In line with Alcock and Weber (2010), each of these tasks refers to general objects and their properties. These tasks were chosen to provide a range of tasks that I thought would be amenable to intuitive, semantic, and syntactic reasoning strategies, both in the decision-making and construction processes. The following hypotheses were based on my task analysis prior to data collection:

- The Global Maximum Task should lend itself to an intuitive reaction based on visualization.

- The Composite Function Task was not likely to produce an intuitive reaction due to the backward thinking needed. The task provided information about the composite function and asked for information about one of the component functions. However, students who are familiar with the fact that a composite function is one-to-one if its component functions are one-to-one may have an intuitive reaction based on similarity.
- The Injective Function Task should lend itself to analysis due to the logic needed to unpack the statement.
- The Monotonicity Task was of high interest because each student in my exploratory study committed a systematic intuitive error on this task that led them to believe that the statement in the task was true. Thus, this task was likely to produce an intuitive reaction that would lead participants to an incorrect designation of its truth value.

A distinguishing characteristic of the chosen tasks that affected students' approaches to the tasks is the *task complexity*. For this study, the task complexity was determined by the structure of the proof or counterexample for the task and was an indicator of the difficulty of the task (Selden & Selden, 2009). Relatively speaking, the Injective Function and Global Maximum Tasks are less complex and the Monotonicity and Composite Function Tasks are more complex.

The structure of a counterexample for the Injective Function Task requires only a direct application of the definition of one-to-one. A counterexample consists of any single non-injective function. On the other hand, the structure of a counterexample for the Monotonicity Task is more complex. It requires two distinct functions that satisfy a given property on a restricted interval such that when they are composed, they satisfy a different property on the same restricted interval. Additionally, the key to this counterexample is to choose an inner function (in the composite) with a range that is outside the given restricted interval.

The structure of a proof for the Global Maximum Task has some complexity because it is a proof by cases, but each case is straightforward. In the case in which the domain of the given function is all real numbers, the statement is true and a proof by contradiction can be used. The assumption for contradiction that there exists a global maximum directly contradicts the given assumption that the function is increasing. The other case, in which the domain is restricted to a closed interval, is false. A counterexample consists of any increasing function on any closed interval since it will attain a global maximum at the right hand endpoint of the interval.

The structure of the proof of the Composite Function Task is relatively more complex compared to the proof of the Global Maximum Task. The proof by contradiction involves an assumption for contradiction that requires the negation of an implication, a logical process that can be confusing. Through this assumption, the existence of two specific points is posited. The proof then requires the application of the given assumption that the composite function is one-to-one on these specific points. Note that because the given assumption is not used until after the assumption for contradiction, the construction of this proof is not a linear process. Finally, the application of the given assumption results in an expression that contradicts the outer function (in the composite) being a function. This is not an obvious contradiction to the definition of function, and it does not contradict the given assumption in the task that is often the expected contradiction.

Data Analysis

The data were analyzed according to uses of intuition and analysis during the participants' processes of deciding whether each assertion was true or false and constructing proofs and counterexamples. Additionally, students' decision-making and construction processes were analyzed to determine students' decision-making pathways and the connections between these processes. Finally, systematic intuitive and analytical errors were identified, and the correctness of proofs and counterexamples was determined.

Intuitive and analytical reasoning. Originally, reasoning used during the proving process was to be classified as intuitive or analytical, with analytical reasoning further classified as semantic or syntactic. However, through the process of attempting to classify students' reasoning, many subtypes of intuitive, semantic, and syntactic reasoning emerged. In order to accommodate these subtypes, I developed a framework (Table 2) that incorporated aspects of the following theories/frameworks: (a) dual-process theory (Evans, 2008, 2012b), (b) semantic and syntactic reasoning (Alcock & Weber, 2010; Weber & Alcock, 2004, 2009), (c) warrant-types (Inglis et al., 2007), and (d) proof schemes (Harel, 2007; Harel & Sowder, 1998).

The reasoning classification framework has four main categories: intuitive, semantic-empirical, semantic-deductive, and syntactic. The intuitive category includes six subtypes and corresponds directly to intuition in dual-process theory and roughly to Inglis et al.'s structural-intuitive warrant-types. Analytical reasoning includes semantic-empirical, semantic-deductive and syntactic reasoning. Semantic reasoning was divided into two main categories – semantic-empirical and semantic-deductive – to reflect the differences in semantic reasoning based on empirical evidence versus semantic reasoning based on informal definitions and deductions. The semantic-empirical category has two subtypes and roughly corresponds to Inglis et al.'s (2007) inductive warrant-type and Harel's (2007) empirical proof scheme class. The semantic-deductive category has eight subtypes and roughly corresponds to Harel's (2007) transformational proof schemes. Finally, syntactic reasoning includes six subtypes and corresponds directly to syntactic reasoning in Alcock and Weber's theory as well as roughly to Harel's (2007) modern axiomatic proof schemes. Further, if grouped together, my semantic-deductive and syntactic reasoning types closely correspond to Inglis et al.'s (2007) deductive warrant-types and Harel's (2007) deductive proof scheme class (that includes transformational and modern axiomatic proof schemes as subclasses).

The four main categories in my classification scheme were chosen to distinguish both among intuitive, semantic, and syntactic reasoning and between reasoning that is acceptable as a basis for a decision versus reasoning that is acceptable as support for a decision. By "acceptable," I mean what is acceptable to a given mathematical community. Examples of mathematical communities include an elementary school classroom, an undergraduate discrete mathematics course, or the community of contemporary mathematicians. Each mathematical community sets its own standards for what is acceptable as proof of a mathematical assertion.

Although each type of reasoning – intuitive, semantic-empirical, semantic-deductive, and syntactic – can form the basis for a decision, intuitive and semantic-empirical reasoning are not acceptable as support for a decision. On the other hand, semantic-deductive and syntactic reasoning are sufficient to support a decision, but the level of acceptable formality will vary depending of the given community. In some communities (such as secondary school classrooms), semantic-deductive reasoning is acceptable as proof of a statement's truth value, but in other communities (such as the community of contemporary mathematicians), only syntactic reasoning constitutes acceptable proof.

For the purposes of this study, it was important to distinguish between semantic-deductive and syntactic reasoning because only syntactic reasoning is acceptable as proof to the mathematical community to which the students belong (the community of students taking upper-level undergraduate mathematics courses). This became clear based on students' professed ideas about proof as well as their general behavior to proceed into a syntactic argument although they had already convinced themselves of a statement's truth value with a semantic-deductive argument.

Table 2

Reasoning Classification Framework

Category	Subtypes
Intuitive	(a) Memory-based (b) Property-based (c) Similarity-based (d) Understanding-based (e) Unjustified (f) Visualization-based
Semantic-empirical	(a) Example-based (b) Graph-based
Semantic-deductive	(a) Definition-based informal argument (b) Diagram-based informal argument (c) Generalization (d) Graph-based informal argument (e) Inconclusive-based informal argument (f) Informal definition-based (g) Kinaesthetic-based informal argument (h) Visualization-based informal argument
Syntactic	(a) Counterexample (b) Failed counterexample (c) Failed proof (d) Formal definition (e) Need for assumption (f) Proof/Disproof

Intuitive reasoning. I classified reasoning as intuitive if the student was unable to fully justify the reasoning or used similarity to make an assessment of the task (Evans, 2010; Kahneman & Frederick, 2002; Witteman & van Greenen, 2010). Descriptions of the six subtypes of intuitive reasoning are given in Table 3.

Table 3

Subtypes of Intuitive Reasoning

Subtype	Description
Memory-based	Based on a personal memory regarding the concepts
Property-based	Based on vague ideas about properties of concepts
Similarity-based	Based on similarity with other concept(s)
Understanding-based	Based on personal understanding of the concepts
Unjustified	No basis
Visualization-based	Based on visualization of the concepts

Analytical reasoning. I classified reasoning as analytical if it was semantic-empirical, semantic-deductive, or syntactic reasoning that was fully justified by the student (Evans, 2010). This included reasoning that was justified at the time of the reasoning or during post-task questioning.

Semantic-empirical reasoning. Semantic-empirical reasoning is based on empirical evidence. Descriptions of the two sub-types of semantic-empirical reasoning are in Table 4.

Table 4

Subtypes of Semantic-empirical Reasoning

Subtype	Description
Example-based	Based on an example as a test case
Graph-based	Based on a graph as a test case

Semantic-deductive reasoning. Semantic-deductive reasoning is based on informal definitions or deductions. Descriptions of the eight subtypes of semantic-deductive reasoning are given in Table 5.

Table 5

Subtypes of Semantic-deductive Reasoning

Subtype	Description
Definition-based informal argument	Informal deductive argument based on informal definitions
Diagram-based informal argument	Informal deductive argument based on a diagram
Generalization	Informal deductive argument generalized from an example
Graph-based informal argument	Informal deductive argument based on a graph
Inconclusive-based informal argument	Informal deductive argument based on inconclusiveness of information
Informal definition-based	Based on informal representations or rephrasing of formal definitions
Kinaesthetic-based informal argument	Informal deductive argument based on kinaesthetic movement
Visualization-based informal argument	Informal deductive argument based on visualization of concepts

Syntactic reasoning. Syntactic reasoning is formal deductive reasoning that adheres to the guidelines for a syntactic proof (Weber & Alcock, 2009) or accepted refutation. Descriptions of the six subtypes of syntactic reasoning are in Table 6.

Table 6

Subtypes of Syntactic Reasoning

Subtype	Description
Counterexample	Based on an example that is recognized as refutation for a statement
Failed counterexample	Based on a failed attempt at constructing a counterexample
Failed proof	Based on a failed attempt at constructing a proof
Formal definition	Based on a formal definition
Need for assumption	Based on the need for an assumption that was not given
Proof/disproof	Formal argument based on syntactic reasoning that proves or refutes a statement

Pathways and connections. Students' decision-making and proof/counterexample construction processes were analyzed in order to determine their decision-making pathways as well as the connections between the decision-making and construction processes. My original pathways framework based on decision-making in dual-process theory turned out to be too restrictive in terms of its focus on intuitive versus analytical reasoning as well as the possible pathways. A more general framework was needed because so few decisions were intuitive. The resultant pathways depend on only the basic flow of reasoning (of any type) and decisions. The pathway structure is linear and the emphasis is on the order in which reasoning and decisions occur. Thus, the pathways indicate whether reasoning preceded or succeeded each decision. With this new framework, any number of pathways can emerge. In this study, the students used eight distinct pathways (Figure 3). In the pathway structure, blunt and pointed arrows represent reasoning and boxes represent decisions.

The naming scheme for the pathways indicates the order of reasoning (R) and decisions (D). For example, Pathway RDR indicates reasoning followed by a decision, followed by further reasoning. The classification of the reasoning used to make a decision is based on the reasoning that precedes the decision. In pathways such as DRD or DRDR, multiple decisions are made and the reasoning that occurs between the decisions is often a combination of reasoning that (a) attempts to support the first decision, (b) overturns the first decision, and (c) supports the second decision (in Pathway DRD, but not DRDR). Pathways that end in decisions may include full support for the decision offered before the decision was made. Additionally, pathways can begin with decisions not based on any type of reasoning.

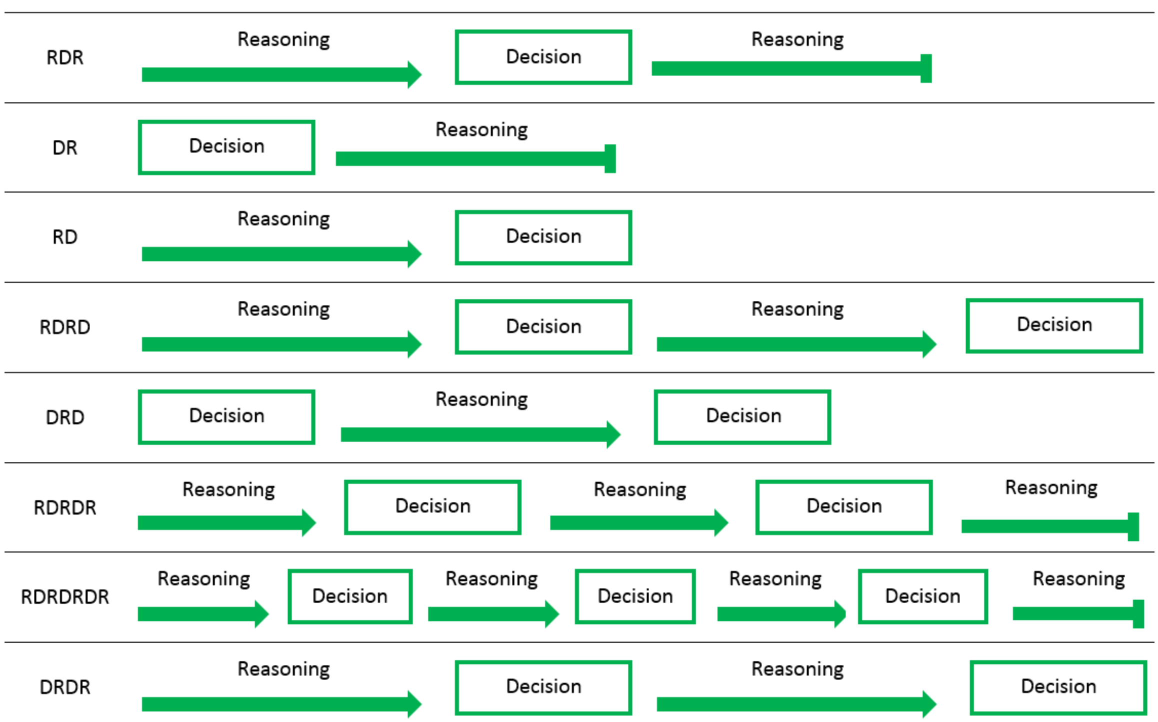


Figure 3. Decision-making pathways.

The analysis of the connections between students' decision-making and proof/counterexample construction processes resulted in two distinct types of both connections and disconnections. Connections between construction and decision-making were categorized as either construction based on decision-making or simultaneous construction and decision-making. Disconnections were constructions not based on decision-making and no decision-making. Some of these differences can be seen in the decision-making pathways. Pathways that begin with decisions represent disconnections in which no decision-making actually occurred. Pathways that end in decisions represent connections with simultaneous decision-making and construction processes. Pathways including reasoning that occurs both before and after a decision represent situations in which the construction was either based or not based on the decision-making process. This distinction requires deeper analysis to identify and is indicated on a case-by-case basis in the results section.

Errors. I analyzed errors that were made during the proving process to determine if they were systematic intuitive or analytical errors. Additionally, I identified cases in which students were able to overcome errors. Errors were classified as systematic if they were committed by at least three students. Systematic intuitive errors were categorized as relevance errors according to the definition given in the literature review. The students in this study did not commit any attribute substitution errors and no new intuitive errors were identified.

Systematic analytical errors included both mathematical and logical errors. Mathematical errors can be contributed to faulty or incomplete mathematical knowledge, such as incorrectly graphing a given function or incorrectly identifying an example of a given concept. Logical errors include: incorrect logical inferences, misinterpretation of logical notation, and use of inappropriate proof frameworks.

Search for patterns. The data were analyzed for emerging patterns in terms of both participants and tasks (Ginsburg, 1997; Patton, 2002). One level of analysis focused on describing how each participant completed the tasks. This allowed for identification of patterns that emerged across participants on each task as well as patterns regarding each participant's performance on the tasks. A broader level of analysis focused on patterns that emerged across tasks and participants (so that they were not task- or participant-dependent).

Credibility and Trustworthiness

A variety of strategies can be used to increase the level of credibility and trustworthiness in qualitative research, and I used: (a) triangulation of (i) interview transcripts and documents, and (ii) tasks within and across participants; and (b) first-level member checks (Bratlinger, Jimenez, Klingner, Pugach, & Richardson, 2005; Glesne, 2011; Patton, 2002).

Triangulation. Two types of triangulation of data sources were used: triangulation of verbal protocols and documents, and triangulation of tasks within and across participants (Bratlinger et al., 2005; Ginsburg, 1997; Glesne, 2011; Patton, 2002). I used both the verbal protocols from the interviews and the written work from the interviews to provide multiple data sources. Additionally, I compared participants' strategies across two interviews that occurred roughly a week apart. These triangulation methods were considered both within and across participants (Ginsburg, 1997). Thus, the data were analyzed in terms of what each participant did across all tasks (the tasks acted as multiple data sources for the participant) as well as what was done on each task across all participants (the participants acted as multiple data sources across tasks).

First-level member checks. First-level member checks of transcripts allowed participants to confirm the accuracy of their responses to the interview questions (Bratlinger et al., 2005; Rubin & Rubin, 2012). Within a few days of each interview, I provided participants with a short summary of their responses to each question from the interview. I did not address the correctness of their responses or provide any opinion or judgment on their responses. I simply asked them if the summary accurately reflected their responses in the interview. In the few situations in which a student indicated that something was inaccurate, I changed it to reflect what they thought was the correct version of their response. These changes were minor and did not affect the data in this study.

Chapter 4: Results and Discussion

In this chapter, I present the results of this study and conclude with a discussion related to my research questions. The chapter is organized by task, with the following order: Monotonicity Task, Composite Function Task, Injective Function Task, and Global Maximum Task. Within each task, the results are organized by research question so that they address:

1. In what ways and to what extent do students use intuition and analysis to decide on the truth value of mathematical statements?
2. What are the connections between students' process of deciding on the truth value of mathematical statements and their ability to construct associated proofs and counterexamples?
3. What types of systematic intuitive, mathematical, and logical errors do students make during the proving process, and what is the impact of these errors on the proving process?

The results of Research Question 1 (RQ1) orient the reader to each task and provide an overview of the students' reasoning as categorized by the reasoning classification framework (Table 2). The results of Research Question 2 (RQ2) include detailed summaries of students' work on each task to illustrate their decision-making and construction processes. Finally, the results of Research Question 3 (RQ3) detail students' errors and their impact on the proving process.

Monotonicity Task

The Monotonicity Task considers what happens when two decreasing functions are composed (Figure 4). The statement in this task is false because of the interval restriction. Thus, a $f: \mathbb{R} \to \mathbb{R}$ counterexample for this task must include a function g with outputs that are not elements of the interval *I*. This task would have been true if it had been stated without the interval restriction. This is why I expected students to commit systematic intuitive errors that would lead them to an incorrect decision.

Prove or disprove: If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ decreasing on an interval *I*, then the composite function $f \circ g$ is increasing on *I*.

Relevant Definitions from Definition List:

If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are two functions, then the **composite function** $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$.

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be **decreasing** if and only if for all $x_1, x_2 \in \mathbb{R}$, ($x_1 < x_2$ implies $f(x_1) > f(x_2)$).

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be **increasing** if and only if for all $x_1, x_2 \in \mathbb{R}$, ($x_1 < x_2$ implies $f(x_1) < f(x_2)$).

Figure 4. The Monotonicity Task and relevant definitions.

The results of this task are organized by approaches students used: correct counterexample, incorrect proof, and over-generalization of an examples. The students were generally unsuccessful on this task with only 4 of the 12 providing a *correct* *counterexample* or demonstrating a correct understanding of why the statement was false. Five students provided an *incorrect proof* of the task based on the definitions of increasing and decreasing on ℝ (a correct proof for the related task without the interval restriction). Three students constructed an *incorrect generalization* based on an example that indicated that the statement may be true.

Decisions and justifications for decisions (RQ1). The 12 students made a total of 17 decisions on the Monotonicity Task – five made two decisions each whereas the other seven students made only one decision. Of the five students who made two decisions, two students overturned incorrect true decisions for correct reasons that led to a correct decision of false supported by correct counterexamples.

I classified the reasoning of the decisions based on the type of reasoning that preceded each. The students justified their decisions on the Monotonicity Task using intuitive, semantic-empirical, semantic-deductive, and syntactic reasoning. Of the 17 decisions on this task, three were intuitive and 13 were analytical. The following subtypes of reasoning were used for decision-making on the Monotonicity Task:

- Intuitive: Property-based (3)
- Semantic-empirical: Example-based (4)
- Semantic-deductive: Definition-based informal argument (2)
- Syntactic: Proof (2) and disproof (2)
- Combination: Syntactic failed proof and syntactic counterexample (1)
- Combination: Semantic-deductive, inconclusive-based, informal argument, syntactic failed proof, and syntactic need for assumption (1)

- Combination: Semantic-empirical example-based and semantic-deductive generalization (1)

One student simply assumed that the statement was true, thus her decision was not justified by either intuition or analysis.

Of the 12 students who made a total of 17 decisions– five made two decisions each whereas the other seven students made only one decision. All five students who made two decisions on this task overturned their first decision and the reasoning that was used to overturn the first decision also acted as full support for their second decision. Three students initially thought the statement was true but overturned those incorrect decisions. Aurelia overturned an incorrect true decision through a failed proof attempt that led her to re-examine an example and find a counterexample. Edward's failed proof attempt due to a needed assumption led him to realize why this statement was actually false and overturn his original true decision in favor of a correct false decision. Jay's intuition led him to think that the statement would be true, but a mathematical mistake in his attempted proof resulted in a disproof that overturned his incorrect true decision. Two students initially thought the statement was false but overturned those correct false decisions to ultimately decide incorrectly that it was true. Jalynn overturned a correct decision supported by an incorrect disproof by noticing a mathematical mistake in her disproof. However, her revised proof was also incorrect. Tina's intuition was that the statement was false, but over generalizing an example convinced her to overturn her correct false decision to claim that the statement was true instead.

Participants used both intuitive and analytical reasoning to make decisions, but they only used analytical reasoning to support their decisions. Table 7 provides an overview of the reasoning used by each student on this task for both decision-making and supporting decisions. Each student in the *incorrect proof group* provided either a syntactic proof or disproof of the statement. Three students in the *correct solution group* constructed syntactic counterexamples, and the fourth student provided an informal argument for why the statement would be false. Each student in the *generalization group* constructed an informal argument based on generalizing an example. None of the students in the generalization group believed that their example was sufficient to prove the statement, and all attempted to provide syntactic justification for why the phenomena in the example would generalize.

Table 7

Types of Reasoning Used on Monotonicity Task Organized by Approach and Student

Student	**Types of Reasoning Used**			
	Intuitive	Semantic-empirical	Semantic-deductive	Syntactic
		Incorrect Proof		
Elliot			Definition-based informal argument	Proof
Emily Evan				Proof
Jalynn				Proof Disproof
Jay	Property-based			Disproof
		Correct Solution		
Aurelia		Example-based		Failed proof, Counterexample
Edward		Example-based	Inconclusive-based informal argument	Failed proof, Need assumption
Inigo	Property-based			Counterexample
Michael			Definition-based informal argument	Counterexample
		Generalization		
Julie Louis		Example-based	Generalization	
Tina	Property-based	Example-based	Generalization	

All analytical reasoning–semantic and syntactic–used on this task to both make and support decisions was deliberate and justified. To support decisions, eight students used syntactic reasoning in the form of either a proof, disproof, or counterexample, and four students constructed semantic-deductive informal arguments, including three who generalized examples.

All intuitive reasoning used on this task was non-deliberate and only partially justified. The three intuitive decisions were based on vague ideas regarding the properties of the functions in the task. Jay's intuition was based on the idea that the composition of two decreasing functions would result in an increasing function, similar to the way in which the product of two negative quantities results in a positive quantity. Inigo had a vague idea about how two negatives would behave with a quadratic function. Tina based her intuitive decision on the idea that graphs of decreasing functions should continue to decrease when the functions are composed.

Connecting decisions and constructions – Decision-making pathways (RQ2). In this section, I will discuss students' decision-making pathways and the connections between the decision-making and construction processes. There were four different decision-making pathways used on this task (Figure 5). Pathways RDR and RDRD were used by five students each, and Pathways DR and RD were each used once. There were six students whose pathways ended with a decision (Pathways RD and RDRD) and six students whose pathways ended with follow-up justification (Pathways RDR and DR).

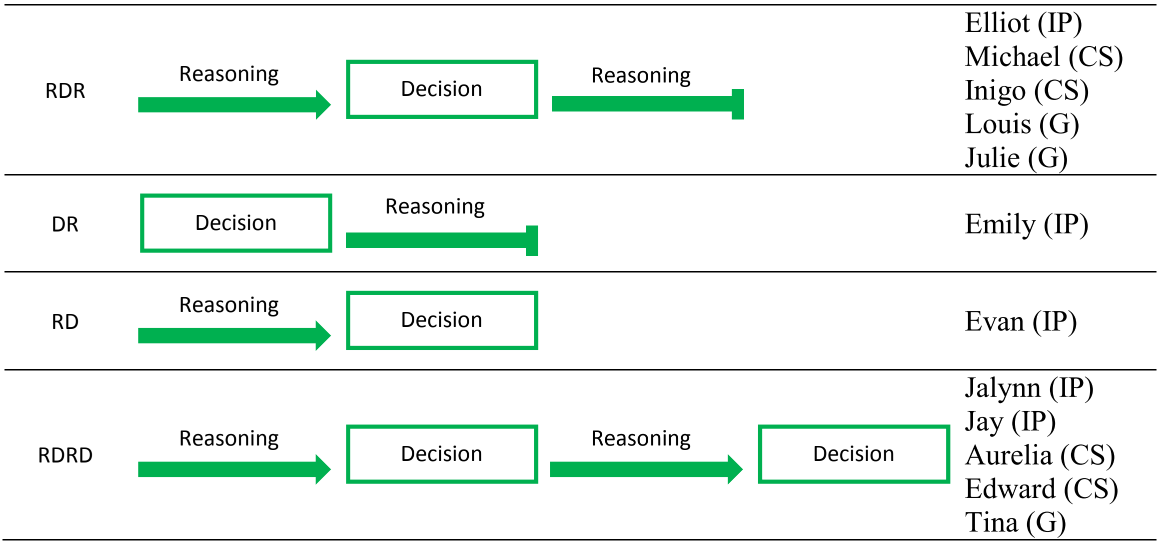


Figure 5. Decision-making pathways for Monotoncity Task. (IP) indicates incorrect proof group, (CS) indicates correct solution group, and (G) indicates generalization group.

This section will be organized by solution groups. Within each solution group, further organization will be provided by pathway, and within pathways, organization is by reasoning type.

Incorrect proof group. Each student constructed the same basic incorrect proof based on the inequality inference in the definitions of decreasing and increasing with limited or no consideration of the domains of the functions in the task. An example of the incorrect proof is as follows:

> Assume $x_1 < x_2$. Since g is decreasing, $g(x_1) > g(x_2)$. Then, because f is decreasing, $f(g(x_1)) < f(g(x_2))$. Thus, $x_1 < x_2$ implies $f(g(x_1)) < f(g(x_2))$. Hence, the composite function $f\big(g(x)\big)$ is increasing.

Although this is not a proof of the given statement, it is a correct proof for the following related statement: If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are decreasing, then the composite function

$f \circ g$ is increasing. Thus, the incorrect proof does not incorporate the interval restriction that is a key part of the Monotonicity Task.

Elliot. Elliot's decision-making followed Pathway RDR with a semantic-deductive, definition-based, informal argument leading to a true decision supported by a syntactic proof (Figure 6). Elliot began by constructing an informal argument based on an informal representation of the definitions of increasing and decreasing. He used vertical arrows to indicate whether the elements of the domain or range of the functions f and g were increasing or decreasing (Figure 7). Elliot noticed that the arrows switched directions when the composite function was formed, and decided the statement was true:

> So that means that as x increases, $g(x)$ decreases and for f...as x decreases on f, $f(x)$ increases 'cause it would just be going the opposite way on this interval *I*. And since the values $g(x)$ in the range of g are what's being plugged into f, that means the values of $f(g(x))$, the values of $g(x)$ are decreasing, so $f(g(x))$ would be increasing. So I'm going to say that's true, and be, go about proving that.

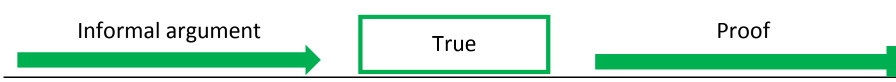


Figure 6. Elliot's decision-making pathway.

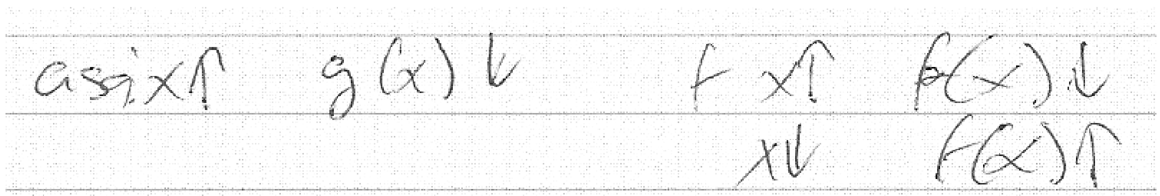


Figure 7. Elliot's semantic work.

Elliot translated his informal argument into a syntactic proof by connecting the changing direction of the arrows to the changing direction of the inequalities in the formal definitions of increasing and decreasing. Because Elliot's proof was based on his informal argument, his decision-making and construction processes were connected.

Emily. Emily's decision-making followed Pathway DR with a true decision supported afterward by a syntactic proof (Figure 8). Emily did not engage in a legitimate decision-making process because she was "assuming it was true from the get-go." Due to her experiences in her mathematics courses in which she was "used to being handed a statement and told to prove it, and told that it's true," she did not consider the statement's truth value being in question.

Figure 8. Emily's decision-making pathway.

Emily began by writing the definition of decreasing for the functions f and g. Unlike the other students in this solution group, Emily eventually recognized that the inputs in the definition belonged to the interval I. However, when she applied the definition of f decreasing to $g(x_1)$ and $g(x_2)$ she failed to verify that they were elements of the interval *I*, resulting in a variation on the incorrect proof. Because Emily simply assumed that the statement was true, her decision-making and construction processes were not connected.

Evan. Evan's decision-making followed Pathway RD with a syntactic proof leading to and supporting a true decision (Figure 9). Evan did not have an initial sense of whether the statement was true or false:

> As of now, and I'm not actually sure which way I think this is going to go, so I'm going to come back to my definition of the composition function and see if there's just a better way to rewrite this so I can see what, kind of, what direction it's going.

Evan algebraically composed the functions to determine the direction of the final inequality because this determined whether the composite function was increasing or decreasing. By ignoring the interval restriction, Evan's work led him to conclude that the composite function was increasing and the statement was true. Because Evan's proof simultaneously led to and supported his decision, his decision-making and construction processes were one in the same.

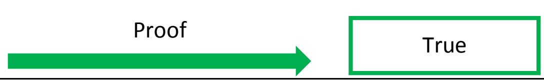


Figure 9. Evan's decision-making pathway.

Jalynn. Jalynn's decision-making followed Pathway RDRD with a syntactic disproof leading to a false decision that was overturned by a syntactic proof that resulted in and supported a true decision (Figure 10). In the beginning, Jalynn was uncertain of the truth value of the statement, so she algebraically composed the functions and applied the definition of decreasing to determine the direction of the inequalities. However, Jalynn made a mathematical mistake based on a substitution she had made that led her final inequality to indicate that the composite function was decreasing (Figure 11). She stated, "So that means it's a decreasing function. I mean, it's still decreasing. So then it's false then. I think. That's what, I mean, I don't know if I confused myself somewhere."

Figure 10. Jalynn's decision-making pathway.

Due to her concern, Jalynn reviewed her work:

> Oh, wait, but hold on. So it is increasing....Let me look at my definition. Yeah. Because $x_1 < x_2$, and we found for it to be increasing that the function of that [x_1] is less than that [x_2]. So to rewrite, it's $f\big(g(x_1)\big) < f\big(g(x_2)\big)$ where $x_1 < x_2$.

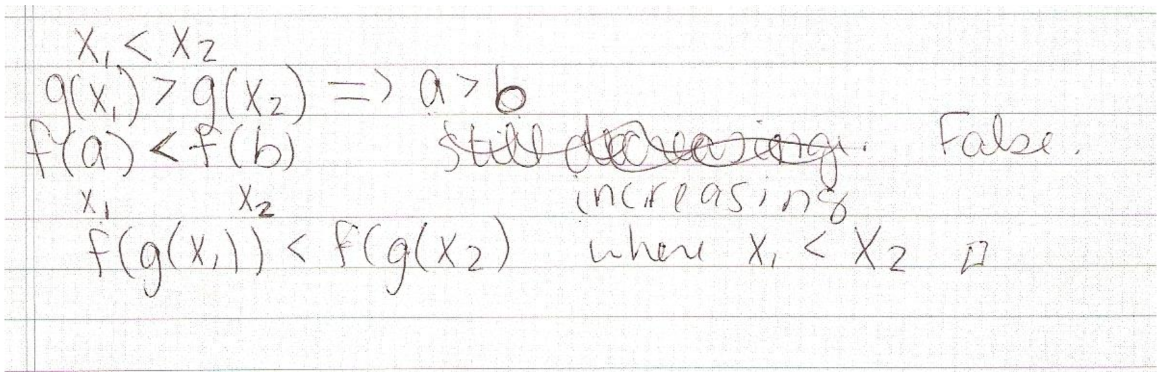


Figure 11. Jalynn's proof illustrating her initial error based on her substitution of a for $g(x_1)$ and b for $g(x_2)$.

Thus, Jalynn identified and overcame her mathematical mistake and decided that the statement was true. However, because she did not attend to the interval restriction in the task, her final proof was the incorrect proof. Jalynn's decision-making and proof construction processes were synonymous because each decision came after she had a disproof or proof of the statement, respectively.

Jay. Jay's decision-making followed Pathway RDRD with a property-based intuition leading to a true decision that was overturned by a syntactic disproof that resulted in and supported a false decision (Figure 12). At first, Jay thought that the statement was true based on a vague idea about how the functions should behave. He described, "Initially, I thought it was probably true because if you take the composite of a decreasing function, it seems like it should be increasing if both are decreasing....Sort of, like, the double negative."

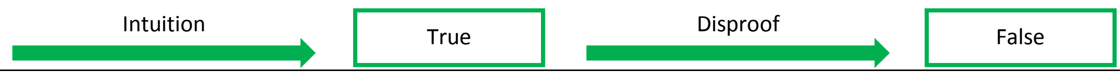


Figure 12. Jay's decision-making pathway.

Jay used the definition of decreasing and algebraically composed f and g in an effort to support his intuitive decision. However, a mathematical mistake with his inequalities (the same one that Jalynn made) coupled with ignoring the interval restriction led Jay to construct a variation of the incorrect proof that disproved the statement. Jay's decision that the statement was false resulted from having a disproof of the statement so that his decision-making and construction processes were one in the same for his second decision.

Jay's initial intuitive decision was based on a vague idea about how decreasing functions should behave, but the relationship between this decision and his disproof construction is unclear. It is possible that either his reference to the "double negative" was an attempt at a retrospective justification of his intuition based on his disproof or he actually had this idea in mind before he wrote his disproof. Jay did comment that he was expecting the inequalities to switch in his work: "plug that into f, then that should turn it around." This could indicate that he initially thought the inequalities would flip "like, the double negative." This expectation that the inequalities would switch could have been based on Jay's intuitive idea that composing two decreasing functions may work like multiplying two negative quantities, but because of his mathematical mistake, he did not get the anticipated result. However, the data do not indicate whether this relationship to the "double negative" resulted from Jay's work with the inequalities and was not a component of his original intuitive decision or if his idea that the inequalities should switch stemmed from his intuitive idea about how these functions should behave.

Correct solution group. There were four students who determined correctly that the statement in the Monotonicity Task is false. Aurelia, Inigo, and Michael provided a correct counterexample. Edward demonstrated a correct understanding of why the task is false, but did not provide a counterexample. Each student produced at least one example, but the examples were used and constructed in various ways. Six decisions were made by this group – one intuitive, two semantic-empirical, one semantic-deductive, one syntactic, and one semantic-deductive/syntactic combination. This section is organized by pathway and reasoning.

Michael. Michael's decision-making followed Pathway RDR with a semantic-deductive, definition-based, informal argument leading to a false decision supported by a syntactic counterexample (Figure 13). Michael began by considering the formal definitions of decreasing for f and g and taking the derivative of the composite function $f(g(x))$. He was aware that he was making an unjustified assumption about the differentiability of the functions, but thought it would help him make progress on the task. He realized that in order for the composite function to be increasing, the outputs of $g(x)$ would have to be in the interval *I* where the function f was decreasing:

> Because if that's the case [that the functions are differentiable], then its $[g(x)]$ derivative is, um, less than or equal to zero....Then since f is also, it $[f'(g(x))]$ should be...less than or equal to zero, which means that their product [the derivative, $f'(g(x))g'(x)$] should be greater than or equal to zero, but that only

holds if $g(x)$ is in *I*. I'm trying to think of a counterexample because it wouldn't work if, if $g(x)$ weren't in *I*, then it [$f(x)$] might not be decreasing on that interval.

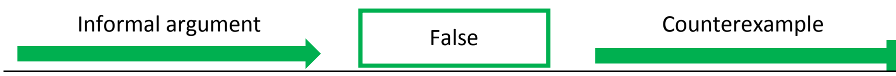


Figure 13. Michael's decision-making pathway.

Michael's informal argument not only led him to decide that the statement was false, but also provided him with specific conditions for a counterexample to satisfy. He started with the function $g(x) = -x - 1$ and restricted it to $x \in [0,1]$. He then noted that the outputs of $g(x)$ would be in the interval $[-2, -1]$. He wanted to find a function f that was decreasing on $[0,1]$ and increasing on $[-2, -1]$. He tried the function $f(x) = -x^2$ multiple times before he realized that it satisfied his requirements. Michael was unique in his approach to constructing the counterexample in that he used the interval restriction and the properties that he wanted in the function f in order to determine an appropriate function. This is a sophisticated example-construction strategy not often used by undergraduate students (Iannone, P., Inglis, M., Mejia-Ramos, J. P., Simpson, A., & Weber, K., 2011). Michael described his process, "I went about trying to find a function that sent, sent the elements of, say, the unit interval to somewhere where the function f wasn't decreasing." When describing his counterexample, Michael said:

because g takes this out of the interval itself, f doesn't necessarily have to be decreasing on any place outside of that interval because that's all they're saying.

> They're saying if it's [f] decreasing on one interval that would mean that g would have to take it to the interval itself in order for f's decreasing-ness [sic] to even apply.

Inigo. Inigo's decision-making followed Pathway RDR with a property-based intuition leading to a false decision supported by a syntactic counterexample (Figure 14). Though Inigo had "never actually thought about this [task] before," initially he did not think it was true. Later, he said, "I definitely looked at it and knew it was false, and I'm not sure how...It didn't feel true...I just felt like a negative and a negative with functions is not going to do something like that because of something squared." This property-based intuition led directly to a counterexample. He showed that an example with linear functions would indicate that the statement was true, but a more complicated example would tell a different story: "If f is that [$-x^2 + 2$] and then g is $-x$, then...[$f(g(x))$ is] $-x^2 + 2$. So that one is still decreasing. So this is not true."

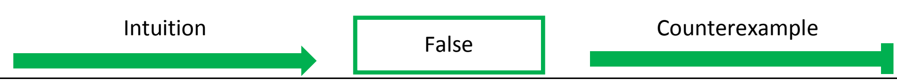


Figure 14. Inigo's decision-making pathway.

Inigo's counterexample was incorrect because he did not provide an interval restriction, and both f and $f(g(x))$ were not decreasing on their entire domains. However, Inigo's uncertainty regarding f and g as decreasing functions prompted him to correct this error. After mentally considering the functions' derivatives, Inigo wrote the derivative of f, drew a set of axes for a graph, then drew a partial graph of f on $[0, \infty)$.

He noted that both *f* and *g* were decreasing on $[0, \infty)$, and chose this as his interval *I*. By partially graphing his counterexample, Inigo realized his error and provided a complete and correct counterexample. Because Inigo's counterexample involved a quadratic and two negatives, the intuition that led to Inigo's decision also led to his counterexample construction, connecting his decision-making and construction processes.

Aurelia. Aurelia's decision-making followed Pathway RDRD with a semantic-empirical example leading to a true decision that was overturned by the combination of a syntactic failed proof attempt and a syntactic counterexample, resulting in a false decision (Figure 15). Aurelia began with the relevant definitions in the task, but she made no progress, so she tried an example "to test it out and see, like, which way it worked." She chose $f(x) = -x$ and $g(x) = -x^2$, resulting in $f(g(x)) = x^2$. She did not restrict these to an interval, committing an error because g and $f \circ g$ are both increasing and decreasing, depending on the interval. However, when she drew the partial graph of $g(x) = -x^2$ from $(-\infty, 0)$ she decided to "just deal with this from zero to infinity," indicating that she realized the function was only decreasing on $(0, \infty)$. As with Inigo, a graph allowed Aurelia to overcome her error and restrict appropriately the function. Because the composite function was increasing on the restricted interval, she decided that the statement was true.

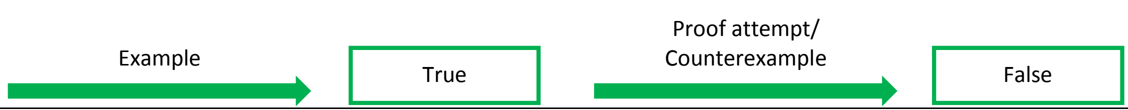


Figure 15. Aurelia's decision-making pathway.

Aurelia did not analyze her example for an indication of why the statement may be true; rather she attempted a direct proof and then a proof by contradiction using the formal definition of decreasing. Thus, her decision-making and construction processes for her first decision were not connected. When she was unable to make progress on a proof, rather than change her decision, she tried to construct a counterexample. She returned to her example, drawing a pair of coordinate axes and wondering "What if I put these [*f* and *g*] in the opposite? Ah!" She interchanged the functions f and g in her example, making $f(x) = -x^2$ and $g(x) = -x$ so that the composite was $f(g(x)) = -x^2$. She hesitated, drew a partial graph of $f\left(g(x)\right) = -x^2$ on $(0, \infty)$, and said "that's right, because I want it to be decreasing."

Aurelia's failed proof attempt prompted her to search for a counterexample that led to the overturn of her initial decision and the new decision that this task was false. Although Aurelia's failed proof attempt did not inform her search for a counterexample, the counterexample led directly to her decision that the statement was false, connecting her decision-making and construction processes for her second decision.

Edward. Edward's decision-making followed Pathway RDRD with a semantic-empirical example leading to a true decision. That decision was overturned by the combination of a syntactic failed proof attempt, syntactic need for an assumption, and a semantic-deductive inconclusive-based informal argument that led to a false decision (Figure 16). Edward began with an example, choosing $f(x) = -x + 2$ and $g(x) = -x^2$, and indicating his chosen restricted interval: "So let's say our *I* is zero to positive infinity. Both are decreasing." Upon composing the functions, $f\left(g(x)\right) = x^2 + 2$, Edward said, "That is increasing. All right, so it could be true. Is it necessarily true? On this practice interval, it is increasing." Because Edward's example did not inform his subsequent proof attempt, his first decision-making process was unconnected to his proof construction process.

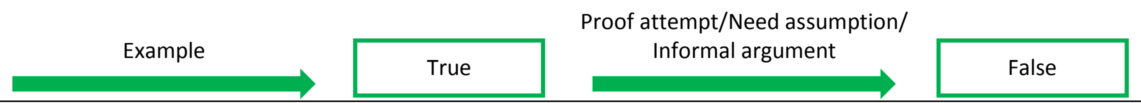


Figure 16. Edward's decision-making pathway.

Edward's proof attempt was similar to the incorrect proof with one significant difference – Edward knew that in order for his proof to be correct, he had to assume that the range of the function g was in the interval *I*. In fact, he made this assumption in his proof, but upon completing his proof, noted "I'll say it's increasing on *I*. Although I didn't do a good job at all of proving where *I* is or working with where *I* is." This concern over the interval restriction led Edward to determine that his proof was invalid:

> If they are both decreasing on an interval *I*, that doesn't necessarily mean the intervals overlap...Because we would need the range of g to be in *I*...we would need the domain of f to be the same decreasing interval as the range of *g,* and we'd need to, the domain of *g* to be decreasing. So, and I didn't prove that connection. I should have.

In response to further prompting, he explained, "I would not necessarily believe this proof because I didn't match up the range to the domain."

When I asked about the assumption in his proof that the range of g was in the interval *I*, he indicated that it was a necessary but illegitimate assumption, "If I make that assumption, yes...that the range of g then is also in this interval *I*,...it does work...But without making that assumption, yes, I don't think it holds....I don't think that's an assumption I can legitimately make." However, it was not until he finally made sense of why that assumption was necessary that he decided that this statement was actually false. He concluded:

> Without this [the assumption], f…could be increasing or decreasing *I*. I mean, depending on where the range of g is mapped onto the domain of f and what, whether it's increasing or decreasing at that interval...'cause the interval...doesn't necessarily line up at f and g. That makes this statement false.

For Edward's second decision, his decision-making and construction processes were connected because they coincided. His syntactic failed proof attempt and need for an assumption led to his semantic-deductive, informal argument regarding the assumption's necessity, all of which both led to and supported his decision that the statement was false.

Generalization group. Three students constructed a semantic-deductive generalization of an example based on the idea that the product of two negative quantities is a positive quantity. Each student in this group looked at an example of the composition of two decreasing functions. In each case, the students chose negative functions as the decreasing functions, and through the composition, the negatives canceled, resulting in a positive function that was classified as increasing. Each student determined that negativity was a defining feature of decreasing functions, and that this phenomena would happen any time two decreasing functions were composed, resulting in an increasing function. This section is organized by pathway and students with the same pathway and reasoning are discussed together.

Julie and Louis. The decision-making of Julie and Louis followed Pathway RDR with a semantic-empirical example leading to a true decision that was supported by a semantic-deductive generalization of the example (Figure 17). Although Julie began with an example, Louis first considered the definition of decreasing and used it to construct his example:

> The definition, it's saying from one y term to the next of, by the definition of a decreasing function, is smaller even though the x's are increasing. So by looking at this, by plugging in a decreasing number into your function, does that make the overall function increase?

Louis was the only student to think of decreasing in terms of "plugging in a decreasing number." Louis chose $g(x) = -3x$ and $f(x) = -2x$, but he did not form the general composite function. Instead, he chose a set of increasing x-values, 1, 2, and 3, and formed the composite function for each specific x-value (6, 12, and 18, respectively) (Figure 18).

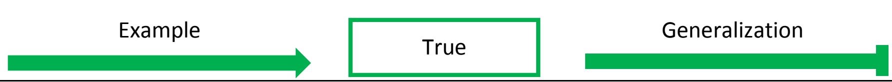


Figure 17. Julie and Louis' decision-making pathway.

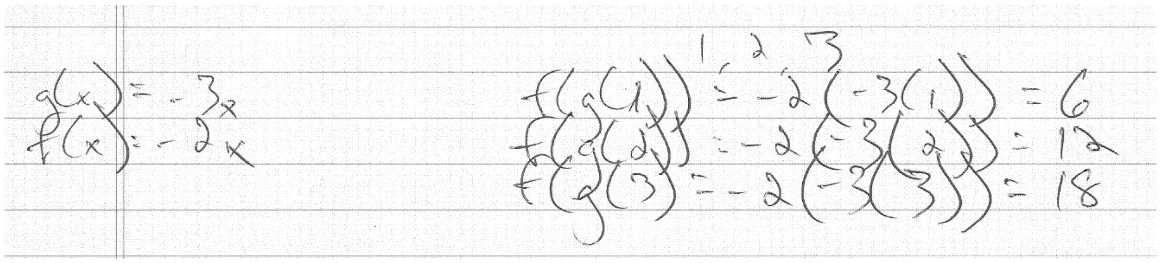


Figure 18. Louis' computational composite functions.

In response to his results, he said "So in that specific case, the function was increasing." Julie chose $f(x) = -x$ and $g(x) = -x^3$ so that $f(g(x)) = x^3$. She said that this composite function was positive and therefore increasing. Julie concluded, "just based off of that, I'm going to assume that this is true."

Both students' examples led them to decide that the statement was true, and they supported this decision with a generalization of what happened in the example. Julie explained:

> It is going to be true because anytime you have a negative and if the, if both *f* and *g* are decreasing, then that means that they're both negative...so for what I would assume is that decreasing functions are negative, and when you compose the two, you're going to have a negative times a negative."

Similarly, Louis claimed, "So you're multiplying your decreasing value, which in these cases were negative by the negative slope to give you a positive number, which overall was increasing." Although both Julie and Louis tried, neither student was able to use the definition of decreasing to construct a syntactic argument, so they settled on their generalizations. For both students, the example that led to the decision also formed the basis for their generalization that supported the decision, so their decision-making and construction processes were connected.

Tina. Tina's decision-making followed Pathway RDRD with a property-based intuition leading to a false decision that was overturned by a semantic-empirical example and its semantic-deductive generalization, resulting in a false decision (Figure 19). Tina started with the relevant definitions and constructed an example (a mathematical error resulted in a decreasing composite function.), but said "I don't think that helped me at all." Then, she expressed the following intuitive idea about properties of decreasing functions:

> You would think if, like, the graphs were already decreasing, I don't know how they can just automatically increase unless that something dramatic changed. Like, they're both on the same interval, so wouldn't they just both stay decreasing if you're putting a decreasing function into a decreasing function?...I'm pretty sure that it's going to be decreasing.

Although Tina expressed this intuition after her preliminary work with the definitions and example, it is unclear whether her intuition related to that work because Tina's intuition referenced graphs and intervals rather than the definitions and example in her preliminary work. On the other hand, even though she did not say anything about it, her example showed that the composite function was decreasing, so her intuition may have stemmed from the example.

Figure 19. Tina's decision-making pathway.

Tina attempted to analyze her example by substituting specific numbers into the functions, and a mathematical error suggested that the numbers were decreasing in the composite function. Thus, Tina began rewriting her example to provide clearer support for her decision. However, after writing "if $x_1 = 5, x_2 = 6, g = -x^2$," she said:

> If I have two decreasing functions and I do its composite, the negatives will cancel out, making it a positive function. Which, if that's true, then it would be increasing then. So would the composite be increasing then? I would say that maybe it's right.

She did not write the example that led her to this change in decision, but since she was rewriting her earlier example, it is likely that she correctly composed the functions this time, as this would support what she said. This combination of an example and its generalization led to her second decision and sufficed as both the decision-making and construction process for that decision

Errors/Difficulties (RQ3). The students in this study committed intuitive, mathematical, and logical errors on the Monotonicity Task (Table 8). The only systematic intuitive error was a relevance error related to the interval restriction in the task. The students committed two systematic mathematical errors: defining decreasing functions as negative and overgeneralizing the idea that "two negatives make a positive." Additionally, students made various nonsystematic mathematical and logical errors. In this section, I will discuss the three systematic errors and then summarize the nonsystematic errors.

Table 8

Error Types on Monotonicity Task Organized by Approach and Student

Student	**Error Types**
	Incorrect Proof
Elliot Evan	Intuitive – relevance
Emily	Intuitive – relevance* Logical – antecedent not satisfied
Jay Jalynn	Intuitive – relevance Mathematical – inequalities (Jalynn)*
	Correct Solution
Aurelia Inigo	Intuitive – relevance*
Edward	Logical – illegal assumption*
Michael	None
	Generalization
Julie	Intuitive – relevance Mathematical – decreasing means negative Mathematical – overgeneralization
Louis	Mathematical – decreasing means negative Mathematical – overgeneralization
Tina	Mathematical – decreasing means negative Mathematical – overgeneralization Mathematical – composition of functions* Mathematical – multiplication
* error was overcome	

Relevance errors. The only systematic intuitive errors committed on this task were relevance errors regarding the interval restriction. These errors took on two forms: (a) ignoring the meaningfulness of the interval restriction, and (b) completely ignoring the interval restriction. This error is especially significant in this task because the interval restriction is the key to the falsity of the statement. Eight students made this error, including all students in the incorrect proof group, Aurelia and Inigo in the correct solution group, and Julie in the generalization group. Only Emily (incorrect proof group), Aurelia, and Inigo (correct solution) were able to overcome this error.

Ignoring meaningfulness. Evan, Elliot, and Jalynn from the incorrect proof group did not place value on the interval restriction beyond being a trivial component of the task. All three students wrote the interval restriction in the beginning of their proof, but never used it in a meaningful way. Both Evan and Jalynn rewrote the definitions for increasing and deceasing, but did not include the domain for the functions. They only wrote the inequality implication (thus they did not indicate what set x_1 and x_2 were elements of, and only wrote, for example, $x_1 < x_2$ implies $f(x_1) > f(x_2)$ for the definition of f decreasing). By not attending to the interval restriction, Evan, Elliot, and Jalynn essentially changed the task to the related task without the interval restriction for which the incorrect proof is a correct proof. Because these students intuitively deemed the interval restriction irrelevant, they were led to an incorrect solution for the task. Although the students' reasoning on this task was mathematically sound (since Jalynn corrected her mathematical mistake), they ended up with an incorrect solution because they were reasoning about a different task than the one given to them. For Evan, Elliot, and Jalynn, an incorrect determination by their intuition that the interval restriction was not meaningfully relevant led to the incorrect solution. Their intuitive representation of the task excluded the interval restriction before these students had the chance to reason about it. Furthermore, because their syntactic work seamlessly led to a proof, no red flags went up to signal that they needed to take it into account in their reasoning.

Aurelia's intuition also ignored the meaningfulness of the interval restriction in this task, but she overcame this error with a partial graph of a quadratic example function. Aurelia began with an example of two decreasing functions and their composite function, but she used a quadratic as one of the decreasing functions and did not initially provide an interval restriction. Since quadratics are only decreasing on part of their domains, this is an incorrect example without the associated interval restrictions. After composing $f(x) = -x$ and $g(x) = -x^2$, Aurelia drew the portion of the graph of $g(x) = -x^2$ from (- ∞, 0) and said "Oh, we'll just deal with this from zero to infinity," indicating that the function was increasing in her graph and only decreasing, as desired, on (0, ∞). Thus, sketching this partial graph of a quadratic function led Aurelia to realize that the interval restriction was necessary and allowed her to overcome her relevance error.

Ignoring completely. Julie and Inigo completely ignored the interval restriction, but Inigo was able to overcome it through an experience similar to Aurelia's. Upon proposing the counterexample of $f(x) = -x^2 + 2$, $g(x) = -x$, and $f(g(x)) = -x^2 + 2$, Inigo offered no interval restriction. However, he expressed uncertainty regarding whether f and g were both decreasing. This uncertainty prompted him to consider their derivatives and then sketch a pair of coordinate axes. He asked me "Am I allowed to choose the interval?" I said that he could, and he drew the portion of the graph of f on $[0, \infty)$ and said "that's [g] decreasing all the time, and that one's [f] decreasing there [on $[0, \infty)$]. So at least there [$[0, \infty)$], it overlaps, and you said for any interval, so I'm going to stand by this." Like for Aurelia, Inigo's sketching of a partial graph of a quadratic helped him realize that he needed the interval restriction and allowed them to overcome his relevance error. Unlike Aurelia and Inigo, Julie chose linear and cubic functions in her example – functions that are decreasing on their entire domains. Thus, when she drew the graphs of her example functions, there was no red flag that went up as was the case when Aurelia and Inigo drew the graphs of their quadratic functions. So, like most of students in the incorrect proof group who committed a relevance error, there was nothing that caused conflict with Julie's intuitive choice to ignore the interval restriction.

Emily also ignored the interval restriction at first, but overcame this error by carefully reading the task. At first, Emily defined f as a decreasing function as follows: $\forall x_1, x_2 \in \mathbb{R}, x_1 < x_2,\ f(x_1) > f(x_2)$. Then, as she was writing the definition for g, she looked back at her definition for f and said "I'm actually going to change my x_1 and x_2 to belonging in the interval because I guess I only know it's decreasing on it." She crossed out $\mathbb{R}$ in her definition for f and replaced it with *I*. She then completed her definition for g, also with x_1 and x_2 elements of *I*.

Finally, Jay also completely ignored the interval restriction on this task because he only mentioned it while initially reading the statement aloud. Thus, Jay's intuitive representation deemed the interval restriction completely irrelevant to the task.

Decreasing functions are always negative. Four students committed the mathematical error of deciding that decreasing functions would always be negative (in the sense that the leading coefficient would be negative). This led to an incorrect solution for Julie, Louis, and Tina (the entire generalization group) who used this association as the basis for their overgeneralization of their example. Inigo correctly solved this task, but used this relationship as the basis for a semantic-deductive generalization of a counterexample. Each student expressed uncertainty about this idea, but used it anyway.

Julie and Inigo both indicated that decreasing functions have negative slopes by definition, as Inigo noted: “Those are decreasing because their slope is negative, and that’s the definition of a decreasing function, I believe.” There is the additional issue that they specifically said that decreasing functions have negative slopes, and only linear functions have slopes. Students often use the word “slope” inappropriately, such as when they say a function has a negative slope although they mean that it has a negative derivative. The students in this study typically indicated that the leading coefficient of the function was negative when they said the function had a negative slope. Tina and Louis did not specifically indicate that the relationship was by definition, but did express their hesitant belief in it, as evidenced by Tina’s statement: “So if it’s true that for a function to be decreasing, then it’s negative...”

Generalization of “two negatives make a positive.” Although this mathematical error only caused difficulties for the three students in the generalization group (Julie, Louis, and Tina), seven of the students in this study stated a version of this idea at some point during their work on this task. Julie, Louis, and Tina used the idea that decreasing functions are always negative as the basis for their generalization that when composing two decreasing functions, the two negatives (from the decreasing functions) would multiply and cancel, resulting in a positive, increasing, composite function. Each student saw this phenomenon occur in the example they chose to explore and decided that it would occur anytime two decreasing functions were composed. Thus, the idea that "two negatives make a positive" was generalized from their example into a semantic-deductive informal argument for the truth of the statement. Although this was the key to their generalization, there were other issues involved such as the fact that the students generalized from a single example and did not take into consideration a variety of functions or the order of the composition.

Nonsystematic errors. Of the nonsystematic errors on this task Jay and Jalynn each wrote their inequalities in the wrong direction in their proofs for the composite function. Both errors stemmed from substitutions the students made that seemed to confuse them. Jay did not double check his work although he originally thought that the inequalities would work out the other way. On the other hand, Jalynn immediately thought that she had confused herself with the inequalities, double-checked her work, and fixed her mistake. During Tina's generalization, she made algebraic mistakes by incorrectly composing her example functions and incorrectly multiplying negative numbers.

Emily and Edward each made logical errors. Emily assumed the conclusion of an implication without checking that the antecedent was satisfied (and it was not). When she used the assumption that f was decreasing, she did not check that the inputs (the g outputs) were in the interval I even though she had written that they must be for f to be decreasing. She did not realize this error. Edward assumed a statement in a proof that was not a given assumption and did not follow from other statements in the proof.

However, his uncertainty about this assumption led him to conclude that he could not legitimately make the assumption. Thus, he overcame the error, leading to his correct decision that the statement was false.

Monotonicity Task summary. The students in this study provided three distinct solutions for this task. The incorrect proof was based heavily on syntactic reasoning and relevance errors regarding the interval restriction. Although each student in this group committed a relevance error, Emily was the only one to overcome the error. Students mostly engaged in reasoning prior to making decisions, and then followed their decision with a proof or disproof.

Students who provided correct solutions to the task engaged in intuitive, semantic-empirical, semantic-deductive, and syntactic reasoning. Aurelia and Inigo (in the correct solution group) overcame relevance errors through the use of a partial graph of a quadratic function. All students in this group engaged in reasoning prior to an initial decision. Inigo and Michael followed their decisions with counterexamples. Aurelia and Edward followed their decisions with proof attempts that led to the overturning of their initial decision and a second, correct decision.

The generalization group used mostly semantic-empirical and semantic-deductive reasoning, and committed the systematic errors of assuming decreasing functions are negative and overgeneralizing. Julie and Louis based their decision on an example and supported it with a subsequent generalization of the example. Tina began with an intuitive decision that was overturned by an example and generalization that led to her second decision.

Composite Function Task

The Composite Function Task asks if a one-to-one composite function implies that the inner function is also one-to-one (Figure 20). The statement in this task is true and can be proved by contradiction (Figure 21).

Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions. Determine, with proof or refutation, whether the following statement is true or false: If the composite function $f \circ g$ is one-to-one, then g is one-to-one.

Relevant Definition(s) from Definition List

If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are two functions, then the **composite function** $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$.

A function $f: \mathbb{R} \to \mathbb{R}$ is called **one-to-one** if and only if for all $x_1, x_2 \in \mathbb{R}$, $(f(x_1) = f(x_2)$ implies $x_1 = x_2)$.

Figure 20. Composite Function Task and relevant definitions.

Assume $(f \circ g)(x) = f\big(g(x)\big)$ is one-to-one. Assume for contradiction that g is not one-to-one. Then, there exists $a_1, a_2 \in \mathbb{R}$ such that $a_1 \neq a_2$ and $g(a_1) = g(a_2)$. Since $f(g(x))$ is one-to-one and $a_1 \neq a_2$, we have $f(g(a_1)) \neq f(g(a_2))$. But, this is a contradiction because $g(a_1) = g(a_2)$. Thus, g is one-to-one.

Figure 21. Proof of the Composite Function Task.

The students were mostly unsuccessful on this task (only two constructed a correct proof of the statement) despite the fact that nine decided correctly that the statement was true. Students' incorrect proofs and arguments resulted from a variety of logical and mathematical errors. The three students who thought that the statement was false constructed counterexamples that were incorrect based on not using the absolute value when taking a square root.

The analysis for this task is based on only 9 of the 12 students in this study. Louis, Julie, and Tina each had significant difficulty on this task, so I excluded them from the analysis. Each student thought the statement was true, but their confusion on this task led to incoherent arguments based on incorrect understandings of the concept of one-to-one and an inability to meaningfully link the function g to the composite function $f \circ g$. These struggles made it impossible for me to categorize meaningfully their decision-making and construction processes, so their work on this task was irrelevant to my research questions.

Decisions and justifications for decisions (RQ1). The students in this study justified their decisions on the Composite Function Task using both intuitive and analytical reasoning. The nine students made a total of 13 decisions – six students made one decision each, two students made two decisions each, and one student made three decisions. All three students who made multiple decisions (Emily, Inigo, and Michael) were concerned about the lack of information on the function f and had failed proof and/or counterexample attempts.

Of the 13 decisions made on the Composite Function Task, five were intuitive, six were analytical, and one was based on a combination of intuitive and analytical reasoning. The following subtypes of reasoning were used for decision-making on this task:

- Intuitive: Similarity-based (3), understanding-based (1), and unjustified (1)
- Syntactic: Need for assumption (1) and failed counterexample (2)
- Combination: Syntactic need for assumption and failed proof (2)
- Combination: Intuitive memory-based and syntactic need for assumption (1)
- Combination: Semantic-empirical example-based and semantic-deductive informal definition (1)

Additionally, one student simply assumed that the given statement was true.

All nine of the students who completed this task constructed a syntactic proof or counterexample in support of their final decision on this task. Because students' decisions were based mostly on intuition or syntactic needs for assumptions or failed proof or counterexample attempts, it makes sense that these would be followed up with syntactic proofs or counterexamples. No students based a decision on an informal argument for this task, and only one student used semantic reasoning for decision-making.

All analytical reasoning used on this task during the decision-making and construction processes was deliberate and justified. In addition to the semantic reasoning already discussed that led to decisions, Elliot used a semantic-deductive diagram/definition-based informal argument in combination with a syntactic proof to support his decision. The remaining eight students supported their decisions with a syntactic proof or counterexample.

Students' intuitive reasoning on this task was non-deliberate and either unjustified or partially justified by similarity, understanding, or memory. Jay noted that "Initially, I thought it was probably true," but provided no justification for this intuitive decision. Elliot and Jalynn simply noted that their decision was based on having seen something similar to this task before whereas Edward specifically noted that this task was similar to the true statement that if f and g are one-to-one, then $f \circ g$ is one-to-one. Evan's intuition was understanding-based because his decision was based on the idea that the statement seemed to "make sense" to him. Finally, Emily's memory-based intuition was rooted in the idea that she remembered the true statement that if f and g are one-to-one, then $f \circ g$ is one-to-one was not an if and only if statement. The chart in Table 9 provides an overview of the reasoning used by each of the nine students included in the analysis on this task for both decision-making and supporting decisions.

Table 9

Types of Reasoning Used on Composite Function Task Organized by Approach and Student

Student	**Types of Reasoning Used**			
	Intuitive	Semantic-empirical	Semantic-deductive	Syntactic
		Counterexample		
Aurelia		Example-based	Informal definition	Counterexample
Emily	Memory-based			Need assumption Failed counterexample Failed proof Counterexample
Inigo				Need assumption Failed proof Counterexample
		Proof		
Edward Jalynn	Similarity-based			Proof
Elliot	Similarity-based		Diagram/ definition-based informal argument	Proof
Evan	Understanding-based			Proof
Jay	Unjustified			Proof
Michael				Need assumption Failed counterexample Proof

Connecting decisions and constructions – Decision-making pathways (RQ2). Four different pathways were used on this task (Figure 22). Inigo, who used Pathway DRDR, was the only student whose pathway began with a decision as he simply assumed that the given statement was true. Each pathway used ended in support for a decision.

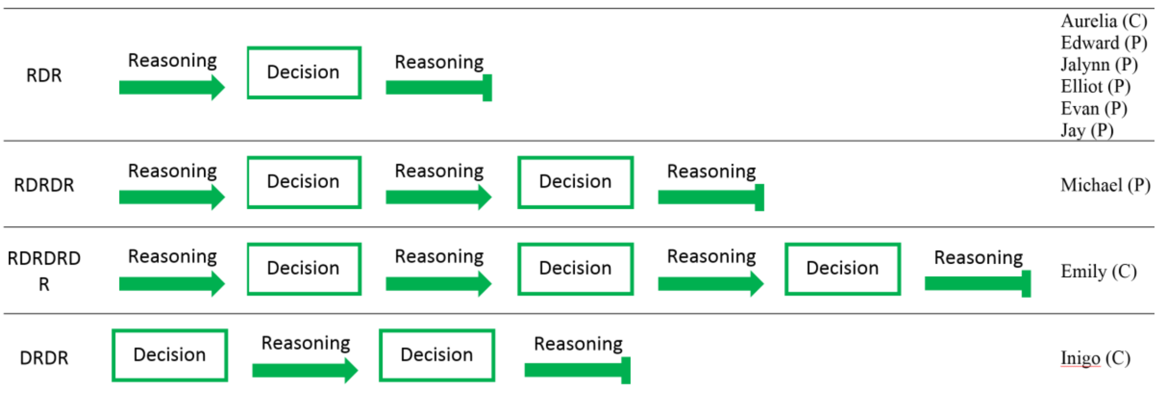


Figure 22. Decision-making pathways used on Composite Function Task. (C) indicates counterexample group and (P) indicates proof group.

All nine students provided syntactic support for their decisions on this task in the form of proofs and counterexamples. Many students did not engage in semantic reasoning on this task, using only syntactic reasoning or intuition followed by syntactic reasoning. The solutions to this task are organized by solution type, resulting in two categories: *syntactic counterexample* and *syntactic proof*. Within each solution group, additional organization will be provided by pathway, and within pathways, organization is by reasoning type.

Syntactic counterexample group. The three students in this group, Aurelia, Emily, and Inigo, decided incorrectly that this statement was false and provided a counterexample to support their decision. Each student in this group used a different pathway.

Aurelia. Aurelia's decision-making followed Pathway RDR with the combination of a semantic-deductive informal definition and a semantic-empirical example leading to a false decision that was supported by a syntactic counterexample (Figure 23). Aurelia began by considering the definitions of composite function and one-to-one, but she struggled to make sense of the definition of one-to-one. Then she decided to look at an example. She chose $g(x) = x^2 - 1$, and $f(x) = 2x$ so that $f(g(x)) = 2x^2 - 2$. However, she said that these were all one-to-one functions and she needed an example of a function that was not one-to-one. She said that a vertical line was not one-to-one, indicating her confusion regarding the definition of one-to-one and the definition of a function. Finally, she tried to look at a piece-wise defined function, but only constructed part of it and a partial corresponding graph.

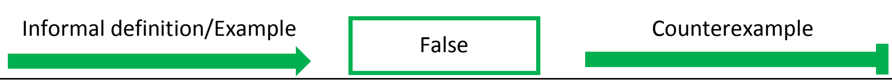


Figure 23. Aurelia' decision-making pathway.

Due to her struggles with trying to find an example of a function that was not one-to-one, Aurelia decided to determine exactly what it meant to not be one-to-one based on the definition. She wrote that $f(x_1) = f(x_2)$ when $x_1 \neq x_2$. At this point, she realized that it was the horizontal line test that corresponded to one-to-one, and she said that both $\cos(x)$ and x^2 were not one-to-one. She then provided the following counterexample: $g(x) = x^2$, $f(x) = \sqrt{x}$, and $f(g(x)) = x$. However, this is an incorrect counterexample because $f(g(x)) = |x|$.

After Aurelia constructed her counterexample, I asked her at what point she thought that the statement was false, and she said it was once she realized "one-to-one is horizontal lines, not vertical lines, where it has to pass through both." She then choose x^2 because it "is really easy to manipulate, and it's not one-to-one." So it was the combination of realizing that the horizontal line test corresponded with the definition of one-to-one and recognizing that x^2 was not one-to-one that led Aurelia to decide that this was a true statement. Her decision-making was connected to her counterexample construction since her counterexample involved x^2.

Emily. Emily's decision-making followed Pathway RDRDRDR with the combination of memory-based intuition and a syntactic need for an assumption leading to a false decision that was overturned by a syntactic failed counterexample attempt. The failed counterexample attempt led to a true decision that was overturned by the same syntactic need for an assumption leading to another false decision that was supported by a syntactic counterexample (Figure 24). Initially, Emily thought that this statement was false because she thought that she needed additional information about the function f, and her memory of a similar statement did not include the statement in the task: "I remember a proof that when both functions are one-to-one,...the composite function is also one-to-one, but I seem to remember that that was not an if and only if statement." However, Emily struggled to construct a counterexample, trying to determine an appropriate f function that would satisfy the properties of a counterexample when coupled with $g(x) = x^2$.

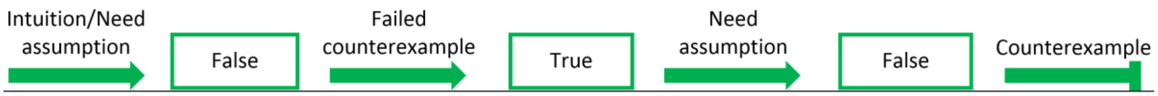


Figure 24. Emily's decision-making pathway.

This failed attempt at constructing a counterexample led Emily to consider that the statement may be true. Emily attempted to set up a proof structure, but another change in decision was prompted when she stated:

> I don't know anything about f. I don't know if f is one-to-one or not. And so the inputs could, theoretically, be different, even if I have the same, the outputs could, theoretically, be different even if I have the same output, I think, so I really think that this is going to be false.

Her final decision was supported by the following counterexample $g(x) = x^2$, $f(x) = x^{3/2}$, and $f(g(x)) = x^3$ in which $f \circ g$ is one-to-one but g is not one-to-one. However, this counterexample is incorrect because $f(g(x)) = (|x|)^3$ is not one-to-one. Although Emily decided to stick with this counterexample, she expressed hesitation that this case was similar to taking the square root of a squared function.

Emily's reasoning was not connected because the reasoning that led to each decision was not mirrored in her attempts to support the decisions. However, this makes sense because her reasoning for why she thought the statement was false was based on a lack of information on the function f that would inhibit a proof attempt rather than support a counterexample search. Additionally, a failed counterexample attempt led her think the statement may be true rather than reasoning that may suggest why the statement would be true.

Inigo. Inigo's decision-making followed Pathway DRDR with an initial true decision that was overturned by a syntactic failed proof attempt, resulting in a false decision that was supported by a syntactic counterexample (Figure 25). Inigo simply assumed that this statement was true from the beginning, as he did on other tasks as well. After looking at the definitions for one-to-one and composite functions, Inigo was concerned about the relationship between the domain of a composite function and the domains of the component functions. He constructed two examples in which he composed two functions, found the domains of the functions, and tried to determine the relationship between the domains. He thought that he needed this information in order to prove the given statement. However, he did not think he was making progress and decided that perhaps he was approaching the task in the wrong way.

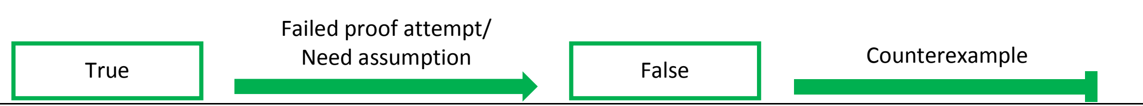


Figure 25. Inigo's decision-making pathway.

Inigo's lack of progress made him consider trying a proof by contradiction, and he considered the possibility of the function g not being one-to-one. However, he still thought that the lack of information on the relationship between the domain of the composite function and the domain of g prohibited him from proving this statement, so he finally considered whether he could find a counterexample. His first counterexample involved the situation in which g was one-to-one and the composite function $f \circ g$ was not one-to-one. He illustrated his example with graphs and showed they were and were not one-to-one, respectively, with the horizontal line test. Afterward, he said "wait...what did I just show?" He realized he had not constructed an appropriate counterexample and that what he actually needed to disprove this statement was a function g that was not one-to-one and a corresponding composite function $f \circ g$ that was one-to-one. He constructed the same counterexample as Aurelia, $g(x) = x^2,\ f(x) = \sqrt{x}$, and $f\big(g(x)\big) = x$ that is not a correct counterexample because $f(g(x)) = |x|$. Inigo's decision-making and construction processes were not connected. He assumed the statement was true, so his initial decision-making did not involve reasoning that could be connected to his proof attempt. Additionally, his lack of progress on his proof is what overturned his first decision and led to his second decision, and the construction of his counterexample did not include reasoning related to his failed proof attempt.

Syntactic proof group. Six students constructed a syntactic proof in support of their decision that this statement was true. Two of the six constructed correct proofs – Edward and Elliot – and the other four constructed incorrect proofs – Evan, Jay, Jalynn, and Michael. The five students (Edward, Elliot, Jalynn, Jay, and Evan) in this group who used Pathway RDR made intuitive decisions. These were the only intuitive decisions made on this task, and include the two students who constructed correct proofs. Michael is the only student in this group who did not use Pathway RDR and did not make an intuitive decision. In this section, I will analyze the work of the students with the correct proofs first, followed by the students with the incorrect proofs, with further organization by pathway.

Edward and Jalynn. The decision-making of Edward and Jalynn followed Pathway RDR with similarity-based intuition leading to a true decision that was supported by a syntactic proof (Figure 26). Edward thought that this statement was true because "I know, I've seen the proof once before...that if both *f* and *g* are one-to-one, then the composite f of g is also one-to-one, and that seems very similar in nature to the question being asked here." Although Edward struggled initially, he constructed a correct proof by contradiction (Figure 27).

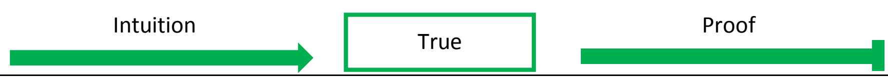


Figure 26. Edward and Jalynn's decision-making pathway.

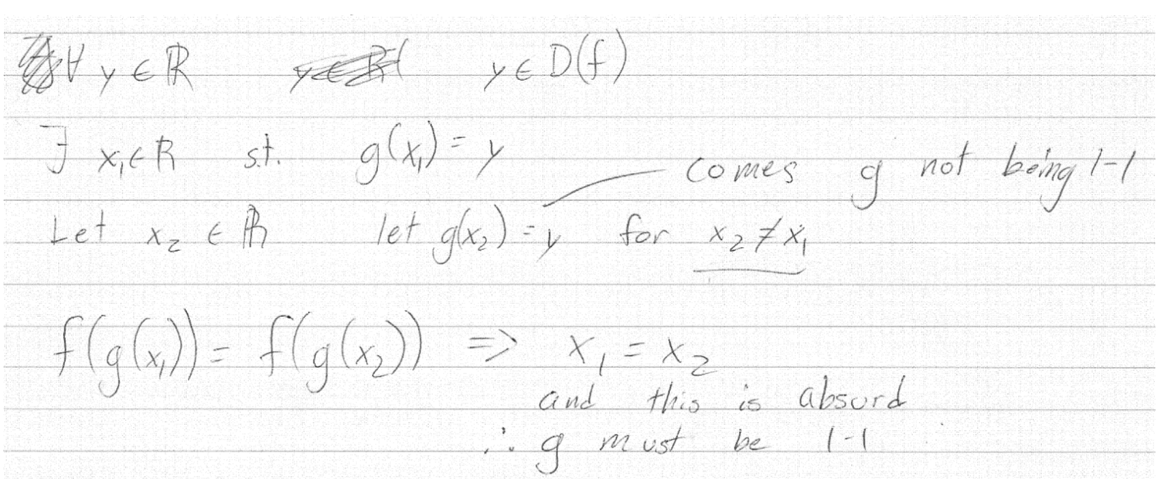


Figure 27. Edward's correct proof of the Composite Function Task.

Edward's decision-making and construction processes were not connected as his proof was not based on his intuitive decision. On the other hand, Jalynn constructed an incorrect proof in which she made mathematical and logical errors, including assuming that *f* was one-to-one. When she wrote what she wanted to show, she said "So, we want to show that $g(x_1) = g(x_2)$, given that $x_1 = x_2$." Although she did not use $x_1 = x_2$ as an assumption, she did conclude her proof with $g(x_1) = g(x_2)$. (Figure 28). Because Jalynn's intuition was unrelated to her proof, her decision-making and construction processes were not connected.

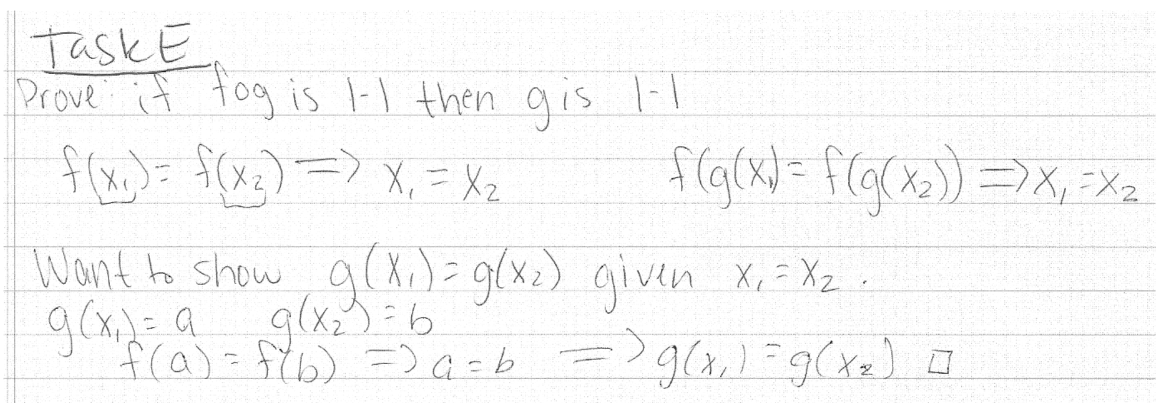


Figure 28. Jalynn's incorrect proof of the Composite Function Task.

Elliot. Elliot's decision-making also followed Pathway RDR with similarity-based intuition leading to a true decision that was supported by the combination of a semantic-deductive, diagram/definition-based, informal argument and a syntactic proof (Figure 29). After Elliot constructed his proof, he noted that he thought the statement was true from the start: "I was pretty sure that it was true at the beginning because I think I remember doing something like this a couple years back. I just didn't really remember how to show that it was true." Thus, Elliot began by trying to determine why this statement would be true. He drew a general diagram of a composite function, wrote the definitions of one-to-one for $f \circ g$ and g, and then related his diagram to the definitions by noting that since $f \circ g$ was one-to-one, "for any given point in domain C, there's a unique point in domain A that maps to it" (Figure 30).

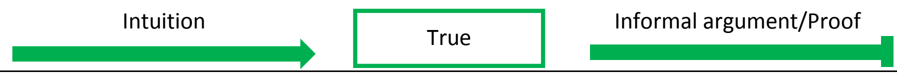


Figure 29. Elliot's decision-making pathway.

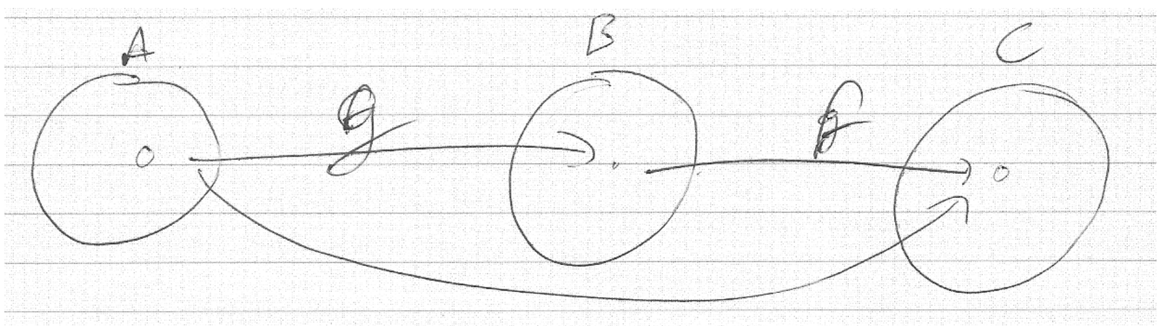


Figure 30. Elliot's general diagram for the composition of functions.

However, Elliot was unable to progress using this line of reasoning, so he decided to try a proof by contradiction instead. He drew a new diagram, this time illustrating how assuming that g was not one-to-one (based on the definition) would lead to a contradiction. He then translated this argument into a correct syntactic proof (Figure 31).

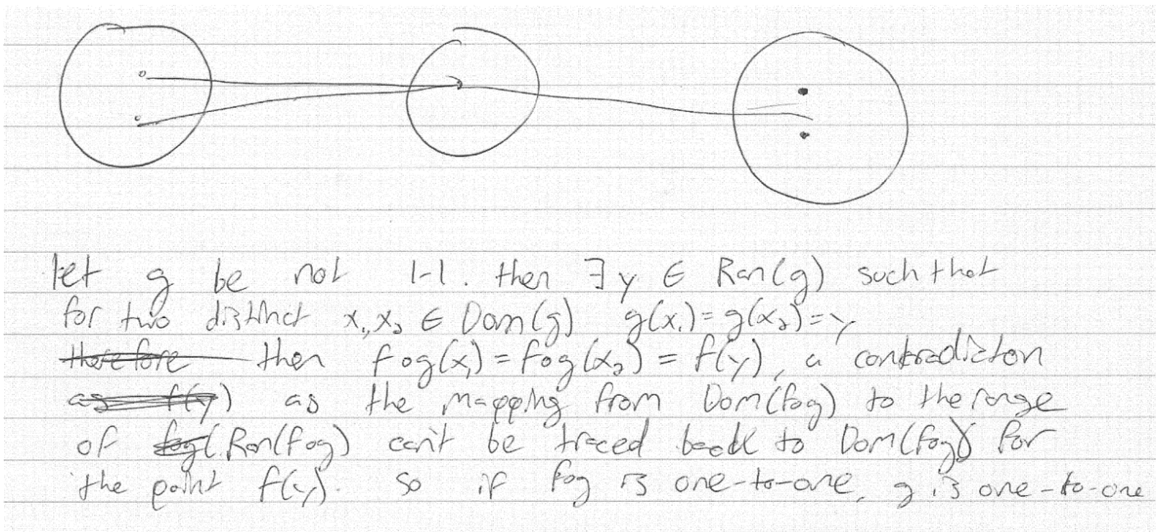


Figure 31. Elliot's diagram and his associated correct proof showing that if g was not one-to-one, then a contradiction would occur.

Although Elliot's decision-making and construction processes were not connected, he did base his proof on the informal argument he constructed, connecting his semantic and syntactic reasoning.

Jay. Jay's decision-making followed Pathway RDR with an unjustified intuition leading to a true decision that was supported by a syntactic proof (Figure 32). Jay thought this statement was true was from the beginning, but provided no justification for his decision, only noting "Initially, I thought it was probably true." Jay began his work by incorrectly writing what it meant for the composite function to be one-to-one, writing that $f(g(x_1)) = f\big(g(x_2)\big)$ then $g(x_1) = g(x_2)$. He thought that this alone implied that g was one-to-one, but he decided to construct a proof by contradiction to make sure. However, his proof contained numerous errors, including incorrectly stating what it meant for g to not be one-to-one and contradicting $g(x_1) = g(x_2)$. Additionally, Jay's proof had notational issues that made him "think there is something missing." Although he did not know how to fix his proof, he maintained that he thought that the statement was true. Because Jay's intuition was unjustified, there was no connection between his decision-making and construction processes.

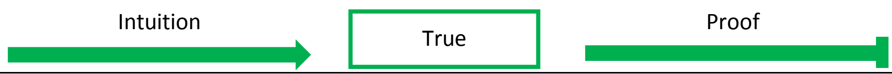


Figure 32. Jay's decision-making pathway.

Evan. Evan's decision-making followed Pathway RDR with an understanding-based intuition leading to a true decision that was supported by a syntactic proof (Figure 33). Evan's intuition was that the statement was true because it "look[ed] like [it] made sense" and he "had some ideas floating around that I, at least, wanted to try into the proof." Evan used the definitions of composite function and one-to-one to construct an incorrect direct proof that included the errors of assuming that f was one-to-one and misuse of the hypothesis of the statement (Figure 34).

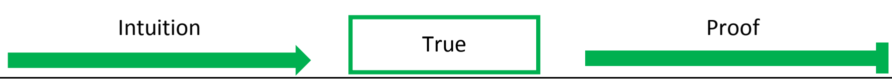


Figure 33. Evan's decision-making pathway.

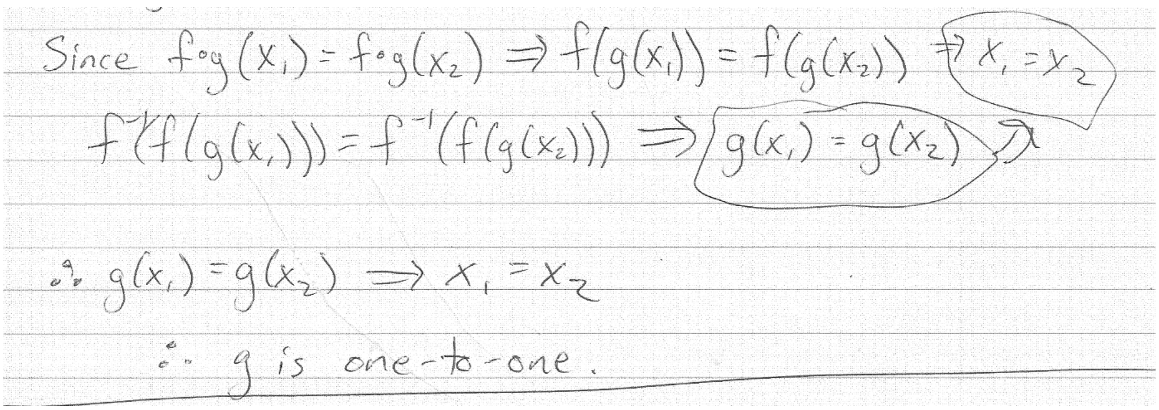


Figure 34. Evan's incorrect proof of the Composite Function Task.

It is unclear whether Evan's decision-making and construction processes were connected as he did not say specifically why he thought the statement made sense or what ideas he wanted to try in his proof.

Michael. Michael's decision-making followed Pathway RDRDR with a syntactic need for an assumption leading to a false decision. This decision was overturned by a syntactic failed counterexample attempt that led to a true decision that was supported by a syntactic proof (Figure 35). At first, Michael decided that this statement was false because "if you have a composite function being one-to-one, then I think the outer function has to be one-to-one." However, Michael was unable to construct a counterexample to support this decision. He chose $g(x) = x^2$ because it is a simple function that is not one-to-one, but was unable to find an appropriate $f(x)$ to compose it with that would result in a one-to-one composite function. He thought that $f(x) = \sqrt{x}$ should work, but finally realized that the composition would not be one-to-one.

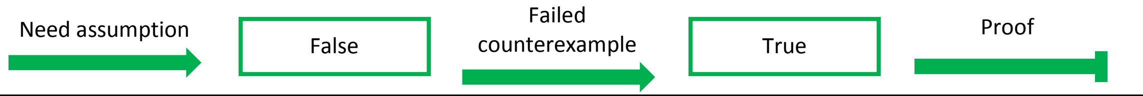


Figure 35. Michael's decision-making pathway.

This failed attempt at constructing a counterexample led Michael to change his decision, considering that the statement may be true. He drew a composite function diagram, but said that this was not helpful and that he was having a difficult time visualizing the task. After a short break, Michael returned to this task and wrote what it meant for g to not be one-to-one. Unfortunately, he incorrectly negated the definition of one-to-one, writing $g(x) = g(y)$ iff $x \neq y$. Then he said "I might have a proof by contradiction, maybe. But it's fishy." He wrote his proof, based on his incorrect interpretation of g not being one-to-one, but was concerned about the "if and only ifs" in the proof (Figure 36). He mulled over his proof for a while and constructed an example in which f was not one-to-one, g was one-to-one, and the composite function was not one-to-one. He said that this example increased his confidence in his proof, and he decided that he was content with it.

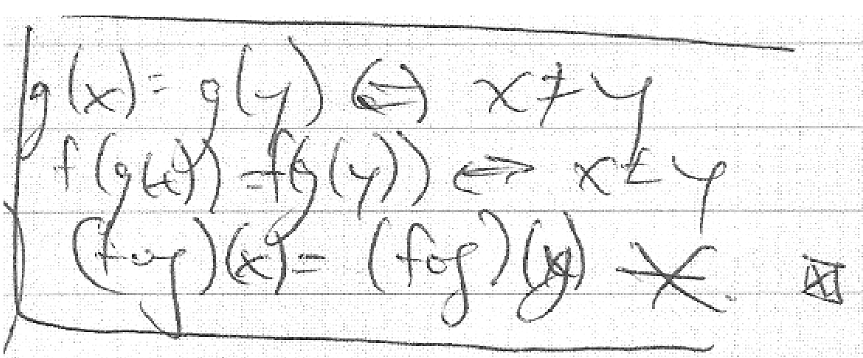


Figure 36. Michael's incorrect proof of the Composite Function Task.

Michael's first decision was based on thinking that he needed the assumption that f was one-to-one, but his counterexample search did not seem to depend on that. His inability to construct a counterexample led to his second decision, but was not connected to his construction of his proof by contradiction. Thus, for both of Michael's decisions, his decision-making was unconnected to his construction process.

Errors/Difficulties (RQ3). The students in this study did not commit any intuitive errors on this task, but they did commit both mathematical and logical errors (Table 10). There was one systematic mathematical error on this task, committed by three students. The systematic mathematical error of claiming the square root of a squared function did not need absolute value was committed by the three students who provided incorrect counterexamples for this task. Furthermore, students made various nonsystematic mathematical and logical errors. No student was able to overcome a systematic error, and only one student overcame any error on this task. Additionally, the errors were significant in that they formed the basis for students' incorrect proofs and counterexamples. In this section, I will discuss the three systematic errors and then provide a summary of the nonsystematic errors.

Table 10

Error Types on Composite Function Task Organized by Approach and Student

Student	**Error Types**
	Counterexample
Emily Aurelia	Mathematical – missing absolute value
Inigo	Mathematical – missing absolute value Mathematical – incorrect domain of composite Logical – incorrect counterexample structure*
	Proof
Edward Elliot	None
Evan	Logical – assumed f was one-to-one Logical – if-then issues
Jay	Logical – incorrect negation of one-to-one Logical – if-then issues Mathematical – incorrect definition of one-to-one composite function Mathematical – notational issues
Jalynn	Logical – incorrect proof structure Logical – assumed f was one-to-one
Michael	Logical – incorrect negation of one-to-one

* error was overcome

Missing absolute value. Aurelia, Emily, and Inigo each made the mathematical error of not considering absolute value when taking the square root of a squared function. These three students were the only three that decided that this task was false and provided counterexamples. However, each of their counterexamples were incorrect due the same mistake. Aurelia and Inigo each used the same counterexample: $g(x) = x^2$,

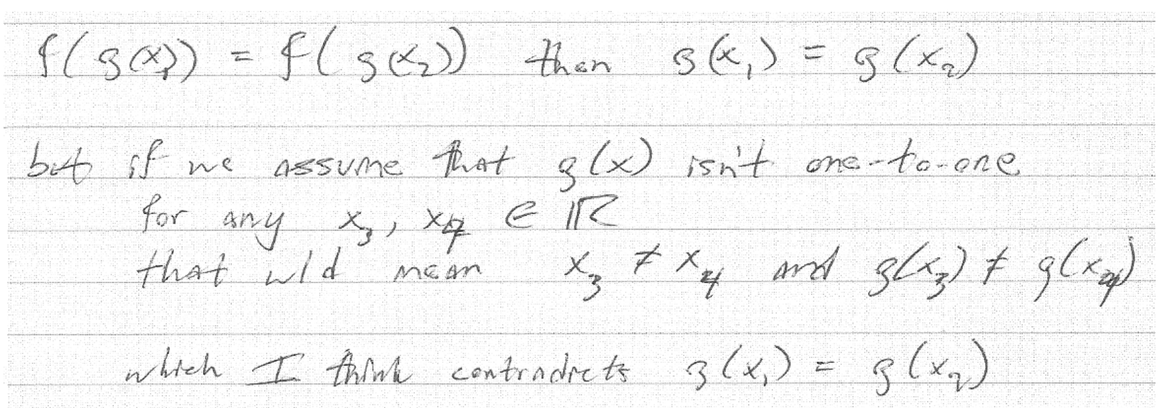
$f(x) = \sqrt{x}$, and $f(g(x)) = x$. Neither student questioned their counterexample although the correct composite function is $f(g(x)) = |x|$. The counterexample that Emily settled on had the same problem: $g(x) = x^2$, $f(x) = x^{3/2}$, and $f(g(x)) = x^3$. Unlike Aurelia and Inigo, Emily was hesitant about her counterexample, and noted that she was uncertain about:

> …taking [an] exponent and raising [it] to a three half power. I feel like there might be a caveat where I'm not getting x^3 perfectly....I still think that this is false and I think that this works, but it's kind of, like, square root of x^2, you think it should be x, but it's really absolute value of x, and so I don't know if something funky like that is, should be happening here.

Nonsystematic errors. Five types of nonsystematic logical errors were made on this task. Jay and Michael each incorrectly negated the definition of one-to-one when assuming for contradiction that the function g was not one-to-one. Jay incorrectly wrote "for any $x_3, x_4 \in \mathbb{R}$, $x_3 \neq x_4$ and $g(x_3) \neq g(x_4)$. Although Jay determined the correct operation here, he did not negate the components correctly or use the correct quantifier. On the other hand, Michael negated the components correctly, but did not use the correct operation or clarify what quantifier he used: $g(x) = g(y)$ iff $x \neq y$. For both students, these incorrect negations formed the basis for their incorrect proofs on this task.

Evan and Jalynn each assumed that the function f in the task was a one-to-one function. Although no information was given on f, the key step in both students' proofs depended on f being one-to-one. Evan used the inverse function f in his proof in order to eliminate the f function. He assumed that the inverse function existed, but this is only the case if f is one-to-one. Thus, it was an implicit assumption. Jalynn's assumption that f was one-to-one was also implicit as she may have been thinking that she was using her assumption that the composite function was one-to-one, but with different variables (see Figure 28 – Jalynn's proof). However, what she actually used, and assumed, was that f was one-to-one.

Evan and Jay each made errors in their work with the if-then statements in the task. Because the hypothesis in the task was that the composite function was one-to-one, this was an if-then statement. Evan incorrectly used the implication in this hypothesis to create a new implication that g was one-to-one (see Figure 34 – Evan's proof). Jay incorrectly used the consequent of the hypothesis to contradict a later statement in his proof, pulling the consequent out of its if-then structure (Figure 37).


$f(g(x_1)) = f(g(x_2))$ then $g(x_1) = g(x_2)$

but if we assume that $g(x)$ isn't one-to-one

for any $x_3, x_4 \in \mathbb{R}$

that w/d mean $x_3 \neq x_4$ and $g(x_3) \neq g(x_4)$

which I think contradicts $g(x_1) = g(x_2)$

Figure 37. Jay's error with the if-then statement (misuse of his assumption).

The final logical errors are related to proof and counterexample structure. Although Jalynn correctly wrote the definition of one-to-one, when she set up her direct proof, she wrote that she wanted to show that $g(x_1) = g(x_2)$ when $x_1 = x_2$. Although she did not use $x_1 = x_2$ as an assumption in her proof, she did conclude her proof with $g(x_1) = g(x_2)$. Inigo incorrectly structured his counterexample for this task, but he overcame this error. At first, he constructed a counterexample in which g was one-to-one and the composite function $f \circ g$ was not one-to-one. However, he immediately realized that this was not what he needed to show to disprove this statement and constructed a counterexample with the correct structure, that g was not one-to-one and the composite function $f \circ g$ was one-to-one.

Three nonsystematic mathematical errors were made on this task. Jay misinterpreted what it meant for the composite function to be one-to-one, writing that if $f\big(g(x_1)\big) = f\big(g(x_2)\big)$ then $g(x_1) = g(x_2)$. Additionally, Jay had notational issues related to his use of multiple variables. He was uncertain about the arbitrariness of the variables and whether he could interchange them. Inigo made the mathematical mistake of determining incorrectly the domain of a composite function in an example he constructed. However, this mistake did not affect his work on this task as he abandoned the need to know about the domains of the functions.

Composite Function Task summary. The students in this study solved this task with either a syntactic proof or counterexample, but experienced numerous difficulties. Of the six students who constructed proofs, only two constructed correct proofs. Students' incorrect proofs were based mostly on logical errors related to using if-then statements and the definition of one-to-one. Additionally, Jay's incorrect proof contained mathematical errors. The three students who constructed incorrect counterexamples all made the error of not considering absolute value when taking the square root of a function. Despite the large quantity of errors on this task, only one student was able to overcome one error.

Aurelia was the only student whose decision-making and construction processes were connected on this task. For every other student, the reasoning that led to their decision was unrelated to their construction of a corresponding proof or counterexample. It is possible that this lack of connection, along with the general lack of semantic reasoning, promoted the students' difficulties on this task. Additionally, although I did not expect intuition to be a factor on this task because of the backward thinking needed, four students made intuitive decisions. However, none of these intuitions were related to why this statement was true, so these intuitions were not translatable into or connected to students' proofs or counterexamples.

Edward and Elliot were the only students who made both correct decisions on this task and constructed correct proofs. Both students made similarity-based intuitive decisions on this task. One of the keys to their success was the ability to correctly negate the definition of one-to-one. Although Edward's reasoning in support of his decision was syntactic, Elliot engaged in both semantic-deductive and syntactic reasoning.

Injective Function Task

The Injective Function Task deals with the question of whether a general function is also a one-to-one function (Figure 38). Although the word "one-to-one" is not mentioned in the task, the statement refers to the definition of one-to-one. The statement in this task is false because not every function is one-to-one. Many students in this study realized this rather quickly. Any function that is not one-to-one is suitable as a counterexample for this task. Intuition was not used by any student on this task. This was expected due to the need for analysis to unpack the statement in the task.

Let $f: A \to B$ be a function and suppose that $a_0 \in A$ and $b_0 \in B$ satisfy $f(a_0) = b_0$.

Prove or disprove: If $f(a) = b$ and $a \neq a_0$, then $b \neq b_0$.

Relevant Definition from Definition List

A function $f: \mathbb{R} \to \mathbb{R}$ is called one-to-one if and only if for all $x_1, x_2 \in \mathbb{R}$,

$(f(x_1) = f(x_2)$ implies $x_1 = x_2)$.

Figure 38. Injective Function Task and relevant definition.

The majority of the students were successful on this task, with 8 of the 12 students providing a correct counterexample and/or demonstrating a correct understanding of why the statement was false. All of the students' arguments centered on the definitions of function and one-to-one. Three students assumed that the given function was one-to-one, but two of them realized that they were not allowed to do so and provided a correct counterexample. Other students assumed that being a function implied that f was also one-to-one. Furthermore, there was some general misunderstanding about the relationship between "function" and "one-to-one."

Decisions and justifications for decisions (RQ1). The students justified their decisions on the Injective Function Task using various types of analytical reasoning. No student used intuition on this task, and all reasoning was deliberate and justified. The 12 students made a total of 15 decisions – nine students made one decision each, and three students made two decisions each. All three of the students who made two decisions overturned incorrect decisions. The following subtypes of reasoning were used for decision-making on this task:

- Semantic-deductive: Informal definition (2) and definition-based, informal argument (2)
- Syntactic: Formal definition (5) and proof (1)
- Combination: Semantic-deductive informal definition and syntactic failed proof (1)
- Combination: Syntactic failed proof, syntactic need for assumption, and syntactic counterexample (2)

There were two decisions that were not based on reasoning because the students simply assumed the statement was true.

Of the three students who overturned incorrect decisions, Aurelia decided that the statement was true based on the formal definition of function and the assumption that she was dealing with different y-values. However, as she was constructing her proof, she realized that she was actually given different x-values in the task and overturned her original decision because a function can have different x-values corresponding to the same y-value. Inigo and Jalynn also thought that the statement was originally true, but each overturned this decision upon realizing that a proof required the assumption that the function f was one-to-one, and they could not legitimately make that assumption because it was not a given in the task.

Almost every student provided syntactic support for their decisions on the Injective Function Task. Every student in the correct solution group except Edward provided a counterexample, and some provided additional informal arguments. Edward was the only student who did not provide a fully syntactic argument. Although Edward provided a counterexample, he did not see it as sufficient to refute the statement and longed for a general argument that disproved it. However, Edward was unable to provide a syntactic argument for why the statement was false, instead providing an inconclusive-based informal argument. Every student in the incorrect solution group provided a "proof" of the statement. Table 11 provides an overview of the reasoning used by each student on this task for both decision-making and supporting decisions.

Table 11

Types of Reasoning Used on Injective Function Task Organized by Correctness and Student

Student	Types of Reasoning Used			
	Intuitive	Semantic-empirical	Semantic-deductive	Syntactic
		Correct Solution		
Elliot Emily			Informal definition	Counterexample
Michael Evan			Inconclusive-based informal argument	Formal definition Counterexample
Edward		Example-based	Inconclusive-based informal argument	Formal definition Proof
Jalynn Inigo				Failed proof Need assumption Counterexample
Aurelia			Informal definition	Formal definition Failed proof Counterexample
		Incorrect Solution		
Tina Louis			Definition-based informal argument	Proof
Julie				Formal definition Proof
Jay				Proof

Connecting decisions and constructions – Decision-making pathways (RQ2).

In this section, I will discuss students' decision-making pathways and the connections between their decision-making and construction processes. Five different pathways were used on this task (Figure 39). Ten students used pathways that ended with follow-up justification and only two students used pathways that ended in decisions.

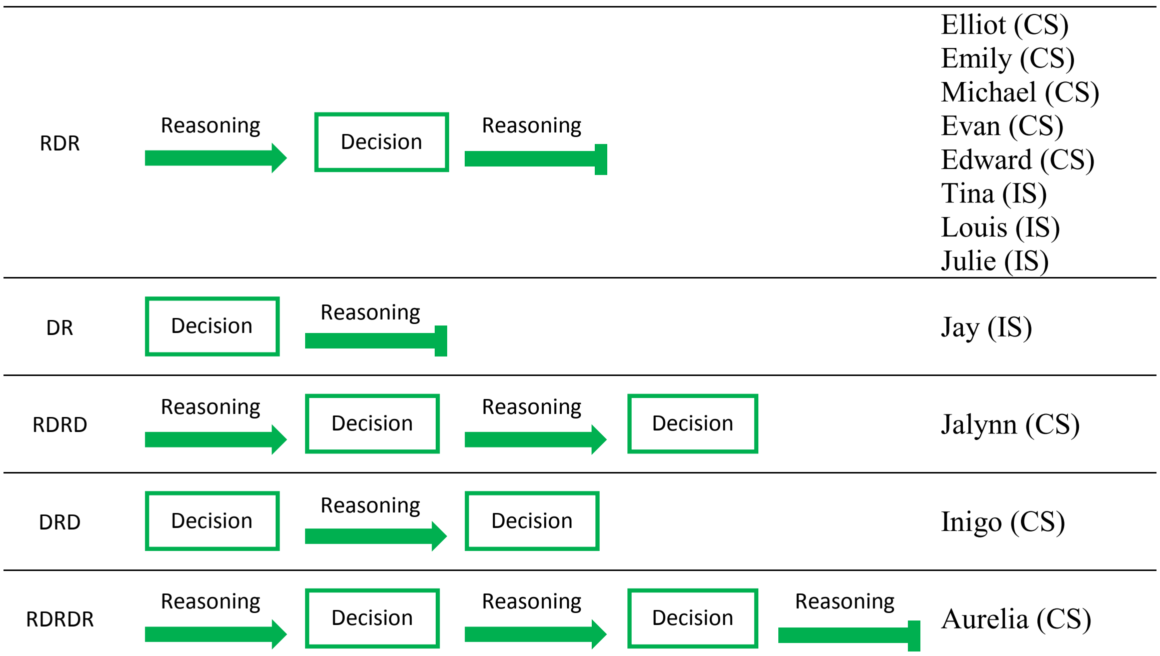


Figure 39. Decision-making pathways used on Injective Function Task. (CS) indicates correct solution group and (IS) indicates incorrect solution group.

The solutions to this task are organized based only on whether they are incorrect or correct because there were not any clearly distinct solutions within these broad groups. All of the students' reasoning on these tasks centered on the definitions of function and one-to-one. These definitions provided the connections between their decision-making and construction processes. Within each solution group (correct and incorrect), the results are organized by pathway with further organization by reasoning type.

Correct solution group. The students in the *correct solution* group realized at some point in their work that the condition of being one-to-one is necessary for this to be a true statement and provided counterexamples and informal arguments to justify their decisions. Three students in this group originally thought that the statement was true, but overturned their decisions. This section is organized by pathway and reasoning type.

Emily and Elliot. The decision-making of Emily and Elliot followed Pathway RDR with a semantic-deductive informal definition leading to a false decision that was supported by a syntactic counterexample (Figure 40). Both students based their decision that the statement was false on an informal definition of function. Emily said, "Because f is a function, if the inputs are different, the outputs aren't necessarily different. So, I feel like it is going to be false." Emily and Elliot both connected this idea to their counterexamples by constructing a function that mapped two inputs to the same output. Each constructed the counterexample $f(x) = x^2$, indicating that unique inputs, $2 \neq -2$ yield non-unique outputs, $4 = 4$.

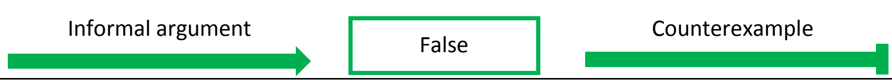


Figure 40. Emily and Elliot's decision-making pathway.

Michael and Evan. The decision-making of Michael and Evan followed Pathway RDR with a syntactic formal definition leading to a false decision that was supported by a semantic-deductive, inconclusive-based, informal argument and a syntactic counterexample (Figure 41). Both Michael and Evan noticed quickly that the statement would be false because they are not given that the function is one-to-one. After this decision, both provided an inconclusive-based informal argument based on having insufficient information to reach the conclusion of the statement. This argument connects to the idea that the given function is not one-to-one, thus linking the decision-making process to the construction of the argument. Evan argued that a counterexample would be a function that maps a to b_0 because "nowhere in the statement does it say b can't equal b_0...there's nothing directly preventing $f(a)$ from equaling b_0 when a's $\neq a_0$." Thus, this argument also directly connected to the construction of a counterexample. However, both Michael and Evan were hesitant to commit to a specific counterexample.

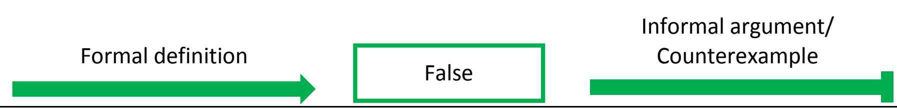


Figure 41. Michael and Evan's decision-making pathway.

Michael seemed content to stop with the idea that any non-injective function would suffice as a counterexample, but he eventually said that $f(x) = x^2$ was the "ultimate" example of non-injectivity and that $f(x) = x^2, a = 1, a_0 = -1, b = b_0 = 1$ would be a specific counterexample. On the other hand, Evan wavered due to the abstractness of the task because the domain and range were given as generic sets rather than the real numbers. He was concerned that $f(x) = x^2$ could not be used as a counterexample in this abstract setting and that anything that mapped two different x-values to a single y-value would show the statement was false. However, Evan finally realized that it was okay to use a specific counterexample in this situation and provided the same example as Emily and Elliot. Thus, both Michael and Evan eventually constructed specific counterexamples that connected back to their informal arguments and original idea that a non-injective function would refute the statement.

Edward. Edward's decision-making followed Pathway RDR with a syntactic formal definition leading to a false decision that was supported by a semantic-deductive, inconclusive-based, informal argument and a syntactic proof (Figure 42). He stated, "I think, from the beginning this is false, reading that, because we do not know if the function is one-to-one." Although Edward understood quickly why this statement was false, he struggled to disprove this statement. Edward's original plan for showing that this statement was false was to use a proof by contradiction in which he assumed that the statement was true, but reached a contradiction based on the function not being one-to-one. He had difficulty with this approach, so he decided to use an example "just to get my thoughts together." He use the same quadratic example provided by others to show that the statement did not hold in a particular case, but he thought a general argument was still necessary to disprove the statement. Because Edward did not consider this a counterexample, and, hence, sufficient to disprove a statement, I categorized it as a semantic-empirical example.

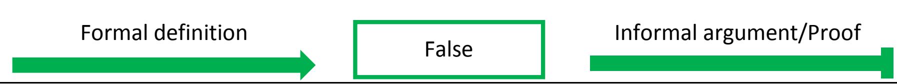


Figure 42. Edward's decision-making pathway.

Edward then attempted to construct a general disproof of this statement by trying to find statements equivalent to the definition of one-to-one, but he was unsuccessful. Finally, he provided an argument constructed of two cases based on whether or not f was one-to-one: (a) a proof by contradiction that the statement is true if f is one-to-one, and (b) an inconclusive-based argument that the statement is false if f is not one-to-one because "we can't say anything about the relationship of b and b_0... the relationship is still ambiguous of b to b_0." All of Edward's work connected back to his original idea that the statement was false because the function was not one-to-one, thus linking his decision-making and construction processes.

Jalynn. Jalynn's decision-making followed Pathway RDRD with a syntactic proof leading to a true decision. This decision was overturned by a combination of a syntactic failed proof attempt, need for an assumption, and counterexample, resulting in a false decision (Figure 43). From the beginning, Jalynn thought that solving this task was related to the concept of one-to-one, but she was unsure how. She was confused about the notation $f: A \rightarrow B$, wondering whether it only indicated the domain and range of the function or if it also implied that the function was onto or one-to-one. She began by writing the givens, indicating that she wanted to show that $b \neq b_0$, and noting that $f(a_0) \neq f(a) \rightarrow b_0 \neq b$. However, her confusion about one-to-one continued: "I don't know if I can just say that it's one-to-one. I can assume that it's one-to-one. Okay. I'll go with that...there would just be a condition for it then." With this assumption that the function f was one-to-one, Jalynn constructed a proof for the statement that led to her initial decision that the statement was true.

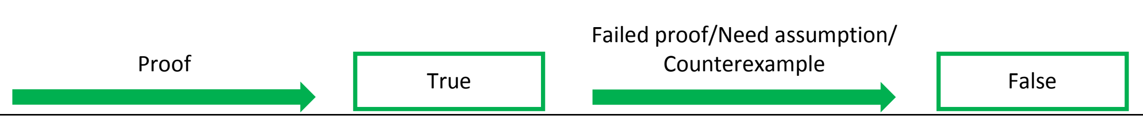


Figure 43. Jalynn's decision-making pathway.

After she completed her proof, I asked Jalynn if she thought that the assumption that f was one-to-one was a necessary condition for her proof. She said that she was unsure because she was still confused about whether the notation indicated that the function was one-to-one. So, I asked her what she thought if we just assumed that the notation only indicated the domain and range of the function, and she replied that "…probably would change it. But I don't, I mean, I'm just trying to think of, like, a [sic] example." She wrote $f(x) = x^2$, and showed that $f(3) = f(-3) = 9$. She said:

> I know that that function isn't one-to-one. Right, so, but if I made the function one-to-one, like, by restricting the domain to just positive numbers, then it would work because there's no positive number that has the same value for that. So I think that in a general case, this would have to work for it to be one-to-one.

Thus, she determined that one-to-one was a necessary condition for this statement to be true.

Finally, I asked Jalynn to clarify whether she thought this statement was true or false, and she replied "it's true if it's one-to-one and it, it's false if, overall it would be false in any case, just like how here [referring to her example of $f(x) = x^2$]...I mean, I guess it just asks for the general case." By analyzing the assumption that the function f was one-to-one in the context of an example, Jalynn overturned her original decision, determining that this statement was false. Jalynn used the concept of one-to-one throughout her work on this task, linking her decision-making and construction processes.

Inigo. Inigo's decision-making followed Pathway DRD with a true decision that was overturned by a combination of a syntactic failed proof attempt, need for an assumption, and counterexample, resulting in a false decision (Figure 44). Inigo assumed from the beginning that this statement was true, saying "for some reason, I always assume all of your things are true." After clearing up a couple questions regarding notation, Inigo said "So essentially this is asking if it's one-to-one." Thus, Inigo knew this statement was related to the concept of one-to-one, and he attempted to construct a proof by contradiction in order to prove the statement. In the proof, in order to claim that ($f(a) = f(a_0)$ implies $a = a_0$), Inigo noted that he needed to assume that f was one-to-one, did so, and finished the proof.

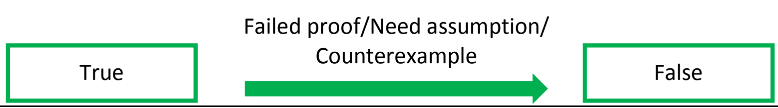


Figure 44. Inigo's decision-making pathway.

Upon completing his proof, Inigo said "I know there's a flaw in some logic there because of this [underlining his assumption that f is one-to-one], but I'm finished." I asked him if he could tell me why he thought it was wrong, and he said "I am assuming that this is one-to-one. And it's not necessarily one-to-one....And I know you can't actually make that assumption here....Wait. Maybe that's a counterexample." He then wrote $f(x) = x^2 - 2, 2 \to 4 = 4$ and said "Yeah. Yep. So when it's one-to-one, that holds [indicating proof]; and then when it's not, there [underlining counterexample]." Like Jalynn, Inigo realized that making the assumption that f was one-to-one in his proof was problematic, and this led him to construct a counterexample and overturn his decision. Although he did not engage in a legitimate decision-making process for his first decision, all of Inigo's work related back to the concept of one-to-one, leading to his failed proof attempt, counterexample, and the overturn of his decision.

Aurelia. Aurelia's decision-making followed Pathway RDRDR with a syntactic formal definition leading to a true decision that was overturned by a combination of a syntactic failed proof attempt and a semantic-deductive informal definition. This led to a false decision that was supported by a syntactic counterexample (Figure 45). After writing the key elements of the task, Aurelia said "So it's a function, so, yeah, that's definitely true." Aurelia began constructing a proof of the statement, but partway through she wrote $a \neq a_0$and realized that she was thinking incorrectly about the task. She was originally thinking that she was given two different *y*-values, but then recognized that she was actually given two different x-values. Then she said "Hm, no, it's a counterexample. So, and so, we're assuming we have two different x-values, but they can definitely have the same *y*-value." Thus, she overturned her original decision through a realization that she was considering the task incorrectly that led to a failed proof attempt and a semantic-deductive informal definition of function. Aurelia constructed the following counterexample: $f(x) = \cos(x), 0 \neq 2\pi,$ but $\cos(0) = \cos(2\pi) = 1$. She drew a graph of $f(x) = \cos(x)$ to help her determine which specific values to use in her counterexample.

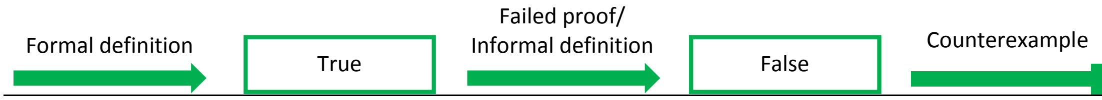


Figure 45. Aurelia's decision-making pathway.

Aurelia's first decision was based thinking that she was given two distinct y-values, and that "if you have a function, you can have an x-value that has two different x-values." Her proof attempt was also based on f being a function, linking her first decision and attempt to support it. Her second decision was based on the realization that she was dealing with distinct x-values rather than distinct y-values. This led to her informal definition that different x-values can correspond to the same y-value. She then constructed a counterexample that exhibited this behavior, again connecting her decision-making and construction processes.

Incorrect solution group. The four students in this group thought that the statement in the Injective Function Task was true and proved it. Tina and Louis assumed that f being a function implied that it was also one-to-one although they did not realize that this was an assumption. Julie's incorrect proof structure led her to a simple "proof" of the task. Finally, Jay knowingly assumed that f was one-to-one in his "proof," but I was unable to coax an explanation from him regarding this assumption. This section is organized by pathway with further organization by reasoning type.

Tina and Louis. The decision-making of Tina and Louis followed Pathway RDR with a semantic-deductive, definition-based, informal argument leading to a true decision that was supported by a syntactic proof (Figure 46). Both Tina and Louis were confused by the relationship between the concepts of function and one-to-one, assuming that being a function implied being one-to-one, but not realizing that this was an assumption. Both students used the definition of one-to-one, but they thought that because f was a function, it was also one-to-one. Tina and Louis each based their decision that this statement was true on the following incorrect definition-based informal argument: "you know that there's only one output for every input, then you know that another input can't produce that same output or else it wouldn't be a function." This reasoning indicates their confusion about the concepts of function and one-to-one. Both seem to think that not being one-to-one actually violates the definition of a function. Tina and Louis both produced a syntactic proof as follow-up support for their decisions (Figure 47). Although there were differences in their proofs, both incorporated the informal argument that led to their decisions, connecting their decision-making and construction processes.

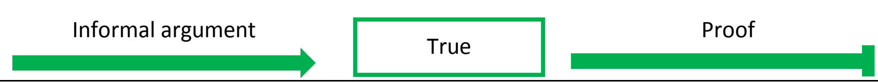
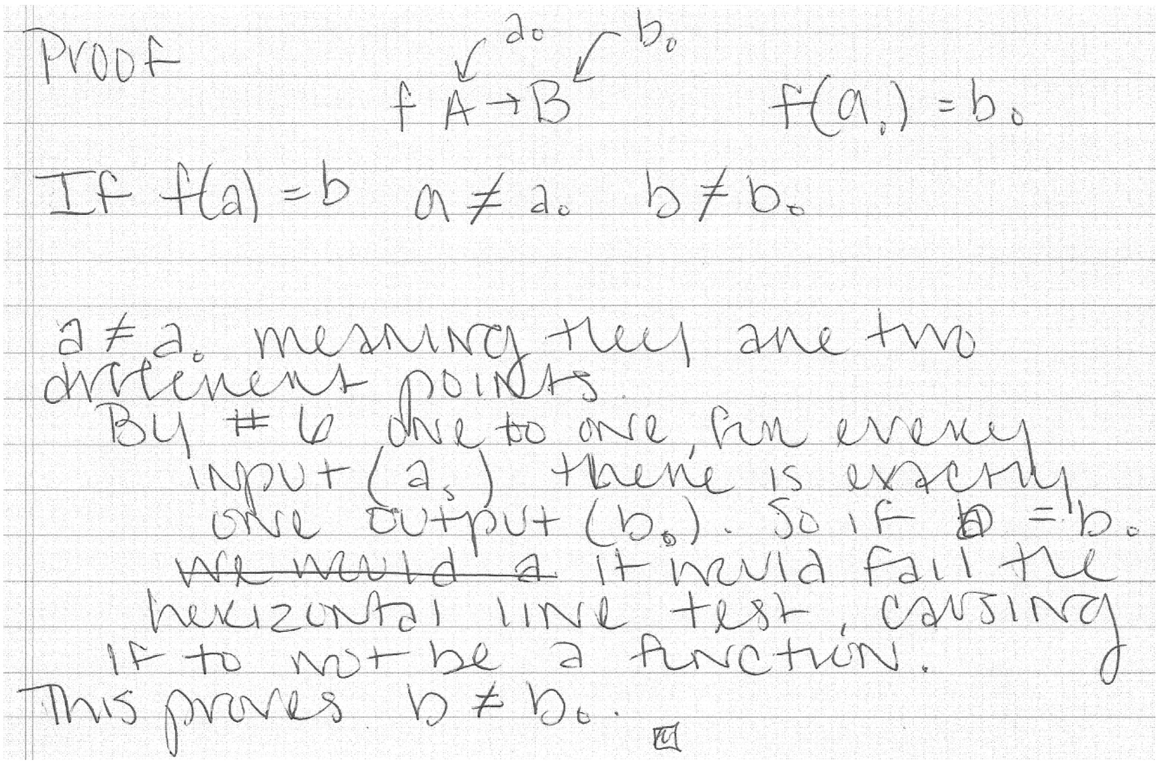


Figure 46. Tina and Louis' decision-making pathway.


Proof

$f\ A \to B$ (a_0, b_0) $f(a_0) = b_0$

If $f(a) = b$ $a \neq a_0$ $b \neq b_0$

$a \neq a_0$ meaning they are two different points.
By #6 one to one, for every input (a_0) there is exactly one output (b_0). So if $b = b_0$ ~~we would a~~ it would fail the horizontal line test, causing it to not be a function.
This proves $b \neq b_0$. ☒

Figure 47. Tina’s incorrect proof of the Injective Function Task.

Julie. Julie’s decision-making followed Pathway RDR with a syntactic formal definition leading to a true decision that was supported by a syntactic proof (Figure 48). Julie’s decision that this was a true statement was based on the definition of a function, “for every x there's one and only and one y.” Julie’s proof is an attempt to contradict this definition of function by showing that a given x-value maps to distinct y-values, thereby linking the decision-making and construction processes. She begins her proof by contradiction by assuming that $f(a) = b_0$. Because she was given that $f(a) = b$, together these implied that $f(a) = b = b_0$. She claimed that this was a contradiction to f being a function because a was mapped to two distinct b-values. However, she assumed that b was equal to b_0, so these are not distinct values.

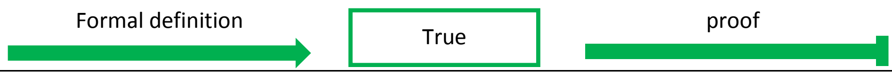


Figure 48. Julie's decision-making pathway.

Jay. Jay's decision-making followed Pathway DR with a true decision supported afterward by a syntactic proof (Figure 49). Jay's decision was not based on reasoning as he simply assumed the statement was true and constructed a direct proof (Figure 50). After Jay completed his proof, I asked him what the key step was, and he said "Well, just, for me, the idea since $a \neq a_0$, then, I, sort of, made a jump and assumed that the, that $f(a)$ then is not equal to $f(a_0)$." I inquired about making this jump, and he replied "That'll only be true if the function was one-to-one, but from just the given information, I don't know exactly if it is one-to-one." I continued attempting to draw information out of him about his use of one-to-one despite being uncertain whether f was one-to-one, but I was unable to gather additional information. He repeated that his proof would work if he knew the function was one-to-one, but he never indicated decisively whether he knew this nor suggested that it was not given in the statement of the task. Because Jay did not engage in a legitimate decision-making process, it could not be linked to his construction process.

Figure 49. Jay's decision-making pathway.

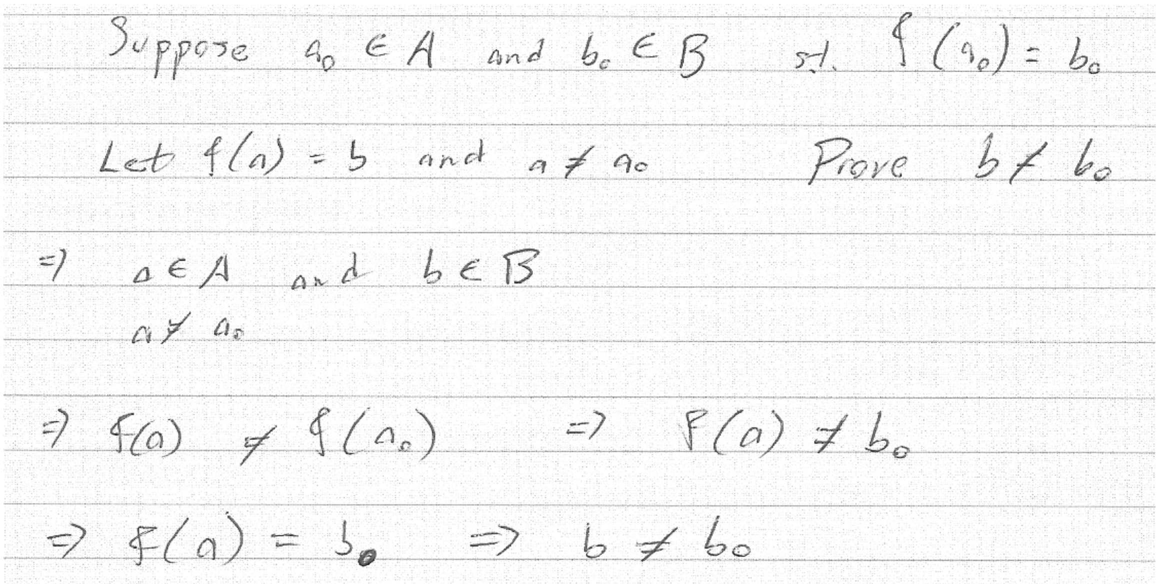


Figure 50. Jay's incorrect proof of the Injective Function Task.

Errors/Difficulties (RQ3). Although there were no intuitive errors committed on the injective-definition task, the students made various mathematical and logical errors, including two systematic errors (Table 12). Three students committed the logical systematic error of making the illegal assumption in their proof that the given function was one-to-one. Some students were able to overcome this error. Additionally, three students made the mathematical systematic error of assuming that functions are one-to-one. Both of these systematic errors were committed by students who correctly and incorrectly solved the task. Furthermore, students made various nonsystematic mathematical and logical errors. All of the errors were committed by only 8 of the 12 students with Emily, Elliot, Michael, and Evan making no errors. In this section, I will discuss the two systematic errors and then provide a summary of the nonsystematic errors.

Table 12

Error Types on Injective Function Task Organized by Correctness and Student

Student	**Error Types**
Correct Solution	
Emily Elliot Michael Evan	None
Edward	Logical – incorrect proof structure Logical – power of counterexample
Aurelia	Mathematical – function implies one-to-one Mathematical – misinterpret statement of task*
Inigo	Logical – incorrect proof structure* Logical – assumed *f* was one-to-one*
Jalynn	Logical – assumed *f* was one-to-one* Mathematical – misunderstanding notation
Incorrect Solution	
Tina	Mathematical – function implies one-to-one
Louis	Logical – incorrect proof structure* Mathematical – function implies one-to-one
Julie	Logical – assumed conclusion Mathematical – different labels imply distinct objects
Jay	Logical – assumed *f* was one-to-one
* error was overcome	

Assuming f is one-to-one. Inigo, Jalynn, and Jay each made the logical error of assuming in their respective proofs that the given function f was one-to-one. All three students knew that this assumption was problematic, but only Inigo and Jalynn were able to overcome this error and correctly determine that the statement was false.

Inigo knew his proof was flawed because of this assumption, saying “I am assuming that this is one-to-one. And it’s not necessarily one-to-one....And I know you can’t actually make that assumption here.” Thus, Inigo knew that this was an illegal move, but he made it anyway. However, he then realized that he could construct a counterexample in which f was not one-to-one and did so correctly.

Jalynn’s nonsystematic error relating to her confusion about the notation $f: A \to B$ contributed to her systematic error of assuming that the given function was one-to-one. Jalynn was uncertain whether the notation $f: A \to B$ implied that f was one-to-one in addition to specifying its domain and range. She knew that she needed f to be one-to-one in order to complete her proof, so she settled on making it an assumption. However, upon pressing from me, she realized that this task considered a general case and that she could not assume that the function was one-to-one. Thus, Jalynn was able to overcome her error by considering the necessity of her assumption that the function was one-to-one and how it related to her assumptions about the notation.

Jay assumed this statement was true and constructed a proof that included that assumption that f was one-to-one. However, he “made a jump” in his proof that would “only be true if the function was one-to-one, but from just the given information, I don’t know exactly if it is one-to-one.” Although I pressed Jay, I was unable to get him to consider this assumption in a way that would help him overcome it.

Assuming functions are one-to-one. Aurelia, Louis, and Tina each made the mathematical error in their respective proofs of assuming that since f was a function, then f was one-to-one. For Aurelia, this assumption became moot because she realized that she had misinterpreted the statement and abandoned her proof attempt. However, this assumption formed the crux of Tina and Louis' proofs that the statement was true. Unlike the students who assumed that f was one-to-one, none of these students seemed to recognize that they had made an assumption. They believed that being one-to-one was a property of all functions.

Aurelia originally thought that this statement was true and began a proof with the assumption that functions are one-to-one: "We're gonna go ahead and use one-to-one. So we are told that it is a function, so, by definition six [one-to-one]..." However, this assumption became irrelevant for Aurelia as another aspect of the task led her to realize that the statement was actually false.

Tina and Louis based their proofs on the argument that different inputs must have different outputs because each input has exactly one output in a function. Neither seemed to realize that they were conflating the concepts of function and one-to-one, but both indicated that functions are one-to-one. Tina used the given definition of one-to-one in her proof, and the crux of her proof was that "if b were to be equal to b_0, we would have, or it would fail the horizontal line test, I guess I can say that, causing f to not be a function." Louis' first line in his proof was "we are given a function, which means there is only one output for every input (def. of one-to-one)." Additionally, when I asked Louis about the key idea of his proof, he said that since $f(a_0) = b_0, f(a) = b$, and $a \neq a_0$, "then you can also know that b is not equal to b_0 by the definition of a function, which is a one-to-one relationship."

Nonsystematic errors. Of the nonsystematic errors made on this task, there were five logical errors and three mathematical errors. Both Louis and Inigo incorrectly assumed for contradiction that $a = a_0$, but each overcame this error upon reconsidering their desired conclusions and made the correct assumption for contradiction that $b = b_0$. Edward wanted to prove that the statement was false, so he attempted to do so with a proof by contradiction in which he assumed that the statement was true. However, he assumed $b \neq b_0$ as a separate given rather than assuming the statement as a whole if-then statement. This error became irrelevant as Edward changed his approach to the task. Additionally, Edward did not recognize the power of a counterexample – that it is sufficient to disprove a statement. This led Edward to construct a general argument for the falsity of the statement. Lastly, Julie assumed what she was trying to prove when she claimed that she had reached a contradiction due to b and b_0 being distinct objects. This error corresponded to Julie's mathematical error of assuming that objects with different labels (b and b_0) are in fact different. This error was costly as it formed the crux of Julie's proof.

Two other students made mathematical errors: Aurelia and Jalynn. Aurelia misinterpreted the task, initially assuming that she was given two distinct y-values rather than two distinct x-values. However, she was able to overcome this mistake simply by recognizing that she had switched the x- and y-values when she wrote $a \neq a_0$ in her proof. Overcoming this error is what allowed Aurelia to correctly solve this task.

Finally, Jalynn's confusion about the notation $f: A \to B$ is a mathematical error that she was not able to overcome on her own. However, I provided her an opportunity to consider it correctly by asking her to assume that the notation only indicated the domain and range of the function. This assumption, coupled with a consideration of her assumption that the function was one-to-one led her to overcome an incorrect decision and construct a correct counterexample.

Injective Function Task summary. The students in this study based their solutions to this task around the definitions of function and one-to-one. Most students recognized that this task was false because the given function was not one-to-one and were able to provide a correct counterexample. However, a few students were unable to overcome various mistakes that led them to incorrect "proofs" for this task. All reasoning on this task with either semantic-deductive or syntactic, with the lone exception of Edward's counterexample that he did not recognize as such.

Students who provided a correct solution to this task mostly based their decisions on informal or formal definitions. Every student in this group constructed a correct counterexample except for Edward who did not recognize his example as a counterexample. The only three students who overturned decisions were in this group, with all overturning incorrect decisions. Inigo was the only student in this group who did not engage in reasoning prior to an initial decision. Despite ending up with correct solutions, half of the students in this group made a variety of mathematical and logical errors that were either overcome or became irrelevant due to a change in direction.

All students who provided an incorrect solution to this task supported their decision with a syntactic proof. Tina and Louis based their decision on an informal argument that conflated the definitions of function and one-to-one. Julie based her decision on the formal definition of a function, but her proof was derailed by mistakes. Jay was the only student in this group who did not engage in reasoning prior to an initial decision.

Global Maximum Task

The Global Maximum Task asks whether an increasing function has a global maximum (Figure 51). This task was ambiguous because it did not specify the domain of the given function. If the domain of the given function is taken to be the real numbers, then the statement in this task is true because a function increasing on the real numbers does not a have a global maximum. However, if the domain of the given function is restricted to a closed interval, then the statement in the task is false because an increasing function on a closed interval will achieve a global maximum at the right hand endpoint of the interval. The correctness of students' solutions was based on their assumptions or inferences about the domain of the function. I expected intuition to play a role in this task because the task lends itself to visualization.

Prove or disprove: If f is an increasing function, then there is no real number c that is a global maximum for f.

Relevant Definitions from Definition List

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to have a **global maximum at a real number** c if and only if for all $x \in \mathbb{R}$ such that $x \neq c$, $f(x) < f(c)$.

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be **increasing** if and only if for all $x_1, x_2 \in \mathbb{R}$, ($x_1 < x_2$ implies $f(x_1) < f(x_2)$).

Figure 51. Global Maximum Task and relevant definitions.

The majority of the students, 11 out of 12, correctly decided that the statement in this task was true when assuming that the domain of the function was all real numbers. One student unknowingly changed the statement to: If f is an increasing function, then there *is* a global maximum for f. He provided two cases depending on the domain of the function. However, because of his change of the statement, he decided that it was false when the domain was all real numbers and true when the domain was restricted to a closed interval. These are correct for his changed statement, but not for the given statement. The students' decisions on this task were based on the following types of reasoning: intuitive, semantic-deductive, syntactic, intuitive/semantic-deductive combination, and semantic-empirical/semantic-deductive combination.

Many of the students quickly made decisions on this task, likely due to their familiarity with the concepts in this task. Based on observations alone, I could not determine what led to their decisions without follow-up questioning after they completed the task. Most of the students' informal arguments that led to their decisions were not expressed until after they had constructed their proof when I asked them about what led to the decision. Thus, I often had to rely on their reflections after the fact in order to determine their initial thought processes and what led to their decisions. The time line and classifications for the decision-making were somewhat unclear as they were often based on their after-the-fact reflections rather than what happened in real time. This means that it is unclear whether these post hoc descriptions actually described what the students were originally thinking or were retroactive justifications based on the proofs or arguments they had already constructed. However, I had no reason to believe they were not accurate because students' post hoc descriptions in other instances, where their real-time reasoning was clear, were accurate.

Decisions and justification for decisions (RQ1). The students in this study justified their decisions using both intuitive and analytical reasoning, with the analytical reasoning taking the form of semantic-empirical reasoning, semantic-deductive informal arguments, and syntactic reasoning. The 12 students made a total of 15 decisions – 10 students made one decision each, one student made two decisions, and one student made three decisions. The following subtypes of reasoning were used for decision-making on the Global Maximum Task:

- Intuitive: Unjustified (1)
- Semantic-deductive: Definition-based, informal argument (4)
- Syntactic: Need for assumption (1)

- Combination: Semantic-deductive, visualization/definition-based, informal argument (5)
- Combination: Intuitive visualization-based and semantic-deductive, definition-based, informal argument (1)
- Combination: Semantic-empirical example/graph-based and semantic-deductive, visualization/kinaesthetic-based, informal argument (1)
- Combination: Semantic-empirical example/graph-based and semantic-deductive, definition-based, informal argument (1)
- Combination: Semantic-deductive, visualization/definition-based, informal argument and semantic-deductive, graph/definition-based, informal argument (1)

Edward and Evan were the only two students who made multiple decisions. Each of Edwards' three decisions was based on a definition-based, informal argument. He went back and forth on this task as he attempted to orient himself to the concept of global maximum (that he originally confused with the concept of boundedness). Upon clarifying the concepts, he decided that the statement was true for an increasing function on the real numbers. Evan, on the other hand, did not overturn a decision although he made two decisions. He originally determined that the statement was false (thinking that the statement said that there *was* a global maximum), but realized that he had made an assumption about the domain of the given function. He added this assumption to original disproof, keeping the decision that it was false in that case, and determined that he would have a second case in which the statement was true.

All analytical reasoning used on this task during the decision-making and construction processes was deliberate and justified. In addition to the semantic reasoning already discussed that led to decisions, three students used semantic reasoning in the form of definition-based informal arguments to support their decisions. Syntactic reasoning used to support decisions took the form of proofs, with the exception of Evan's disproof.

Students' intuitive reasoning on this task was non-deliberate and either unjustified or partially justified. Louis' intuition was unjustified as he simply thought it was true upon reading the statement, but didn't know why until he began constructing his proof. Jay noted that "just off the top of my head, you know, when you think of increasing function, it'll be going to infinity, and you can't find a constant." Table 13 provides an overview of the reasoning used by each student on this task for both decision-making and supporting decisions.

Table 13

Types of Reasoning Used on Global Maximum Task Organized by Approach and Student

Student	**Types of Reasoning Used**			
	Intuitive	Semantic-empirical	Semantic-deductive	Syntactic
		Proof by Contradiction		
Inigo Emily Aurelia Elliot			Visualization/ definition-based informal argument	Proof
Michael Edward			Definition-based informal argument	Proof
Louis	Unjustified			Proof
Jalynn			Visualization/ definition-based informal argument Graph/definition-based informal argument	Proof
Evan			Visualization/ definition-based informal argument	Disproof Need assumption Proof
		Informal Argument		
Tina		Example/graph-based	Definition-based informal argument	
Jay	Visualization/ property-based		Definition-based informal argument	
Julie		Example/graph-based	Visualization/ kinaesthetic-based informal argument Definition-based informal argument	

Connecting decisions and constructions – Decision-making pathways (RQ2).

In this section, I will discuss students' decision-making pathways and the connections between their decision-making and construction processes. Three different pathways were used on this task (Figure 52). Evan used a linked combination of a pair of pathway RDRs in which the reasoning that supported the first pathway was linked to the reasoning that led to the decision in the second pathway (see Evan's summary). Tina was only student whose pathway ended in a decision.

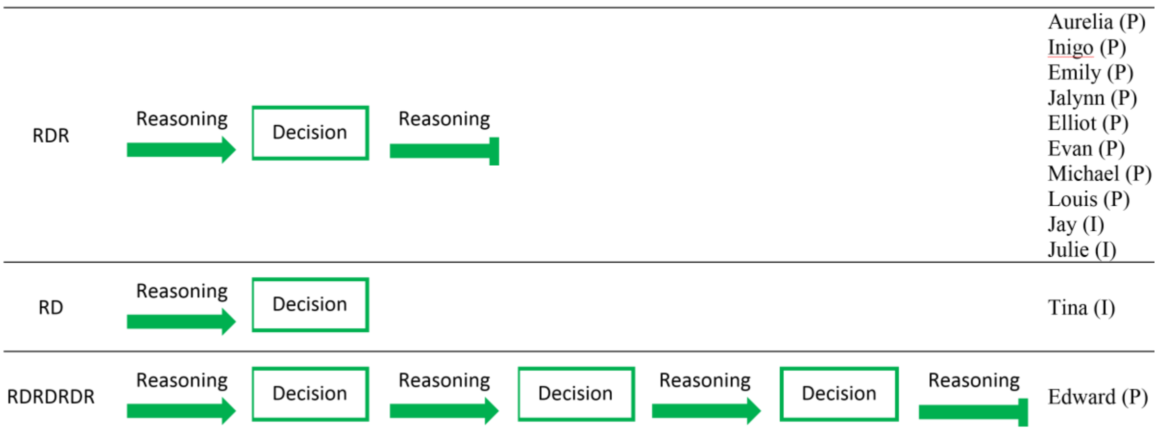


Figure 52. Decision-making pathways used on Global Maximum Task. (P) indicates proof by contradiction group and (I) indicates informal argument group.

The solutions on this task are organized by their level of rigor, resulting in two categories: the syntactic proof by contradiction and the semantic-deductive informal argument. Nine students constructed some version of the proof by contradiction and three students constructed some type of informal argument. Students' reasoning centered on the definitions of increasing and global maximum, as well as visualization or graphs related to increasing functions. The remainder of this section will be organized by solution groups. Within each solution group, further organization will be provided by pathway, and within pathways, organization is by reasoning type.

Proof by contradiction group. The statement in the Global Maximum Task is true if the domain of the function is all real numbers and false if the domain is restricted to a closed interval. The proof by contradiction is based on the domain of the function being all real numbers. An example of the proof by contradiction is as follows:

> Assume for contradiction that $c = f(a)$, $a \in \mathbb{R}$, is a global maximum for f. Then for all $x \in \mathbb{R}$ such that $x \neq a$, $f(x) < f(a)$. Consider $a + 1 > a$. Since f is increasing, $f(a + 1) > f(a)$, a contradiction to $c = f(a)$ being the global maximum. Thus, there is no real number *c* that is a global maximum for f.

The students in this solution group either assumed or inferred that the domain of the given function was all real numbers, decided that the statement was true, and constructed a proof by contradiction. Edward and Evan are two non-typical members of this group – Edward overturned a couple of decisions before taking the proof by contradiction approach, and Evan was the only student whose final solution consisted of both cases. Evan constructed both the proof by contradiction and an additional proof for the case in which the domain was restricted to a closed interval. However, Evan's proof by contradiction was actually a disproof because he changed the statement to read that the given function had a global maximum (rather than did *not* have one).

Aurelia and Inigo. The decision-making of Aurelia and Inigo followed Pathway RDR with a semantic-deductive, visualization/definition-based, informal argument leading to a true decision that was supported by a syntactic proof (Figure 53). Aurelia and Inigo based their decisions on the informal argument that an increasing function defined on the real numbers cannot have a global maximum. Both students had images of increasing functions in their heads that related to this idea that they did not draw until I asked if they had related images (Figure 54). Aurelia was thinking about vague increasing functions and noted that:

> If you're just increasing, you can never decrease to, like, make that a global maximum. Because it has to be some sort of, like, high point. But if you're just increase-, unless it stops, there is no way that there's not a higher point.

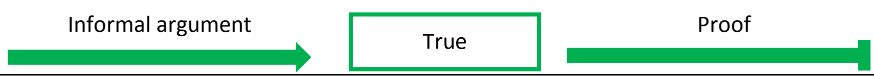


Figure 53. Aurelia and Inigo's decision-making pathway.

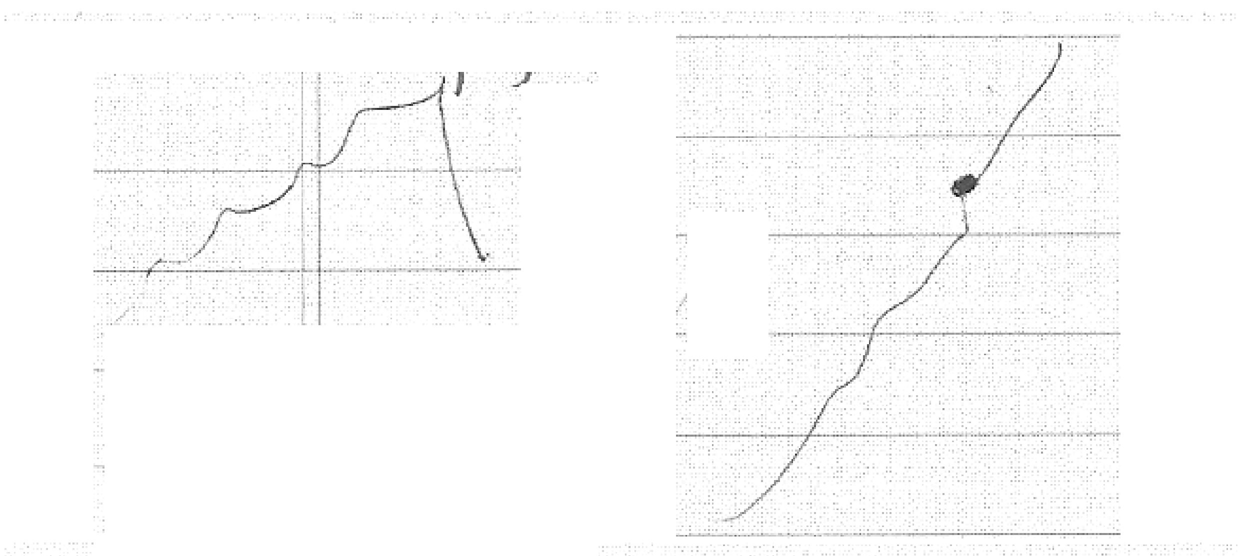


Figure 54. Aurelia's drawings of her images of increasing functions.

Inigo, on the other hand, was specifically thinking about linear functions, saying that linear functions couldn't have a global maximum because "any increasing function, they're always increasing, so there's never, because the range of any increasing function is almost always just to infinity. And then if something goes to infinity, there's no maximum for it."

Inigo seemed to assume that for this task, the domain of the function was all real numbers. On the other hand, Aurelia struggled throughout her work on this task with whether she could assume that the domain was all real numbers and actually completely stopped working on the task at several points. Upon first reading the statement, Aurelia said "So, I'm assuming that means if f is increasing throughout the whole entire function? So, this is obviously not true if you have, like, an [sic] composite function that stops at a certain point. And that might be a counterexample." She attempted to construct a piece-wise function that had a restricted domain and said "it's just the definition of a function that I'm getting caught up with right now. Like, can a function actually just stop or does it have to keep going further?" She drew a graph of $f(x) = x^2$ restricted to $[0,2]$. She indicated that if that was considered a function, then it was a counterexample, but if not, then she should be able to prove the statement. Then she said that she did not think that I was trying to trick her, so that probably was not a function, and she would try to prove the statement again.

Aurelia and Inigo supported their decisions with the proof by contradiction. Aurelia's informal argument included the idea that the global maximum was the highest point and that an increasing function defined on the real numbers will always have a higher point, so her argument is reflected in the proof by contradiction, linking her decision-making and construction processes. On the other hand, Inigo's informal argument was based on the idea that "if something goes to infinity, there's no maximum for it," and this does not explicitly translate into the proof by contradiction. Thus, there is no clear connection between Inigo's decision-making and construction processes.

Michael. Michael's decision-making followed Pathway RDR with a semantic-deductive, definition-based, informal argument leading to a true decision that was supported by a syntactic proof (Figure 55). Michael inferred that the domain of the function would be all real numbers because the task dealt with a global maximum that, for him, ruled out the case of the restricted domain. His informal argument was based on the idea that an increasing function defined on all real numbers would not have a global maximum, and he constructed the proof by contradiction. Michael's decision-making and construction processes were not connected explicitly because the idea of an increasing function on the real numbers not having a global maximum is not connected explicitly to the definitions of increasing and global maximum.

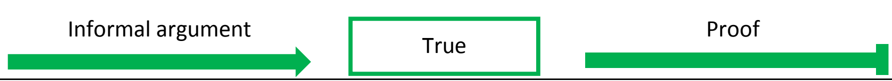


Figure 55. Michaels's decision-making pathway.

Elliot and Emily. The decision-making of Elliot and Emily followed Pathway RDR with a semantic-deductive, visualization/definition-based, informal argument leading to a true decision that was supported by a syntactic proof (Figure 56). Both Elliot and Emily's decisions that this statement was true were based on the informal argument that one can always find a larger function value on an increasing function. Upon reading the question and writing the definition for increasing, Emily asked herself, "Do I need to prove or disprove this?" She replied to herself, "I believe that it's true because if it's an increasing function, then I'll always be able to find an *x*-value that gives me an $f(x)$ value that's larger than it." Later, Emily said that that she had an image of $y = x$ in her head when she was originally thinking about an increasing function. Although Elliot also had an image of an increasing function in his head, it was not a specific function (Figure 57). He explained that "for any point *c*, there would be a larger point further on the graph."

Figure 56. Elliot and Emily's decision-making pathway.

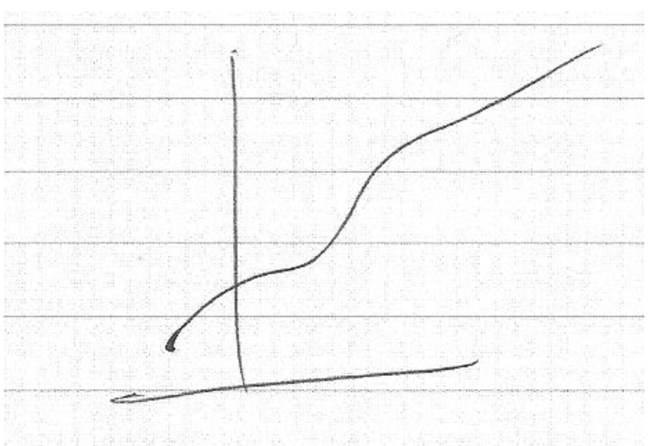


Figure 57. Elliot's drawing of his image of a generic increasing function.

Elliot's and Emily's arguments depended on the idea that the given function was increasing infinitely. Emily simply assumed that the domain of $f(x)$ was all real numbers although she did not mention the domain of the function. However, Elliot inferred that the domain was all real numbers because the function was strictly increasing. Elliot and Emily's construction process was connected to their decision-making process by the definitions of increasing and global maximum because the proof by contradiction is based on the idea that for an increasing function on the real numbers, one can find a greater function value than an assumed global maximum.

Jalynn. Jalynn's decision-making followed Pathway RDR with the combination of a semantic-deductive, visualization/definition-based, informal argument and a semantic-deductive, graph/definition-based, informal argument leading to a true decision that was supported by a syntactic proof (Figure 58). Upon reading the statement, Jalynn asked, "I can assume that it's an infinitely increasing function?" I said that she could (although this restricted the task not only to a function defined on the real numbers, but also to an unbounded function, assuming she meant that $f(x) \to \infty$ as $x \to \infty$). Later, she said that she had an image in her head of the graph of a function that had the property that any point you tried to call a global maximum would have a point higher because it was increasing. Jalynn wrote the definitions for increasing and global maximum used these to expand her visualization/definition-based informal argument into a graph/definition-based informal argument. She drew the graph she had visualized and elaborated on her initial argument (Figure 59). She said:

> I would want to show that there is another number that's bigger than c [the global maximum]. I know that if we have this function [drawing graph], no matter what

it's doing, if it keeps going like that [increasing] then we can't [have a global maximum] because it's going off to infinity.

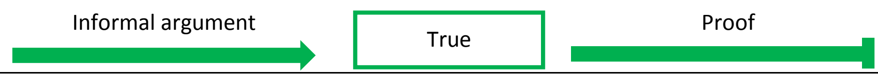


Figure 58. Jalynn's decision-making pathway.

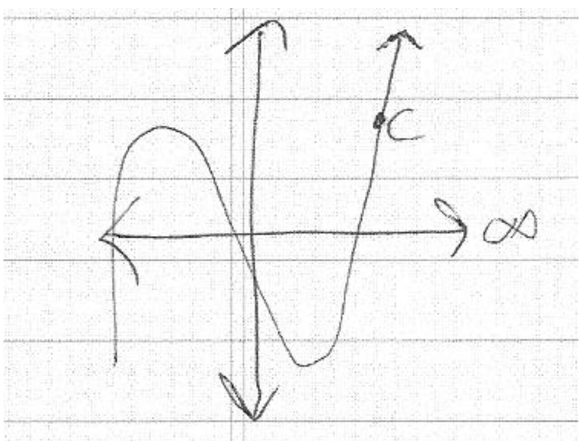


Figure 59. Jalynn's drawing of her image of why an increasing function has no global maximum.

Finally, Jalynn constructed the proof by contradiction. Her reasoning is connected throughout as her first informal argument formed the basis for her second informal argument that in turn provided the idea for her proof.

Louis. Louis' decision-making followed Pathway RDR with an unjustified intuition leading to a true decision that was supported by a syntactic proof (Figure 60). After Louis completed his proof, I asked him when he thought that this statement was true. He replied "Well, when I first read it, I kind of thought that it was a true statement." However, he said that he did not really know why until he began constructing his proof.

Louis used the definition of global maximum to infer that the function would be increasing on the real numbers. He noted that a global maximum was different from a local maximum because "it's not like you could set an interval for which one point would be the highest." Although Louis essentially constructed the proof by contradiction, it was embedded in an inductive argument that Louis explained as follows: "if you could show 1 to 2, 2 to 3. And then, you could continue that logic on to show that if you pick any number, $f(x_n)$, it can be shown that there is a greater number or whatever." Thus, Louis had the idea that he could always find a larger number than any number that he picked as a potential global maximum. His proof began inductively, but the only part of his argument that he actually used to prove the statement consisted of the proof by contradiction (Figure 61). Because Louis did not know why he initially thought this statement was true, his decision-making and construction processes were not connected.

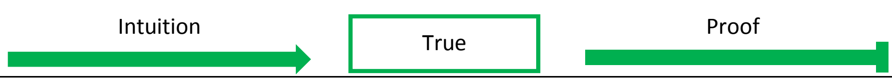


Figure 60. Louis' decision-making pathway.

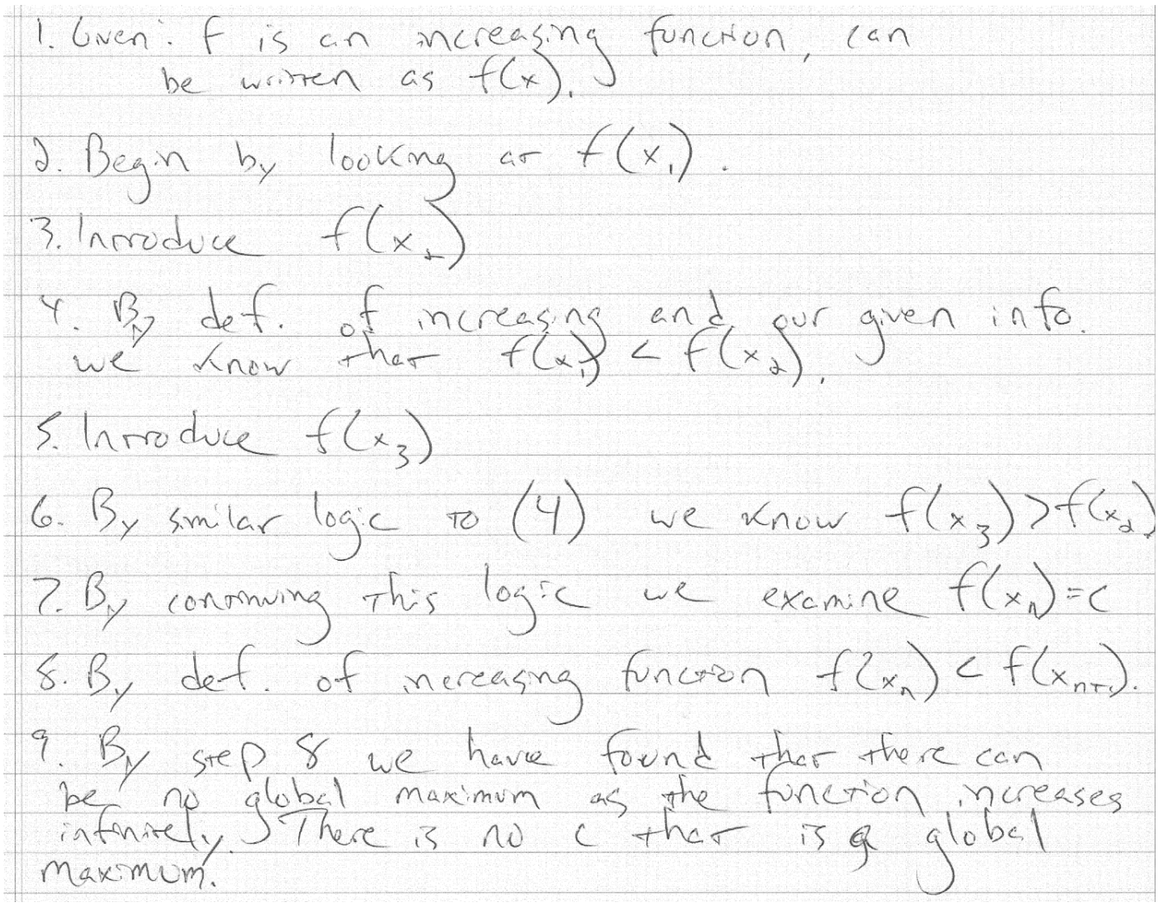

1. Given: f is an increasing function, can be written as f(x).
2. Begin by looking at f(x1).
3. Introduce f(x2)
4. By def. of increasing and our given info. we know that f(x1) < f(x2).
5. Introduce f(x3)
6. By similar logic to (4) we know f(x3) > f(x2)
7. By continuing this logic we examine f(xn) = c
8. By def. of increasing function f(xn) < f(xn+1).
9. By step 8 we have found that there can be no global maximum as the function increases infinitely. There is no c that is a global maximum.

Figure 61. Louis' correct proof (steps 7, 8 and 9) embedded in an inductive "proof."

Edward. Edward's decision-making followed Pathway RDRDRDR with a semantic-deductive, definition-based, informal argument leading to a true decision. This decision was overturned by a second semantic-deductive, definition-based, informal argument that led to a false decision. This decision was overturned by a yet a third semantic-deductive, definition-based, informal argument that led to a true decision that was supported by syntactic proof (Figure 62). Initially, Edward thought that this statement was true because increasing functions cannot have a maximum. He said, "At first I was thinking about just crossing some artificial threshold, and I needed to prove that I would cross some artificial threshold *c,* that was a global maximum, to show that I was passing it." He began a proof by contradiction by assuming that the function $f(x)$ was bounded by some number *M*. After getting stuck, he said "It doesn't say how much it increases...I've come to the conclusion that this is false." Later he explained that, "I had the realization that increasing functions can converge, and there is a limit. There can be a limit to a bounded increasing function." However, as he tried to talk through this idea, he changed his mind again:

> If f is an increasing function that is bounded, then there is a real number *c* that is a global maximum for f...but it wouldn't be a global maximum because you could keep increasing. OK, I think this is true again. 'Cause no real number *c* will be a global maximum for f as f is always increasing by the nature of it being an increasing function.

Edward realized that he had confused the ideas of a bound and a global maximum, and he then constructed the proof by contradiction. Edward was able to overturn his decisions by thinking through the concepts in the given task and making sense of the task situation. In the end, he had a correct understanding of the concepts that led to a correct proof, assuming the domain of the function was all real numbers. Edward's ideas in both his decision-making and construction processes were connected.

Figure 62. Edward's decision-making pathway.

Evan. Evan's decision-making followed a linked pair of Pathway RDRs with a semantic-deductive, visualization/definition-based, informal argument leading to a false decision that was supported by a syntactic disproof. This disproof required an assumption that led to a second case and true decision that were supported by a syntactic proof (Figure 63). Although Evan wrote the statement correctly on his paper, all of his work indicates that he was thinking that the statement said that the function *did* have a global maximum rather than saying that it did *not* have a global maximum. Thus, although Evan decided that the statement was false in the case in which the domain of the given function was all real numbers, his disproof was the proof by contradiction. Before he constructed his proof, he wrote the definitions for increasing and global maximum and provided the following informal argument that led to his decision that this statement was false: "from what I know about a global maximum, there has to be a value on either side of it that's smaller than it," but since f is increasing, "it'll be increasing indefinitely." Evan later said that he had the image of $y = x$ in mind when thinking about an increasing that "never goes back down." Evan noted that the key idea to his disproof was that for an increasing function, "if you pick a point, then next point's going to be larger....I knew that if I assigned a global max, c, anywhere, as long as f was unbounded, you'd be able to find a larger value." Thus, Evan connected his decision-making to his disproof construction process.

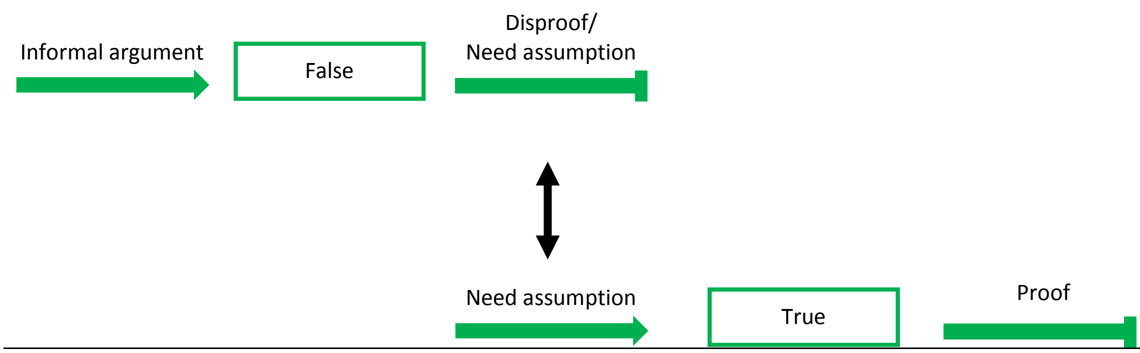


Figure 63. Evan's decision-making pathway.

After Evan had constructed the proof by contradiction, he added the assumption that the domain of the function was all real numbers to his disproof and created a second case in which the domain was a closed interval. He knew that in this case, the statement would be true (the function would have a global maximum) and provided the following proof: if the domain was [a, b], then for all $x < b$, $f(x) < f(b)$, so that $f(b)$ was the global maximum. The need for the assumption in his disproof led to the need for the second case and his decision that the second case was true. His proof was based on having a closed interval for the domain, connecting his decision-making and construction processes on the second case.

Informal argument group. The three students in this group did not support their decision with the proof by contradiction although they all decided that the statement was true. Instead, each constructed an informal argument as support for their decisions. Tina and Jay seemed to think that what they had constructed was sufficient for a proof, but Julie knew that her argument was not a proof. Julie struggled to translate her informal argument into a syntactic proof, and I told her that an explanation to support her decision was sufficient. Although the students in this group were unable to provide syntactic proofs, they each had the basic idea as to why the statement was true.

Jay. Jay's decision-making followed Pathway RDR with the combination of visualization-based intuition and a semantic-deductive, definition-based, informal argument leading to a true decision that was supported by a semantic-deductive, definition-based, informal argument (Figure 64). Although Jay did not think that this statement was true until after he developed his informal argument, he had an intuitive idea about it: "just off the top of my head, you know, when I, when you think of increasing function, it'll be going to infinity, and you can't find a constant." His intuition corresponded with a function defined on all real numbers, and Jay simply assumed the function in the task had a domain of all real numbers. After looking at the definitions for increasing and global maximum, Jay provided an informal argument for how to show that this statement was true: "The formal way of doing this would be finding an interval for x_1 and then finding a c greater than that, but then also finding that $f(x_2)$ is greater than c." So Jay's idea seemed to be that he could find a function value greater than a supposed global maximum. Although Jay had the right idea, his argument in support of this idea falls short of being complete (Figure 65). Jay attempted an inductive argument involving repeatedly finding greater function values that could never be global maximums because there were even greater function values. Thus, Jay's support for his decision reflected the same idea that led to the decision, connecting his decision-making and construction processes.

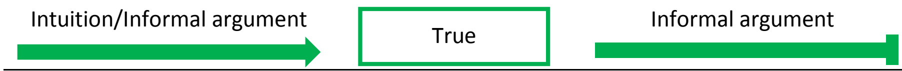


Figure 64. Jay's decision-making pathway.

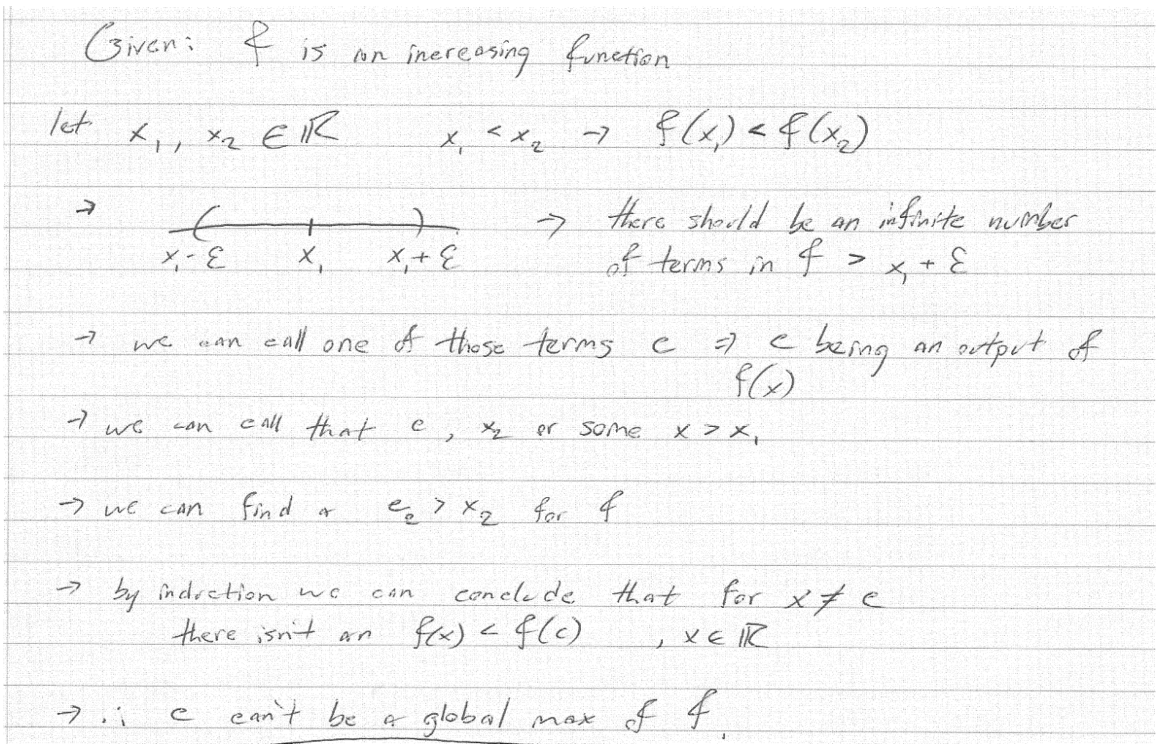


Figure 65. Jay's incorrect proof for the Global Maximum Task.

Julie. Julie's decision-making followed Pathway RDR with the combination of a semantic-deductive, visualization/kinesthetic-based, informal argument and a semantic-empirical example/graph leading to a true decision that was supported by a semantic-deductive, definition-based, informal argument (Figure 66). After reading the statement, Julie said:

> When it says increasing function, I'm thinking about a linear function that just keeps increasing, but doesn't just have to be a linear function. There are, for example, quadratic functions that increase, but then, at some point, they also decrease if there's a maximum.

She made an upward motion with her hand when she said the linear function just keeps increasing and later said that she had an image in her mind of a line such as $y = x$. At first, Julie was uncertain as to whether the quadratic function she had in mind satisfied the hypothesis of the task, so she moved on to reading the definitions for increasing and global maximum. Although she said "I keep picturing f as a linear function," she was still concerned about other types of functions. She drew a graph of a quadratic and a cubic and indicated that the cubic was always increasing (Figure 67).

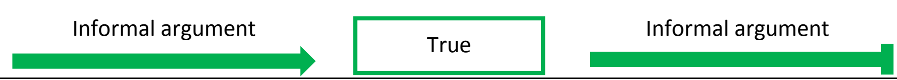


Figure 66. Julie's decision-making pathway.

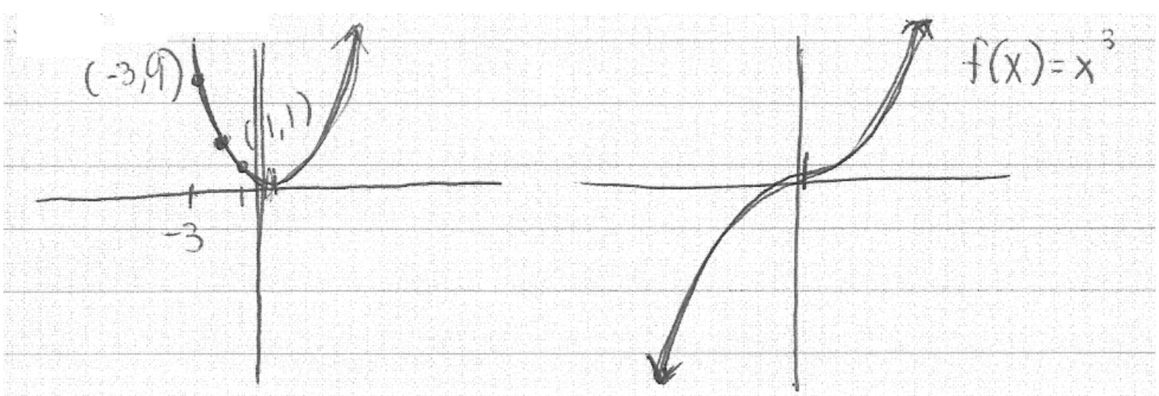


Figure 67. Julie's graphs on the Global Maximum Task.

She played around with her quadratic for some time, but eventually said that she did not think that it was an increasing function and focused on the cubic instead. She continually reread her definitions, especially for global maximum. She said that for $f(x) = x^3$ there was no "c" from the definition of global maximum, but "that's just one example. You also have linear functions, which are increasing and have no maximum." Julie did not realize that she had assumed that the domain of the given function was all real numbers.

This reasoning ultimately led her to conclude this statement was true and introduced her own definition based on her graphs for a global maximum. This definition formed the basis for the semantic-deductive, definition-based, informal argument she offered in support of her true decision. She explained:

> From what I remember, a maximum is where the, I guess, the turning point from where the slope is positive and then it turns negative. So an increasing function, then, has to have a negative slope at some point, which would make it decreasing. But if we go by that, then there is no real number, *c*, that is a global maximum for f.

She said that she did not know how to write this argument formally, so it stood as the support for her decision. All of Julie's arguments are based on the relationships between increasing, decreasing, and a global maximum on a graph, connecting her decision-making and construction processes.

Tina. Tina's decision-making followed Pathway RD with the combination of a semantic-empirical example/graph and a semantic-deductive, definition-based, informal argument leading to and supporting a true decision (Figure 68). Tina began by graphing the function $f(x) = x^2$ and noting that there was no maximum because it was increasing "on both sides." She then looked at the definition of increasing, noting that increasing x-values will result in increasing corresponding function values. Based on this idea, she constructed an informal argument that led to her decision that the statement was true and simultaneously supported the decision:

> So if my initial x, starting value for x is always going to be less than the next one and I'm going up, that means I can't have a maximum because I'm going to keep going up. So it is saying that there is no real number c that's a global maximum, so I'm gonna say true, or the statement is true.

She explained that the key idea to her argument was that "if I'm going up [because increasing], I can't go up to a maximum because there would be no maximum if it kept, like, going up." Tina's argument is based on the assumption that the domain of the function is all real numbers, but she did not realize that she had made this assumption. Her semantic-deductive, informal argument links to her semantic-empirical example and graph, and both led to and supported the decision, connecting her decision-making and construction processes.

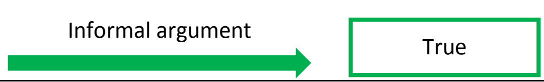


Figure 68. Tina's decision-making pathway.

Errors/Difficulties (RQ3). The students did not commit any intuitive errors on this task, but they did commit various mathematical and logical errors, including two systematic errors (Table 14). Both systematic errors relate to the fact that no domain was provided for the function in the task. The logical systematic error was the assumption that the domain of the function was all real numbers, and eight students made this error. The second systematic error was mathematical, in that students inferred that the domain of the function was all real numbers from other given information. Three students made this error. Jalynn was the only student who did not commit one of these systematic errors because she had asked me if she could assume that the domain of the function was all real numbers, and I said that she could. Furthermore, students made various nonsystematic mathematical and logical errors. In this section, I will discuss the two systematic errors and then provide a summary of the nonsystematic errors.

Table 14

Error Types on Global Maximum Task Organized by Approach and Student

Student	**Error Types**
Proof by Contradiction	
Jalynn	None
Emily Aurelia	Logical – assumed $\mathrm{dom}(f) = \mathbb{R}$
Inigo Edward	Logical – assumed $\mathrm{dom}(f) = \mathbb{R}$ Mathematical – global maximum = upper bound* (Edward only)
Evan	Logical – assumed $\mathrm{dom}(f) = \mathbb{R}$* Logical – incorrect proof structure Logical – Changed task statement
Elliot Michael	Mathematical – inferred $\mathrm{dom}(f) = \mathbb{R}$
Louis	Mathematical – inferred $\mathrm{dom}(f) = \mathbb{R}$ Logical – incorrect proof structure
Informal Argument	
Julie	Logical – assumed $\mathrm{dom}(f) = \mathbb{R}$
Jay	Logical – assumed $\mathrm{dom}(f) = \mathbb{R}$ Logical – incorrect proof structure
Tina	Logical – assumed $\mathrm{dom}(f) = \mathbb{R}$ Mathematical – misunderstanding of concept of increasing
* error was overcome	

Assuming $\boldsymbol{\mathrm{dom}(f) = \mathbb{R}}$. Aurelia, Edward, Emily, Evan, Inigo, Jay, Julie, and Tina each assumed that the domain of the given function was all real numbers. Each student made some type of comment to the effect of increasing functions always getting bigger or going to infinity. Julie noted that "when it says increasing function, I'm thinking about a linear function." Thus, the first function that came to mind was an increasing function defined on the real numbers. Jay's intuition on this task was that "when you think of increasing function, it'll be going to infinity." Thus, I think that the prototypical example of an increasing function is a function defined on the real numbers that approaches infinity as x approaches infinity that may have contributed to the students' assumption.

On the other hand, Tina, Inigo, Aurelia, and Evan mentioned that if the domain was restricted to a closed interval, then the function would have a maximum. Tina and Inigo did not think that this case applied to this task. Aurelia struggled throughout the entire task with whether a function could have a restricted domain, and even, had a counterexample ready in case she determined that it could. In the end, she thought that was a special case that would only be considered if I was trying to "trick" her, so she decided that the case in which the domain was all real numbers was the only applicable case. Evan was the only student who overcame his initial assumption that the domain was all real numbers and provided two cases in his solution depending on whether the domain was a closed interval or all real numbers. However, he only realized he had made this assumption because I asked him to reread the statement and ensure that he was saying what he wanted to say, indicating that he may have made a mistake although this was not the mistake I was trying to get him to see.

Inferring $\boldsymbol{dom(f) = \mathbb{R}}$. Elliot, Louis, and Michael each inferred from given information that the domain of the given function was all real numbers. Louis and Michael claimed that the domain would be all real numbers because the task concerned a global maximum. Louis said that there could not be a restricted domain because "this is dealing with a global max" rather than a local maximum for which "you could set an interval." Michael said that "the key word here is global...because it's global and then it's for the entire real line." Michael also said that if the domain was a closed interval, then there would be a global maximum, but this was not the situation at hand. Thus, both Louis and Michael considered the alternate case in which the domain was restricted to a closed interval, but thought the concept of a global maximum would not apply in that case.

Elliot determined that the domain was all real numbers because the function was strictly increasing: "When I saw that it was strictly increasing, I realized that...the function would have to be defined on all real numbers." Elliot did have the image of an increasing function that approached infinity as x approached infinity in his head for this task, so that may have influenced his inference. Perhaps he was confusing the domain issue with whether the function could have been a constant function, but this was ruled out by the strictly increasing definition.

Nonsystematic errors. Of the non-systematic errors on this task, two types of logical errors were committed, both related to proof structures. Evan used a proof by contradiction to disprove his altered version of the statement (that there *was* a global maximum). Evan wanted to disprove the following statement: If f is an increasing function, then there is a global maximum for f. In order to disprove this, he would need to show that f is increasing and f does not have a global maximum. However, he showed that if f is increasing, then f does not have a global maximum. This does not logically disprove the statement if f is increasing, then f has a global maximum. Louis and Jay each made the logical error of trying to construct an inductive proof for a function defined on the real numbers. Each attempted to choose a sequence of points that were potential global maximums and argue that there would always be a greater point. Their sequences were not well-defined and neither had a legitimate inductive structure.

Nonsystematic mathematical errors were made by Inigo, Edward, and Tina. Edward and Inigo both confused the concept of a global maximum with an upper bound. In particular, Inigo claimed that the concepts of global maximum and least upper bound were the same. However, because Inigo used the definition of global maximum in his work, this error was moot. Edward's original decision that the statement was true was based on his thinking that he was dealing with an upper bound rather than a global maximum. However, as he thought more about this idea, he considered the case that "increasing functions can converge...[but] it doesn't come to a specific number, that converging limit, which the distance between the function is always getting closer but never actually hits, so there is not a max that is seen as a maximum." Thus, he eventually saw the difference between an upper bound and a maximum. Through this thinking, Edward was able to overcome this error and construct a proof based on the concept of a global maximum.

Tina had a couple of mathematical misunderstandings about the concept of increasing. When she drew the graph of $f(x) = x^2$, she said "this has a minimum at a point, but it is constantly increasing on both sides, so there's no maximum." Thus, she seems to confuse the idea of the limit of a function approaching infinity with the function increasing. Additionally, she continued the idea that an increasing function would have a minimum in her proof, although she confused x- and y-values, by saying "each x_n is less than x_{n+1} value causing there to only be a minimum." Finally, she claimed that the definition of an increasing function said that "it's going to start at a point, and the numbers are just going to keep growing."

Evan made an error that was not intuitive, logical, or mathematical when he somehow got it in his mind that the statement read that the given function had a global maximum. When he wrote the statement on his paper, he wrote it correctly, but then when talking about the task, he was thinking about it in its altered version the entire time. After he wrote his proof by contradiction, he said that the statement was false, even though he produced a correct proof of the truth of the correct statement. To ensure he had said what he intended, I asked him to reread the statement. But rather than correct his mistake, he instead noticed his assumption regarding the domain of the function. I do not consider this a relevance error because he clearly was still concerned about the relevance of whether the function had a global maximum based on his proof by contradiction.

Global Maximum Task summary. The students in this study based their solutions to this task around the idea that an increasing function will increase infinitely and therefore cannot have a global maximum. All but one student provided an argument dependent on the domain of the given function being all real numbers although this was not given in the task. Evan was the only student who determined that there would be two cases depending on whether the domain was restricted to a closed interval or was all real numbers. The students used intuitive, semantic-empirical, semantic-deductive, and syntactic reasoning for decision-making on this task as well as semantic-deductive and syntactic reasoning to support their decisions.

The students who provided the proof by contradiction based their proofs on the idea that if they assumed they had a global maximum, they could always find a greater function value because the function was increasing infinitely. This proof falls out easily from the definitions of increasing and global maximum, but most students had the idea in the form of a semantic-deductive informal argument before constructing their proofs. Edward was the only student who overturned decisions on this task, and he did so through consideration of his initial thoughts regarding upper bounds and how these actually relate to the concept of global maximum. Once he realized that these were different concepts, he was able to prove the statement correctly, assuming the domain was all real numbers. Although Evan made two decisions, he did not overturn any decisions. Instead, he provided a solution that included two cases that included consideration of the domain of the function.

The students who supported their decisions with semantic-deductive informal arguments generally had the right idea of why this statement was true assuming the domain of the function was all real numbers. Jay argued that there would always be a greater number than a supposed global maximum, but his inductive argument missed the mark for this task. Julie and Tina each used graphs of specific examples to help them make sense of this task and constructed corresponding informal arguments.

Discussion

In this section, I discuss patterns across the four tasks regarding: (a) students' reasoning types to address RQ1, (b) connections between decision-making and proof/counterexample construction to address RQ2, and (c) students' errors and difficulties in addressing RQ3.

Students' reasoning types. In this section, I will discuss students' use of intuitive and analytical reasoning for decision-making and relationships between reasoning type and students' performance. There were 60 decisions made in this study – 11 intuitive and 49 analytical. Students made various types of intuitive decisions, and within the analytical decisions, students used semantic-empirical, semantic-deductive, and syntactic reasoning, as well as subtypes of these, for decision-making.

Intuitive decision-making. The students in this study made 11 intuitive decisions. Half of the decisions on the Composite Function Task were intuitive. There were three and two intuitive decisions on the Monotonicity and Injective Function Tasks, respectively. Intuition was not used on the Injective Function Task. Aurelia, Michael and Julie did not make any intuitive decisions, Jay made three intuitive decisions, and the other eight students each made one intuitive decision.

There was a variety of types of intuition used that helped students decide whether to pursue a proof or refutation of the task. However, only one type of intuition provided a basis for such pursuits. In three of the four cases of property-based intuitive decisions, the students' support for their decision was connected to their intuition. For example, on the Monotonicity Task, Inigo's counterexample was based on his property-based intuition about how a pair of negatives would interact with a quadratic function. Each other type of intuition – unjustified, memory-based, understanding-based, and similarity-based – was unconnected to subsequent reasoning, such as when Elliot decided that the statement in the Composite Function Task was true because "I remember doing something like this a couple years back. I just didn't really remember how to show that it was true." Although Elliot's intuition provided a decision on the statement's truth value, it did not suggest why it was true or suggest how he could prove it. This was a common experience for all of the students who made non-property-based intuitive decisions.

Analytical decision-making. Every decision on the Injective Function Task, the vast majority of decisions on the Monotonicity and Global Maximum Tasks, and half the decisions on the Composite Function Task were analytical. On the Injective Function, Global Maximum, and Composite Function Tasks, the decisions were based on mostly on semantic-deductive or syntactic reasoning. On the Injective Function Task, most decisions were based on either informal (semantic-deductive) or formal (syntactic) definitions of function or one-to-one that formed the basis for students' proofs or counterexamples. On the Global Maximum Task, decisions were based mostly on visualization-based and/or definition-based informal arguments (semantic-deductive) that formed the basis for students' proofs. However, on the Composite Function Task, students' analytical decisions were based mostly on failed proof or counterexample attempts (syntactic) that did not inform students' subsequent reasoning.

The Monotonicity Task was the only task for which semantic-empirical reasoning was used significantly for decision-making. Five students based their decisions on an example (four students used $f(x) = x^2$), leading to incorrect decisions. Of those five students, three (Louis, Julie, and Tina) overgeneralized their example to construct an informal argument in support of their decision. However, the other two students (Aurelia and Edward) did not base their proof attempt on their examples. This allowed them to overturn their example-based decisions and construct correct counterexamples for the task.

Reasoning types and performance. A few patterns emerged with respect to students' use of different reasoning types for decision-making and their membership in the three performance groups. The students in the strong performance group (Edward, Elliot, Inigo, and Michael) used mostly semantic-deductive or syntactic reasoning for decision-making. The students in the average performance group (Aurelia, Emily, Evan, and Jalynn) also used mostly semantic-deductive and syntactic reasoning, but much of it was in combination. The weak performance group (Louis, Jay, Julie, and Tina) used mostly intuitive and semantic-empirical reasoning, sometimes in combination with semantic-deductive reasoning. Overall, the strong and average groups preferred semantic-deductive and syntactic reasoning for decision-making whereas the weak group favored intuitive and semantic-empirical reasoning.

Across performance groups, most students used the same types of reasoning on the Injective Function, Global Maximum, and Composite Function Tasks. However, on the Monotonicity Task, there were distinct patterns of reasoning type per performance group. The strong group used mostly semantic-deductive reasoning, the average group used mostly syntactic reasoning, and the weak group used mostly semantic-empirical reasoning. This loosely corresponds to the three solution types on this task – the correct solution, standard incorrect proof, and generalization – respectively.

Connections between students' decision-making and proof or counterexample construction processes. Students' decision-making and construction processes were mostly connected on the Injective Function, Monotonicity, and Global Maximum Tasks, but they were mostly disconnected on the Composite Function Task. When analyzing the connectedness of students' decision-making and construction processes, I identified two distinct types of both connections and disconnections. Connections between construction and decision-making were categorized as either *construction based on decision-making* or *simultaneous construction and decision-making*. Disconnections were construction not based on decision-making and no decision-making. Table 15 shows the type and frequency of connections and disconnections by task. Simultaneous decisions and constructions and no decisions occurred less frequently than construction based on decisions and constructions not based on decisions. Reasoning on the Injective Function, Monotonicity, and Global Maximum Tasks was mostly connected, and reasoning on the Composite Function Task was mostly disconnected.

Table 15

Type and Frequency of Connections and Disconnections by Task

	Injective Function	Monotonicity	Global Maximum	Composite Function
Construction based on decisions	10	7	11	1
Simultaneous decisions & construction	3	6	1	0
Construction *not* based on decisions	0	3	3	10
No decisions	2	1	0	1

Connections or disconnections between decision-making and construction seem to be related to the following: (a) reasoning type, (b) task complexity, (c) solution correctness, and (d) overturned decisions.

Reasoning type. When considering the relationship between connections/disconnections and reasoning type, I only considered situations where the construction was based on the decision-making and where the construction was not based on the decision-making because both decision-making and construction take place and are separate components of the proving process. Across all tasks, students mostly connected semantic-deductive and syntactic reasoning. This usually took the form of proofs based on informal arguments such as when Elliot based his proof for the Global Maximum Task on the informal argument that there is always a larger function value on an increasing function defined on the real numbers.

Students linked other types of reasoning as well, such as when Jay and Inigo based their syntactic proofs for the Monotonicity Task on their intuitions. Julie and Louis connected their semantic-empirical and semantic-deductive reasoning on the Monotonicity Task by generalizing their examples. Other students linked different types of syntactic reasoning such as when Julie based her proof for the Injective Function Task on the formal definition of function. Additionally, on the Composite Function Task, Evan based his proof for his second case on his need for the assumption in the first case that the domain of the function was all real numbers.

Constructions that were not based on the decision-making occurred on the monotonicity, global maximum, and Composite Function Task. Both cases of disconnect on the Monotonicity Task involved semantic-empirical reasoning not connected to syntactic reasoning: Aurelia and Edward constructed an example that illustrated the statement was true, but then did not use their example to inform their proof attempts. In both cases, their failed proof attempts led them to determine that the statement was actually false and they were able to construct correct counterexamples. Thus, in this case, the disconnect aided their ability to overturn an incorrect decision.

On the Global Maximum Task, Inigo and Michael did not connect their semantic-deductive informal arguments to their syntactic proofs. Their informal arguments were based on the fact that functions that approach infinity do not have global maximums, but this idea did not translate explicitly into their proofs. Across the Global Maximum and Composite Function Tasks, six students failed to link their intuition to their syntactic reasoning. These intuitions were either unjustified, based on a memory about the task, or based on a similar task. Because these intuitions lie outside the properties of the task itself, they provide no basis for connecting them to subsequent construction processes.

An example would be Edward or Jalynn's similarity-based intuitions on the Composite Function Task that suggested the statement was true because the similar statement in which the component functions are one-to-one implies the composite function is one-to-one is true. Additionally on the Composite Function Task, there were five cases in which syntactic reasoning was not connected to other syntactic reasoning. In each case, the situation involved a counterexample or proof that was not connected to a failed proof or counterexample or a need for an assumption. For example, Michael did not base his counterexample search on his need for the assumption that the function *f* was one-to-one, and when his counterexample search failed, he did not consider why it failed when constructing his follow-up proof.

Task complexity. The less complex tasks seemed to correlate with constructions based on decision-making unlike the more complex tasks. On the less complex tasks that only required a straightforward application of a definition, the Injective Function and Global Maximum Tasks, the vast majority of the students based their construction on their decision-making. These connections mostly took the form of proofs or counterexamples based on informal arguments or informal definitions such as when Emily based her counterexample on the Injective Function Task on the informal definition of a function that different inputs do not necessarily result in different outputs.

On the more complex tasks that involved complex proof or counterexample structures, (the Monotonicity and Composite Function Tasks), students were less likely to base their construction process on their decision-making. Almost half of the decisions in the Monotonicity Task were made simultaneously with the constructions. In these cases, students' decision-making pathways ended in decisions because they were unable to come to a final decision on the statements' truth value until after they already had a complete argument in support of its truth or falsity. Such was the case with Evan who did not have an idea of whether the statement was true or false, so he used his definitions to construct a proof that simultaneously led to and supported his decision.

The Composite Function Task was the only task in which the majority of the students did not connect their decision-making and construction processes. It is unclear how much of this related to the complexity of the task rather than the types of reasoning the students used. The disconnections are most likely based on a combination of these factors. Although many students had an intuition on this task, it did not set up potential reasoning for its truth or falsity. Additional disconnections were related to failed proof and counterexample attempts. The frequency of these failed attempts is probably based on the complexity of the task. Unfortunately, the simplicity of the failed attempts did not reveal conditions that could inform a subsequent proof or counterexample attempt, such as Inigo's failed proof attempt based on the lack of a relationship between the domain of the composite function and the domain of g. Thus, the students did not learn anything from these failed attempts that would aid further work on the task.

Solution correctness. The analysis of a relationship between solution correctness and connections is limited in scope. Because the Global Maximum Task was true or false based on cases, and no student actually correctly solved the task, I will not consider it in this section. Additionally, I have eliminated all cases in which a student made multiple decisions and experienced both connections and disconnections on the same task. This is simply because I need an injective function in order to determine frequencies. With these restrictions, this part of the discussion is based only on 35 of the 60 total decisions made by the students in this study. Table 16 shows the frequencies of connections and disconnections by solution correctness.

Table 16

Frequency Distribution of Connections and Disconnections by Solution Correctness

	Correct	Incorrect
Connected Decision-making & Construction	11	12
Disconnected Decision-making & Construction	3	9

As can be seen in Table 16, there does not seem to be a relationship between connectedness of construction and decision-making and correct solutions as connections between decision-making and construction occurred with about the same frequency for correct and incorrect solutions. However, when it comes to disconnections, these are more likely to correspond with incorrect solutions than correct solutions. Thus, connected decision-making and construction processes coincides with a higher likelihood of a correct solution than disconnected processes.

Overturned decisions. There were a total of 14 overturned decisions across the four tasks. Mostly syntactic reasoning was involved with the overturned decisions, and in particular, half of the overturned decisions were overturned due to a need for assumption or a failed proof or counterexample attempt. Of the 14 overturned decisions, 10 of them related to either connected, simultaneous decision-making and construction or disconnected no decision-making.

Of the ten total cases of simultaneous decision-making and construction, eight of these involved overturned decisions: three on the Injective Function Task and five on the Monotonicity Task. For example, on both the Injective Function and Monotonicity Tasks, Jalynn made two decisions with simultaneous decision-making and construction processes. All of her reasoning was syntactic. On the Injective Function Task, a proof led to the decision that the statement was true. However, the need for an assumption in the proof led her to consider whether the statement was false. After constructing a counterexample, she decided that it was in fact false in the general case. On the Monotonicity Task, a mathematical error in a proof led her to a decision that the statement was false, but her uncertainty prompted her to double check her work. She corrected the error that resulted in a proof that supported the subsequent decision that the statement was true.

Of the four cases of disconnections between construction and decision that involved no decision, two of these included overturned decisions (both by Inigo). On the injective function and composite tasks, Inigo did not initially engage in decision-making and simply assumed that the given statement was true. This caused a disconnect involving no decision for Inigo's first decision on these tasks. However, both decisions were overturned due to failed proof attempts.

Students' errors and difficulties. In this section, I will discuss logical and mathematical errors that occurred across tasks as well as a couple of recurring themes that caused students difficulties in the tasks. There were no intuitive errors that occurred across tasks as relevance errors were the only types of intuitive errors committed by the students and these only occurred on the Monotonicity Task. The two key logical errors were making illegal assumptions and using incorrect proof structures. The only key mathematical errors were conceptual misunderstandings. The two recurring difficulties students experienced were related to function domains and the use of $f(x) = x^2$ as the preferred example in any situation. I will also discuss the situations in which students were able to overcome errors.

Logical errors. The students in this study made illegal assumptions across all tasks. Between the Injective Function and Composite Function Tasks, five students assumed that a given function was one-to-one although this information was not given and could not be inferred from the conditions in the task. This assumption changed the truth value of the statement in the Injective Function Task. On the Global Maximum Task, eight students assumed that the domain of the given function was all real numbers although its domain was not stated. Thus, the students failed to consider the case in which the domain was restricted. Finally, on the Monotonicity Task, one student assumed that the range of the function g was in the interval *I* where the function *f* was increasing. This was a necessary assumption for the statement to be true, but it was not a legitimate assumption.

The other key logical error that students experienced in this study was incorrect proof structures. These errors only occurred on the Injective Function, Global Maximum, and Composite Function Tasks, with eight instances total. On the Injective Function and

Global Maximum Task, respectively, Edward and Evan each attempted to disprove a statement using proof by contradiction. This strategy in itself is not viable, but they also made structural mistakes within these attempts. On the Injective Function Task, Inigo and Louis each made an incorrect assumption for contradiction by negating a given rather than the conclusion. On the Global Maximum Task, Louis and Jay each tried to construct an inductive proof although this was inappropriate for the task. On the Composite Function Task, Jalynn incorrectly used the definition of one-to-one to structure her proof and Inigo incorrectly structured his counterexample by switching the desired properties of two functions.

Mathematical errors. The most prominent mathematical errors were conceptual misunderstandings, but the concepts differed based on the task. On the Injective Function Task, three students (Aurelia, Tina, and Louis) thought that all functions were one-to-one. These concepts are closely related, so their misunderstandings are reasonable to some degree. On the Global Maximum Task, Tina confused the concept of increasing with infinite limits at infinity. She referred to the graph of $f(x) = x^2$ and indicated that it was increasing because it was "constantly increasing on both sides." Additionally, three students inferred that the given function f was defined on all real numbers on the Global Maximum Task. Louis and Michael thought that the domain was all real numbers because they were dealing with a global maximum versus a local maximum. Michael said "the key word here is global...because it's global, and then it's for the entire real line." Elliot inferred that the function was defined on the real numbers because it was strictly increasing rather than increasing.

On the Monotonicity Task, some students thought that decreasing functions were always negative and many students overgeneralized the concept of “two negatives make a positive.” Four students specifically used the idea that decreasing functions are negative to justify a generalization from an example. Furthermore, every student who constructed an example of a decreasing function chose a negative function. It is possible that students’ prior mathematical experiences with increasing and decreasing functions – such as linear functions and derivatives – shaped their view of increasing and decreasing functions. Seven students overgeneralized the idea that multiplying two negative quantities results in a positive quantity. This idea manifested itself in two distinct ways – it was generalized from examples and from the switching of inequalities when using the definition of decreasing. Aurelia, Julie, Louis, Inigo, and Tina each constructed examples that served as the basis for informal arguments to support their decisions. On the other hand, Jay and Jalynn referred to the switching of the inequalities in their proofs as similar to two negatives resulting in a positive, as Jalynn explained: “So it made it opposite, and then whatever you put into f, it’s going to make it opposite again. So it’s kind of like a double negative.”

Recurring difficulties. The students in this study had various issues with respect to function domains on three of the four tasks. On the Monotonicity Task, although the properties of increasing and decreasing only applied on a restricted interval, the students often assumed the properties applied to all real numbers. This assumption may have stemmed from the fact that the function had a domain of all real numbers or the fact the definitions for the properties were based on all real numbers. Either way, this assumption changed the task and resulted in four incorrect proofs.

On the Injective Function Task, the given function was defined on an abstract domain A, causing difficulties for Evan and Jalynn. Evan did not think that he could use $f(x) = x^2$ as a counterexample because he did not know how to reconcile it with the abstract domain and thought that he had to construct an abstract example. Jalynn thought that the domain and range designation on this task had additional meaning, such as that the function was one-to-one or onto, although she did not struggle with this on other tasks. This seemed to be because the other tasks were defined on the reals rather than on abstract sets.

The domain and range of the function in the Global Maximum Task are not provided. This made the task somewhat ambiguous, and a complete solution required consideration of the cases in which the domain was all real numbers (or an open interval) and when it was restricted (or a closed interval). However, Evan was the only student who considered both cases, and this was only because I asked him to reconsider his answer due to his inadvertent change in the statement. Seven students simply assumed that the given function was defined on all real numbers, and three students inferred that the domain was all real numbers. Jalynn asked if she could assume that the function was defined on all real numbers. Because most students determined that the function was defined on all real numbers, no student correctly solved this task. It is likely that this determination either stemmed from the fact that the definitions were based on all real numbers or that many students are used to assuming that they are dealing with functions defined on all real numbers unless stated otherwise.

Another recurring theme that caused some difficulties for students was the use of the function $f(x) = x^2$ in almost every situation. In general, the students repeatedly turned to $f(x) = x^2$, thinking that if it did not work, then nothing would. Michael noted that it was the "ultimate" example of a non-injective function, and that in general, "if there were to be a counterexample, it would probably be x^2, just because it's so simple." On the Injective Function Task, $f(x) = x^2$ was used appropriately by the majority of students as a counterexample, but students' insistence that this example should always work led them to try to force its appropriateness in any situation. As an example, on the Composite Function Task, Aurelia, Emily, and Inigo pushed themselves to make $f(x) = x^2$ work as a counterexample, resulting in the mathematical error of not considering absolute value when taking the square root of a squared function.

Overcoming errors. The students in this study committed 67 total errors and only overcame 14 of these. See Table 17 for a breakdown of total and overcome errors by task. Each student made some type of error on at least two tasks, and most students made errors on all tasks. Eight of the 12 students were able to overcome at least one error that they made, and half of these students overcame multiple errors.

Table 17

Frequency Table of Errors and Overcome Errors by Task

Task	Number of Errors	Number of Overcome Errors
Injective Function	15	5
Monotonicity	20	6
Global Maximum	18	2
Composite Function	14	1
Total	67	14

The most prominent error types that were overcome were relevance errors, illegal assumptions, and incorrect proof structures. In general, students overcame errors either because they were uncertain about their work and wanted to double-check it or they were indirectly prompted by me to reconsider their work. For example, on the Injective Function Task, Inigo and Louis each corrected their proof structures after uncertainty propelled them to reconsider the desired conclusion of the task. Also on the Injective Function Task, Jalynn overcame the error of making the illegal assumption that a given function was one-to-one upon prompting from me to rethink her notation confusion.

In a few cases, further consideration of the mathematics involved led to the discovery of the error. When Inigo and Aurelia each considered the partial graph of a quadratic function on the Monotonicity Task, they realized that they needed an interval restriction and overcame their relevance errors. On the Global Maximum Task, Edward's ongoing contemplation regarding global maximums and upper bounds led him to overcome his mathematical error confusing the two concepts.

Based on these results, in the following chapter I make some concluding remarks regarding my research questions. Additionally, I offer conclusions related to students' performance on these tasks and their mathematical backgrounds as well as their understanding of the culture of proof in mathematics. Finally, I discuss implications of this research for the teaching and learning of proof and proving in undergraduate mathematics and suggest directions for future research.

Chapter 5: Conclusions

This research explored students' use of intuitive, semantic, and syntactic reasoning during the processes of deciding on the truth value of mathematical statements and constructing proofs and counterexamples. Additionally, this study investigated connections between the decision-making and construction processes and systematic errors of reasoning. Because of the emphasis on syntactic reasoning and *prove this* statements in the undergraduate curriculum (Davis & Hersh, 1981; Durand-Guerrier et al., 2012; Weber, 2004), little is known about how students approach mathematical statements with unknown truth values, what types of reasoning are useful for deciding on their truth value, and what types of systematic errors may inhibit success in the proving process. This study shed light on these topics by considering the following research questions:

1. In what ways and to what extent do students use intuition and analysis to decide on the truth value of mathematical statements?
2. What are the connections between students' process of deciding on the truth value of mathematical statements and their ability to construct associated proofs and counterexamples?
3. What types of systematic intuitive, mathematical, and logical errors do students make during the proving process, and what is the impact of these errors on the proving process?

Although undergraduate students are rarely involved in evaluating the truth value of mathematical statements, this is an essential aspect of mathematical practice that deserves attention. Researchers hypothesize that students who are required to evaluate a conjecture before proving it, and who are able to connect these processes, are more likely to succeed at constructing a proof (Garuti et al., 1998). Thus, experiences with statements in which the truth value is unknown could alleviate some of the difficulties that undergraduate students have with the proving process (Alcock, 2010; Dreyfus, 1999; Harel & Sowder, 1998, 2009; Moore, 1994; Selden & Selden, 1987, 2003, 2007; Solomon, 2006; Weber, 2001).

In this study, I used a combination of cognitive psychology and mathematics education theories to develop a framework for distinguishing among different types of reasoning based on both their cognitive and mathematical properties. Dual-process theory emphasizes the distinction between intuitive and analytical reasoning based on whether the reasoning can be fully justified (Evans, 2010; Fischbein, 1982). Alcock's and Weber's theory of semantic and syntactic reasoning differentiates semantic reasoning based on intuitive or informal representations of concepts from syntactic reasoning based on logic and structure (Alcock & Weber, 2010; Weber & Alcock, 2004, 2009). The category of semantic reasoning was too broad to be useful, so intuitive reasoning was separated completely (to align with dual-process theory) and the remaining types of semantic reasoning were separated into semantic-empirical and semantic-deductive to distinguish between reasoning based on empirical evidence versus informal arguments. This resulted in a framework with four main categories of reasoning – intuitive, semantic-empirical, semantic-deductive, and syntactic – each with subcategories.

Previous research indicates that each of these types of reasoning can be useful for deciding on the truth value of mathematical statements and constructing proofs or counterexamples. Intuition creates a meaningful representation of a task that can assist with understanding, providing a starting point, suggesting a direction to pursue, or guiding action on the task (Burton, 2004; Fischbein, 1982, 1987; Smith & Hungwe, 1998; Wilder, 1967). Additionally, intuition can help students notice similarities that connect the current task to prior knowledge and experiences (Burton, 2004; Fischbein, 1987). Semantic reasoning can support understanding and can provide an informal basis for a proof or counterexample (de Villiers, 2010; Raman, 2002; Weber & Alcock, 2004, 2009). Syntactic reasoning can assist with understanding and provide structure for a proof or counterexample (AMATYC, 1995; Selden & Selden, 2009; Weber & Alcock, 2009).

The qualitative methods of clinical task-based interviews and the think-aloud method revealed students' reasoning processes in depth. The participants completed four mathematical tasks in which they were required to decide on the truth value of a given mathematical statement and either prove or disprove the statement accordingly. Students spoke aloud as they worked on the task so that I could follow their thought processes. After each task, I asked them to reflect on their decision regarding the truth value of the statement in order to gain additional insight into their decision-making processes. Analysis of the students' written work and transcripts of their spoken words focused on the types of reasoning the students used during the decision-making and construction processes, connections between these processes, and systematic errors that occurred during the processes. In the remainder of this chapter, I discuss conclusions regarding (a) my research questions, (b) students' performance on the tasks and their mathematical backgrounds, and (c) students' understanding of the culture of proof. Additionally, I suggest implications for teaching and learning and recommendations for future research.

Conclusions With Respect to RQ1

My first research question addressed the use of intuitive and analytical reasoning for deciding on the truth value of mathematical statements. Of the 60 decisions made in this study, 11 were intuitive and 49 were analytical. Both intuitive and analytical reasoning were used in numerous ways for decision-making, although intuition was not used as often as I had anticipated.

Use of intuition for decision-making. The students in this study used intuition in a variety of ways to decide on the truth value of mathematical statements. The six distinct types of intuition that I identified correspond to different ways intuition provided students with a starting point when approaching the tasks. Students used memory-based and understanding-based intuitions that drew on their prior experiences with the concepts in the task to indicate a direction to pursue. Similarity-based intuition was used when students identified a statement with a known truth value that was similar enough to the given statement to suggest the truth value of the given statement. When using property-based or visualization-based intuition, the students based their decisions on vague ideas about function properties or visualizations of the functions in the task. Finally, even unjustified intuitions provided the students with a sufficient indication of a statement's truth value to allow them to begin on the task.

Limited use of intuition. Intuition was not used as often as I had expected for deciding on the truth value of the mathematical statements in this study. I had expected intuition to play a larger role due to its usefulness in providing a direction to pursue when a statement's truth value is unknown. I theorize that intuition did not play the role I expected due to methodological limitations and a poor choice of a working definition of intuition.

Methodological limitations. In dual-process theory, intuition is defined by its independent operation from working memory. In order to verify whether intuition is used based on this definition, special procedures, such as functional magnetic resonance imaging (fMRI), are needed to determine which areas of the brain are active during the decision-making process. In this study, I did not have access to such procedures. In lieu of these procedures, a working definition of intuition needs to be used in order to distinguish intuitive and analytical reasoning based on students' expressed words and actions. I used a working definition of intuition that did not seem to capture intuition as expected (see the next section).

In this study, I provided the students with a list of definitions of the terms in the tasks. Many students immediately turned to this definition list upon reading each task statement and wrote the relevant definitions. It is possible that the availability of these definitions alone hindered students' use of intuition by suggesting that syntactic reasoning was the preferred starting point.

Working definition of intuition. Based on the idea that an intuitive response is formed at least partially without awareness and cannot be fully explained (Evans, 2010;

Fischbein, 1982), I used justification as my sole criteria for distinguishing intuition from analysis in this study. If a students' reasoning was fully justified – even if it was justified later rather than in the moment – then I considered it analytical. If the reasoning was partially or not justified, I classified it as intuitive. Based on this limited working definition, I classified few instances of reasoning as intuitive.

There is no consensus on a working definition of intuition in mathematics, and other versions have the potential to capture intuition differently than the definition I used. Two variations that I think would have been better (in terms of defining intuition consistently with what mathematicians and mathematics educators would want to call *intuition* (Davis & Hersh, 1981)) would have (a) distinguished justifications made in the moment from post hoc justifications, and (b) considered various expressions of intuition.

In the moment versus post hoc justifications. In this study, after the students completed their work on a task, I inquired as to when they made their decisions and what led to those decisions. They often offered justifications for their decisions that were not expressed at the time they made the decision. These were frequently tied to their support for the decisions, and it is reasonable to consider whether these were post hoc explanations of their work rather than delayed expressions of their thoughts at the time the decision was made.

In particular, this change in working definition would have allowed me to classify three decisions on the Global Maximum Task as intuitive rather than analytical. Elliot, Jalynn, and Inigo each provided post hoc justifications for their decisions, but it is likely these were based on their proofs rather than representing their thought processes during decision-making. Furthermore, these students made comments that indicated that their decisions were not fully justified at the time. For example, Elliot noted that it was "immediately implausible" for an increasing function to have a global maximum, and Jalynn said "I just knew that if it was infinitely increasing, you couldn't have a global maximum."

On the Injective Function Task, four students (Aurelia, Edward, Evan, and Michael) made decisions classified as analytical based on post hoc justifications. For example, Evan decided that the Injective Function Task was false because "it doesn't say anything about the function being one-to-one." However, he said that this was just a "suspicion," and later justified his decision by fully explaining that "nowhere in the statement does it say b can't equal b_0 ...there's nothing directly preventing $f(a)$ from equaling b_0 when a's $\neq a_0$."

Expression of intuition. Working definitions of intuition may need to consider that students' expression of intuition may vary by task (possibly related to task complexity). It may be that intuition on a less complex task is expressed quickly, clearly, and with confidence. On the other hand, intuition on a more complex task may be expressed slowly, vaguely, and with uncertainty.

On the Injective Function and Global Maximum Tasks (the less complex tasks), many students had quick, confident responses with a clear, yet incomplete justification. Due to delayed expression of an explanation, these were classified as analytical decisions. However, it could be that these decisions were intuitive, especially because they were made quickly and the students had high confidence in them. Although quickness and confidence are not always associated with intuition, they are common correlates (Evans, 2008; Thompson 2009).

By contrast, students' intuitive decisions on the Monotonicity and Composite Function Tasks (the more complex tasks) were typically slow, vague, and coupled with uncertainty. This manifestation of intuition was easier to classify as intuition using my working definition. It is possible that the complexity of these tasks made it more difficult for students to have a clear intuition on the task, and also reduced their certainty in it. Often, students seemed hesitant about their intuitions on these tasks and were uncertain about how their intuitions corresponded with a solution to the task.

Use of analysis for decision-making. Most decisions in this study were made using analytical reasoning in the form of either semantic-empirical, semantic-deductive, or syntactic reasoning. Students who used semantic-empirical reasoning used an example or a graph as a test case and based their decision on the result of the test case. Most semantic-deductive decisions stemmed from informal arguments based on examples, graphs, diagrams, visualization, kinaesthetic movement, informal definitions, or inconclusiveness of given information. The other semantic-deductive decisions were based on informal definitions. These informal arguments or definitions provided students with sufficient evidence of a statement's truth value to warrant a decision.

Syntactic reasoning was used in similar ways as other reasoning types, but it was also used in unique ways. Similar ways include syntactic decisions based on formal definitions or a suspected need for an assumption. Unique ways that syntactic reasoning was used for decision-making include situations in which a decision was not made until after a student had full support for the decision in the form of a proof, disproof, or counterexample. In these cases, no additional reasoning was needed after the decision. Additionally, failed proof or counterexample attempts, often coupled with a lack of a needed assumption, led many students to overturn decisions and make new decisions.

Significance of RQ1 conclusions. This study contributes to the limited research on the use of intuition for evaluating conjectures and expands the types of intuitive reasoning that can be used successfully for decision-making in mathematics. Similarity-based and memory-based intuitions are new types of intuition that, to my knowledge, have not been discussed in research on deciding on the truth value of mathematical statements. If the mathematics education community can come to an agreed upon definition of intuition, we may be able to better understand how intuition can help guide students on proof tasks, and how to develop intuition in our students.

The wide variety of semantic reasoning used to make decisions in this study corresponds with the reasoning used by mathematicians in prior research studies (Alcock & Inglis, 2008; de Villiers, 2010; Inglis et al., 2007). Prior research on students' reasoning has indicated that they accept semantic-empirical arguments as proof (Harel & Sowder, 1998, 2007; Inglis et al., 2007), but the reasoning of the students in this study moved beyond the semantic-empirical use of examples or diagrams to more sophisticated semantic-deductive reasoning. Furthermore, students' use of syntactic reasoning in this study to make decisions mirrored mathematicians' use of syntactic reasoning for evaluating conjectures in prior research (Inglis et al., 2007; Weber 2009) and expanded the research on students' uses of syntactic reasoning for decision-making (Buchbinder &

Zaslavsky, 2007; Durand-Guerrier et al., 2012). The results of this study indicate that students can learn to use intuitive, semantic, and syntactic reasoning to evaluate conjectures appropriately and successfully. Additionally, this suggests the value in encouraging students to learn a variety of reasoning types in order to assist with deciding on the truth value of mathematical statements.

Conclusions With Respect to RQ2

My second research question was concerned with what connections exist between the processes of deciding on the truth value of mathematical statements and constructing associated proofs or counterexamples. In this section, I will discuss the following findings (a) the types of connections and disconnections that I identified, (b) the disconnect between intuition and analysis, and (c) the benefits and hindrances of both connections and disconnections.

Types of connections and disconnections. I identified two types of both connections and disconnections between the processes of deciding on the truth value of mathematical statements and constructing associated proofs or counterexamples. Connections between these processes took the form of either a construction based on decision-making or simultaneous construction and decision-making. Disconnections occurred when a construction was not based on decision-making or no decision-making took place. Students' ability to connect these processes seemed to depend on reasoning type and task complexity.

Disconnections between intuition and analysis. In general, most connections occurred between semantic-deductive and syntactic reasoning in the decision-making and construction processes. The students in this study only connected one type of intuitive decision-making to the corresponding analytical construction process – property-based intuition. The other types of intuition did not suggest a reason for the decision, so it is not surprising that they were not connected to analytical reasoning in the construction process. However, because connections supposedly facilitate proving (Garuti et al., 1998), certain types of intuition may be undesirable bases for decision-making.

Benefits and hindrances of connections and disconnections. In general, situations in which decision-making and construction were connected corresponded to a higher likelihood of a correct solution. However, this may only apply to connections for which the construction is based on the decision-making. In this study, of the 10 occurrences of connected simultaneous decision-making and construction, eight of these decisions were overturned, suggesting that simultaneous decision-making and construction processes may not facilitate proving.

On the other hand, disconnections did not always hinder proving. For example, on the Monotonicity Task, Aurelia and Edward used an example to decide that the statement was true. However, their corresponding proofs were disconnected from their examples. This allowed them to realize that their proof attempts would not be successful, and they overturned their incorrect decisions. Julie, Tina, and Louis also used an example to decide that the Monotonicity Task was true, but they constructed incorrect proofs based on a generalization of the example.

Significance of RQ2 conclusions. Connections in which the construction process was based on the decision-making process mostly facilitated proving, offering some evidence to support the conjecture by Garuti et al. (1998). Thus, experiences with proving that involve both a decision-making and construction process have significant potential to help students succeed in the proving process. When students experience proving tasks in which the truth value of the statement is known, they do not engage in a decision-making process. The fact that this situation (no decision-making) results in disconnections may suggest a reason for some of the documented difficulties students have with proof.

Although intuition can be useful for decision-making, only property-based intuition led to connections between decision-making and construction in this study. This suggests that the development of certain types of intuition, specifically those based on mathematical concepts, should be encouraged over the general development of any type of intuition.

Conclusions With Respect to RQ3

My third research question addressed systematic errors that students made during the proving process and the impact those errors had. The only systematic intuitive errors made in this study were relevance errors, and these were only committed on the Monotonicity Task. Although a variety of systematic logical and mathematical errors occurred on each task, these conclusions focus on errors that were systematic across tasks. Systematic logical errors made across tasks included making illegal assumptions and using incorrect proof structures. Systematic mathematical errors that occurred across tasks were related to conceptual misunderstandings. Although some errors were overcome and a few ended up being irrelevant, most led to incorrect decisions on the truth value of the mathematical statements or incorrect proofs, disproofs, or counterexamples.

Similar to the limited use of intuition in this study, I was surprised by the limited amount of systematic intuitive errors that occurred. Not only did relevance errors occur only on the Monotonicity Task, but attribute substitution errors did not occur at all. In my pilot work, attribute substitution errors occurred on the Monotonicity Task, including substituting the concept of *multiplying two negative numbers results in a positive number* for the concept of *composing two decreasing functions results in an increasing function*. Although many students in this study made this association, it was not an attribute substitution error because the students did not make an unknown substitution of these concepts. Instead, they acknowledged the similarity of the situations, but did not completely replace one concept by the other. This may have been due to the fact that the students in my pilot study were novice undergraduate provers whereas the students in this study were more experienced undergraduate provers. In general, the students' experience with proof and limited use of intuition are the most likely causes for the limited systematic intuitive errors.

I expected that students would make intuitive relevance errors on the Monotonicity Task, and seven of the twelve students in this study did so. The errors took the form of either completely ignoring the interval restriction or ignoring the meaningfulness of the interval restriction. However, of the seven students who made this error, three overcame it. Aurelia and Inigo did so through the use of partial graph of a quadratic, and this was the key to their ability to make a correct decision on this task. Emily overcame the error by rereading the statement and noticing the interval restriction.

The systematic logical errors made by the students in this study – making illegal assumptions and using incorrect proof structures – are documented errors that students make in the proving process (Moore, 1994; Selden & Selden, 1987). Previous research indicates that incorrect proof structures are common in proof tasks (Alcock, 2010; Moore, 1994), but I am unaware of research indicating that making illegal assumptions is a frequently occurring error. Of the 14 errors that were overcome in this study, seven were systematic logical errors. Four students overcame the error of making an illegal assumption – two due to uncertainty related to the assumption and two due to indirect prompting from me. Additionally, three students overcame the error of using incorrect proof structures, and all did so due to uncertainty that led to them double-check their structure. Thus, it seems that some level of uncertainty in the proving process can be beneficial in prompting students to review their work and overcome errors.

The systematic mathematical errors made across tasks in this study were related to conceptual misunderstandings. There was much confusion surrounding the concepts of function and one-to-one. Some students thought that all functions were one-to-one or assumed that a given function was one-to-one. On some level, this is not surprising due to the similarity in the definitions of these concepts, but it should be a major concern that some fourth year mathematics and mathematics education majors fail to recognize that not all functions are one-to-one.

Another overextension of mathematical properties was the assumption that decreasing functions are negative and increasing functions are positive. This assumption has some merit because decreasing linear functions have negative slopes and increasing linear functions have positive slopes, and the procedure for determining whether a differentiable function is decreasing or increasing is to check whether the derivative is negative or positive, respectively. However, this misunderstanding led some students to choose example functions that supported an incorrect decision on the Monotonicity Task and to overgeneralize the idea that multiplying two negative quantities results in a positive quantity into a justification for the (incorrect) truth of the statement in the Monotonicity Task.

A final conceptual misunderstanding that arose in this study was that the concept of a global maximum of a function inferred that the domain of the function must be all real numbers. This confusion most likely arose from the distinction between local and global maximums and minimums and the overgeneralization of a global maximum being the maximum on a function's entire domain to it being the maximum on the real numbers.

Significance of RQ3 conclusions. The systematic errors made by the students in this study had a significant impact on the correctness of their solutions because few of them were overcome. However, some level of uncertainty may assist students in overcoming errors. Thus, if we encourage students to act with a sense of skepticism, which can be supported with the use of proof tasks in which the truth value of the statement in the task is unknown, then students may be more likely to overcome errors.

Even simply asking follow-up questions that prompt students to review their work could lead to the correction of many errors that otherwise might suggest that students have more difficulties with proof than they truly have.

The conceptual misunderstandings exhibited by the students in this study are of concern due to the centrality of functions in mathematics. The students should have had extensive experience with the concepts of functions, one-to-one functions, composite functions, and increasing and decreasing functions. However, their work on these tasks showed significant misunderstandings. This suggests that substantial work is needed in the area of helping undergraduate students develop strong conceptual understandings of these topics.

Conclusions With Respect to Students' Performance and Mathematical Background

As was discussed in the methods section, the four tasks in this study can be ordered based on the complexity of a proof or counterexample for the task. The Injective Function and Global Maximum Tasks are less complex due to the structure of a proof or counterexample requiring straightforward applications of definitions. The Monotonicity and Composite Function Tasks are more complex because the structure of a counterexample or proof is more complex. The Monotonicity Task requires a counterexample with two functions that satisfy specific properties in relation to each other. The proof for the Composite Function Task is not linear, requires applying the given assumption to a function created in the proof, and involves an unexpected contradiction.

The students in this study were grouped loosely according to their performance on the tasks. Members of the *strong performance group* include Edward, Elliot, Inigo, and Michael, who met the criteria of correctly solving at least one of the more complex tasks and at least one of the less complex tasks with no errors or overcame their only errors. Members of the *average performance group* include Aurelia, Emily, Evan, and Jalynn, who did not meet the criteria for the strong performance group, but correctly solved at least one task with no or minor errors. Finally, the members of the *weak performance group* include Louis, Jay, Julie, and Tina, who met the criteria of incorrectly solving or being unable to solve all of the tasks.

The students in this study had one of the following majors: (a) mathematics, (b) Adolescent-to-Young Adult (AYA) integrated mathematics education who earned secondary mathematics teacher certification, (c) dual majors in mathematics and AYA integrated mathematics education, or (d) economics. The strong performance group included two mathematics majors, one dual mathematics and AYA integrated mathematics education, and one economics major. The average performance group included two mathematics majors and two AYA integrated mathematics education majors. Finally, the weak performance group included two AYA integrated mathematics education majors, one mathematics major, and one dual mathematics and AYA integrated mathematics education major.

Although there are some general trends regarding the relationships between performance groups and students' mathematical backgrounds, there are no absolutes. The combined number of proof-based courses that the students had taken or were enrolled in ranged from two to eight courses. The strong group contained the three students with the most number of proof-based courses and the student (Edward, the sole economics major) with the least number of proof-based courses. The students in the average and weak groups had between three and five proof-based courses. Every student in the strong group had passed or was currently enrolled in Advanced Calculus, but this was also the case with two students (Emily and Evan) in the average group and one student in the weak group (Jay). Only one student in the strong group had taken College Geometry, but three students (Aurelia, Emily, Jalynn) in the average group had, as had all students in the weak group. Most students across all groups had taken Number Theory, Linear Algebra, and either Abstract or Modern Algebra. The lack of a clear connection between students' background (in terms of major and prior and current coursework) and their performance on the tasks in this study may indicate that some other feature of a student's background or approach to problem solving may influence their performance on prove or disprove tasks.

Conclusions With Respect to Students' Understanding of the Culture of Proof

The students in this study understood that a proof entails syntactic reasoning. Across all tasks the students attempted to provide syntactic support for their decisions and most were successful in reaching this level of formality. However, there was at least one case on each task where a student was unable to construct an entirely syntactic argument for their proof or refutation. For example, on the Monotonicity Task, each student in the generalization group constructed an informal argument based on generalizing an example. However, none of the students in the generalization group believed that their example was sufficient to prove the statement, and all attempted to provide syntactic justification for why the phenomena in the example would generalize. Contrary to previous research (Dreyfus, 1999; Harel & Sowder, 1998; Inglis et al., 2007), the students in this study were aware that an example was insufficient to prove a statement and a general argument was needed to accompany the example. When students struggled to construct a syntactic argument, it was usually due to their inability to either understand or use the given definitions. Whenever students used intuitive, semantic-empirical, or semantic-deductive reasoning to make a decision on the truth value of a statement, they always attempted to follow it up with syntactic reasoning. When I inquired about students' understanding of proof, every student mentioned at least one of the following: use of assumptions to reach conclusions, use of definitions, use of logical reasoning, and use of proof structures. The fact that these students understood what a proof was is encouraging. This suggests that they know what the end goal is, and that we should focus on helping them develop the skills they need to reach that goal.

Although the students understood what a proof was even when they were unable to construct one, they did not demonstrate the same level of understanding when it came to refutations. This is evidenced by (a) students' thinking proof by contraction is used for disproving statements (3 times), (b) students' assumptions that the statements were true (4 times), and (c) students' assumptions of non-given statements that were necessary for the given statement's truth (14 times). Evan, Edward, and Jay stated that proof by contradiction was used to disprove statements, and Edward and Evan actually used proof by contradiction to try to disprove the Injective Function and Composite Function Tasks, respectively. Edward constructed a counterexample on the Injective Function Task, but did not find it sufficient to disprove the statement. He seemed to think that a general argument supporting the falsity of a statement, akin to a proof supporting the truth of a statement, was necessary.

In four cases, students simply assumed that the given statement was true and did not consider its truth value being in question. For Emily, this phenomenon was a result of her experiences in her mathematics courses in which she was "used to being handed a statement and told to prove it, and told that it's true." It is possible that this was case for the other two students, Inigo and Jay, who also failed to question a statement's truth value.

In the face of a needed assumption in a proof that was not given, the students in this study always made the illegal assumption rather than consider that the statement they were trying to proof may be false. In some cases, and in particular on the Global Maximum Task, the students seemed unaware that they had made an illegal assumption. In other cases, students intentionally made an assumption knowing that it was not given. Inigo and Jalynn on the Injective Function Task, and Edward on the Monotonicity Task, made explicit assumptions in their proof attempts that they knew were assumptions. Each student completed a "proof" that included the illegal assumption. Prompting from me to consider their assumptions was necessary and sufficient for each student to overcome these errors. However, their hesitance in making the assumptions did not stop them from making them and did not prompt them to consider that the statement may be false.

These difficulties with refutations seem to point to the idea that students may lack experience dealing with statements in which the truth value is in question. Some students seem to have the wrong idea about how to disprove statements and the role of proof by contradiction. Some students may have the wrong idea about mathematics if they believe that every statement thrust in front of them is true. Furthermore, students seem to do whatever it takes to construct a proof a statement, even if that means making illegal assumptions, before they seem to consider that the statement may be false. In each of these cases, students lack key strategies for thinking about, identifying, and refuting false statements. This suggests that students simply need experience refuting statements so that they can develop these strategies.

Implications for Teaching and Learning

The results of this research suggest implications for the teaching and learning of proof at the undergraduate level. In general, it seems that students could benefit from additional experience with prove-or-disprove tasks that require consideration of the truth value of mathematical statements. The emphasis on tasks involving mathematical statements with known truth values helps students develop their syntactic reasoning skills, but does not encourage the development of semantic or intuitive reasoning skills. Although prior research suggests that students use semantic reasoning as a substitute for proof (Harel & Sowder, 1998, 2007; Inglis et al., 2007), the students in this study understood the appropriate use of semantic and intuitive reasoning and used these successfully to evaluate conjectures. Thus, instruction in proving should encourage the use of semantic and intuitive reasoning as valuable tools in exploring and deciding on the truth value of mathematical statements.

Prior research has indicated that students and mathematicians have preferred reasoning types (Alcock & Weber, 2010; Weber, 2009; Weber & Alcock, 2004). However, the results of this study, as well as previous research (Alcock, 2010; Raman, 2001, 2003) indicates that all types of reasoning are important for successful proving. Thus, it is important to encourage students to use reasoning types that they are less inclined to use without encouragement. Through the classification of a student's reasoning on proving tasks, an instructor may determine if a student has a preference for intuitive, semantic, or syntactic reasoning. This will help the instructor determine the type(s) of reasoning with which the student needs more experience. Armed with this information, the instructor may give the student proving tasks that encourage the use of a specific type of reasoning. Thus, by identifying students' preferred reasoning types, instructors may encourage the development of non-preferred reasoning types through thoughtful task selection.

Although the students in this study were comfortable with what a proof was and how to prove statements, many students lacked an understanding of how to disprove statements. Additional experience with prove-or-disprove tasks would provide students with opportunities to develop their refutation skills and understandings. Engagement with tasks that have the possibility of being false could also curb students' tendency to weaken statements by making illegal assumptions. Prove-or-disprove statements can be enhanced to include requirements to extend true statements or salvage false statements such as in the textbook *Extending the Frontiers of Mathematics: Inquiries Into Proof and Argumentation* (Burger, 2007). In this case, students need to alter the assumptions or conclusions in order to generalize a true statement or turn a false statement into a true one. Such tasks should help students see when additional assumptions are needed to make a false statement true rather than make illegal assumptions in an attempt to force a statement to be true.

In addition to providing students opportunities to engage in both decision-making and construction processes in mathematics, instructors should monitor the ways students are connecting the processes. This research suggests that a separate, yet connected, decision-making phase may be preferable to simultaneous decision-making and construction. Thus, an exploratory phase should be encouraged allowing students to seek evidence that would support a decision. In the subsequent construction phase, students could attempt to transform their evidence into a proof or counterexample. In this way, students can see that although proving is a complex process, they can use a variety of reasoning strategies – intuitive, semantic, and syntactic – to connect the creative and rigorous aspects of proving. This again points the importance of tasks in which the truth value of the statement is unknown. When students are given a statement with a known truth value, the decision-making phase becomes non-existent. There is no possibility of connections when there is no decision-making. It is imperative that students have the opportunity to connect the decision-making and construction processes, which means that they need the opportunity to engage in the decision-making process.

In this study, students were often able to overcome errors when they felt uncertain about their work and explored these feelings. Instructors should encourage students to consider feelings of uncertainty as potential tools that may suggest that something is wrong, and to follow-up on these feelings with review of their work. Some students in this study expressed feelings of uncertainty when in fact they had made errors, but they did not act on those feelings, resulting in incorrect solutions. Because mathematics is often presented in a way that suggests that everything is already known, prove-or-disprove tasks can support the development of students' skepticism about mathematics. When students have the opportunity to explore the truth-value of mathematical statements, they can learn to use uncertainty to their advantage rather than view it as a weakness.

The systematic conceptual misunderstandings reported in this study suggest that even advanced undergraduate mathematics and mathematics education majors have difficulties with certain mathematical concepts, even when they have significant experience with the concepts. Thus, there is work to be done to address these misconceptions and to assist students in developing correct understandings of these concepts. This is especially important for the AYA integrated mathematics education majors who will be teaching these concepts to their own grade 7-12 students.

Though the sample size is limited, most of the weakest performers in this study were AYA integrated mathematics education majors. This is consistent with research showing that preservice and inservice teachers have considerable difficulty with proof and proving (see Harel & Sowder, 2007 for an overview). This is a serious stumbling block for the incorporation of the teaching and learning of proof throughout the entire K–16 curriculum, as suggested by CUPM (2004), NCTM (2000), and Common Core State Standards (National Governors' Association & Council of Chief State School Officers, (NGA & CCSSO), 2010) as well as researchers studying proof and proving such as Moore (1994) and Selden and Selden (2009). Extensive teacher training in proof and proving seems necessary to develop the understanding teachers need in order to cultivate appropriate concepts of mathematical proof in their students (Harel & Sowder, 2007). Thus, teacher preparation and professional development programs should consider the attention they give to developing preservice and inservice teachers' skills with proof and proving.

Recommendations for Future Research

This study points us in certain directions regarding the intuitive and analytical reasoning students use to decide on the truth value of mathematical statements, the ways students connect the decision-making process to the process of constructing a proof or counterexample, and the systematic errors that prohibit students' success in the proving process. Additional research would be beneficial with respect to each of these topics.

Although intuition did not play a significant role in this study, this may be due to methodological limitations and an inadequate working definition of intuition. Because the definition of intuition that I used relied on working memory and I did not use methods that could determine if working memory was engaged, I was forced to rely on a working definition of intuition. Repeating this study with methods from neuroscience, such as functional imaging, that would reveal whether working memory was engaged could assist with distinguishing intuition from analysis and help determine an appropriate working definition of intuition for mathematical proof tasks. Additionally, the availability of the definition list in this study potentially circumvented students' intuitive responses. Thus, I think the study could be repeated without the use of the definition list, and I hypothesize that intuition would play a larger role.

Theoretical research on developing a standard working definition of intuition in mathematics would benefit the mathematics education community. The limited research on intuition in mathematics indicates that there may be a variety of types of intuition that need to be distinguished along with differentiating intuition and analysis (Davis & Hersh, 1981). In this study, I chose a specific working definition based on dual-process theory, but it did not fully capture all of the cases that I thought should be classified as intuition. Thus, this research could benefit from additional analysis using different working definitions of intuition such as excluding retrospective justifications of reasoning or taking into account various expressions of intuition.

The results of this study suggest that task complexity may play a role in the types of reasoning students use as well as their connections between the decision-making and construction processes. Additional research on task complexity, including how best to determine it, would improve our understanding of the relationship between a given task and students' reasoning. Selden and Selden (2009) suggest a variety of factors, such as the length of a proof, the hierarchical structure of a proof, and the applicability of an expected intuition to a proof, that may contribute to task complexity. These should be explored with respect to the difficulty students' have on a given proof task. Additionally, such research could help identify types of reasoning that facilitate proving independently of the task. It also could suggest situations in which connections between the decision-making and construction processes may be difficult to achieve.

The results of this study indicate that connections between the decision-making and construction processes are not always beneficial and disconnections are not always harmful. Additional research into the types of reasoning that are associated with the different types of connections and disconnections could suggest key subtypes of intuitive, semantic, or syntactic reasoning that could facilitate the proving process. Additionally, further identification of factors that affect connections could suggest situations in which connecting (or not connecting) the decision-making and proving processes may facilitate proving rather than hinder it.

The conceptual misunderstandings displayed by the students in this study are troubling considering the level and majors of the students. Research into the development of these misunderstandings and instructional interventions that can help alleviate these misunderstandings could improve students' understanding of these key mathematical concepts and processes. Finally, the results of this study suggest that students can use their feelings of uncertainty to help them overcome errors. Metacognitive research on students' feelings of uncertainty during the proving process could help reveal when uncertainty could be beneficial to students and how they can harness it as tool to facilitate proving.

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Appendix A: Questionnaire for Participant Recruitment

1. Are you interested in participating in a research study on mathematical proof?

2. Are you an undergraduate student?

3. Have you passed at least one proof-based mathematics course with a grade of B or better? If *yes*, please list the course name, when it was taken, and your grade.

<u>Consent</u>

If you answered *yes* to each of these questions, please provide me with your email address so that I may contact you to provide additional information and hopefully set up an interview.

Name:______________________________________

Email:______________________________________

Thank you!

Kelly M. Bubp

bubp@ohio.edu

Appendix B: Mathematical Background Questionnaire

1. What is your current year (1st, 2nd, etc.) and major in school?

Current year (1st, 2nd, etc.)	Major

2. What mathematics course(s) are you currently enrolled in? Why are you taking these courses at this particular time?

3. List all college level mathematics courses you have taken (separate by year/level) and the grades you earned in them.

Course Title	Term/Year	Grade

Appendix C: Protocol for First Interview

At the beginning of the first interview, I will talk to the participants in order to make sure they understand why I am conducting this study, what the interview process will entail, and the contents of the consent form. I will ensure that they sign the consent form before I begin recording the interview. What follows is a rough sketch of what I will say.

> Hello. It's nice to see you. Thank you so much for agreeing to participate in this study. Before we begin, I just want to make sure that you understand the nature of the study, what you are agreeing to as a participant, and what will happen during the interview process. I will also need to get your signature on the consent form.
>
> I am conducting this study for my dissertation in mathematics education. I have a Bachelor's and Master's degree in mathematics and have always been interested in proof and proving. Mathematics never came naturally to me, so I have always had to work hard at it, especially when it came to constructing proofs. So, don't worry if you have any struggles with any of these problems. The point of this is not for you to get every problem right. I don't care about whether your proofs are right or wrong, all I am interested in is the reasoning that you engage in while completing the tasks.
>
> Even though I recruited you from a mathematics class, this study has nothing to do with that class. Your instructor has no idea that you have agreed to participate and will not know anything about these interviews. So you don't need

to worry about this having any impact on your grade or anything else for that class.

At this time, let me review what is on the consent form in front of you and let you know what is expected of you during the interview process. Recall that I also sent the consent form to you in my last email so that you have a copy. If you have any questions about anything, please don't hesitate to ask.

As I have already explained, you will be asked to complete five proof tasks over the course of two interviews. While you are working on the tasks, I need to you to think aloud. This means that I need you to say out loud everything that you are thinking as you are working on the tasks. If you stop talking, I will keep reminding you to talk out loud. Other than that, I will not interfere with what you are doing. You will have a list of definitions of terms in the tasks that you can reference at any time. You are not allowed to use any other materials. If you have questions during the interview, you are free to ask me anything.

In this first interview, you will work on three tasks, one at a time, and then complete two more in the second interview. I will ask you follow-up questions to the tasks regarding the reasoning you used on the tasks and any difficulties you had on the tasks. In the end of the second interview next week, I will also ask you some general questions about proof and proving.

Although I appreciate your participation in this study, it is entirely voluntary and you are free to quit the interview process at any time. The only people that will see the information from your interviews is me and my dissertation committee. All of our names and contact information on the consent form in case you have any questions or concerns later on.

All of your information will be kept in the strictest confidence. A pseudonym of your choice will identify all of your information so that no one will be able to link your written work or the interview transcripts to you. Your information will be stored in a password protected PC and/or a secure cabinet and the recordings will be destroyed upon completion of the project.

Do you have any questions about anything? Please take a moment to read through the consent form, and if you feel comfortable participating in this project, sign the form and we will get started.

(Example task selection – choices and order will be randomly selected for each participant).

Task A: Let $f: A \to B$ be a function and suppose that $a_0 \in A$ and $b_0 \in B$ satisfy $f(a_0) = b_0$. Prove or disprove: If $f(a) = b$ and $a \neq a_0$, then $b \neq b_0$.

Task B: Prove or disprove: If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ decreasing on an interval *I*, then the composite function $f \circ g$ is increasing on *I*.

Task C: Let *D* be a set. Define a relation ≈ on functions with domain *D* as follows: $f \approx g$ if and only if there exists $x \in D$ such that $f(x) = g(x)$. Prove or disprove: The relation ≈ is an equivalence relation.

Follow-up questions:

1. At what point did you decide that you wanted to prove or disprove each statement? What specifically led you to that decision?
2. Describe any difficulties you had in deciding whether to prove or disprove the statements in this interview.

Appendix D: Protocol for Second Interview

(Example task selection – the two tasks not chosen for the first interview will be used on the second interview).

Task D: Determine, with proof or refutation, whether the following statement is true or false: If f is an increasing function, then there is no real number c that is a global maximum for f.

Task E: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be functions. Determine, with proof or refutation, whether the following statement is true or false: If the composite function $f \circ g$ is one-to-one, then g is one-to-one.

Follow-up questions:

1. At what point did you decide that you wanted to prove or disprove each statement? What specifically led you to that decision?
2. Describe any difficulties you had in deciding whether to prove or disprove the statements in this interview.

General questions on proof and intuition:

1. Describe any general strategies that you use when deciding whether to prove or disprove a given mathematical statement.
2. Is your approach to prove or disprove tasks different than your approach to tasks in which you are told to prove or told to disprove a statement?
3. Describe the process of proving. What is a proof? Is it different from the proving process?

4. Do you think you have a mathematical intuition? If so, describe how you obtained it and how you use it. Does it help you decide whether you think a statement is true or false?

Appendix E: Definition List

List of relevant definitions that will be provided to participants during interviews. Definitions are based on those found in Alcock and Weber (2010), Burger (2007), and Smith, Eggen, and St. Andre (1997).

Definitions

1. If $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ are two functions, then the **composite function** $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$.
2. A function $f: \mathbb{R} \rightarrow \mathbb{R}$is said to be **decreasing** if and only if for all x_1,$x_2 \in \mathbb{R}$, ($x_1 < x_2$ implies $f(x_1) > f(x_2)$).
3. A relation *T* on a set *D* is called an **equivalence relation** if and only if *T* is reflexive, symmetric, and transitive.
4. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to have a **global maximum at a real number** *c* if and only if for all $x \in \mathbb{R}$ such that $x \neq c$, $f(x) < f(c)$.
5. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be **increasing** if and only if for all $x_1, x_2 \in \mathbb{R}$, ($x_1 < x_2$ implies $f(x_1) < f(x_2)$).
6. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called **one-to-one** if and only if for all $x_1, x_2 \in \mathbb{R}$, ($f(x_1) = f(x_2)$ implies $x_1 = x_2$).
7. A relation *T* on a set *D* is called **reflexive** if and only if for all $x \in D$, xTx.
8. A relation *T* on a set *D* is called **symmetric** if and only if for all $x, y \in D$, if xTy, then yTx.
9. A relation *T* on a set *D* is called **transitive** if and only if for all $x, y, z \in D$, if xTy, and yTz, then xTz.

Appendix F: Enrollment – Proof-Based Courses

Detailed Listing of Proof-based Courses Passed or Enrolled in by Student

Name	Major	Proof-based course numbers
Aurelia	AYA	300, 301, 302
Edward	Econ	303, 403
Elliot	Math	301, 304, 305, 401, 403, 404, 410, 411
Emily	Math	301, 302, 303, 304, 403
Evan	Math	301, 303, 401, 403
Inigo	Both	300, 301, 302, 303, 304, 403
Jalynn	AYA	301, 302, 304
Jay	Math	302, 303, 304, 403
Julie	AYA	300, 301, 302, 304
Louis	AYA	301, 302, 303
Michael	Math	301, 303, 401, 402, 403, 404, 405, 601
Tina	Both	301, 302, 303

Course Titles – four digit numbers represent semester courses and three digit numbers represent quarter courses.

Course Number	Course Title
300	Introduction to Proof
301	Introduction to Number Theory
302	College Geometry
303	Principles of Linear Algebra
304	Introduction to Abstract Algebra
305	Introduction to Advanced Calculus
401/402	Abstract Algebra I & II
403/404	Advanced Calculus I & II
405	Introduction to Topology
601	Real Analysis I
410	Special Topics – Topology
411	Special Topics – Set Theory

Appendix G: Choice, Origin, and Modification of Interview Tasks

The Global Maximum and Equivalence Relation Tasks are from Alcock and Weber (2010). These were both proof production tasks in that study. The Equivalence Relation Task is exactly the same as it was in Alcock and Weber (2010), however, I changed the Global Maximum Task from a prove task to a prove-or-disprove task. Alcock and Weber (2010) indicate that these tasks are "general assertions about classes of mathematical objects with certain properties" (p. 97), and due to this general nature, "an individual may reasonably approach the question either by working syntactically with the appropriate definitions or by examining particular instances of the general objects discussed" (p. 98). Additionally, they indicate that that by using a range of tasks with a variety of objects, the participants had the opportunity to "display a range of strategies if they were so inclined" (p. 98). This may address the limitation that the findings of the study may be dependent on the chosen tasks. Finally, the participants in this study were current students in an introduction to proof course, so these tasks should be appropriate for my planned participants.

The Composite Function and Monotonicity Tasks are modified from tasks in Smith, Eggen, and St. Andre (1997, pp. 178, 185), a textbook for a transition-to-proof course. The Composite Function Task was altered to reflect its informal statement in the book: "if the composite of two functions is one-to-one, then the first function applied must be one-to-one" (p. 183). Additionally, I eliminated the domain restrictions stated in the text to allow both functions to be real-valued. For the Monotonicity Task, I included the definitions for increasing and decreasing that they had provided in the previous task in the book, added the domain and codomain to the functions, and defined increasing and decreasing on $\mathbb{R}$ rather than just on a proper subinterval $I \subset \mathbb{R}$.

The Injective Function Task is from Burger (2007, p. 47), a textbook for a transition-to-proof course. However, the notation was changed from its original set-theoretic version that was used in my exploratory study, to function notation. I changed this because the undergraduate participants in the exploratory study had a difficult time making sense of the notation, and I wanted to make it more accessible to them. Thus, I changed it to function notation because they should be more likely used to seeing this notation at this stage in their mathematical career.