ON THE DESIGN OF NONLINEAR

GAIN SCHEDULED CONTROL SYSTEMS

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# Chapter 1: Introduction

Today, the design of linear control systems is very advanced. But in the real world, most control problems involve nonlinear plants where a linear control system will not give desired performance. Thus nonlinear control systems are needed. There are several methods used to design nonlinear control systems such as the linearization approach, robust control, adaptive control, decomposition of nonlinear systems, exact linearization via feedback, etc. However, they all have their limitations and may not be suitable for controlling system behavior over a wide operating range.

Gain scheduling, which originated in the design of flight control systems, is a commonly practiced nonlinear design method. The basic idea in gain scheduling is to extend the validity of the linearization approach to a range of constant operating points. When a nonlinear plant can be modeled in such a way that the operating points are parameterized by several scheduling variables, a gain scheduling controller can be designed by linearizing the nonlinear plant at several constant operating points, designing a linear feedback controller at each point, and implementing the resulting family of linear controllers as a single controller whose parameters are changed by monitoring the scheduling variables. Gain scheduling technique has been applied to many applications, especially in flight control problems [2] , [3], [5] - [7], [9], [12] and [15]. The theoretical studies of gain scheduling have mainly focused on two aspects. In [11], the interpolation of linear controllers which led to linear parameter-varying controllers was given. In [8], linear parameter-varying controllers by optimization methods were designed. In [13], the properties of linear parameter-varying systems were analyzed. Additional effort is related to the nonlinear aspect. In [14], 'schedule on a slow variable' for certain classes of nonlinear problems was established. In [10], implementation of gain scheduling static state feedback laws was analyzed.

A recently developed framework for gain scheduled controller design is proposed in [4] and is listed here:

- Compute a nonlinear plant's family of constant operating points, parameterized by a set of scheduling variables. Construct the corresponding family of linearized plants.
- For this family of linearized plants, design a family of linear controllers to meet specific design goals at each constant operating points throughout the operating range.
- Construct a gain scheduled controller that linearizes to the corresponding linear controller at each constant operating point.
- Check nonlocal performance of the resulting nonlinear control system.

The performance of such a nonlinear controller in a region close to any constant operating point is similar to the performance of the corresponding linear controller. In [16], a pitch-axis missile autopilot is designed by applying the above gain scheduling method. In this methodology, the third step is critical. A particular gain scheduled controller is constructed in [4] and the existence conditions of such a nonlinear controller is also addressed.

This thesis describes the application of the above gain scheduling technique to the design of nonlinear feedback controllers for a nonlinear tracking problem and a nonlinear reduced-order observer. The existence conditions for each problem will be addressed. It also will be proven that they can be designed separately. An autopilot for the missile pitch-axis model described in [16] will be designed as an example of utilizing the nonlinear feedback controller and the nonlinear reduced-order observer. This is a reduced-order nonlinear gain scheduling system design as opposed to the full-order nonlinear gain scheduling system design of [15].

The following summarizes the notation and definitions used in this thesis.

- Matrices and sub-matrices will be denoted by capital letters, e.g. $A$ denotes a matrix, while $A_{11}$ denotes the upper left sub-matrix of $A$.
- Vectors will be denoted by lower case letters, e.g. $x$.
- Constant operating points are denoted with superscript $^o$, e.g. $x^o$.
- Vector of scheduling variables is denoted by $\alpha$.
- A matrix that is a function of the scheduling variables $\alpha$ is denoted by, for example, $A(\alpha)$.

- Constant operating points that are a function of the scheduling variables $\alpha$ are denoted with $\alpha$ in parenthesis, e.g. $x^o(\alpha)$.
- The Jacobian matrix is defined as follows: If $f:R^n \to R^m$, then $\frac{\partial f}{\partial x}$ denotes the $m \times n$ Jacobian matrix whose $(i,j)$-entry is the partial derivative $\frac{\partial f_i}{\partial x_j}$.
- Variations about constant operating point are denoted with $\delta$-subscript, e.g. $x_\delta$.

# Chapter 2: Nonlinear Feedback Controller

## 2.1 Introduction

There are many control tasks such as stabilization, tracking, disturbance rejection, etc. which require the use of feedback control. In these control problems, there may be additional goals for the design, like meeting certain requirements on the transient response. The design technique of such feedback controllers is highly advanced for linear systems. Thus it is advantageous to use the design procedures already developed for linear systems to design nonlinear feedback controllers for nonlinear systems. The nonlinear feedback controller must be able to perform well over a wide range and not just near some constant operating points.

## 2.2 Problem Description

Consider an $m$-input, $p$-output, $n$-dimensional nonlinear plant of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{2.1}$$

where $f(\cdot,\cdot)$ and $h(\cdot)$ are smooth functions, $x(t)$ is the state vector, $u(t)$ is the control input vector and $y(t)$ is the measured output available for various purposes. Assume there is a parameterized family of constant operating points defined by smooth functions $[x^o(\alpha), u^o(\alpha), y^o(\alpha)]$, $\alpha \in \Gamma$, where $\Gamma$ is an open set containing the origin in $\Re^l$ and $\alpha$ depends on $u(t)$, $x(t)$ and $y(t)$. That is,

$$\begin{aligned} f(x^o(\alpha), u^o(\alpha)) &= 0 \\ h(x^o(\alpha)) &= y^o(\alpha) \end{aligned} \qquad \alpha \in \Gamma \tag{2.2}$$

Given any $\alpha \in \Gamma$, the corresponding linearization of the nonlinear plant (2.1) can be written as

$$\begin{aligned} \dot{x}_\delta(t) &= A(\alpha)x_\delta(t) + B(\alpha)u_\delta(t) \\ y_\delta(t) &= C(\alpha)x_\delta(t) \end{aligned} \tag{2.3}$$

where the deviation variables are defined by

$$\begin{aligned} x_\delta(t) &= x(t) - x^o(\alpha) \\ u_\delta(t) &= u(t) - u^o(\alpha) \\ y_\delta(t) &= y(t) - y^o(\alpha) \end{aligned} \qquad \alpha \in \Gamma \tag{2.4}$$

The coefficients in (2.3) are given by the partial derivative calculation on (2.1),

$$\begin{aligned} A(\alpha) &= \frac{\partial f}{\partial x}(x^o(\alpha), u^o(\alpha)) \\ B(\alpha) &= \frac{\partial f}{\partial u}(x^o(\alpha), u^o(\alpha)) \\ C(\alpha) &= \frac{\partial h}{\partial x}(x^o(\alpha)) \end{aligned} \qquad \alpha \in \Gamma \tag{2.5}$$

Assume that at each $\alpha \in \Gamma$, the linearization satisfies the following conditions:

1. The pair $[A(\alpha), B(\alpha)]$ of (2.3) is controllable in the sense of linear systems.
2. $A(\alpha)$ is nonsingular.
3. $\begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & 0 \end{bmatrix}$ has full row rank.

Under these assumptions, a nonlinear feedback controller, which forms a servo system together with (2.1) to track input $r(t)$,

$$\begin{aligned} \dot{\xi}(t) &= c(\xi(t), x(t), y(t), r(t)) \\ u(t) &= d(\xi(t), x(t)) \end{aligned} \tag{2.6}$$

will be built such that the corresponding linearization of (2.6) is a linear feedback controller of (2.3) such that the closed-loop system tracks step input $r_\delta(t)$ at any $\alpha \in \Gamma$. It also will be shown that the nonlinear controller (2.6) does not always exist. There are other conditions that must be satisfied.

## 2.3 Parameterized Family of Linear Feedback Controller

In this section, a parameterized family of linear feedback tracking controllers for the linearization family (2.3) will be constructed.

The design goal of the parameterized family of linear feedback controller is to get a family of closed-loop systems with zero steady state error response for step reference inputs and to provide acceptable transient response characteristics.

This design involves designing controller and feedback law that yield a stable closed-loop system capable of tracking step reference inputs with zero error. To obtain zero error for step inputs, an integrator will be added to obtain a Type I system. Thus, the parameterized linear feedback controller for (2.3) is

$$\begin{aligned} \dot{\xi}_\delta(t) &= r_\delta(t) - y_\delta(t) \\ u_\delta(t) &= F(\alpha)x_\delta(t) + K_I(\alpha)\xi_\delta(t) \end{aligned} \tag{2.7}$$

where $m \times n$ smooth function $F(\alpha)$ and $m \times p$ smooth function $K_I(\alpha)$ are to be determined to stabilize the system and satisfy the required transient response specifications at each constant operating point. The block diagram of the parameterized linear closed-loop system is shown in Figure 2.1.

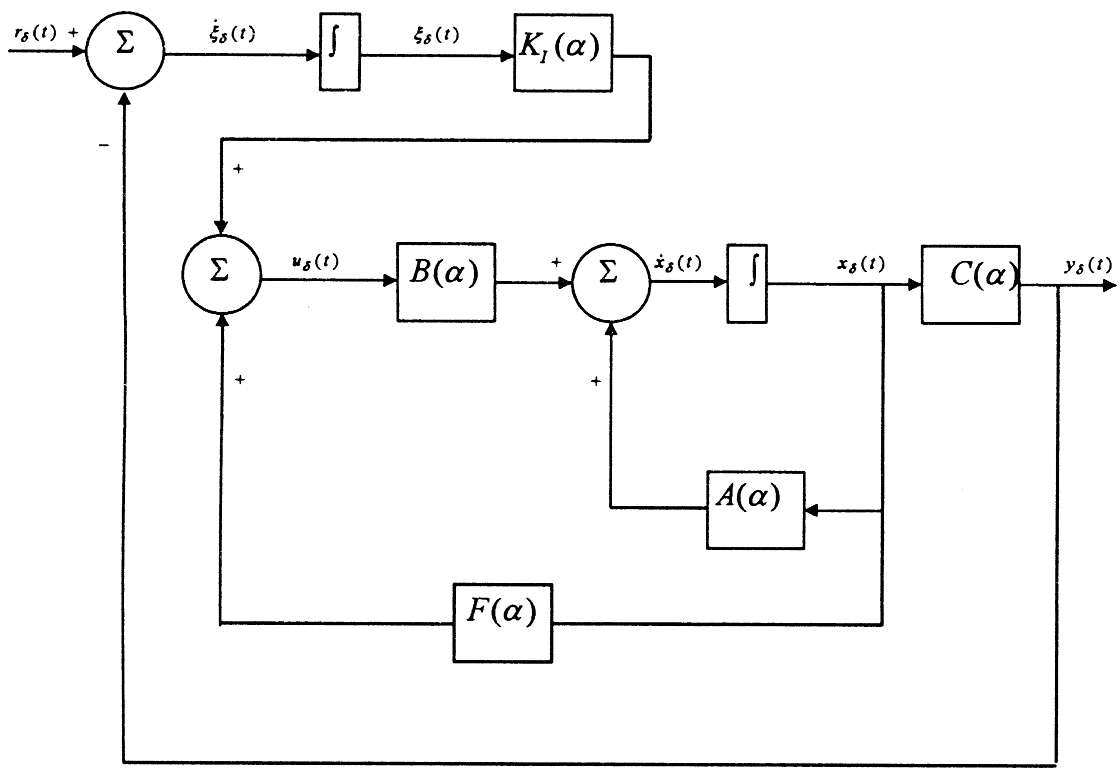


**Figure 2.1** Parameterized Linear Closed-Loop System

The expression of the parameterized linear closed-loop system is

$$\begin{aligned}
\begin{bmatrix} \dot{x}_\delta(t) \\ \dot{\xi}_\delta(t) \end{bmatrix} &= \begin{bmatrix} A(\alpha) & 0 \\ -C(\alpha) & 0 \end{bmatrix} \begin{bmatrix} x_\delta(t) \\ \xi_\delta(t) \end{bmatrix} + \begin{bmatrix} B(\alpha) \\ 0 \end{bmatrix} u_\delta(t) + \begin{bmatrix} 0 \\ I_p \end{bmatrix} r_\delta(t) \\
u_\delta(t) &= \begin{bmatrix} F(\alpha) & K_I(\alpha) \end{bmatrix} \begin{bmatrix} x_\delta(t) \\ \xi_\delta(t) \end{bmatrix} \\
y_\delta(t) &= \begin{bmatrix} C(\alpha) & 0 \end{bmatrix} \begin{bmatrix} x_\delta(t) \\ \xi_\delta(t) \end{bmatrix}
\end{aligned} \tag{2.8}$$

That is

$$\begin{aligned}
\begin{bmatrix} \dot{x}_\delta(t) \\ \dot{\xi}_\delta(t) \end{bmatrix} &= \begin{bmatrix} A(\alpha)+B(\alpha)F(\alpha) & B(\alpha)K_I(\alpha) \\ -C(\alpha) & 0 \end{bmatrix} \begin{bmatrix} x_\delta(t) \\ \xi_\delta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I_p \end{bmatrix} r_\delta(t) \\
y_\delta(t) &= \begin{bmatrix} C(\alpha) & 0 \end{bmatrix} \begin{bmatrix} x_\delta(t) \\ \xi_\delta(t) \end{bmatrix}
\end{aligned} \tag{2.9}$$

where $I_p$ is a $p \times p$ identity matrix.

Assumption 1 and 2 implies that $\left( \begin{bmatrix} A(\alpha) & 0 \\ -C(\alpha) & 0 \end{bmatrix} \begin{bmatrix} B(\alpha) \\ 0 \end{bmatrix} \right)$ is controllable at any $\alpha \in \Gamma$. Thus $[F(\alpha), K_I(\alpha)]$ can be chosen to arbitrarily locate the $n+p$ eigenvalues of

$$\begin{aligned}
&\begin{bmatrix} A(\alpha) & 0 \\ -C(\alpha) & 0 \end{bmatrix} + \begin{bmatrix} B(\alpha) \\ 0 \end{bmatrix} \begin{bmatrix} F(\alpha) & K_I(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} A(\alpha)+B(\alpha)F(\alpha) & B(\alpha)K_I(\alpha) \\ -C(\alpha) & 0 \end{bmatrix}
\end{aligned} \tag{2.10}$$

which governs the stability and dynamic characteristics of the parameterized linear closed-loop system (2.9). To get a stable system, the eigenvalues must be placed in the open left half of the $s$-plane. The position of these eigenvalues in the $s$-plane determines the transient response of (2.9) at each $\alpha \in \Gamma$.

## 2.4 Nonlinear Feedback Controller

Now a nonlinear feedback controller of the form (2.6), which has a parameterized family of linearization (2.7), will be constructed.

For such a nonlinear feedback controller (2.6) to exist, there must be a smooth function $\xi^o(\alpha)$, $\alpha \in \Gamma$, such that at any constant operating point the following must be satisfied

$$\begin{aligned} &c(\xi^o(\alpha), x^o(\alpha), y^o(\alpha), r^o(\alpha)) = 0 \\ &d(\xi^o(\alpha), x^o(\alpha)) = u^o(\alpha) \end{aligned} \qquad \alpha \in \Gamma \tag{2.11}$$

Furthermore, the controller (2.6) should linearize to (2.7) at each constant operating point, while allowing for variations in all variables, including variations in the parameter variables $\alpha$ due to variations in the variables $u(t)$, $x(t)$ and $y(t)$. That is,

$$\begin{aligned} &\frac{\partial c}{\partial \xi}(\xi^o(\alpha), x^o(\alpha), y^o(\alpha), r^o(\alpha)) = 0 \\ &\frac{\partial c}{\partial x}(\xi^o(\alpha), x^o(\alpha), y^o(\alpha), r^o(\alpha)) = 0 \\ &\frac{\partial c}{\partial y}(\xi^o(\alpha), x^o(\alpha), y^o(\alpha), r^o(\alpha)) = -I_p \\ &\frac{\partial c}{\partial r}(\xi^o(\alpha), x^o(\alpha), y^o(\alpha), r^o(\alpha)) = I_p \\ &\frac{\partial d}{\partial \xi}(\xi^o(\alpha), x^o(\alpha)) = K_I(\alpha) \\ &\frac{\partial d}{\partial x}(\xi^o(\alpha), x^o(\alpha)) = F(\alpha) \end{aligned} \qquad \alpha \in \Gamma \tag{2.12}$$

Construct a nonlinear feedback controller of the form (2.6) by

$$\begin{aligned} c(\xi(t), x(t), y(t), r(t)) &= [r(t) - r^o(\alpha)] - [y(t) - y^o(t)] \\ d(\xi(t), x(t)) &= K_I(\alpha)[\xi(t) - \xi^o(\alpha)] + F(\alpha)[x(t) - x^o(\alpha)] + u^o(\alpha) \end{aligned} \tag{2.13}$$

It is obvious that this controller satisfies requirements (2.11). This form of the nonlinear feedback controller is not unique, though it is quite reasonable in that the coefficients of the parameterized family of linear feedback controllers appear directly, and the deviation variables are included. In general (2.13) is nonlinear in $u(t)$, $x(t)$ and $y(t)$, since $\alpha$ depends on these variables. Note that the constant operating point function $\xi^o(\alpha)$, $\alpha \in \Gamma$ is not specified, the selection of which plays an important role.

Linearization of (2.13) according to the parameterized family of constant operating points gives

$$\begin{aligned} \dot{\xi}_\delta(t) &= r_\delta(t) - y_\delta(t) \\ &\quad - [\frac{\partial r^o(\alpha)}{\partial \alpha} - \frac{\partial y^o(\alpha)}{\partial \alpha}] \\ &\quad \times [\frac{\partial \alpha}{\partial y} y_\delta(t) + \frac{\partial \alpha}{\partial u} u_\delta(t) + \frac{\partial \alpha}{\partial x} x_\delta(t)] \\ u_\delta(t) &= K_I(\alpha)\xi_\delta(t) + F(\alpha)x_\delta(t) \\ &\quad - [K_I(\alpha)\frac{\partial \xi^o(\alpha)}{\partial \alpha} + F(\alpha)\frac{\partial x^o(\alpha)}{\partial \alpha} - \frac{\partial u^o(\alpha)}{\partial \alpha}] \\ &\quad \times [\frac{\partial \alpha}{\partial y} y_\delta(t) + \frac{\partial \alpha}{\partial u} u_\delta(t) + \frac{\partial \alpha}{\partial x} x_\delta(t)] \end{aligned} \tag{2.14}$$

Comparing (2.14) with (2.7) shows that there are extra hidden coupling terms in the parameterized linearization (2.14) which are not accounted for in the design of parameterized linear feedback controller (2.7). If the square-bracketed coefficients in (2.14) vanish, that is,

$$\frac{\partial r^o(\alpha)}{\partial \alpha} = \frac{\partial y^o(\alpha)}{\partial \alpha} \tag{2.15}$$

$$K_I(\alpha)\frac{\partial \xi^o(\alpha)}{\partial \alpha} + F(\alpha)\frac{\partial x^o(\alpha)}{\partial \alpha} - \frac{\partial u^o(\alpha)}{\partial \alpha} = 0 \tag{2.16}$$

then these extra terms will be gone and (2.13) will be a desired nonlinear feedback controller.

The requirement (2.15) can be satisfied by choosing

$$r^o(\alpha) = y^o(\alpha) \tag{2.17}$$

**2.5 Revised Nonlinear Feedback Controller**

Though there are instances where the configuration of the linear feedback controller family is such that the solution to (2.16) is apparent, generally, it is very hard to find a solution to (2.16). In fact, in most cases a solution to (2.16) does not exist. Thus controller (2.13) will not be the desired nonlinear feedback controller because its linearization family will exhibit additional terms of the type shown in square brackets in (2.14) that are not accounted for in the linear design process.

To solve this problem, an alternate feedback controller will be constructed based on the above design.

### 2.5.1 Revised Parameterized Family of Linear Feedback Controller

The block diagram of the parameterized linear feedback controller (2.7) is shown in Figure 2.2.

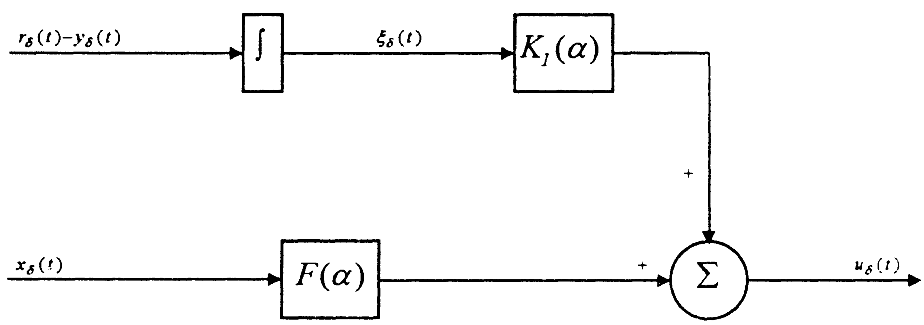


**Figure 2.2** Block Diagram of Parameterized Linear Controller (2.7)

This block diagram can be redrawn as shown in Figure 2.3 using the rules for linear block diagram manipulation and the relationship $x_\delta(t) = \int \dot{x}_\delta(t)dt$.

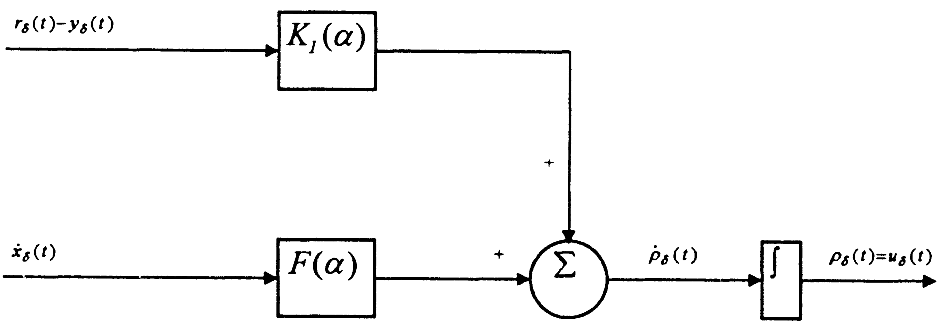


**Figure 2.3** Equivalent Block Diagram of Figure 2.2

The expression of the block diagram in Figure 2.3 is

$$\begin{aligned} \dot{\rho}_\delta(t) &= F(\alpha)\dot{x}_\delta(t) + K_I(\alpha)[r_\delta(t) - y_\delta(t)] \\ u_\delta(t) &= \rho_\delta(t) \end{aligned} \tag{2.18}$$

Substituting (2.3) into (2.18) yields

$$\begin{aligned} \dot{\rho}_\delta(t) &= F(\alpha)B(\alpha)\rho_\delta(t) + F(\alpha)A(\alpha)x_\delta(t) + K_I(\alpha)[r_\delta(t) - y_\delta(t)] \\ u_\delta(t) &= \rho_\delta(t) \end{aligned} \tag{2.19}$$

Thus (2.18) is a parameterized linear feedback controller which is equivalent to (2.7) from a linear system's viewpoint.

### 2.5.2 Revised Nonlinear Feedback Controller

Rewrite (2.19) with $\xi_\delta(t)$ replacing $\rho_\delta(t)$

$$\begin{aligned}\dot{\xi}_\delta(t) &= F(\alpha)B(\alpha)\xi_\delta(t) + F(\alpha)A(\alpha)x_\delta(t) + K_I(\alpha)[r_\delta(t) - y_\delta(t)] \\ u_\delta(t) &= \xi_\delta(t)\end{aligned} \tag{2.20}$$

Now the existence of a nonlinear feedback controller of the form (2.6), which has a parameterized family of linearization (2.20) will be investigated.

For such nonlinear feedback controller (2.6) to exist, there must be a smooth function $\xi^o(\alpha)$, $\alpha \in \Gamma$, such that at any constant operating point (2.11) holds. Furthermore, the controller (2.6) should linearize to (2.20) at each constant operating point, while allowing for variations in all variables, including variations in the parameter variables $\alpha$ due to variations in the variables $u(t)$, $x(t)$ and $y(t)$. That is,

$$\left.\begin{aligned}
\frac{\partial c}{\partial \xi}(\xi^o(\alpha), x^o(\alpha), y^o(\alpha), r^o(\alpha)) &= F(\alpha)B(\alpha) \\
\frac{\partial c}{\partial x}(\xi^o(\alpha), x^o(\alpha), y^o(\alpha), r^o(\alpha)) &= F(\alpha)A(\alpha) \\
\frac{\partial c}{\partial y}(\xi^o(\alpha), x^o(\alpha), y^o(\alpha), r^o(\alpha)) &= -K_I \\
\frac{\partial c}{\partial r}(\xi^o(\alpha), x^o(\alpha), y^o(\alpha), r^o(\alpha)) &= K_I \\
\frac{\partial d}{\partial \xi}(\xi^o(\alpha), x^o(\alpha)) &= I_m \\
\frac{\partial d}{\partial x}(\xi^o(\alpha), x^o(\alpha)) &= 0
\end{aligned}\right. \qquad \alpha \in \Gamma \tag{2.21}$$

From the second equation of (2.20), it is reasonable to choose

$$\xi^o(\alpha) = u^o(\alpha) \tag{2.22}$$

Consider a nonlinear feedback controller of the form (2.6) given by

$$\begin{aligned} c(\xi(t),x(t),y(t),r(t)) &= F(\alpha)B(\alpha)[\xi(t)-\xi^o(\alpha)] \\ &\quad + F(\alpha)A(\alpha)[x(t)-x^o(t)] \\ &\quad + K_I(\alpha)[r(t)-y(t)] \\ d(\xi(t),x(t)) &= \xi(t) \end{aligned} \tag{2.23}$$

It is obvious that this nonlinear feedback controller satisfies requirements (2.11). There is still one more requirement that (2.23) must be met. The linearization family of (2.23) according to the parameterized family of constant operating points with (2.17)

$$\begin{aligned} \dot{\xi}_\delta(t) &= F(\alpha)B(\alpha)\xi_\delta(t) + F(\alpha)A(\alpha)x_\delta(t) + K_I(\alpha)[r_\delta(t)-y_\delta(t)] \\ &\quad -[F(\alpha)B(\alpha)\frac{\partial\xi^o(\alpha)}{\partial\alpha} + F(\alpha)A(\alpha)\frac{\partial x^o(\alpha)}{\partial\alpha}] \\ &\quad \times[\frac{\partial\alpha}{\partial u}u_\delta(t) + \frac{\partial\alpha}{\partial y}y_\delta(t) + \frac{\partial\alpha}{\partial x}x_\delta(t)] \\ d(\xi(t),x(t)) &= \xi_\delta(t) \end{aligned} \tag{2.24}$$

must be the same as (2.20), which means that the square-bracketed coefficients in (2.24) must vanish. But from (2.2) and (2.5) it is known that

$$A(\alpha)\frac{\partial x^o(\alpha)}{\partial\alpha} + B(\alpha)\frac{\partial u^o(\alpha)}{\partial\alpha} = 0 \tag{2.25}$$

Along with (2.22), it yields

$$F(\alpha)A(\alpha)\frac{\partial x^o(\alpha)}{\partial\alpha} + F(\alpha)B(\alpha)\frac{\partial\xi^o(\alpha)}{\partial\alpha} = 0 \tag{2.26}$$

Thus, (2.23) with (2.17) and (2.22) is a desired nonlinear feedback controller.

# Chapter 3: Nonlinear Reduced-Order Observer

## 3.1 Introduction

The state variables are often required to be available for various purposes such as feedback control. But in most cases, it is not possible to directly measure all the state variables. So an observer, which is a dynamic system whose state variables approach those of the actual system, is constructed. A full order observer has the same order as the actual system no matter how many observed outputs the actual system produces. From the fact that if all the state variables can be measured, then a dynamic observer is not needed, it is reasonable and possible to construct an observer with a lower order depending on the outputs of the actual system. Such a device is called a reduced-order observer. For a nonlinear system, the observer must be able to estimate the states of actual system over a wide range, not just near some constant operating points.

## 3.2 Problem Description

Consider a nonlinear plant described in Section 2.2. Assume that for each $\alpha \in \Gamma$, the linearization (2.3) satisfies

$$rank C(\alpha) = p \text{ and } p < n \tag{3.1}$$

$$rank \frac{\partial x^o(\alpha)}{\partial \alpha} = rank \frac{\partial y^o(\alpha)}{\partial \alpha} = l \tag{3.2}$$

and further, the pair $[C(\alpha), A(\alpha)]$ of (2.3) is observable in the sense of linear systems at each $\alpha \in \Gamma$.

Under these assumptions, a nonlinear observer of (3.1) with dynamic order $n-p$

$$\begin{aligned} \dot{z}(t) &= a(z(t), y(t), u(t)) \\ \hat{x}(t) &= b(z(t), y(t)) \end{aligned} \tag{3.3}$$

will be constructed such that (3.3) has a parameterized family of constant operating points satisfying $\hat{x}^o(\alpha) = x^o(\alpha)$ and the corresponding linearization of (3.3) is a linear reduced-order observer of the linearization (2.3) at any $\alpha \in \Gamma$. It also will be shown that the nonlinear observer (3.3) does not always exist, there are other conditions that must be satisfied.

### 3.3 Parameterized Family of Linear Reduced-Order Observer

In this section, a parameterized family of linear reduced-order observers for the linearization family (2.3) will be constructed. The development here closely follows the constructions derived in [1] for linear time-invariant systems.

From linear theory, a full-order parameterized observer for (2.3) is

$$\dot{\hat{x}}_\delta(t) = A(\alpha)\hat{x}_\delta(t) + B(\alpha)u_\delta(t) + L(\alpha)[y_\delta(t) - C(\alpha)\hat{x}_\delta(t)] \tag{3.4}$$

where $L(\alpha)$ is a gain matrix to be selected to place the eigenvalues of $\hat{A}(\alpha) = A(\alpha) - L(\alpha)C(\alpha)$.

First, suppose that the state variables of (2.3) can be grouped into two sets: those that can be measured directly and those that depend indirectly on the former. That is:

$$x_\delta(t) = \begin{bmatrix} x_{\delta 1}(t) \\ x_{\delta 2}(t) \end{bmatrix}$$

Then (2.3) can be rewritten as following:

$$\begin{aligned} \dot{x}_{\delta 1}(t) &= A_{11}(\alpha)x_{\delta 1}(t) + A_{12}(\alpha)x_{\delta 2}(t) + B_1(\alpha)u_\delta(t) \\ \dot{x}_{\delta 2}(t) &= A_{21}(\alpha)x_{\delta 1}(t) + A_{22}(\alpha)x_{\delta 2}(t) + B_2(\alpha)u_\delta(t) \\ y_\delta(t) &= C_1(\alpha)x_{\delta 1}(t) \end{aligned} \tag{3.5}$$

where $C_1(\alpha)$ is a nonsingular matrix at each $\alpha \in \Gamma$. Also (3.4) can be rewritten as following:

$$\begin{aligned}\dot{\hat{x}}_{\delta 1}(t) &= A_{11}(\alpha)\hat{x}_{\delta 1}(t) + A_{12}(\alpha)\hat{x}_{\delta 2}(t) + B_1(\alpha)u_\delta(t) \\ &\quad + L_1(\alpha)(y_\delta(t) - C_1(\alpha)\hat{x}_{\delta 1}(t)) \\ \dot{\hat{x}}_{\delta 2}(t) &= A_{21}(\alpha)\hat{x}_{\delta 1}(t) + A_{22}(\alpha)\hat{x}_{\delta 2}(t) + B_2(\alpha)u_\delta(t) \\ &\quad + L_2(\alpha)(y_\delta(t) - C_1(\alpha)\hat{x}_{\delta 1}(t))\end{aligned} \tag{3.6}$$

But since $x_{\delta 1}(t)$ can be solved directly from (3.5), there is no need to implement the observer equation for $\hat{x}_{\delta 1}(t)$ and

$$\hat{x}_{\delta 1}(t) = x_{\delta 1}(t) = C_1^{-1}(\alpha)y_\delta(t) \tag{3.7}$$

Hence the parameterized observer (3.6) becomes

$$\dot{\hat{x}}_{\delta 2}(t) = A_{21}(\alpha)C_1^{-1}(\alpha)y_\delta(t) + A_{22}(\alpha)\hat{x}_{\delta 2}(t) + B_2(\alpha)u_\delta(t) \tag{3.8}$$

which is a dynamic system of the same order as the number of state variables that cannot be measured directly. The eigenvalues of $A_{22}(\alpha)$ govern the dynamic behavior of the parameterized linear reduced-order observer (3.8). Since the eigenvalues of $A_{22}(\alpha)$ are not assured to be suitable for (3.8) to be a satisfactory observer, a more suitable general parameterized linear reduced-observer for (2.3) is built in the following way:

$$\begin{aligned}\hat{x}_{\delta 1}(t) &= x_{\delta 1}(t) = C_1^{-1}(\alpha)y_\delta(t) \\ \hat{x}_{\delta 2}(t) &= L(\alpha)y_\delta(t) + z_\delta(t) \\ \dot{z}_\delta(t) &= G(\alpha)z_\delta(t) + K(\alpha)y_\delta(t) + H(\alpha)u_\delta(t)\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}G(\alpha) &= A_{22}(\alpha) - L(\alpha)C_1(\alpha)A_{12}(\alpha) \\ K(\alpha) &= [A_{21}(\alpha) - L(\alpha)C_1(\alpha)A_{11}(\alpha)]C_1^{-1}(\alpha) + G(\alpha)L(\alpha) \\ H(\alpha) &= B_2(\alpha) - L(\alpha)C_1(\alpha)B_1(\alpha)\end{aligned} \tag{3.10}$$

and the matrix $L(\alpha)$ is to be selected to place the eigenvalues of $G(\alpha)$. The eigenvalues of $G(\alpha)$ can be placed anywhere because pair $[C_1(\alpha)A_{12}(\alpha), A_{22}(\alpha)]$ is observable if the pair $[C(\alpha), A(\alpha)]$ is observable, a fact that has been proven by Luenberger [1].

To investigate the error between the outputs of this parameterized linear reduced-order observer and the real state variables, define the estimation error

$$\tilde{x}_\delta(t) = x_\delta(t) - \hat{x}_\delta(t) = \begin{bmatrix} x_{\delta 1}(t) - \hat{x}_{\delta 1}(t) \\ x_{\delta 2}(t) - \hat{x}_{\delta 2}(t) \end{bmatrix} = \begin{bmatrix} \tilde{x}_{\delta 1}(t) \\ \tilde{x}_{\delta 2}(t) \end{bmatrix} \tag{3.11}$$

From (3.9)

$$\tilde{x}_{\delta 1}(t) = x_{\delta 1}(t) - \hat{x}_{\delta 1}(t) = 0 \tag{3.12}$$

Thus only $\tilde{x}_{\delta 2}(t)$ need be considered. From (3.11)

$$\dot{\tilde{x}}_{\delta 2}(t) = \dot{x}_{\delta 2}(t) - \dot{\hat{x}}_{\delta 2}(t) \tag{3.13}$$

Substituting (3.5) and (3.9) into (3.13) yields

$$\dot{\tilde{x}}_{\delta 2}(t) = A_{21}(\alpha)x_{\delta 1}(t) + A_{22}(\alpha)x_{\delta 2}(t) + B_2(\alpha)u_\delta(t) - L(\alpha)\dot{y}_\delta(t) - \dot{z}_\delta(t) \tag{3.14}$$

But, from (3.5) and (3.9)

$$\dot{y}_\delta(t) = C_1(\alpha)\dot{x}_{\delta 1}(t) = C_1(\alpha)(A_{11}(\alpha)x_{\delta 1}(t) + A_{12}(\alpha)x_{\delta 2}(t) + B_1(\alpha)u_\delta(t)) \tag{3.15}$$

And from (3.9) and (3.11)

$$\begin{aligned} \dot{z}_\delta(t) &= G(\alpha)z_\delta(t) + K(\alpha)y_\delta(t) + H(\alpha)u_\delta(t) \\ &= G(\alpha)[\hat{x}_{\delta 2}(t) - L(\alpha)y_\delta(t)] + K(\alpha)y_\delta(t) + H(\alpha)u_\delta(t) \\ &= G(\alpha)[x_{\delta 2}(t) - \tilde{x}_{\delta 2}(t)] - [G(\alpha)L(\alpha) - K(\alpha)]C_1(\alpha)x_{\delta 1}(t) \\ &\quad + H(\alpha)u_\delta(t) \end{aligned} \tag{3.16}$$

Hence (3.14) becomes

$$\begin{aligned}\dot{\tilde{x}}_{\delta 2}(t) &= G(\alpha)\tilde{x}_{\delta 2}(t) \\ &\quad +[A_{21}(\alpha)-L(\alpha)C_1(\alpha)A_{11}(\alpha)+G(\alpha)L(\alpha)C_1(\alpha) \\ &\qquad -K(\alpha)C_1(\alpha)]x_{\delta 1}(t) \\ &\quad +[A_{22}(\alpha)-L(\alpha)C_1(\alpha)A_{21}(\alpha)-G(\alpha)]x_{\delta 2}(t) \\ &\quad +[B_2(\alpha)-L(\alpha)C_1(\alpha)B_1(\alpha)-H(\alpha)]u_\delta(t)\end{aligned} \tag{3.17}$$

With the definition of $G(\alpha)$, $K(\alpha)$ and $H(\alpha)$ in (3.10), (3.17) can be simplified as

$$\dot{\tilde{x}}_{\delta 2}(t) = G(\alpha)\tilde{x}_{\delta 2}(t) \tag{3.18}$$

The dynamic error response (3.18) is governed by the eigenvalues of $G(\alpha)$ which are determined by the choice of $L(\alpha)$. In particular, if $L(\alpha)$ is chosen to yield the eigenvalues of $G(\alpha)$ lying in the open left half of the $s$-plane, then $\tilde{x}_{\delta 2}(t) \to 0$ as $t \to 0$ which implies that the $\hat{x}_{\delta 2}(t)$ will exponentially approach $x_{\delta 2}(t)$. Moreover, the further into the left-half of the $s$-plane, the observer poles are placed, the faster $\hat{x}_{\delta 2}(t)$ will approach $x_{\delta 2}(t)$.

The parameterized linear reduced-order observer (3.9) is for the case in which the state variables of (2.3) can be divided in a way that allows $x_{\delta 1}(t)$ to be solved. When this is not true, a more general parameterized linear reduced-order observer is needed.

Assumption (3.1) implies that there is a smooth full column rank $n \times (n-p)$ matrix function $Q(\alpha)$ that is a basis for the null space of $C(\alpha)$ at each $\alpha \in \Gamma$. From the second equation of (2.2) and (2.5), it follows that

$$\frac{\partial y^o(\alpha)}{\partial \alpha} = \frac{\partial h(x^o(\alpha))}{\partial x}\frac{\partial x^o(\alpha)}{\partial \alpha} = C(\alpha)\frac{\partial x^o(\alpha)}{\partial \alpha} \tag{3.19}$$

The rank assumption in (3.2) gives

$$rank\begin{bmatrix} Q(\alpha) & \dfrac{\partial x^o(\alpha)}{\partial \alpha}\end{bmatrix} = n-p+l \qquad \alpha \in \Gamma \tag{3.20}$$

which implies that there exists a smooth $(n-p)\times n$ matrix function $M(\alpha)$ with full row rank at each $\alpha \in \Gamma$ satisfying

$$M(\alpha)\begin{bmatrix} Q(\alpha) & \dfrac{\partial x^o(\alpha)}{\partial \alpha}\end{bmatrix} = \begin{bmatrix} I_{n-p} & R(\alpha)\end{bmatrix} \tag{3.21}$$

where $I_{n-p}$ is an $(n-p)\times(n-p)$ identity matrix and $R(\alpha)$ is an arbitrary smooth $(n-p)\times l$ matrix function. The existence of such an $M(\alpha)$ can be seen from the fact that the null space of $\begin{bmatrix} Q(\alpha) & \dfrac{\partial x^o(\alpha)}{\partial \alpha}\end{bmatrix}$ is contained in the null space of $\begin{bmatrix} I_{n-p} & R(\alpha)\end{bmatrix}$ at each $\alpha \in \Gamma$ because $\begin{bmatrix} Q(\alpha) & \dfrac{\partial x^o(\alpha)}{\partial \alpha}\end{bmatrix}$ has full column rank.

Define a parameterized linear transformation matrix as follows

$$T(\alpha) = \begin{bmatrix} C(\alpha) \\ M(\alpha)\end{bmatrix} \tag{3.22}$$

The choice of $M(\alpha)$ from (3.21) guarantees that $T(\alpha)$ is invertible at each $\alpha \in \Gamma$. Denote its inverse by

$$T^{-1}(\alpha) = \begin{bmatrix} P(\alpha) & Q(\alpha) \end{bmatrix} \tag{3.23}$$

where $P(\alpha)$ is $n \times p$ and $Q(\alpha)$ is $n \times (n-p)$. Note that $Q(\alpha)$ defined here is the same as that in (3.21). By construction, the following relations must hold among $C(\alpha)$, $M(\alpha)$, $P(\alpha)$ and $Q(\alpha)$

$$\begin{bmatrix} C(\alpha) \\ M(\alpha) \end{bmatrix} \begin{bmatrix} P(\alpha) & Q(\alpha) \end{bmatrix} = \begin{bmatrix} C(\alpha)P(\alpha) & C(\alpha)Q(\alpha) \\ M(\alpha)P(\alpha) & M(\alpha)Q(\alpha) \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_{n-p} \end{bmatrix} \tag{3.24}$$

and

$$\begin{bmatrix} P(\alpha) & Q(\alpha) \end{bmatrix} \begin{bmatrix} C(\alpha) \\ M(\alpha) \end{bmatrix} = P(\alpha)C(\alpha) + Q(\alpha)M(\alpha) = I_n \tag{3.25}$$

where $I_k$ is a $k \times k$ identity matrix.

Define a new state vector $w_\delta(t)$ by

$$w_\delta(t) = T(\alpha)x_\delta(t) \tag{3.26}$$

and divide it into two parts

$$w_\delta(t) = \begin{bmatrix} w_{\delta 1}(t) \\ w_{\delta 2}(t) \end{bmatrix} \tag{3.27}$$

where $w_{\delta 1}(t)$ is $p \times n$ and $w_{\delta 2}(t)$ is $(n-p) \times n$. From (3.22), (3.26) and (3.27), it follows

$$\begin{aligned} w_{\delta 1}(t) &= C(\alpha)x_\delta(t) = y_\delta(t) \\ w_{\delta 2}(t) &= M(\alpha)x_\delta(t) \end{aligned} \tag{3.28}$$

Applying the parameterized linear transformation (3.26) to (2.3) yields

$$\dot{w}_\delta(t) = T(\alpha)A(\alpha)T^{-1}(\alpha)w_\delta(t) + T(\alpha)u_\delta(t)$$
$$y_\delta(t) = C(\alpha)T^{-1}(\alpha)w_\delta(t) \tag{3.29}$$

Rewrite (3.29) with (3.22), (3.23), (3.27) and (3.28):

$$\begin{aligned}
\dot{w}_{\delta 1}(t) &= \overline{A}_{11}(\alpha)w_{\delta 1}(t) + \overline{A}_{12}(\alpha)w_{\delta 2}(t) + \overline{B}_1(\alpha)u_\delta(t) \\
\dot{w}_{\delta 2}(t) &= \overline{A}_{21}(\alpha)w_{\delta 1}(t) + \overline{A}_{22}(\alpha)w_{\delta 2}(t) + \overline{B}_2(\alpha)u_\delta(t) \\
y_\delta(t) &= \overline{C}_1(\alpha)w_{\delta 1}(t) = w_{\delta 1}(t)
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
\overline{A}(\alpha) &= \begin{bmatrix} \overline{A}_{11}(\alpha) & \overline{A}_{12}(\alpha) \\ \overline{A}_{21}(\alpha) & \overline{A}_{22}(\alpha) \end{bmatrix} = \begin{bmatrix} C(\alpha)A(\alpha)P(\alpha) & C(\alpha)A(\alpha)Q\alpha) \\ M(\alpha)A(\alpha)P(\alpha) & M(\alpha)A(\alpha)Q(\alpha) \end{bmatrix} \\
\overline{B}(\alpha) &= \begin{bmatrix} \overline{B}_1(\alpha) \\ \overline{B}_2(\alpha) \end{bmatrix} = \begin{bmatrix} C(\alpha)B(\alpha) \\ M(\alpha)B(\alpha) \end{bmatrix} \\
\overline{C}(\alpha) &= \begin{bmatrix} \overline{C}_1(\alpha) & \overline{C}_2(\alpha) \end{bmatrix} = \begin{bmatrix} C(\alpha)P(\alpha) & C(\alpha)Q(\alpha) \end{bmatrix} = \begin{bmatrix} I_p & 0 \end{bmatrix}
\end{aligned} \tag{3.31}$$

The reason for the choice of the parameterized linear transformation (3.26) is now very clear: the output $y_\delta(t)$ is a direct measurement of $w_{\delta 1}(t)$, which allows the previous parameterized linear reduced-order observer derived for (3.9) to be used here for (3.30). After obtaining the estimates $\hat{w}_{\delta 1}(t)$ and $\hat{w}_{\delta 2}(t)$, (3.23), (3.26) and (3.27) can be used to get $\hat{x}_\delta(t)$, that is,

$$\begin{aligned}
\hat{x}_\delta(t) &= T^{-1}(\alpha)\hat{w}_\delta(t) = \begin{bmatrix} P(\alpha) & Q(\alpha) \end{bmatrix} \begin{bmatrix} \hat{w}_{\delta 1}(t) \\ \hat{w}_{\delta 2}(t) \end{bmatrix} \\
&= P(\alpha)\hat{w}_{\delta 1}(t) + Q(\alpha)\hat{w}_{\delta 2}(t)
\end{aligned} \tag{3.32}$$

According to (3.9), the parameterized linear reduced-order observer for (3.30) is as follows

$$\begin{aligned}
\hat{w}_{\delta 1}(t) &= y_\delta(t) \\
\hat{w}_{\delta 2}(t) &= L(\alpha)y_\delta(t) + z_\delta(t) \\
\dot{z}_\delta(t) &= G(\alpha)z_\delta(t) + K(\alpha)y_\delta(t) + H(\alpha)u_\delta(t)
\end{aligned} \tag{3.33}$$

where

$$\begin{aligned}
G(\alpha) &= \overline{A}_{22}(\alpha) - L(\alpha)\overline{A}_{12}(\alpha) \\
K(\alpha) &= \overline{A}_{21}(\alpha) - L(\alpha)\overline{A}_{11}(\alpha) + G(\alpha)L(\alpha) \\
H(\alpha) &= \overline{B}_2(\alpha) - L(\alpha)\overline{B}_1(\alpha)
\end{aligned} \tag{3.34}$$

Note that observability of the pair $[C(\alpha), A(\alpha)]$ is equivalent to the observability of the pair $[\overline{C}(\alpha), \overline{A}(\alpha)]$ at each $\alpha \in \Gamma$, which implies that the pair $[\overline{A}_{12}(\alpha), \overline{A}_{22}(\alpha)]$ is observable at each $\alpha \in \Gamma$. Thus, $L(\alpha)$ can be chosen to place the eigenvalues of $G(\alpha)$ so that (3.33) can have desirable dynamic behavior.

Finally, combine (3.32) with (3.33) to yield

$$\hat{x}_\delta(t) = Q(\alpha)z_\delta(t) + (P(\alpha) + Q(\alpha)L(\alpha))y_\delta(t) \tag{3.35}$$

Hence, (3.34) and (3.35) describe a parameterized family of linear reduced-order observers for (2.3).

### 3.4 Nonlinear Reduced-Order Observer

Now a nonlinear reduced-order observer of the form (3.3), which has a parameterized family of linearization (3.34) and (3.35), will be constructed.

For such nonlinear reduced-order observer (3.3) to exist, there must be smooth functions $z^o(\alpha)$, $\alpha \in \Gamma$, where $\Gamma$ is the same open set as before, such that

$$\begin{aligned} &a(z^o(\alpha), y^o(\alpha), u^o(\alpha)) = 0 \\ &b(z^o(\alpha), y^o(\alpha)) = x^o(\alpha) \end{aligned} \qquad \alpha \in \Gamma \tag{3.36}$$

Furthermore, the observer (3.3) should linearize to (3.34) and (3.35) at each constant operating point, while allowing for variations in all variables, including variations in the scheduling variables $\alpha$ due to variations in the variables $u(t)$, $x(t)$ and $y(t)$. Thus, it is required that

$$\begin{aligned} &\frac{\partial a}{\partial z}(z^o(\alpha), y^o(\alpha), u^o(\alpha)) = G(\alpha) \\ &\frac{\partial a}{\partial y}(z^o(\alpha), y^o(\alpha), u^o(\alpha)) = K(\alpha) \\ &\frac{\partial a}{\partial u}(z^o(\alpha), y^o(\alpha), u^o(\alpha)) = H(\alpha) \\ &\frac{\partial b}{\partial z}(z^o(\alpha), y^o(\alpha)) = Q(\alpha) \\ &\frac{\partial b}{\partial y}(z^o(\alpha), y^o(\alpha)) = P(\alpha) + Q(\alpha)L(\alpha) \end{aligned} \qquad \alpha \in \Gamma \tag{3.37}$$

Consider a nonlinear reduced-order observer of the form (3.3) by

$$\begin{aligned} a(z(t), y(t), u(t)) &= G(\alpha)[z(t) - z^o(\alpha)] + K(\alpha)[y(t) - y^o(\alpha)] \\ &\quad + H(\alpha)[u(t) - u^o(t)] \\ b(z(t), y(t)) &= Q(\alpha)[z(t) - z^o(\alpha)] \\ &\quad + (P(\alpha) + Q(\alpha)L(\alpha))[y(t) - y^o(\alpha)] + x^o(\alpha) \end{aligned} \tag{3.38}$$

It is obvious that this observer (3.38) satisfies requirements (3.36). This form for the nonlinear reduced-order observer is not unique, though it is quite reasonable in that the coefficients of the parameterized family of linear reduced-order observers appear directly, and the deviation variables are included. In general (3.43) is nonlinear in $u(t)$, $x(t)$ and $y(t)$, since $\alpha$ depends on these variables. Note that the constant operating point function $z^o(\alpha)$, $\alpha \in \Gamma$ is not specified, the selection of which plays an important role.

Linearization of (3.38) about the parameterized family of constant operating points gives

$$\begin{aligned}
\dot{z}_\delta(t) &= G(\alpha)z_\delta(t) + K(\alpha)y_\delta(t) + H(\alpha)u_\delta(t) \\
&\quad -[G(\alpha)\frac{\partial z^o(\alpha)}{\partial \alpha} + K(\alpha)\frac{\partial y^o(\alpha)}{\partial \alpha} + H(\alpha)\frac{\partial u^o(\alpha)}{\partial \alpha}] \\
&\quad \times[\frac{\partial \alpha}{\partial y}y_\delta(t) + \frac{\partial \alpha}{\partial u}u_\delta(t) + \frac{\partial \alpha}{\partial x}x_\delta(t)] \\
\hat{x}_\delta(t) &= Q(\alpha)z_\delta(t) + (P(\alpha)+Q(\alpha)L(\alpha))y_\delta(t) \\
&\quad -[Q(\alpha)\frac{\partial z^o(\alpha)}{\partial \alpha} + (P(\alpha)+Q(\alpha)L(\alpha))\frac{\partial y^o(\alpha)}{\partial \alpha} - \frac{\partial x^o(\alpha)}{\partial \alpha}] \\
&\quad \times[\frac{\partial \alpha}{\partial y}y_\delta(t) + \frac{\partial \alpha}{\partial u}u_\delta(t) + \frac{\partial \alpha}{\partial x}x_\delta(t)]
\end{aligned} \tag{3.39}$$

Comparing (3.39) with (3.34) and (3.35), there are extra terms in the parameterized linearization (3.39) which are not accounted for in the design of parameterized linear reduced-order observer (3.34) and (3.35). If the square-bracketed coefficients in (3.39) vanish, that is,

$$\begin{bmatrix} G(\alpha) \\ Q(\alpha) \end{bmatrix} \frac{\partial z^o(\alpha)}{\partial \alpha} = \begin{bmatrix} 0 \\ \frac{\partial x^o(\alpha)}{\partial \alpha} \end{bmatrix} - \begin{bmatrix} K(\alpha) & H(\alpha) \\ P(\alpha) + Q(\alpha)L(\alpha) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial y^o(\alpha)}{\partial \alpha} \\ \frac{\partial u^o(\alpha)}{\partial \alpha} \end{bmatrix} \tag{3.40}$$

then these extra terms will be gone and (3.38) will be the desired nonlinear reduced-order observer. Requirements (3.40) can be rewritten as

$$G(\alpha)\frac{\partial z^o(\alpha)}{\partial \alpha} + K(\alpha)\frac{\partial y^o(\alpha)}{\partial \alpha} + H(\alpha)\frac{\partial u^o(\alpha)}{\partial \alpha} = 0 \tag{3.41}$$

$$Q(\alpha)\frac{\partial z^o(\alpha)}{\partial \alpha} + [P(\alpha) + Q(\alpha)L(\alpha)]\frac{\partial y^o(\alpha)}{\partial \alpha} - \frac{\partial x^o(\alpha)}{\partial \alpha} = 0 \tag{3.42}$$

Multiply both sides of (3.42) by $M(\alpha)$ and use relationships (3.19) and (3.24) to obtain

$$\frac{\partial z^o(\alpha)}{\partial \alpha} = [M(\alpha) - L(\alpha)C(\alpha)]\frac{\partial x^o(\alpha)}{\partial \alpha} \tag{3.43}$$

On the other hand, multiply both sides of (3.43) by $Q(\alpha)$ and use relationships (3.19) and (3.25) to obtain (3.42). So requirement (3.42) is equivalent to (3.43). Substituting (3.43) into (3.41) and using (3.34), the other requirement becomes

$$\begin{aligned} &[A_{22}(\alpha)M(\alpha) - L(\alpha)A_{12}(\alpha)M(\alpha)]\frac{\partial x^o(\alpha)}{\partial \alpha} \\ &+[A_{21}(\alpha)C(\alpha) - L(\alpha)A_{11}(\alpha)C(\alpha)]\frac{\partial x^o(\alpha)}{\partial \alpha} \\ &+[B_2(\alpha) - L(\alpha)B_1(\alpha)]\frac{\partial u^o(\alpha)}{\partial \alpha} = 0 \end{aligned} \tag{3.44}$$

Using (3.25) and (3.31), the requirement (3.44) can be further simplified as following

$$[M(\alpha)-L(\alpha)C(\alpha)][A(\alpha)\frac{\partial x^o(\alpha)}{\partial \alpha}+B(\alpha)\frac{\partial u^o(\alpha)}{\partial \alpha}]=0 \tag{3.45}$$

But from (2.2) and (2.5)

$$\begin{aligned} 0 &= \frac{\partial f(x^o(\alpha),u^o(\alpha))}{\partial x}\frac{\partial x^o(\alpha)}{\partial \alpha}+\frac{\partial f(x^o(\alpha),u^o(\alpha))}{\partial u}\frac{\partial u^o(\alpha)}{\partial \alpha} \\ &= A(\alpha)\frac{\partial x^o(\alpha)}{\partial \alpha}+B(\alpha)\frac{\partial u^o(\alpha)}{\partial \alpha} \end{aligned} \tag{3.46}$$

which means that requirement (3.45) is always satisfied. Therefore, requirement (3.40) is equivalent to (3.43).

Separate $z^o(\alpha)$ into two parts by

$$z^o(\alpha)=z_1^o(\alpha)-z_2^o(\alpha) \tag{3.47}$$

such that requirement (3.43) along with (3.19) and (3.21) becomes

$$\frac{\partial z_1^o(\alpha)}{\partial \alpha}=M(\alpha)\frac{\partial x^o(\alpha)}{\partial \alpha}=R(\alpha) \tag{3.48}$$

and

$$\frac{\partial z_2^o(\alpha)}{\partial \alpha}=L(\alpha)\frac{\partial y^o(\alpha)}{\partial \alpha} \tag{3.49}$$

There will be a solution to (3.48) if $R(\alpha)$ is appropriately chosen. For example, if $R(\alpha)=0$ is chosen, then $z_1^o(\alpha)=0$ is a solution to (3.48). Thus the requirement (3.43) can be simplified to (3.49).

Therefore, when choosing $z_2^o(\alpha)$, $\alpha\in\Gamma$ that satisfies (3.49), system (3.3) with (3.38) is the desired nonlinear reduced-order observer which has a parameterized family of linearization (3.34) and (3.35).

### 3.5 A Special Case

Generally, it is hard to find a $z_2^o(\alpha)$ that satisfies requirement (3.49). A solution may even not exist. In most cases the existence of a solution to (3.49) is a restrictive condition, as can be seen by considering the standard mixed partial derivative characterization for the existence of solutions to a total differential equation, and the fact that (3.49) is an implicit total differential equation (Hartman, 1982; Wang and Rugh, 1987). This means that the observer of the form (3.3) with (3.38) will typically not be the desired nonlinear reduced-order observer. It is parameterized family of linearizations will have extra terms of the type shown in square brackets in (3.39) that do not appear in the design of the parameterized family of linear reduced-order observers. However, there are situations that the existence of a solution to (3.49) is obvious. In this section, a special case for which (3.49) has an obvious solution will be discussed.

There are instances where $\overline{A}_{22}(\alpha) = M(\alpha)A(\alpha)Q(\alpha), \quad \alpha \in \Gamma$ has suitable eigenvalues so that they need not be relocated via $L(\alpha)$. Thus

$$L(\alpha) = 0 \qquad\qquad \alpha \in \Gamma \tag{3.50}$$

Then (3.33) and (3.35) become

$$\begin{aligned} \hat{w}_{1\delta}(t) &= y_\delta(t) \\ \dot{z}_\delta(t) &= G(\alpha)z_\delta(t) + K(\alpha)y_\delta(t) + H(\alpha)u_\delta(t) \\ \hat{x}_\delta(t) &= P(\alpha)\hat{w}_{\delta 1}(t) + Q(\alpha)z_\delta(t) \end{aligned} \tag{3.51}$$

where

$$G(\alpha) = \overline{A}_{22}(\alpha)$$
$$K(\alpha) = \overline{A}_{21}(\alpha) \qquad (3.52)$$
$$H(\alpha) = \overline{B}_2(\alpha)$$

In this situation,

$$z_2^o(\alpha) = 0 \qquad (3.52)$$

is a solution to (3.49).

## 3.6 Separation Property

In the previous chapter and this chapter, a nonlinear reduced-order observer and a nonlinear feedback controller are constructed separately. Now a nonlinear controller that combines these constructions will be investigated. There are two ways to build such a nonlinear controller. One way is to combine a nonlinear reduced-order observer and a nonlinear feedback controller directly to form a nonlinear controller. Another way is to design a parameterized linear controller by combining a parameterized linear reduced-order observer and a parameterized linear feedback controller, then construct a nonlinear controller based on the parameterized linear controller. It will be shown that the results of these two methods are the same which implies that nonlinear reduced-order observer and feedback controller of a nonlinear controller can be designed separately as in the linear case.

Consider the nonlinear plant described in Section 3.2. Assume that for each $\alpha \in \Gamma$, the linearization (2.3) also satisfies the three assumptions in Section 2.2.

### 3.6.1 Nonlinear Controller by Serializing Nonlinear Reduced-Order Observer and Nonlinear Feedback Controller

With the Assumptions 1, 2 and 3 in Section 2.2, a nonlinear feedback controller can be constructed as shown in Chapter 2. That is,

$$\begin{aligned} \dot{\xi}(t) = c(\xi(t), x(t), y(t), r(t)) = F(\alpha)B(\alpha)[\xi(t) - \xi^o(\alpha)] \\ + F(\alpha)A(\alpha)[x(t) - x^o(t)] \\ + K_I(\alpha)[r(t) - y(t)] \\ u(t) = d(\xi(t), x(t)) = \xi(t) \end{aligned} \tag{3.53}$$

where $\xi^o(\alpha)$ must satisfy

$$\xi^o(\alpha) = u^o(\alpha) \tag{3.54}$$

And $F(\alpha), K_I(\alpha)$ are chosen as indicated in Section 2.3.

In the cases that the state vector $x(t)$ of (2.1) is not measurable, the outputs $\hat{x}(t)$ of the nonlinear reduced-order observer (3.3) with (3.38) will be used to estimate the state vector $x(t)$ in (3.53). Thus, combining (3.3), (3.38) and (3.53), replacing $x(t)$ in (3.53) by $\hat{x}(t)$ in (3.3) with (3.38), yields

$$
\begin{aligned}
\dot{z}(t) &= H(\alpha)[\xi(t) - u^o(\alpha)] + G(\alpha)[z(t) - z^o(\alpha)] \\
&\quad + K(\alpha)[y(t) - y^o(\alpha)] \\
\dot{\xi}(t) &= F(\alpha)B(\alpha)[\xi(t) - u^o(\alpha)] \\
&\quad + \mathrm{F}(\alpha)\mathrm{A}(\alpha)\mathrm{Q}(\alpha)[z(t) - z^o(\alpha)] \\
&\quad + \mathrm{K}_\mathrm{I}(\alpha)[r(t) - y(t)] \\
&\quad + F(\alpha)A(\alpha)[P(\alpha) + Q(\alpha)L(\alpha)][y(t) - y^o(\alpha)] \\
u(t) &= \xi(t)
\end{aligned}
\tag{3.55}
$$

The system (3.55), assumed to satisfy the conditions (3.49) and (3.54), is the desired nonlinear controller.

### 3.6.2 Nonlinear Controller Based on Parameterized Linear Controller with Parameterized Linear Reduced-Order Observer

In this section, a parameterized linear controller will be designed by combining a parameterized linear reduced-order observer and a parameterized linear feedback controller, then a nonlinear controller will be constructed based on the parameterized linear controller. A condition under which this nonlinear controller exists will also be given.

### 3.6.2.1 Parameterized Linear Controller

From Chapter 2, a parameterized linear feedback controller can be designed as follows under the Assumptions 1, 2 and 3 in Section 2.2:

$$\begin{aligned} \dot{\xi}_\delta(t) &= F(\alpha)B(\alpha)\xi_\delta(t) + F(\alpha)A(\alpha)x_\delta(t) + K_I(\alpha)[r_\delta(t) - y_\delta(t)] \\ u_\delta(t) &= \xi_\delta(t) \end{aligned} \tag{3.56}$$

where $\{F(\alpha), K_I(\alpha)\}$ are chosen as indicated in Section 2.3.

In the cases that the state vector $x(t)$ of (2.1) is not measurable, the state vector $x_\delta(t)$ of (2.3) is not measurable and the outputs $\hat{x}_\delta(t)$ of the parameterized linear reduced-order observer (3.33) with (3.35) will be used to estimate $x_\delta(t)$. Thus the combination of (3.33), (3.35) and (3.56) with $\hat{x}_\delta(t)$ replacing $x_\delta(t)$ yields a parameterized linear controller for (2.3) as following:

$$\begin{aligned} \dot{z}_\delta(t) &= H(\alpha)\xi_\delta(t) + G(\alpha)z_\delta(t) + K(\alpha)y_\delta(t) \\ \dot{\xi}_\delta(t) &= F(\alpha)B(\alpha)\xi_\delta(t) + F(\alpha)A(\alpha)Q(\alpha)z_\delta(t) \\ &\quad + K_I(\alpha)[r_\delta(t) - y_\delta(t)] \\ &\quad + F(\alpha)A(\alpha)[P(\alpha) + Q(\alpha)L(\alpha)]y_\delta(t) \\ u_\delta(t) &= \xi_\delta(t) \end{aligned} \tag{3.57}$$

### 3.6.2.2 Nonlinear Controller

Now a nonlinear controller of the form

$$\begin{aligned} \dot{z}(t) &= m(\xi(t), z(t), y(t)) \\ \dot{\xi}(t) &= n(\xi(t), z(t), y(t)), r(t)) \\ u(t) &= s(\xi(t), z(t), y(t)) \end{aligned} \tag{3.58}$$

which has a parameterized family of linearization (3.57) will be constructed. For such nonlinear controller (3.58) to exist, there must exist smooth functions $[z^o(\alpha), \xi^o(\alpha)]$, $\alpha \in \Gamma$, such that

$$\begin{aligned} m(\xi^o(\alpha), z^o(\alpha), y^o(\alpha), r^o(\alpha)) &= 0 \\ n(\xi^o(\alpha), z^o(\alpha), y^o(\alpha)), r^o(\alpha)) &= 0 \\ s(\xi^o(\alpha), z^o(\alpha), y^o(\alpha)) &= u^o(\alpha) \end{aligned} \tag{3.59}$$

Furthermore, the nonlinear controller (3.58) should linearize to (3.57) at each constant operating point, while allowing for variations in all variables, including variations in the parameter variables $\alpha$ due to variations in the variables $u(t)$, $x(t)$ and $y(t)$. Thus at each $\alpha \in \Gamma$ it is required

$$\frac{\partial m}{\partial \xi}(\xi^o(\alpha),z^o(\alpha),y^o(\alpha),r^o(\alpha)) = H(\alpha)$$
$$\frac{\partial m}{\partial z}(\xi^o(\alpha),z^o(\alpha),y^o(\alpha),r^o(\alpha)) = G(\alpha)$$
$$\frac{\partial m}{\partial y}(\xi^o(\alpha),z^o(\alpha),y^o(\alpha),r^o(\alpha)) = K(\alpha)$$
$$\frac{\partial m}{\partial r}(\xi^o(\alpha),z^o(\alpha),y^o(\alpha),r^o(\alpha)) = 0$$
$$\frac{\partial n}{\partial \xi}(\xi^o(\alpha),z^o(\alpha),y^o(\alpha),r^o(\alpha)) = F(\alpha)B(\alpha)$$
$$\frac{\partial n}{\partial z}(\xi^o(\alpha),z^o(\alpha),y^o(\alpha),r^o(\alpha)) = F(\alpha)A(\alpha)Q(\alpha)$$
$$\frac{\partial n}{\partial y}(\xi^o(\alpha),z^o(\alpha),y^o(\alpha),r^o(\alpha)) = F(\alpha)A(\alpha)[P(\alpha)+Q(\alpha)L(\alpha)] - K_I(\alpha)$$
$$\frac{\partial n}{\partial r}(\xi^o(\alpha),z^o(\alpha),y^o(\alpha),r^o(\alpha)) = K_I(\alpha)$$
$$\frac{\partial s}{\partial \xi}(\xi^o(\alpha),z^o(\alpha),y^o(\alpha)) = I_p$$
$$\frac{\partial s}{\partial z}(\xi^o(\alpha),z^o(\alpha),y^o(\alpha)) = 0 \tag{3.60}$$
$$\frac{\partial s}{\partial y}(\xi^o(\alpha),z^o(\alpha),y^o(\alpha)) = 0$$

Consider a nonlinear controller of the form (3.58) defined by

$$\begin{aligned} m(\xi(t),z(t),y(t),r(t)) &= H(\alpha)[\xi(t)-\xi^o(\alpha)] \\ &\quad + G(\alpha)[z(t)-z^o(\alpha)] \\ &\quad + K(\alpha)[y(t)-y^o(\alpha)] \\ n(\xi(t),z(t),y(t),r(t)) &= F(\alpha)B(\alpha)[\xi(t)-\xi^o(\alpha)] \\ &\quad + F(\alpha)A(\alpha)Q(\alpha)[z(t)-z^o(\alpha)] \\ &\quad + K_I(\alpha)[r(t)-y(t)] \\ &\quad + F(\alpha)A(\alpha)[P(\alpha)+Q(\alpha)L(\alpha)][y(t)-y^o(\alpha)] \\ s(\xi(t),z(t),y(t)) &= \xi(t) \end{aligned} \tag{3.61}$$

It is obvious that this controller (3.61) satisfies requirements (3.59) by choosing

$\xi^o(\alpha)$ satisfying (3.54) and $r^o(\alpha) = y^o(\alpha)$ for any $\alpha \in \Gamma$. Note that the constant operating point function $z^o(\alpha)$ hasn't been specified, the selection of which plays an important role.

Linearization of (3.61) about the parameterized family of constant operating points gives

$$
\begin{aligned}
\dot{z}_\delta(t) &= H(\alpha)\xi_\delta(t) + G(\alpha)z_\delta(t) + K(\alpha)y_\delta(t) \\
&\quad -[H(\alpha)\frac{\partial \xi^o(\alpha)}{\partial \alpha} + G(\alpha)\frac{\partial z^o(\alpha)}{\partial \alpha} + K(\alpha)\frac{\partial y^o(\alpha)}{\partial \alpha}] \\
&\quad \times [\frac{\partial \alpha}{\partial y} y_\delta(t) + \frac{\partial \alpha}{\partial u} u_\delta(t) + \frac{\partial \alpha}{\partial x} x_\delta(t)] \\
\dot{\xi}_\delta(t) &= F(\alpha)B(\alpha)\xi_\delta(t) + F(\alpha)A(\alpha)Q(\alpha)z_\delta(t) + K_I(\alpha)(r_\delta(t) - y_\delta(t)) \\
&\quad + F(\alpha)A(\alpha)(P(\alpha) + Q(\alpha)L(\alpha))y_\delta(t) \\
&\quad -[F(\alpha)B(\alpha)\frac{\partial \xi^o(\alpha)}{\partial \alpha} + F(\alpha)A(\alpha)Q(\alpha)\frac{\partial z^o(\alpha)}{\partial \alpha} \\
&\qquad + F(\alpha)A(\alpha)(P(\alpha) + Q(\alpha)L(\alpha))\frac{\partial y^o(\alpha)}{\partial \alpha}] \\
&\quad \times [\frac{\partial \alpha}{\partial y} y_\delta(t) + \frac{\partial \alpha}{\partial u} u_\delta(t) + \frac{\partial \alpha}{\partial x} x_\delta(t)] \\
u_\delta(t) &= \xi_\delta(t)
\end{aligned}
\tag{3.62}
$$

Comparing (3.62) with (3.57), there are extra hidden coupling terms in the parameterized linearization (3.62) which are not accounted for in the design of parameterized linear controller (3.57). If the square-bracketed coefficients in (3.62) vanish, that is

$$
G(\alpha)\frac{\partial z^o(\alpha)}{\partial \alpha} + H(\alpha)\frac{\partial u^o(\alpha)}{\partial \alpha} + K(\alpha)\frac{\partial y^o(\alpha)}{\partial \alpha} = 0 \tag{3.63}
$$

$$
\begin{aligned}
&F(\alpha)B(\alpha)\frac{\partial u^o(\alpha)}{\partial\alpha} + F(\alpha)A(\alpha)Q(\alpha)\frac{\partial z^o(\alpha)}{\partial\alpha} \\
&\qquad + F(\alpha)A(\alpha)(P(\alpha)+Q(\alpha)L(\alpha))\frac{\partial y^o(\alpha)}{\partial\alpha} = 0
\end{aligned}
\tag{3.64}
$$

Using the relationship (3.19), (3.25) and (3.46), (3.64) can be simplified as

$$
F(\alpha)A(\alpha)Q(\alpha)[\frac{\partial z^o(\alpha)}{\partial\alpha} - (M(\alpha)-L(\alpha)C(\alpha))\frac{\partial x^o(\alpha)}{\partial\alpha}] = 0 \tag{3.65}
$$

which will hold when (3.49) holds for any $\alpha \in \Gamma$. In Section 3.4, it has been shown that (3.64) is also equivalent to (3.49).

Thus, a nonlinear controller that linearizes to (3.57) at each $\alpha \in \Gamma$ exists under the conditions (3.49) and (3.54). If a nonlinear controller exists, then (3.58) with (3.61) defines one possible choice.

### 3.6.3 Conclusion

It can be seen that nonlinear controller (3.55) is the same as nonlinear controller (3.58) with (3.61). Furthermore, these two nonlinear controller must satisfy the same conditions. Thus the methods described in Section 3.6.1 and 3.6.2 give the same results, which means that nonlinear reduced-order observer and feedback controller of a nonlinear controller can be designed independently as in the linear case.

# Chapter 4: Missile Autopilot Design Example

## 4.1 Introduction

In this chapter, a nonlinear autopilot will be designed for a pitch-axis missile model, which utilizes the nonlinear feedback controller and the nonlinear reduced-order observer described in previous chapters. The objective is to design the nonlinear autopilot in such a way that it generates a control signal so that the missile tracks a normal acceleration command over a wide operating range of mach number and angle of attack. From the design procedure it shows that the nonlinear autopilot can achieve various requirements by designing a suitable parameterized family of linear controllers for the parameterized family of linearizations of the nonlinear plant.

## 4.2 Pitch-Axis Missile Model

Figure 4.1 illustrates the pitch-axis missile model and the essential variables. The missile's pitch-axis behavior [5] is characterized by the following variables:

$V_m$ missile speed in *feet/sec*

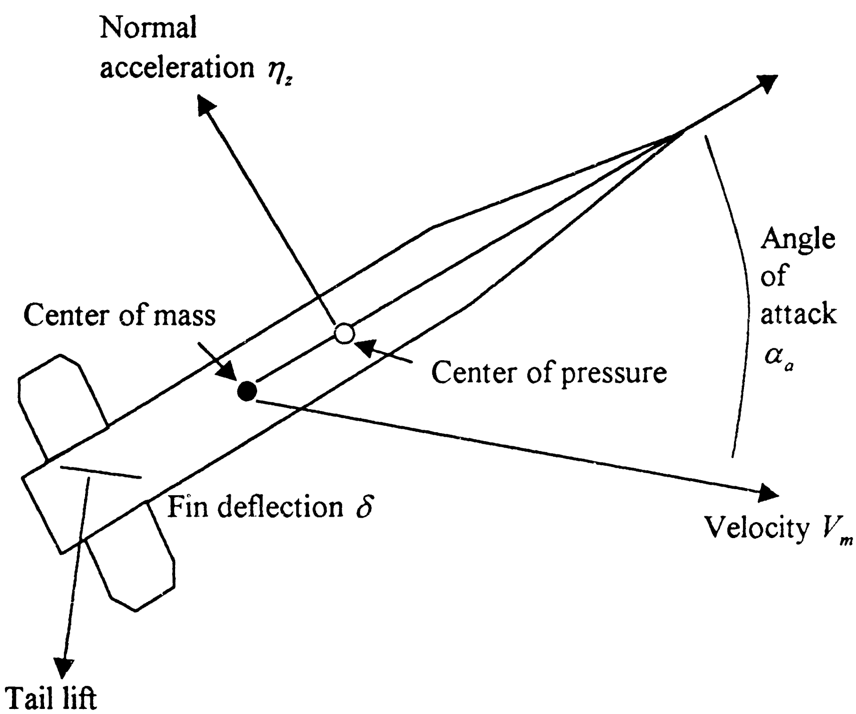


**Figure 4.1** Pitch-axis missile

| | |
|---|---|
| $a$ | speed of sound in *feet/sec* |
| $\alpha_a$ | angle-of-attack in *radians* |
| $q$ | pitch rate in *radians/sec* |
| $\delta$ | tail fin deflection in *radians* |
| $\theta$ | pitch attitude angle in *radians* |
| $\eta_z$ | normal acceleration in *gees* |
| $h$ | altitude in *feet* |
| $M = \dfrac{V_m}{a}$ | mach number (unitless) |

The short-period longitudinal aerodynamics of the missile are described by

$$\begin{aligned}\dot{\alpha}_a &= \frac{\rho a M S}{2m}[C_N \cos\alpha_a - C_A \sin\alpha_a] + \frac{g}{aM}\cos(\theta - \alpha_a) + q \\ \dot{q} &= \frac{\rho a^2 M^2 S d}{2I_Y} C_M\end{aligned} \tag{4.1}$$

and the remaining longitudinal aerodynamics are described by

$$\begin{aligned}\dot{V}_m &= \frac{\rho V_m^2 S}{2m}[C_A \cos(\alpha_a) + C_N \sin(\alpha_a)] - g\sin(\theta - \alpha_a) \\ \dot{\theta} &= q \\ \dot{h} &= V_m \sin(\theta - \alpha_a)\end{aligned} \tag{4.2}$$

where $a$ is the speed of sound (*feet/sec*) which is given by

$$a = a(h) = \begin{cases} a_{sl} - K_a h & 0 \le h \le 36000 \text{ ft} \\ a_{tr} & 36000 \le h \le 66000 \text{ ft}\end{cases} \tag{4.3}$$

and $\rho$ is the air density ($lb - \sec^2 / ft^4$) which is given by

$$\rho = \rho(h) = \begin{cases} \rho_{0sl} e^{-K_{\rho sl} h} & 0 \le h \le 36000 \text{ ft} \\ \rho_{0tr} e^{-K_{\rho tr}(h-36000)} & 36000 \le h \le 66000 \text{ ft} \end{cases} \tag{4.4}$$

with atmospheric constants listed in Table 4.1.

**Table 4.1** Atmospheric Constants

| Atmospheric Constants | |
|---|---|
| $\rho_{0sl} = 0.002377\ lb - \sec^2 / ft^4$<br>$K_{\rho sl} = -0.0000336174\ \ 1/ft$<br>$a_{sl} = 1116.4\ ft/\sec$<br>$K_a = 0.00410833\ \ 1/ft$ | $\rho_{0tr} = 0.00708554\ lb - \sec^3 / ft^4$<br>$K_{\rho tr} = -0.0000480377\ \ 1/ft$<br>$a_{tr} = 968.1\ ft/\sec$ |

The tail fin actuator is modeled by

$$\frac{d}{dt}\begin{bmatrix} \delta \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_a^2 & -2\zeta\omega_a \end{bmatrix}\begin{bmatrix} \delta \\ \dot{\delta} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_a^2 \end{bmatrix}\delta_c \tag{4.5}$$

where $\delta_c$ is the commanded tail fin deflection in *radians*. Additional missile and actuator parameters are listed in Table 4.2.

The normal acceleration is given by

$$\eta_z = \frac{\rho a^2 M^2 S}{2mg} C_N + \cos(\theta) \tag{4.6}$$

The aerodynamic coefficients $C_A$, $C_N$ and $C_M$ characterize the aerodynamic axial force, normal force and pitching moment, respectively, of the missile. For the particular missile model used here, these aerodynamic coefficients are given by

$$\begin{aligned} C_A &= a_a \\ C_N &= a_n\alpha_a^3 + b_n|\alpha_a|\alpha_a + c_n(2-\frac{M}{3})\alpha_a + d_n\delta \\ C_M &= a_m\alpha_a^3 + b_m|\alpha_a|\alpha_a + c_m(-7+\frac{8M}{3})\alpha_a + d_m\delta + e_m q \end{aligned} \tag{4.7}$$

with polynomial coefficient values listed in Table 4.3 [16].

**Table 4.2** Missile and Actuator Parameters

| Symbol | Description | Value |
|---|---|---|
| $S$ | surface area | $0.44\ ft^2$ |
| $m$ | mass | $13.98\ slug$ |
| $g$ | acceleration due to gravity | $32.2\ ft/\sec^2$ |
| $d$ | reference length | $0.75\ ft$ |
| $I_Y$ | pitch moment of inertia | $182.5\ slug-ft^2$ |
| $\zeta$ | actuator damping ratio | 0.7 |
| $\omega_a$ | actuator natural frequency | $150\ radians/\sec$ |

**Table 4.3** Aerodynamic Coefficient Parameters

| Axial Force Coefficient | Normal Force Coefficient | Pitch Moment Coefficient |
|---|---|---|
| $a_a = -0.300$ | $a_n = 19.373$<br>$b_n = -31.023$<br>$c_n = -9.717$<br>$d_n = -1.948$ | $a_m = 40.440$<br>$b_m = -64.015$<br>$c_m = 2.922$<br>$d_m = -11.803$<br>$e_m = -1.719$ |

### 4.3 Parameterized Family of Constant Operating Points

Calculating the missile's parameterized family of constant operating points meets an immediate difficulty in that true constant operating points cannot exist for a ballistic missile because there is no thrust to maintain the condition $\dot{V}_m = 0$. Even if this is ignored and constant airspeed is assumed, the normal acceleration values corresponding to achievable constant operating points for the remaining variables are small, which is not desirable since it will cause large deviations from such constant operating points to track relatively large normal acceleration commands such that the underlying assumption upon which a linear approximation is based will not be satisfied. As an alternative, only the short-period longitudinal aerodynamics of the missile will be considered here. Variables $V_m$, $\theta$ and $h$ in the short-period longitudinal aerodynamics (4.1) will be regarded as constant parameters.

There is one control input $\delta_c$ and two measurable outputs $q$ and $\eta_z$ of this nonlinear missile plant. The missile autopilot will be designed for $h = 30{,}000\ ft$ and $\theta = 0\ radians$.

Define

$$\begin{aligned} x(t) &= [\alpha_a(\mathrm{t})\ \ q(t)\ \ \delta(t)\ \ \dot{\delta}(t)]^T \\ u(t) &= \delta_c(t) \\ y(t) &= [q(t)\ \ \eta_z(t)]^T \end{aligned} \tag{4.8}$$

Then the missile model described above can be written in the following form:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{4.9}$$

For constant operating points, set $\dot{x}(t) = 0$. Choose parameter vector $\alpha = [\alpha_a, M]$, $\alpha \in \Gamma$ where $\Gamma = \{(\alpha_a, M) \mid -45^o \le \alpha_a \le 45^o,\ 1 \le M \le 3\}$, then the parameterized family of constant operating points is

$$\begin{aligned} x^o(\alpha) &= [\alpha_a\ \ q^o(\alpha)\ \ \delta^o(\alpha)\ \ 0]^T \\ u^o(\alpha) &= \delta^o(\alpha) \\ y^o(\alpha) &= h(x^o(\alpha)) = [q^o(\alpha)\ \ \eta_z^o(\alpha)]^T \end{aligned} \tag{4.10}$$

where $q^o(\alpha)$ and $\delta^o(\alpha)$ are calculated from $\dot{\alpha}_a = \dot{q} = 0$:

$$
\begin{aligned}
0 &= \frac{\rho a M S}{2m}[C_N^o \cos\alpha_a - C_A \sin\alpha_a] + \frac{g}{aM}\cos(\theta - \alpha_a) + q^o(\alpha) \\
0 &= \frac{\rho a^2 M^2 S d}{2 I_Y} C_M^o
\end{aligned}
\tag{4.11}
$$

Along with (4.7), it yields

$$
\begin{bmatrix} q^o(\alpha) \\ \delta^o(\alpha) \end{bmatrix} = \begin{bmatrix} \dfrac{2m}{\rho a M S} & \cos(\alpha_a) d_n \\ e_m & d_m \end{bmatrix}^{-1}
\times \begin{bmatrix} C_A \sin\alpha_a - \cos\alpha_a [a_n \alpha_a^3 + b_n |\alpha_a| + c_m (2 - \frac{M}{3})\alpha_a] - \dfrac{2mg}{\rho a^2 M^2 S}\cos(\theta - \alpha_a) \\ -a_m \alpha_a^3 - b_m |\alpha_a| \alpha_a - c_m(-7 + \frac{8M}{3})\alpha_a \end{bmatrix}
\tag{4.12}
$$

## 4.4 Parameterized Family of Linearizations

Define

$$
\begin{aligned}
x_\delta(t) &= x(t) - x^o(\alpha) \\
u_\delta(t) &= u(t) - u^o(\alpha) \\
y_\delta(t) &= y(t) - y^o(\alpha)
\end{aligned}
\tag{4.13}
$$

Then the parameterized family of linearizations of the missile model (4.9) is

$$
\begin{aligned}
\dot{x}_\delta(t) &= A(\alpha) x_\delta(t) + B(\alpha) u_\delta(t) \\
y_\delta(t) &= C(\alpha) x_\delta(t)
\end{aligned}
\tag{4.14}
$$

with

$$A(\alpha)=\frac{\partial f}{\partial x}(x^o(\alpha),u^o(\alpha))$$

$$=\begin{bmatrix} a_{11} & 1 & \frac{\rho a M S d}{2m}d_n\cos\alpha_a & 0 \\ \frac{\rho a^2M^2Sd}{2I_Y}C^o_{M\alpha_a} & \frac{\rho a^2M^2Sd}{2I_Y}e_m & \frac{\rho a^2M^2Sd}{2I_Y}d_m & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_a^2 & -2\zeta\omega_a \end{bmatrix} \tag{4.15}$$

$$B(\alpha)=\frac{\partial f}{\partial u}(x^o(\alpha),u^o(\alpha))=[0\ \ 0\ \ 0\ \ \omega_a^2]^T$$

$$C(\alpha)=\frac{\partial h}{\partial x}(x^c(\alpha))=\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\rho a^2M^2S}{2mg}C^o_{N\alpha_a} & 0 & \frac{\rho a^2M^2S}{2mg}d_n & 0 \end{bmatrix}$$

where

$$a_{11}=\frac{\rho a M S}{2m}(C^o_{N\alpha_a}\cos\alpha_a - C^o_N\sin\alpha_a - C_A\cos\alpha_a)+\frac{g}{aM}\sin(\theta-\alpha_a)$$

$$C^o_{N\alpha_a}=\frac{\partial C_N}{\partial\alpha_a}(\alpha_a,M,\delta^o)=3a_n\alpha_a^2+2b_n|\alpha_a|+c_n(2-\frac{M}{3}) \tag{4.16}$$

$$C^o_{M\alpha_a}=\frac{\partial C_M}{\partial\alpha_a}(\alpha_a,M,\delta^o,q^o)=3a_m\alpha_a^2+2b_m|\alpha_a|+c_m(-7+\frac{8M}{3})$$

It is easy to verify that at any $\alpha\in\Gamma$, linearization (4.12) satisfies

1. $C(\alpha)$ has full row rank.

2. $rank\frac{\partial x^o(\alpha)}{\partial\alpha}=rank\frac{\partial y^o(\alpha)}{\partial\alpha}=l=2$

3. the pair $[C(\alpha),A(\alpha)]$ is observable in the sense of linear systems.

4. the pair $[A(\alpha),B(\alpha)]$ is controllable in the sense of linear systems.

5. $A(\alpha)$ is nonsingular.

6. $\begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & 0 \end{bmatrix}$ has full row rank.

### 4.5 Missile Autopilot

The control problem of a pitch-axis missile involves determining a tail fin deflection that will generate the proper normal acceleration vector $\eta_z$ to track the commanded normal acceleration $\eta_c$. Thus, it has

$$r(t) = \eta_c(t) \tag{4.17}$$

In Chapter 3, it has been shown that there are two ways to design such a missile autopilot leading to the same result. Here, the first method will be used. That is, the missile autopilot will be constructed by combining a nonlinear feedback controller and a nonlinear reduced-order observer, which are designed separately. For consistency with the assumptions in the autopilot design process, all simulation results presented in the remainder of this chapter correspond to constant $V_m$, $\theta$ and $h$ as opposed to the aerodynamics given by (4.2).

#### 4.5.1 Nonlinear Feedback Controller

Since the parameterized linearization family (4.14) of the pitch-axis missile model (4.10) satisfies all the requirements stated in Chapter 2, the parameterized linear feedback controller designed in Chapter 2 can be used here to let the system track the given normal acceleration $\eta_c$. Since there are two outputs of the parameterized linearization family (4.14) and only one input $\eta_c$ to be tracked, an adjusted output of (4.14) is defined as

$$y_{\delta ad}(t) = C_{ad} y_\delta(t) \tag{4.18}$$

where $C_{ad} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ for any $\alpha \in \Gamma$. Thus the parameterized linear feedback controller is

$$\begin{aligned} \dot{\xi}_\delta(t) &= r_\delta(t) - C_{ad} y_\delta(t) \\ u_\delta(t) &= F(\alpha) x_\delta(t) + K_I(\alpha) \xi_\delta(t) \end{aligned} \tag{4.19}$$

where $F(\alpha)$ and $K_I(\alpha)$ are chosen to place the poles of the linear closed-loop system of (4.14) and (4.19)

$$\begin{aligned} \begin{bmatrix} \dot{x}_\delta(t) \\ \dot{\xi}_\delta(t) \end{bmatrix} &= \begin{bmatrix} A(\alpha) + B(\alpha)F(\alpha) & B(\alpha)K_I(\alpha) \\ -C_{ad}C(\alpha) & 0 \end{bmatrix} \begin{bmatrix} x_\delta(t) \\ \xi_\delta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r_\delta(t) \\ y_\delta(t) &= \begin{bmatrix} C(\alpha) & 0 \end{bmatrix} \begin{bmatrix} x_\delta(t) \\ \xi_\delta(t) \end{bmatrix} \end{aligned} \tag{4.20}$$

The poles of the tail fin actuator (4.4) are

$$p_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_a = -105.00 \pm 107.12j \tag{4.21}$$

which are also poles of the parameterized linearization family (4.14). Since they are suitable in this case, only other three poles need to be placed so that the linear closed-loop system (4.20) has acceptable rise time, settling time and no significant overshoot for a step input, the other three poles are chosen to be placed at

$$p_{3,4} = -\omega_0(\cos 30^o \pm j\sin 30^o) = -\omega_0(\frac{\sqrt{3}}{2} \pm \frac{1}{2}j)$$
$$p_5 = -\omega_0 \tag{4.22}$$

with $\omega_0 = 10$. Note that the poles are chosen to be placed at the same location for all $\alpha \in \Gamma$.

Figure 4.2 shows the unit step response of the linear closed-loop system (4.20) under these configurations.

Then the nonlinear feedback controller can be constructed as in Chapter 2

$$\begin{aligned} \dot{\xi}(t) = c(\xi(t), x(t), y(t), r(t)) = F(\alpha)B(\alpha)[\xi(t) - \xi^o(\alpha)] \\ + F(\alpha)A(\alpha)[x(t) - x^o(t)] \\ + K_I(\alpha)[r(t) - y(t)] \\ u(t) = d(\xi(t), x(t)) = \xi(t) \end{aligned} \tag{4.23}$$

where

$$\xi^o(\alpha) = u^o(\alpha) = \delta^o(\alpha) \tag{4.24}$$

Figure 4.3 shows the response of the nonlinear closed-loop system to track a $\eta_c = -30g$ step input when the system is initially in a steady state corresponding to

$$\alpha_a = 0 \ radians$$
$$M = 2.5$$

which gives the initial conditions for Figure 4.3 as follows:

$$\alpha_a(0) = 0 \ radians$$
$$q(0) = -0.0121 \ radians/\sec$$
$$\xi(0) = u(0) = 0.0018 \ radians$$
$$\eta_z(0) = 0.9326 \ gees$$
$$r(0) = 0.9326 \ gees$$

Figure 4.4 shows the response of the system with a linear feedback controller designed at above constant operating point to track a $\eta_c = -30g$ step input with same initial conditions.

Figure 4.5 shows the response of the nonlinear closed-loop system for the following initial condition

$$\begin{aligned}
&\alpha_a(0) = 0 \ radians \\
&q(0) = 1 \ radians/\sec \\
&\xi(0) = u(0) = 0 \ radians \\
&\delta(0) = 0.5 \ radians \\
&r(0) = 0.9326 \ gees
\end{aligned}$$

which dose not initially correspond to a short-period equilibrium.

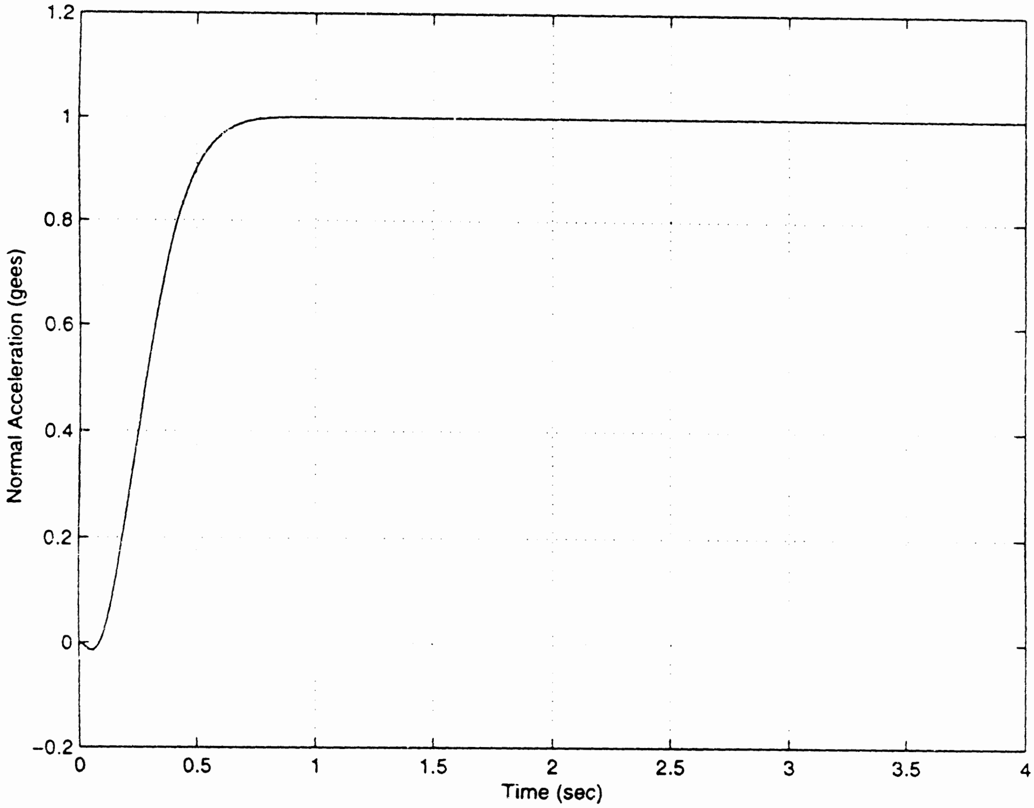


**Figure 4.2** Linear Closed-Loop Unit Step Response

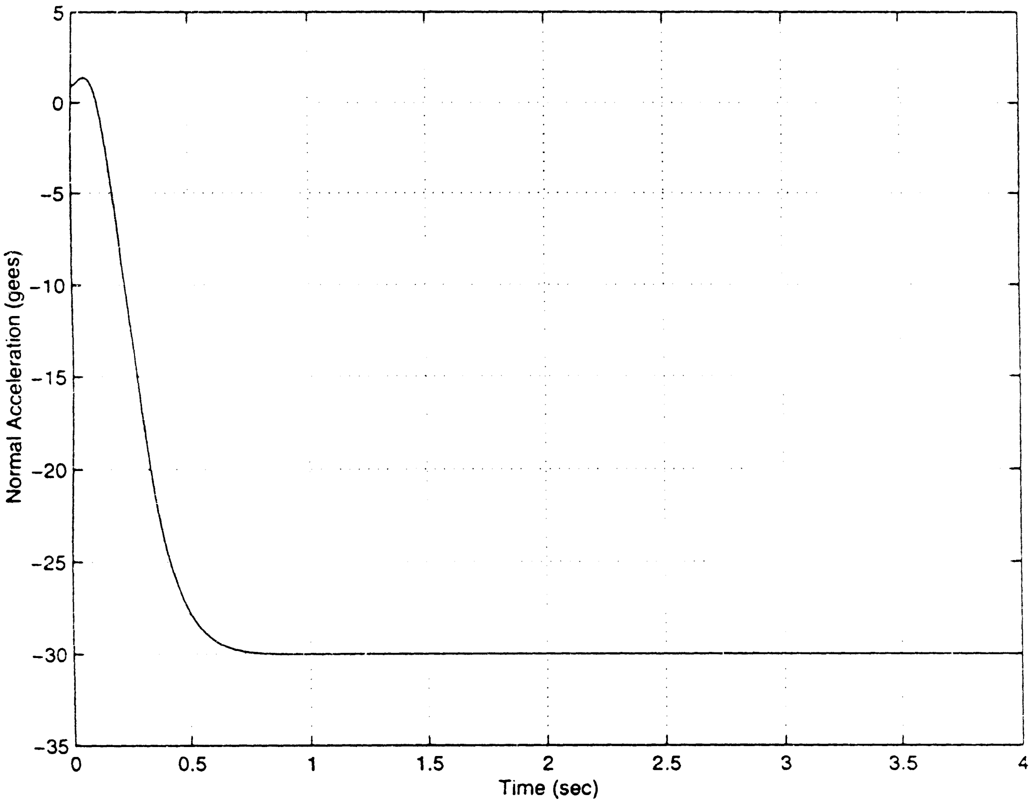


**Figure 4.3** Response of Nonlinear Closed-Loop System with State-Feedback Controller to -30g Normal Acceleration Command

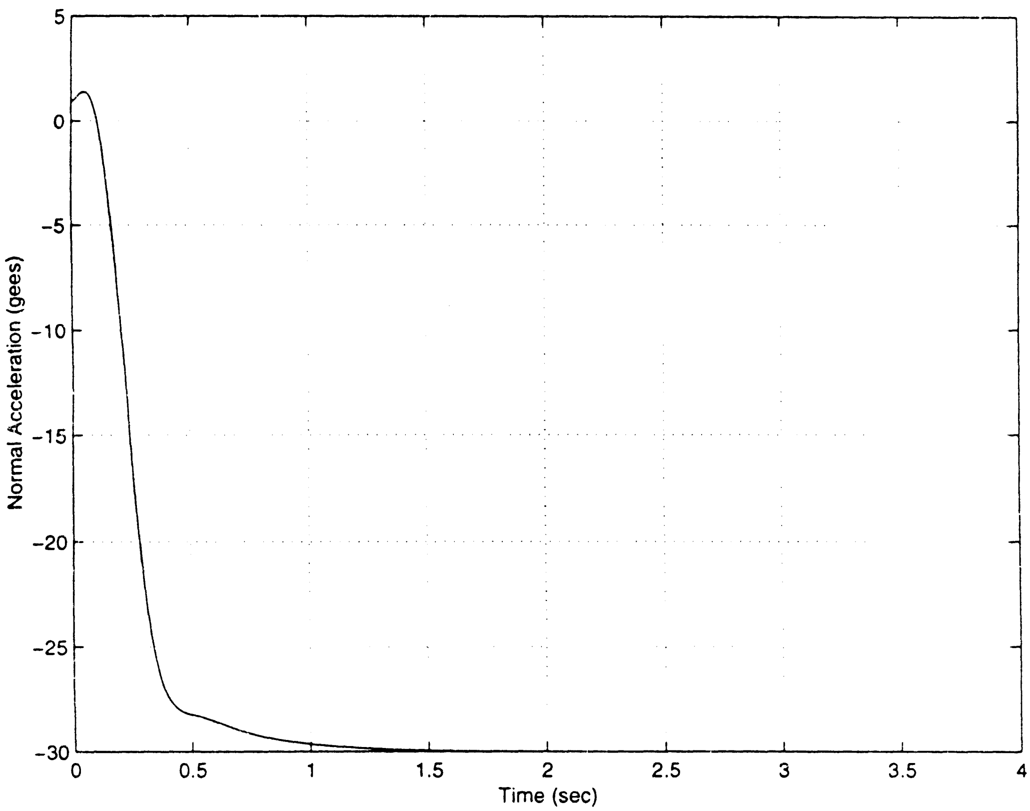


**Figure 4.4** Response of the System with Linear State-Feedback Controller to -30g Normal Acceleration Command

### 4.5.2 Nonlinear Reduced-Order Observer

Since the parameterized linearization family (4.14) of pitch-axis missile model (4.10) satisfies all the requirements stated in Chapter 3, the following parameterized linear reduced-order observer can be constructed for (4.14)

$$\begin{aligned} \dot{z}_\delta(t) &= G(\alpha)z_\delta(t) + K(\alpha)y_\delta(t) + H(\alpha)u_\delta(t) \\ \hat{x}_\delta(t) &= Q(\alpha)z_\delta(t) + (P(\alpha) + Q(\alpha)L(\alpha))y_\delta(t) \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} G(\alpha) &= \overline{A}_{22}(\alpha) - L(\alpha)\overline{A}_{12}(\alpha) \\ K(\alpha) &= \overline{A}_{21}(\alpha) - L(\alpha)\overline{A}_{11}(\alpha) + G(\alpha)L(\alpha) \\ H(\alpha) &= \overline{B}_2(\alpha) - L(\alpha)\overline{B}_1(\alpha) \end{aligned} \tag{4.26}$$

Choose $R(\alpha)$ in (3.26) to be

$$R(\alpha) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{\partial x^o(\alpha)}{\partial \alpha} \tag{4.27}$$

then

$$M(\alpha) = M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.28}$$

is a solution to (3.26). It is easy to compute that $Q(\alpha)$ in (3.28) is

$$Q(\alpha) = \begin{bmatrix} -\dfrac{d_n}{C^o_{N\alpha_a}} & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.29}$$

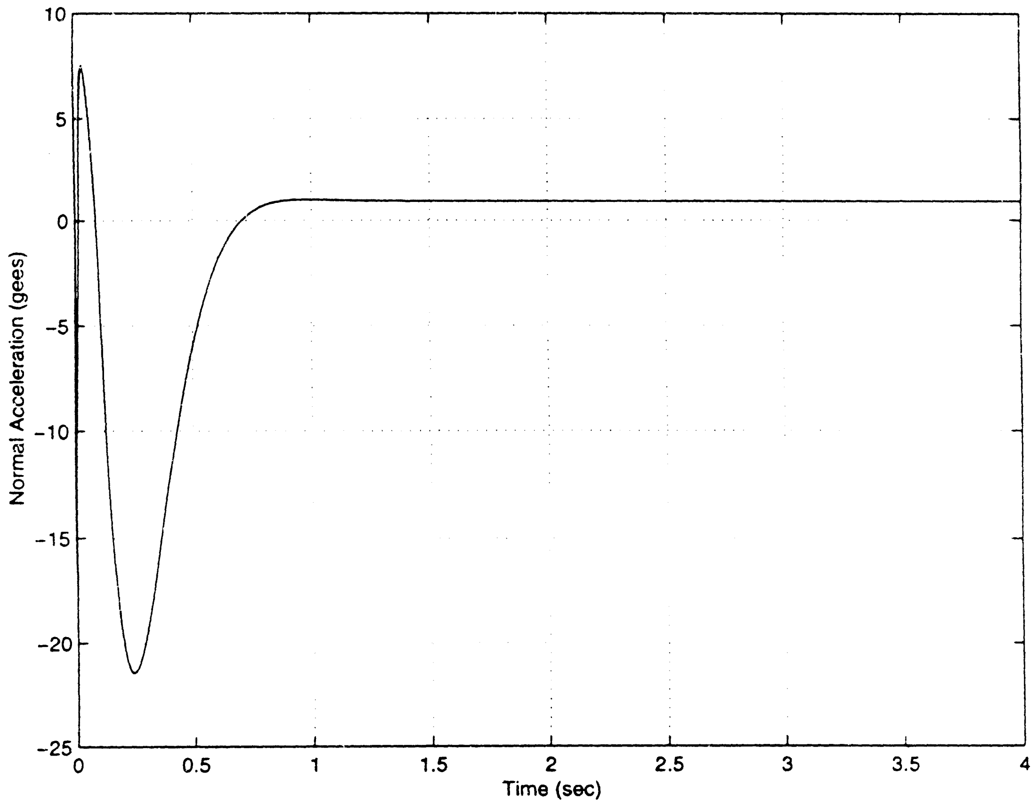


**Figure 4.5** Response of the Nonlinear Closed-loop System to the disturbance of the states

Thus $\overline{A}_{22}(\alpha)$ in (4.26) will be

$$\overline{A}_{22}(\alpha) = M(\alpha)A(\alpha)Q(\alpha) = \begin{bmatrix} 0 & 1 \\ -\omega_a^2 & -2\zeta\omega_a \end{bmatrix} \tag{4.30}$$

which has the same eigenvalues as the tail fin actuator and is suitable for the parameterized linear reduced-order observer (4.25). Thus, $L(\alpha)$ in (4.25) is set to 0.

Therefore, the parameterized linear reduced-order observer (4.25) becomes

$$\begin{aligned} \dot{z}_\delta(t) &= \overline{A}_{22}(\alpha)z_\delta(t) + \overline{A}_{21}(\alpha)y_\delta(t) + \overline{B}_2(\alpha)u_\delta(t) \\ \hat{x}_\delta(t) &= Q(\alpha)z_\delta(t) + P(\alpha)y_\delta(t) \end{aligned} \tag{4.31}$$

where

$$P(\alpha) = \begin{bmatrix} 0 & \dfrac{2mg}{\rho a^2 M^2 S}\dfrac{1}{C^o_{\alpha_a}} \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\overline{A}_{21}(\alpha) = M(\alpha)A(\alpha)P(\alpha) = 0$$

$$\overline{B}_2(\alpha) = M(\alpha)B(\alpha) = \begin{bmatrix} 0 \\ \omega_a^2 \end{bmatrix}$$

And the corresponding nonlinear reduced-order observer is

$$\begin{aligned} \dot{z}(t) = a(z(t), y(t), u(t)) &= \overline{A}_{22}(\alpha)[z(t) - z^o(\alpha)] + \overline{A}_{21}(\alpha)[y(t) - y^o(\alpha)] \\ &\quad + \overline{B}_2(\alpha)[u(t) - u^o(t)] \\ \hat{x}(t) = b(z(t), y(t)) &= Q(\alpha)[z(t) - z^o(\alpha)] + P(\alpha)[y(t) - y^o(\alpha)] + x^o(\alpha) \end{aligned} \tag{4.32}$$

with

$$z^o(\alpha) = Mx^o(\alpha) = [\delta^o(\alpha) \;\; 0]^T \tag{4.33}$$

### 4.5.3 Missile Autopilot

The missile autopilot is formed by combining the nonlinear feedback controller (4.23) and the nonlinear reduced-order observer (4.32) to obtain

$$\begin{aligned}
\dot{z}(t) &= \overline{A}_{22}(\alpha)[z(t) - z^o(\alpha)] + \overline{A}_{21}(\alpha)[y(t) - y^o(\alpha)] + \overline{B}_2(\alpha)[\xi(t) - \delta^o(t)] \\
\dot{\xi}(t) &= F(\alpha)A(\alpha)Q(\alpha)[z(t) - z^o(\alpha)] + F(\alpha)B(\alpha)(\xi(t) - \delta^o(t)) \\
&\qquad + F(\alpha)A(\alpha)P(\alpha)[y(t) - y^o(\alpha)] + K_I(\alpha)[r(t) - y(t)] \\
u(t) &= \xi(t)
\end{aligned} \tag{4.34}$$

where $z^o(\alpha)$ satisfies (4.33).

Figure 4.6 shows the response of the nonlinear closed-loop system with missile autopilot to track $\eta_c = -30g$ step input when the system is initially in a steady state corresponding to

$$\begin{aligned}
\alpha_a &= 0 \ \textit{radians} \\
M &= 2.5
\end{aligned}$$

which gives the initial conditions for Figure 4.6 as follows:

$$\begin{aligned}
&\alpha_a(0) = 0 \ \textit{radians} \\
&q(0) = -0.0121 \ \textit{radians}/\sec \\
&z(0) = \begin{bmatrix} 0.0018 \ \textit{radians} \\ 0 \ \textit{radians}/\sec \end{bmatrix} \\
&\xi(0) = \delta(0) = 0.0018 \ \textit{radians} \\
&\dot{\delta}(0) = 0 \ \textit{radians}/\sec \\
&\eta_z(0) = 0.9326 \ \textit{gees} \\
&r(0) = 0.9326 \ \textit{gees}
\end{aligned}$$

Figure 4.7, 4.8 and 4.9 show the error dynamics of the nonlinear closed-loop system due to the mismatch between the initial value of states of the reduced-order observer and the actual states of the plant. The initial conditions for these figures are

$$\alpha_a(0) = 0 \ radians$$
$$q(0) = -0.0121 \ radians/\sec$$
$$z(0) = \begin{bmatrix} 0.1 \ radians \\ 0.1 \ radians/\sec \end{bmatrix}$$
$$\xi(0) = \delta(0) = 0.0018 \ radians$$
$$\dot{\delta}(0) = 0 \ radians/\sec$$
$$r(0) = 0.9326 \ gees$$

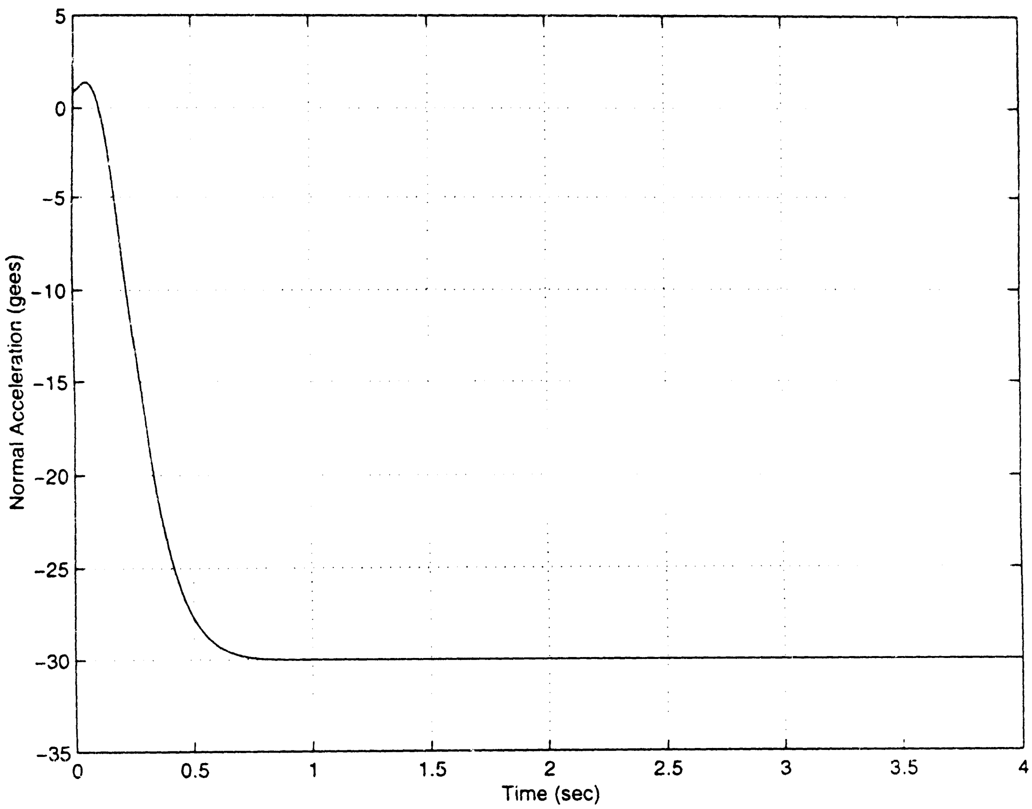


**Figure 4.6** Response of Nonlinear Closed-Loop System with Observer-Based Controller to -30g Normal Acceleration Command

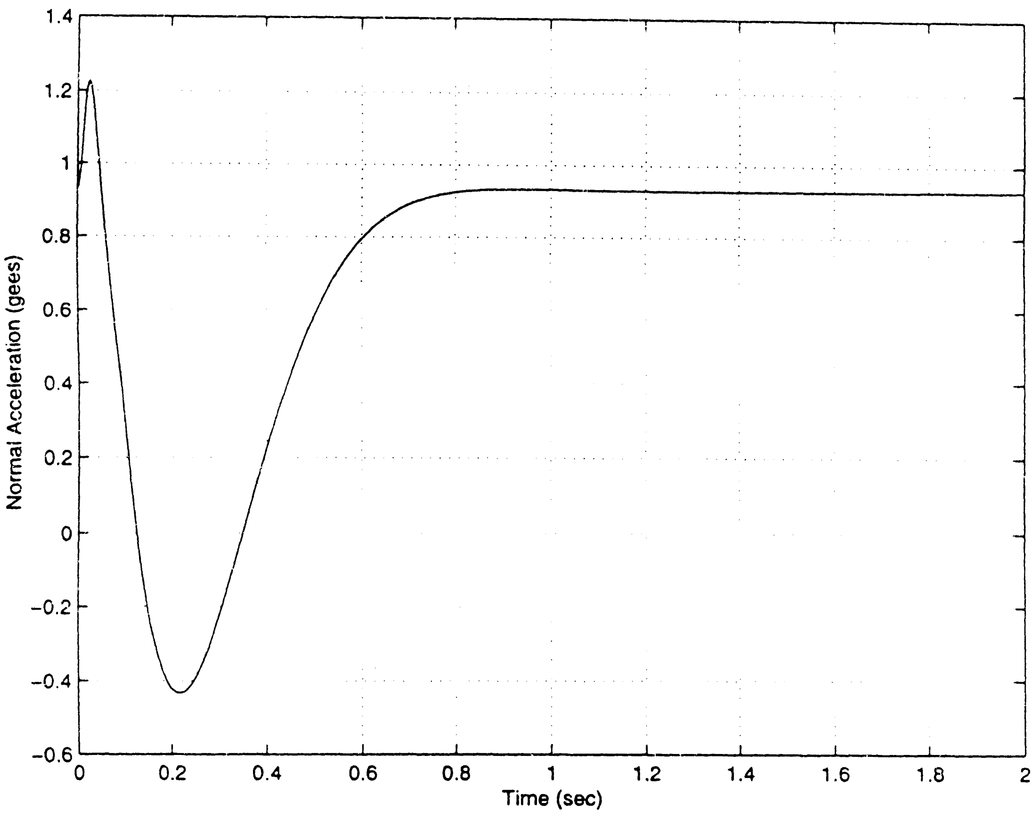


**Figure 4.7** Response of Nonlinear Closed-Loop System with Observer-Based Controller to Disturbance in the Reduced-Order Observer States

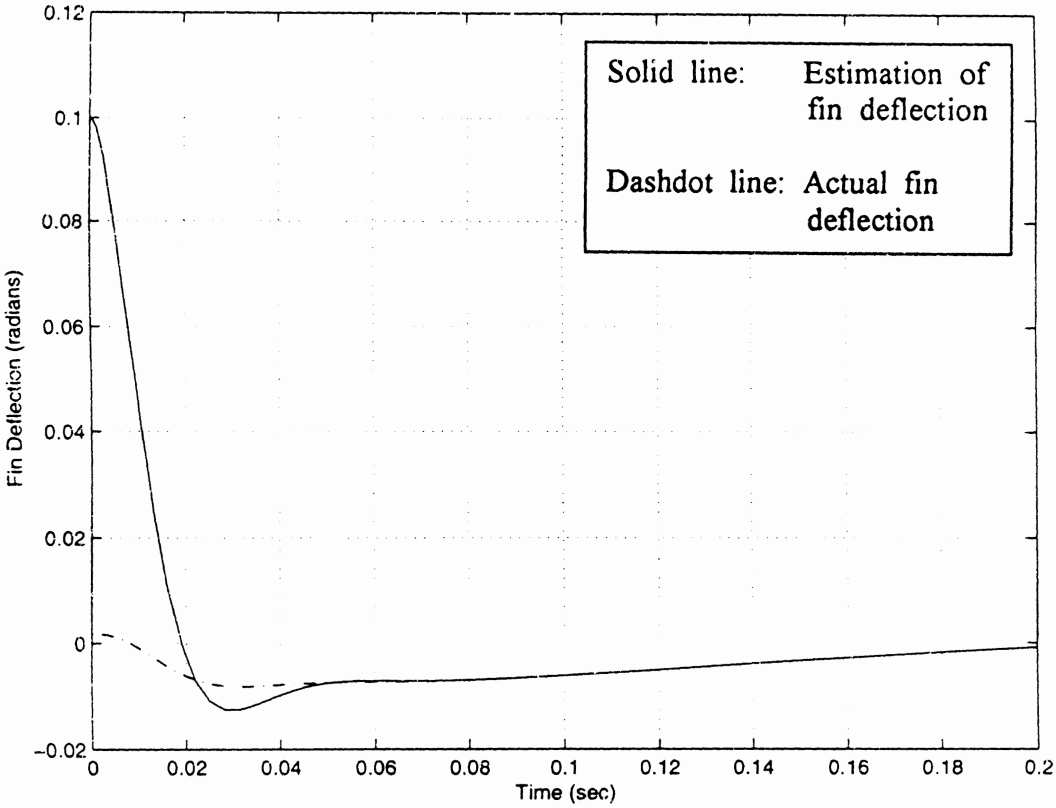


**Figure 4.8** Response of the Fin Deflection and its Estimation due to the Disturbance in the Reduced-Order Observer States

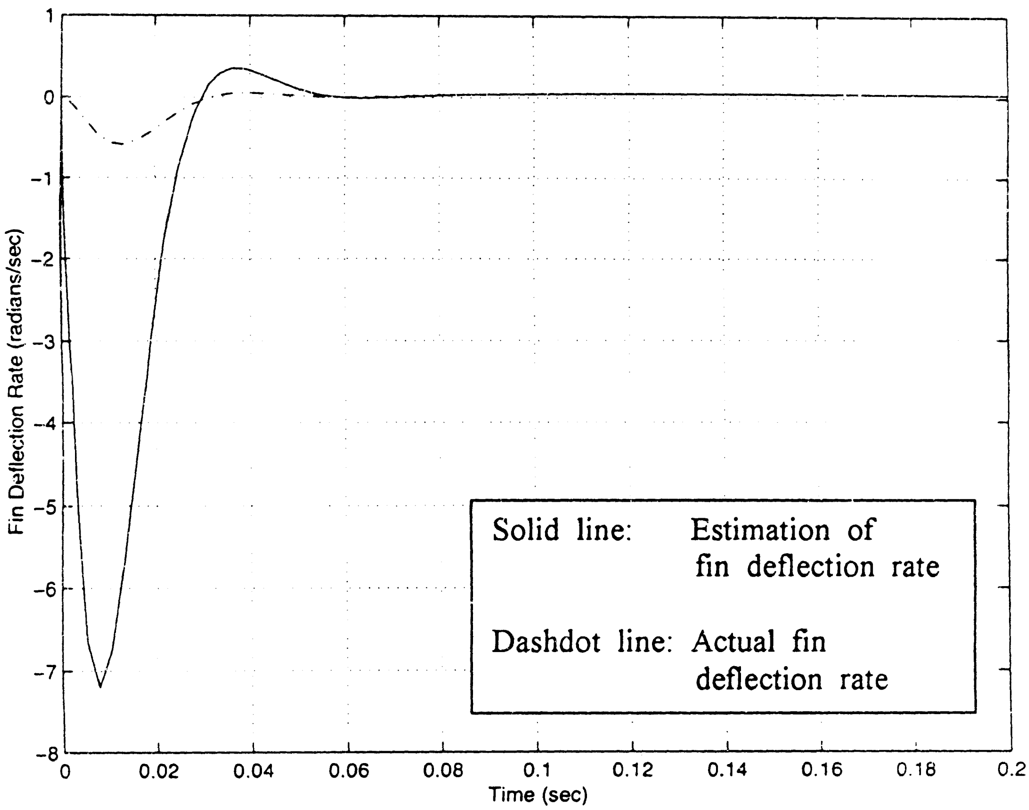


**Figure 4.9** Response of the Fin Deflection Rate and its Estimation due to the Disturbance in the Reduced-Order Observer States

# Chapter 5: Conclusions

In this thesis a nonlinear feedback controller and a nonlinear reduced-order observer were designed for a nonlinear plant in the form of (2.1). As an example, an autopilot for pitch-axis missile model utilizing them was developed. In Chapter 2, a nonlinear feedback controller was constructed and corresponding requirements that must be met were also addressed. Since in most cases the requirements are very hard to be satisfied, a revised nonlinear feedback controller was built to avoid the difficulties. In Chapter 3, a nonlinear reduced-order observer with certain requirements was constructed. A nonlinear controller was built by combining the nonlinear feedback controller and the nonlinear reduced-order observer. A separation property has been proven to ensure that these two parts can be designed separately. In Chapter 4, a pitch-axis missile model was introduced. Then a nonlinear missile autopilot was designed based on the previous two chapters. This involves the following steps:

- Select scheduling variables and calculate the parameterized family of constant operating points, construct corresponding parameterized family of linearized plants
- Set performance goals and design corresponding parameterized family of linear feedback controllers and reduced-order observers

- Build nonlinear feedback controller and reduced-order observer and combine them to form a nonlinear missile autopilot
- Check the performance of resulting nonlinear control system.

Simulations of the linear feedback controller, nonlinear feedback controller and missile autopilot showed that the performance of the missile autopilot reflected the performance of the parameterized family of linear feedback controllers and was desirable.

Further research is necessary to improve the method proposed in this thesis. Generally, a solution to (3.49) is hard to get and even may not exist. Thus further study is needed to solve this difficulty. Another area that needs further work is the incorporation of more sophisticated linear design techniques, as opposed to simple pole placement, into the gain scheduling framework described in this thesis.

# References


[1] Bernard Friedland, "Control System Design: An Introduction to State-Space Methods". McGraw-Hill, Inc., 1976.

[2] R. A. Hyde and K. Glover, "The Application of Scheduled H-Infinity Controllers to a VSTOL Aircraft". *IEEE Transactions on Automatic Control*, Vol. AC-38, No. 7, pp. 1021-1039, 1993.

[3] M. G. Kellett, "Continuous Scheduling of H-Infinity Controllers for a MS760 Paris Aircraft". *Robust Control System Design Using H-Infinity and Related Methods*, P.H. Hammond (Ed.), pp. 197-223, Institute of Measurement and Control, London, 1991.

[4] D. A. Lawrence and W. J. Rugh, "Gain Scheduling Dynamic Linear Controllers for a Nonlinear Plant". *Automatica*, Vol. 31, No. 5, pp. 381-390, 1995.

[5] C. P. Mracek and J. R. Cloutier, "Full envelope Missile Longitudinal Autopilot Design Using the State-Dependent Riccati Equation Method". *Proceedings of the 1997 AIAA Guidance, Navigation and Control Conference*, New Orleans, Louisiana, 1997.

[6] R. Nichols, R. T. Reichert and W. J. Rugh, "Gain Scheduling for H-Infinity Controllers: A Flight Control Example". *IEEE Transactions on Control Systems Technology*, Vol. 1, No. 2, pp. 69-79, 1993.

[7] A. Ostroff, "High-Alpha Application of Variable-Gain Output Feedback Control". *AIAA Journal of Guidance, Control and Dynamics*, Vol. 15, pp. 491-497, 1992.

[8] A. Pachard, "Gain Scheduling via Linear Fractional Transformations". *Syst. Control Lett.*, Vol. 22, pp. 79-92, 1994.

[9] R. T. Reichert, "Dynamic Scheduling of Modern-Robust-Control Autopilot Designs for Missiles". *IEEE Control Systems Magazine*, Vol. 12, No. 5, pp. 35-42, 1992.

[10] W. J. Rugh, "Analytical Framework for Gain Scheduling". *IEEE Control Systems Magazine*, Vol. 11, pp. 79-84, 1991.

[11] S. Shahruz and S. Bethash, "Design of Controllers for Linear Parameter-Varying Systems by the Gain Scheduling Technique". *Journal of Math. Anal. Applic.*, Vol. 168, pp. 185-217, 1992.

[12] J. S. Shamma and J. R. Cloutier, "Gain-Scheduled Missile Autopilot Design Using Linear Parameter Varying Transformations". *AIAA Journal of Guidance, Control and Dynamics*, Vol. 16, No. 2, pp. 256-263, 1993.

[13] J. S. Shamma and M. Athans, "Guaranteed Properties of Gain Scheduled Control of Linear Parameter-varying Plants". *Automatica*, Vol. 27, pp. 559-564, 1991.

[14] J. S. Shamma and M. Athans, "Analysis of Gain Scheduled Control for Nonlinear Plants". *IEEE Transaction on Automatic Control*, Vol. 35, pp. 898-907, 1990.

[15] Metin Songchaikul, "Nonlinear Control System Design Using a Gain Scheduling Technique". *M.S. Thesis, Ohio University, Dec, 1992*

[16] D. P. White, W. J. Wozniak, D. A. Lawrence, "Missile Autopilot Design Using a Gain Scheduling Technique". *Proceedings of the 26th Southeastern Symposium on System Theory*, pp. 606-610, Athens, Ohio, 1994.