RINGS CHARACTERIZED BY DIRECT SUMS OF CS MODULES

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A module M is called a CS module if every submodule of M is essential in a direct summand of M. In this dissertation certain classes of rings characterized by direct sums of CS modules are considered. It is proved that for a ring R for which either $soc(R_R)$ or $E(R_R)$ is finitely generated, the following hold: (i) R is a right Artinian ring and all uniform right R-modules are Σ-quasi-injective iff for every CS right R-module M, $M^{(\mathbf{N})}$ is CS, and; (ii) R is a right Artinian ring and all uniform right R-modules have composition length at most two iff the direct sum of any two CS right R-modules is again CS. Partial answers are obtained to a question of Huynh whether a semilocal ring or a ring with finite right uniform dimension that is right countably Σ-CS is right Σ-CS. The results obtained in this dissertation yield, in particular, new characterizations of QF-rings. These results extend previously known results due to several authors such as Clark, Dung, Gómez Pardo, Guil Asensio, Huynh, Jain, López-Permouth, Müller, Oshiro and others.

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Contents

Chapter 1

Preliminaries

In this dissertation R will stand for a noncommutative and associative ring with identity and all rings considered will be as such. Our modules are unitary right modules unless otherwise specified. R_R $({}_RR)$ stands for R considered as a right (left) R-module over itself. For all the unproved basic properties in this section, the reader may refer to [2], [13], [24], [26]. Also note that, throughout the text, the bold type letters $\mathbf{N}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{C}$ will denote the sets of natural, rational, real and complex numbers respectively.

For two right R-modules M and N, $Hom_R(M, N)$ denotes the set of R-homomorphisms from M to N, and $End_R(M)$ denotes the set of R-endomorphisms of M. A monomorphism $f : M \to N$ is said to split if its image $Im(f)$ is a direct summand of N. Similarly, if f is an epimorphism, then it is said to split if its kernel $Ker(f)$ is a direct summand of M. Let A_i $(i \epsilon I)$, A and B be submodules of M. We will use the notation $\sum_{i \epsilon I} A_i$ to denote the sum of A_i, i.e. the submodule of M generated by A_i. Also $A \oplus B$ and $\bigoplus_{i \epsilon I} A_i$ stand for the direct sum of A and B, and the direct sum of the submodules A_i, respectively. We will use the same notation for arbitrary direct sums of modules, where the context will reflect the distinction clearly. For a cardinal I, $M^{(I)}$ means a direct sum of I-copies of M.

N is called an essential submodule of M if N intersects nontrivially with any nonzero submodule of M. The socle of M is the sum of all simple submodules of M and is denoted $soc(M)$. It is also equal to the intersection of all essential submodules of M. N is said to be a small submodule of M if for any submodule A of M, $N + A = M$ implies $A = M$. The radical of M is the intersection of all maximal submodules of M and it is denoted $rad(M)$. If M contains no maximal submodules, $rad(M)$ is then defined to be equal to M. It is well known that $rad(R_R)$ and $rad(_RR)$ are both equal to the Jacobson radical $J(R)$. Also, $rad(M)$ coincides with the sum of all small submodules of M. If M is finitely generated, then $rad(M)$ itself is a small submodule of M.

$Z(M)$ denotes the submodule of M consisting of the elements of M whose annihilators are essential right ideals of R, and it is called the singular submodule of M. $Z_2(M)$ is the submodule of M containing $Z(M)$ with $Z_2(M)/Z(M) = Z(M/Z(M))$. Note that $Z(M/Z_2(M)) = 0$. It is clear that $Z(R_R)$ and $Z_2(R_R)$ are ideals of R.

A nonzero module M is said to be (directly) indecomposable if its only direct summands are 0 and M. A nonzero module M is called a uniform module if every nonzero submodule of M is essential in M. We say that M has finite uniform

(Goldie) dimension if it does not contain a direct sum of infinitely many nonzero submodules, otherwise M is said to have infinite uniform dimension. If M has finite uniform dimension, then there exists a smallest finite number n such that M does not contain a direct sum of more than n nonzero submodules. The number n is then called the uniform (Goldie) dimension of M. M has uniform dimension n if and only if there exists a direct sum of n uniform submodules which is essential in M. Any essential extension of M has the same uniform dimension as M.

For two submodules A and B of M, B is said to be a complement in M of A if B is a maximal element in the set of submodules of M having zero intersection with A. This is equivalent to the condition that $A \cap B = 0$ and $(A \oplus B)/B$ is essential in M/B. If C is a submodule of M with $A \cap C = 0$, there exists, by Zorn's Lemma, a complement B in M of A containing C. A is called a (essentially) closed submodule of M if it is not contained as a proper essential submodule of any other submodule of M. Closed submodules and complements in M coincide.

M is said to be N-injective if every homomorphism f from a submodule A of N to M can be extended to some element of $Hom_R(N, M)$. For instance M is N-injective if N is a semisimple module. If M is N-injective then M is A-injective and N/A-injective for any submodule A of N. M is called a quasi-injective module if it is M-injective and an injective module if it is N-injective for every module N. It is known that M is an injective module if and only if M is R_R-injective (Baer's criterion). $E(M)$ denotes the injective hull of M. Similarly to the injective hull, for any M, there exists a minimal quasi-injective extension (unique up to isomorphism), called the quasi-injective hull $\hat{M}$ of M. More precisely we have $\hat{M} = Tr(M, E(M))$, where $Tr(M, E(M)) = \sum\{f(M) | f : M \to E(M)\}$ is the trace of M in $E(M)$. M is quasi-injective iff $M = \hat{M}$ iff M is a fully invariant submodule of $E(M)$. The proofs of the statements in this paragraph and other details about N-injectivity can be found in [27].

The Loewy series (or socle series) of a module M is defined as the ascending chain

$$0 = S_0(M) \subseteq S_1(M) \subseteq ... \subseteq S_\alpha(M) \subseteq S_{\alpha+1}(M) \; ...,$$

where $S_{\alpha+1}(M)/S_\alpha(M) = soc(M/S_\alpha(M))$ for each (non-limit) ordinal $\alpha \geq 0$, and $S_\alpha(M) = \bigcup_{0 \leq \beta < \alpha} S_\beta(M)$ for each limit ordinal α.

M is called a semi-Artinian module if every nonzero factor module of M has essential socle. M is a semi-Artinian module if and only if $S_\alpha(M) = M$ for some ordinal $\alpha \geq 0$ (see [7, 3.12]). A ring R is called a right semi-Artinian ring if R_R is semi-Artinian, equivalently, if every right R-module has essential socle. R is a local ring if $J(R)$ is its largest proper right (left) ideal or, equivalently, if $R/J(R)$ is a division ring. R is a semilocal ring if the factor ring $R/J(R)$ is semisimple Artinian. R is called a semiperfect ring if it is a semilocal ring and its idempotents lift modulo $J(R)$. R is called a right perfect ring if it is a left semi-Artinian ring containing no infinite set of orthogonal (nonzero) idempotents or, equivalently, if every descending chain of cyclic left ideals terminates. Right/ left perfect rings are semiperfect and obviously, local rings are both semilocal and semiperfect.

R is called a QF (quasi-Frobenius) ring if R is right and left self-injective and Artinian. This is equivalent to the condition that R is a right self injective ring which is right or left Noetherian.

A module M is called a uniserial module if all submodules of M are linearly ordered by inclusion. R is called a serial ring if both R_R and ${}_RR$ are direct sums of uniserial submodules.

Throughout the text, for any module M and $x \epsilon M$, $ann_R(x)$ will stand for the annihilator of x in R. In case $M = R$, $ann_R(x)$ will always denote the right annihilator of x in R.

1.1 CS modules

In this section we give the definition of CS modules and some examples which motivate the open questions considered in this dissertation. Consider the following conditions on a module M:

C_1 : Every closed submodule of M is a direct summand of M, equivalently every submodule of M is essential in a direct summand of M,

C_2 : Every submodule of M isomorphic to a direct summand of M is itself a direct summand of M,

C_3 : If A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is also a direct summand of M.

A module M satisfying C_1 is called a CS module. If M satisfies both C_1 and C_2, then it is called a continuous module. C_2 implies C_3, and a module satisfying both C_1 and C_3 is called a quasi-continuous module. Uniform modules are precisely the indecomposable CS modules. Also, if M is a uniform continuous module then $End_R(M)$ is a local ring. For any continuous module M with $S = End_R(M)$, $J(S)$ consists of the R-endomorphisms of M with essential kernel. In particular for a right continuous ring R, we have $J(R) = Z(R_R)$. See [27] and [7] for proofs of the above statements and other details about CS modules.

CS modules have been extensively studied by many authors in the last three decades. The three types of modules defined above are generalizations of injective modules. For further generalizations of these concepts the reader may refer to [9], [10], [31] and [32]. The following hierarchy holds and is strict:

Injective $\Longrightarrow$ quasi-injective $\Longrightarrow$ continuous $\Longrightarrow$ quasi-continuous $\Longrightarrow$ CS.

We refer the reader to [27] for counterexamples to the reverse implications in the above hierarchy.

Contrary to injective modules, direct sums of CS modules may not be CS.

Example 1.1.1 The $\mathbf{Z}$-modules $\mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}/p^3\mathbf{Z}$ are both uniform, hence CS, but $\mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p^3\mathbf{Z}$ is not CS (see [7, p.56]).

Example 1.1.2 $\mathbf{Z}^n$ is a CS $\mathbf{Z}$-module for each n by [7, Corollary 12.10], but $\mathbf{Z}^{(\mathbf{N})}$ is not CS, since if it were, then we would obviously have an epimorphism $f : \mathbf{Z}^{(\mathbf{N})} \rightarrow \mathbf{Q}$ with nonessential kernel. Then by CS property, $Ker(f)$ is essential in some direct summand K of $\mathbf{Z}^{(\mathbf{N})}$. Hence $\mathbf{Q} \cong K/Ker(f) \oplus T$ for some direct summand T of $\mathbf{Z}^{(\mathbf{N})}$. Since $\mathbf{Q}$ is nonsingular, $K = Ker(f)$. It is now easy to see that $\mathbf{Q}$ embeds in $\mathbf{Z}$, which is a contradiction.

M is called (countably) Σ-CS or Σ-(quasi-)injective if every (countable) direct sum of copies of M is CS or (quasi-)injective. For any property (P) of modules, a ring R is said to be a right (P)-ring if the module R_R has the property (P). M is Σ-(quasi-)injective if and only if it is countably Σ-(quasi-)injective. However, not every countably Σ-CS module is Σ-CS. Before giving an example separating these two conditions, we give a definition: A ring R is called (von Neumann) regular if for each element $a \epsilon R$, there exists some $x \epsilon R$ such that $axa = a$. This is equivalent to the condition that every cyclic (or finitely generated) right ideal of R is a direct summand of R_R. For a continuous module M with $S = End_R(M)$, $S/J(S)$ is a regular ring (see [27, Theorem 3.11]).

Example 1.1.3 A regular ring R is right countably Σ-CS if and only if it is right self-injective (Dung and Smith [8, Proposition 3]). In this case, R is right Σ-CS if and only if it is semisimple Artinian. Consider, in particular, $R = End(V_K)$, where V_K is an infinite dimensional vector space over a field K. Then, R is a right countably Σ-CS ring which is not right Σ-CS.

1.2 Some useful results

In this section we give some known results which we will use or cite throughout the dissertation. The proofs of the results with no references specified can be found in [2], [13], [24], or [26]. The following is a key lemma.

Lemma 1.2.1 ([7, Lemma 7.5]) Let M and N be two modules and $X = M \oplus N$. The following conditions are equivalent:

(i) M is N-injective;

(ii) For every submodule A of X with $A \cap M = 0$, there exists a submodule K of X containing A such that $M \oplus K = X$.

A family $\{M_i | i \epsilon I\}$ of modules is said to have the property (A_2) if, for any $x \epsilon M_i (i \epsilon I)$ and $m_k \epsilon M_{i_k}$ for distinct $i_k \epsilon I (k \epsilon \mathbf{N})$ with $ann_R(x) \subseteq ann_R(m_k)$ for all k, the ascending sequence $\bigcap_{k \geq n} ann_R(m_k)$ becomes stationary.

Proposition 1.2.1 ([27, Proposition 1.18]) Let $M = \bigoplus_{i \epsilon I} M_i$ be a direct sum. The following conditions are equivalent:

(i) M is quasi-injective;

(ii) M_i is M_j-injective for any $i, j \epsilon I$ and (A_2) holds.

Proposition 1.2.2 (Azumaya) Let $M = \bigoplus_{i \epsilon I} M_i$ be a direct sum of indecomposable modules M_i with local endomorphism rings. Let $M = A \oplus B$ where B is indecomposable. Then $A \oplus M_j = M$ for some $j \epsilon I$, whence $B \cong M_j$.

Proposition 1.2.3 The following conditions are equivalent for a ring R:

(i) R is right Noetherian;

(ii) Every injective right R-module is Σ-injective;

(iii) Every injective right R-module is a direct sum of indecomposable submodules;

(iv) Every countable direct sum of injective hulls of simple modules is injective.

Proposition 1.2.4 The following conditions are equivalent for a ring R:

(i) R is right Artinian;

(ii) Every injective right R-module is a direct sum of injective hulls of simple submodules.

Proposition 1.2.5 M is Σ-(quasi-)injective if and only if it is countably Σ-(quasi-)injective.

Proposition 1.2.6 R is a QF ring if and only if R_R is (countably) Σ-injective.

Theorem 1.2.1 (Okado [28]) A ring R is right Noetherian if and only if every CS right R-module is a direct sum of uniform modules.

Proposition 1.2.7 (Dung [5, Corollary 3.6]) Let $M = \bigoplus_{i \epsilon I} M_i$ where M_i are indecomposable quasi-injective modules. The following conditions are equivalent:

(i) M is quasi-injective;

(ii) M_i is M_j-injective for all distinct pairs of indices $i, j \epsilon I$ and every uniform submodule of M is essential in a direct summand of M.

Proposition 1.2.8 (Dung [6, Proposition 3.9]) Let R be a right Noetherian ring. Then every CS right R-module is Σ-CS if and only if every uniform right R-module is quasi-injective.

Note that under the conditions of the above proposition, every uniform right R-module is in fact Σ-quasi-injective (Proposition 1.2.1).

Proposition 1.2.9 (Dung [6, Theorem 2.4]) A direct sum $M = \bigoplus_{i \epsilon I} M_i$ of modules M_i with local endomorphism ring is CS if and only if the direct sum of every countable subfamily of $\{M_i | i \epsilon I\}$ is CS.

It is well known that every Σ-(quasi-)injective module is a direct sum of uniform submodules. The following result improves this considerably. First note that $\aleph_1$ stands for the first uncountable cardinal and M is said to be $\aleph_1$-Σ-CS if $M^{(\aleph_1)}$ is CS.

Theorem 1.2.2 (Gómez Pardo and Guil Asensio [15, Theorem 2.6]) If M is $\aleph_1$-Σ-CS then M is a direct sum of uniform submodules.

Corollary 1.2.1 (Gómez Pardo and Guil Asensio [14, Corollary 2.7]) If all CS right R-modules over a ring R are Σ-CS, then R is right Noetherian.

Proof. By Theorem 1.2.2 and Theorem 1.2.1.

Proposition 1.2.10 (Huynh and Rizvi [21, Proposition 2.5]) Let $M = \bigoplus_{i\epsilon I} M_i$ where each M_i is uniform. Assume that M is continuous and every uniform submodule in $M^{(\mathbf{N})}$ is essential in a direct summand. Then M is Σ-quasi-injective.

Lemma 1.2.2 ([7, 13.1]) The following conditions are equivalent for a ring R:

(i) The direct sum of any two uniform right R-modules is CS;

(ii) Any uniform right R-module has length at most two;

(iii) Any direct sum of uniform right R-modules is CS.

Theorem 1.2.3 (Huynh and Muller [20, Theorem 2]) If R is a right CS ring over which every direct sum of CS right R-modules is CS, then every right R-module is CS.

Theorem 1.2.4 ([7, 13.5]) For a ring R, the following conditions are equivalent:

(i) Every right R-module is CS;

(ii) R is an Artinian serial ring with $J(R)^2 = 0$.

Proposition 1.2.11 (Al-Attas and Vanaja [1, Proposition 3.5]) Let $M = \bigoplus_{i\epsilon I} M_i$ be a countably Σ-CS module, where M_i are indecomposable modules. The following conditions are equivalent:

(i) M is Σ-CS;

(ii) Each M_i is Σ-CS;

(iii) Each M_i is quasi-injective;

(iv) Each M_i has local endomorphism ring;

(v) Each M_i has ACC (DCC) on submodules isomorphic to M_i.

Lemma 1.2.3 (Al-Attas and Vanaja [1, Lemma 3.8]) Let M be a continuous uniform module such that M^2 is CS. Then M is quasi-injective.

Proposition 1.2.12 (Al-Attas and Vanaja [1, Lemma 2.7]) A nonsingular uniform module M is countably Σ-CS if and only if it is Σ-CS.

Proposition 1.2.13 (Oshiro [30]) A right Σ-CS ring is (two sided) Artinian.

Proposition 1.2.14 (Dinh and Huynh [4, Theorem 3.2]) A finitely generated CS module M over a right semi-Artinian ring has finite uniform dimension.

Proposition 1.2.15 (Jonah [23]) If R is a right perfect ring then every right R-module has the ascending chain condition on cyclic submodules.

Chapter 2

Rings over which every CS module is countably Σ-CS

It is known that a ring R is right Noetherian if and only if every injective right R-module is Σ-injective (Proposition 1.2.3). Dung considered rings over which CS right R-modules are Σ-CS (Proposition 1.2.8). Gómez Pardo and Guil Asensio proved that such a ring R is right Noetherian (Corollary 1.2.1) and Huynh, Jain and López-Permouth [9] later showed that R is in fact right Artinian. From these results and Proposition 1.2.8, the following conditions on a ring R are equivalent:

(i) All CS right R-modules are Σ-CS;

(ii) R is a right Artinian ring whose uniform right R-modules are Σ-quasi-injective.

Recall that not every countably Σ-CS module is Σ-CS (Example 1.1.3). Thus it is natural to ask if, in the above equivalence, Σ-CS can be replaced by countably

Σ-CS. Moreover, it would be interesting to describe rings whose CS modules are countably Σ-CS. In this chapter we show that such a ring is right semi-Artinian with all uniform right R-modules Σ-quasi-injective (Corollary 2.1.2 and Corollary 2.2.3). We also answer the former question affirmatively for a ring with finitely generated right socle. (Theorem 2.1.2).

We also consider some conditions for uniform modules to be quasi-injective. Concerning this we prove that if R is either

(i) a right or left perfect ring such that for each cyclic uniform right R-module U, U^2 is CS, or;

(ii) a right semi-Artinian ring such that for each uniform right R-module U, U^2 is CS, then a direct sum M of indecomposable right R-modules is Σ-CS if and only if M is countably Σ-CS.

2.1 When every CS module is countably Σ-CS

Consider the following conditions on a ring R:

(P): Every CS right R-module is countably Σ-CS,

Condition (P) also motivates the study of the following weaker conditions (see the results following Proposition 2.2.1):

(Q): For every uniform right R-module U, U^2 is CS,

(T): For every cyclic uniform right R-module U, U^2 is CS.

Clearly (P) implies (Q) implies (T).

First we give a result which will be used in the proof of our key result in this chapter:

Lemma 2.1.1 (Huynh, Jain and López-Permouth [19]) Let R be a semiprime right Goldie ring with every uniform right ideal countably Σ-CS. Then R is semisimple Artinian.

Proof. By assumption $R_R = soc(R_R) \oplus T$ for some right ideal T of R. Assume T is nonzero. Then there exists some non-simple uniform right ideal U contained in T properly containing a nonzero right ideal V. Pick some $v \epsilon V$ for which vV is nonzero. Obviously $ann_R(v) \cap U = 0$, since R is right nonsingular. Thus $vU \cong U$, whence U embeds in itself properly, which is a contradiction by Proposition 1.2.11 and Proposition 1.2.12.

Now we can prove the key result of this chapter:

Theorem 2.1.1 Let R be a ring with finite right uniform dimension and property (P). Then R is right Artinian and every uniform right R-module is Σ-quasi-injective.

Proof. First we prove that R is right Noetherian: Since R_R has finite uniform dimension, so does $E(R_R)$, whence $E(R_R)$ is a direct sum of (finitely many) uniform submodules. By assumption $E(R_R)^{(\mathbf{N})}$ is CS. Thus, by Proposition 1.2.10 $E(R_R)$ is Σ-quasi-injective, whence Σ-injective. Now let M be any injective right R-module. We claim that M is a direct sum of uniform submodules:

Let S be the set consisting of all submodules of M which are isomorphic to uniform direct summands of $E(R_R)$ and the zero submodule. We can choose, by Zorn's Lemma, a maximal family F of submodules in S whose sum (say T) is direct. Since $E(R_R)$ is Σ-injective, T is obviously injective. So $M = T \oplus V$ for some submodule V of M. By maximality of F, V does not have any submodule isomorphic to a uniform direct summand of $E(R_R)$.

If we can prove that V is a direct sum of uniform modules, then M is also a direct sum of uniform modules. This implies by Proposition 1.2.3 that R is right Noetherian.

Now let E be a uniform direct summand of $E(R_R)$ and $D = V^{(\mathbf{N})} \oplus E$. Being a direct summand of $(V \oplus E)^{(\mathbf{N})}$, D is CS. Let B be a nonzero submodule of D with $B \cap V^{(\mathbf{N})} = 0$, and let C be a closed submodule of D essentially containing B. Then $D = C \oplus C'$ for some submodule C' of D. It is easy to see that $C \cap E$ is nonzero, since otherwise $V^{(\mathbf{N})}$ and E would contain isomorphic nonzero submodules, whence V would contain a copy of E; a contradiction. It follows that $E \cap C' = 0$, hence E is embedded in C. Since C is uniform, $E \cong C$, whence C is injective.

Now by modularity $V^{(\mathbf{N})} \oplus C = V^{(\mathbf{N})} \oplus C_0$, where C_0 is a submodule of E which is isomorphic to C. Hence $C_0 = E$, so that $D = V^{(\mathbf{N})} \oplus C$. Thus, by Lemma 1.2.1, $V^{(\mathbf{N})}$ is E-injective. Since $E(R_R)$ is a direct sum of uniform modules, we conclude that $V^{(\mathbf{N})}$ is $E(R_R)$-injective, hence injective. Thus V is Σ-injective by Proposition 1.2.5. Then by Theorem 1.2.2 V is a direct sum of uniform modules, as desired.

Thus by Proposition 1.2.3 R is right Noetherian. Now we show that R is right Artinian: Let N be the prime radical of R. Then, clearly, R/N is a semi-prime right Noetherian ring whose uniform right ideals are countably Σ-CS as right R-modules, hence as right R/N-modules. Then by Lemma 2.1.1 R/N is a semi-simple Artinian ring. Since N is nilpotent, this implies that R is right Artinian.

Now let V be any cyclic uniform right R-module. Since V is indecomposable with finite composition length, its endomorphism ring is local. Hence V is quasi-injective by assumption and Proposition 1.2.11. Now let U be any uniform right R-module and recall that a module is quasi-injective if and only if it is a fully invariant submodule of its injective hull. Write $U = \sum_{u \epsilon U} uR$. Let $f \epsilon End(E(U))$. Note that $E(U) = E(uR)$ for each nonzero $u \epsilon U$. Then for each nonzero $u \epsilon U$, $f(uR) \subseteq uR$. Thus, $f(U) = \sum_{u \epsilon U} f(uR) \subseteq U$. Therefore U is quasi-injective (and countably Σ-CS), hence Σ-quasi-injective by Proposition 1.2.11. Now the result follows.

Corollary 2.1.1 Let R be a ring such that every CS right R-module is $\aleph_1$-Σ-CS.

Then R is a right Artinian ring with all uniform right R-modules Σ-quasi-injective.

Proof. By Theorem 1.2.2 every $\aleph_1$-Σ-CS right R-module is a direct sum of uniform modules. Then $E(R_R)$ is a direct sum of uniform modules by this fact and the assumption. Thus R_R has finite uniform dimension as well as property (P). Now the conclusion follows by Theorem 2.1.1.

Lemma 2.1.2 Let R be a ring with property (P). Then the following assertions hold:

(i) The direct sum of any countable family of injective right R-modules is CS;

(ii) The direct sum of any family of uniform injective right R-modules is quasi-injective;

(iii) $Soc(S_S)$ is nonzero for any nonzero factor ring S of R;

(iv) $Soc(R_R)$ is finitely generated if and only if R is a right Artinian ring with all uniform right R-modules Σ-quasi-injective.

Proof. (i) Let $\{M_n\}$ be a countable family of injective right R-modules and let $E = E(\bigoplus M_n)$. Then $E^{(\mathbf{N})}$ is CS by assumption and $\bigoplus M_n$ is isomorphic to a direct summand of $E^{(\mathbf{N})}$. Hence $\bigoplus M_n$ is CS.

(ii) Let $\{U_i : i \in I\}$ be a family of uniform injective right R-modules where I is an arbitrary index set. Then the direct sum of any countable subfamily of $\{U_i : i \in I\}$ is CS by (i) above. This shows by Proposition 1.2.9 that the family $\{U_i : i \in I\}$ satisfies the condition (A_2). Now the result follows by Proposition 1.2.1.

(iii) We first prove that $Soc(R_R)$ is nonzero: Assume $Soc(R_R) = 0$. Let $E = \bigoplus E_n$ be a countable direct sum of injective hulls of simple right R-modules. Now $E(R_R)$ and E have no nonzero isomorphic submodules and the module $M = E(R_R) \oplus E$ is CS by (i) above. Now let A be a submodule of M with $A \cap E = 0$ and let K be a complement in M of E containing A. Then $K \cap E(R_R)$ is essential in K, since otherwise K would have a nonzero submodule which could be embedded in both E and $E(R_R)$; a contradiction. Now since M is CS, there exists a submodule L of M such that $K \oplus L = M$. Let $\nu : K \oplus L \to K$ be the obvious projection, and let D be an injective submodule of $E(R_R)$ essentially containing $K \cap E(R_R)$. Obviously, the restriction of ν to D is a monomorphism. This proves that $D \cong K$, whence K is injective. Note that $(E \oplus K)/E$ is essential in M/E. Thus $E \oplus K = M$. Now by Lemma 1.2.1 E is $E(R_R)$-injective, hence injective. This proves by Proposition 1.2.3 that R is right Noetherian, hence right Artinian by Theorem 2.1.1 a contradiction. Therefore $Soc(R_R)$ is nonzero. Now let S be any nonzero factor ring. Since for any right S-module B, the S-submodules and the R-submodules of B coincide, S has the property (P). Thus $Soc(S_S)$ is nonzero by the above argument.

(iv) Assume $Soc(R_R) = \bigoplus_{i=1}^{n} S_i$ where S_i are simple modules. Let E and E_n be as in (iii) above. $E(R_R) = (\bigoplus_{i=1}^{n} E(S_i)) \oplus B$ for some injective module B with $Soc(B) = 0$. Obviously, B and E have no isomorphic nonzero submodules and $E \oplus (\bigoplus_{i=1}^{n} E(S_i))$ is quasi-injective by (ii) above, whence E is $\bigoplus_{i=1}^{n} E(S_i)$-injective. Now let $N = E \oplus B$. Applying the same argument as in (iii) above to

N yields that E is also B-injective. Thus E is $E(R_R)$-injective, hence injective. Then R is right Noetherian. The result now follows by Theorem 2.1.1.

Corollary 2.1.2 A ring R with property (P) is right semi-Artinian.

Proof. Consider the Loewy series (S_α) of the module R_R. Then $S = \cup S_\alpha$ is an ideal such that the right R-module R/S has zero right socle. But then $R/S = 0$ by Lemma 2.1.2. Thus $R = S$. Now the result follows.

Before proving the following result, note that Dung proved that for a right Noetherian ring R, every CS right R-module is Σ-CS if and only if every uniform right R-module is quasi-injective (see Proposition 1.2.8).

Theorem 2.1.2 The following assertions are equivalent for a ring R:

(i) Every CS right R-module is $\aleph_1$-Σ-CS;

(ii) R has property (P) and $Soc(R_R)$ is finitely generated;

(iii) R is a right Artinian ring with all uniform right R-modules Σ-quasi-injective;

(iv) R is a right Noetherian ring with all uniform right R-modules quasi-injective;

(v) Every CS right R-module is a direct sum of quasi-injective uniform right R-modules.

Proof. (i) $\Rightarrow$ (ii) By Corollary 2.1.1.

(ii) $\Rightarrow$ (iii) By Lemma 2.1.2.

(iii) $\Rightarrow$ (iv) Clear, and (iv) $\Rightarrow$ (i) by Proposition 1.2.8.

(iv) $\Leftrightarrow$ (v) By Theorem 1.2.1, a ring R is right Noetherian if and only if every CS right R-module is a direct sum of indecomposable (hence uniform) modules. Now the conclusion follows.

Proposition 2.1.1 Assume that R is a ring with property (P) and R_R is contained in a finitely generated CS right R-module (in particular $E(R_R)$ is finitely generated). Then R satisfies the conditions of Theorem 2.1.2.

Proof. Let M be a finitely generated CS right R-module containing R_R. Note that R is right semi-Artinian by Corollary 2.1.2. Then M has finite uniform dimension by Proposition 1.2.14. Hence $soc(R_R)$ is finitely generated. Now the result follows by Theorem 2.1.2.

2.2 When are uniforms quasi-injective?

In this section we discuss some sufficient conditions on a ring R for all uniform right R-modules to be quasi-injective.

Proposition 2.2.1 Let U be a uniform module with finite composition length n. Then U is quasi-injective if and only if U^2 is CS.

Proof. Let U^2 be CS and $\hat{U}$ be the quasi-injective hull of U. Assume U is not equal to $\hat{U}$. Since $\hat{U}$ is U-generated, there exists some homomorphism $f : U \to \hat{U}$ such that $f(U)$ is not contained in U. Then we have an obvious epimorphism $g : U^2 \to U + f(U)$ with non-essential kernel. By CS assumption $U^2 = K \oplus B$ where K contains $Ker(g)$ essentially. But since $\hat{U}$ is uniform, $K = 0$ and $B \cong U + f(U)$. It can be easily seen that B is embedded in U via one of the canonical epimorphisms $U^2 \to U$, a contradiction. Therefore $U = \hat{U}$.

Note that Proposition 2.2.1 also follows from [7, Corollary 8.9].

Corollary 2.2.1 Every uniform right R-module with finite composition length over a ring R with property (Q) is quasi-injective.

Proposition 2.2.2 Let R be a left or right perfect ring and U be a cyclic uniform right R-module. Then U is quasi-injective if and only if U^2 is CS.

Proof. If U is quasi-injective then U^2 is quasi-injective, hence CS. Conversely, let U be a cyclic uniform right R-module with U^2 CS. Let V be a copy of U and $M = V \oplus U$. Let A be a submodule of M with $A \cap U = 0$. Let K be a complement in M of U containing A. K is obviously nonzero. Since M is CS K is a direct summand of M hence $K \oplus L = M$ for some submodule L of M. Let $\alpha_V : U \oplus V \to V$, $\beta_K : K \oplus L \to K$ be the canonical epimorphisms. Assume $\alpha_V(K)$ is distinct from V. If $L \cap U = 0$ the U embeds in K. Otherwise we must have $L \cap V = 0$, in which case V embeds in K. In either case U embeds in K. Then by assumption U has a proper submodule isomorphic to itself, say W. Now if R is left perfect/ resp. right perfect then every right R-module has DCC/ resp. ACC on cyclic submodules by Proposition 1.2.15. In the former case, we immediately have a contradiction. In the latter case; we can extend an isomorphism $\phi : W \to U$ to some monomorphism $\varphi : U \to \hat{U}$ (where we have U properly contained in $\varphi(U)$), and iterating this process we obtain in $\hat{U}$ a properly ascending sequence of modules each isomorphic to U; a contradiction. Thus in each case $\alpha_V(K) = V$. Hence $U \oplus K = U \oplus \alpha_V(K) = U \oplus V = M$. This implies, by Lemma 1.2.1, that U is V-injective, hence quasi-injective.

Corollary 2.2.2 If R is a left or right perfect ring with property (T), then all uniform right R-modules are quasi-injective.

Proof. By Proposition 2.2.2, all cyclic uniform right R-modules are quasi-injective. Now the fact that every uniform right R-module is quasi-injective follows in the same way as in Theorem 2.1.1.

Recall that a left perfect ring is right semi-Artinian. Now we prove

Proposition 2.2.3 If R is a right semi-Artinian ring with property (Q), then all uniform right R-modules are quasi-injective.

Proof. Let U be a uniform right R-module and V be the sum of all quasi-injective submodules of U. Since R is right semi-Artinian V is nonzero. Let C be a nonzero quasi-injective submodule of U and $f \epsilon End(E(V))$. Since $E(V) = E(C) = E(U)$, we have $f(C) \subseteq C$. Hence V is a quasi-injective submodule of U. Assume V is distinct from U. Then by semi-Artinian assumption, there exists a submodule T of U such that V is a maximal submodule of T. Now let $g : T \rightarrow T$ be a monomorphism. We may assume without loss of generality that $g \epsilon End(E(T))$. Since $E(T) = E(V)$ and V is nonzero quasi-injective, $g(V) \subseteq V$, hence $g(V) = V$. Then $T/V \cong g(T)/g(V) = g(T)/V$. This implies $g(T) = T$. Hence T is a continuous module with T^2 CS. Now by Lemma 1.2.3, T is quasi-injective; a contradiction. Thus $V = U$, hence U is quasi-injective.

Corollary 2.2.3 If R is a ring with property (P) then all uniform right R-modules are Σ-quasi-injective.

Proof. By Proposition 2.2.3 and Corollary 2.1.2 every uniform right R-module U is quasi-injective. Now let $\{E_i : i \in I\}$ be a family of copies of $E(U)$. Then $\bigoplus_{i \in I} E_i$ is quasi-injective by (ii) of Lemma 1. Thus $\{E_i : i \in I\}$ has the property (A_2) by Proposition 1.2.1, whence so does a family of $|I|$ copies of U. Now the result follows again by Proposition 1.2.1.

2.3 When are countably Σ-CS modules Σ-CS?

It has been proved by Gómez Pardo and Guil Asensio (Theorem 1.2.2) that a $\aleph_1$-Σ-CS module is a direct sum of uniform modules. On the other hand, a countably Σ-CS module does not necessarily have an indecomposable decomposition (Example 1.1.3). Following Dung [6], Attas and Vanaja [1] presented some conditions for a countably Σ-CS module which is a direct sum of uniforms to be Σ-CS. Thus it would be interesting to know the rings over which all countably Σ-CS modules are Σ-CS. In the next two results we are able to provide partial answers to this question:

Corollary 2.3.1 Let R be a left or right perfect ring and M be a countably Σ-CS right R-module which is a direct sum of cyclic uniform modules. Then M is Σ-CS.

Proof. Let $M = \bigoplus_{i \epsilon I} U_i$ be a countably Σ-CS module, where U_i are cyclic uniform modules. Each U_i is quasi-injective by assumption and Proposition 2.2.2. Now the conclusion follows by Proposition 1.2.11.

Corollary 2.3.2 Let R be a ring such that R is either

(i) a left or right perfect ring with property (T), or;

(ii) a right semi-Artinian ring with property (Q), or;

(iii) a ring with property (P),

and let M be a countably Σ-CS module which is a direct sum of uniform modules. Then M is Σ-CS.

Proof. (i) and (ii) follows by Corollary 2.2.2, Proposition 2.2.3 and the proof of the Corollary 2.3.1. (iii) then follows from Corollary 2.1.2 and (ii).

2.4 Remarks

Example 2.4.1 Consider the following rings:

$$R = \mathbf{Q}[x]/(x^2),\ S = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix} \text{ (where } K \text{ is a field), and } A = \begin{pmatrix} \mathbf{C} & \mathbf{C} \\ 0 & \mathbf{R} \end{pmatrix}.$$

(1)R is a quasi-Frobenius ring all of whose cyclic right R-modules are quasi-injective (see [26, Remark 6.84]).

(2) It is easy to see that S is a right Artinian ring with all proper cyclic right R-modules quasi-injective, and S_S is not uniform. Thus R and S both satisfy the conditions of Theorem 2.1.2.

(3) A is not a right-CS ring, but it satisfies the conditions of Theorem 2.1.2 (see [19]).

Note that any right semi-Artinian right V-ring (i.e. a ring R with all simple right R-modules injective) trivially satisfies Proposition 2.2.3 and that such a ring is not necessarily right Noetherian even if we assume $R/Soc(R_R)$ is a division ring (see [18]).

We close the chapter with an open question.

Question Is a ring R with property (P) right Artinian ?

By Theorem 2.1.2 it would suffice to see that for such a ring R, $Soc(R_R)$ is finitely generated.

Chapter 3

Rings over which finite direct sums of CS modules are CS

In this chapter we will discuss rings over which the direct sum of any two CS modules is again CS. Huynh and Müller proved in [20] that a right nonsingular ring over which all direct sums of CS right modules are CS is right Artinian. Later the nonsingularity condition was removed by Huynh, Jain and López-Permouth in [18]. However the structure of rings over which finite direct sums of CS modules are CS is not known. In this chapter we show in Theorem 3.1.1 that a ring R with this property is right Artinian if $soc(R_R)$ is finitely generated, or; if R_R is contained in a finitely generated CS module (in particular, if the injective hull of R_R is finitely generated). As a corollary we show that all right R-modules are CS if R is a right CS ring over which finite direct sums of CS right R-modules are CS. Such a ring is Artinian serial by Theorem 1.2.4.

3.1 When finite direct sums of CS modules are CS

In Chapter 2 we proved that a ring R for which either $soc(R_R)$ or $E(R_R)$ is finitely generated is right Artinian if all CS right R- modules are countably Σ-CS. In this section we prove, in particular, that the same conclusion holds for a ring R over which all finite sums of CS modules are again CS (Theorem 3.1.1).

Consider the following property for a ring R:

(W): The direct sum of any two CS right R-modules is CS

Proposition 3.1.1 Let R be a ring with finitely generated right socle and property (W). Then R is right Artinian.

Proof. Let $soc(R_R) = S_1 \oplus S_2 \oplus ... \oplus S_n$ $(n \in \mathbf{N})$, where S_i are simple right ideals, $E_i = E(S_i)$ and $\{V_\alpha\}_{\alpha \in I}$ be any nonempty family of injective hulls of simple modules. Call $V = \bigoplus_{\alpha \in I} V_\alpha$. Then the module

$$M = E_1 \oplus E_2 \oplus ... \oplus E_n \oplus V$$

is CS by Lemma 1.2.2, hence quasi-injective by Proposition 1.2.7. Thus V is a quasi-injective module which is $\bigoplus_{i=1}^n E_i$-injective. We have

$$E(R_R) = E \oplus E_1 \oplus E_2 \oplus ... \oplus E_n,$$

where E is an injective module with $soc(E) = 0$. Assume, without loss of generality, that E is nonzero and let $A = E \oplus V$. Then A is CS by assumption and the above argument.

Now let X be a submodule of A with $X \cap V = 0$. Also let B be a complement in A of the submodule V containing X. Then there exists a submodule B' of A such that $B \oplus B' = A$. Since E and V have no mutually isomorphic nonzero submodules, we have $B \cap E$ essential in B. Choose a closed submodule D of E essentially containing $B \cap E$. It is easy to see that D embeds in B essentially. Since D is injective this implies $D \cong B$, whence B is injective. Thus $B \oplus V = A$. Now, by Lemma 1.2.1 V is E-injective. Thus V is $E(R_R)$-injective, hence injective. We have established that the direct sum of any family of injective hulls of simple modules is injective. This implies that R is right Noetherian.

By the above argument, every injective module is a direct sum of uniform modules. Since by Lemma 1.2.2 each uniform module has length at most two, every injective is a direct sum of injective hulls of simple modules. This proves by Proposition 1.2.4 that R is right Artinian. Hence the proof is complete.

Recall that a ring R is called right semi-Artinian if every right R-module has essential socle.

Corollary 3.1.1 A ring R with property (W) is right semi-Artinian.

Proof. By Proposition 3.1.1, a nonzero ring R with property (W) can not have a zero right socle. Also, it is easy to verify that the property (W) is inherited by factor rings. Thus, every nonzero factor ring of R has nonzero right socle. Now let S be the union of the socle series of R_R. Then S is an ideal with $soc((R/S)_{R/S}) = 0$. By the preceding argument, $R = S$. Then, as in Corollary 2.1.2, R is right semi-Artinian.

Note that Corollary 3.1.1 can also be deduced from Theorem 8 of [20] and Lemma 1.2.2.

Now we prove the main theorem of this chapter.

Theorem 3.1.1 The following conditions are equivalent for a ring R:

(i) R has property (W) and $soc(R_R)$ is finitely generated;

(ii) R has property (W) and $E(R_R)$ is finitely generated;

(iii) R has property (W) and R_R is contained in a finitely generated CS module;

(iv) R has finite right uniform dimension and the direct sum of any two uniform right R-modules is CS;

(v) R is a right Artinian ring whose uniform right R-modules have length at most two.

Proof. (i) $\Rightarrow$ (v) By Proposition 3.1.1 R is right Artinian, the rest follows by Lemma 1.2.2.

(v) $\Rightarrow$ (ii) First note that by Lemma 1.2.2 and the assumption any direct sum of uniform modules is CS. Also, since R is right Artinian, every CS module is a direct sum of uniform submodules. Thus $E(R_R)$ is a finite direct sum of uniform modules which are of finite length. Hence the conclusion follows.

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (i) The conclusion follows by Proposition 1.2.14 and Corollary 3.1.1 in the same way as in the proof of Proposition 2.1.1.

(iv) $\Rightarrow$ (v) By Lemma 1.2.2 and the assumption, uniform modules have finite length. Also, since $E(R_R)$ has finite uniform dimension, it is a (finite) direct sum of uniforms. By the preceding argument, $E(R_R)$, hence R_R is Artinian.

(v) $\Rightarrow$ (iv) by Lemma 1.2.2.

It was proved by Gómez-Pardo and Guil-Asensio that any Σ-CS module is a direct sum of uniform modules, whence they also showed that a ring whose CS modules are Σ-CS is right Noetherian. Using this result, Huynh, Jain and López-Permouth proved the following statement, which follows easily from Theorem 3.1.1.

Corollary 3.1.2 (Huynh, Jain and López-Permouth [19]) The following conditions are equivalent for a ring R:

(i) R is a right Artinian ring and all uniform right R-modules have length at most two;

(ii) Any direct sum of CS right R-modules is CS;

Proof. (i) $\Rightarrow$ (ii) Any CS module is a direct sum of uniforms by Artinian assumption and Theorem 1.2.1. Also, any direct sum of uniform modules is CS by Lemma 1.2.2 and the assumption. Now the conclusion follows immediately.

(ii) $\Rightarrow$ (i) By assumption $E(R_R)$ is Σ-CS, hence it is a (finite) direct sum of uniform modules by Theorem 1.2.2. Now the result follows by Lemma 1.2.2, Theorem 3.1.1 and the assumption.

Theorem 1.2.3 and Theorem 3.1.1 yield the following

Corollary 3.1.3 Let R be a right CS ring with property (W). Then every right R-module is CS.

3.2 Remarks

Example 3.2.1 Consider the rings S and A of Example 2.4.1. Then, S is a right CS, (two sided) Artinian and serial ring with $J(S)^2 = 0$, hence S satisfies Corollary 3.1.3 (see Theorem 1.2.4). A satisfies Theorem 3.1.1, but it is not right CS (see [18] and [20]).

We do not know the answer to the following question in general:

Question Is a ring R with property (W) right Artinian?

By Theorem 3.1.1 it would suffice to prove that $soc(R_R)$ or $E(R_R)$ is finitely generated.

Chapter 4

Σ-CS rings and QF rings

In this chapter we address some questions raised by Huynh in [16] and [17]. Recall that M is called a (countably) Σ-CS module if every direct sum of (countably many) copies of M is CS. Σ-CS rings were first introduced by Oshiro [29] under the name *co-H rings*.

By Proposition 1.2.5 a module M is countably Σ-(quasi)-injective if and only if it is Σ-(quasi)-injective. For $M = R_R$ these two conditions are equivalent to R being a QF (quasi-Frobenius) ring (Proposition 1.2.6). Also, it is known that a right Σ-CS ring is right and left Artinian (Proposition 1.2.13). However, a von Neumann regular right self-injective ring is right countably Σ-CS but not Σ-CS unless it is semisimple Artinian (Example 1.1.3). Hence it is clear that a right countably Σ-CS ring need not even have finite right uniform dimension. Huynh raised the question if a right countably Σ-CS ring with finite right uniform dimension is right Σ-CS ([16]). This question has been studied by several authors such as Huynh [16], Huynh and Rizvi [21], and in the more general terms of modules by Dung [6] and Al-Attas and Vanaja [1]. However, a definitive answer has not been obtained thus far. In this chapter we prove a theorem characterizing, in terms of radicals, when countably Σ-CS and Σ-CS are equivalent for a ring R (Theorem 4.1.1). We also prove a result (Theorem 4.1.2) which considers the case as to when a semilocal right countably Σ-CS ring is right Σ-CS.

Finally we give new characterizations of QF rings which extend some results due to Huynh and Tung [22] and Clark and Huynh [3].

4.1 Countably Σ-CS rings with finite uniform dimension

For any module M, countably Σ-(quasi-)injectivity and Σ-(quasi-)injectivity are equivalent properties (Proposition 1.2.5). However, as pointed out earlier, this is no longer the case when (quasi-)injectivity is replaced by the CS property in the above assertion.

Huynh raised the question if a finite right uniform (Goldie) dimensional or semilocal right countably Σ-CS ring is Σ-CS. In Theorem 4.1.1 and Theorem 4.1.2 we characterize when the respective situations hold.

The following proposition is a summary of some known results on this subject:

Proposition 4.1.1 Let R be a right countably Σ-CS ring with finite right uniform dimension. Then the following statements are equivalent:

(i) R is a right Σ-CS ring;

(ii) Every uniform direct summand of R_R has a local endomorphism ring;

(iii) R has ACC (DCC) on projective uniform principal right ideals;

(iv) $E(R_R)$ is countably generated;

(v) Every uniform submodule of $E(R_R)^{(\mathbf{N})}$ is essential in a direct summand.

In the above proposition, the equivalence (i) $\Leftrightarrow$ (ii) is due to Al-Attas and Vanaja [1], (i) $\Leftrightarrow$ (iii) was obtained by Huynh [16], and (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (i) by Huynh and Rizvi [21].

First we give a preliminary characterization which will be useful in the proof of the main results. Recall that an element $x \epsilon R$ is called right regular if $ann_R(x) = 0$.

Proposition 4.1.2 The following statements are equivalent for a right countably Σ-CS ring R with finite right uniform dimension:

(i) R_R is a right Σ-CS ring;

(ii) Every right regular element of R is right invertible;

(iii) Every countably generated projective uniform right R-module U with nonzero $Z(U)$ has local endomorphism ring;

(iv) Every uniform summand U of R_R with nonzero $Z(U)$ has local endomorphism ring;

(v) Every uniform summand U of R_R with nonzero $Z(U)$ satisfies ACC (DCC) on submodules isomorphic to U.

Proof. (i) $\Rightarrow$ (ii) R is Artinian by Proposition 1.2.13. If $x \in R$ is right regular then $xR \cong R_R$, so that $xR = R$.

(ii) $\Rightarrow$ (i) Since R_R is CS with finite uniform dimension, $R_R = \bigoplus_{i=1}^n U_i$ where U_i are uniform. By Proposition 1.2.11 it suffices to see that each U_i has DCC on submodules isomorphic to U_i: Fix i, and assume V is a submodule of U_i which is isomorphic to U_i. Then $R_R \cong (\bigoplus_{j \neq i} U_j) \oplus V$. Call the latter I. Then there exists a right regular element $x \in I$ such that $xR = I$. By assumption $I = R$. Hence by modular law $V = U_i$.

(i) $\Rightarrow$ (iii) Let U be any projective uniform module. U is isomorphic to a direct summand of $R_R^{(\Lambda)}$ for some index set Λ, hence U is Σ-CS. So by Proposition 1.2.11 U has local endomorphism ring.

(iii) $\Rightarrow$ (iv) This is obvious.

(iv) $\Rightarrow$ (i) Let $R_R = \bigoplus_{i=1}^n U_i$ where U_i are uniform modules, as in the proof of (ii) $\Rightarrow$ (i) above. Fix $i :$ If U_i is nonsingular then U_i is Σ-CS by Proposition 1.2.12. Else, if $Z(U_i)$ is nonzero, then by assumption U_i has local endomorphism ring, hence is Σ-CS by Proposition 1.2.11. Now the result follows by Proposition 1.2.11.

(i) $\Rightarrow$ (v) Any uniform direct summand of R_R is Σ-CS. Now the result follows by Proposition 1.2.11.

(v) $\Rightarrow$ (i) Any uniform summand U of R_R with ACC (DCC) on submodules isomorphic to U is Σ-CS by Proposition 1.2.11. The rest of the proof follows in the same way as in (iv) $\Rightarrow$ (i).

Next we prove a key lemma:

Lemma 4.1.1 If M is a CS (resp. quasi-injective) module, then so is any fully invariant submodule N of M. In particular $rad(M)$ and $Z(M)$ are CS (resp. quasi-injective).

Proof. First we do the proof for the CS case. Let M be CS and let A be a submodule of N. Then $M = K \oplus T$ for some submodules K and T, where A is essential in K. It is easy to see that by assumption $N = (N \cap K) \oplus (N \cap T)$. Thus A is essential in $N \cap K$, where the latter is a direct summand of N. Therefore N is CS. As for the quasi-injective case, one can see easily that N is invariant under homomorphisms of $E(N)$. Hence N is quasi-injective.

For convenience we will call a module M semilocal if $M/rad(M)$ is semisimple. Now we can prove the main results of this section:

Theorem 4.1.1 Let R be a right countably Σ-CS ring with finite right uniform dimension. Then the following statements are equivalent:

(i) R is a right Σ-CS ring;

(ii) R is a semilocal ring and $J(R)_R$ is Σ-CS;

(iii) $Z_2(R_R)_R$ is semilocal and $rad(Z_2(R_R))_R$ is Σ-CS;

(iv) $Z_2(R_R)_R$ is semilocal and $rad(Z_2(R_R))_R$ is a direct sum of uniform modules with local endomorphism rings;

(v) $Z_2(R_R)_R$ is semilocal and, for each uniform direct summand U of $Z_2(R_R)_R$, $rad(U)$ satisfies ACC (DCC) on submodules isomorphic to $rad(U)$;

(vi) $Z(R_R) \subseteq J(R)$ and $(Z_2(R_R)/Z(R_R))_R$ has finite composition length;

(vii) $Z_2(R_R)_R$ is Noetherian.

Proof. (i) $\Rightarrow$ (ii) R is Artinian by Proposition 1.2.13, hence semilocal. Also, for any index set I, $rad(R^{(I)})_R \cong J(R)_R^{(I)}$, hence $rad(R^{(I)})_R$ is CS by Lemma 4.1.1. Therefore $J(R)_R$ is Σ-CS.

(ii) $\Rightarrow$ (iii) R_R is CS and $Z_2(R_R)$ is clearly a closed submodule, hence a direct summand of R_R. Thus $Z_2(R_R)_R$ is a semilocal module if R is semilocal. Also $rad(Z_2(R_R))$ is a direct summand of $J(R)_R$. Hence $rad(Z_2(R_R))_R$ is Σ-CS.

(iii) $\Rightarrow$ (iv) $rad(Z_2(R_R))_R$ is a Σ-CS module with finite uniform dimension, thus it is a direct sum of Σ-CS uniform modules. A Σ-CS uniform module has local endomorphism ring by Proposition 1.2.11. Hence the result follows.

(iv) $\Rightarrow$ (v) Let $rad(Z_2(R_R))_R = \bigoplus_{i=1}^{n} V_i$ where V_i are uniform modules with local endomorphism rings. If U is a uniform direct summand of $Z_2(R_R)_R$, then $V = rad(U)$ is a direct summand of $rad(Z_2(R_R))_R$. Without loss of generality we may assume that V is nonzero. Since V_i have local endomorphism rings, by Proposition 1.2.2 $V \cong V_i$ for some i. Thus V is a countably Σ-CS uniform module with local endomorphism ring by Lemma 4.1.1 (see the proof of (i) $\Rightarrow$ (ii) above), and since $Z_2(R_R)$ is a direct summand of R_R. Now the result follows by Proposition 1.2.11.

(v) $\Rightarrow$ (i) By Proposition 4.1.2, it suffices to show that every uniform direct summand U of R_R with nonzero $Z(U)$ is continuous (hence has local endomorphism ring): Now let U be as such. Then since $U/Z(U)$ is singular, $U = Z_2(U)$. Hence U is a direct summand of $Z_2(R_R)$. It is easily seen that $U/rad(U)$ is finitely generated semisimple, hence has finite composition length. Also U is countably Σ-CS, which implies by Lemma 4.1.1 that $rad(U)$ is countably Σ-CS. Assume, without loss of generality, that $rad(U)$ is nonzero (otherwise U would be semisimple, hence trivially continuous). Thus $rad(U)$ is uniform. Then by assumption and Proposition 1.2.11 $rad(U)$ is continuous. Now let $f : U \to U$ be a monomorphism. We have $f(rad(U)) \subseteq rad(U)$. Since f is 1-1 and $rad(U)$ is continuous uniform, this means $f(rad(U)) = rad(U)$. Then we have $U/rad(U) \cong f(U)/rad(U)$, which implies that both have the same composition length, hence $f(U) = U$. This proves U is continuous, as was required.

(i) $\Rightarrow$ (vi) R is Artinian, so that $R_R = \bigoplus_{i=1}^n U_i$ where U_i are local uniform modules. Then we have $Z(R_R) = \bigoplus_{i=1}^n Z(U_i)$ and $Z(U_i) \subseteq rad(U_i) \subseteq J(R)$ for each i. Hence the result follows.

(vi) $\Rightarrow$ (i) Let U be a uniform direct summand of R_R with nonzero $Z(U)$. Then as in the proof of (v) $\Rightarrow$ (i) U is a direct summand of $Z_2(R_R)$. We will prove that U is continuous: Let $f : U \to U$ be a monomorphism. Note that $Z(U)$ is a direct summand of $Z(R_R)$ so that $U/Z(U)$ has finite composition length by assumption. Now, $(f(U) + Z(U))/Z(U) \cong f(U)/(f(U) \cap Z(U)) = f(U)/Z(f(U)) \cong U/Z(U)$. Thus $(f(U)+Z(U))/Z(U)$ and $U/Z(U)$ have the same composition length. Hence

$f(U) + Z(U) = U$. But $Z(U)$ is a small submodule of U by assumption and since U is a direct summand of R_R, whence $f(U) = U$. Therefore the result follows by Proposition 4.1.2.

(i) $\Rightarrow$ (vii) R is Artinian, hence the result follows immediately.

(vii) $\Rightarrow$ (i) Any uniform direct summand U of R_R with nonzero $Z(U)$ is contained in $Z_2(R_R)$ by the proof of (v) $\Rightarrow$ (i). Thus U is a countably Σ-CS uniform module with ACC on submodules isomorphic to U. The result now follows from Proposition 4.1.2.

Note that for a right CS ring R, the conditions "$Z(R_R) \subseteq J(R)$" and "$Z(R_R) \subseteq rad(Z_2(R_R))$" are equivalent since $Z_2(R_R)$ is a direct summand of R_R.

Every finitely generated semilocal CS module can easily be seen to be a direct sum of uniforms. It is not known whether a semilocal right countably Σ-CS ring is Σ-CS. The following result characterizes when the above holds:

Theorem 4.1.2 Let R be a semilocal right countably Σ-CS ring. Then the following statements are equivalent:

(i) R is right Σ-CS;

(ii) $rad(Z_2(R_R))_R$ is Σ-CS;

(iii) $rad(Z_2(R_R))_R$ is a direct sum of uniform modules with local endomorphism rings;

(iv) For any uniform direct summand U of $Z_2(R_R)_R$, $rad(U)$ has ACC (DCC) on submodules isomorphic to $rad(U)$;

(v) $Z(R_R) \subseteq J(R)$ and $rad(Z_2(R_R))/Z(R_R)$ has finite composition length;

(vi) $rad(Z_2(R_R))_R$ is Noetherian.

Proof. By Theorem 4.1.1.

4.2 QF rings

Oshiro proved in [29] that R is a QF ring if and only if R is right Σ-CS and $Z(R_R) = J(R)$. Also Huynh proved in [16, Corollary 2] the following result:

A ring R is QF if and only if R is a semiperfect, right countably Σ-CS ring with no nonzero projective right ideals contained in $J(R)$.

Huynh also raised the question if *semilocal* could replace *semiperfect* in the above statement, or generally, if every semilocal right countably Σ-CS ring is right Σ-CS (see [17]). Note that for a right CS ring R , the condition "no nonzero projectives in $J(R)$" is clearly equivalent to the condition "$J(R) \subseteq Z(R_R)$". Also since R is semiperfect, $Z(R_R) \subseteq J(R)$, hence the condition $Z(R_R) = J(R)$ is implicit in Huynh's assumption in the above result. Thus the following result is a partial answer to Huynh's question and it extends Oshiro's result:

Corollary 4.2.1 The following are equivalent for a ring R:

(i) R is a QF ring;

(ii) R is a semilocal right countably Σ-CS ring with $Z(R_R) = J(R)$;

(iii) R is a semilocal right countably Σ-CS ring with no nonzero projective right ideals contained in $J(R)$ and $Z(R_R) \subseteq rad(Z_2(R_R))$.

Proof. (i) $\Rightarrow$ (ii) holds by Oshiro‘s result mentioned above, and (ii) $\Leftrightarrow$ (iii) follows from the arguments preceding Theorem 4.1.2 and this corollary.

(ii) $\Rightarrow$ (i) Assume (ii). Then $rad(Z_2(R_R))/Z(R_R) = 0$. Now the conclusion follows by part (v) of Theorem 4.1.2 and Oshiro's result.

In the next proposition we answer Huynh's question affirmatively for yet another case than those accounted for in Theorem 4.1.2, namely when $J(R)_R$ is quasi-injective.

Proposition 4.2.1 If R is a semilocal right countably Σ-CS ring and $J(R)_R$ is quasi-injective then R is a right Σ-CS ring.

Proof. R is a direct sum of uniforms, as expressed in the paragraph preceding Theorem 4.1.2. By this fact and Proposition 1.2.11, it suffices to prove that every uniform direct summand of R_R is quasi-injective. So let U be a uniform direct summand of R_R and $A = A_1 \oplus A_2$, where $A_i \cong U$. Let B be a submodule of A with $B \cap A_1 = 0$, and let K be a complement in A of A_1 containing B. By semilocality assumption $U/rad(U)$ is semisimple, so that we can assume without loss of generality that $rad(U)$ is nonzero. By assumption K is a (nonzero) direct summand of A. Hence $rad(K)$, being a direct summand of $rad(A)$, is nonzero since otherwise $rad(A)$ would be uniform, a contradiction. Now let $\mu : A \to A_2$ be the obvious projection. Also since $A_2 \cong U$ and U embeds in K, there exists a monomorphism $f : A_2 \to K$. Now by the preceding arguments and since $J(R)_R$ is quasi-injective, $\mu(rad(K)) = rad(A_2)$ and $f(rad(A_2)) = rad(K)$. Hence $rad(A_2) = \mu f(rad(A_2))$. Thus the composition lengths of $A_2/rad(A_2)$, $\mu f(A_2)/\mu f(rad(A_2))$ and $\mu f(A_2)/rad(A_2)$ are equal, whence $\mu f(A_2) = A_2 = \mu(K)$. Then we have $A_1 \oplus K = A_1 \oplus \mu(K) = A$. Then by Lemma 1.2.1 A_1 is A_2-injective. Hence U is quasi-injective, as required.

Note that while every QF ring satisfies the conditions of Proposition 4.2.1 by Lemma 4.1.1, the converse is not true, as the following example shows

Example 4.2.1 Let K be a field and consider

$$R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}.$$

Then R is a right Σ-CS ring with quasi-injective radical, hence it satisfies the conditions of Proposition 4.2.1, but R is not a QF ring.

Consider a module decomposition $M = \bigoplus_{i \in I} M_i$. For a submodule A of M, we will say that A is finitely contained in $\bigoplus_{i\in I} M_i$ if $A \subseteq \bigoplus_{i\in F} M_i$ for a finite subset F of I. Next, we extend some results in [22] and [3] characterizing QF rings.

Theorem 4.2.1 Let R be a semiperfect right self-injective ring. The following statements are equivalent:

(i) R is a QF ring;

(ii) $Z(R_R)^{(\mathbf{N})}$ has all uniform submodules finitely contained;

(iii) No uniform closed submodule of $R_R^{(\mathbf{N})}$ is singular.

Proof. (i) $\Rightarrow$ (ii) Assume (i). Then $R_R^{(\mathbf{N})}$ is an injective module, whence $Z(R_R)^{(\mathbf{N})}$ is quasi-injective by Lemma 4.1.1. Now let U be a uniform submodule of $Z(R_R)^{(\mathbf{N})}$. There exists a direct summand K of $Z(R_R)^{(\mathbf{N})}$ essentially containing U. K is complemented by any direct sum decomposition of $Z(R_R)^{(\mathbf{N})}$ into uniforms, since $Z(R_R)^{(\mathbf{N})}$ is quasi-injective. Thus K is finitely generated, hence finitely contained in the sum $Z(R_R)^{(\mathbf{N})}$. Then so does U.

(ii) $\Rightarrow$ (iii) Assume (ii). Let U be a uniform closed submodule of $R_R^{(\mathbf{N})}$ which is singular. Then U is in $Z(R_R)^{(\mathbf{N})}$. Thus, by assumption, U is in a finite sub-sum of $Z(R_R)^{(\mathbf{N})}$, whence it is in a finite sub-sum of $R_R^{(\mathbf{N})}$. Since U is closed and R is right self-injective, U is a direct summand of $R_R^{(\mathbf{N})}$. But then U is projective, contradicting its singularity. Now the conclusion follows.

(iii) $\Rightarrow$ (i) Assume (iii). Let $M = R_R^{(\mathbf{N})} \oplus V$, where V is any uniform direct summand of R_R. Let A be a submodule of M with $A \cap R_R^{(\mathbf{N})} = 0$. Take a complement K in M of $R_R^{(\mathbf{N})}$ containing A. Since K embeds in V it is uniform. Also since K is closed it is not singular, by assumption. Consider the canonical projection $\mu : M \to V$. Then $K \cong \mu(K)$. Since R is right self-injective $Z(R_R) = J(R)$, whence $rad(V) = Z(V)$. Also note that V is a local module by semiperfect assumption on R. These arguments together yield $\mu(K) = V$, whence $R_R^{(\mathbf{N})} \oplus K = R_R^{(\mathbf{N})} \oplus \mu(K) = M$. Now, by Lemma 1.2.1 $R_R^{(\mathbf{N})}$ is V-injective. Since V is an arbitrary direct summand of R_R, this implies that $R_R^{(\mathbf{N})}$ is injective. Therefore R is a QF ring by Proposition 1.2.6.

Corollary 4.2.2 (Huynh and Tung [22])The following statements are equivalent for a ring R:

(i) R is a QF ring;

(ii) R is a semiperfect ring with finite right uniform dimension, $J(R)$ contains no nonzero projective right ideals and every closed uniform submodule of $R_R^{(\mathbf{N})}$ is a direct summand.

Proof. (i) $\Rightarrow$ (ii) is trivial. So assume (ii). Let $R_R = \bigoplus_{i=1}^n U_i$, where U_i are local modules. By finite uniform dimension and the assumption on $R_R^{(\mathbf{N})}$, it is easy to see that each U_i must be uniform. Let M, V, A, K and μ be as in the proof of (iii) $\Rightarrow$ (i) of Theorem 4.2.1. By the assumption on $R_R^{(\mathbf{N})}$, K is a direct summand of M. Thus K is projective and $\mu(K)$ is a nonzero submodule of V which is not contained in $rad(V)$. Since V is local this implies that $\mu(K) = V$. Thus, in the same way as in Theorem 4.2.1, $R_R^{(\mathbf{N})}$ is injective, whence R is a QF ring.

Corollary 4.2.3 (Clark and Huynh [3]) The following statements are equivalent for a ring R:

(i) R is a QF ring;

(ii) R is a semiperfect right self-injective ring such that every uniform submodule of any projective right R-module M is contained in a finitely generated submodule of M.

Proof. By Theorem 4.2.1-(ii).

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