OBERLIN COLLEGE

The one place we're trying to get to is just where we can't get: algebraic speciality and gravito-electromagnetism in Bianchi type IX

by

Benjamin Kurt Lemberger

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Advised by
Rob Owen
Oberlin College Department of Physics and Astronomy

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Abstract

Einstein's theory of General Relativity, put forward in 1915, predicts that space and time do not form a fixed background, but instead are malleable and dynamic quantities themselves. Their union forms something called spacetime, which when curved causes gravitational effects. This framework has led to models of the universe which match observations that the entire universe is expanding. Running these models backwards in time leads to a 'big bang', which is a single point from which the entire known universe came from. This single point is a singularity, a place where the theory breaks down, rendering questions like 'what happened before the big bang' meaningless. However, we can use General Relativity to study what happens *near* these singularities, which can have profound implications for whatever theory will succeed General Relativity, which will need to explain singularities, and will presumably be a quantum theory of gravity.

In 1970, Belinsky, Khalatnikov, and Lifshitz made a conjecture about the nature of spacetime near any singularity. They proposed that as one asymptotically approaches a singularity, each spatial point decouples from the points around it, and therefore acts like an independent homogeneous universe. An important homogeneous universe is the 'Mixmaster Universe', and in many cases, numerical simulations show that, on approaches to singularities, each point begins to act like its own Mixmaster Universe. The Mixmaster universe features chaotic, oscillatory behavior known as 'Mixmaster Dynamics'.

Mixmaster dynamics are fairly well understood, but in this thesis I will study them in a new way, utilizing an alternative language for understand the curvature of spacetime called Gravito-Electromagnetism. In electromagnetism, the electric and magnetic fields are decomposed from a single quantity which contains all the information of the electromagnetic field. A similar decomposition can be done to the gravitational analogue of the full field quantity, giving rise to the Gravito-Electric and Gravito-Magnetic fields, which have relatively simple physical interpretations, making them ideal for the visualization of spacetimes. Additionally, I will explore the Mixmaster Universe using a related algebraic classification commonly used in General Relativity called the Petrov Classification. While the Mixmaster Universe is known to not be algebraically special according to this classification, we use a recently developed measurement of the 'nearness' of a spacetime to algebraic speciality to gain more insight into Mixmaster Dynamics.

"The introduction of numbers as coordinates... is an act of violence."

Hermann Weyl

" Mordor... the one place in Middle-Earth we don't want to see any closer... the one place we're trying to get to... is just where we can't get. Let's face it, Mr. Frodo. We're lost! "

Samwise Gamgee

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Chapter 1

Introduction

Arguably the first unification of the forces was when Newton succeeded in describing the falling of objects toward Earth and the movement of celestial bodies within the same framework. Aided by his laws of gravity, astronomers began to make precise measurements and predictions of the solar system. Newton's laws were usually very accurate, with a few noteable exceptions. These discrepancies between Newton's theory and observation could be explained by positing the existence of additional planets, and indeed some planets were discovered first not by direct observation but by the deviation of other planets from their predicted Newtonian trajectories. After both Neptune [2] and Pluto [3] were discovered in this fashion, many were tempted to also explain the abormal precession of Mercury by a similar mechanism [4]. However, Albert Einstein proposed an alternative explanation, which involved a dramatic revision of the notions of space, time, and gravity. Following his succesful theory of special relativity in 1905, Einstein put forward a more general theory in 1915 which included gravity. Special Relativity does away with the distinction between space and time, and instead works in terms of a new entity called spacetime, which can be sliced differently into space and time with equal validity by observers moving at different speeds relative to one another. The new theory, called General Relativity, describes spacetime not as some flat, fixed entity upon which everything takes place, but instead as a dynamic entity itself that can bend and curve. The energy distribution affects the shape of spacetime, and the shape of spacetime affects the evolution of the energy distribution. Free particles travel along straight lines, but straight lines in a curved spacetime can appear to an observer as a curved line, as if the particle were attracted to another massive body. Although radical, General Relativity is quite elegant, and was soon verified experimentally in the low gravity limit. The rest of the century would see ever more precise experimental confirmations of the theory. One of the theory's greatest successes was an explanation of the expansion of

the universe. Today, much research is done on gravitational radiation, as many believe that we may be less than 10 years away from directly measuring a gravitational wave.

One of the most interesting aspects of Einstein's general theory of relativity is the prediction of the existence of singularities. These are regions of spacetime in which the theory has nothing to say about what occurs, because the spacetime curvature becomes infinite. However, the theory does have something to say about what is happening *near* singularities. In 1970, Belinksy, Khalatnikov, and Lifshitz conjectured that, as one approaches any singularity, each spatial point asymptotically decouples from nearby points, and therefore acts like its own little homogeneous universe. Intuitively, this occurs because near a singularity, nearby spatial points become causally disconnected because any event that could have affected both points would have had to have been before the singularity. This conjecture, caled the BKL conjecture, has gained a significant amount of numerical and analytical evidence since its proposition, and its validity would have profound implications for the search of a quantum theory of gravity [5], which is largely concerned with the nature of singularities, where both the quantum and relativistic regimes are relevant. Because spacetime dynamics becomes effectively decoupled at distinct spatial points, the dynamics of spacetime near a singularity becomes, ironically, simpler to study than in generic spacetime regions. Along any worldline asymptotically approaching the singularity, one can ignore the spatial dependence of the fields and instead focus on the dynamics of a homogeneous universe. A classification due to Luigi Bianchi catalogues all four dimensional homogeneous universes, and a particularly relevant case for the BKL cojecture is Bianchi type IX, also called the Mixmaster Universe [1].

Arguably the second unification of the forces was in 1873 when Maxwell treated the interaction between charged particles with a single force called electromagnetism [6]. Although proposed well before Einstein's theory of relativity, Maxwell's theory was actually relativisticially consistent; as a direct consequence of this, the distinction between Electric and Magnetic Fields became geometrically meaningless, as they are both decomposed from a single quantity in an observer-dependant way. However, these quantities are still taught and used in physics classrooms all around the world, over 100 years after Einstein's special relativity was published. This is because these quantities are easy to calculate and intuitive to visualize. Although Newtonian gravity is similarly simple to work with, General Relativity is much more complex. However, it has long been known that one can decompose the gravitational field tensor in an analogous way to the electromagnetic field tensor, in order to form the so-called 'gravito-electric' and 'gravito-magnetic' fields. Recently, Nichols et. al [1] have breathed new life into this formalism by providing an intuitive physical interpretation for these fields, and providing a way to

[1] So-named because the early universe gets causally 'mixed up' due to phases of expansion and contraction. Also possibly due to the Mixmaster blender, which was introduced around the same time.

calculate and visualize the 'field lines'. The Gravito-Electric field is called the Tendex field, and describes the stretching of observers. The Gravito-Magnetic field is called the Frame-Drag field or Vortex field, and describes the twisting of observers. These quantities provde a new way to visualize and understand the results of both analytic and numerical calculations.

A related tool used in General Relativity is called the Petrov classification. This involves the study of what are known as the 'principal null directions' at each point in a spacetime. If a circular beam of light were to travel through curved spacetime, the presence of spacetime curvature would focus rays along one axis, and defocus them along a perpendicular axis, deforming the circular beam into an ellipse. However, through each point in any spacetime, it can be shown that there exist four directions through which a circle of light could pass and not change shape, although it may expand or shrink uniformly. These four special directions are closely related to the algebraic structure of the spacetime curvature tensor. Like the roots of a polynomial (in fact, *very* much like the roots of a polynomial, as we will see), these directions can be degenerate, and in the special case of flat spacetime, every direction is a principal null direction. The Petrov Cclassification is a way of classifying spacetimes based on the degeneracy of their principal null directions, and proves to be important in a number of cases. It defines a kind of generalized symmetry of spacetime, namely a symmetry of the 'internal structure' of the gravitational field. Historically, it has been useful in constructing and interpreting exact solutions of Einstein's field equations [7]. For example, simple gravitational wave spacetimes are generall Petrov Type N (all four principal null directions coincide) or Type III (three principal null directions coincide). The Schwarzschild black hole is Type D (two pairs of coinciding principal null directions). A rotating black hole, known as Kerr geometry, is also Type D, a fact which motivated Teukolsky to develop a useful method to carry out perturbation theory on such spacetimes [8] [9]. Recently, work has been done to measure how 'near' these principal null directions are to one another, which provides a measurement of how 'near' a spacetime is to a certain Petrov type [10].

In this thesis we will seek to gain new understanding of the Mixmaster Universe by using Gravito-Electromagnetism and the Petrov classification system. It is structured as follows: first, I will give a brief summary of the math background needed for General Relativity. An extended exposition is given in Appendix I. In Chapter 3, familiar electromagnetism is cast in the language of tensors on spacetime. Here we see how to decompose the Electromagnetic field tensor into its electric and magnetic parts for future comparison. In Chapter 4, a brief introduction to General Relativity is given, with motivation for Einstein's equation, as well as the definition of some important quantities. At the end of this section, the Gravito-Electromagnetism formalism is introduced. Chapter 5 contains a standard treatment of the Mixmaster Universe. Chapter 6 presents

our analysis of the Mixmaster Universe using Gravito-Electromagnetism and the Petrov classification. Finally, Chapter 7 sums up the main results and suggests directions for future work.

Chapter 2

Brief Math Background

Vectors are abstract geometrical objects, but often in physics it is most convenient to express them with respect to some basis, such as $\vec{v} = v^1\vec{e}_1 + v^2\vec{e}_2 + v^3\vec{e}_3$. This is often written as $\vec{v} = (v^1, v^2, v^3)$, but this notation is misleading. If we were to write v in terms of the xyz basis, and also in terms of the $r\theta\phi$ spherical coordinates, the triplets (v^x, v^y, v^z) and (v^r, v^θ, v^ϕ) will look very different, yet they are supposed to be representing the same vector. In general, for a set of basis vectors $\{\vec{e}_i\}$, we can write $\vec{v} = \sum_i v^i\vec{e}_i$. Einstein summation convention states that there is an implied sum over an repeated index that appears once 'upstairs' and once 'downstairs'. Adhering to this convention, then, which we shall for the remainder of this thesis, we have that $\vec{v} = v^i\vec{e}_i$. The $\vec{e}_i$ form a basis for the vector space of all vectors, and we can use something called the tensor product, denoted by $\otimes$, to create larger vector spaces (formally, a vector space is a set of objects which can be added together and multiplied by constants, and traditional vectors are a good example of this, hence the name). The space spanned by all elements of the form $\vec{e}_i \otimes \vec{e}_j$ creates a new vector space. Members of this space are caled rank 2 tensors, and in general can be written as $T = T^{ij}\vec{e}_i \otimes \vec{e}_j$, where Einstein summation convention is implied over both indices on the right. Similarly, rank n tensors are elements of the vector space spanned by all possible combinations of n tensor products of the basis vectors.

The derivative of a vector with respect to some coordinate x can be denoted by $\frac{\partial\vec{v}}{\partial x} = \partial_x\vec{v}$, and is given by $\partial_x\vec{v} = \partial_x(v^i\vec{e}_i) = (\partial_x v^i)\vec{e}_i + v^i\partial_x\vec{e}_i$. But $\partial_x\vec{e}_i$ is just a new vector and can therefore be expressed in the basis $\{\vec{e}_i\}$: we denote this by $\partial_x\vec{e}_i = \Gamma^j_{xi}\vec{e}_j$, where the Γ are called the connection coeffecients. Therefore, $\partial_x\vec{v} = (\partial_x v^i + \Gamma^i_{xj}v^j)\vec{e}_i$. This can easily be generalized to all tensors, with $n+1$ terms for the derivative of a rank n tensor. Notice that this derivative operator is coordinate independent: for this reason, we call it the

covariant derivative, and denote it ∇. In general,

$$\nabla_\sigma T^{\mu...\nu}_{\alpha...\beta} = \partial_\sigma T^{\mu...\nu}_{\alpha...\beta} + \Gamma^\mu_{\sigma\lambda} T^{\lambda...\nu}_{\alpha...\beta} + ... + \Gamma^\nu_{\sigma\lambda} T^{\mu...\lambda}_{\alpha...\beta} - \Gamma^\lambda_{\sigma\alpha} T^{\mu...\nu}_{\lambda...\beta} - ... - \Gamma^\lambda_{\sigma\beta} T^{\mu...\nu}_{\alpha...\lambda} \tag{2.1}$$

The dot product of two vectors is $\vec{v} \cdot \vec{w} = (v^i \vec{e}_i) \cdot (w^j \vec{e}_j) = v^i w^j \vec{e}_i \cdot \vec{e}_j$. In normal euclidean space, $\vec{e}_i \cdot \vec{e}_j = diag(1,1,1) = \delta_{ij}$. In four dimensional flat spacetime, for an orthormal coordinate basis, as we shall see, $\vec{e_\mu} \cdot \vec{e_\nu} = \text{diag}(-1,1,1,1) = \eta_{\mu\nu}$, where the greek indices run from 0 to 3. This means thatr the time-like unit vector has negative length, as $\vec{e}_0 \cdot \vec{e}_0 = -1$. There is a deep connection between partial derivatives and vectors (see Appendix I), and for this reason we can denote some vectors as $\vec{e_x} = \vec{\partial_x}$. In general, for some basis associated with a coordinate system in this way, $\partial_\mu \cdot \partial_\nu = g_{\mu\nu}$, and we call $g_{\mu\nu}$ the metric. The dot product between two vectors is now $\vec{v} \cdot \vec{w} = v^\mu w^\nu g_{\mu\nu}$, and as a shorthand we write $v^\mu g_{\mu\nu} = v_\nu$ so that $\vec{v} \cdot \vec{w} = v^\mu w_\mu$.

The metric has many other uses. It allows us define the connection coeffecients, given by the following formula:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu}) \tag{2.2}$$

It also allows us to define the length of a path by taking the dot product of the tangent vector to a path with itself, and integrating along the path. If the tangent vector to the path is given by t^μ and the path is parameterized by λ which runs from 0 to 1, then the length is given by

$$\int_0^1 \sqrt{\pm g_{\mu\nu} t^\mu(\lambda) t^\nu(\lambda)} d\lambda \tag{2.3}$$

Finally, the metric gives us a notion of parallel transport, a process of dragging a tensor along a path in such a way that is does not change over unfunutesimal steps. Mathematically, this constraint is enforced by saying that the covariant derivative of the tensor along the tangent vector to the path at each point must be zero:

$$t^\sigma \nabla_\sigma T^{\alpha...\beta}_{\mu...\nu} = 0 \tag{2.4}$$

The connection coeffecients follow from the metric by what is called the metric compatibility condition, which is essentially that the metric is parallel transported along any path:

$$\nabla_\lambda g_{\mu\nu} = 0 \tag{2.5}$$

Lastly, I must define the hodge star operator. Rank p tensors which are completely antisymmetric (meaning it is equal to minus itself under the exchange of any two indicies,

e.g. $T_{ijk} = -T_{jik}$) are called p-forms. When working in n dimensions, the hodge star operator is denoted by $\star$, takes p-forms to $(n-p)$-forms as follows:

$$(\star T)_{\mu\ldots\nu} = \frac{1}{p!}\epsilon_{\mu\ldots\nu}{}^{\sigma\ldots\rho}T_{\sigma\ldots\rho} \tag{2.6}$$

where there are $n-p$ indices between and including μ and ν, and p indices between and including σ and ρ. ϵ is a totally antisymmetric tensor with as many indices as there are dimensions, and in four dimensions $\epsilon_{0123} = \det(g)$ in whatever basis ϵ is being represented in.

In the next section, we will recast Electromagnetism in the language of tensors, which will prepare us to treat General Relativity and to understand the analogies necessary to motivate Gravito-Electromagnetism.

Chapter 3

Electromagnetism Reboot

3.1 Special Relativity

Consider a 4 dimensional manifold that has the same global structure as $\mathbb{R}^4$, and is equipped with a metric. Because it is globally $\mathbb{R}^4$, there exist global coordinates, and suppose the metric is such that there exists some global coordinates, call them x^0, x^1, x^2, and x^3, such that when expressed in those coordinates the metric components are

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

This is the metric of what is called Minkowski spacetime, and when the metric takes this form we call it $\eta_{\mu\nu}$. Expressed in these coordinates, then, $\eta = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. By convention, the coordinate indices range from 0 to 3, instead of from 1 to 4.

3.2 Geodesics

The paths that free particles follow through spacetime are called geodesics (free meaning not being acted on by any forces, so geodesics will serve as our notion of straight lines). A geodesic is a path that parallel transports its own tangent vector. If we are given some point in the manifold, and some element of that point's tangent space, called the initial vector, we can calculate a geodesic as follows. We parallel transport the initial vector by some infinitesimal amount in the direction that it's pointing. Then we parallel transport

that new vector by some infinitesimal amount in the direction it's pointing, and so on. The equation that describes this is essentially the parallel transport equation, applied to its own tangent vector, i.e. if a path is geodesic than its tangent vector t^μ satisfies

$$t^\mu \nabla_\mu t^\nu = 0 \tag{3.2}$$

at each point along the path. If one thinks of the tangent vector as a velocity to the path, then its derivative is like its acceleration. In this sense, the geodesic equation is a special case of Newton's second law, saying that in the absense of force the acceleration is 0, i.e. $F = 0 = ma \Rightarrow a = 0$. If an observer (such as a person) is following a geodesic, we call them an inertial observer. It is precisely for these inertial observers, in the coordinates that they naturally construct, that the metric takes the form of η. For these observers, the vector associated with the one-form dx^0 appears to be pointing in the time direction, while the others span three dimensional space. This observer can think of $dx^0 = cdt$, where c is the speed of light, or rather just $dx^0 = dt$ as from now on we will set $c = 1$. A different inertial observer can only differ from the first by having a different velocty. Moving between these two observers' rest frames is called a Lorentz Transformation, a special kind of coordinate transformation. The nature of these transformations means that different observers will disagree about what the time direction and what the spatial directions are, but because their differences are only a coordinate transformation, they still agree on the underlying geometry, meaning they agree on what the metric tensor is, but simply have expressed it in different coordinates.

We could use many different quantities to parameterize a geodesic path, such as time (as long as the observer never goes back in time!), but the simplest parameterization turns out to be the total length of the path travelled so far as measured by the metric, using the definition for the length of a path in the previous section. Although we have expressed the metric in a coordinate dependent way, when it acts on the two tangent vectors, the resulting scalar does not depend on coordinate choice, and so in any coordinate system the path will have the same length (although different observers may disagree as to how much time has passed versus how far in space the particle has travelled). We call this length the proper time, τ, because it is how much time passes in the rest frame of the particle travelling on the path, and its value at some point p along the path is defined by

$$\tau = \int_0^s \sqrt{-\eta_{\mu\nu} t^\mu t^\nu} d\lambda \tag{3.3}$$

where λ is any arbitrary parameterization of the curve, t is the tangent vector to the curve, and s is the value of λ such that the curve is at p at parameter value s. Here I'm assuming that the quantity $\eta_{\mu\nu} t^\mu t^\nu$ is negative, is the case for the tangent vector of the

geodesic of any massive particle. We call these geodesics timelike. Massless particles, such as photons, follow null geodesics, meaning that their tangent vector is always length zero, because along this path $c^2dt^2 = dx^2 + dy^2 + dz^2$. There exist spacelike geodesics, for which $\eta_{\mu\nu}t^\mu t^\nu > 0$, but this corresponds to faster than light travel, for which there is no experimental evidence. When a path is expressed using this parameterization, it follows from the above definition that its tangent vector always satisfies $t^\mu t_\mu = -1$. This means that any inertial observer is always travelling at the same 'speed' through spacetime, and in their rest frame, they are not moving through space at all, so it is entirely through time.

3.3 The Electromagnetic Field Tensor

In this picture of spacetime as a manifold, electric and magnetic fields lose their absolute signifigance, as different observers would disagree on what they are. For instance, a line of charge at rest, resulting in only an electric field for one observer, would appear to create a magnetic field to another intertial observer who is moving along the direction of the line of charge, as in this frame the line would be a current. These two quantities are melded into one observer-independent (read: coordinate independent) object, called the electromagnetic field tensor. We define the electromagnetic field tensor as follows: if in some inertial coordinate system, the electric field is measured to have components E_i and the magnetic field is measured to have components B_j, then the components of the electromagnetic field tensor $F^{\mu\nu}$ in those coordinates are

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_z & 0 \end{pmatrix} \tag{3.4}$$

In these coordinates, the component of our observer's velocity u is $u^\mu = (1, 0, 0, 0)$, i.e. we are in that observer's rest frame. The electric field, as measured by this observer, can be reconstructed from F by

$$E_\mu = F_{\mu\nu}u^\nu \tag{3.5}$$

Additionally, we can use the hodge star operator to compactly express the magnetic field in the frame of this observer:

$$B_\mu = (\star F)_{\mu\nu}u^\nu = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}u^\nu \tag{3.6}$$

Since the determinant of diag$(-1,1,1,1)$ is simply -1, ϵ above is just the completely antisymmetric $(0,4)$ tensor with $\epsilon_{0123} = +1$. The decomposition of the field tensor into an electric and magnetic part is a useful computational tool, since it is easier to visualize and calculate with electric and magnetic fields than the full field tensor. Notice how the first component of each, E_0 and B_0, is automatically zero. However, we see here how the quantities themselves are frame dependent. Specifically, in some other frame, the new electric and magnetic fields would be linear combinations of the original electric and magnetic fields. The get 'mixed together', similar to how space and time get 'mixed together' when changing coordinates between the rest frames of observes moving at different speed through space. The elcectric and magnetic fields, much like space and time, are two sides of the same coin.

3.4 Maxwell's Equations

Maxwell's equations can be written elegantly in this formalism. First, we'll rewrite their familiar form in terms of the tensors B_μ and E_μ, and then condense these four equations into two equations for $F_{\mu\nu}$. To simplify them a bit, I'll first define the current four-vector, which in the frame of our observer is given by $J^\mu = (\rho, J_1, J_2, J_3)$. Then,

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \rightarrow \partial_i E^i = \mu_0 J^0 \tag{3.7}$$

$$\nabla \cdot B = 0 \rightarrow \partial_i B^i = 0 \tag{3.8}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \rightarrow \epsilon_i{}^{jk} \partial_j E_k = -\partial_0 B_i \tag{3.9}$$

$$\nabla \times B = \mu_0 (J + \epsilon_0 \frac{\partial E}{\partial t}) \rightarrow \epsilon_i{}^{jk} \partial_j B_k = \mu_0 (J_i + \epsilon_0 \partial_0 E_i) \tag{3.10}$$

where μ_0 and ϵ_0 are constants, and in this context $\epsilon_i{}^{jk}$ is completely antisymmetric, $\epsilon_{ijk} = +1$, and its indices run from 1 to 3 (excluding 0. In all future instances, latin letters such as i and j will indicate a range of 1 to 3, while greek letters such as μ and ν will range from 0 to 3). It turns out that we can repackage these four equations of E and B as two equations of F as follows:

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \Leftrightarrow \partial_i E^i = \mu_0 J^0 \quad , \quad \epsilon_i{}^{jk} \partial_j B_k = \mu_0 (J_i + \epsilon_0 \partial_0 E_i) \tag{3.11}$$

$$\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0 \Leftrightarrow \partial_i B^i = 0 \quad , \quad \epsilon_i{}^{jk} \partial_j E_k = -\partial_0 B_i \tag{3.12}$$

It can be tricky to see just how these equations are equivalent, but we will show now that they are. For the first pair, we split $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$ into two equations: $\partial_\mu F^{\mu 0} = \mu_0 J^0$ and $\partial_\mu F^{\mu i} = \mu_0 J^i$. The first of these simply says that $\partial_\mu E^\mu = \mu_0 J^0$, and because $E^0 = 0$, this is just $\partial_i E^i = \mu_0 J^0$. To treat the second equation, we note that $\epsilon^{ijk} \partial_j B_k =$

$\frac{1}{2}\epsilon^{ijk}\epsilon_{klm}\partial_j F^{lm} = \frac{1}{2}\delta^{[i}_{[l}\delta^{j]}_{m]}\partial_j F^{lm} = \partial_j F^{ij}$. Therefore, $\mu_0 J^i = \partial_\mu F^{\mu i} = \partial_0 F^{0i} i \partial_j F^{ij} = -\partial_0 E^i + \epsilon^{ijk}\partial_j B_k$. Since the indices only run from 1 to 3, and we are working in flat spacetime, we can raise and lower indices without changing anything else. Therefore, the equation above implies that $\epsilon_i{}^{jk}\partial_j B_k = \mu_0 J_i + \partial_0 E_i$, and by setting $c = 1$ in the definition of $F_{\mu\nu}$ we have lost the correct constants, but the form is still the same. For the second pair, first note that $\partial_i B^i = \frac{1}{2}\epsilon^{ijk}\partial_i F_{jk}$. But ϵ^{ijk} is completely anti-symmetric, and F_{jk} is anti-symmetric, so $\epsilon^{ijk}\partial_i F_{jk} = \epsilon^{ijk}(\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij})$ which is just zero, so we have that $\partial_i B^i = 0$. Additionally, we have that $0 = \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} \Rightarrow 0 = \epsilon_k c^{ij}(\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i}) = 2\partial_0 B_k + \epsilon_k{}^{ij}(\partial_i F_{j0} - \partial_j F_{i0}) = 2\partial_0 B_k + 2\epsilon_k{}^{ij}\partial_i E_j \Rightarrow \epsilon_k{}^{ij} = -\partial_0 B_k$.

Now, these equations apply only on flat four-dimensional spacetime. But since we've written them in such a way, in order to generalize these equations to any manifold, we need only to exchange the derivative operator on flat spacetime, ∂_μ, with the derivative operator on our manifold, ∇_μ. We will also make a simplification, by utilising a commonly used convention in general relativity. If we wish to take the symmetric part of a tensor $T_{\mu\nu}$, we simply have to 'kill off' its anti-symmetric part, by constructing the quantity $\frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$. The factor of $\frac{1}{2}$ is a normalization constant, thrown in so that if $T_{\mu\nu}$ is already symmetric, then its symmetric part is just itself. Since we often want to consider just the symmetric or just the anti-symmetric part of a tensor, we introduce the compactifying notation $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$ and $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$. The act of putting parentheses or brackets around tensor indices is called 'symmetrizing' or 'anti-symmetrizing', respectively. If one is anti-symmetrizing more than two indices, one simply adds all the even permutations together, and subtracts all odd permutations of the indices, and then uses (number of terms)$^{-1}$ as a normalization factor. The symmetrization process is identical, except odd permutations of the indices are subtracted instead of added. As a particularly relevant example, $T_{[\mu\nu\sigma]} = \frac{1}{6}(T_{\mu\nu\sigma} - T_{\mu\sigma\nu} + T_{\sigma\mu\nu} - T_{\sigma\nu\mu} + T_{\nu\sigma\mu} - T_{\nu\mu\sigma})$.

Consider now $\partial_{[\mu}F_{\nu\sigma]}$. We can see from the definition that $F_{\mu\nu}$ is anti-symmetric, and so $\partial_{[\mu}F_{\nu\sigma]} = \frac{1}{3}\partial_\mu F_{\nu\sigma} + \partial_\mu F_{\nu\sigma} + \partial_\mu F_{\nu\sigma}$, which is just the left side most side of (3.12). With this convenient abbreviation, and taking $\partial \to \nabla$, Maxwell's equations on any manifold are given by

$$\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu \tag{3.13}$$

$$\nabla_{[\mu}F_{\nu\sigma]} = 0 \tag{3.14}$$

Therefore, given some some vector J representing the charge distribution on the manifold, using Maxwell's equations one can work out F, which in turn yields the force on any charged particle due to that charge distribution. For a particle of charge q with a velocty v^μ, the tensor version of the Lorentz Force Law, $\vec{\text{Force}} = q(\vec{E} + \vec{v} \times \vec{B})$, is $(\text{Force})^\mu = q u_\nu F^{\mu\nu}$.

Motivated by Maxwell's equations, the magnetic field is often written as $\vec{B} = \vec{\nabla} \times \vec{A}$ and the electric field as $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$, where ϕ is a scalar field and $\vec{A}$ is a vector field, since this gaurantees two of Maxwell's equations, namely $\vec{\nabla} \cdot B = 0$ and $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. If we combine ϕ and $\vec{A}$ into a single four-vector by placing ϕ in the time component, creating the vector four-potential $A^\mu = (\phi, A^1, A^2, A^3)$, then it turns out that in flat spacetime $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. There is an important operator from p-forms to $p+1$-forms, called the exterior derivate (see the Appendix section **??** for more information), which is denoted by d, and for a p-form $T_{\alpha\beta\ldots\gamma}$, $(dT)_{\sigma\alpha\beta\ldots\gamma} = \nabla_{[\sigma} T_{\alpha\beta\ldots\gamma]}$. We see then, that $F = dA$, for $(dA)_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\lambda{}_{\mu\nu} A_\lambda - \partial_\nu A_\mu + \Gamma^\lambda{}_{\nu\mu} A_\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$, because the connection coeffecients are symmetric in the bottom two indices by construction (and by symmetry of the metric). This means $\Gamma^\lambda{}_{\nu\mu} = \Gamma^\lambda{}_{\mu\nu}$, or to use our fancy new notation, simply that $\Gamma^\lambda{}_{[\nu\mu]} = 0$. This implies, then, that $dF = 0$, because $F = dA$, and $d^2 = 0 \Rightarrow 0 = d^2 A = dF$. But by the definition of d, $(dF)_{\mu\nu\lambda} = \nabla_{[\mu} F_{\nu\lambda]} = 0$ is half of Maxwell's equations! I told you $d^2 = 0$ had profound physical implications; from this viewpoint, we can view two of Maxwell's equation as a consequence of describing the force as tensor on a manifold.

The other half of the equations, $\nabla_\mu F^{\mu\nu} = J^\nu$, imply conservation of charge. In 3-dimensional vector calculus language, conservation of charge is expressed by the continuity equations, $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$, where ρ is the charge density and J is the 3-current. This essentially says the rate of charge of charge in some infinitesimal volume is equal to the amount that flows out of it. This can be expressed neatly in terms of the current four-vector: $\partial_\mu J^\mu = \partial_\mu J_\nu \eta^{\mu\nu} = -\partial_0 \rho + \partial_i J^i = 0$. For the observer whose rest frame is these coordinates, ∂_0 is a time derivative and ∂_i are spatial derivatives, so this is equivalent to the continuity equation. The generalized continuity equation, then, is

$$\nabla_\mu J^\mu = 0 \tag{3.15}$$

It follows from the anti-symmetry of F, and the definition of the connection coeffecients, that $\nabla_\mu \nabla_\nu F^{\mu\nu} = 0$, which means that conservation of charge follows from the first pair of Maxwell's equations.

There is another way to view this equation, which is more analogous to the first equation. The algebra gets a bit messy, but it's pretty straightforward to show that

$$\nabla_\mu F^{\nu\mu} = (\star d \star F)^\nu \tag{3.16}$$

In these forms, Maxwell's equations are simply

$$dF = 0 \tag{3.17}$$

$$\star d \star F = J \tag{3.18}$$

Here the symmetries of the equations are obvious, and we gain a useful clue for constructing gravity. Maxwell's equations tie the field, F, to the source, J, via the exterior derivative. But the exterior derivative of the field tensor is 0; one must look to the exterior derivative of the dual of the field tensor in order to find a relationship to the source. Keep this in mind in the next chapter, when, after doing a tiny bit more math, we construct a geometric theory of gravity.

Chapter 4

Gravity

4.1 The Riemann Tensor

Before we turn our attention to gravity, we must develop a little bit more math. Suppose we've got a manifold with a metric defined on it. As defined in Appendix **??**, we know how to take a derivative, denoted by ∇, of a tensor field:

$$\nabla_\sigma T^{\mu\ldots\nu}_{\alpha\ldots\beta} = \partial_\sigma T^{\mu\ldots\nu}_{\alpha\ldots\beta} + \Gamma^\mu_{\sigma\lambda} T^{\lambda\ldots\nu}_{\alpha\ldots\beta} + \ldots + \Gamma^\nu_{\sigma\lambda} T^{\mu\ldots\lambda}_{\alpha\ldots\beta} - \Gamma^\lambda_{\sigma\alpha} T^{\mu\ldots\nu}_{\lambda\ldots\beta} - \ldots - \Gamma^\lambda_{\sigma\beta} T^{\mu\ldots\nu}_{\alpha\ldots\lambda} \tag{4.1}$$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}(\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu}) \tag{4.2}$$

This derivative operator will be referred to as the covariant derivative because it does not depend on a choice of coordinates; this is because it includes the derivatives of the components of the tensor, as well as the derivatives of the basis vectors, which is contained in the Γ terms. It allows us to define parallel transport of a tensor field along a path, meaning physically that, for a vector, its tangent vector at any point is parallel to its tangent vector at all other points on the path. Mathematically, this means that the derivative of the vector field vanishes along the path, and this expressions is easily generalized to all tensors:

$$t^\sigma \nabla_\sigma T^{\mu\ldots\nu}_{\alpha\ldots\beta} = 0 \tag{4.3}$$

where t^σ is the tangent vector field of the path.

Partial derivatives commute, i.e. $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$, but what about our covariant derivative? In general, $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) T^{\sigma\ldots\rho}_{\alpha\ldots\beta} \neq 0$, so we will define a new tensor which quantifies

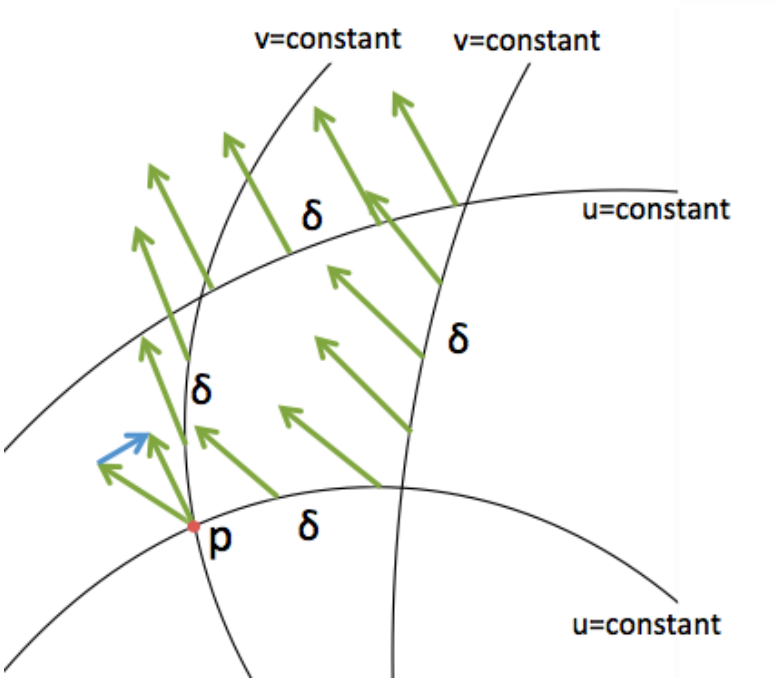

FIGURE 4.1: a geometric interpretation of the riemann tensor

the failure of the covariant derivative to commute when acting on a dual vector:

$$R^{\sigma}{}_{\rho\mu\nu}v^{\rho} = (\nabla_{\nu}\nabla_{\mu} - \nabla_{\mu}\nabla_{\nu})v_{\sigma} \tag{4.4}$$

This tensor R is called the Riemann Curvature Tensor (or just the Riemann tensor for short), and will turn out to be quite important in gravity over all, as well as particularly relevant to this thesis. Geometrically, the vector $R^{\sigma\rho}{}_{\mu\nu}w_{\sigma}u^{\mu}v^{\nu}$ at some point p can be interpereted as the vector that is the difference between w^{ρ} at p, and the new vector $w^{'\rho}$ at p, which is simply w^{ρ} at p after being parallel transported about an infinitesimally small rectangle generated by moving some infinitesimal amount δ along the vector field u, then δ along v, then $-\delta$ along u, and finally $-\delta$ along v. By expanding out the covariant derivatives, one can derive the following expression for $R^{\sigma}_{\rho\mu\nu}$ [11]:

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_{\mu}\Gamma^{\sigma}{}_{\nu\rho} - \partial_{\nu}\Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\lambda}\Gamma^{\lambda}_{\mu\rho} \tag{4.5}$$

This tensor obeys a number of important symmetries, some obvious from construction, and others more subtle. These include:

$$R_{\mu\sigma\nu\rho} = -R_{\sigma\mu\nu\rho} \tag{4.6}$$

$$R_{\mu\sigma\nu\rho} = -R_{\mu\sigma\rho\nu} \tag{4.7}$$

$$R_{[\mu\sigma\nu]\rho} = 0 \tag{4.8}$$

$$\tag{4.9}$$

from which it is possible to derive another useful property, called 'pair exchange symmetry':

$$R_{\mu\sigma\nu\rho} = R_{\nu\rho\mu\sigma} \tag{4.10}$$

From the Riemann tensor, we define a few more useful quantities. We have the Ricci tensor, $R_{\mu\nu}$, defined by

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \tag{4.11}$$

and the Ricci scalar R defined by

$$R = R^{\lambda}{}_{\lambda} \tag{4.12}$$

I will take this opportunity to define the trace of a rank 2 tensor. For any rank 2 tensor $T_{\mu\nu}$, its trace is $T^{\lambda}{}_{\lambda}$, and is denoted by $Trace(T)$ or simply T. This coincides with the definition of the trace of a matrix, which is simply the sum of the diagonal elements.

Additionally, we will define the Weyl tensor $C^{\mu\nu}{}_{\rho\sigma}$ by [12]

$$C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} - 2\delta^{[\mu}{}_{[\rho}R^{\nu]}{}_{\sigma]} + \frac{1}{3}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}R \tag{4.13}$$

Conceptually, the Riemann tensor completely describes the curvature of spacetime at a point. The Weyl tensor describes the part of the spacetime curvature that is purely gravitational, i.e. caused by distant matter rather than local matter. These are a lot of definitions at once, but all of these quantities will prove important shortly.

By construction, the Riemann tensor also obeys what is called the Bianchi identity [11], which is that

$$\nabla_{[\lambda}R_{\mu\sigma]\nu\rho} = 0 \tag{4.14}$$

Notice that this looks much like (3.14), except with two extra indices. We will use this similarity as hint for constructing a geometric theory of gravity.

4.2 General Relativity

General Relativity erases the notion of gravity as a force, and instead describes it through geometry. According to General Relativity, energy (including mass, the electromagnetic field, and other field, including the gravitational field itself) warps spacetime around it, and so when it appears to us as if an object is acting under the influence of 'gravity', in reality that object is being acted on by no force, and is instead following a straight line, or a geodesic, just as one would expect it to if it was not being acted on by a force. The apparent gravitational interaction between distant objects is caused by the curving of spacetime that they each affect. The question is, what is the precise relationship between energy distribution and curvature?

First of all, how do we quantify our energy distribution? We will utilize the stress-energy tensor T, which is a symmetric rank 2 tensor on spacetime. Expressed in the coordinates of some observers rest frame, it is interpereted as follows:

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} \tag{4.15}$$

where in this observer's rest frame T^{00} is the energy density, T^{0i}, or equivalently T^{j0}, are the momentum density, T^{ij} is the shear stress, for $i \neq j$, and T^{ii} is the pressure. For the electromagnetic field, $T_{00} = E^2 + B^2$, and T_{0i} is the poynting vector. This tensor encodes all of the information about the distrubution of energy (including mass) throughout the manifold (universe) on which it is defined. What we seek, then, is a relationship between the metric and the stress energy tensor. In this way, the distribution of energy dictates the shape of spacetime, causing a gravitational effect. Because energy anywhere affects spacetime everywhere, we know that equations governing the relationship between $T_{\mu\nu}$ and $g_{\mu\nu}$ must be partial differential equations. By comparison with Newtonian gravity, one is lead to believe this equation is second order. There are many possible forms this equation could take, so we must use elegance as a guiding principle. Of course, there's no reason there ought to be an 'elegant' relationship between field and source, but this notion has proved useful in many areas of physics, and gravity turns out to be no exception.

We have a restriction on T, which is that energy must be conserved. Much like charge conservation for J, this is expressed as

$$\nabla_\mu T^{\mu\nu} = 0 \tag{4.16}$$

This is a very important feature of our universe, so it would be sure be nice if this wasn't a constraint that we had to add in later, but was instead a consequence of gravity. Suppose we have found some rank 2 tensor (when we have a metric, the distinction between vectors and dual vectors becomes less important because we can raise and lower indices), so rather than call a tensor type (p, q), we simply say it is rank $p + q$) that can be calculated from the metric, call it G, which when related to T by $G_{\mu\nu} \propto T_{\mu\nu}$ yields an accurate theory of gravity. Then, if it were the case that $\nabla_\mu G^{\mu\nu} = 0$, energy conservation would be automatically implied by our theory of gravity. So let's try to find some tensor G that satisfies this condition.

We have an additional clue from electromagnetism; the electromagnetic field tensor is the thing that is tied to the source, and its anology in gravity is the Riemann tensor. The rank 2 tensor that we can make from the Riemann tensor is the Ricci tensor, but it is not true that $\nabla_\mu R^{\mu\nu} = 0$. But this is okay, because, taking a cue from electromagnetism, it's not the field tensor that we want to tie to the source, but instead its dual. The Riemann tensor is rank 4, but the symmetry $R_{\mu\nu\sigma\rho} = R_{\sigma\rho\mu\nu}$ hints that we should take the dual of each pair of indices separately, resulting in another tensor of rank 4. This is denoted by

$$(\star R \star)_{\mu\nu\sigma\rho} = \frac{1}{4}\epsilon_{\mu\nu}{}^{\lambda\theta}\epsilon_{\sigma\rho}{}^{\phi\psi}R_{\lambda\theta\phi\psi} \tag{4.17}$$

If we contract this tensor in an anlogous way as to form the Ricci tensor from the Riemann tensor, we get the Einstein Tensor, denoted G. In terms of the Ricci tensor and scalar,

$$G_{\mu\nu} = (\star R \star)^\sigma{}_{\mu\sigma\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{4.18}$$

But, recall the Bianchi identity, $\nabla_{[\lambda}R_{\mu\nu]\sigma\rho} = 0$. What happens if we contract this identity twice? We get that

$$\begin{aligned}
0 &= g^{\mu\sigma}g^{\lambda\rho}\nabla_{[\lambda}R_{\mu\nu]\sigma\rho} && (4.19)\\
&= g^{\mu\sigma}g^{\lambda\rho}(\nabla_\lambda R_{\mu\nu\sigma\rho} + \nabla_\mu R_{\nu\lambda\sigma\rho} + \nabla_\nu R_{\lambda\mu\sigma\rho}) && (4.20)\\
&= \nabla_\rho R^\sigma{}_{\nu\sigma}{}^\rho + \nabla_\sigma R_\nu{}^{\rho\sigma}{}_\rho + \nabla_\nu R^{\rho\sigma}{}_{\sigma\rho} && (4.21)\\
&= 2\nabla_\rho R_\nu{}^\rho - \nabla_\nu R && (4.22)
\end{aligned}$$

or equivalently, that $\nabla_\rho R^{\nu\rho} - \frac{1}{2}\nabla^\nu R = 0$, because the derivative is metric compatible, so we can take g in and out of derivatives freely. But this means that

$$\nabla_\nu G^{\nu\mu} = \nabla_\nu (R^{\nu\mu} - \frac{1}{2} g^{\mu\nu} R) = \nabla_\nu R^{\nu\mu} - \frac{1}{2}\nabla^\mu R = 0 \tag{4.23}$$

by the Bianchi identity! This is quite exciting; by analogy with electromagnetism, we have been led directly to a tensor whose divergence vanishes, suggesting that curvature and source should be tied together by $G_{\mu\nu} \propto T_{\mu\nu}$. If we use the constant 8π, yielding what is usually called the Einstein Equation,

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{4.24}$$

(in geometrized units, $G = c = 1$), then we get a theory of gravity that reduces to familiar Newtonian gravity in the correct limits, and also predicts testable deviations from Newtonian gravity, which have been confirmed many times by experiment [13].

So what kind of solutions exist to Einstein's equation? It's a set of ten coupled second order nonlinear differential equations, so exact solutions are few and far between. Of course, flat spacetime, where $g_{\mu\nu} = \eta_{\mu\nu}$ is an exact solution to the vacuum equation ($T = 0$). But it is not the only vacuum solution; we will explore some other vacuum solutions in great detail shortly. However, first I must push the analogy between Electromagnetism and Gravity a bit farther, in order to give us an additional and less widely used tool when studying these solutions.

4.3 Gravito-Electromagnetism

We used the analogy between the electromagnetic field tensor F and the Riemann tensor R in order to help motivate general relativity in the previous section. F can be decomposed into its Electric and Magnetic parts, which are easier to calculate with and visualize. It is natural to ask, then, can the same be done to the Riemann tensor? Let us try; the electromagnetic field tensor is decomposed into its electric (E) and magnetic (B) parts relative to some observer with four-velocity u by

$$E^\mu = u_\nu F^{\mu\nu} \qquad B^\mu = -u_\nu (\star F)^{\mu\nu} = \frac{1}{2} u_\nu \epsilon^{\nu\mu\sigma\rho} F_{\sigma\rho} \tag{4.25}$$

We would now like to construct similar quantities for gravity. Instead of the Riemann tensor, we will use the Weyl tensor, but this makes little difference to us because the spacetimes that we will study in this thesis are vacuum solutions, and under vacuum $R_{\mu\nu\sigma\rho} = C_{\mu\nu\sigma\rho}$. By analogy, we define the Gravito-Electric E and Gravito-Magnetic B

fields, relative to an observer with velocity v, by [1]

$$E_{\mu\nu} = C_{\mu\sigma\nu\rho}u^\sigma u^\rho \tag{4.26}$$

$$B_{\mu\nu} = -(\star C)_{\mu\sigma\nu\rho}u^\sigma u^\rho = \frac{1}{2}u^\sigma \epsilon_{\mu\sigma\lambda\theta}C^{\lambda\theta}{}_{\nu\rho}u^\rho \tag{4.27}$$

We have a condition on $R_{\mu\nu\rho\sigma}$, the Bianchi identity, which translates to a condition on $C_{\mu\nu\rho\sigma}$. We also have a relationship between $R_{\mu\nu}$ and $T_{\mu\nu}$, Einstein's equation ($G_{\mu\nu} = 8\pi T_{\mu\nu}$). We can therefore recast both of these equations in terms of the Gravto-Electric (GE) and Gravito-Magnetic (GM) fields, and what we find is, surprisingly, equations much like Maxwell's equations for electromagnetism. In curved spacetime, Maxwell's equations for the Electric and Magnetic field, in the reference frame of some inertial observer with velocity v, is given by [14]

$$\nabla_\mu E^\mu = -2\omega^\lambda B_\lambda - J_\sigma v^\rho \tag{4.28}$$

$$\nabla_\mu B^\mu = 2\omega^\lambda E_\lambda \tag{4.29}$$

$$v^\mu \nabla_\mu E_\rho - v^\mu \epsilon_{\mu\rho\lambda\sigma}\nabla^\lambda B^\sigma = -\frac{2}{3}\Theta E_\rho + \sigma_{\rho\lambda}E^\lambda - v^\theta \epsilon_{\theta\rho\sigma\mu}\omega^\sigma E^\mu - P^\lambda_\rho J_\lambda \tag{4.30}$$

$$v^\mu \nabla_\mu B_\rho + v^\mu \epsilon_{\mu\rho\lambda\sigma}\nabla^\lambda E^\sigma = -\frac{2}{3}\Theta B_\rho + \sigma_{\rho\lambda}B^\lambda - v^\theta \epsilon_{\theta\rho\sigma\mu}\omega^\sigma B^\mu \tag{4.31}$$

where $P_{\sigma\rho} = g_{\sigma\rho} + n_\sigma n_\rho$, where n is a unit time-like vector, is a projection operator, which takes a tensor and kills off its time part, projecting it down into the space of the observer who'se time direction is n; in an observers rest frame, then, $n = u$, where u is the four-velocity of the observer. Additionally, J is the current four-vector, $\Theta = P^{\mu\nu}\nabla_\mu v_\nu$, $\sigma_{\mu\nu} = (P^\theta_{(\mu}P^\phi_{\nu)} - \frac{1}{3}P^{\theta\phi}P_{\mu\nu})P^\lambda_\theta \nabla_\lambda v_\phi$, and $\omega_\mu = -\frac{1}{2}v^\lambda \epsilon_{\lambda\mu\sigma\rho}\nabla^\sigma v^\rho$. In flat spacetime, where $\Theta = \omega = \sigma = 0$, these equations take the more familiar form. The Bianchi Identity and Einstein's equation, expressed in terms of the GE and GM fields, take the following form [14]:

$$\nabla_\sigma E^\sigma_\mu = -3w^\sigma B_{\mu\sigma} + v^\theta \epsilon_{\theta\mu\phi\rho}\sigma^\phi{}_\lambda B^{\rho\lambda} + \frac{1}{3}P^\sigma_\mu \nabla_\sigma \rho \tag{4.32}$$

$$\nabla_\sigma B^\sigma_\mu = 3w^\sigma E_{\mu\sigma} - v^\theta \epsilon_{\theta\mu\phi\rho}\sigma^\phi{}_\lambda E^{\rho\lambda} + (\rho + p)\omega_\mu \nabla_\sigma \rho \tag{4.33}$$

$$v^\lambda \nabla_\lambda E_{\mu\nu} - v^\theta \epsilon_{\theta\lambda\rho(\mu}\nabla^\lambda B_{\nu)}{}^\rho \tag{4.34}$$

$$= -\Theta E_{\mu\nu} + 3\sigma_{\lambda(\mu}E_{\nu)}{}^\lambda - \omega^\lambda \epsilon_{\lambda\sigma(\mu}E_{\nu)}{}^\sigma - \frac{1}{2}(\rho + p)\sigma_{\mu\nu} \tag{4.35}$$

$$v^\lambda \nabla_\lambda E_{\mu\nu} + v^\theta \epsilon_{\theta\lambda\rho(\mu}\nabla^\lambda B_{\nu)}{}^\rho \tag{4.36}$$

$$= -\Theta B_{\mu\nu} + 3\sigma_{\lambda(\mu}B_{\nu)}{}^\lambda - \omega^\lambda \epsilon_{\lambda\sigma(\mu}B_{\nu)}{}^\sigma \tag{4.37}$$

where ρ is the energy density as measured by the observer, i.e. $\rho = u^\mu u^\nu T_{\mu\nu}$. These equations are much more complex, but comparison with the analogous equations for Electromagnetism reveals some obvious similarities. In flat spacetime, where $\Theta = \omega =$

$\sigma = 0$, and in vacuum, where $J = \rho = 0$, the Electromagnetic equations are

$$\nabla_\mu E^\mu = 0 \tag{4.38}$$

$$\nabla_\mu B^\mu = 0 \tag{4.39}$$

$$v^\mu \nabla_\mu E_\rho - v^\mu \epsilon_{\mu\rho\lambda\sigma} \nabla^\lambda B^\sigma = 0 \tag{4.40}$$

$$v^\mu \nabla_\mu B_\rho + v^\mu \epsilon_{\mu\rho\lambda\sigma} \nabla^\lambda E^\sigma = 0 \tag{4.41}$$

The Gravitational equations, in this case, are

$$\nabla_\mu E^\mu_\nu = 0 \tag{4.42}$$

$$\nabla_\mu B^\mu_\nu = 0 \tag{4.43}$$

$$v^\mu \nabla_\mu E_{\rho\nu} - v^\mu \epsilon_{\mu\rho\lambda\sigma} \nabla^\lambda B^\sigma_\nu = 0 \tag{4.44}$$

$$v^\mu \nabla_\mu B_{\rho\nu} + v^\mu \epsilon_{\mu\rho\lambda\sigma} \nabla^\lambda E^\sigma_\nu = 0 \tag{4.45}$$

And now the parallel between these equations becomes striking. Evidentally, the GE and GM fields behave similarly near energy to the way the Electric and Magnetic fields behave near charge, but what about their physical interpretation?

Following [1], the GE field essentially describes the relative acceleration of two inertial observers. If observer A has four-velocty u, and ξ is an infinitesimally small vector, orthogonal to u (so it is entirely spatial), and at the tip of ξ is an inertial oberver B (they are infinitesimally separated, so that a vector can be thought of as traversing the space between them), then the relative acceleration of these observers is given intuitively $u^\mu \nabla_\mu (u^\nu \nabla_\nu \xi^\sigma)$, and this acceleration is given nicely in terms of the GE field:

$$u^\mu \nabla_\mu (u^\nu \nabla_\nu \xi^\sigma) = -E^\sigma{}_\lambda \xi^\lambda \tag{4.46}$$

This is similar to the tidal field of Newtonian gravity, which is

$$E_{ij} = \nabla_i \nabla_j \Phi \tag{4.47}$$

where the ∇_i is the gradient in Cartesian coordinates, and Φ is the Newtonian gravitational potential. This quantity is automatically trace free in vacuum, by the poisson equation $\nabla^2 \Phi = 0$.

The GM field has a similar, although less well known, physical interpretation. Imagine each observer has a unit spatial vector, call it σ^ν. The difference in these vectors is given by $\xi^\mu \nabla_\mu \sigma^\nu$, which is initially zero, but will evolve as the observers progress along their respective geodesics. The change of this diffence vector along the geodesics is given by $u^\nu \xi^\mu \nabla_\nu \nabla_\mu \sigma$. The relative frame dragging velocity, $\Delta\Omega$, is defined, in anlogy to classical

mechanics, as $u^\nu \xi^\mu \nabla_\nu \nabla_\mu \sigma = \Delta\Omega \times \sigma$. From this and the definition of σ, it follows that $\Delta\Omega = \sigma \times u^\nu \xi^\mu \nabla_\nu \nabla_\mu \sigma$. It turns out that the GM field describes this relative fram dragging:

$$\Delta\Omega_\mu = B_{\mu\nu}\xi^\nu \tag{4.48}$$

In a sense, rotating objects 'drag' nearby objects into co-rotation with them, and the GM field describes this dragging. For this reason, we call the GM field the frame-drag field.

Unlike the Electric and Magnetic fields, which are fields of vectors (rank 1 tensors), the GE and GM fields are rank 2 tensors. The GE and GM fields depend on the slicing of time and space anyway (meaning they are coordinate dependent, much like the Electric and Magnetic fields in Electromagnetism), and by construction the time components are zero in the rest frame of the observer who defines them, i.e.

$$E^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E_{11} & E_{12} & E_{13} \\ 0 & E_{12} & E_{22} & E_{23} \\ 0 & E_{13} & E_{23} & E_{33} \end{pmatrix} \qquad B^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & B_{11} & B_{12} & B_{13} \\ 0 & B_{12} & B_{22} & B_{23} \\ 0 & B_{13} & B_{23} & B_{33} \end{pmatrix} \tag{4.49}$$

as these tensors are also symmetric. Therefore, we consider these quanties as living in space, or more specifically, in some specific time slice, and think of them not as three-dimensional vectors at each point like the Electric and Magnetic fields, but as 3×3 matrices at each point, by throwing out the time components of the matrices above (which are the ones that are identically zero). Moreover, these matrices are symmetric by construction, so they are guaranteed to have a complete eigenbasis. Then the field can be thought of as its eigenvectors at each point, field strengths given by the associated eigenvalue (because eigenvectors are only a direction and don't carry a length with them). These are the eigendirections for the relative acceleration of inertial observers, and the magnitude of this acceleration is proportional to the eigenvalue. We call this field the tidal field.

We can use the eigenvectors at each point to create the 'field lines' of the GE and GM fields. Unsurprisingly, due to the Bianchi equations, which closely resemble Maxwell's equations of electromagnetism, the field lines from simple sources look much like electric and magnetic fields. Figure 1 shows the GE field lines from a central mass at rest. Red lines have negative eigenvalues, and blue lines have positive eigenvalues. The figures demonstrate how an extended observer would be stretched or squished, depending on her orientation with respect to the field lines. Figure 2 shows the GM field lines for a central mass which is spinning. Work in developing the GEM formalism and its field-line

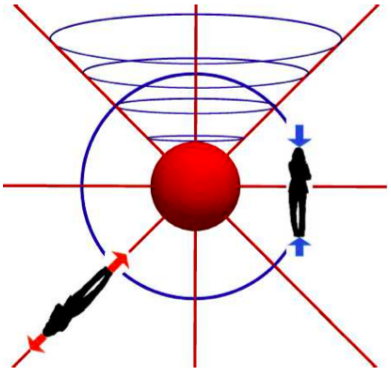

FIGURE 4.2: Gravito-Electric field lines from a central mass. Red lines indicate negative tendicity and blue lines indicate positive tendiciy. Image taken from [1]

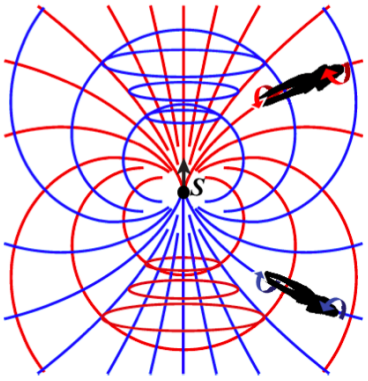

FIGURE 4.3: Gravito-Magnetic field lines from a spinning central mass. Red lines indicate negative vortivity and blue lines indicate positive vorticity. Image taken from [1]

visualizations is carried out in [1].

4.4 Petrov Classification

A related language used to analyze curvature is called the Petrov Classification. At each point in spacetime, there exist special null directions called 'principal null directions' [10]. Physically, they correspond to geometrically special directions through that point. If a cylinder of light were travelling through spacetime and through that point, its shape would be squished in some directions and stretched in others, making it non-circular at some places. However, cylinders of light that travel in the direction of any of these principle null directions will experience no shape change. It can be shown, although it is not obvious, that at least one of these directions must exist at any point in any spacetime. In fact, in general four such directions must exist, although they may

coincide. The defining equation for a principal null vector k is

$$k^{\lambda}k^{\xi}k_{[\sigma}C_{\rho]\lambda\xi[\mu}k_{\nu]} = 0 \tag{4.50}$$

There are always four solutions to the above equation, although they may not all be distinct. The degeneracy of the principal null directions constitute what is called the Petrov Classification. A point in a spacetime with four distinct principal null directions is called Petrov Type I. If two coincide, then it is Petrov Type II. If there are two pairs of principal null directions which coincide, it is Petrov Type D. If three coincide, then it is Petrov Type III. If all four coincide, it is Petrov Type N, and in the case of (a conformally) flat spacetime, where the Weyl tensors vanishes and every null direction is a principle null direction, then it is said to be Petrov Type O.

The degeneracy of the principal null vectors are known to match the degeneracy of the solutions λ to the quartic polynomial

$$\Psi_4\lambda^4 + 4\Psi_3\lambda^3 + 6\Psi_2\lambda^2 + 4\Psi_1\lambda + \Psi_0 = 0 \tag{4.51}$$

where the Ψ_i are called the Weyl Scalars, and are defined as follows. First choose a complex null tetrad, meaning choose two real vectors, l^α and n^α, and one complex vector, m^α such that they are each null ($l_\alpha l^\alpha = 0$, $n_\alpha n^\alpha = 0$, $m_\alpha m^\alpha = 0$) and such that $l^\alpha n_\alpha = -1$ and $m^\alpha \bar{m}_\alpha = 1$. Then the Weyl Scalars can be taken from the Weyl tensor by

$$\Psi_0 = C_{\alpha\beta\gamma\delta}l^\alpha m^\beta l^\gamma m^\delta \tag{4.52}$$

$$\Psi_1 = C_{\alpha\beta\gamma\delta}l^\alpha n^\beta l^\gamma m^\delta \tag{4.53}$$

$$\Psi_2 = C_{\alpha\beta\gamma\delta}l^\alpha m^\beta \bar{m}^\gamma n^\delta \tag{4.54}$$

$$\Psi_3 = C_{\alpha\beta\gamma\delta}l^\alpha n^\beta \bar{m}^\gamma n^\delta \tag{4.55}$$

$$\Psi_4 = C_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta \tag{4.56}$$

Notice that, in four dimensions, the Weyl tensor has 10 free real components, and so the five Weyl scalars, with their real and imaginary parts, carry the same amount of information. Indeed, none of this information is redundant, so the Weyl Scalars encode all the information in the Weyl tensor. Of course, picking a different null tetrad would change the scalars and therefore the solutions to the quartic, but it will not change the degeneracy of the solutions, which is all we are looking for. In other contexts, such as studying gravitational waves, picking the the right null tetrad can lead to significant physical meaning of the scalars [15], but we are not concerned with such things here.

A natural question to ask is whether or not a spacetime can be 'almost' some Petrov

Type, if two principal null directions are not the same but are 'near' to one another. The short answer is no, because one cannot measure the angle between null vectors in a coordinate independent way. However, [16] defined

$$\Delta_{ij} =\mid \lambda_i - \lambda_j \mid \tag{4.57}$$

the differences between the roots, and interpreted this naturally as the 'nearness' of two roots, and therefore as the 'nearness' of the spacetime to some Petrov type. As noted, though, this quantity depends on the choice of coordinates. This means that at some point one observer might find Δ_{ij} very close to zero for two distinct i and j, while another observer may find Δ_{ij} to be very large at that same point for those same roots. Owen partially fixed this problem by introducting the new quantity [10]

$$\Theta_{ij} = 2\text{arcsin}\left[\frac{\mid \lambda_i - \lambda_j \mid}{\sqrt{(1+\mid \lambda_i \mid^2)(1+\mid \lambda_j \mid^2)}}\right] \tag{4.58}$$

which are invarant under spatial coordinate transformations, transformations which leave the time-like vector unchanged. In general, the dependence of this quantity on the choice of slicing of space and time makes these quantities observer dependent. However, for a homogeneous universe, this is not a problem, as in general there is only one choice of time coordinate for which the universe is homogeneous. This particular time coordinate is therefore a 'preferred' slicing of space and time, in which case there is a 'preffered', and most importantly consistent, way to measure the nearness of the spacetime to some Petrov type, when working in a homogeneous universe.

The roots of (4.51) can be used to find the spatial projection of the principal null directions. This is done by reverse stereographic projection of the two-sphere on the complex plane. In this way, each complex root of the polynomial (4.51) corresponds to a point on a two-sphere, which can be interpreted as a direction. The Θ_{ij} then correspond to the angles between these direction. Algebraic speciality then corresponds to the two of these directions becoming the same, or equivalently, one or more of the Θ_{ij} becoming 0.

Chapter 5

The Mixmaster Universe

5.1 The Kasner metric

All three-dimensional Lie algebras were classified by Bianchi in 1897 [12], and it can be shown that these are in a none-to-one correspondence with four-dimensional homogeneous universes. These universes are classified by the Lie group of symmetries of each spatial slice, and so a choice of a three-dimensional Lie group leads to a homogeneous universe (see [12] for a much more detailed discussion on this). Mixmaster is Bianchi type IX, and has symmetry group SU(2). However, for reasons that will become apparent shortly, we will first study the homogeneous universe with symmetry group $\mathbb{R}^{\nVdash}$, which is named after Edward Kasner. The Kasner metric is of the form

$$g = -dt^2 + t^{2p_1}(dx^1)^2 + t^{2p_2}(dx^2)^2 + t^{2p_3}(dx^3)^2 \tag{5.1}$$

and take $T_{\mu\nu} = 0$ (vacuum). From the definition of the connection coeffecients Γ, we find that

$$\Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \frac{p_i}{t}\delta^i_j \qquad \Gamma^0{}_{ij} = p_i t^{2p_i - 1}\delta_{ij} = p_i t^{-1} g_{ij} \tag{5.2}$$

where no sums are implied, and all other components are zero. From these, we can calculate the nonzero componenents of the Riemann tensor:

$$R^i{}_{0j0} = -R^i{}_{00j} = \frac{p_i(p_i - 1)}{t^2}\delta^i_j \tag{5.3}$$

$$R^0{}_{i0j} = -R^0{}_{ij0} = \frac{p_i(p_i - 1)}{t^{2p_i - 2}}\delta_{ij} \tag{5.4}$$

$$R_{ijkl} = p_i p_j t^{2p_j - 2}(\delta_{ik}\delta_{lj} - \delta_{il}\delta_{kj}) = p_i p_j t^{-2}(g_{ik}g_{lj} - g_{il}g_{kj}) \tag{5.5}$$

In vacuum, Einstein's equation is

$$G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu} \tag{5.6}$$

But from constracting the second expression, we find that $R = 2R$ (because $g^{\lambda}{}_{\lambda} = 4$), which means $R = 0$, so in vacuum Einstein's equation is simply $R_{\mu\nu} = 0$. From the Riemann Tensor, we find that

$$R_{\mu\nu} = \begin{pmatrix} \frac{\sum_i p_i(1-p_1)}{t^2} & 0 & 0 & 0 \\ 0 & \frac{p_1}{t^{2-2p_1}}(p_1+p_2+p_3-1) & 0 & 0 \\ 0 & 0 & \frac{p_2}{t^{2-2p_2}}(p_1+p_2+p_3-1) & 0 \\ 0 & 0 & 0 & \frac{p_3}{t^{2-2p_3}}(p_1+p_2+p_3-1) \end{pmatrix} \tag{5.7}$$

from which it is obvious that, in order to satisfy Einstein's equations in vacuum,

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \tag{5.8}$$

These constraints mean that either one p_i is one, and the other two are zero (in which case one can show that this is just flat spacetime in Rindler coordinates), or else two are positive and one is negative. Since the coeffecient of each spatial direction dx^i is t^{2p_i}, this means that the two spatial directions with positive associated p_i are expanding, while the third direction is shrinking (meaning that the arclength of paths between fixed coordinate locations is shrinking, or growing). Minus the determinant of the metric provides a measure of the overall scale of the universe [13], and in this case

$$-\det(g) = -(-1)(t^{2p_1})(t^{2p_2})(t^{2p_3}) = t^{2(p_1+p_2+p_3)} = t^2 \tag{5.9}$$

so overall this is an expanding universe, with a big bang singularity at $t = 0$.

There are three parameters and two constraint equations, so we can parameterize the entire solution set with one parameter, which we will call u. We define $p_i(u)$ as follows [17]:

$$p_1 = \frac{-u}{1+u+u^2} \qquad p_2 = \frac{1+u}{1+u+u^2} \qquad p_3 = \frac{u(1+u)}{1+u+u^2} \tag{5.10}$$

By restricting $u \in (1, \infty)$, we eliminate the redundant solutions which correspond to a permution of the p_i's.

From the Riemann tensor, we can also calculate the GE and GM field. In vacuum, $R_{\mu\sigma\nu\rho} = C_{\mu\sigma\nu\rho}$, so we have

$$E_{ij} = \begin{pmatrix} \frac{(p_1-1)p_1}{t^2} & 0 & 0 \\ 0 & \frac{(p_2-1)p_2}{t^2} & 0 \\ 0 & 0 & \frac{(p_3-1)p_3}{t^2} \end{pmatrix} \qquad B_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.11}$$

$(p_i - 1)p_i < 0$ for $p_i > 0$, and $(p_i - 1)p_i > 0$ for $p_i < 0$, so two components of the GE field are negative, and one is positive. $E < 0$ corresponds to stretching and $E > 0$ corresponds to squeezing, so this is reinforces with the picture of Kasner as growing in two spatial directions while shrinking in the third.

Using the identies (5.8), it can be shown that $p_1(1 - p_1) = p_2p_3$, as well as its cyclical counterparts, and so for future comparison we rewrite the components of the GE field as

$$E_{11} = \frac{-p_2p_3}{t^2} \tag{5.12}$$

$$E_{22} = \frac{-p_1p_3}{t^2} \tag{5.13}$$

$$E_{33} = \frac{-p_1p_2}{t^2} \tag{5.14}$$

5.2 The Mixmaster metric

The Kasner solution represents an important limiting case for the Mixmaster Universe. Mixmaster is defined similarly to Kasner, except a different time coordinate is used, and it is defined on $S^3 \times \mathbb{R}$ instead of $\mathbb{R}^4$ (see ref [17] for the original discussion of Mixmaster). The metric is simplest when written in terms of the one-forms σ^i which are generators of the Lie algebra of the group $SU(2)$, whose group manifold is S^3. These one-forms are not coordinate one-forms, which we have previously been working with. This is because there is no scalar function x such that $\sigma_i = dx$. Such a condition would require $d\sigma_i = 0$, because $d^2 = 0$, but $d\sigma_i \neq 0$. In this non-coordinate basis, the metric is given by

$$g = -l_1(t)^2 l_2(t)^2 l_3(t)^2 dt^2 + \sum_i l_i(t)^2 (\sigma^i)^2 \tag{5.15}$$

but the σ^i can be rewritten in terms of the euler angles, ψ, θ, ϕ, which parameterize rotations in the group $SU(2)$. The σ_i are defined by

$$\sigma^1 = \sin(\psi)d\theta - \cos(\psi)\sin(\theta)d\phi \tag{5.16}$$

$$\sigma^2 = -\cos(\psi)d\theta - \sin(\psi)\sin(\theta)d\phi \tag{5.17}$$

$$\sigma^3 = -d\psi - \cos(\theta)d\phi \tag{5.18}$$

from which one can derive the 'structure equationa' for the rotation group, $d\sigma_i = \epsilon_i^{jk}\sigma$. In the ψ, θ, ϕ coordinates the metric is

$$g_{\mu\nu} = \begin{pmatrix} -l_1^2 l_2^2 l_3^2 & 0 & 0 & 0 \\ 0 & l_3^2 & 0 & cos(\theta)l_3^2 \\ 0 & 0 & \cos^2(\psi)l_2^2 + \sin^2(\psi)l_1^2 & \cos(\psi)\sin(\theta)\sin(\psi)(l_2^2 - l_1^2) \\ 0 & \cos(\theta)l_3^2 & \cos(\psi)\sin(\theta)\sin(\psi)(l_2^2 - l_1^2) & \cos^2(\theta)l_3^2 + \sin^2(\theta)(\cos^2(\psi)l_1^2 + \sin^2(\psi)l_2^2) \end{pmatrix} \tag{5.19}$$

However, it is adventageous to work in an orthormal basis, for which $g_{\mu\nu} = \eta_{\mu\nu}$. From the one-forms σ^i, we are automatically given this basis, up to a scale factor of l_i. Using the metric to find the vectors associated with the σ^i, and normalizing by dividing by l_i, we have the orthonormal basis

$$\vec{e}_0 = \frac{1}{l_1 l_2 l_3}\partial_t \tag{5.20}$$

$$\vec{e}_1 = \frac{1}{l_1}(\cos(\psi)\cot(\phi)\partial_\psi + sin(\psi)\partial_\theta - \cos(\psi)\csc(\theta)\partial_\phi) \tag{5.21}$$

$$\vec{e}_2 = \frac{1}{l_2}(\sin(\psi)\cot(\theta)\partial_\psi - \cos(\psi)\partial_\theta - \sin(\psi)\csc(\theta)\partial_\phi) \tag{5.22}$$

$$\vec{e}_3 = -\frac{1}{l_3}\partial_\psi \tag{5.23}$$

In this case, the 'structure constants' are just the completely antisymmetric levi-civita tensor ϵ, meaning that $[\vec{e}_i, \vec{e}_j] = \epsilon^k{}_{ij}\vec{e}_k$. Here the bracket denotes a commuter, and the vectors are interpereted as derivatve operators: $\vec{e}_i \leftrightarrow \vec{(}e_i)^\mu \nabla_\mu$. Notice that the latin subscripts denote which vector we're using, while the greek super- and subscripts denote the components of that vector. For more on the deep connection between vectors and derivative operators, see the first Appendix.

Following [17], we introduce new coordinates β_i which are defined by

$$l_i = e^{\beta_i} \tag{5.24}$$

so that these β_i are also only functions of the coordinate t. Expressed in the orthonormal basis, and in these new coordinates, the Ricci tensor can be found to be

$$R_{00} = e^{-2(\beta_1+\beta_2+\beta_3)}(2\dot{\beta}_1\dot{\beta}_2 + 2\dot{\beta}_1\dot{\beta}_3 + 2\dot{\beta}_2\dot{\beta}_3 - \ddot{\beta}_1 - \ddot{\beta}_2 - \ddot{\beta}_3) \tag{5.25}$$

$$R_{11} = \frac{1}{2}e^{-2(\beta_1+\beta_2+\beta_3)}(e^{4\beta_1} - e^{4\beta_2} - e^{4\beta_3} + 2e^{2(\beta_2+\beta_3)} + 2\ddot{\beta}_1) \tag{5.26}$$

$$R_{22} = \frac{1}{2}e^{-2(\beta_1+\beta_2+\beta_3)}(-e^{4\beta_1} + e^{4\beta_2} - e^{4\beta_3} + 2e^{2(\beta_1+\beta_3)} + 2\ddot{\beta}_2) \tag{5.27}$$

$$R_{33} = \frac{1}{2}e^{-2(\beta_1+\beta_2+\beta_3)}(-e^{4\beta_1} - e^{4\beta_2} + e^{4\beta_3} + 2e^{2(\beta_1+\beta_2)} + 2\ddot{\beta}_3) \tag{5.28}$$

and $R_{\mu\nu} = 0$ for $\mu \neq \nu$. As previously discussed, Einstein's equations in vacuum are simply $R_{\mu\nu} = 0$, implying that all of the expressions above must equal zero at all times for some solution $\beta_i(t)$. In order to guarantee three of these constraints, as well as make the remaining contstraint first order in derivatives of β_i, we use the $R_{ii} = 0$ to solve for $\ddot{\beta}_i$ as a function of β_j. Then R_{ii} is identically zero, and

$$R_{00} = \frac{1}{2}e^{-2(\beta_1+\beta_2+\beta_3)}(-e^{4\beta_1} - e^{4\beta_2} - e^{4\beta_3} + 2e^{2(\beta_1+\beta_2)} + 2e^{2(\beta_1+\beta_3)} + 2e^{2(\beta_2+\beta_3)} \tag{5.29}$$

$$+ 4\dot{\beta}_1\dot{\beta}_2 + 4\dot{\beta}_1\dot{\beta}_3 + 4\dot{\beta}_2\dot{\beta}_3) \tag{5.30}$$

which implies that

$$-e^{4\beta_1} - e^{4\beta_2} - e^{4\beta_3} + 2e^{2(\beta_1+\beta_2)} + 2e^{2(\beta_1+\beta_3)} + 2e^{2(\beta_2+\beta_3)} + 4\dot{\beta}_1\dot{\beta}_2 + 4\dot{\beta}_1\dot{\beta}_3 + 4\dot{\beta}_2\dot{\beta}_3 = 0 \tag{5.31}$$

This is a particular example of something called the 'Hamiltonian constraint', which acts as the Hamiltonian for gravitational dynamics. Note that it has quadratic 'kinetic' terms, much like the Hamiltonian for a massive particle. It also has potential terms which are 'velocity' independent, and grow very sharply with the parameters β_i. The Hamiltonian is always equal to a constant, and when there is freedom in the choice of the time coordinate, as is the case here, it can be shown that the Hamiltonian is zero. Our approach to studying Mixmaster dynamics will be to approximate solutions to this equation. Imagine that, at some time, the β_i's are all negative and reasonable large, so that the terms involving $e^{4\beta_i}$ become negligible. Then we have that

$$\dot{\beta}_1\dot{\beta}_2 + \dot{\beta}_1\dot{\beta}_3 + \dot{\beta}_2\dot{\beta}_3 \approx 0 \tag{5.32}$$

and from $R_{ii} = 0$ that

$$\ddot{\beta}_i \approx 0 \tag{5.33}$$

Evidently, then, the β_i are linear in time, and obey the following constraint, which is simply (5.31), rewritten in a more suggestive way:

$$(\dot{\beta}_1 + \dot{\beta}_2 + \dot{\beta}_3)^2 = \dot{\beta}_1{}^2 + \dot{\beta}_2{}^2 + \dot{\beta}_3{}^2 \tag{5.34}$$

Besides an overall normalization, this condition is exactly that which was imposed on the Kasner indices p_i. This means that one β_i will shrink linearly in t, while the other two grow linearly in time. In turn, this means two l_i grow while one shrinks. In this way, we say that Mixmaster is approximately Kasner when the β_i are all significantly negative. Of course, this is by no means true for all time. But suppose we started at some time for which it was; what would happen? Two β_i are growing, so eventually two of the $e^{4\beta_i}$ terms will become non-negligible. But e^{4x} is such a steeply growing function that one of these terms will suddenly become non-negligible before any other. For concreteness, let us suppose it is β_1 that becomes relevant first. In this case, then, we consider the equations

$$-e^{4\beta_1} + 4\dot{\beta}_1\dot{\beta}_2 + 4\dot{\beta}_1\dot{\beta}_3 + 4\dot{\beta}_2\dot{\beta}_3 = 0 \tag{5.35}$$

$$\ddot{\beta}_1 = -\frac{1}{2}e^{4\beta_1} \tag{5.36}$$

$$\ddot{\beta}_2 = \frac{1}{2}e^{4\beta_1} \tag{5.37}$$

$$\ddot{\beta}_3 = \frac{1}{2}e^{4\beta_1} \tag{5.38}$$

These are not as simple to solve as in the Kasner case, but do indeed have an analytic solution. A solution, given by [18], is

$$\beta_1(t) = \beta_1(0) - \frac{1}{2}\log[\cosh(\frac{k}{\sqrt{6}}t) + \frac{2\sqrt{6}\dot{\beta}_1(0)}{k}\sinh(\frac{k}{\sqrt{6}}t)] \tag{5.39}$$

$$\beta_2(t) = \beta_2(0) + \frac{1}{2}\log[\cosh(\frac{k}{\sqrt{6}}t) + \frac{2\sqrt{6}\dot{\beta}_1(0)}{k}\sinh(\frac{k}{\sqrt{6}}t)] + (\dot{\beta}_1(0) + \dot{\beta}_3(0))t \tag{5.40}$$

$$\beta_3(t) = \beta_3(0) + \frac{1}{2}\log[\cosh(\frac{k}{\sqrt{6}}t) + \frac{2\sqrt{6}\dot{\beta}_1(0)}{k}\sinh(\frac{k}{\sqrt{6}}t)] + (\dot{\beta}_1(0) + \dot{\beta}_2(0))t \tag{5.41}$$

where

$$k = (24\dot{\beta}_1(0)^2 + 6e^{4\beta_1(0)}) \tag{5.42}$$

and we require that

$$\dot{\beta}_3(0) = \frac{\frac{1}{4}e^{4\beta_1(0)} - \dot{\beta}_1(0)\dot{\beta}_2(0)}{\dot{\beta}_1(0) + \dot{\beta}_2(0)} \tag{5.43}$$

and the remaining five parameters $\beta_1(0)$, $\beta_2(0)$, $\beta_3(0)$, $\dot{\beta}_1(0)$, and $\dot{\beta}_2(0)$ are free to be

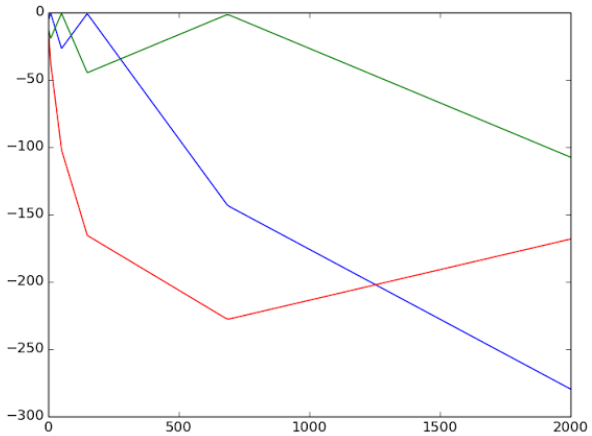

FIGURE 5.1: The β_i as a function of time (in arbitrary units) for about 4 Kasner eras. An epoch change occurs around $t = 700$ when the smallest β_i changes slope from negative to positive.

specified. Time '0' here just indicates some arbitrary starting point. This solution corresponds to a geometry called the 'Taub Cosmology'. The functional forms are rather opaque, but their behavior is easy to understand qualitatively. As expected from the Kasner case, when each of the β_i are significantly negative, each β_i is linear in time to an excellent approximation. Then, as β_1 approaches and passes zero, each of the three β_i simultaneously change to become approximately linear with some new slope. In particular, the slope β_1 changes from positive to negative, so that the Kasner approxamation becomes good again. Another β_i is now headed towards relevance, and by simply by permuting the solutions given above, we can find a good approximate solution to the next permutation. These changes in the slope of the β_i are called 'bounces', and so Mixmaster dynamics is well approximated by a series of Kasner 'eras', linked by bounces which change the Kasner indices. The slopes of the linear parts of the β_i correspond to the p_i, with normalization, so that

$$p_i = \frac{\dot{\beta}_i}{\dot{\beta}_1 + \dot{\beta}_2 + \dot{\beta}_3} \tag{5.44}$$

We can describe the evolution of one set of kasner parameters p_i to another set $p_i^{'}$ rather simply using the parameter u introduced above [19]. For $u > 1$, $p_i^{'}(u) = p_i(u-1)$. Otherwise, $p_i^{'} = p_i(\frac{1}{u-1})$. Notice that u increments down near one, and then shoots up to some large value depending on how close to an integer it was. This gives rise to the distinction between a Kasner 'era' and a Kasner 'epoch'. An era refers simply to the section of the Mixmaster solution which corresponds to a Kasner solution for some value of u. An epoch change occurs when u becomes less than 2, necessitating

the use of the second bounce rule. Epochs can be very short if, say, $u = 2.6$, for then $2.6 \to 1.6 \to 2.5 \to 1.5 \to 2$ and the middle values 2.5 and 1.5 constitute an entire epoch. However, these nice values are rare. More often, u is irrational (as there are many more irrational than rational numbers), in which case epochs can be quite long. If $u = 1.0001$, then $1.001 \to 10000$ and the epoch will last for 9999 eras. For an initial u with non-zero digits far to the right of the decimal point, epochs like this and much longer are common. A series of Kasner epochs, each containing a series of Kasner eras, completely describes Mixmaster dynamics.

Chapter 6

Results

6.1 Gravito-Electromagnetism in Mixmaster

The Gravito-Electric field for the Mixmaster universe is shown to be the following, by simple decomposition of the Weyl tensor, taking the symmetric and trace-free part, and substituting our expressions for $\ddot{\beta}_i$ from three of Einstein's equations.

$$E_{11} = \frac{1}{3}e^{-2(\beta_1+\beta_2+\beta_3)}(2e^{4\beta_1} - e^{4\beta_2} - e^{4\beta_3} - e^{2(\beta_1+\beta_2)} + 2e^{2(\beta_2\beta_3)} + \dot{\beta}_1\dot{\beta}_2 + \dot{\beta}_1\dot{\beta}_3 - 2\dot{\beta}_2\dot{\beta}_3) \tag{6.1}$$

$$E_{22} = \frac{1}{3}e^{-2(\beta_1+\beta_2+\beta_3)}(-e^{4\beta_1} + 4e^{4\beta_2} - e^{4\beta_3} + 2e^{2(\beta_1+\beta_2)} - e^{2(\beta_2\beta_3)} + \dot{\beta}_1\dot{\beta}_2 - 2\dot{\beta}_1\dot{\beta}_3 + \dot{\beta}_2\dot{\beta}_3) \tag{6.2}$$

$$E_{33} = \frac{1}{3}e^{-2(\beta_1+\beta_2+\beta_3)}(-e^{4\beta_1} + 2e^{4\beta_2} + 2e^{4\beta_3} - e^{2(\beta_1+\beta_2)} - e^{2(\beta_2\beta_3)} - 2\dot{\beta}_1\dot{\beta}_2 + \dot{\beta}_1\dot{\beta}_3 + \dot{\beta}_2\dot{\beta}_3) \tag{6.3}$$

with all other components 0. Similarly, the Gravito-Magnetic field is

$$B_{11} = \frac{1}{2}e^{-2(\beta_1+\beta_2+\beta_3)}(2e^{2\beta_1}\dot{\beta}_1 - (e^{2\beta_1} + e^{2\beta_2} - e^{2\beta_3})\dot{\beta}_2 - (e^{2\beta_1} - e^{2\beta_2} + e^{2\beta_3})\dot{\beta}_3) \tag{6.4}$$

$$B_{22} = \frac{1}{2}e^{-2(\beta_1+\beta_2+\beta_3)}(-(e^{2\beta_1} + e^{2\beta_2} - e^{2\beta_3})\dot{\beta}_1) + 2e^{2\beta_2}\dot{\beta}_2 + (e^{2\beta_1} - e^{2\beta_2} - e^{2\beta_3})\dot{\beta}_3) \tag{6.5}$$

$$B_{33} = \frac{1}{2}e^{-2(\beta_1+\beta_2+\beta_3)}(-(e^{2\beta_1} - e^{2\beta_2} + e^{2\beta_3})\dot{\beta}_1 + (e^{2\beta_1} - e^{2\beta_2} - e^{2\beta_3})\dot{\beta}_2 + e^{2\beta_3}\dot{\beta}_3) \tag{6.6}$$

with all other components zero. Both of these component expressions are with respect to the same orthonormal basis introduced previously. It is an interesting feature of Mixmaster that its GE and GM fields share an eigenbasis, along with the Ricci tensor.

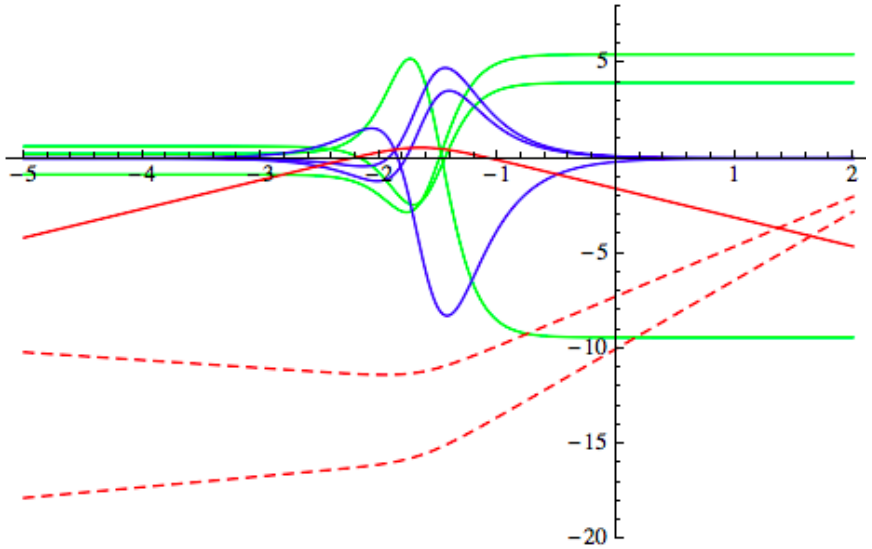

FIGURE 6.1: The eigenvalues of $(l_1 l_2 l_3)^2 E_{ii}$ (in green) and $(l_1 l_2 l_3)^2 B_{ii}$ (in blue) as a function of time. Additionally, the β_i are plotted in red, the dotted trace representing the β_j whose term was added to the Kasner approximation. The horizontal axis is the coordinate time

For visualization purposes, we will multiple the GE and GM fields by the lapse $(l_1 l_2 l_3)$ squared, as otherwise the exponential growth/decay dominates any plot of them. These quantities are plotted versus time in figure 6.1, along with the β_i during a bounce transition, displaying two separate Kasner eras before and after the bounce. Notice that the GM field is approximately zero during a Kasner era, as one would expect since the GM field is identically zero in Kasner. It then becomes active during a bounce, and shuffles around the components of the GE field, which come out changed but constant again for the next era.

During a Kasner era, when we neglect terms with extra factors of $e^{2\beta_i}$ and let $\beta_i \approx b_i t + c_i$, The GM field is zero and the GE field becomes

$$E_{11} \approx \frac{-b_2 b_3}{N^2} \tag{6.7}$$

$$E_{22} \approx \frac{-b_1 b_3}{N^2} \tag{6.8}$$

$$E_{33} \approx \frac{-b_1 b_2}{N^2} \tag{6.9}$$

where $N = l_1 l_2 l_3 = e^{\beta_1+\beta_2+\beta_3}$ is the lapse function. The time dependence is apparently different from that of the Kasner solution, whose GE field drops off like t^2, but we are using non-comparable time coordinates, as in Kasner the lapse is just 1. If we switch to a time coordinate t' which eliminates the lapse in Mixmaster, we see that the time dependencies do match. Letting $dt' = N dt$, during a Kasner era when $\beta_i = b_i t + c_i$, we

have that

$$t' = \frac{e^{c_1+c_2+c_3}}{b_1+b_2+b_3} e^{(b_1+b_2+b_3)t} + C' \tag{6.10}$$

or

$$t = \frac{1}{b_1+b_2+b_3} \log\left(\frac{b_1+b_2+b_3}{e^{c_1+c_2+c_3}}\right) \tag{6.11}$$

where C′ is an integration constant. As the GE field has no time component, and all of the time-space components of the metric are zero, transforming time coordinates amounts to simply rewriting the components of the field in terms of the new time coordinate. During a Kasner era, we have that

$$N = e^{\beta_1+\beta_2+\beta_3} = e^{(b_1+b_2+b_3)t+c_1+c_2+c_3} = (b_1+b_2+b_3)(t'-C') \tag{6.12}$$

which means that the components of the GE field are

$$E_{11} = \frac{-b_2 b_3}{(t'+C')^2(b_1+b_2+b_3)^2} \tag{6.13}$$

$$E_{22} = \frac{-b_1 b_3}{(t'+C')^2(b_1+b_2+b_3)^2} \tag{6.14}$$

$$E_{33} = \frac{-b_1 b_2}{(t'+C')^2(b_1+b_2+b_3)^2} \tag{6.15}$$

which upon making the previous identification of $p_i = \frac{\dot{\beta}_i}{\dot{\beta}_1+\dot{\beta}_2+\dot{\beta}_3}$, and noting that $\dot{\beta}_i = b_i$, yields

$$E_{11} = \frac{-p_2 p_3}{(t'+C')^2} \tag{6.16}$$

$$E_{22} = \frac{-p_1 p_3}{(t'+C')^2} \tag{6.17}$$

$$E_{33} = \frac{-p_1 p_2}{(t'+C')^2} \tag{6.18}$$

which is identical to the GE field for Kasner, (5.12), up to an arbitrary time translation.

6.2 Petrov Type in Mixmaster

For Mixmaster, $\Psi_1 = \Psi_3 = 0$, which greatly simplifies the solutions to (4.51), which is now simply a quadratic equation for λ^2. The four solutions, then, are

$$\lambda = \pm\left(-3\frac{\Psi_2}{\Psi_4} \pm \frac{\sqrt{9\Psi_2^2 - \Psi_4\Psi_0}}{\Psi_4}\right)^{1/2} \tag{6.19}$$

During a Kasner era, the Weyl Scalars will be approximately equal to the Weyl Scalars in Kasner, for some triplet p_i. The Weyl Scalars in Kasner are given by

$$\Psi_0 = \frac{p_1(p_3 - p_2)}{t^2} \tag{6.20}$$

$$\Psi_1 = 0 \tag{6.21}$$

$$\Psi_2 = \frac{p_2 p_3}{2t^2} \tag{6.22}$$

$$\Psi_3 = 0 \tag{6.23}$$

$$\Psi_4 = -\frac{p_1(p_3 - p_2)}{t^2} = \Psi_0 \tag{6.24}$$

Using the parameterization of all possible triplets p_i which satisfy Einstein's equations, (5.10), the nonzero scalars become

$$\Psi_0 = \Psi_4 = \frac{u(1-u)(1+u)}{2t^2(1+u+u^2)^2} \tag{6.25}$$

$$\Psi_2 = \frac{u(1+u)^2}{2t^2(1+u+u^2)^2} \tag{6.26}$$

which make the roots of the characteristic polynomial

$$\lambda = \pm\left[\frac{1}{1-u}\left(3(1+u) \pm \sqrt{9(1+u)^2 - (1-u)^2}\right)\right]^{1/2} \tag{6.27}$$

Remember that $u \in [1, \infty)$, and notice that, for $u > 1$, $\sqrt{9(1+u)^2 - (1-u)^2} < \sqrt{9(1+u)^2} = 3(1+u)$, so that for $u > 1$, the roots are all real, and all distinct. This demonstrates that, for $u > 1$, Kasner is Petrov type I. When $u = 1$, two roots go to zero, and two roots go out to plus or minus real infinity, making this spacetime Petrov type D. Thus we see that, as Mixmaster undergoes era changes, taking $u \to u - 1$, it is getting closer and closer to type D (through stereographic projection, the roots going to $\pm\infty$ are actually approaching one another). When $u < 2$ and a bounce occurs, $u \to (u-1)^{-1}$, u becomes much larger, and therefore the spacetime becomes far away from type D, and then begins to slowly approach it once again.

The dependence of u on the nearness to Type D can be seen another way by looking at the quanties Θ_{ij} (6.2). Their functional form as functions of u is quite complex, but their behavior can be seen in figure 6.2. Each trace corresponds to two of the six Θ_{ij}. Notice how as $u \to 0$, two of the Θ_{ij} go to zero, demonstrating that the spacetime approaches type D. If u begins as a rational number, it will eventually reach $u = 1$ according to the bounce rule [20], but if the initial u is irrational, this will never occur, because if u is irrational, $u-1$ and $(u-1)^{-1}$ are also irrational. Therefore, given initial conditions which correspond to a Kasner era of irrational u, Mixmaster is Type I, moves towards Type D, gets arbitrarily close and then bounces far away to begin another approach to Type

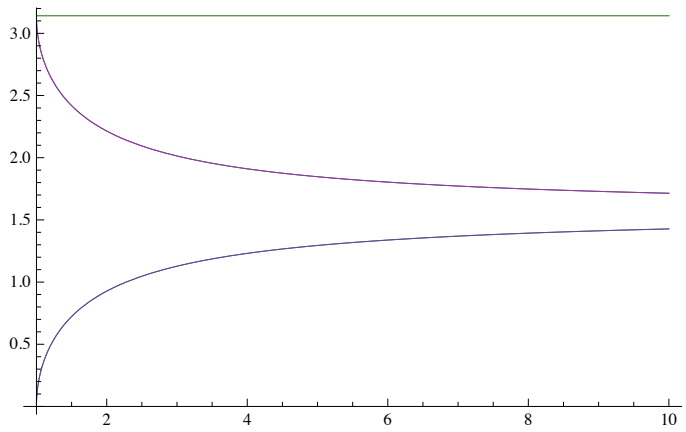

FIGURE 6.2: Θ_{ij} as a function of the parameter u. Each trace is doubly degenerate. As u approaches one, two Θ_{ij} approach zero, which means two pairs of principal null directions approach one another, meaning the spacetime approaches Petrov Type D.

D, and this process repeats indefinitely, without ever acheiving Type D. Furthermore, because the closer u is to 1, the larger $(1-u)^{-1}$ becomes, the closer to Type D Mixmaster becomes, the farther away from Type D it is propelled during the epoch change. Kind of tragic, really.

The principal null directions, when projected into spatial vectors (which can be done in a canonical way for a homogeneous universe such as Mixmaster), form interesting patterns. As can be seen by figure (6.2), as u decreases, the directions approach one another in pairs. Because all the roots are real, all four directions lie in the same plane, which corresponds to the direction associated with a certain p_i via the metric. The p_i evolve as follows: as eras change, the negative p_i gets smaller and smaller, approaching $-1/3$. The two positive p_i alternate with era between being the largest, with one above $2/3$ and one below $2/3$, and as u decreases they both approach $2/3$. When an epoch change occurs, the smalles p_i becomes the largest, the one just below $2/3$ becomes negative, and the one just above $2/3$ stays positive but moves nears 0. During each of these eras, the principle null directions lie in the plane which is perpendicular to the negative p_i. They form a non-orthogonal 'X' shape, with the smaller angle bisected by the axis corresponding to the largest p_i. As the spacetime evoles, then, not only does the small angle get smaller, but they shape also rotates 90 degrees. When an epoch change occurs, the angles spread back out, and the four principal directions lie in a new plane, perpendicular to the new smallest p_i. In the limit as $u \to \infty$, the angle between any two adjacent principal null directions approaches $\frac{\pi}{2}$, making two orthogonal lines. At the turning point between an era change and an epoch change, which corresponds to $u = 2$, the small angle between principal null directions is about $53.1301°$. The significance of this angle is not obvious, and invites further investigation.

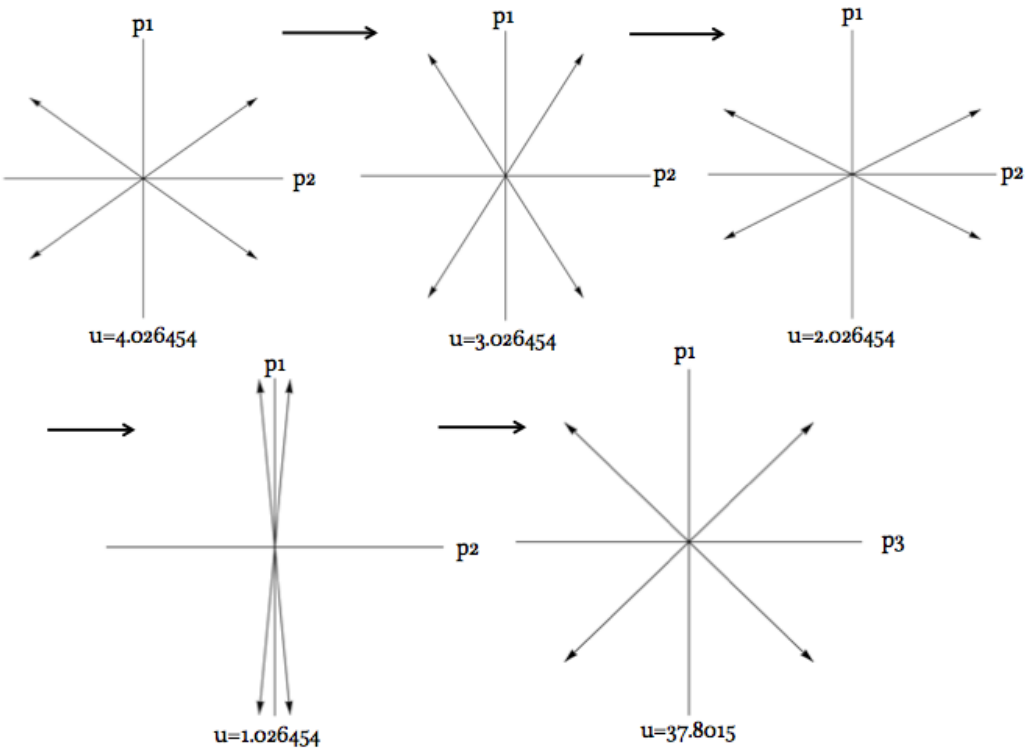

FIGURE 6.3: The principle null directions over a series of Kasner eras, including one epoch change. After the epoch change, the axes change.

The principal null directions do not all lie in a plane during a bounce. The angles between the principal null directions are plotted above for a wide variety of bounces. Notice the sharp spike within the bounce, which becomes more prevelant for smaller initial values of u. This structure of the evolution of the principal null directions within the bounce is unexpectedly rich, and warrants further investigation. It does not appear that, in general, spacetime becomes algebraically special at any point during a bounce, as none of the Θ_{ij} go to zero, but it is possible that they do for a bounce that begins with a u sufficiently close to 1, as in this regime the mid-bounce spike becomes extremely pronounced. In the graphs it does not appear to go to zero, but the features are so sharp that perhaps this is appearance is due to finite resolution. This should be investigated further.

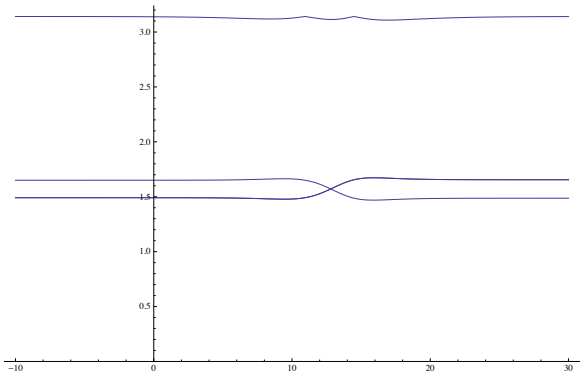

FIGURE 6.4: Θ_{ij} as a function of time, in arbitrary units, during the bounce $u = 18.356 \to 17.356$.

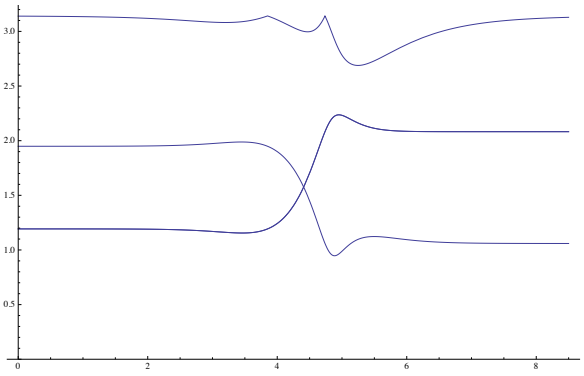

FIGURE 6.5: Θ_{ij} as a function of time, in arbitrary units, during the bounce $u = 3.564 \to 2.564$.

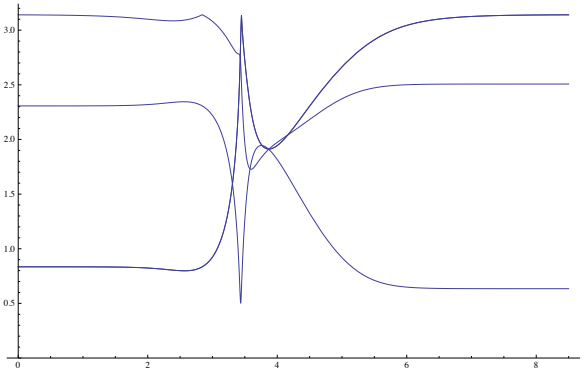

FIGURE 6.6: Θ_{ij} as a function of time, in arbitrary units, during the bounce $u = 1.73 \to 1.4$.

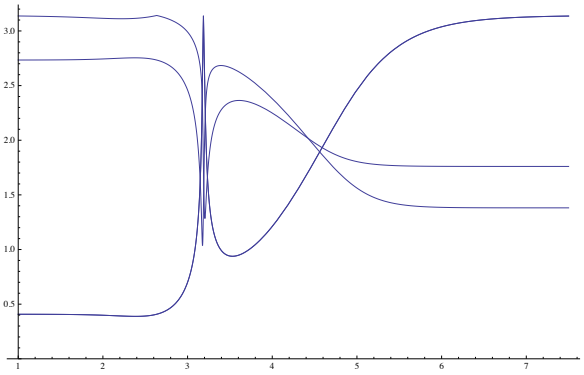

FIGURE 6.7: Θ_{ij} as a function of time, in arbitrary units, during the bounce $u = 1.134 \to 7.46$.

Chapter 7

Conclusion

The BKL conjecture, which has a significant amount of numerical and analytical evidence, suggests that homogeneous spacetimes are crucial for understanding spacetime dynamics near a singularity. The behavior of Bianchi type IX, or Mixmaster, which is the homogeneous universe with symmetry group $SU(2)$, is especially relevant in this context. While Mixmaster dynamics are well understood, in this thesis we attempted to understand the dynamics in a new way by looking at them using the language of Gravito-Electromagnetism and the Petrov Classification. We found the Gravito-Magnetic field is zero in between 'bounces', and that during the bounces the Magnetic field becomes non-zero, and the Gravito-Electric field values shuffle around. This is evocative of an induction-type process in familiar electromagnetism, but I have searched in vain for a solution to Maxwell's equations which exhibit analogous behavior. We also found that we can understand the familiar 'bounce law' transitions as slow movement towards, punctuated by quick jumps away from, a certain symmetry called Petrov Type D. This process never ends, and the closer to Type D spacetime becomes, the farther away from Type D it transitions to. The movement of the principal null directions during a bounce also displays interesting structure, which should be investigated further. In particular, the question of whether or not there exists a bounce through which Mixmaster passes through algebraic speciality should be answered.

Appendix A

Math Background

Much of this exposition follows ref [21]

A.1 Topology

Consider a set of points, X, called a space. A Topology of X, called $\mathcal{T}$, is a subset of the power set of X, i.e. a set of subsets of X, which satisfies four conditions:

$$1 : X \in \mathcal{T} \tag{A.1}$$

$$2 : \emptyset \in \mathcal{T} \tag{A.2}$$

$$3 : U_1, ..., U_n \in \mathcal{T} \Rightarrow U_1 \cap U_2 \cap ... \cap U_n \in \mathcal{T} \tag{A.3}$$

$$4 : U_\alpha \in \mathcal{T} \Rightarrow \underset{\alpha}{\cup} U_\alpha \in \mathcal{T} \tag{A.4}$$

where $\emptyset$ is the empty set, and $U_i \subset X$. We call the subsets of X that are in $\mathcal{T}$ 'open sets', and an open set containing a point is called a neighborhood of that point (for a more complete treatment of topology, see [22]). A space together with its topology is called a topological space, and although the definition is abstract, it does allow us to make precise a notion of 'nearness' to a point, without explicitly defining the distance between points (see later for plenty more on metrics). A topology specifies overall features of a space, such as the difference between a sphere and a torus, but does not encode all information about a space; it does not change as the space deforms without changing fundamental structure. For example, a cube and a cylinder have the same topology. The topology essentially tells you which points are 'inescapably' close to one another, something that would change if one point were glued (identified) with another. For examle, consider the closed interval $[0, 1]$, with open sets being given by arbitrary unions of sets of the form

$\{x \in [0,1] \mid |x-y| < \epsilon\}$ for any $y \in [0,1]$ and any $\epsilon > 0$. There are open sets containing 0 that do not contain 1. However, if we identify 0 with 1, effectively turning the interval into a circle, the topology has changed, because there is now no open set containing 0 and not 1 (actually, since 0 is now 1 this is a meaningless statement. More precisely, the topology has changed because any open set around 0/1 necessarily contains some point very close to but slightly less than 1, unlike before the identification was made). If the circle is morphed into a square, its topology does not change, but the distance between points has. A metric described the distance between points, and so it specifies the shape of the space much more rigidly. Therefore a metric implies a topology but not vice-versa. However, a topology happens to be enough to define continuity.

Recall the definition of continuity for a function from $\mathbb{R}^n$ to $\mathbb{R}^m$; a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous $\iff$ for each $x \in \mathbb{R}^n$ and for all $\epsilon \in \mathbb{R}, \epsilon > 0$, there exists a $\delta \in \mathbb{R}$, $\delta > 0$ such that, given any $y \in \mathbb{R}$, $| x - y | < \delta \Rightarrow | f(x) - f(y) | < \epsilon$. Roughly speaking, this definition says that, given any two points in the domain that are 'close enough', their images under the function are also 'close', which is exactly how one would expect a continuous function to behave. Of course, in the case of $\mathbb{R}^n$, we know how to find the distance between two points, so this idea of closeness can easily be made rigorous. The topology of a space allows us to make a generalization of this statement for spaces without a defined distance between two points, with continuity defined as so: consider a space X, with topology $\mathcal{T}_X$, a space Y with topology $\mathcal{T}_Y$, and a function $f : X \to Y$. f is continuous $\iff$ for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$. By comparison with the more specific statement for euclidean space, we see here how open sets substitute in for a more abstract notion of nearness.

The distance between two points $x, y \in \mathbb{R}^n$, denoted by $d(x,y)$, is given by $d(x,y) = ((x_1 - y_1)^2 + ... + (x_n - y_n)^2)^{1/2}$. If we define the topology on $\mathbb{R}^n$ by the arbitrary union of sets of the form $\{x \mid d(x,y) < \epsilon\}$ for all $y \in \mathbb{R}^n$ and $\epsilon > 0$, then we see that the normal definition of open sets on $\mathbb{R}^n$ is recovered, and the open set definition of continuous functions coincides with the epsilon-delta definition. Whenever we refer to $\mathbb{R}^n$, we assume it has this topology.

A.2 Manifolds

The types of topological spaces that this thesis will most concern itself with are called differentiable manifolds. These spaces have an extra condition, in addition to the structure of a topology. In order to be a manifold, each point p in a topological space $\mathcal{M}$ must have a neighborhood (call it U) that has a bicontinuous map, called a chart, $\psi : U \to V$, where V is some open subset of $\mathbb{R}^n$, for some n. This n must be the same for each point

p, and we call n the dimension of the manifold. In a phrase, a manifold must locally look like $\mathbb{R}^n$. Wherever the domain of two different charts over lap, we can define a transition function from a subset of $\mathbb{R}^n$ to $\mathbb{R}^n$. Let's call these charts $\psi_1 : U_1 \to \mathbb{R}^n$ and $\psi_2 : U_2 \to \mathbb{R}^n$, with $U_1 \cap U_2 \neq \emptyset$. Then we can define $\psi_1 \circ \psi_2^{-1} : \psi_2(U_1 \cap U_2) \to \mathbb{R}^n$ and $\psi_2 \circ \psi_1^{-1} : \psi_1(U_1 \cap U_2) \to \mathbb{R}^n$. But the domain of each of these functions is a subet of $\mathbb{R}^n$, so we can evaluate the continuity of these functions in the normal euclidean sense. If each of these transition functions are differentiable, then the manifold is call differentiable. We won't be considering non-differential manifolds, so from here on out we'll just call them 'manifolds', and assume that the differentiability condition holds.

A good example of a manifold is the two-sphere, S^2, which can be described as an embedding of $\mathbb{R}^3$ as the set of all points equidistant from the origin. It is not too difficult to see that, surrounding each point on the sphere, there is a small patch of the sphere that is homeomorphic to $\mathbb{R}^2$, i.e. it can be transformed into $\mathbb{R}^2$ by stretching and deforming. More formally, a homeomorphism is a bicontinuous function, and these homeomorphisms are simply our charts. In fact, any proper subset of S^2 is homeomorphic to a subset of $\mathbb{R}^2$, but all of S^2 cannot be mapped to $\mathbb{R}^2$. Therefore, S^2 requires at least two charts to cover it. If we were to change the topology of S^2 (open subsets of S^2 are the intersections of S^2 with opens sets of $\mathbb{R}^3$, when S^2 is considered a subset of $\mathbb{R}^3$), say by glueing the north and south pole together, it would no longer be a manifold. Think of a donut shape, but without the hole through the center, just a pinched point. Most points would still have a neighborhood homeomorphic to $\mathbb{R}^2$, but any neighborhood of the north/south pole would contain both what used to be a neighborhood of the north pole, and what used to be a neighborhood of the south pole, which is two copies of $\mathbb{R}^2$ glued together at a point, which is not homeomorphic to $\mathbb{R}^2$. Therefore, this set of a points with the new topology after glueing is not a manifold.

A.3 Functions

A real (or complex) function on a space is a rule assigning to each point in the space a real (or complex) number (from here on we'll use real functions, but everything we say about them can easily be generalized to complex functions). If this space happens to have a topology, and satisfies the condition above for a manifold, then it is a function on a manifold, $\mathcal{M}$, denoted $f : \mathcal{M} \to \mathbb{R}$ (or $f : \mathcal{M} \to \mathbb{C}$). This structure on a set of points allows us to define an astonishing amount of useful objects on the space. Notice first that our generalized definition of continuous allows us to evalute the continuity of f. Additionally, a function on a manifold, restricted to an open set which is contained in the domain of some chart, automcatically yields a function on a subset of euclidean space by

precomposition with the chart function: suppose $f : \mathcal{M} \to \mathbb{R}$, $U \subset \mathcal{M}$, and $\psi : U \to \mathbb{R}^n$ is a chart about some point. Then define $\tilde{f} : \psi(U) \subset \mathbb{R}^n \to \mathbb{R}$ by $\tilde{f} = f \circ \psi^{-1}$. We can also go the other way, and similarly produce functions on the manifold from functions on $\mathbb{R}^n$. In particular, consider the coordinate functions, $x^i : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto x^i$, where the symbol x^i is serving double duty as a function and as a number that is the ith component of some point x, but these uses are so similar we won't bother to distinguish. We automatically have the new functions $\tilde{x}^i : \mathcal{M} \to \mathbb{R}$, $\tilde{x}^i = x^i \circ \psi$. These new functions serve as local coordinates on the manifold, and allow us to write $f \mid_U = f(x^i) \mid_U$. In this way we see the choice of a chart is equivalent to a choice of a coordinate system on the manifold. As coordinates are want to do, we now have two notions of the same function: as an abstract rule assigning a number to each point of the manifold, and as a concrete function of the coordinates. The latter form is the more familiar form of a function, called its *functional form.* This form is often more useful; if the manifold is two dimensional and we pick some chart (coordinate system) with coordinates named x and y, then the functional form of some function f might look something like $f(x, y) = xy^2$. This is expression is easy to visualize and calculate with. However, there are many other valid coordinates to use on the manifold, and the functional form of f may look quite different when expressed in these coordinates, and this obscures the fact that these are really the same function. If we have f as a function of some coordinate system x^i, and we have some new coordinate system y^i, and we know how these two coordinate systems are related (i.e. we can write one set of coordinates as a function of the other, $x^i(y^j)$), then $f(y^j) = f(x^i(y^j))$.

Suppose we want to differentiate a function (there's no supposing here, really, it's definitely something we want to do). If we want the derivative of f at some point $p \in \mathcal{M}$ in some direction, then we simply create a parameterized path $\gamma : [0, 1] \to \mathcal{M}$ such at $\gamma(1/2) = p$ and that at p, γ is travelling in the direction that we want to differentiate in. Then the derivative we want is given by

$$\frac{df}{d\gamma} \mid_p = \lim_{\epsilon \to 0} \frac{f(\gamma(\frac{1}{2} + \epsilon)) - f(\gamma(\frac{1}{2}))}{\epsilon} \tag{A.5}$$

A function which is continuous and has continuous derivatives of all orders is called infinitely differentiable, and we denote the space of all infinitely differentiable functions on a manifold $\mathcal{M}$ by $C^\infty(\mathcal{M})$.

A.4 Vectors

Normally a vector does the job of specifying a direction, so we will use the context of differentiating functions in some direction as a way to define a vector on a manifold. A

vector is an element of an abstract mathematical structure called a vector space, and we'll soon see that the directions (or rather directional derivatives) at a point form a vector space, making directional derivatives a viable candidate for vectors. This approach will prove very fruitful. In euclidean space, one can think of a vector as a line from one point to another, and so a vector is defined by a choice (and ordering) of two points. This concept is not well defined for manifolds in general: for instance, consider the two-sphere. If we pick the north and south pole as two points to define a vector, which vector is it that they define? There are infinitely many possibilities. To remedy this, we will not consider a vector as lying in the manifold; rather, a vector at some point will lie in some other space, called a tangent space, attached to the manifold at that point. The tangent space to $\mathcal{M}$ at a point p is called $T_p(\mathcal{M})$ and forms a vector space. Often, in euclidean space, we often consider two vectors of the same magnitude and direction at different points to be the 'same' vector. This is because there is one and only one way to drag vectors around euclidean space, and so we can compare vectors at two different points in an obvious way. Although it's easy to visualize a vector moving around euclidean space without changing length or direction, let's take a minute to think about the precise condition. If we have a 'traditional' vector $v \in \mathbb{R}^n$, $v = (v^1, v^2, ..., v^n)$, and this vector is based at some point $x = (x^1, x^2, ..., x^n)$, then we can drag this vector to a new point $y = (y^1, y^2, ..., y^n)$ along any path $\gamma : [0, 1] \to \mathbb{R}^n$, $\gamma(0) = x$, $\gamma(1) = y$, by making sure that $\frac{dv^i}{dt} \mid_{\gamma(t)} = 0$ for each i at all $t \in [0, 1]$. This does not work in general for manifolds. As we will soon see, there is a generalization of the expression above, which is a well-defined notion of 'parallel transporting' a vector that drags a vector at one point to become a vector at another point without changing its orientation in a certain sense. However, in general this process depends on the path taken between two points, and so there is not a well-defined way of comparing vectors at one point in a manifold to vectors at another point of the same manifold. This means that a vector lives in the tangent space of just one point on our manfold, and doesn't 'exist' anywhere else. There is no canonical way to compare $T_p(\mathcal{M})$ and $T_q(\mathcal{M})$ (yet!).

The tangent space at a point p is spanned by the derivatives of parameterized paths through p, since these paths can specify a direction (and magnitude) at p. Notice that changing the shape of the path changes the direction that the vector points, while reparameterization of the path changes the 'length' (in a certain sense) of the vector defined by the path. Take a chart whose domain contains p, i.e. choose local coordinates x^i on the manifold, and consider the set of n paths γ_j given by holding each x^i constant except for one, x^j, and using x^j as the parameter for the path. These paths form a basis for all paths through p, and therefore the derivatives of these paths at p form a basis for $T_p(\mathcal{M})$. We can then write any element $v_p \in T_p(\mathcal{M})$ as $v_p = v_p^1 \gamma_1^{'}(p) + ... + v_p^n \gamma_n^{'}(p) = v_p^i \gamma_i^{'}(p)$, where the prime denotes an absolute derivative with respect to the

path parameter, and the last expression uses Einstein summation notation, which implies a sum over the hopefully obvious range of an index that is repeated, once 'upstairs' and once 'downstairs' (in this case the range is from $i = 1$ to $i = n$). From this expression, it is obvious that the tangent space at a point of an n-dimensional manifold is n dimensional.

We will take this approach one step further and actually define a vector at a point by the derivative of functions taken along the direction of the vector at that point. We can define directional derivatives adopted to a coordinate system by $\frac{\partial}{\partial x^i} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ by $\frac{\partial}{\partial x^i} f = D_i \widetilde{f}$, where D_i is the normal derivative on euclidean space in the x^i direction. The space of all directional derivatives at some point is therefore n-dimensional, and as noted above, so is the tangent space, which means this association can be made one-to-one. In addition, we want $T_p(\mathcal{M})$ to have vector space structure, so that we can we can compare two vectors at the same point, among other reasons. The directional derivatives at a point already have the desired structure, so it is automatically inherited by $T_p(\mathcal{M})$ due to the identification of vectors with directional derivatives. Observe that, because of it's definition in terms of the 'normal' derivative operator,

$$\frac{\partial}{\partial x^i}(f+g) = \frac{\partial f}{\partial x^i} + \frac{\partial g}{\partial x^i} \tag{A.6}$$

$$\frac{\partial}{\partial x^i}(af) = a\frac{\partial f}{\partial x^i} \tag{A.7}$$

for all $f, g \in C^\infty$ and $a, b \in \mathbb{R}$. Recall the set of paths γ_i previously introduced, that held all coordinates constant except x^i, and used x^i as the parameter, and make the obvious assocation of $\gamma_i^{'}(p)$ with $\frac{\partial}{\partial x^i} \mid_p$. Then any vector can be written in terms of partial derivatives; $v_p = v_p^i \gamma_i^{'}(p) = v_p^i \frac{\partial}{\partial x^i} \mid_p$, where the v_p^i are just numbers, and now we can add vectors and multiply vectors by scalars, as in equations 6 and 7, simply by defining vectors by their action on functions. For example,

$$(aw_p + bv_p)f = (aw_p^i \frac{\partial}{\partial x^i} \mid_p + bv_p^i \frac{\partial}{\partial x^i} \mid_p)f = aw_p(f) + bv_p(f) \tag{A.8}$$

Notice that vectors have adopted an additional and important feature, the liebnitz law:

$$v_p(fg) = v_p^i \frac{\partial(fg)}{\partial x^i} \mid_p \tag{A.9}$$

$$= v_p^i (f(p)\frac{\partial g}{\partial x^i} \mid_p + g(p)\frac{\partial f}{\partial x^i} \mid_p) \tag{A.10}$$

$$= f(p)v_p^i \frac{\partial g}{\partial x^i} \mid_p + g(p)v_p^i \frac{\partial f}{\partial x^i} \mid_p \tag{A.11}$$

$$= f(p)v_p(g) + g(p)v_p(f) \tag{A.12}$$

where $fg \in C^\infty$ is defined by $(fg)(p) = f(p)g(p)$. Although we've described the identification between vectors and partial derivatives in terms of coordinates, note that the

construction (a path in the manifold and its derivative at a point) is purely geometric in nature, and therefore the identification does not depend on the choice of coordinates. If we have two coordinate systems, x^i and x'^i, and they are related by $x'^i = x'^i(x^j)$, then the coordinate expression of some vector $v = v^i \frac{\partial}{\partial x^i} = v'^i \frac{\partial}{\partial x'^i}$ is given by $v'^j = v^i \frac{\partial x'^j}{\partial x^i}$.

From this viewpoint, a vector $v_p \in T_p(\mathcal{M})$ can be seen as a function $v_p : C^\infty(\mathcal{M}) \to \mathbb{R}^n$, where v_p eats a function and spits out that the value of that function's derivative at p along the path who'se derivative at p is v_p. This function is linear (eqs 6-7) and obeys the leibniz law (eq 8). From the tangent spaces at each point, we can construct the tangent space of the entire manifold $T(\mathcal{M}) = \bigcup_{p \in \mathcal{M}} T_p(\mathcal{M})$. An element of $T(\mathcal{M})$ is called a vector field, and intuitively amounts to the choice of one vector at each point of a manifold. A vector field is then a function from functions to functions (yikes!), that is , it is linear and obeys the liebniz law. In fact, if one considers the space of all functions $v : C^\infty \to C^\infty$ such that

$$v(\alpha f + \beta g) = \alpha v(f) + \beta v(g) \tag{A.13}$$

$$v(fg) = fv(g) + gv(f) \tag{A.14}$$

then not only do partial derivatives satisfy these conditions (when they are viewed as a function from C^∞ to C^∞, eating a vector and spitting out its derivative), but it can be shown that partial derivatives span this space! Therefore, vector fields are all such functions, and the value of that vector field at any point is the vector such that, when a function is acted on by the vector field, its value at that point is the derivative of that function, at that point, in the direction of that vector.

A.5 Dual Vectors

To sum up so far, we've got a set of points that locally looks like euclidean space. We considered infinitely differentiable functions on this space, $f : \mathcal{M} \to \mathbb{R}$, which assigned a real number to each point. Then we created a vector field, which conceptually assigns a vector to each point, but is seen as a function $v : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ that is bilinear and obeys the liebniz law, and we interperet vectors as directional derivatives to act on functions. Another notion that euclidean vectors have that we have not yet found an analogue for in our generalized vectors is length. We won't define the length of a vector quite yet, but what form will it take? It would need to be a machine that eats a vector and spits out its length, which is just a number (actually we'll end up defining it slightly differently, but this gives you the idea). Therefore, we need functions from vectors to numbers, i.e. some objects $\omega : T_p(\mathcal{M}) \to \mathbb{R}^n$. But no combination of what we've got

already does such a thing, so we will construct a new object: dual vectors, which have the desired domain and co-domain, and are linear: for some dual vector ω_p at p,

$$\omega_p(av_p + gw_p) = a\omega_p(v_p) + b\omega(w_p) \tag{A.15}$$

for all $v_p, w_p \in T_p(\mathcal{M})$, and $a, b \in \mathbb{R}$. The space of all such functions is called the cotangent space, $T_p^*(\mathcal{M})$, and is another vector space. The reader may be becoming weary of these constructions - what's to stop us from going on forever, first by defining a new space $T_p^{**}(\mathcal{M})$, with objects as functions $\gimel : T_p^*(\mathcal{M}) \to \mathbb{R}$? First of all, these objects are not tiresome, but exciting, for their construction came for free with the definition of a manifold. But unfortunately, there are no new spaces to make; $\gimel$ turns out to be a boring old vector, i.e. $T_p^{**}(\mathcal{M}) = T_p(\mathcal{M})$. Again, one can take the union of the dual tangent spaces at each point and create $T^*(\mathcal{M})$, whose elements are dual vector fields and are functions from $T(\mathcal{M})$ to $C^\infty(\mathcal{M})$. This is obvious when one views vectors as linear functions on dual vectors, defined by $v(\mu) = \mu(v)$ for all vectors v and dual vectors μ.

So what are these dual vector fields? If the reader is not already familiar with them, and wishes to dramatically revise and simplify his or her understanding of vector calculus, then the author strongly recommends reading all about them (Baez). However, there is far too much to say in this space here, which I am trying (but failing) to keep brief, so I will neglect most of the elegance of these dual vectors, and explain them instead only the context that is of most immediate concern to this thesis, which is their relationship to vectors with regards to a metric. In euclidean space, there is an operator called the dot product, which eats two vectors and spits out a number, which is supposed to represent their magnitude along one another. The square root of the dot product of a vector with itself corresponds to the length of the vector. But what is the length of a vector on a manifold? Unfortunately, there is no analogous dot product operator: one could define a dot product like on euclidean space, where we simply take a coordinate basis and sum the components squared, but this operator depends on the choice of coordinates, and is therefore useless. This is where dual vectors come in; they allow us to get from vectors to real numbers in a coordinate independent way. A metric, among other things, gives a unique correspondence between vectors and dual vectors, which is why we don't realize dual vectors exist in euclidean space, because whenever we work with what is secretely a dual vector, we simply work with its vector counterpart. The reader may have encountered these creatures before, in quantum mechanics. In bra-ket notation, a ket $|\psi\rangle$ represents a vector in a Hillbert space. A bra $\langle\phi|$ is then a dual vector, because it acts on kets to produce a number: $\langle\phi\,|\,\psi\rangle$. This is considered to be the inner product, or dot product of these two functions, which are also vectors because they live in a vector space, Hillbert space. These bras are linear: $(\langle a\psi| + \langle b\xi|)(|c\phi\rangle + |d\theta\rangle) =$

$ac\langle\psi|\phi\rangle + ad\langle\psi|\theta\rangle + bc\langle\xi|\phi\rangle + bd\langle\xi|\theta\rangle$ for any complex numbers a, b, c, d. Kets *are* dual vectors, just expressed in very different notation.

Suppose we have a basis for vectors at some point, $\frac{\partial}{\partial x^i} = \partial_i$, where the second symbol is an abbreviation of the first that we will use commonly, and I've omitted the $|_p$ because I don't think there's much room for confusion and we'll want to generalize to vector and dual vector fields soon anyways. Then the functions dx^i defined by $dx^i(\partial_j) = \delta^i_j$ form a basis for the dual vectors at that point. For a dual vector $\omega = \omega_i dx^i$ and vector $v = v^i\partial_i$, $\omega(v) = \omega_i dx^i(v^j\partial_j) = \omega_i v^j dx^i(\partial_j) = \omega_i v^j \delta^i_j = \omega_i v^i$. Now the reader sees why it is so convenient to put indices upstairs for dual vectors, and downstairs for vectors. Using dx^i to represent dual vectors actually contains some other meaning as well. The operator $d : C^\infty \to T^*(\mathcal{M})$, called the exterior derivative, is defined by $df(v) = v(f)$, and is what one normally thinks of as the gradient, seeing as it takes a function and returns an object that, when acted on by a vector, yields the derivative of the function in that direction. Notice that the gradient is fundamentally a dual vector, since it requires a vector to yield a function (or a number if we're evaluating the gradiant of a vector field at some point). In euclidean space, we have a metric and therefore simply think of the gradient as its associated vector, $\nabla = (\partial_x, \partial_y, \partial_z)$, but deep down it's secretely a dual vector. If we take the coordinate functions $\widetilde{x}^i$, and take their exterior derivatives, we have the one forms $d\widetilde{x}^i$, but to unclutter the notation we'll simply call them dx^i. Notice, then, that $dx^i(\partial_j) = \partial_j x^i = \delta^i_j$ automatically. Without explaining precisely how, hopefully at this point the reader will see that when one performs integration, it is really over dual vectors, because dx is now seen to be a dual vector. When one integrates over more than one variable, it's really integration over a higher dimensional dual vector called a p-form, which I will now define, but please don't confuse the point p on our manifold with the order p of a form. Since we have the vector space $T^*_p(\mathcal{M})$, let us consider $T^*_p(\mathcal{M}) \otimes T^*_p(\mathcal{M})$, whose elements will take two vectors and spit out a number (this is what a dot product does, and this is what our metric will be!). Another name for a dual vector is a one-form, and more generally a p-form is an element of $T^*_p(\mathcal{M}) \otimes$...p times...$\otimes T^*_p(\mathcal{M})$ that is completely antisymmetric, meaning that a p-form switches sign under exchange of any two of its arguments. For a two form T, this means that $T(v, w) = -T(w, v)$ for any vectors v and w.

A function is a 0-form, because it accepts zero vectors and spits out a function, itself. It turns out that the exterior derivative can be naturally extended from a function from 0-forms to 1-forms to a function from p-forms to (p+1)-forms. As previously stated, the exterior derivative acting on zero-forms is the gradient, but when it acts on one-forms it is the curl, and when it acts on 2-forms it is the divergence (this doesn't really make sense yet, but at the end of this section I'll introduce the hodge star operator, after which I will be able to demonstrate this claim). The content of the divergence theorem,

stokes' theorem, and the fundamental theorem of calculus can all be expressed in one relationship, called the generalized stokes' theorem:

$$\int_M d\omega = \int_{\partial M} \omega \tag{A.16}$$

where ω is a p-form, M is the manifold that it lives on, and ∂M is the manifold that is the boundary of the manifold M. I won't explain here how to integrate over p-forms [21] but there is an important consequence of this equation that is worth noting. It can be shown relatively easily that the exterior derivative applied twice to any form is 0, i.e. that

$$d^2 = 0 \tag{A.17}$$

This fact essentially boils down to the commutivity of partial derivatives. This useful fact, combined with Stokes' theorem, allows us to prove an important fact about manifolds, that the boundary of its boundary is zero: for any form ω,

$$\int_M d^2\omega = \int_{\partial M} d\omega = \int_{\partial\partial M} \omega = 0 \tag{A.18}$$

$$\Rightarrow \partial\partial M = 0 \tag{A.19}$$

This simple equation has enormous physical consequences, including but not limited to two of maxwell's equations in Electromagnetism, and the conservation of energy and momentum in General Relativity. I hope the reader is convinced that p-forms have an incredible unifying power, and although perhaps unfamiliar, are a natural and useful way to re-imagine many old ideas.

A.6 Tensors

We've taken products of dual tangent spaces to create new vector spaces ('vector space' here referring to the abstract mathematical structure, not to the tangent space where vectors live. Confusing, I know.), so we may as well throw tangent spaces into the mix and define a more general object called tensors, which have a type (p, q), and lives in the vector space $\mathcal{T}^{p,q}(\mathcal{M})$. $\mathcal{T}^{p,q}(\mathcal{M})$ is formed by taking the product of p tangent spaces and q cotangent spaces. We now see that functions are $(0, 0)$ tensors, vectors are $(1, 0)$ tensors, and p-forms are antisymmetric $(0, p)$ tensors.

Recall when we wrote vectors in terms of a basis, for example $v = v^i\partial_i$. The coeffecients v^i are given by v acting on the coordinate functions, i.e. $v^i = v(x^i)$. When working with vectors in euclidean space, one often mistakes a vector for its coordinates, making

a statement such as $v = (v_x, v_y, v_z)$, which we now see to be wrong, but is something that we will continue to do sometimes, that is to confuse a tensor with its components. More generally, the components of a tensor T in some basis are given by $T^{\mu\nu...\rho}{}_{\alpha\beta...\gamma} = T(\partial_\alpha, \partial_\beta, ..., \partial_\gamma, dx^\mu, dx^\nu, ..., dx^\rho)$, so that $T = T^{\mu\nu...\rho}{}_{\alpha\beta...\gamma}\partial_\mu \otimes \partial_\nu \otimes ... \otimes \partial_\rho \otimes dx^\alpha \otimes dx^\beta \otimes ... \otimes dx^\gamma$. Because tensors are linear in each slot, knowing its components in some basis allows us to know its value when acting on anything. For instance, if we have a $(0, 2)$ tensor T, and we know $T_{\mu\nu} = T(\partial_\mu, \partial_\nu)$ for some coordinate system x^μ, then we know how T acts on any two arbitrary vectors, say v and w; $T(v, w) = T(v^\mu\partial_\mu, w^\nu\partial_\nu) = v^\mu w^\nu T_{\partial_\mu,\partial_\nu} = v^\mu w^\nu T_{\mu\nu}$. This is much like how, if we measure, say, the magnitude of an electric field at some point in three linearly independent directions, then we know the entire vector that represents the electric field at that point.

There is an important operation on tensors, called contraction, which is most easily explained in terms of a coordinate expression of a tensor, but does not actually depend on which coordinate system we are using, thanks to the relationship between vectors and dual vectors. A type (p, q) tensor can be turned into a type $(p-1, q-1)$ tensor by inserting a sum over a vector and dual vector basis into a vector and dual vector slot respectively. This idea is best understood through example; If we have a $(2, 2)$ tensor T, then it can be written as $T = T^{\mu\nu}{}_{\sigma\rho}\partial_\mu \otimes \partial_\nu \otimes dx^\sigma \otimes dx^\rho$. As remarked, in relativity one is often sloppy and confuses a tensor with its coordinates, mistakenly calling $T^{\mu\nu}{}_{\sigma\rho}$ the tensor T. Expressed in a coordinate basis like so, contraction is simple; the contraction of T on the second and fourth indices is given by $T^{\mu\nu}{}_{\sigma\nu}$, where summation is implied by Einstein's summation convention.

A.7 Derivatives of Tensors

At this point, we've got a differentiable manifold, on which were able to define tensor fields. We can differentiate $(0, 0)$ tensors, because the value of a function at different points is just a number, which are trivial to compare. Vectors fields, however, are vector-valued at a point, and as previously discussed, there is no canonical way to compare vectors at different points, which means any derivative operator will have some arbitrariness built into it. Let's go ahead and build one anyways, even if it's arbitrary. We want a derivative operator, call it ∇, to turn a (p, q) tensor into a $(p+1, q)$ tensor, with the extra vector slot to specifiy the direction of the derivative, and the remaining (p, q) tensor after eating the direction vector will represent the derivative of the original (p, q) tensor along that direction. The following exposition is based off of [11]. First of all, we want our derivative operator to work on functions like a normal partial derivative,

i.e.

$$\nabla_\mu f = \partial_\mu f \tag{A.20}$$

Notice here how the operator as turned a $(0,0)$ tensor into a $(1,0)$ tensor. We would also like this operator to be linear, and obey the leibnitz law:

$$\nabla_\sigma(aT^{\mu\ldots\nu}{}_{\alpha\ldots\beta} + bS^{\mu\ldots\nu}{}_{\alpha\ldots\beta}) = a\nabla_\sigma T^{\mu\ldots\nu}{}_{\alpha\ldots\beta} + b\nabla_\sigma S^{\mu\ldots\nu}{}_{\alpha\ldots\beta}) \tag{A.21}$$

$$\nabla_\sigma(T^{\mu\ldots\nu}{}_{\alpha\ldots\beta}S^{\rho\ldots\lambda}{}_{\gamma\ldots\delta}) = (\nabla_\sigma T^{\mu\ldots\nu}{}_{\alpha\ldots\beta})S^{\rho\ldots\lambda}{}_{\gamma\ldots\delta} + T^{\mu\ldots\nu}{}_{\alpha\ldots\beta}(\nabla_\sigma S^{\rho\ldots\lambda}{}_{\gamma\ldots\delta}) \tag{A.22}$$

for all tensors T and S, and all real numbers a and b (the tensor product is ommitted here, writing $T^{\mu\ldots\nu}{}_{\alpha\ldots\beta}S^{\rho\ldots\lambda}{}_{\gamma\ldots\delta}$ instead of $T^{\mu\ldots\nu}{}_{\alpha\ldots\beta} \otimes S^{\rho\ldots\lambda}{}_{\gamma\ldots\delta}$). We are still a long way from completely specifying our derivative operator because there are many operators which satisfy the conditions posed so far, and so we introduce a couple more:

$$\nabla_\sigma T^{\mu\ldots\nu\lambda}{}_{\rho\ldots\gamma\lambda} = (\nabla T)^{\mu\ldots\nu\lambda}{}_{\sigma\rho\ldots\gamma\lambda} \tag{A.23}$$

$$\nabla_\mu\nabla_\nu f = \nabla_\nu\nabla_\mu f \tag{A.24}$$

The first one essentially says that it doesn't matter if we contract before or after taking a derivative - this aligns with our intuitive notion of a derivative, where we're simply evluating the tensor at two points and taking the limit as the distance between those points shrink to zero. Whether we contract and then subrtract, or subtract and then contract, should make no difference. The second condition is called the Torsion Free condition, and although there author is unaware of any good reason it must be true mathematically, when one does ultimately build a theory of gravity in this language, one finds no evidence for it experimentally. Finally, using these five conditions, we can say something concrete about our derivative; one can show that the difference between the action of any two derivative operators on a vector must be $(1,2)$ tensor, i.e. $\nabla_\mu v_\nu - \nabla'_\mu v_\nu = C^\sigma_{\mu\nu}v_\sigma$ for some tensor C, where ∇ and ∇' are any two operators which obey the conditions enforced so far. In some coordinate basis, ∂_μ can serve as a derivative operator, but the trouble is in some other coordinate basis, $x^{\mu'}$, this derivative operator will not transform into the derivative operator associated with the primed basis, i.e. $\partial'_\mu \neq \partial_{\mu'}$ (otherwise we would just use it as the derivative operator!). However, if we have one specific operator chosen, then we do know that in any basis the derivative operator can be expressed as $\nabla_\mu v_\nu = \partial_\mu v_\nu + C^\sigma_{\mu\nu}v_\sigma$, because ∂_μ is a valid operator. Notice that, by defining how the operator works on vectors defines how it works on all tensors, because we can work out its action on dual vectors by $\nabla(v^i\omega_i) = \partial(v^i\omega^i) = v^i\nabla(\omega_i) + \omega_i\nabla(v^i)$, and use induction to get its action on any tensor (as we also know how it acts on functions). Therefore, using the five conditions we have narrowed down but not completely specified

our derivative operator. Even though it is arbitrary, we could simply pick one of the many valid operators, and therefore have a nice derivative operator on our manifold.

Equipped with a derivative operator, we are now able to make precise this method of moving vectors around that I have previously referred to as parallel transport. Essentially, we want to take a vector v (or more generally a tensor) and transport it along some path in our manifold γ, all the while mainting its orientation and length in a certain sense. The sense in which the vector is not changing is that its derivative is zero along the path; recall the equation given above for dragging a vector around $\mathbb{R}^n$. Similarly to this expression, If t^μ is the tangent vector along the path, then a selection of vectors v^ν out of the tangent space at each point the path crosses that can be thought of as one vector being parallel transported along the path is the set of vectors such that at each point along the path, $t^\mu \nabla_\mu v^\nu = 0$. Or more generally, for any tensor,

$$t^\sigma \nabla_\sigma T^{\mu\ldots\nu}_{\alpha\ldots\beta} \tag{A.25}$$

A.8 The Metric

Notice that up until this point, everything has been defined without the use or choice of a metric. It is surprising that so much can be said about a manifold who'se shape we don't even know! But at last, it has come time to formally introduce our soon-to-be best friend, the metric. For with the aid of parallel transport, and the choice of a metric on our manifold, there is one final, necessary condition on our derivative operator that will specify it uniquely. Fortunately, we have built much of the machinery already, so defining the metric will be easy. A metric g is a symmetric, non-degenerate rank $(0, 2)$ tensor on our manifold. Symmetric means that $g(v, w) = g(w, v)$, and non-degenerate means $g(v, w) = 0$ for all vectors w only in the case where $v = 0$, meaning that for any function f, $v(f) = 0$. Notice that, our tangent space being a vector space, a zero vector must exist at each point. We physically interperet the metric as a generalized dot product, and therefore the length of a vector v is given by $\sqrt{\pm g(v, v)} = \sqrt{\pm g_{\mu\nu} v^\mu v^\nu}$. The plus or minus sign is in there to make sure we're not trying to take the square root of a negative number. In three-dimensional euclidean space, the metric is given by $g = dx^2 + dy^2 + dz^2$, and is positive definite, meaning it always yields positive numbers or zero, and so we choose a plus sign for the dot product. This means that $v \cdot w = g(v, w) = (dx^2 + dy^2 + dz^2)(v^x\partial_x + v^y\partial^y + v^z\partial_z)(w^x\partial_x + w^y\partial^y + w^z\partial_z) = v^x w^x + v^y w^y + v^z w^z$ because all terms of the form $dxdx(\partial_x\partial_y)$ are zero, because $dx^i(\partial_j) = \delta^i_j$.

As promised, the metric gives us a way to find an isomorphism between $T_p(\mathcal{M})$ and $T^*_p(\mathcal{M})$. The metric can be seen as a function from two vectors to numbers, or as a

function from one vector to a dual vector, since when it eats one vector it's still got one vector slot left open before it becomes a number. This the way we will identify vectors with dual vectors, and the non-degenerate condition ensures that this identification is an isomorphism. We denote the one-form associated with the vector v by $g(v)$. We can also use the metric to define the length of a path; the length of some path $\gamma : [0,1] \to \mathcal{M}$, with tangent vector field t, is given by

$$\int_0^1 \sqrt{\pm g(t(\gamma(s)), t(\gamma(s)))} ds \tag{A.26}$$

Notice that, while this definition is independent of parameterization, it is not independent of choice of path. Therefore we can't actually define the distance between two points (which I kind of hinted before that we could do, sorry about that), we can only find the length of some path connecting the two points (spoiler alert/physics digression - just like how we extremize the action in classical mechanics to find the path that a particle takes, in general relativity we will extremize the length of a path that the particle takes through spacetime!).

With our metric in hand, there is now another condition that we would like our derivative operator to have. When we parallel transport vectors, not only should their orientations stay constant, but so should their lengths. This means that, for any two parallel transported vectors v and w, we want

$$0 = t^\sigma \nabla_\sigma g(v,w) = t^\sigma \nabla_\sigma (g_{\mu\nu} v^\mu w^\nu) = v^\mu w^\mu t^\sigma \nabla_\sigma g_{\mu\nu} + g_{\mu\nu} w^\nu t^\sigma \nabla_\sigma v^\mu + g_{\mu\nu} v^\mu t^\sigma \nabla_\sigma w^\nu \tag{A.27}$$

But since v and w are parallel transported, the second and third terms in the last expression are zero. If we want this condition to hold for any vectors v, w, and t, then we require that

$$\nabla_\sigma g_{\mu\nu} = 0 \tag{A.28}$$

This is called metric compatibility, and is enough to uniquely determine the tensor $C^\sigma_{\mu\nu}$ in our definition of the derivative operator. We call these special metric-compatible objects connection coeffecients, and they are

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu}) \tag{A.29}$$

Therefore, once we pick a metric on our manifold, we are also given a derivative operator on tensors, which can be shown to be defined by

$$\nabla_\sigma T^{\mu\ldots\nu}_{\alpha\ldots\beta} = \partial_\sigma T^{\mu\ldots\nu}_{\alpha\ldots\beta} + \Gamma^\mu_{\sigma\lambda} T^{\lambda\ldots\nu}_{\alpha\ldots\beta} + \ldots + \Gamma^\nu_{\sigma\lambda} T^{\mu\ldots\lambda}_{\alpha\ldots\beta} - \Gamma^\lambda_{\sigma\alpha} T^{\mu\ldots\nu}_{\lambda\ldots\beta} - \ldots - \Gamma^\lambda_{\sigma\beta} T^{\mu\ldots\nu}_{\alpha\ldots\lambda} \tag{A.30}$$

This is exciting indeed, because as we will soon see, much of physics can be done by assuming spacetime is a manifold, with physical quantities given by tensors on the manifold. This viewpoint has the principal of relativity built into it, because it states that interactions in spacetime are geometric in nature, and any coordinates that we introduce to quantify them are arbitrary and a different choice of coordinate ought to lead to the same physical predictions. The coordinate independence of the definition of tensors ensures this will work. But before we get to apply the mathematical ideas we have just introduced to physics, there is one last operation I would like to define, called the hodge star operator, but don't worry, it's pretty straightforward, and quite neat.

A.9 The Hodge Star Operator

The hodge star operator is essentially a function from p-forms to (n-p)-forms, where n is the dimension of the manifold that the forms live on. We define the n-form $\epsilon_{\alpha\beta...\gamma}$ to be the completely antisymmetric $(0, n)$ tensor with $\epsilon_{12...n} = +\sqrt{Det[g]}$ where $Det[g]$ is the normal sense of the determinant of a matrix, when the components of g in some basis are written out in an n by n matrix, and the numbers in coordinate slots indicate to evaluate the tensor with the first basis vector in the first slow, the second basis vector in the second slot, et cetera. Notice that specifying one element of an n-form specifies each element, so the space of n-forms is 1 dimensional, and in fact any p-form with $p > n$ must be completely zero, due to antisymmetry. The dimensions of the space of all p-forms in an n-dimensional manifold is a simple combinatorics problem, and one finds that the space of all p-forms is the same size as the space as all $(n-p)$-forms. The hodge star operator gives us a one-to-one map between these two spaces, and is defined as follows: for any p-form ω on an n-dimensional manifold,

$$(\star\omega)_{\mu...\nu} = \frac{1}{p!}\epsilon_{\mu...\nu}{}^{\sigma...\rho}\omega_{\sigma...\rho} \tag{A.31}$$

where the indices $\sigma...\rho$ run over p indices, and $\mu...\nu$ runs over $n-p$. Notice that this map contains an arbitrary minus sign in, which was built into the definition of ϵ. This means that while an identification can be made between forms, it is arbitrary up to a minus sign. Some very cool things happens in $n = 3$, which you will recognize as regular vector calculus, but can now be seen to hinge on the qualities of forms in three dimensions. First, the space of all 2-forms is the same size as the space of all 1-forms, so we can use the hodge star to turn one into the other, which as I stated before, allows us to take the exterior derivative of a one form, which is a two form, but then re-map it back onto a one-form so that it can be interpereted as a vector when a metric is given. This vector that the exterior derivative of a one-form is interpereted as is what is usually

known as the curl of the original vector version of the one-form that had its exterior derivative taken. Symbolically, this means that for some vector v in three-dimensional euclidean space, $g(\nabla \times v) = \star dg(v)$, where g acting on a vector is that vector's associated one form. The cross product requires an arbitrary choice of being right or left-handed, which amounts to a minus sign, and we see this arbitrariness manifest in the hodger star operator. The advantage of the new notation is immediate; not only do we understand where the arbitrary minus sign is coming from, but we can now have a curl on any manifold of any dimensions, which would not have been possible with the old definition. Additionally, the space of all 3-forms is one dimensional in $n = 3$, which is the same dimensions as the space of real numbers. This allows us to take the exterior derivative of a two-form, which is a three-form, and interperet this three-form as a number. This corresponds to our usual notion of the divergence: $\nabla \cdot v = \star d \star g(v)$. We now also have the ability to take divergences on any manifold.

A.10 Math Summary

That was a whole lot of math condensed to a very small space. For the sake of brevity, I omitted many details which would aid in the reader's understanding, if he or she is not already familiar with the material. Fortunately, while much of what I have said here is elegent and interesting, it is not all necessary for understanding the application of manifold structure to physics. I record here a quick recap of what we just did, highlighting the important points to remember; we took a set of points and gave it a topology and assumed it was a (differentiable) manifold. This structure (most importantly, the charts from being a manifold) allowed us to introduce local coordinates anywhere we pleased on the manifold, and these coordinates allowed us to take directional derivatives of functions on the manifold. At any given point, the directional derivatives at that point form a vector space, called the tangent space, and we call elements of the tangent space vectors, which act on functions to yield numbers. We were then able to define another vector space, the cotangent space, the elements of which act on vectors to yield numbers. By taking arbitrary products of tangent spaces and cotangent spaces we can build a legion of new vector spaces at each point, each of which contain tensors of different types. By choosing a tensor at each point in the manifold, we can create a tensor field. We wish to be able to differentiate tensor fields, but finding a derivative operator proved difficult, so we defined a metric on our manifold which is a type $(0, 2)$ tensor. It eats two vectors and spits out their generalized dot product, and by ensuring that our derivative operator was compatible with this metric, in addition to a few other

necessary conditions, we were able to pin down our derivative operator on tensors. Finally, we defined the hodge star operator, which identifies antisymmetric $(0, p)$ tensors with $(0, n-p)$ tensors, where n is the dimension of our manifold.

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