LARGE CARDINALS

OLIVER PECHENIK

Abstract. Infinite sets are a fundamental object of modern mathematics. Surprisingly, the existence of infinite sets cannot be proven within mathematics. Their existence, or even the consistency of their possible existence, must be justified extra-mathematically or taken as an article of faith. We describe here several varieties of large infinite set that have a similar status in mathematics to that of infinite sets, i.e. their existence cannot be proven, but they seem both reasonable and useful. These large sets are known as *large cardinals*. We focus on two types of large cardinal: *inaccessible cardinals* and *measurable cardinals*. Assuming the existence of a measurable cardinal allows us to disprove a questionable statement known as the Axiom of Constructibility ($V = L$).

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1. Introduction

The Ancient Greeks were very wary of infinite sets. With only one extant clear exception, all of their mathematics was developed using only finite objects and finite processes.[1] For modern mathematics, such an approach is no longer tenable. We necessarily use infinite sets all the time, especially given our set-based approach to continuous objects in geometry and topology.

Yet it is not possible to prove the existence of an infinite set. If there were to be a proof that such a set exists, it would be in the branch of mathematics known as set theory. However, in the most widely used axiomatization of set theory, the Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC), infinite sets are created by dictum: ZFC contains a statement known as the Axiom of Infinity (Inf), which postulates nothing more than the existence of an infinite set. Our first goal in this paper will be to see that the remaining axioms, in the absence of Inf, do not allow us to prove that there are any infinite sets.

The remainder of this paper is devoted to considering stronger axioms of infinity. Like Inf, each axiom will postulate the existence of sets larger than can be demonstrated without the axiom. Again like Inf, none of these axioms will be statements we can prove from the other axioms. We must accept or reject these stronger axioms according to our intuitions about the shape of the mathematical universe and our intuitions about the validity of results obtained from them.[2]

Just as the Ancient Greeks rejected infinite sets and carried out all their mathematics in a finitist framework, we can choose to reject these large infinities and perform our mathematics without them. Indeed, nearly all of contemporary mathematics outside of set theory proper is built without the use of these large sets. However, developing comfort with these large sets may allow us to enrich the edifice of mathematics in wonderful ways, just as the Greeks might well have developed true differential and integral calculus, if only they had been willing to work with infinity in a more open way.

[1]A decade ago, I could have safely written this sentence without the concession at front. Recently, however, it has been discovered that Archimedes explicitly used infinite sets in his work. In particular, he is known to have considered the set of all cross-sections of a geometric object, and further to have considered performing an operation on each section. See the popular accounts [17], [18], and the academic paper [19] for a reinterpretation of Archimedes' work in light of these newly discovered techniques. It seems, however, that the use of infinite sets died with Archimedes in the Siege of Syracuse, not to reappear for over a thousand years.

[2]The philosophical considerations governing our acceptance or rejection of these axioms are covered at length in [14], [15], and [21]

2. Basic Objects of Set Theory

Set theory is the branch of mathematics that focuses on working rigorously with infinite sets. Set theory developed gradually out of analysis, where such concerns first became prominent, until eventually becoming a mathematical field in its own right. Because infinite sets now pervade mathematics, set theory now has applications to all other fields of mathematics.

The first important results of set theory are the following theorems, now well-known.

Theorem 2.1 (Cantor)**.** *There is no injection* $f : \mathbb{R} \to \mathbb{N}$*.*

Theorem 2.2 (Cantor)**.** *There is no injection* $g : P(X) \to X$ *for any set* X*, where* $P(X)$ *represents the set of all subsets of* X*.*

These results showed that infinite sets come in different sizes. Indeed, since every set has more subsets than elements, there are infinitely many different sizes of infinite set and there is no largest size a set can have. The structure of this infinite class of sizes is a central object of study in set theory. The sizes are called *cardinal numbers* or just *cardinals.*

Cantor thought of cardinals as being abstracted properties of infinite sets with a different sort of existence from the underlying sets.[3] It is more useful however to define a cardinal to be a particular set of that cardinality. To determine which set should represent the cardinality of a class of equinumerous sets, we need first to look at a class of objects called *ordinals.*

Definition 1. A set α is an *ordinal* if

(1) elements of elements of α are always elements themselves of α, and
(2) α is well-ordered by $\in$ (set membership).

Sets with property (1) are called *transitive.* If we write this property of α as $(y \in x$ and $x \in \alpha) \implies y \in \alpha$, then the name 'transitive' makes a lot of sense. We will see transitivity in a different context later.

The class of ordinals is linearly-ordered, indeed well-ordered, by set membership: every ordinal is the set of all smaller ordinals. For example, the smallest ordinals are $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, and $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$. We identify these ordinals with the natural numbers $0, 1, 2, 3$, and 4, respectively. Similarly, there is an ordinal corresponding to every natural number n; we call the corresponding ordinal n as well. If we want to build arithmetic out of set theory, we can define the natural numbers to *be* these ordinals. Alternately, we can assume that we already know what the natural numbers are, and merely note that these ordinals can function isomorphically. For our purposes, it does not matter whether these ordinals actually *are* the natural numbers or whether they just behave like

[3]Both [16] and the introduction to [10] have interesting material about Cantor's mathematical philosophy.

the natural numbers; the important point is that we can use them in place of the natural numbers whenever convenient.

If we take the union

$$\omega = \bigcup_{n \in \mathbb{N}} n,$$

we obtain another ordinal. This ordinal ω functions like the set of all natural numbers $\mathbb{N}$.[4] Unlike the natural numbers however, the ordinals do not stop at ω. For example, we may take $\{\omega, \{\omega\}\}$ to form the ordinal $\omega + 1$. Indeed, it is clear that for every ordinal α, there is a larger *successor ordinal* $\{\alpha, \{\alpha\}\} = \alpha + 1$. Also, a union of ordinals is itself an ordinal, at least as large as any of the ordinals in the union. Ordinals, such as ω, that are not successor ordinals, are called *limit ordinals.*

When we were looking at finite ordinals, each one had more elements than the one before; the ordinal n was a set with n elements. This is no longer the case with infinite ordinals. It is easy to find a bijection, for example, from ω to $\omega + 1$ (send 0 to ω, and all other $n \in \omega$ to $n - 1$). Every infinite ordinal is equinumerous with its successor. However, limit ordinals can be equinumerous with each other as well. The ordinal

$$2\omega = \bigcup_{n \in \mathbb{N}} (\omega + n)$$

is a limit ordinal, but it is equinumerous with ω since it looks just like two copies of ω. If we take, however, the union

$$\omega_1 = \bigcup_{\alpha \text{ is countable}} \alpha,$$

then we get a limit ordinal with uncountably many elements. (If ω_1 were countable, it would have to be an element of itself, which is not the case for any ordinal.) This fact inspires our definition of cardinals as the least ordinals of each size.

Definition 2 (von Neumann)**.** A set κ is a *cardinal* if κ is an ordinal that cannot be injected into any of its elements.

Every finite ordinal is a cardinal. The cardinality of all countable sets is ω, which goes by the name of $\aleph_0$ when we are thinking of it as a cardinal. The cardinals are well-ordered by $\in$, just like the ordinals. The next largest cardinal after $\omega = \aleph_0$ is ω_1, which we also call $\aleph_1$. Similarly, every infinite cardinal κ has a *successor cardinal*

$$\kappa^+ = \bigcup_{|\alpha| = \kappa} \alpha.$$

[4]The n's in the union are ordinals, whereas the indexing n's are natural numbers, which are possibly ontologically different. However, as we said before, this distinction has no significance for us. Similarly, we may identify $\mathbb{N}$ with ω as convenient.

Note that the successor cardinal of $\aleph_0 (= \omega)$ is $\aleph_1$, which is not the same as $\omega + 1$, the successor ordinal of $\omega (= \aleph_0)$. Indeed, every infinite cardinal is a limit ordinal. However, not every cardinal is a successor cardinal. Cardinals, such as

$$\aleph_\omega = \bigcup_{n \in \mathbb{N}} \aleph_n,$$

that are not successor cardinals are called *limit cardinals*. The limit cardinals are precisely the cardinals $\aleph_\alpha$, where α is a limit ordinal.

We write $2^{\aleph_\alpha}$ for the number of subsets of a cardinal $\aleph_\alpha$. Cantor's Theorem 2.2 says that $2^{\aleph_\alpha} > \aleph_\alpha$, so $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$. The *Generalized Continuum Hypothesis (GCH)* is that this second inequality is always actually an equality. We will remain agnostic about the truth of GCH.[5]

Definition 3. A cardinal κ is a *strong limit cardinal* if $2^\lambda < \kappa$ for all $\lambda < \kappa$.

Every strong limit cardinal is a limit cardinal. If in fact GCH holds, then the two concepts are not distinct.

The distinction between successor and limit cardinals is very important. Equally important is the related distinction between *regular* and *singular* cardinals.

Definition 4. A infinite cardinal κ is *singular* if it can be written as $\bigcup_{\alpha<\lambda} \xi_\alpha$, where $\lambda < \kappa$ and every $\xi_\alpha < \kappa$. That is to say that singular cardinals are those that can be built out of a smaller number of smaller sets. If κ is not singular, then we say it is *regular*.

All singular cardinals are limit cardinals and all successor cardinals are regular cardinals. However, limit cardinals need not be singular. The cardinal

$$\aleph_\omega = \bigcup_{n \in N} \aleph_n$$

is an example of a singular limit cardinal, while $\aleph_0$ is an example of a regular limit cardinal.

	Successor	Limit
Regular	$\aleph_1, \aleph_{\omega+2}$	$\aleph_0$
Singular	none	$\aleph_\omega, \aleph_{2\omega^4}$

TABLE 1. Examples of cardinals that are regular successors, regular limits, and singular limits. Singular successor cardinals do not exist.

[5]By work of Gödel and Cohen, GCH is independent of ZFC: it can neither be proven nor disproven from the ZFC axioms. Later, we will discuss additional axioms that can be added to ZFC. *Large cardinal axioms* will leave GCH independent, while the *Axiom of Constructibility* allows GCH to be proved. It is also possible to add GCH itself as an axiom. If we assume all the axioms of ZFC except for Choice, and also assume GCH, then it is possible to prove the Axiom of Choice. However, from the assumptions ZFC + GCH, it is not possible to prove any of the large cardinal axioms or the Axiom of Constructibility.

3. Zermelo-Fraenkel-Choice Set Theory

Although the axioms of ZFC continued to be tweaked for about 50 years, the core of the axiomatization was developed by Zermelo in 1908. Zermelo's goal at the time was, not to give a foundation to all of set theory, but rather to isolate the assumptions he was using in the proof of one particular theorem.[6] Nonetheless, with slight modifications, his axiomatization became accepted as the canonical one for the general theory of sets.

Below, I describe the axioms of ZFC in informal language. For a more formal but still elementary exposition, see [13] or [16]. The historical circumstances and philosophical motivations underlying the acceptance of each axiom are admirably discussed in [14].

Axioms of ZFC:

- *Extensionality* A set is determined by its elements.
- *Pairing* Any two objects can be collected together into a set.
- *Separation* Given any set X and any property, the elements of X that have that property form a subset of X.[7]
- *Union* If X is a set of sets, then the elements of X may be unioned together to form a set.
- *Power Set* For any set X, there is a set $P(X)$ containing precisely the subsets of X.
- *Replacement* For any operation O and set X, there is a set $O(X) = \{O(x) : x \in X\}$.[8]
- *Choice* The Cartesian product of non-empty sets is non-empty.
- *Regularity* All objects are sets. Sets form a hierarchy of complexity where every set is more complex than any of its elements. In particular, no set contains itself as a member.
- *Infinity* There is a set that can be injected into a proper subset of itself.

All these axioms are patently true to the majority of contemporary mathematicians, though they haven't all been thought obvious in the past. The sole possible exception to this is the Axiom of Regularity, which is probably false. There are certainly interesting models of the

[6]Specifically, Zermelo was interested in Cantor's Well-Ordering Conjecture, which claimed that every set can be put in one to one correspondence with some ordinal. This would assure that every set has a cardinality according to our definitions. Zermelo's proof of the Well-Ordering Conjecture is valid, but marred by the later discovery that one of his axioms (the Axiom of Choice) is actually equivalent to the Well-Ordering Conjecture.

[7]The question of what exactly we mean by "property" was a contentious issue for many years. The modern understanding of a "property" is that it is a unary relation defined in the first-order language of set theory. Fraenkel himself thought that second-order language should be allowed. The history of this axiom can be found in [14]. Note that we cannot formalize Separation as a single axiom because we cannot formally say "for all properties P." Rather, we must view Separation as an infinite collection of axioms (an axiom schema), one for each property.

[8]As with Separation, formalizing Replacement requires understanding "operations" to be functions describable in the first-order language of set theory and making a schema of axioms, one for each operation.

rest of ZFC where Regularity fails spectacularly (for example, Aczel's antifounded universes described in [1] and [16]). There are few compelling reasons to believe that sets cannot contain themselves. There are also interesting models with objects that are not sets, and good reasons to think that there are such objects.[9] Our justifications for assuming Regularity are that (1) it makes our lives a lot easier, (2) it is true for sets as applied almost everywhere outside of set theory, and (3) in a well-defined sense, we will not get into trouble by assuming it. We use the Axiom of Regularity as a matter of convenience.

Using Regularity, we can construct the set theoretic universe recursively, adding sets in order of their complexity. Level zero of the universe, V_0, is just the empty set. Recursively, we define V_{n+1} to be the power set of V_n. So, $V_1 = \{\emptyset\}, V_2 = \{\emptyset, \{\emptyset\}\}, V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, etc. Once we have constructed V_n for every n, we can define V_ω as the union $\bigcup_{n\in\mathbb{N}} V_n$, and continue the process with $V_{\omega+1} = P(V_\omega), V_{\omega+2} = P(V_{\omega+1})$, etc., taking power sets or unions as necessary forever. Note that if $\alpha \leq \beta$, then $V_\alpha \subseteq V_\beta$. By the Axiom of Regularity, the union of all of these transfinitely many levels is precisely the set theoretic universe V.[10]

4. ZFC without Infinite Sets

Now imagine we've built a time machine and gone to visit Euclid in 300 BCE. We'd like to speed the history of mathematics by convincing him to consider a line segment as a infinite set of points. He, of course, finds this idea very problematic because of the infinite set involved. How are we to convince him that infinite sets exist?

Let us consider the axiomatic system ZFC – Inf, where we accept all the axioms of ZFC, except for failing to postulate the existence of infinite sets. We can think of ZFC – Inf as a formalized version of "Ancient Greek set theory." If we are lucky, the Axiom of Infinity will prove to be redundant in ZFC, i.e. we will be able to prove Inf from the other axioms. Then we can show Euclid a proof that infinite sets exist, using only ideas that he believes. Unfortunately, as we will see, this is not possible. *Without postulating infinite sets, we cannot prove that they exist!*

The easiest way to show that a statement ϕ is not a consequence of a certain set of axioms, is to find an object that satisfies the axioms, but where ϕ is false. For example, we can show that Euclid's Parallel Postulate is not a consequence of his other four axioms by exhibiting

[9] Set theoretic objects that are not sets are known equivalently as *atoms*, *individuals*, or *urelements*. Sometimes we talk about proper classes as if they were objects too large to be sets. This is just a linguistic convention. Proper classes are not actually objects, although it is sometimes convenient to pretend in English that they are. The existence of proper classes does not violate the Axiom of Regularity.

[10] That is to say,

$$V = \bigcup_{\alpha \text{ is an ordinal}} V_\alpha.$$

spherical geometry. Under a suitable interpretation of "point" and "line," spherical geometry violates the Parallel Postulate while satisfying the other axioms. Therefore, we cannot prove the Parallel Postulate from the others.

Theorem 4.1. *The existence of infinite sets cannot be proven from the axioms of ZFC – Inf.*

Proof. We will exhibit an object that satisfies all the axioms, but does not have any infinite sets. In fact, we have already made such an object. All we need is V_ω!

We need to check that V_ω satisfies all the axioms of ZFC – Inf.

- *Extensionality* Holds in V, so holds in V_ω. ✓
- *Pairing* If $X, Y \in V_\omega$, then $X, Y \in V_n$ for some n. So then $\{X, Y\} \in V_{n+1} \subset V_\omega$. ✓
- *Separation* If $X \in V_n$, then the elements of X are also in V_n. Therefore the set $\{x \in X : x \text{ has some particular property } \phi\}$ is an element of V_{n+1}. ✓
- *Union* If $X \in V_n$, then the elements of X are also in V_n, as are the elements of the elements of X. Hence, $\bigcup X \in V_{n+1}$. ✓
- *Power Set* If $X \in V_n$, then again the elements of X are in V_n. So every subset of X is in V_{n+1}. The power set of X is in V_{n+2}. ✓
- *Replacement* Let $X \in V_\omega$ and let F be a function $X \to V_\omega$. Like all sets in V_ω, X is finite, while $|V_\omega| = |\mathbb{N}|$. Therefore, $|F(X)|$ is also finite, and so $F(X) \subset V_n$ for some n. Hence, $F \in V_\omega$ and so the range of F is also an element of V_ω. ✓
- *Choice* The Cartesian product of finitely many non-empty finite sets is always itself non-empty and finite. Non-emptiness of the product can be shown by induction. To show the product is finite, suppose our finitely many non-empty finite sets, $S_1, S_2, \ldots, S_n$, are in $V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}$ respectively. Then, they are all in $V_\beta = V_{\max\{\alpha_i\}_{1\le i\le n}}$. Now, note that if $a, b \in V_\alpha$, for some α, then $\{a, b\} \in V_{\alpha+1}$, so $(a, b) = \{a, \{a, b\}\} \in V_{\alpha+2}$. Therefore, the Cartesian product lies in $V_{\beta+2n+1}$.✓[11]
- *Regularity* By construction. ✓

On the other hand, as we noticed when considering Replacement, every element of V_ω is finite. If the existence of an infinite set could be proven from ZFC – Inf, then infinite sets would have to exist in every object satisfying ZFC – Inf, and in particular in V_ω. Since V_ω has no infinite sets, their existence cannot be proven! □

We aren't going to let this problem stop *us* from believing in infinite sets, but it does make it difficult for us to convince Euclid to believe in them. Perhaps, though, Euclid's fears would be allayed if we could show him that ZFC – Inf is consistent with the existence of infinite sets, i.e. that ZFC is consistent. Unfortunately, this is not possible either.

[11]The definition of the ordered pair (a, b) as $\{a, \{a, b\}\}$ is due to Kuratowski. It is not the only possible way to define ordered pairs in term of unordered sets, but it is perhaps the simplest and has become standard.

Theorem 4.2. *If ZFC – Inf is consistent, then the consistency of infinite sets cannot be shown in ZFC – Inf.*

Proof. By constructing V_ω, we can show in ZFC that ZFC – Inf is consistent. So if we could prove from ZFC – Inf that ZFC was consistent, then we could put these two proofs together to show in ZFC that ZFC is consistent. However, by Gödel's Second Incompleteness Theorem, this is not something we can do, unless in fact ZFC is *inconsistent*.[12] But we supposed we could prove from ZFC – Inf that ZFC was consistent, so that would mean that ZFC – Inf was itself inconsistent. Certainly we don't believe that, and Euclid won't believe it either. So our assumption must have been invalid, that we could prove from ZFC – Inf that ZFC was consistent. □

So, we can't show that infinite sets exist, and we can't show that infinite sets *might* exist! Euclid is not very impressed, and we go home, leaving the past unchanged.

5. Large Cardinals

Going back in time was not a great success. But maybe going forward in time would work better. We decide to visit the great Martian mathematician Zftdctad in the year 4321. He doesn't seem very interested in talking to us, but he *does* say, before sending us back home, that our mathematics would be a lot better if we were smart enough to believe in "large cardinals."

Well, what are large cardinals? There is no general formal definition of a large cardinal. Essentially, we say that a cardinality is large, if it is bigger than anything whose existence can be proven in ZFC. The first large cardinals were introduced by Hausdorff in 1908. Since then, large cardinals of diverse types have become an important part of set theory, without finding much acceptance in the wider realm of mathematics.

We begin our investigation by considering *inaccessible cardinals*, which are one of the smaller types of large cardinal.

Definition 5. An uncountable cardinal κ is *inaccessible* if it is a regular strong limit cardinal.

We call such cardinals inaccessible because they cannot be reached from below. ZFC provides us with two tools to obtain larger sets from smaller sets: Union and Power Set. If κ is a regular cardinal, then any union of fewer than κ sets of cardinality less than κ, is itself smaller than κ. Hence, we cannot build any regular cardinal out of smaller sets by using the Axiom of Union. If κ is a strong limit cardinal, then $2^\lambda < \kappa$ for all $\lambda < \kappa$. Hence, we cannot get any strong limit cardinal from a smaller set by using the Axiom of Power Set. Therefore,

[12]The background for Gödel's Incompleteness Theorems appears in Appendix A.

inaccessible cardinals cannot be constructed from smaller sets by using either of the tools at our disposal. We can postulate their existence, but we cannot find them otherwise.

Notice that if we had not required inaccessible cardinals to be uncountable, then $\aleph_0$ would count as an inaccessible: a union of finitely-many finite sets is finite, and the power set of a finite set is finite. Hence postulating the existence of an inaccessible cardinal is analogous to postulating the existence of infinite sets. To postulate the existence of infinite sets is precisely to claim the existence of a regular strong limit cardinal. To postulate the existence of an inaccessible is merely to claim the existence of more than one regular strong limit cardinal.

Just as in ZFC – Inf it was impossible to show that infinite cardinals existed or even that their existence was consistent, we cannot show in ZFC that inaccessibles exist, nor that their existence is consistent with ZFC. Our arguments for or against inaccessibles must be based on intuition about the mathematical universe, just like our arguments in favor of infinite sets.

Theorem 5.1. *The existence of an inaccessible cardinal is not provable in ZFC.*

Proof. If inaccessible cardinals do not exist, then we certainly cannot prove their existence. Therefore, suppose there exists an inaccessible cardinal κ. We may assume that κ is the smallest inaccessible cardinal. We will use κ to find a model of ZFC where there are no inaccessibles. Instead of V_ω as before, we want to look at V_κ.

Except for Choice and Replacement, the proof that the axioms of ZFC – Inf are true in V_κ is exactly the same as the proof that they are true in V_ω. To prove that Choice holds in V_κ, let $Z \in V_\kappa$ be a collection of non-empty sets and let $\zeta = |Z|$. By the Axiom of Choice, we know that there is some x in the Cartesian product $\prod Z$. All we need to show is that $x \in V_\kappa$.

Certainly, every coordinate of x is in V_{α_i} for some $\alpha_i < \kappa$. Therefore, all the coordinates of x lie in $V_\lambda = \bigcup_i V_{\alpha_i}$. Since κ is regular and there are $\zeta(< \kappa)$ sets in the union, we have $\lambda < \kappa$.

Now, note that if $a, b \in V_\alpha$, for some α, then $\{a, b\} \in V_{\alpha+1}$, so $(a, b) = \{a, \{a, b\}\} \in V_{\alpha+2}$.

In our situation, we had ζ coordinates all lying in V_λ. Therefore, $x \in V_{\lambda+2\zeta}$. Since $V_{\lambda+2\zeta} \subset V_\kappa$, Choice holds in V_κ.

For Replacement, we can adapt the proof for V_ω. Let $X \in V_\kappa$ and let F be a function $X \to V_\kappa$. Like all sets in V_κ, $|X| < \kappa$, since κ is an inaccessible cardinal. On the other hand, $|V_\kappa| = \kappa$. Therefore, $|F(X)| < \kappa$ as well. Every element of $F(X)$ lies in some V_{α_i} with $\alpha_i < \kappa$, so since κ is regular, $F(X) \subset V_\alpha$ for some $\alpha < \kappa$. Hence, $F \in V_\kappa$ and so the range of F is also an element of V_κ.

Since $\kappa > \aleph_0$, it is clear that V_κ satisfies the Axiom of Infinity. So all of the axioms of ZFC are true in V_κ.

However, as κ is the least inaccessible cardinal, and $\kappa \notin V_\kappa$, V_κ is a model of ZFC where there are no inaccessibles. $\square$

Theorem 5.2. *The consistency of inaccessible cardinals cannot be shown in ZFC.*

Proof. Let ZFC + I be the axioms of ZFC plus the assumption that at least one inaccessible cardinal exists. By constructing V_κ where κ is inaccessible, we can show in ZFC + I that ZFC is consistent. So if we could prove from ZFC that ZFC + I was consistent, then we could put these two proofs together to show in ZFC + I that ZFC + I is consistent. By Gödel's Second Incompleteness Theorem, we cannot do this, unless ZFC + I is *inconsistent*. But we supposed we could prove from ZFC that ZFC + I was consistent, so that would mean that ZFC was itself inconsistent. We may take that as a contradiction, so our assumption must have been invalid, that we could prove from ZFC that ZFC + I was consistent. □

These results should not prevent us from using inaccessible cardinals any more than the analogous results should prevent us from using infinite sets. On the other hand, they don't give us a lot of reason *to* believe in inaccessible cardinals.

There are a few quick arguments for why inaccessibles should exist.[13] The first is that the mathematical universe ought to contain all the objects we can reason about mathematically. Since, as we will see, lots of beautiful, intricate mathematics can be constructed around inaccessible cardinals, they should exist in the same way that other mathematical objects do. A second argument is that union and power set are fairly weak operations. It would seem strange if we could build all mathematical objects out of small sets, using only such weak tools. A third argument is that, if inaccessible cardinals did not exist, then $\aleph_0$ would be the unique regular strong limit cardinal. Perhaps this is so. However, it seems intuitively more plausible that if P is a "generic" and "natural" property sets can have, there should be arbitrarily large sets with property P.[14] The best argument, however, for working with large cardinals is that they are mathematically useful, just like infinite sets. This utility is the topic of the next section.

6. An Application for Large Cardinals

We will now embark on a whirlwind tour of a small area of the large cardinal landscape. Hopefully, this tour will give us a sense of the diversity and intricacy of large cardinal techniques. It will also show us how large cardinals can resolve other sorts of mathematical problems. Even more than before, we will gloss over technical details as we sketch a brief intuitive map of this small region of mathematics.

[13]These are covered at greater length in [14], [15], and [21].

[14]Certainly, we cannot expect this of every property. For example, if P is the property of being the least infinite cardinal, then P uniquely picks out $\aleph_0$. I would say that such a property, however, is not "generic." Undoubtably, there are similar examples of properties that less obviously pick out a small number of cardinals. However, there does not seem to be a compelling reason why being a regular strong limit cardinal should be such a property.

Ideally, we would like to see large cardinals resolve a known problem that does not involve large cardinals in its statement. There are many examples of large cardinals doing exactly that. Many questions of infinitary combinatorics and model theory, in particular, are resolved by assuming sufficiently strong large cardinal axioms. Even in intuitively more distant branches of mathematics, such as braid theory, large cardinals can be used to answer questions.[15]

Unfortunately, the resolutions of these problems do not lend themselves to the sort of whirlwind explication that is appropriate here. Therefore, we will turn to a problem that is probably unfamiliar, although it is classical within set theory.

Recall that the Generalized Continuum Hypothesis (GCH) is the statement that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all α. Many mathematicians in the early 20th century struggled to either prove or disprove GCH, but no real progress was made until the work of Kurt Gödel in the late '30s.[16] In similar fashion to things we have done before, Gödel built a construct, L, that satisfies the axioms of ZFC and also satisfies GCH. Thus, he showed that GCH cannot be disproven from the axioms of ZFC.[17] In fact, Gödel proved rather more than this. His construction of L does not use the Axiom of Choice, but in L it is very clear that the Axiom of Choice is true. Hence, Gödel also demonstrated that the Axiom of Choice is consistent with the other axioms of ZFC.

The class L is an example of what is called an *inner model*. An inner model is a subclass of the universe V that contains all the ordinals and is also transitive. Not only is L an inner model of the universe, but it is the smallest inner model, in the sense that every inner model contains L as a subclass.

Gödel built L by analogy to our earlier construction of V as a sequence of levels. Naturally, we take $L_0 = \emptyset$. And at limit ordinals α, we set $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$, analogously to the limit step in the construction of V. The difference between V and L takes place at successor steps. Rather than let $L_{\alpha+1}$ be the power set of L_α, we only allow into $L_{\alpha+1}$ those subsets of L_α

[15] I am thinking here of [12], where a very strong large cardinal axiom is used to prove a braid theoretic result. In [4], it was shown that this proof can be modified so as to eliminate the large cardinal assumption, i.e. the theorem is in fact a theorem of ZFC. However, even the author of [4] admits that this proof would probably never have been discovered without consideration of large cardinals. He writes, "The point here is that considerations of highly infinite objects ... have led to an *intuition* that ended in results of the most constructive spirit ... The fact that the set-theoretical assumption has been subsequently eliminated does not diminish its role in the matter: there would have been probably little chance to embark on the proof ... if some 'exotic' axiom of set theory had not given previously some evidence that the involved property could be true."

[16] Gödel announced his results on GCH in 1938, but a detailed proof did not appear until 1940.

[17] The reverse, that GCH cannot be *proven* from ZFC, was eventually demonstrated as well, through Paul Cohen's 1963 method of forcing.

that can be uniquely defined by a first-order formula. We can think of L as a "narrower" version of V, and we call it the *Constructible Universe*.

On the other hand, if we repeat this construction inside L to get a "narrower" L', we find that in fact $L = L'$, so L' is not any narrower. This leads us to wonder if perhaps L was not actually any narrower than V either. The statement $L = V$ is called the *Axiom of Constructibility*, although very few people would claim it is a self-evident truth. The status of the Axiom of Constructibility is the problem that we will be considering.[18]

In fact, the consensus of set theorists is that $L = V$ is false.[19] Why do they think that, when L has such nice properties (e.g. GCH and obvious satisfaction of Choice)?[20] Well, the problem is that L is rather too nice. There is no compelling reason why every set should be definable from simpler sets by a first-order formula. Indeed, given the weakness of first-order languages, this seems an unlikely prospect. Furthermore, while the Axiom of Choice is true in L, it is true for the wrong reasons. Choice is true in constructible universes because we can well-order the entire universe![21] We want Choice to be a statement of the richness of the universe; given an infinite collection of non-empty sets, there is some exotic object in their direct product. However, in constructible universes Choice turns into a statement of poverty; the universe is so impoverished that we can well-order the entire thing!

We are not going to be able to prove from the axioms of ZFC that $L \neq V$. This is because L is a model of ZFC where all sets are in fact constructible. However, if we assume the existence of a sufficiently large cardinal, we will be able to disprove the Axiom of Constructibility.

7. Measurable Cardinals

The large cardinals we wish to consider are called *measurable cardinals*.[22] They come naturally out of consideration of measures, in the analysis sense. Once we have figured out what measurable cardinals are, we will use one to construct an inner model M that is a proper

[18]Notice that if $L = V$, then there are no proper inner models. (All inner models contain L and are contained in V.)

[19]One of the few advocates for belief in $L = V$ was the philosopher Quine, who based his position on his inability to think up scientific applications for $L \neq V$. See [14] and [15] for further discussion of Quine's views on set theory.

[20]To see that the Axiom of Choice holds in L, we need merely see that L can be well-ordered. Start by ordering L according to which level L_α each element first appears in. Many elements will appear for the first time at the same level, so we need some way to break ties. But each element is named by a first-order formula, and it is very easy to well-order the relevant set of formulas, so we can well-order L by carrying over this well-ordering on formulas.

[21]This property is known as the Axiom of Global Choice.

[22]Measurable cardinals were proposed in 1930 by the young Stanislaw Ulam. In fact, we can disprove Constructibility from somewhat weaker large cardinal axioms, but it is less convenient. When we are feeling a bit impudent, we like to refer to those large cardinals that don't contradict $V = L$ as *small large cardinals*.

subclass of V. Since L is a subclass of every inner model, it will follow that L is a proper subclass of V, and hence $V \neq L$.

It is well-known that there is no non-trivial σ-additive translation-invariant measure defined on all subsets of $[0, 1]$. However, in some situations we could dispense with the translation invariance. Hence, it is natural to ask whether there is any non-trivial σ-additive measure on all subsets of $[0, 1]$ at all. Since we no longer care about translation-invariance, we quickly see that the only relevant feature of $[0, 1]$ is its cardinality. This motivates the more general question: Is there any cardinal κ with a non-trivial σ-additive measure defined on all subsets of κ?

This question is equivalent to the existence of some sort of large cardinal, and hence it is not resolvable in ZFC. The particular sort of large cardinal implied by the existence of such a measure depends on more specific properties of the measure. One such potential property of a measure is only taking on the values 0 and 1. We call such a measure *two-valued.*

Theorem 7.1. *There exists a set S with a non-trivial σ-additive two-valued measure defined on all subsets of S if and only if there exists a* measurable cardinal.

Unfortunately, we don't yet know what a measurable cardinal is. To define them, we must first discuss *ultrafilters.*

Definition 6. An *ultrafilter* on a set S is a subset U of $P(S)$ such that

(1) $\emptyset \notin U$,
(2) $S \in U$,
(3) if $A \in U$ and $A \subset B$, then $B \in U$,
(4) if $A, B \in U$, then $A \cap B \in U$,
(5) if $A \subset S$ and $A \notin U$, then $S \backslash A \in U$.

If further $\bigcap U$ is non-empty, then we say U is a *principal ultrafilter.* Principal ultrafilters are uninteresting. We will be concerned entirely with non-principal ultrafilters.

Property (4) implies that U is closed under finite intersections. If U is closed under intersections of less than λ-many elements of U, then we say that U is λ*-complete.*

The correct intuitive picture of an ultrafilter is that it consists of the "large" subsets of S. Since the word "large" describes a rather fuzzy concept, it may be possible to define several distinct ultrafilters on S, in which different sets are counted as "large." However, all ultrafilters on S have much in common, and they all line up reasonably well with our fuzzy idea of the "large" subsets of S.

Now, if we had a measure on S, there would be another canonical interpretation of a subset of S being "large": we could say that the large subsets are those of full measure. These two meanings of "large" are not unrelated. If I have an ultrafilter U on S, I can turn it into a two-valued measure on S by assigning measure 1 to all subsets in U and measure 0 to all subsets not in U. Conversely, the full measure sets in any non-trivial two-valued measure form an ultrafilter. So, although we define measurable cardinals in terms of ultrafilters, the resulting cardinals will turn out to be closely connected by the above correspondence to the existence of two-valued measures, thereby justifying the name 'measurable.'

Definition 7. An uncountable cardinal κ is *measurable* if there exists a κ-complete non-principal ultrafilter on κ.

So, now we know what measurable cardinals are. However, it is not clear how large these measurable cardinals are. And it is certainly not apparent what they have to do with the Axiom of Constructibility.

When Stanislaw Ulam introduced measurable cardinals in 1930, he demonstrated that they are also inaccessible.[23] However, until 1963 it was unknown whether the least inaccessible cardinal could be measurable. It cannot. In fact, there is a large hierarchy of inaccessible cardinals below the least measurable cardinal.[24] If κ is measurable, then κ is the κth inaccessible cardinal. Indeed, κ must be the κth cardinal with this property. Inaccessible cardinals are large, but they are very small in comparison to measurable cardinals!

In order to show that measurable cardinals contradict the Axiom of Constructibility we will have to put their ultrafilters to good use. We will use the ultrafilter on a measurable cardinal to construct an *ultrapower* of the universe.

Ultrapowers are perhaps most familiar from their use in the development of nonstandard analysis.[25] To rigorously introduce infinitesimals into analysis we use a non-principal ultrafilter U on $\mathbb{N}$ and consider the set of functions $f : \mathbb{N} \to \mathbb{R}$. We put an equivalence class on this set by equating any two functions that agree on a element of U. The equivalence classes are the hyperreal numbers $*\mathbb{R}$, the fundamental objects of nonstandard analysis. Every $r \in \mathbb{R}$ is canonically embedded in $*\mathbb{R}$ as $\hat{r} = (r, r, r, r, \dots)$. However, $*\mathbb{R}$ also contains many new elements. For example, $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ is a rational infinitesimal in $*\mathbb{R}$.

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[23]This result requires the Axiom of Choice. Without Choice, it is possible for $\aleph_1$ (and many others) to be measurable!

[24]The first work in this direction was by William Hanf, with various pieces of the picture quickly added by Tarski, Erdős, Keisler, and Hajnal. Some of the history of this discovery may be found on page 39 of [10].

[25][8] is a very friendly introduction to ultrapowers in nonstandard analysis. [7] and [20] are more sophisticated but still readable accounts.

Theorem 7.2 (Łoś)**.** *If $\mathbb{U}$ is an ultrapower of any mathematical structure $\mathbb{S}$, then $\mathbb{U}$ and $\mathbb{S}$ satisfy the same first-order sentences.*[26]

Corollary 7.3. *Every first-order sentence is true in $\mathbb{R}$ if and only if it is true in $*\mathbb{R}$.*

Łoś's Theorem is what makes nonstandard analysis effective. Instead of proving something directly in $\mathbb{R}$, we can prove it in $*\mathbb{R}$, where it is often easier to prove, and then apply Corollary 7.3.

We are now ready to show that the existence of a measurable cardinal contradicts the Axiom of Constructibility. Let U be a κ-complete non-principal ultrafilter on a measurable cardinal κ. Instead of functions $f : \mathbb{N} \to \mathbb{R}$, we want to look at functions $g : \kappa \to V$. As before, we equate functions that agree on an element of U.[27] The set of equivalence classes is an ultrapower of the universe; call it M. Since we built M inside the universe, M is a subclass of V. In fact, it is not hard to show that M is an inner model of V.[28] However, just as we embedded $\mathbb{R}$ in $*\mathbb{R}$, we can embed V inside M by sending every $x \in V$ to the constant function $f_x : \kappa \to V$ with value x. We will write j for this map $x \mapsto f_x$.

Łoś's Theorem applies in this context as well, so the first-order sentences true in V are the first-order sentences true in $j(V)(= M)$. Therefore, if α is an ordinal in V, then $j(\alpha)$ is an ordinal in $j(V)$. But $j(V) \subseteq V$, so in fact $j(\alpha)$ is an ordinal in V too.[29] Also, if $\alpha < \beta$ then $j(\alpha) < j(\beta)$. Putting these two facts together, we have by an easy induction that $\alpha \leq j(\alpha)$ for every ordinal α. In fact, since U is κ-complete, we have equality for all $\alpha < \kappa$, i.e. j fixes all the ordinals below κ.[30]

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[26]Łoś actually proved a stronger result about a more general kind of construction, known as an *ultraproduct* construction. An ultrapower is a special ultraproduct where all factors are equal.

[27]There are technical issues here coming from the fact that V is a proper class. However, they can all be resolved unproblematically.

[28]Actually, M might not be transitive, but if it isn't, we can just apply a simple technique called the *Mostowski collapse* to produce an isomorphic model that *is* transitive.

[29]Properties that carry over in this fashion are called *upwards absolute*. It is not difficult to show that being an ordinal is upwards absolute. It is also easily shown to be *downwards absolute*, meaning that it carries over from V to all inner models. Such properties that are both upwards absolute and downwards absolute are called merely *absolute*.

[30]Suppose $\alpha < \kappa$ and some $m \in M$ is less than α. Remember m is an equivalence class of functions f. Since $m < \alpha$, each function f takes on values less than α over a subset of the domain that is in U. Note that this subset can be written as $\bigcup_{\beta<\alpha} f^{-1}(\beta)$. Since this union is in U and U is κ-complete, one of the terms of the union must be in U. But this means that there is some $\beta < \alpha$ such that f takes on the value β on a set in U, and hence f and $j(\beta)$ are both in the same equivalence class m.

We would like to know what j does with κ. To determine this, we focus instead on the identity function $i \in M$ defined by $i(\alpha) = \alpha$ for all α. Since every coordinate of i is an ordinal, i certainly represents some ordinal in M. Since every coordinate is less than κ, $i < j(\kappa)$. However, for every $\gamma < \kappa$, i is greater than γ on a set of coordinates in U. (This comes from κ-completeness.) Therefore, $i \geq \kappa$.

For our purposes, we now know all about i that we need to. The important thing to note is that $\kappa \leq i < j(\kappa)$, and therefore $\kappa < j(\kappa)$.[31]

Theorem 7.4. *If measurable cardinals exists, then $V \neq L$.*

Proof. Let κ be the least measurable cardinal. Use a nonprincipal κ-complete ultrafilter on κ to construct the inner model $M = j(V)$. Since L is the smallest inner model, $L \subseteq M \subseteq V$.

By assumption, κ is the least measurable cardinal in V. And $j(\kappa)$ satisfies the same sentences in M as κ does in V, so $j(\kappa)$ must be the least measurable cardinal in M.

Now suppose, by way of contradiction, that $L = V$. Then M is equal to both, since it is sandwiched in between. Then, since κ is the least measurable cardinal in $V(= M)$ and $j(\kappa)$ is the least measurable cardinal in $M(= V)$, it must be that $\kappa = j(\kappa)$. But this is a contradiction, since we know $\kappa < j(\kappa)$. □

So, there we have it. Belief in measurable cardinals is sufficient to disprove the silly property of Constructibility. This is good news for the richness of our mathematical universe!

To the extent that set theory is a foundational subject, its goal should be to provide as rich a foundation for the rest of mathematics as possible. Large cardinals are certainly helpful towards that goal. If a topologist wants to study a topology where all sets are constructible, she can work inside L. And if she wants to allow more complicated sets, she can work outside of L in a universe that contains measurable cardinals. The point is that large cardinals aren't only useful when we wish to study extraordinarily large sets. No, just as belief in infinite sets helps us develop finitist results, large cardinals help us in the study of classical areas of mathematics. Belief in such sets will guide us to a richer and more powerful mathematics for the third millennium. Plus, our time travel will have paid off. Won't Zftdctad be proud!

[31]This is strange. Do not expect this to seem natural.

Appendices

Appendix A. Gödel's Incompleteness Theorems

Anything can be proven from an inconsistent set of axioms. Gödel's First Incompleteness Theorem (1931) says that if an axiomatic system is consistent and sufficiently powerful, then there are sentences that are *undecidable*, i.e. sentences σ such that neither σ nor $\sim\sigma$ can be proven from the axioms.[32] By 'sufficiently powerful,' we mean that it is possible to model Peano arithmetic inside the system.[33] For example, ZFC is necessarily incomplete, unless of course it is inconsistent (in which case we have bigger problems). Since ZFC is incomplete, there must be statements that we can consistently add as axioms, but that we can not prove. Candidates for such new axioms include the Generalized Continuum Hypothesis (§2), the large cardinal axioms (§5), and the Axiom of Constructibility (§6).

Gödel's Second Incompleteness Theorem (also 1931) names a particular sentence that is undecidable in sufficiently powerful axiomatic systems.[34]

Theorem A.1 (Gödel's Second Incompleteness Theorem, 1931)**.** *If T is a consistent and sufficiently powerful set of axioms, then the statement that T is consistent can neither be proven nor disproven from the axioms.*

The statement that a theory T is consistent is usually written as $\mathrm{Con}(T)$. The immediate effect of this theorem was to crush Hilbert's dream of proving via mathematics that mathematics itself is consistent. However, as shown in §4 and §5, it is also useful as a lemma in proving other results.

Theorems such as Theorem 5.2 ('The consistency of inaccessible cardinals cannot be shown in ZFC') are relative consistency results. We tacitly assume that ZFC is consistent. If ZFC is not consistent, then we can both prove and disprove the existence of inaccessible cardinals in ZFC. This sort of result is the best that we can hope for. We can only show that a mathematical theory is consistent *relative* to the theory that we are using in the proof. For

[32]If σ is a undecidable sentence of the form "for all x, ϕ," then σ is in some sense true, since it must be impossible to find a counterexample to ϕ. On the other hand, it is possible that ϕ has counterexamples, but that these counterexamples cannot be constructed in the axiomatic system. For example, in ZFC − Choice the statement "All sets of real numbers are Lebesgue integrable" is undecidable. This means it is not possible to construct a non-Lebesgue measurable set without using the Axiom of Choice. However, most mainstream mathematicians accept Choice and claim that the sentence "All sets of real numbers are Lebesgue integrable" is false.

[33]The theory of the natural numbers under addition alone is *not* 'sufficiently powerful.' In this simple mathematical theory, known as Presburger arithmetic, every statement can either be proven or disproven.

[34]The Second Incompleteness Theorem was independently discovered by von Neumann in the brief interval between Gödel's announcements of the two theorems.

example, the construction of L in §6 demonstrates

$$\mathrm{Con}(\mathrm{ZFC} - \mathrm{Choice}) \implies \mathrm{Con}(\mathrm{ZFC} + \mathrm{GCH}).$$

Both [22] and [23] contain excellent expositions of the Incompleteness Theorems. English translations (along with the original German versions) of Gödel's original papers may be found in [6].

Appendix B. Sources for Proofs

All the proofs given in this paper are standard material. My formulations of the proofs are amalgamations based primarily on my readings of [9], [10], and [11].

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