ABSTRACT

ON MORLEY'S CATEGORICITY THEOREM WITH AN EYE TOWARD FORKING

by Colin Craft

The primary result of this paper is Morley's Categoricity Theorem that a complete theory T which is κ-catecorigal for some uncountable cardinal κ is λ-categorical for every uncountable cardinal λ. We prove this by proving a characterization of uncountably categorical theories due to Baldwin and Lachlan. Before the actual statement and proof of Morley's theorem, we give an overview of the prerequisites from mathematical logic needed to understand the theorem and its proof. After proving Morley's theorem we briefly indicate some possible directions of further study having to do with forking and the related notion of independence of types.

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Chapter 1

Preliminaries

The material in this chapter is based on lectures given by Dr. C. Laskowski at the University of Maryland during the 2007-2008 academic year.

We begin by reviewing basic concepts from mathematical logic which we will need.

1.1 Languages, Formulas and Structures

Definition 1.1 *A* **first-order formal language** $\mathcal{L}$ *consists of finite strings of symbols from the alphabet:*

1. $\wedge, \vee, \neg, \exists, \forall, =, (,)$
2. *An infinite set of "variable symbols"*
   $x_0, x_1, x_2, \ldots$
   $v_0, v_1, v_2, \ldots$
   $y_0, y_1, y_2, \ldots$
   $x, y, z, w, \ldots$
3. *A set (possibly empty) of* **constant symbols** $c, d, \ldots$
4. *For each* $n \geq 1$ *a set (possibly empty) of* n*-ary* **relation symbols** R, S, $\ldots$
5. *For each* $n \geq 1$ *a set (possibly empty) of* n*-ary* **function symbols** $f, g, h \ldots$

The function, relation and constant symbols are the *non-logical symbols* of our language and the remaining symbols are the *logical symbols* of our language. The symbols in point (1) of the definition are intended to be interpreted as follows: $\wedge$ is "and," $\vee$ is "or," $\neg$ is "not," $\exists$ is "there exists," $\forall$ is "for all" and $=$ denotes equality. The symbols $\exists$ and $\forall$ are know as *quantifiers*. The symbols (and) will be used for delimiting the ranges of quantifiers, relation symbols, etc.

Every formal language has the same logical symbols, but the non-logical symbols of various languages vary widely. For this reason, we will specify a language by specifying its non-logical symbols and write, e.g. $\mathcal{L} = \{R_1, R_2, f\}$ for a language with two relation symbols, one function symbol and no constant symbols.

The size of a language is the total of number symbols in the language and is denoted by $|\mathcal{L}|$. Since every language has the same countably infinite set of logical symbols, the size of a language is determined by its set of non-logical symbols: if the set of non-logical symbols has size κ and $\kappa \geq \aleph_1$, then $|\mathcal{L}| = \kappa$; otherwise $|\mathcal{L}| = \aleph_0$. Most languages which we consider will be countable (i.e, of size $\aleph_0$).

In a natural language, like English, some strings of symbols are meaningful ("Morley proved a theorem") and some are not ("Qnfyabxvcretsggfpoine"). It is the same way with our formal language. We now define which strings from our formal language will be considered "meaningful."

Definition 1.2 *The* **$\mathcal{L}$-terms** *are:*

1. *Every variable symbol is an $\mathcal{L}$-term*
2. *Every constant symbol is an $\mathcal{L}$-term*
3. *If $t_1, t_2, \ldots, t_n$ are $\mathcal{L}$-terms and $f \in \mathcal{L}$ is an n-ary function symbol then $f(t_1, t_2, \ldots, t_n)$ is an $\mathcal{L}$-term*

These are the only $\mathcal{L}$-terms

Example: If $f, c \in \mathcal{L}$, f a function symbol, c a constant symbol, then $f(f(x_1, c), f(x_2, f(x_3, c)))$ is an $\mathcal{L}$-term.

Definition 1.3 *The* **$\mathcal{L}$-atomic formulas** *are:*

1. *If t_1, t_2 are $\mathcal{L}$-terms then $(t_1 = t_2)$ is an $\mathcal{L}$-atomic formula*

2. *If $t_1, t_2, \dots, t_n$ are $\mathcal{L}$-terms and $R \in \mathcal{L}$ is an n-ary relation symbol, then $R(t_1, t_2, \dots, t_n)$ is an $\mathcal{L}$-atomic formula.*

Example: For f, c as in the last example and R a relation symbol, $f(x_1, c) = c$ and $R(x_1, f(x_2, x_3))$ are $\mathcal{L}$-atomic formulas.

Definition 1.4 *The* **$\mathcal{L}$-formulas** *are:*

1. *Every $\mathcal{L}$-atomic formula is a $\mathcal{L}$-formula*
2. *If ϕ is an $\mathcal{L}$-formula, then so is $\neg\phi$*
3. *If ϕ, ψ are $\mathcal{L}$-formulas, then so are $(\phi \wedge \psi)$ and $(\phi \vee \psi)$*
4. *If ϕ is an $\mathcal{L}$-formula and x is a variable symbol occurring in ϕ, then $\exists x\phi$ and $\forall x\phi$ are $\mathcal{L}$-formulas as well.*

These are the only $\mathcal{L}$-formulas.

When the formal language $\mathcal{L}$ is clear from context or unimportant, we will refer to $\mathcal{L}$-terms, $\mathcal{L}$-atomic formulas and $\mathcal{L}$-formulas as simply terms, atomic formulas and formulas, respectively.

We will want to differentiate between two kinds of $\mathcal{L}$-formulas: those which contain variable symbols that are "free" and those that do not. To this end, we will define for each formula ϕ a collection $FV(\phi)$ of the *free variables* of ϕ. Just as our definition of formulas was built up from the definitions of terms and atomic formulas, so our definition of $FV(\phi)$ is built up from definitions of FV for terms and atomic formulas.

Definition 1.5 *For t an $\mathcal{L}$-term, $FV(t)$ is*

1. *If t is the variable symbol x, then $FV(t) = \{x\}$*
2. *If t is the constant symbol c, then $FV(t) = \varnothing$*
3. *If $f \in \mathcal{L}$ is an n-ary function symbol and $t_1, t_2, \dots, t_n$ are terms then $FV(f(t_1, \dots, t_n)) = FV(t_1) \cup FV(t_2) \cup \dots \cup FV(t_n)$*

Definition 1.6 *If ϕ is an $\mathcal{L}$-atomic formulas, then $FV(\phi)$ is:*

1. *If t_1, t_2 are terms and ϕ is $(t_1 = t_2)$ then $FV(\phi) = FV(t_1) \cup FV(t_2)$*

2. *If $R \in \mathcal{L}$ is an n-ary relation symbol and $t_1, t_2, \ldots, t_n$ are terms, then $FV(R(t_1, t_2, \ldots, t_n)) = FV(t_1) \cup FV(t_2) \cup \ldots \cup FV(t_n)$*

Definition 1.7 *If ϕ is an $\mathcal{L}$-formula then we define $FV(\phi)$ inductively as follows:*

- *If ϕ is an atomic formula, then $FV(\phi)$ is as above.*
- *If ϕ, ψ are atomic formulas such that $FV(\phi), FV(\psi)$ have been defined, then*
  - $FV(\neg\phi) = FV(\phi)$
  - $FV((\phi \wedge \psi)) = FV((\phi \vee \psi)) = FV(\phi) \cup FV(\psi)$
  - $FV(\exists x\phi) = FV(\forall x\phi) = FV(\phi) \setminus \{x\}$

For an $\mathcal{L}$-formula ϕ we refer to the elements of $FV(\phi)$ as the *free variables* of ϕ.

Definition 1.8 *An* **$\mathcal{L}$-sentence** *is an $\mathcal{L}$-formula with no free variables, i.e. an $\mathcal{L}$-formula ϕ such that $FV(\phi) = \varnothing$.*

Example: Let $\mathcal{L}$ be a formal language containing functions symbols f_1, f_2, constant symbol c and relation symbols R_1, R_2.
Let ϕ be the formula $f_1(x_1, x_2) = f_2(x_1, c)$ and let ψ be the formula $R_1(x_1, x_3) \vee R_2(x_1, x_3) \vee f_2(x_3, x_4) = c$.
Then we have:

- $FV(\phi) = \{x_1, x_2\}$
- $FV(\psi) = \{x_1, x_3, x_4\}$
- $FV(\exists x_2\phi) = \{x_1\}$ and
- $FV(\forall x_1 \forall x_3 \forall x_4 \psi) = \varnothing$ (note that this makes $\forall x_1 \forall x_3 \forall x_4 \psi$ a sentence)

We may say or write, e.g., that "ϕ has free variables x_1 and x_2," that "x_1, x_2 are free in ψ," etc.

Before going on, we need two pieces of notation.

Notation: We will write $\phi(x_1, x_2, \ldots, x_n)$ to denote a formula with free variables *among* $x_1, x_2, \ldots x_n$, i.e. a formula such that $FV(\phi) \subseteq \{x_1, x_2, \ldots x_n\}$.

Notation: For $\mathcal{L}$-formulas ϕ, ψ:

- $(\phi \to \psi)$ is an abbreviation of $(\neg\phi \vee \psi)$
- $(\phi \leftrightarrow \psi)$ is an abbreviation of $((\phi \to \psi) \wedge (\psi \to \phi))$

It is the $\mathcal{L}$-formulas which are the "meaningful" strings of symbols from our formal language. Of course, for these strings to be truly "meaningful," they must have a meaning. To precisely define this meaning, we need the concept of a *structure.*

Definition 1.9 *An* **$\mathcal{L}$-structure***, or* **$\mathcal{L}$-model***,* $\mathfrak{A}$ *for a first-order language* $\mathcal{L}$ *is a pair* $\mathfrak{A} = (A, \mathcal{I})$ *where* A *is a non-empty set, called the* **underlying set** *or* **universe** *of* $\mathfrak{A}$ *and* $\mathcal{I}$ *is an* interpretation *of each non-logical symbol in* $\mathcal{L}$*, i.e.*

- *If* $c \in \mathcal{L}$ *is a constant symbol,* $\mathcal{I}(c) \in A$
- *If* $f \in \mathcal{L}$ *is an* n*-ary function symbol,* $\mathcal{I}(f)$ *is a function:* $A^n \to A$
- *If* $R \in \mathcal{L}$ *is an* n*-ary relation symbol,* $\mathcal{I}(R)$ *is a subset of* A^n

Notation: We will write

- $c^{\mathfrak{A}}$ in place of $\mathcal{I}(c)$
- $f^{\mathfrak{A}}$ in place of $\mathcal{I}(f)$
- $R^{\mathfrak{A}}$ in place of $\mathcal{I}(R)$
- We will generally use capital Fraktur letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{M}, \mathfrak{N}$ to denote structures and the corresponding capital Roman letter to denote the structure's underlying set A, B, C, M, N. The notation $|\mathfrak{A}|$ is also used to denote the underlying set of the structure $\mathfrak{A}$.

The $\mathcal{L}$-formulas obtain a meaning relative to a given structure. For example, consider the language $\mathcal{L} = R$ with a single binary relation symbol and the $\mathcal{L}$-structure $\mathfrak{N} = (\mathbb{N}, \leq)$ where $\mathbb{N}$ is the set of natural numbers (including 0) and $\leq$ is the usual "less than or equal to" relation on

$\mathbb{N}$. (In our above notation, this would be the structure $\mathfrak{N}$ with $N = \mathbb{N}$ and $R^{\mathfrak{N}} = \leq = \{(a, b) \in \mathbb{N}^2 : a \leq b\}$.)

Then $\exists x \forall y (R(x, y))$ is a formula, in fact a sentence, of our language and if we interpret the variable symbols to range over the underlying set of our structure and the relations symbol R to be $R^{\mathfrak{N}} = \leq$, then this sentence becomes true: There is a natural number x ($\exists x$) such that for every natural number y ($\forall y$) x is less than or equal to y ($R(x, y)$). Specifying a structure has given us a set for our variables to be drawn from and an interpretation of our relation symbol, i.e. it has given our sentence a meaning. In the next section, we will make this notion precise with formal definitions.

1.2 The Definition of "Truth"

For this section, we fix a first-order language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathfrak{A}$.

To define our concept of "truth[1]" for $\mathcal{L}$ and $\mathfrak{A}$ it will be easier to work in the expanded language $\mathcal{L} \cup \{c_a : a \in A\}$ where for each element $a \in A, c_a$ is a new constant symbol. We will denote this language by $\mathcal{L}_A$. (In general, if we have a language $\mathcal{L}$ and a structure $\mathfrak{A}$, then for any subset $B \subseteq A$ we will denote by $\mathcal{L}_B$ the language $\mathcal{L} \cup \{c_b : b \in B\}$ where c_b is a new constant symbol for each element of B. We denote by $\mathfrak{A}_B$ the $\mathcal{L}_B$-structure where each c_b is interpreted as b, and the other non-logical symbols of $\mathcal{L}$ are interpreted as in $\mathfrak{A}$.)

We will define what it means for a sentence in the language $\mathcal{L}_A$ to be true in $\mathfrak{A}$. Since every $\mathcal{L}$-sentence is an $\mathcal{L}_A$-sentence, this will define truth for all $\mathcal{L}$-sentences in $\mathfrak{A}$. This definition will also give us a natural way of considering the truth or falsity of general formulas, i.e. those with free variables.

Definition 1.10 *A* **closed** *$\mathcal{L}_A$-term is an $\mathcal{L}_A$-term which contains no variable symbols.*

So the set of closed $\mathcal{L}_A$-terms is the smallest set of finite strings from $\mathcal{L}_A$ that contains the constant symbols for $\mathcal{L}_A$ and is closed under every function symbol $f \in \mathcal{L}_A$.

Definition 1.11 *For each closed $\mathcal{L}_A$-term t we define $t^{\mathfrak{A}}$:*

- *For each constant symbol $c \in \mathcal{L}$, $c^{\mathfrak{A}} = \mathcal{I}(c)$, the interpretation of c in $\mathfrak{A}$.*
- *For each constant symbol $c_a \in \mathcal{L}_A \setminus \mathcal{L}$, $c_a^{\mathfrak{A}} = a$*
- *For every n-ary function symbol f and all $\mathcal{L}_A$-terms $t_1, t_2, \ldots t_n$, $(f(t_1, t_2, \ldots t_n))^{\mathfrak{A}} = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}, t_2^{\mathfrak{A}}, \ldots t_n^{\mathfrak{A}})$*

Definition 1.12 *If $t_1, t_2, \ldots t_n$ are closed $\mathcal{L}_A$-terms, then the $\mathcal{L}_A$-atomic sentences (i.e., atomic formulas with no free variables) which are* **true in** $\mathfrak{A}$ *(or* **satisfied in** $\mathfrak{A}$ *or* **modelled by** $\mathfrak{A}$*) are:*

[1]We place the word *truth* in quotation marks to emphasize that we are defining a formal notion of truth for our mathematical purposes, not a grand philosophical notion of truth in general.

- $(t_1 = t_2)$ *iff* $t_1^{\mathfrak{A}} = t_2^{\mathfrak{A}}$
- $R(t_1, t_2, \ldots t_n)$ *iff* $(t_1^{\mathfrak{A}}, t_2^{\mathfrak{A}}, \ldots t_n^{\mathfrak{A}}) \in R^{\mathfrak{A}}$ *(R an n-ary relation symbol of $\mathcal{L}$)*

In these cases we write:

- $\mathfrak{A} \models (t_1 = t_2)$ *and*
- $\mathfrak{A} \models R(t_1, t_2, \ldots t_n)$ *respectively.*

Before we define truth for general $\mathcal{L}_A$-sentences we need one piece of notation.

Notation: If $\phi(x_1, x_2, \ldots, x_n)$ is an $\mathcal{L}_A$-formula with free variables among $x_1, x_2, \ldots, x_n$ and $a_1, a_2, \ldots a_n \in A$ then $\phi(c_{a_1}, c_{a_2}, \ldots c_{a_n})$ is the $\mathcal{L}_A$-sentence obtained by substituting the constant symbol c_{a_k} for each occurrence of x_k.

Definition 1.13 *In general, for σ an $\mathcal{L}_A$-sentence we define σ* **is true in** $\mathfrak{A}$ *(or* **satisfied in** $\mathfrak{A}$ *or* **modelled by** $\mathfrak{A}$*) and write $\mathfrak{A} \models \sigma$ if:*

- *If σ is an $\mathcal{L}_A$-atomic sentence, then the definition is as in definition 1.12 above.*
- $\mathfrak{A} \models \neg\sigma$ *iff* $\mathfrak{A} \not\models \sigma$
- $\mathfrak{A} \models (\sigma \wedge \psi)$ *iff* $\mathfrak{A} \models \sigma$ *and* $\mathfrak{A} \models \psi$
- $\mathfrak{A} \models (\sigma \vee \psi)$ *iff* $\mathfrak{A} \models \sigma$ *or* $\mathfrak{A} \models \psi$ *(here the word "or" is used inclusively; i.e. we have $\mathfrak{A} \models \sigma \vee \psi$ if either one or both of the conditions $\mathfrak{A} \models \sigma$ and $\mathfrak{A} \models \psi$ are met)*
- $\mathfrak{A} \models \exists x \phi(x)$ *iff* $\mathfrak{A} \models \phi(c_a)$ *for some $a \in A$*
- $\mathfrak{A} \models \forall x \phi(x)$ *iff* $\mathfrak{A} \models \phi(c_a)$ *for all $a \in A$*

Let $\psi(x)$ be the formula $\exists y R(y, x)$ in the language with a singe relation symbol R. In this formula, the variable x is free. Consider the structure $\mathfrak{N} = (\mathbb{N}, <)$ for this language. Then whether or not $\psi(x)$ is a true statement in this structure depends upon what the value of the variable x is: If $x \neq 0$, then ψ is true since $0 < x$; if $x = 0$ then ψ is false since there is no $y \in \mathbb{N}$ such that $y < 0$. Thus, before we can speak of the truth or falsity of a general formula, we must specify the values of its free variables. Formally,

Definition 1.14 *If $\phi(x_1, x_2, \ldots, x_n)$ is an $\mathcal{L}$-formula and we have specified $x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n$ for some $a_1, a_2, \ldots a_n \in A$ then $\phi(x_1, x_2, \ldots, x_n)$ is* **true in $\mathfrak{A}$** *(or* **satisfied in $\mathfrak{A}$** *or* **modelled by $\mathfrak{A}$***) if and only if the $\mathcal{L}_A$-sentence $\phi(c_{a_1}, c_{a_2}, \ldots, c_{a_n})$ is true in $\mathfrak{A}$.*

Notation: If $\phi(x_1, x_2, \ldots, x_n)$ is true in $\mathfrak{A}$ when we specify $x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n$, then we write $\mathfrak{A} \vDash \phi(a_1, a_2, \ldots, a_n)$.

We include one final definition which we will want later.

Definition 1.15 *Given an $\mathcal{L}$-structure $\mathfrak{A}$, we say a subset $D \subseteq A^n$ is* **definable** *if there is an $\mathcal{L}$-formula $\phi(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ and $b_1, b_2, \ldots, b_m \in A$ such that $D = \{(a_1, a_2, \ldots, a_n) \in A^n : \mathfrak{A} \vDash \phi(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)\}$.*

In this case, we say that the formula ϕ **defines** *D.*

1.3 Relations between Structures

For this section, fix a first-order language $\mathcal{L}$.

Notation: If $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures then $f : \mathfrak{A} \to \mathfrak{B}$ denotes a function f with domain A, the underlying set of $\mathfrak{A}$, and codomain B, the underlying set of $\mathfrak{B}$.

It is often the case in mathematics that one wishes to compare two mathematical structures, such as groups or topological spaces, to see if they are essentially "the same." It is no different with our formal structures.

Definition 1.16 *Let $\mathfrak{A}, \mathfrak{B}$ be $\mathcal{L}$-structures. A function $\Phi : \mathfrak{A} \to \mathfrak{B}$ is called an* **isomorphism** *if Φ is a bijection and each non-logical symbol of $\mathcal{L}$ is preserved by Φ, i.e.:*

- *For each constant symbol c in $\mathcal{L}$, $\Phi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$*
- *For each n-ary function symbol f in $\mathcal{L}$, $\Phi(f^{\mathfrak{A}}(a_1, a_2, \dots a_n)) = f^{\mathfrak{B}}(\Phi(a_1), \Phi(a_2), \dots, \Phi(a_n))$ for all $a_1, a_2, \dots a_n \in A$.*
- *For each n-ary relation symbol R in $\mathcal{L}$, $R^{\mathfrak{A}}(a_1, a_2, \dots, a_n)$ iff $R^{\mathfrak{B}}(\Phi(a_1), \Phi(a_2), \dots \Phi(a_n))$.*

In this case we say that $\mathfrak{A}$ and $\mathfrak{B}$ are **isomorphic** *and write $\mathfrak{A} \cong \mathfrak{B}$.*

Example: Let $\mathcal{L}$ be the language with a single binary relation symbol and consider the structures $(\mathbb{N}, \leq)$ and $(\mathbb{N} \cup \{-1\}, \leq)$ where $\leq$ has the usual interpretation. Then the map $\Phi : \mathbb{N} \to \mathbb{N} \cup \{-1\}$ given by $\Phi(n) = n - 1$ witnesses that these structures are isomorphic.

A similar, but weaker, notion of similarity between two structures is that of *elementary equivalence.*

Definition 1.17 *Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are* **elementary equivalent** *if and only if for every $\mathcal{L}$-sentence σ we have $\mathfrak{A} \models \sigma \iff \mathfrak{B} \models \sigma$. In this case, we write $\mathfrak{A} \equiv \mathfrak{B}$.*

We will show that isomorphism implies elementary equivalence by establishing the following proposition.

Proposition 1.18 *If $\Phi : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism of $\mathcal{L}$-structures then for every $\mathcal{L}$-formula $\phi(x_1, x_2, \ldots, x_n)$ and for all $a_1, a_2, \ldots, a_n \in A$, we have $\mathfrak{A} \vDash \phi(c_{a_1}, c_{a_2}, \ldots c_{a_n})$ if and only if $\mathfrak{B} \vDash \phi(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots c_{\Phi(a_n)})$*

proof: Since it is the case that if $\phi(x_1, x_2, \ldots, x_n)$ is an $\mathcal{L}$-formula and $a_1, a_2, \ldots, a_n \in A$ then $\phi(c_{a_1}, c_{a_2}, \ldots c_{a_n})$ is an $\mathcal{L}_A$-sentence, we can (and will) establish this proposition by first establishing it for closed $\mathcal{L}_A$-terms, then using this to show its truth for $\mathcal{L}_A$-atomic sentences, then inducting on the complexity of general sentences to establish the proposition.
Closed $\mathcal{L}_A$-terms: There are three cases to consider

1. The term is constant symbol c of $\mathcal{L}$
2. The term is a constant symbol c_a of $\mathcal{L}_A \setminus \mathcal{L}$
3. We have closed $\mathcal{L}_A$ terms $t_1, t_2, \ldots, t_n$ and the term in question is $f(t_1, t_2, \ldots, t_n)$ for a function symbol f.

These are handled as follows:

1. Since c is constant symbol in $\mathcal{L}$, $\Phi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ by the definition of isomorphism.
2. By definition, if $c_a \in \mathcal{L}_A \setminus \mathcal{L}$ then $c_a^{\mathfrak{A}} = a$ and $c_{\Phi(a)}^{\mathfrak{B}} = \Phi(a)$ so $\Phi(c_a^{\mathfrak{A}}) = c_{\Phi(a)}^{\mathfrak{B}}$.
3. If $t_1(c_{a_1}, c_{a_2}, \ldots c_{a_m}), t_2(c_{a_1}, c_{a_2}, \ldots c_{a_m}), \ldots, t_n(c_{a_1}, c_{a_2}, \ldots c_{a_m})$ are closed $\mathcal{L}_A$-terms such that, for each $1 \leq i \leq n$, we have

$$\Phi(t_i^{\mathfrak{A}}(c_{a_1}, c_{a_2}, \ldots c_{a_m}) = t_i^{\mathfrak{B}}(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots c_{\Phi(a_m)})$$

   and f is a function symbol, then since Φ is an isomorphism $\Phi(f^{\mathfrak{A}}(t_1, t_2, \ldots, t_n)) = f^{\mathfrak{B}}(t_1^{\mathfrak{B}}, t_2^{\mathfrak{B}}, \ldots, t_n^{\mathfrak{B}})$

$\mathcal{L}_A$-atomic sentences: Here there are two cases:

1. $t_1 = t_2$
2. $R(t_1, t_2, \ldots, t_n))$

For R a relation symbol and $t_1, t_2, \ldots t_n$ closed terms. In these cases we have:

1. $\mathfrak{A} \models t_1(c_{a_1}, c_{a_2}, \ldots, c_{a_m}) = t_2(c_{a_1}, c_{a_2}, \ldots, c_{a_m})$ iff $t_1^{\mathfrak{A}}(c_{a_1}, c_{a_2}, \ldots, c_{a_m}) = t_2^{\mathfrak{A}}(c_{a_1}, c_{a_2}, \ldots, c_{a_m})$. From the closed $\mathcal{L}_A$-term case above, this last equation is true iff

$$t_1^{\mathfrak{B}}(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_m)}) = t_2^{\mathfrak{B}}(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_m)})$$

   which is true iff

$$\mathfrak{B} \models t_1(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_m)}) = t_2(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_m)})$$

2. In this case, making use of the closed $\mathcal{L}_A$-term case in a manner similar to its use in 1. above, we have:
   $\mathfrak{A} \models R(t_1(c_{a_1}, c_{a_2}, \ldots c_{a_m}), t_2(c_{a_1}, c_{a_2}, \ldots c_{a_m}), \ldots, t_n(c_{a_1}, c_{a_2}, \ldots c_{a_m}))$ iff $R^{\mathfrak{A}}(t_1^{\mathfrak{A}}(c_{a_1}, c_{a_2}, \ldots c_{a_m}), t_2^{\mathfrak{A}}(c_{a_1}, c_{a_2}, \ldots c_{a_m}), \ldots, t_n^{\mathfrak{A}}(c_{a_1}, c_{a_2}, \ldots c_{a_m}))$. Since Φ is an isomorphism, this last holds iff $R^{\mathfrak{B}}(t_1^{\mathfrak{B}}(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_m)}),$
   $t_2^{\mathfrak{B}}(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_m)}), \ldots, t_n^{\mathfrak{B}}(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_m)}))$ iff
   $\mathfrak{B} \models R(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_m)}), t_2(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_m)}), \ldots,$
   $t_n(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_m)})$

General $\mathcal{L}_A$-sentences: Here we induct on the complexity of a sentence: we assume the truth of the proposition for $\mathcal{L}_A$-sentences ϕ and ψ and then show that it is true for $(\phi \wedge \psi), (\phi \vee \psi), \neg\phi, \exists x\phi$, and $\forall x\phi$. The base case of our induction is the proof of the proposition for $\mathcal{L}_A$-atomic sentences completed above. The cases $(\phi \wedge \psi), (\phi \vee \psi), \neg\phi, \exists x\phi$, and $\forall x\phi$ are all similar, so we do only the $\vee$ case as an example.

We assume that the proposition holds for ϕ and ψ, i.e. for all $a_1, a_2, \ldots, a_n$, $\mathfrak{A} \models \phi(c_{a_1}, c_{a_2}, \ldots, c_{a_n})$ iff $\mathfrak{B} \models \phi(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_n)})$ and $\mathfrak{A} \models \psi(c_{a_1}, c_{a_2}, \ldots, c_{a_n})$ iff $\mathfrak{B} \models \psi(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_n)})$.

Then by the $\vee$ clause of the definition of truth $\mathfrak{A} \models (\phi \vee \psi)(c_{a_1}, c_{a_2}, \ldots, c_{a_n})$ iff $\mathfrak{A} \models \phi(c_{a_1}, c_{a_2}, \ldots, c_{a_n})$ or $\mathfrak{A} \models \psi(c_{a_1}, c_{a_2}, \ldots, c_{a_n})$. By our inductive assumption, this is true iff $\mathfrak{B} \models \phi(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_n)})$ or $\mathfrak{B} \models \psi(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_n)})$. By the $\vee$ clause of the definition of truth again, this last is true iff $\mathfrak{B} \models (\phi \vee \psi)(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_n)})$. Thus, the $\vee$ case is established. The other cases are similar, and so our proposition is proved. □

In the proof above it should have become obvious that expressions such as $\phi(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \ldots, c_{\Phi(a_n)})$ are quite cumbersome. To avoid writing them, we now adopt some notational conveniences.

Notation:

- Given an $\mathcal{L}$-structure $\mathfrak{A}$, for each $a \in A$, in place of the constant symbol c_a in $\mathcal{L}_A$ we shall simply write a. So an expression like $\psi(c_{a_1}, c_{a_2}, \dots, c_{a_n})$ becomes simply $\psi(a_1, a_2, \dots, a_n)$.

- In place of tuples such as $x_1, x_2, \dots, x_n$ or $a_1, a_2, \dots, a_n$ we shall write simply $\bar{x}$ or $\bar{a}$. Thus, we write expressions like $\phi(\bar{x})$ in place of $\phi(x_1, x_2, \dots, x_n)$, $\bar{a} \in A$ in place of $a_1, a_2, \dots, a_n \in A$ and $f(\bar{a}) = \bar{b}$ in place of $f(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$.

For example, with these conventions, the expression

$$\mathfrak{B} \models (\phi \vee \psi)(c_{\Phi(a_1)}, c_{\Phi(a_2)}, \dots, c_{\Phi(a_n)})$$

becomes simply $\mathfrak{B} \models (\phi \vee \psi)(\Phi(\bar{a}))$.

Just as the notion of isomorphism is common in mathematics and has a precise definition in the context of structures, so too are the notions of "embedding" and "substructure."

Definition 1.19 *Given $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ an* **embedding** *is a one-to-one map $\Psi : \mathfrak{A} \to \mathfrak{B}$ such that*

1. *$\Psi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for each constant symbol $c \in \mathcal{L}$*

2. *$\Psi(f^{\mathfrak{A}}(\bar{a})) = f^{\mathfrak{B}}(\Psi(\bar{a}))$ for all function symbols $f \in \mathcal{L}$ and $\bar{a} \in A$*

3. *$\mathfrak{A} \models R(\bar{a})$ iff $\mathfrak{B} \models R(\Psi(\bar{a}))$ for all relations symbols $R \in \mathcal{L}$ and all $\bar{a} \in A$.*

Note that an embedding which is also onto is an isomorphism.

Definition 1.20 *For $\mathfrak{A}$ and $\mathfrak{B}$ $\mathcal{L}$-structures, we say $\mathfrak{A}$ is a* **substructure** *of $\mathfrak{B}$ if $A \subseteq B$ and $id : A \to B$ is an embedding. In this case we write $\mathfrak{A} \subseteq \mathfrak{B}$.*

Of particular interest to us will be a strong kind of embedding known as an *elementary embedding.*

Definition 1.21 *Given $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ an* **elementary embedding** *is a one-to-one map $\Psi : \mathfrak{A} \to \mathfrak{B}$ such that for any $\mathcal{L}$-formula $\phi(\bar{x})$ and any $\bar{a} \in A$ we have $\mathfrak{A} \models \phi(\bar{a})$ if and only if $\mathfrak{B} \models \phi(\Psi(\bar{a}))$.*

If $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures and $C \subseteq A$, we say that a map $f : C \to B$ is **partial elementary** *if for any $\mathcal{L}$-formula $\phi(\bar{x})$ and any $\bar{c} \in C$ we have $\mathfrak{A} \models \phi(\bar{c})$ if and only if $\mathfrak{B} \models \phi(f(\bar{c}))$*

We say that $\mathfrak{A}$ is an **elementary substructure** *of $\mathfrak{B}$ and that $\mathfrak{B}$ is an* **elementary extension** *of $\mathfrak{A}$ if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and $id : \mathfrak{A} \to \mathfrak{B}$ is an elementary embedding. In this case we write $\mathfrak{A} \preceq \mathfrak{B}$ or $\mathfrak{B} \succeq \mathfrak{A}$.*

An elementary embedding is an embedding. To see this, note, for example, that if $c \in \mathcal{L}$ is an constant symbol then if $\phi(x)$ is the formula $c = x$ we have $\mathfrak{A} \models \phi(c^{\mathfrak{A}})$ and thus $\mathfrak{B} \models \phi(\Psi(c^{\mathfrak{A}}))$. By the definition of truth, $\mathfrak{B} \models \phi(\Psi(c^{\mathfrak{A}}))$ implies that $c^{\mathfrak{B}} = \Psi(c^{\mathfrak{A}})$, so Ψ satisfies point 1. of definition 1.19. Similar arguments show that an elementary embedding satisfies points 2. and 3. of definition 1.19 as well.

One way to test whether or not a substructure is an elementary substructure is the following:

Theorem 1.22 (Tarski-Vaught Test) *Suppose $\mathfrak{A} \subseteq \mathfrak{B}$. Then $\mathfrak{A} \preceq \mathfrak{B}$ if and only if for all $\mathcal{L}$-formulas $\phi(x, \bar{y})$ and for all $\bar{a} \in A$, if $\mathfrak{B} \models \exists x \phi(x, \bar{a})$ then $\mathfrak{B} \models \phi(a^*, \bar{a})$ for some $a^* \in A$.*

proof: First suppose $\mathfrak{A} \preceq \mathfrak{B}$. Say $\phi(x, \bar{y})$ is an $\mathcal{L}$-formula and $\bar{a} \in A$ is such that $\mathfrak{B} \models \exists x \phi(x, \bar{a})$. Then by the fact that $\mathfrak{A} \preceq \mathfrak{B}$, $\mathfrak{A} \models \exists x \phi(x, \bar{a})$. So, by the definition of truth, there is $a^* \in A$ such that $\mathfrak{A} \models \phi(a^*, \bar{a})$. Thus, since $\mathfrak{A} \preceq \mathfrak{B}$, $\mathfrak{B} \models \phi(a^*, \bar{a})$.

Conversely, assume that $\mathfrak{A} \subseteq \mathfrak{B}$ and whenever $\phi(x, \bar{y})$ is an $\mathcal{L}$-formula such that $\mathfrak{B} \models \exists x \phi(x, \bar{a})$ for some $\bar{a} \in A$ we have $\mathfrak{B} \models \phi(a^*, \bar{a})$ for some $a^* \in A$. We show that $\mathfrak{A} \preceq \mathfrak{B}$ by induction on the complexity of $\mathcal{L}$-formulas. Before starting the induction, we need a claim.

Claim: For any $\bar{a} \in A$ and any $\mathcal{L}$-term t, $t^{\mathfrak{A}}(\bar{a}) = t^{\mathfrak{B}}(\bar{a})$.

proof: If c is a constant symbol of $\mathcal{L}$ then $\mathfrak{A} \subseteq \mathfrak{B}$ gives us $c^{\mathfrak{A}} = c^{\mathfrak{B}}$; if x is a variable symbol of $\mathcal{L}$, then if x is assigned the value a in $\mathfrak{A}$, we can assign x the same value in $\mathfrak{B}$ since $A \subseteq B$ gives $a \in B$. Now, if $t_1(\bar{x}), t_2(\bar{x}), \dots t_n(\bar{x})$ are terms for which the claim holds and f is a function symbol of $\mathcal{L}$, then $t(\bar{x}) = f(t_1(\bar{x}), t_2(\bar{x}), \dots t_n(\bar{x}))$ is a term of $\mathcal{L}$ and for

$\bar{a} \in A$, we have $t^{\mathfrak{A}}(\bar{a}) = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}(\bar{a}), t_2^{\mathfrak{A}}(\bar{a}), \dots t_n^{\mathfrak{A}}(\bar{a}))$. Since $\mathfrak{A} \subseteq \mathfrak{B}$, $f^{\mathfrak{A}} = f^{\mathfrak{B}}$ so $f^{\mathfrak{A}}(t_1^{\mathfrak{A}}(\bar{a}), t_2^{\mathfrak{A}}(\bar{a}), \dots t_n^{\mathfrak{A}}(\bar{a})) = f^{\mathfrak{B}}(t_1^{\mathfrak{A}}(\bar{a}), t_2^{\mathfrak{A}}(\bar{a}), \dots t_n^{\mathfrak{A}}(\bar{a}))$. By our assumption on $t_1, t_2, \dots t_n$, this last expression is equal to $f^{\mathfrak{B}}(t_1^{\mathfrak{B}}(\bar{a}), t_2^{\mathfrak{B}}(\bar{a}), \dots t_n^{\mathfrak{B}}(\bar{a})) = t^{\mathfrak{B}}(\bar{a})$. Thus, by induction on the complexity of terms, the claim is established for all terms t. ■

We now proceed with our induction:

Base case (atomic formulas): If $\phi(\bar{x})$ is an atomic formula, then ϕ has one of the forms $t_1(\bar{x}) = t_2(\bar{x})$ or $R(t_1(\bar{x}), t_2(\bar{x}), \dots t_n(\bar{x})))$ for R a relation symbol and $t_1, t_2, \dots, t_n$ $\mathcal{L}$-terms. In the first case, the claim gives us that for $\bar{a} \in A$, $\mathfrak{A} \models \phi(\bar{a}) \iff t_1^{\mathfrak{A}}(\bar{a}) = t_2^{\mathfrak{A}}(\bar{a}) \iff t_1^{\mathfrak{B}}(\bar{a}) = t_2^{\mathfrak{B}}(\bar{a}) \iff \mathfrak{B} \models \phi(\bar{a})$.

In the second case, since $\mathfrak{A} \subseteq \mathfrak{B}$, $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^n$. Thus, by the claim for $\bar{a} \in A$, we have

$$\begin{aligned}\mathfrak{A} \models \phi(\bar{a}) \iff R^{\mathfrak{A}}(t_1^{\mathfrak{A}}(\bar{a}), t_2^{\mathfrak{A}}(\bar{a}), \dots t_n^{\mathfrak{A}}(\bar{a})) \iff \\ R^{\mathfrak{B}}(t_1^{\mathfrak{A}}(\bar{a}), t_2^{\mathfrak{A}}(\bar{a}), \dots t_n^{\mathfrak{A}}(\bar{a})) \iff \\ R^{\mathfrak{B}}(t_1^{\mathfrak{B}}(\bar{a}), t_2^{\mathfrak{B}}(\bar{a}), \dots t_n^{\mathfrak{B}}(\bar{a})) \iff \mathfrak{B} \models \phi(\bar{a})\end{aligned}$$

So we have established the base case of our induction.

Induction step (general formulas): We have one case for each of the symbols $\wedge, \vee, \neg, \forall$ and $\exists$. The cases for $\wedge, \vee$ and $\neg$ cases are all similar, so we do only the $\wedge$ case as an example.

$\wedge$ *case*: Assume the theorem holds for $\phi(\bar{x})$ and $\psi(\bar{x})$. Then for $\bar{a} \in A$, $\mathfrak{A} \models (\phi \wedge \psi)(\bar{a}) \iff (\mathfrak{A} \models \phi(\bar{a})$ and $\mathfrak{A} \models \psi(\bar{a})) \iff$ (by the induction hypothesis) $\mathfrak{B} \models \phi(\bar{a})$ and $\mathfrak{B} \models \psi(\bar{a}) \iff \mathfrak{B} \models (\phi \wedge \psi)(\bar{a})$.

$\exists$ *case*: Let $\phi(x, \bar{y})$ be a formula such that for $a^*, \bar{a} \in A$, $\mathfrak{A} \models \phi(a^*, \bar{a}) \iff \mathfrak{B} \models \phi(a^* \bar{a})$. We must show that if $\psi(\bar{y})$ is $\exists x \phi(x, \bar{y})$ then for any $\bar{a} \in A$, $\mathfrak{A} \models \psi(\bar{a}) \iff \mathfrak{B} \models \psi(\bar{b})$.

Well, say $\bar{a} \in A$ and $\mathfrak{A} \models \psi(\bar{a})$. Then $\mathfrak{A} \models \exists x \phi(x, \bar{a})$ so there is $a^* \in A$ such that $\mathfrak{A} \models \phi(a^*, \bar{a})$. By our assumption on ϕ, this gives $\mathfrak{B} \models \phi(a^*, \bar{a})$, so $\mathfrak{B} \models \exists x \phi(x, \bar{a})$, i.e., $\mathfrak{B} \models \psi(\bar{a})$.

Now, let ϕ, ψ be as above and assume $\mathfrak{B} \models \psi(\bar{a})$ for some $\bar{a} \in A$. Then $\mathfrak{B} \models \exists \phi(x, \bar{a})$. By the hypotheses of the theorem, this implies that $\mathfrak{B} \models \phi(a^*, \bar{a})$ for some $a^* \in A$. By our assumption on ϕ, this gives $\mathfrak{A} \models \phi(a^*, \bar{a})$, so $\mathfrak{A} \models \exists x \phi(x, \bar{a})$, i.e., $\mathfrak{A} \models \psi(\bar{a})$.

This leaves only the $\forall$ case, which we handle by noting that $\forall x$ is equivalent to $\neg\exists x\neg$ and we have already handled the $\exists$ and $\neg$ cases. By induction, then, our proof is complete. $\square$

Another result we will want is the Elementary Chain Theorem.

Definition 1.23 *If $(I, \leq)$ is a linearly ordered set, then an* **elementary chain** *of $\mathcal{L}$-structures is a set of $\mathcal{L}$-structures $\{\mathfrak{A}_i : i \in I\}$ such that if $i, j \in I$, $i \leq j$, then $\mathfrak{A}_i \preceq \mathfrak{A}_j$.*

Notation: Given an elementary chain $\{\mathfrak{A}_i : i \in I\}$, we let $\mathfrak{A}^*$ be the $\mathcal{L}$-structure with universe $A^* = \bigcup_{i \in I} A_i$ and

- $c^{\mathfrak{A}^*} = c^{\mathfrak{A}_i}$ for some (equivalently for all) $i \in I$
- $f^{\mathfrak{A}^*}(\bar{a}) = f^{\mathfrak{A}_i}(\bar{a})$ for some (equivalently for all) $i \in I$ such that $\bar{a} \in A_i$
- $R^{\mathfrak{A}^*}(\bar{a}) \iff R^{\mathfrak{A}_i}(\bar{a})$ for some (equivalently for all) $i \in I$ such that $\bar{a} \in A_i$

We write $\mathfrak{A}^* = \bigcup_{i \in I} \mathfrak{A}_i$.

Note that this gives us that for each $i \in I$ we have $\mathfrak{A}_i$ a substructure of $\mathfrak{A}^*$.

Theorem 1.24 (Elementary Chain Theorem) *Let $(I, \leq)$ be a linearly ordered set. If $\{\mathfrak{A}_i : i \in I\}$ is an elementary chain, then $\mathfrak{A}_i \preceq \mathfrak{A}^*$.*

proof: Since we have, as noted above, $\mathfrak{A}_i \subseteq \mathfrak{A}^*$ for each $i \in I$ it suffices, by the Tarski-Vaught Test 1.22, to check that for all $i \in I$ we have that if $\bar{b} \in A_i$ and $\mathfrak{A}^* \models \exists x \psi(x, \bar{b})$ then there is $a \in A_i$ such that $\mathfrak{A}^* \models \psi(a, \bar{b})$. Well, choose some $i_0 \in I$, $\psi(x, \bar{y})$ and $\bar{b} \in A_{i_0}$ such that $\mathfrak{A}^* \models \exists x \psi(x, \bar{b})$. Since $\mathfrak{A}^* \models \exists x \psi(x, \bar{b})$ there is some $a^* \in A^*$ such that $\mathfrak{A}^* \models \psi(a^*, \bar{b})$. But $A^* = \bigcup_{i \in I} A_i$ so there is some $j \in I$ such that $a^* \in A_j$. If $j \leq i_0$, then $A_j \subseteq A_{i_0}$ and so $a^* \in A_{i_0}$ and we have $\mathfrak{A}_{i_0} \preceq \mathfrak{A}^*$, as desired.

On the other hand, say $i_0 \leq j$. Since $a^* \in A_j$, we have $\mathfrak{A}_j \models \psi(a^*, \bar{b})$ and thus $\mathfrak{A}_j \models \exists x \psi(x, \bar{b})$. But now, since $\{\mathfrak{A}_i : i \in I\}$ is an elementary chain, $\mathfrak{A}_{i_0} \preceq \mathfrak{A}_j$ and so $\mathfrak{A}_j \models \exists x \psi(x, \bar{b})$ implies that $\mathfrak{A}_{i_0} \models \exists x \psi(x, \bar{b})$. Thus there is some $a \in A_{i_0}$ such that $\mathfrak{A}_{i_0} \models \psi(a, \bar{b})$. As $\mathfrak{A}_i \subseteq \mathfrak{A}^*$, this means that $\mathfrak{A}^* \models \psi(a, \bar{b})$, so there is $a \in A_{i_0}$ such that $\mathfrak{A}^* \models \psi(a, \bar{b})$ and thus $\mathfrak{A}_{i_0} \preceq \mathfrak{A}^*$ by the Tarski-Vaught Test.

In all cases, then, we have $\mathfrak{A}_{i_0} \preceq \mathfrak{A}^*$, and so as $i_0 \in I$ was arbitrary, we have the desired result. $\square$

We next want two important theorems that tell us that we can find elementary substructures and extensions of minimal and arbitrarily large sizes. The proofs of these theorems would require too great a digression to be given here, but may be found in an introductory textbook on mathematical logic such as [2], §2.3.

Theorem 1.25 (Upward Löwenheim-Skolem Theorem) *For any infinite $\mathcal{L}$-structure $\mathfrak{A}$ and any cardinal $\kappa \geq |\mathcal{L}| + |A|$ there is an $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{B} \succeq \mathfrak{A}$ and $|B| = \kappa$.*

Theorem 1.26 (Downward Löwenheim-Skolem Theorem) *For any $\mathcal{L}$-structure $\mathfrak{A}$ and any set $D \subseteq A$ if κ is a cardinal such that $|\mathcal{L}| + |D| \leq \kappa \leq |A|$ then there is an $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{B} \preceq \mathfrak{A}$, $|B| = \kappa$ and $D \subseteq B$.*

1.4 Theories and their Models

For this section, fix a first-order language $\mathcal{L}$.

Definition 1.27 *An* **$\mathcal{L}$-theory** *T is a set of $\mathcal{L}$-sentences. If $\mathfrak{A}$ is an $\mathcal{L}$-structure such that for all $\sigma \in T$ $\mathfrak{A} \models \sigma$ then we call $\mathfrak{A}$ a* **model** *of T and write $\mathfrak{A} \models T$.*

Not every theory need have a model. For example, if T is the theory $\{\exists x(x \neq x)\}$ then there is no model of T (i.e., no $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models T$) since there is no structure which contains an element not equal to itself.

Definition 1.28 *An $\mathcal{L}$-theory T is* **satisfiable** *if and only if there is an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models T$. A theory T is* **finitely satisfiable** *if and only if for all finite $T_0 \subseteq T$ there is an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models T_0$.*

As it turns out, satisfiability is equivalent to finite satisfiability. This fact is quite useful, as it is often easier to construct a model for a finite subset of theory, than for the whole theory.

Theorem 1.29 (Compactness Theorem) *An $\mathcal{L}$-theory T is satisfiable if and only if it is finitely satisfiable.*

Unfortunately, the proof of the Compactness Theorem would take us too far afield into the theory of deductions to be given here. It may be found in any standard introductory textbook in mathematical logic.

The next result is a good illustration of how the Compactness Theorem is used.

Theorem 1.30 *Let T be an $\mathcal{L}$-theory. If T has arbitrarily large finite models, than T has infinite models.*

proof: Let $\{c_n : n \in \omega\}$ be an infinite set of constant symbols not in $\mathcal{L}$ and let $\mathcal{L}' = \mathcal{L} \cup \{c_n : n \in \omega\}$. Let $T' = T \cup \{c_n \neq c_m : n, m \in \omega, n < m\}$. We show that T' is finitely satisfiable, hence, by Compactness, satisfiable.

Let $T_0' \subseteq T'$ be a finite subset of T'. Then T_0' consists of a finite number of sentences from T and a finite number of sentences of the form $c_n \neq c_m$. Thus, T_0' contains only finitely many of the constant symbols in $\{c_n : n \in \omega\}$, say N of them. Since T has arbitrarily large finite models, there is an $\mathcal{L}$-structure $\mathfrak{A}_N$ of size at least N.

We extend $\mathfrak{A}_N$ to an $\mathcal{L}'$-structure $\mathfrak{A}'_N$ by choosing N distinct elements of the underlying set of $\mathfrak{A}_N$ and interpreting of each of the $c_i \in \{c_n : n \in \omega\}$ that appears in T'_0 as one of these elements. Thus, if $c_n \neq c_m \in T'_0$, we have $\mathfrak{A}'_N \models c_n \neq c_m$. If $\sigma \in T \cap T'_0$, then $\mathfrak{A}_N \models \sigma$ since $\mathfrak{A}_N \models T$. Since the interpretations of the symbols of $\mathcal{L}$ did not change when we passed from $\mathfrak{A}_N$ to $\mathfrak{A}'_N$, we thus also have $\mathfrak{A}'_N \models \sigma$ so in particular $\mathfrak{A}'_N \models T'_0$. Thus, by the Compactness Theorem, there is an $\mathcal{L}'$-structure $\mathfrak{A}'$ such that $\mathfrak{A}' \models T'$.

Since $\mathfrak{A}' \models T'$, $\mathfrak{A}' \models T$ and $\mathfrak{A}' \models c_n \neq c_m$ for all $n, m \in \omega, n < m$. Thus, $\mathfrak{A}'$ is infinite. Now, let $\mathfrak{A}$ be the $\mathcal{L}$-structure obtained from $\mathfrak{A}'$ by simply deleting the constant symbols $\{c_n : n \in \omega\}$ from $\mathcal{L}'$ and leaving everything else unchanged. Then $\mathfrak{A}$ has the same universe as $\mathfrak{A}'$, and is therefore an infinite $\mathcal{L}$-structure. Since no symbol appearing in T was deleted and the interpretation of no symbol appearing in T was changed when we passed from $\mathfrak{A}'$ to $\mathfrak{A}$, we thus have $\mathfrak{A} \models T$, and we have shown that T has an infinite model, as desired. □

So if a theory has "bigger and bigger" finite models, it has an infinite model. This infinite model is not the end of the line, however, as a theory with an infinite model has a model of any given infinite cardinality.

Theorem 1.31 *Let T be a $\mathcal{L}$-theory. If T has an infinite model, then T has models of arbitrarily large infinite cardinality.*

sketch of proof: The proof of this result is entirely similar to the proof of the last result. We let κ be an infinite cardinal for which we wish to show that there is $\mathfrak{A} \models T$ and $|A| = \kappa$. We let $(c_\alpha : \alpha < \kappa)$ be κ-many distinct new constant symbols which we add to our language $\mathcal{L}$ to form a new language $\mathcal{L}'$. We then form the $\mathcal{L}'$-theory $T' = T \cup \{c_\alpha \neq c_\beta : \alpha < \beta < \kappa\}$. Using the existence of an infinite model for T we show that any finite subset of T' has a model.

This gives us T' finitely satisfiable, hence satisfiable by Compactness 1.29, so there is a $\mathcal{L}'$-structure $\mathfrak{B}' \models T'$. This means that $c_\alpha^{\mathfrak{B}'} \neq c_\beta^{\mathfrak{B}'}$ for all $\alpha < \beta < \kappa$. Thus $\mathfrak{B}'$ has size at least κ. Choosing a subset of the universe of $\mathfrak{B}'$ and applying the Downward Löwenheim-Skolem theorem 1.26, we can find a model of T' of size exactly κ. Restricting this structure to make it an $\mathcal{L}$-structure then gives us our desired model of T of size κ. □

We now turn our attention to a class of theories called complete theories. We need one other definition before defining a complete theory.

Definition 1.32 *If T is a $\mathcal{L}$-theory and σ is an $\mathcal{L}$-sentence then we say that T* **models** *σ if whenever $\mathfrak{A}$ is an $\mathcal{L}$-structures such that $\mathfrak{A} \models T$, then $\mathfrak{A} \models \sigma$. In this case, we write $T \models \sigma$.*

Definition 1.33 *We say that an $\mathcal{L}$-theory T is* **complete** *if for all $\mathcal{L}$-sentences σ either $T \models \sigma$ or $T \models \neg\sigma$.*

A complete theory is "complete" in the sense that it "completely" settles the truth or falsity of any proposition expressible in our language $\mathcal{L}$; i.e. for any $\mathcal{L}$-sentence σ T gives that σ is either true ($T \models \sigma$) or false ($T \models \neg\sigma$).

From our definition of truth it follows that for a given $\mathcal{L}$-structure $\mathfrak{A}$ and for any $\mathcal{L}$-sentence σ either $\mathfrak{A} \models \sigma$ or $\mathfrak{A} \models \neg\sigma$. Thus, the set of all sentences modelled by $\mathfrak{A}$ is a complete theory. We denote this theory by $Th(\mathfrak{A})$.

We now introduce the notion of *categoricity*, which will give us a test for completeness.

Definition 1.34 *If κ is an infinite cardinal, we say that an $\mathcal{L}$-theory T is* **κ-categorical** *if T has a model of size κ and all models of T of size κ are isomorphic.*

Theorem 1.35 (Łoś-Vaught Test) *Suppose the $\mathcal{L}$-theory T is κ-categorical for some infinite cardinal $\kappa \geq |\mathcal{L}|$ and T has no finite models. Then T is complete.*

proof: Suppose otherwise. Then there is an $\mathcal{L}$-sentence σ such that $T \not\models \sigma$ and $T \not\models \neg\sigma$. Thus there are $\mathfrak{A}, \mathfrak{B}$ $\mathcal{L}$-structures such that $\mathfrak{A} \models T \cup \{\sigma\}$ and $\mathfrak{B} \models T \cup \{\neg\sigma\}$. By the Löwenheim-Skolem theorems 1.26 and 1.25 we may assume without loss of generality that both $\mathfrak{A}, \mathfrak{B}$ are of size κ. By categoricity, then, $\mathfrak{A} \cong \mathfrak{B}$. But from proposition 1.18 it follows that $\mathfrak{A} \cong \mathfrak{B}$ implies $\mathfrak{A} \equiv \mathfrak{B}$ and therefore that $\mathfrak{A} \models \sigma \iff \mathfrak{B} \models \sigma$, a contradiction. Thus, T must be complete. □

In the Łoś-Vaught Test, the assumption that T has no finite models is necessary because a theory which has both an infinite model of size κ and a finite model, cannot be complete. In particular, say $\mathfrak{A} \models T$ is infinite, $\mathfrak{B} \models T$ is finite of size n, and σ is the sentence $\exists x_1 \ldots \exists x_{n+1} \bigwedge_{i \neq j} x_i \neq x_j$. Then $\mathfrak{A} \models \sigma$ and $\mathfrak{B} \models \neg\sigma$, so T is not complete.

Chapter 2

Types, Saturation and Homogeneity

The material in this chapter primarily comes from [2] *Model Theory: An Introduction* by David Marker, with the exceptions of results 2.4, 2.5, and 2.9. These, along with some of the basic definitions, are based on the same lectures of Laskowski mentioned at the beginning of the previous chapter.

Throughout this chapter, $\mathcal{L}$ will be a countable first-order language.

2.1 Types

Definition 2.1 *Given an n-tuple of variable symbols $\bar{x}$, a n-***type** *$p(\bar{x})$ is a set of formulas all of whose free variables are contained in $\bar{x}$, i.e., if $\phi \in p$, then the free variables of ϕ are among those in $\bar{x}$. (Typically, we will refer simply to types, rather than n-types, unless the length of the tuple is particularly important.)*

Given an $\mathcal{L}$-structure $\mathfrak{A}$ and a tuple $\bar{a} \in A$ we say that $\bar{a}$ **realizes** *p in $\mathfrak{A}$ if $\mathfrak{A} \models \phi(\bar{a})$ for all $\phi \in p$.*

We say $\mathfrak{A}$ **realizes** *p if some tuple from A realizes p in $\mathfrak{A}$. If $\mathfrak{A}$ does not realize p, we say $\mathfrak{A}$* **omits** *p.*

We say that the type $p(\bar{x})$ is **complete** *if for every $\mathcal{L}$-formula $\phi(\bar{x})$ with free variables among those in $\bar{x}$ either ϕ or $\neg\phi$ is in p.*

One way of specifying a complete type it take the set of all formulas satisfied by some tuple.

Definition 2.2 *If $\mathfrak{A}$ is an $\mathcal{L}$-structure and $\bar{a} \in A$, then the* **type of $\bar{a}$ in $\mathfrak{A}$** *is $tp(\bar{a}) = \{\phi(\bar{x}) : \mathfrak{A} \models \phi(\bar{a}),\ \phi$ an $\mathcal{L}$-formula$\}$*

Sometimes, in considering the set of formulas satisfied by a tuple, we will want to allow the formulas to have parameters from some set P. That is, we will want to consider formulas satisfied by the tuple in the language $\mathcal{L}_P$. In this case, we will use the following notation:

Notation: $tp(\bar{a}/P)$ is the set of all $\mathcal{L}_P$-formulas satisfied by the tuple $\bar{a}$.

Note that each $\mathcal{L}$-formula is a $\mathcal{L}_P$-formula so $tp(\bar{a}) \subseteq tp(\bar{a}/P)$ for each tuple $\bar{a}$ and set of parameters P.

Our definition of formulas was inductive, and at each step we built only finite formulas from already constructed finite formulas. Thus, all formulas in our language are finite; in particular, if we have $\phi_i(\bar{x})$ for each $i \in \omega$, an infinite conjunction such as $\bigwedge_{i\in\omega} \phi_i$ is *NOT* a formula of our language. We can, however, consider the type $p(\bar{x}) = \{\phi_i : i \in \omega\}$. Then if p is realized in $\mathfrak{A}$ we have some $\bar{a} \in A$ such that $\mathfrak{A} \models \phi_i(\bar{a})$ for *every* i. Thus, the type p can serve as a "poor man's" infinite conjunction. Just as we looked at the relation between sentences and theories in defining expressions like $T \models \sigma$, then, it is natural to look at the relation between types and theories.

Definition 2.3 *Given a theory T, a type $p(\bar{x})$ is* **consistent with** *T if there is a structure $\mathfrak{A}$ such that $\mathfrak{A} \models T$ and p is realized in $\mathfrak{A}$.*

We say that $p(\bar{x})$ is **finitely satisfiable** *with respect to T if for any finite $p_0 \subseteq p$ there is a structure $\mathfrak{A}$ such that $\mathfrak{A} \models T \cup \{\exists \bar{x} \bigwedge p_0(\bar{x})\}$*

The Compactness theorem 1.29 allows us to show that these two notions are equivalent.

Proposition 2.4 *If T is a theory and p is a type, then p is consistent with T if and only if p is finitely satisfiable with respect to T.*

proof: That p consistent with T implies p finitely satisfiable with respect to T is obvious. For the converse, assume $p = p(\bar{x})$ is finitely satisfiable with respect to T. We add to our language $\mathcal{L}$ a new constants symbol c_i for each $x_i \in \bar{x}$. Call this new language $\mathcal{L}^*$. Consider the $\mathcal{L}^*$-theory $T^* = T \cup p(\bar{c})$ where $p(\bar{c})$ is the set of $\mathcal{L}^*$-sentences obtained by replacing each free occurrence of

x_i in a formula in p with c_i. We show that T^* has a model. Restricting this model to $\mathcal{L}$ will then give us our desired model of T realizing p.

Let $\Delta \subseteq T^*$ be finite. Then $\Delta = T_0 \cup \{\psi_1(\bar{c}), \ldots, \psi_m(\bar{c})\}$ where $T_0 \subseteq T$ is finite and $\psi_1(\bar{x}), \ldots, \psi_m(\bar{x}) \in p(\bar{x})$. Since p is finitely satisfiable, there is $\mathfrak{A} \models T$ such that $\mathfrak{A} \models \exists \bar{x} \bigwedge_i \psi_i(\bar{x})$. Expand $\mathfrak{A}$ to an $\mathcal{L}^*$-structure $\mathfrak{A}^*$ by interpreting c_i as a_i for some $\bar{a}$ such that $\mathfrak{A} \models \bigwedge_i \psi_i(\bar{a})$. Then $\mathfrak{A}^* \models \Delta$. Thus T^* is finitely satisfiable, hence satisfiable by the Compactness theorem 1.29. Let $\mathfrak{B}^* \models T^*$. Then if we restrict $\mathfrak{B}^*$ to the language $\mathcal{L}$, we get an $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{B} \models T$ and $\mathfrak{B} \models p(\bar{b})$ where $\bar{b} = \bar{c}^{\mathfrak{B}^*}$ is the interpretation of $\bar{c}$ in $\mathfrak{B}^*$. So we have p consistent with T, as desired. $\square$

The following is an useful proposition:

Proposition 2.5 *Let T be a complete theory and let $\mathfrak{A}$ be an $\mathcal{L}$-structure such that $\mathfrak{A} \models T$. If $p(\bar{x})$ is a type consistent with T then there is $\mathfrak{B} \succeq \mathfrak{A}$ such that $\mathfrak{B}$ realizes p.*

Unfortunately, the proof of this proposition would requires too many prerequisites to be given here. It may be found in an introductory text on model theory such as [2], §4.1.

Notation: Let T be a complete theory and $\mathfrak{A} \models T$, $B \subseteq A$. Then we denote the collection of all complete n-types in the language $\mathcal{L}_B$ that are consistent with $\mathfrak{A}$ by $S_n^{\mathfrak{A}}(B)$. We denote the set of all complete n-types consistent with T by $S_n(T)$.

We will be interested in theories which realize "few" types. Precisely, we will be interested in theories called ω-stable theories, defined as follows:

Definition 2.6 *Let T be a complete theory in a countable language, κ an infinite cardinal. Then we say T is* **κ-stable** *if whenever $\mathfrak{A} \models T$, $B \subseteq A$ and $|B| = \kappa$, we have $|S_n^{\mathfrak{A}}(B)| = \kappa$.*

If $\kappa = \aleph_0$, then we say T is **ω-stable**.

We will want to know that an ω-stable theory has a particular kind of model called a *prime* model. Formally:

Definition 2.7 *Suppose T is an $\mathcal{L}$-theory. A model $\mathfrak{A}$ of T is* **prime** *if $\mathfrak{A}$ embeds elementarily into any other model of T.*

To accomplish this goal, we will need to develop some additional theory.

Definition 2.8 *Given an $\mathcal{L}$-theory T and a type $p(\bar{x})$, we say $p(\bar{x})$ is* **isolated** *with respect to T, or that T* **locally realizes** *$p(\bar{x})$, if there is an $\mathcal{L}$-formula $\theta(\bar{x})$ such that $T \cup \{\exists \bar{x}\theta(\bar{x})\}$ is satisfiable and for all $\phi \in p$, $T \models \forall \bar{x}(\theta(\bar{x}) \to \phi(\bar{x}))$. In this case, we say that θ* **isolates** *the type p. A theory T* **locally omits** *a type p if it does not locally realize p.*

Proposition 2.9 *If T is a complete theory and the type p is isolated with respect to T, then p is realized in any model of T.*

proof: Let $\theta(\bar{x})$ be a formula witnessing that p is isolated with respect to T. Then $T \cup \{\exists \bar{x}\theta(\bar{x})\}$ is satisfiable. Hence, as T is complete, $T \models \exists \bar{x}\theta(\bar{x})$. Thus, if $\mathfrak{A} \models T$ is an arbitrary model for T, there is $\bar{a} \in A$ such that $\mathfrak{A} \models \theta(\bar{a})$. As $T \models \forall \bar{x}(\theta(\bar{x}) \to \phi(\bar{x}))$ for all $\phi \in p$, we thus have $\mathfrak{A} \models \forall \bar{x}(\theta(\bar{x}) \to \phi(\bar{x}))$ for all $\phi \in p$ and so $\mathfrak{A} \models \phi(\bar{a})$ for all $\phi \in p$. Therefore $\bar{a}$ realizes p in $\mathfrak{A}$, so $\mathfrak{A}$ realizes p. □

The terms "isolated" and "locally" are suggestive of topology. This is no accident. There is an useful topology on $S_n^{\mathfrak{A}}(B)$ and $S_n(T)$ known as the *Stone topology.* While we will not do so here since it is not needed for our purposes, it can be shown that in the Stone topology the words "isolated" and "locally" have their usual topological meanings.

Definition 2.10 *Say $\bar{x}$ is an n-tuple of variable symbols and $FV(\phi) \subseteq \bar{x}$. Let $[\phi] = \{p \in S_n^{\mathfrak{A}}(B) : \phi \in p\}$. The* **Stone topology** *on $S_n^{\mathfrak{A}}(B)$ is the topology generated by taking the sets $[\phi]$ as basic open sets.*

The **Stone topology** *on $S_n(T)$ is generated the same way, only using $[\phi] = \{p \in S_n(T) : \phi \in p\}$ in place of $[\phi] = \{p \in S_n^{\mathfrak{A}}(B) : \phi \in p\}$.*

A feature of the Stone topology which we will need is that it provides a characterization of theories which have prime models.

Theorem 2.11 *If $\mathcal{L}$ is countable and T is a complete $\mathcal{L}$-theory with infinite models, then T has a prime model if and only if the isolated types in $S_n(T)$ are dense for all n.*

proof: The proof of this result is best obtained by showing the equivalence of the two given conditions to a third having to do with so-called *atomic* models of T. As we have no other reason to mention atomic models, other than this result, we forgo developing their theory and instead refer the reader to Marker [2] §4.2 for a proof of this result. □

Another important property of the Stone topology is that it makes $S_n^{\mathfrak{A}}(B)$ a compact space.

Proposition 2.12 *$S_n^{\mathfrak{A}}(B)$ is compact.*

proof: It suffices to show that if $C = \{[\phi_i(\bar{x})] : i \in I\}$ is a cover of $S_n^{\mathfrak{A}}(B)$ by basic open sets then C has a finite subcover. Toward a contradiction, assume that this is not the case. Let $\Gamma = \{\neg\phi_i(\bar{x}) : i \in I\}$. We will show that $Th(\mathfrak{A}_B) \cup \Gamma$ is satisfiable. Let $I_0 \subseteq I$ be finite. Then by our assumption that there is no finite subcover of C, there is a type $p \notin \bigcup_{i \in I_0}[\phi_i]$. Let $\mathfrak{C}$ be an elementary extension of $\mathfrak{A}$ containing a realization $\bar{c}$ of p. Such a $\mathfrak{C}$ exists by proposition 2.5. Then we have $\mathfrak{C} \models Th(\mathfrak{A}_B) \cup \bigwedge_{i \in I_0} \neg\phi_i(\bar{c})$. Thus, for any finite $\Delta \subseteq Th(\mathfrak{A}_B) \cup \Gamma$, we can find $\mathfrak{C} \models \Delta$ and thus $Th(\mathfrak{A}_B) \cup \Gamma$ is finitely satisfiable, hence satisfiable by Compactness 1.29.

Now, let $\mathfrak{D}$ be an elementary extension of $\mathfrak{A}$ containing a realization$\bar{d}$ of Γ. Then we have

$$tp^{\mathfrak{D}}(\bar{d}/B) \in S_n^{\mathfrak{A}}(B) \setminus \bigcup_{i \in I}[\phi_i]$$

contradicting the fact that $C = \{[\phi_i(\bar{x})] : i \in I\}$ is a cover of $S_n^{\mathfrak{A}}(B)$. Thus C must have a finite subcover, so $S_n^{\mathfrak{A}}(B)$ is compact. □

We achieve our goal of showing that ω-stable theories have prime models by establishing the following theorem.

Theorem 2.13 *Suppose that $\mathcal{L}$ is countable and that T is a complete $\mathcal{L}$-theory. Say $\mathfrak{A} \models T$ and $B \subseteq A$ is countable. If $|S_n^{\mathfrak{A}}(B)| < 2^{\aleph_0}$ then the isolated types in $S_n^{\mathfrak{A}}(B)$ are dense.*

Note that in particular, this means that ω-stable theories have prime models.

proof: We proceed by contradiction. Let ϕ be a formula such that $[\phi]$ contains no isolated types. Since ϕ does not isolate a type, there is some ψ such that $\mathfrak{A} \not\models \forall\bar{x}(\phi(\bar{x}) \to \psi(\bar{x}))$ and $\mathfrak{A} \not\models (\phi(\bar{x}) \to \neg\psi(\bar{x}))$. Therefore both $[\phi \wedge \psi]$ and $[\phi \wedge \neg\psi]$ are non-empty. Now, $[\phi \wedge \psi], [\phi \wedge \neg\psi] \subseteq [\phi]$ so, as $[\phi]$ does not contain an isolated type, neither do $[\phi \wedge \psi]$ and $[\phi \wedge \neg\psi]$.

This allows us to build a binary tree of formulas $(\phi_\sigma : \sigma \in 2^\omega)$ such that:

1. each $[\phi_\sigma]$ is non-empty but contains no isolated types
2. if $\sigma \subset \tau$, then $\phi_\tau \vDash \phi_\sigma$
3. $\phi_{\sigma,i} \vDash \phi_{\sigma,1-i}$

Let $\phi_\varnothing = \phi$. Say we have ϕ_σ such that $[\phi_\sigma]$ is non-empty but contains no isolated types. As above, we can find ψ such that $[\phi_\sigma \wedge \psi]$ and $[\phi_\sigma \wedge \neg\psi]$ are both non-empty and contain no isolated types. Let $\phi_{\sigma,0}$ be $\phi \wedge \psi$ and $\phi_{\sigma,1}$ be $\phi \wedge \neg\psi$.

Now, say $f : \omega \to 2$. Then because $[\phi_{f|0}] \supseteq [\phi_{f|1}] \supseteq [\phi_{f|2}] \supseteq \ldots$ and $S_n^{\mathfrak{A}}(B)$ is compact, there is some

$$p_f \in \bigcap_{n=0}^{\infty} [\phi_{f|n}]$$

Let $f, g : \omega \to 2$ be such that $f \neq g$. We will show $p_f \neq p_g$. Since $f \neq g$ there is m such that $f|m = g|m$ but $f(m) \neq g(m)$. By construction $\phi_{f|m+1} \vDash \neg\phi_{g|m+1}$, and so $[\phi_{f|m+1}] \cap [\phi_{g|m+1}] = \varnothing$. Therefore $p_f \neq p_g$, and so the map $f \mapsto p_f$ is one-to-one from 2^ω into $S_n^{\mathfrak{A}}(B)$. This means $|S_n^{\mathfrak{A}}(B)| \geq 2^{\aleph_0}$, contradicting our assumption that $|S_n^{\mathfrak{A}}(B)| < 2^{\aleph_0}$. Thus, the isolated types in $S_n^{\mathfrak{A}}(B)$ must be dense. □

In fact, more is true of ω-stable theories. Given the following notion of relative primality

Definition 2.14 *Given an $\mathcal{L}$-theory T, an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \vDash T$ and $X \subseteq A$, we say $\mathfrak{A}$ is* **prime over** *X if whenever $\mathfrak{B} \vDash T$ and $f : X \to B$ is a partial elementary map, there is an elementary $f^* : \mathfrak{A} \to \mathfrak{B}$ extending f.*

we have the following theorem.

Theorem 2.15 *Say T is a complete, ω-stable theory with infinite models, $\mathfrak{A} \vDash T$ and $X \subseteq A$. Then there is $\mathfrak{A}_0 \vDash T$ such that $\mathfrak{A}_0$ is prime over X. Moreover, $\mathfrak{A}_0$ can be chosen such that every element of A_0 (the underlying set of $\mathfrak{A}_0$) realizes an isolated type over X.*

Before proving theorem 2.15, we need a pair of lemmas.

Lemma 2.16 *Let T be a complete theory with infinite models, $\mathfrak{A}$ a model of T. Suppose that $(\bar{a}, \bar{b}) \in A^{m+n}$ realizes an isolated type in $S_{m+n}(T)$. Then $\bar{a}$ realizes an isolated type in $S_m(T)$. In fact, if $B \subseteq A$ and $(\bar{a}, \bar{b}) \in A^{m+n}$ realizes an isolated type in $S_{m+n}^{\mathfrak{A}}(B)$, then $tp^{\mathfrak{A}}(\bar{a}/B)$ is isolated.*

proof: Let $\phi(\bar{x},\bar{y})$ isolate $tp^{\mathfrak{A}}(\bar{a},\bar{b}/B)$. We claim that $\exists\bar{y}\phi(\bar{x},\bar{y})$ isolates $tp^{\mathfrak{A}}(\bar{a}/B)$. Let $\psi(\bar{x})$ be an $\mathcal{L}_B$-formula such that $\mathfrak{A} \vDash \psi(\bar{a})$. We must show that $Th(\mathfrak{A}_B) \vDash \forall\bar{x}(\exists\bar{y}(\phi(\bar{x},\bar{y})) \to \psi(\bar{x}))$.

Toward a contradiction, suppose otherwise. Then there is $\bar{c} \in A^m$ such that $\mathfrak{A} \vDash \exists\bar{y}(\phi(\bar{c},\bar{y})) \to \neg\psi(\bar{c})$. Let $\bar{d} \in A^m$ such that $\mathfrak{A} \vDash \phi(\bar{c},\bar{d}) \wedge \neg\psi(\bar{d})$. But now, because $\phi(\bar{x},\bar{y})$ isolates $tp^{\mathfrak{A}}(\bar{a},\bar{b}/B)$ we have $Th(\mathfrak{A}_B) \vDash \forall\bar{x}\forall\bar{y}(\phi(\bar{x},\bar{y}) \to \psi(\bar{y}))$ and so $\mathfrak{A} \vDash \phi(\bar{c},\bar{d}) \wedge \neg\psi(\bar{d})$ is a contradiction. Thus, we have $Th(\mathfrak{A}_B) \vDash \forall\bar{x}(\exists\bar{y}(\phi(\bar{x},\bar{y})) \to \psi(\bar{x}))$, as desired. □

Lemma 2.17 *Let $C \subseteq D \subseteq A$ for some $\mathfrak{A} \vDash T$, T a complete theory with infinite models. Suppose that every $\bar{d} \in D^m$ realizes an isolated type in $S^{\mathfrak{A}}_m(C)$. Suppose that $\bar{a} \in A^n$ realizes an isolated type in $S^{\mathfrak{A}}_n(D)$. Then, $\bar{a}$ realizes an isolated type in $S^{\mathfrak{A}}_n(C)$.*

proof: Let $\phi(\bar{x},\bar{y})$ be an $\mathcal{L}$-formula and $\bar{d} \in D^m$ such that $\phi(\bar{x},\bar{d})$ isolates $tp^{\mathfrak{A}}(\bar{a}/D)$. Let $\theta(\bar{y})$ be an $\mathcal{L}_C$-formula isolating $tp^{\mathfrak{A}}(\bar{d}/C)$. We first claim that $\phi(\bar{x},\bar{y}) \wedge \theta(\bar{y})$ isolates $tp^{\mathfrak{A}}(\bar{a},\bar{d}/C)$.

Let $\psi(\bar{a},\bar{d})$ be such that $\mathfrak{A} \vDash \psi(\bar{a},\bar{d})$. Because $\phi(\bar{x},\bar{d})$ isolates $tp^{\mathfrak{A}}(\bar{a}/D)$, we have $Th(\mathfrak{A}_C) \vDash \forall\bar{x}(\phi(\bar{x},\bar{d}) \to \psi(\bar{x},\bar{d}))$. Thus, because $\theta(\bar{y})$ isolates $tp^{\mathfrak{A}}(\bar{d}/C)$, we have $Th(\mathfrak{A}_C) \vDash \forall\bar{x}\forall\bar{y}(\theta(\bar{y}) \to (\phi(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y})))$ and $Th(\mathfrak{A}_C) \vDash \forall\bar{x}\forall\bar{y}((\theta(\bar{y}) \wedge \phi(\bar{x},\bar{y})) \to \psi(\bar{x},\bar{y}))$, as we wished to show.

Now, by the last lemma, because $tp^{\mathfrak{A}}(\bar{a},\bar{d})$ is isolated, so is $tp^{\mathfrak{A}}(\bar{a}/C)$. □

We are now ready to prove theorem 2.15.

proof of 2.15: We will find an ordinal δ and build a sequence of sets $(X_\alpha : \alpha \le \delta)$ where $X_\alpha \subseteq A$ as follows:

- $X_0 = X$
- if α is a limit ordinal, then $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$
- if no element of $A \setminus X_\alpha$ realizes an isolated type over X_α, we stop and let $\delta = \alpha$; otherwise, we choose $a_\alpha \in A \setminus X_\alpha$ realizing an isolated type over X_α and let $X_{\alpha+1} = X_\alpha \cup \{a_\alpha\}$.

We want to let $\mathfrak{A}_0$ be the substructure of $\mathfrak{A}$ with universe X_δ. Before we can do this, we must verify that there is a substructure of $\mathfrak{A}$ with universe X_δ. To do this, it suffices to verify that X_δ is closed under all function symbols of $\mathcal{L}$.

Well, let $f \in \mathcal{L}$ be an n-ary function symbol and let $a_1, \dots, a_n \in X_\delta$. Let $a = f^{\mathfrak{A}}(a_1, \dots, a_n)$. We must show that $a \in X_\delta$. To do this, it suffices to show that a realizes an isolated type over X_β where $\beta \leq \delta$ is least such that $a_1, \dots, a_n \in X_\beta$. If $a \in X_\beta$ then we have $a \in X_\delta$ as desired. So assume $a \notin X_\beta$. Then a is the unique realization in A of the $\mathcal{L}_{X_\beta}$-formula $\phi(x, a_1, \dots, a_n)$ given by $x = f(a_1, \dots, a_n)$. Clearly, ϕ isolates $tp^{\mathfrak{A}}(a/X_\beta)$ so we may, without loss of generality, assume that we chose a as a_β (where a_β is as in the third bullet point above), and thus $a \in X_{\beta+1}$ so $a \in X_\delta$, as desired.

Thus there is a substructure of $\mathfrak{A}$ with universe X_δ and we let $\mathfrak{A}_0$ be this substructure.

Claim 1: $\mathfrak{A}_0 \prec \mathfrak{A}$.
proof of claim 1: We apply the Tarski-Vaught test 1.22. Let $\mathfrak{A} \models \phi(\bar{x}, \bar{a})$ where $\bar{a} \in X_\delta$. By theorem 2.13, the isolated types in $S_n^{\mathfrak{A}}(X_\delta)$ are dense. Therefore, there is $b \in A$ such that $\mathfrak{A} \models \phi(b, \bar{a})$ and $tp^{\mathfrak{A}}(b/X_\delta)$ is isolated. By the choice of δ, $b \in X_\delta$, so by the Tarski-Vaught test 1.22, $\mathfrak{A}_0 \prec \mathfrak{A}$. ■

Claim 2: $\mathfrak{A}_0$ is a prime over X.
proof of claim 2: Let $\mathfrak{B} \models T$ and let $f : X \to \mathfrak{B}$ be partial elementary. We will show by induction that there exists $f = f_0 \subset f_1 \subset f_2 \subset \dots \subset f_\delta$, where $f_\alpha : X_\alpha \to \mathfrak{B}$ is elementary.

Let $f_0 = f$. If α is a limit ordinal, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$.

Say $f_\alpha : X_\alpha \to \mathfrak{B}$ is partial elementary. Let $\bar{a} \in X_\delta$ and let $\phi(x, \bar{a})$ isolate $tp^{\mathfrak{A}_0}(a_\alpha/X_\alpha)$. Let $\psi(x) \in tp^{\mathfrak{A}_0}(a_\alpha/X_\alpha)$. Then we have $(\mathfrak{A}_0)_{X_\delta} \models \forall x(\phi(x, \bar{a}) \to \psi(x))$. Thus, since f_α is partial elementary, we have $(\mathfrak{A}_0)_{X_\delta} \models \forall x(\phi(x, f_\alpha(\bar{a})) \to \psi(x))$ and thus $\phi(x, f_\alpha(\bar{a}))$ isolates $f_\alpha(tp^{\mathfrak{A}_0}(a_\alpha/X_\alpha)) = \{\chi(x, f_\alpha(\bar{a})) : \chi(x, \bar{a}) \in tp^{\mathfrak{A}}(a_\alpha/X_\alpha)\}$. Further, because f_α is partial elementary, there is $b \in B$ with $\mathfrak{B} \models \phi(b, f_\alpha(\bar{a}))$. Thus $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, b)\}$ is elementary with domain $X_{\alpha+1}$.

Inductively, then we obtain $f_\delta : \mathfrak{A}_0 \to \mathfrak{A}$ elementary and extending f and thus $\mathfrak{A}_0$ is prime over X. □

2.2 Saturation and Homogeneity

In the last section, we considered stable theories which realize "few" types. At the other end of the spectrum are models which realize "many" types. These models are know as *saturated*.

Definition 2.18 *Let κ be an infinite cardinal. We say that $\mathfrak{A} \models T$ is κ-***saturated** *if, for all $X \subseteq A$, if $|X| < \kappa$ and $p \in S_n^{\mathfrak{A}}(X)$, the p is realized in $\mathfrak{A}$. We say that $\mathfrak{A}$ is* **saturated** *if $\mathfrak{A}$ is $|A|$-saturated.*

In this work, we will not have a great deal to say about saturated models, but they are more than worth mentioning because of their wide importance in other areas. A weakening of saturation which will be more germane to us is *homogeneity.*

Definition 2.19 *Let κ be an infinite cardinal. We say that $\mathfrak{A} \models T$ is κ-***homogeneous** *if whenever $X \subseteq A$ and $|X| < \kappa$, $f : X \to A$ is partial elementary and $a \in A$ there is $f^* \supseteq f$ such that $f^* : X \cup \{a\} \to A$ is partial elementary.*

We say $\mathfrak{A}$ is **homogeneous** *if it is $|\mathfrak{A}|$-homogeneous.*

Lemma 2.20 *If $\mathfrak{A}$ is κ-saturated, $\mathfrak{A}$ is κ-homogeneous.*

proof: Let $X \subseteq A$, $|X| < \kappa$ and say $f : X \to A$ is partial elementary. Let $b \in A \setminus X$ and let $p(x) = \{\phi(x, f(\bar{a})) : \bar{a} \in X$ and $\mathfrak{A} \models \phi(b, \bar{a})\}$. Now, say $\psi_1, \ldots, \psi_n \in p$, then we have $\mathfrak{A} \models \exists x \bigwedge_i \psi_i(x, \bar{a})$ and thus, by the fact that f is partial elementary, $\mathfrak{A} \models \exists x \bigwedge \psi_i(x, f(\bar{a}))$. Therefore, p is finitely satisfiable, hence satisfiable by Compactness 1.29 and therefore, because $\mathfrak{A}$ is κ-saturated, there is $c \in A$ such that $\mathfrak{A} \models p(c)$. Thus $f \cup \{(b, c)\}$ is elementary, so $\mathfrak{A}$ is κ-homogeneous, as desired. □

Combining homogeneity with types gives us a test for whether or not two countable structures are isomorphic.

Theorem 2.21 *Let T be a complete theory in a countable language. Suppose that $\mathfrak{A}$, $\mathfrak{B}$ are countable homogeneous models of T and $\mathfrak{A}, \mathfrak{B}$ realize the same types in $S_n(T)$ for $n \geq 1$. Then $\mathfrak{A} \cong \mathfrak{B}$.*

proof: We build an isomorphism $f : \mathfrak{A} \to \mathfrak{B}$ by building a sequence of partial elementary maps with finite domain $f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots$ and letting f be the union of the f_i. At the various stages of the construction, we will ensure that the final f will have domain A and range B, so that it will be an isomorphism as desired. Let $a_0, a_1, \dots,$ be an enumeration of A and $b_0, b_1, \dots$ an enumeration of B.

stage 0: Let $f_0 = \emptyset$. Because T is complete, f_0 is partial elementary.

Inductively, assume that f_s is partial elementary. Let $\bar{a}$ be the domain of f_s and let $\bar{b} = f_s(\bar{a})$.

stage $s + 1 = 2i + 1$: Let $p = tp^{\mathfrak{A}}(\bar{a}, a_i)$. Since $\mathfrak{A}$ and $\mathfrak{B}$ realize the same types, there are $\bar{c}, d \in B$ such that $tp^{\mathfrak{B}}(\bar{c}, d) = p$. By the choice of $\bar{c}$, $tp^{\mathfrak{A}}(\bar{a}) = tp^{\mathfrak{B}}(\bar{c})$ and since f_s is partial elementary $tp^{\mathfrak{A}}(\bar{a}) = tp^{\mathfrak{B}}(\bar{b})$. Thus we have $tp^{\mathfrak{A}}(\bar{c}) = tp^{\mathfrak{B}}(\bar{b})$. Since $\mathfrak{B}$ is homogeneous, there is $e \in B$ such that $tp^{\mathfrak{B}}(\bar{b}, e) = tp^{\mathfrak{B}}(\bar{c}, d) = p$. Thus, $f_{s+1} = f_s \cup \{(a_i, e)\}$ is partial elementary with a_i in the domain.

stage $s + 1 = 2i + 2$: As in the previous case, we can find $\bar{c}, d \in A$ such that $tp^{\mathfrak{A}}(\bar{c}, d) = tp^{\mathfrak{B}}(\bar{b}, b_i)$. Since $\mathfrak{A}$ is homogeneous, there is $e \in A$ such that $tp^{\mathfrak{A}}(\bar{c}, d) = tp^{\mathfrak{A}}(\bar{a}, e)$. Letting $f_{s+1} = f_s \cup \{(e, b_i)\}$ we have f_{s+1} partial elementary with b_i in the range.

We let $f = \bigcup_i f_i$. Then f is elementary since each f_i is and $f_i \subseteq f_{i+1}$ for each i. At stage $2i + 1$ of our construction, we ensured that $a_i \in dom(f)$ and at stage $2i + 2$ we ensured that $b_i \in ran(f)$. Thus f is surjective from A onto B, so f is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, as desired. $\square$

The argument given in the last proof is an example of a "back-and-forth" argument. At stages $2i + 2$ we went "forth" to make sure that b_i was in the range of f and at stages $2i + 1$ we went "back" to make sure that a_i was in the domain of f. While we will not have cause to use many back-and-forth arguments here, they are widely applicable in logic as a whole.

Chapter 3

Morley's Categoricity Theorem

The material in this chapter comes from Marker [2].

For this chapter fix a countable first-order language $\mathcal{L}$.

Notation: If $\mathfrak{A}$ is an $\mathcal{L}$-structure and $\phi(\bar{x})$ is an $\mathcal{L}$-formula, then we let $\phi(\mathfrak{A}) = \{\bar{a} \in A : \mathfrak{A} \models \phi(\bar{a})\}$.

Our goal in this chapter is to prove Morley's Categoricity Theorem:

> If T is a complete theory in a countable language L and T has infinite models, then if T is κ-categorical for some uncountable cardinal κ, T is λ-categorical for every uncountable cardinal λ.

We will do this by proving a characterization of uncountably categorical theories due to Baldwin and Lachlan. First, however, we will need a thoerem of Vaught and a theorem of Ramsey.

3.1 Vaught's Two-Cardinal Theorem

Definition 3.1 *Let $\kappa > \lambda \geq \aleph_0$. We say that a complete $\mathcal{L}$-theory T has a* **(κ, λ)-model** *if there is $\mathfrak{A} \models T$ and $\phi(\bar{x})$ an $\mathcal{L}$-formula such that $|A| = \kappa$ and $|\phi(\mathfrak{A})| = \lambda$.*

A theory that has (κ, λ)-model cannot be κ-categorical. The next lemma shows that a theory with a model of size κ has a model of size κ in which every set definable by an $\mathcal{L}$-formula has size κ, and this model is then not isomorphic to the (κ, λ)-model of T, making T not κ-categorical.

Lemma 3.2 *Let T be a complete $\mathcal{L}$-theory with infinite models. If there is $\mathfrak{A} \vDash T$ with $|A| = \kappa$, then there is $\mathfrak{B} \vDash T$ such that $|B| = \kappa$ and every infinite subset of B definable by an $\mathcal{L}$-formula has size κ.*

proof: Since our language $\mathcal{L}$ is countable, there are only countably many $\mathcal{L}$-formulas (with no parameters), hence at most countably many $\mathcal{L}$-formulas $\phi(\bar{x})$ such that T models "there are infinitely many realizations of ϕ" (i.e. T models "there are n distinct realizations of ϕ" for each n). Let $\{\phi_\beta : \beta \in \rho\}$ be the set of these formulas where ρ is some cardinal $\rho \leq \aleph_0$.

Add to $\mathcal{L}$ a tuple of new constant symbols $\bar{c}_{\alpha,\beta}$ for each $(\alpha, \beta) \in \kappa \times \rho$, where the length of $\bar{c}_{\alpha,\beta}$ is equal to the number of free variables in ϕ_β. We denote this expanded language by $\mathcal{L}^*$. Now, expand T to an $\mathcal{L}^*$-theory T^* by adding to T the $\mathcal{L}^*$-sentences $\{\phi_\beta(\bar{c}_{\alpha,\beta}) : \alpha \in \kappa, \beta \in \rho\}$ and $\{\bar{c}_{\alpha,\beta} \neq \bar{c}_{\alpha',\beta'} : \alpha, \alpha' \in \kappa, \beta, \beta' \in \rho, (\alpha, \beta) \neq (\alpha', \beta')\}$.

Let T_0^* be any finite subset of T^*. Then T_0^* contains only finitely many of the sentences from $\{\phi_\beta(\bar{c}_{\alpha,\beta}) : \alpha \in \kappa \text{ and } \beta \in \rho\}, \{\bar{c}_{\alpha,\beta} \neq \bar{c}_{\alpha',\beta'} : \alpha, \alpha' \in \kappa, \beta, \beta' \in \rho, (\alpha, \beta) \neq (\alpha', \beta')\}$. Thus T_0^* asserts only sentences of T and the existence of some finite number of distinct realizations for each ϕ_β appearing in T_0^*. Since T asserts that there are infinitely many distinct realizations for each ϕ_β and $\mathfrak{A} \vDash T$, we thus have $\mathfrak{A} \vDash T_0^*$ (when we expand $\mathfrak{A}$ to an $\mathcal{L}^*$ structure by interpreting the new constant symbols appearing in T_0^* to be distinct realizations in A of the ϕ_β appearing in T_0^*). Thus, T^* is finitely satisfiable, hence by the Compactness theorem 1.29, satisfiable.

Let $\mathfrak{B}^* \vDash T^*$. Then if $\phi(\bar{x})$ is an $\mathcal{L}$-formula such that T asserts the existence of infinitely many realizations of ϕ (i.e. if ϕ is some ϕ_β) then $\mathfrak{B}^*$ contains κ-many distinct realizations of ϕ: one for the interpretation of each $\bar{c}_{\alpha,\beta}$ such that $\alpha \in \kappa$. By the Downward Löwenheim-Skolem theorem we may assume without loss of generality that $|B^*| = \kappa$ (if $|B^*| > \kappa$ then we simply take a set of size κ in B^* containing all realizations of all ϕ_β's and use the Downward Löwenheim-Skolem theorem 1.26 to form a model of size κ containing this set). Letting $\mathfrak{B}$ be the reduction of $\mathfrak{B}^*$ to an $\mathcal{L}$-structure (i.e., leave the universe of $\mathfrak{B}^*$ untouched but deleting from its interpretation $\mathcal{I}$ any constant symbol not part of $\mathcal{L}$) we get a $\mathfrak{B} \vDash T$ as desired. □

Vaught's Two-Cardinal theorem will tell us that a theory with a (κ,λ)-model for some $\kappa > \lambda \geq \aleph_0$ has an $(\aleph_1,\aleph_0)$-model. Before proving this, however, we need some additional theory.

Definition 3.3 *If T is an $\mathcal{L}$-theory with models $\mathfrak{A},\mathfrak{B}$, then we say that $(\mathfrak{B},\mathfrak{A})$ is a* **Vaughtian pair** *of models of T if $\mathfrak{A} \preceq \mathfrak{B}$, $A \neq B$ and there is an $\mathcal{L}_A$-formula ϕ such that $\phi(\mathfrak{A})$ is infinite and $\phi(\mathfrak{A}) = \phi(\mathfrak{B})$.*

Lemma 3.4 *If T has a (κ,λ)-model where $\kappa > \lambda \geq \aleph_0$, then there is $(\mathfrak{B},\mathfrak{A})$ a Vaughtian pair of models of T.*

proof: Let $\mathfrak{B}$ be a (κ,λ)-model of T. Let $X = \phi(\mathfrak{B})$ be such that $|X| = \lambda$. By the Downward Löwenheim-Skolem theorem 1.26, there is $\mathfrak{A} \models T$ such that $X \subseteq A$ and $|A| = \lambda$. Since $X \subseteq A$, $(\mathfrak{B},\mathfrak{A})$ is a Vaughtian pair of models of T. $\square$

Lemma 3.5 *If $(\mathfrak{B},\mathfrak{A})$ is a Vaughtian pair of models of T, then there is a Vaughtian pair $(\mathfrak{B}_0,\mathfrak{A}_0)$ of models of T such that $\mathfrak{B}_0$ is countable.*

proof: We can prove this lemma by using the Löwenheim-Skolem theorems, but first we must expand our language. Let $\mathcal{L}^* = \mathcal{L} \cup \{U\}$ where U is a unary relation symbol. For given $\mathcal{L}$-structures $\mathfrak{A} \subseteq \mathfrak{B}$, we consider the pair $(\mathfrak{B},\mathfrak{A})$ as an $\mathcal{L}^*$-structure by taking the underlying set of $(\mathfrak{B},\mathfrak{A})$ to be B, interpreting the nonlogical symbols of $\mathcal{L}$ as they are interpreted in $\mathfrak{B}$ (hence also as in $\mathfrak{A}$ since $\mathfrak{A} \subseteq \mathfrak{B}$) and interpreting U as A, i.e. $(\mathfrak{B},\mathfrak{A}) \models U(a) \iff a \in A$.

For each $\mathcal{L}$-formula $\phi(\bar{x})$ we inductively define $\phi^U(\bar{x})$, the restriction of ϕ to U, as follows:

1. if ϕ is atomic, ϕ^U is $U(x_1) \wedge U(x_2) \wedge \ldots \wedge U(x_n) \wedge \phi(\bar{x})$
2. if ϕ is $\neg\psi$, ϕ^U is $\neg\psi^U$
3. if ϕ is $\psi \wedge \theta$, then ϕ^U is $\psi^U \wedge \theta^U$
4. if ϕ is $\psi \vee \theta$, then ϕ^U is $\psi^U \vee \theta^U$
5. if ϕ is $\exists x\psi$, then ϕ^U is $\exists x(U(x) \wedge \psi^U)$
6. if ϕ is $\forall x\psi$, then ϕ^U is $\forall x(U(x) \to \psi^U)$

Induction over the complexity of formulas (as used in the proof of the Tarski-Vaught Test, theorem 1.22) shows that for $\bar{a} \in A$, $\mathfrak{A} \models \phi(\bar{a})$ if and only if $(\mathfrak{B}, \mathfrak{A}) \models \phi^U(\bar{a})$.

Now, let ϕ be an $\mathcal{L}_A$-formula such that $\phi(\mathfrak{A})$ is infinite and $\phi(\mathfrak{A}) = \phi(\mathfrak{B})$. Let $\{a_1, \ldots, a_n\}$ be the parameters from A occurring in ϕ. Then by the Downward Löwenheim-Skolem, we can find a countable $\mathcal{L}^*$-structure $\mathfrak{C}^* \prec (\mathfrak{B}, \mathfrak{A})$ such that $\{a_1, \ldots, a_n\} \subseteq C^*$.

We let $\mathfrak{B}_0$ be the $\mathcal{L}$-structure which has the same underlying set as $\mathfrak{C}^*$. The structure $\mathfrak{B}_0$ is thus countable because $\mathfrak{C}^*$ is. We want to let $\mathfrak{A}_0$ be the $\mathcal{L}$-structure with underlying set $\{c \in C^* : \mathfrak{C}^* \models U(c)\}$. Before we can do this, however, we must check that the set $\{c \in C^* : \mathfrak{C}^* \models U(c)\}$ is closed under the functions symbols of $\mathcal{L}$. Well, let $f \in \mathcal{L}$. Since $\mathfrak{A} \leq \mathfrak{B}$, we have that for any $\bar{a} \in A$, $f(\bar{a}) \in A$. This means that for any $\bar{x}$ in the universe of the $\mathcal{L}^*$-structure $(\mathfrak{B}, \mathfrak{A})$, we have $(\mathfrak{B}, \mathfrak{A}) \models U(\bar{x}) \to U(f(\bar{x}))$. Thus, as $\mathfrak{C}^* \leq (\mathfrak{B}, \mathfrak{A})$, we have $\mathfrak{C}^* \models \forall \bar{x}(U(\bar{x}) \to U(f(\bar{x})))$. Since f was an arbitrary function symbol of $\mathcal{L}$, this gives us that $\{c \in C^* : \mathfrak{C}^* \models U(c)\}$ is indeed closed under every function symbol in $\mathcal{L}$, so we may indeed let $\mathfrak{A}_0$ be the $\mathcal{L}$-structure with underlying set $\{c \in C^* : \mathfrak{C}^* \models U(c)\}$. This means that the underlying set of $\mathfrak{A}_0$ is contained in the underlying set of $\mathfrak{B}_0$, so we have $\mathfrak{A}_0 \subseteq \mathfrak{B}_0$.

Now, because $(\mathfrak{B}, \mathfrak{A})$ is a Vaughtian pair, $\mathfrak{A} \prec \mathfrak{B}$, so for any formula $\psi(\bar{x})$ we have

$$(\mathfrak{B}, \mathfrak{A}) \models \forall \bar{x} \left(\left(\bigwedge_{i=1}^{n} U(x_i) \wedge \psi(\bar{x}) \right) \to \psi^U(\bar{x}) \right)$$

Since $\mathfrak{C}^* \prec (\mathfrak{B}, \mathfrak{A})$, these sentences are also true in $\mathfrak{C}^*$, which implies that $\mathfrak{A}_0 \prec \mathfrak{B}_0$.

Finally, note that for each $k \in \mathbb{N}$ the sentence

$$\exists \bar{x}_1 \exists \bar{x}_2 \ldots \exists \bar{x}_k \left(\bigwedge_{i<j} \bar{x}_i \neq \bar{x}_j \wedge \bigwedge_{i=1}^{k} \phi(\bar{x}_i) \right)$$

holds in $(\mathfrak{B}, \mathfrak{A})$ as do the sentences $\exists x \neg U(x)$ and $\forall \bar{x}(\phi(\bar{x}) \to \bigwedge_i U(x_i))$. Thus, these sentences are also true in $\mathfrak{C}^*$, which shows (along with what we have already shown) that the formula ϕ witnesses that $(\mathfrak{B}_0, \mathfrak{A}_0)$ is a Vaughtian pair. Hence, $(\mathfrak{B}_0, \mathfrak{A}_0)$ is a Vaughtian pair of models for T with $\mathfrak{B}_0$ countable, as desired. $\square$

Having found a countable Vaughtian pair, we now show how that pair can be expanded to a countable pair with several desirable properties.

Lemma 3.6 *Suppose that $\mathfrak{A}_0 \prec \mathfrak{B}_0$ are countable models of T such that $(\mathfrak{B}_0, \mathfrak{A}_0)$ is a Vaughtian pair. We can find $(\mathfrak{B}, \mathfrak{A}) \succeq (\mathfrak{B}_0, \mathfrak{A}_0)$ such that $(\mathfrak{B}, \mathfrak{A})$ is a Vaughtian pair and $\mathfrak{B}$, $\mathfrak{A}$ are countable, homogeneous and realize the same types in $S_n(T)$. By theorem 2.21, this will mean that $\mathfrak{A} \cong \mathfrak{B}$.*

proof: In this proof we work in the language $\mathcal{L}^* = \mathcal{L} \cup \{U\}$ of the previous proof. Expressions such as ϕ^U have the same meaning in this proof as in the previous proof.

Claim 1: If $\bar{a} \in A_0$ and $p \in S_n(\bar{a})$ is realized in $\mathfrak{B}_0$, then there is a countable $(\mathfrak{B}', \mathfrak{A}') \succeq (\mathfrak{B}_0, \mathfrak{A}_0)$ such that p is realized in $\mathfrak{A}'$.

proof of claim 1: Let $\Gamma(\bar{x}) = \{\phi^U(\bar{x}, \bar{a}) : \phi(\bar{x}, \bar{a}) \in p\} \cup \{\psi(\bar{b}) : \psi$ an $\mathcal{L}^*_{(B_0,A_0)}$-formula and $(\mathfrak{B}_0, \mathfrak{A}_0) \vDash \psi(\bar{b})\}$. Let $\phi_1, \ldots, \phi_m \in p$. Then $\mathfrak{B}_0 \vDash \exists\bar{x} \bigwedge_i \phi_i(\bar{x}, \bar{a})$ and so, as $\mathfrak{B}_0 \succ \mathfrak{A}_0$, $\mathfrak{A}_0 \vDash \exists\bar{x} \bigwedge_i \phi_i(\bar{x}, \bar{a})$. This gives that $(\mathfrak{B}_0, \mathfrak{A}_0) \vDash \exists\bar{x} \bigwedge_i \phi_i^U(\bar{x}, \bar{a})$. Now, if $\Delta \subseteq \Gamma$ is finite, then $\Delta = \{\phi_1^U, \ldots, \phi_m^U\} \cup \Sigma$ where $\phi_1, \ldots, \phi_m \in p$ and $\Sigma \subseteq \{\psi(\bar{b}) : \psi$ an $\mathcal{L}^*_{(B_0,A_0)}$-formula and $(\mathfrak{B}_0, \mathfrak{A}_0) \vDash \psi(\bar{b})\}$. Clearly, $(\mathfrak{B}_0, \mathfrak{A}_0) \vDash \Sigma$ and the forgoing gives that

$$(\mathfrak{B}_0, \mathfrak{A}_0) \vDash \exists\bar{x} \bigwedge_i \phi_i^U(\bar{x}, \bar{a}).$$

Hence$(\mathfrak{B}_0, \mathfrak{A}_0) \vDash \Delta$, so Γ is finitely satisfiable and hence satisfiable by Compactness 1.29.

Therefore, we can find a pair of $\mathcal{L}$-structures $(\mathfrak{B}', \mathfrak{A}')$ such that $(\mathfrak{B}', \mathfrak{A}') \vDash \Gamma$. By the Downward Löwenheim-Skolem theorem 1.26, we can find a countable elementary submodel of $(\mathfrak{B}', \mathfrak{A}')$ containing a realization of Γ, so, replacing $(\mathfrak{B}', \mathfrak{A}')$ by this model if necessary, we assume without loss of generality that $(\mathfrak{B}', \mathfrak{A}')$ is countable. Since $(\mathfrak{B}', \mathfrak{A}') \vDash \Gamma$, $(\mathfrak{B}', \mathfrak{A}') \vDash \{\psi(\bar{b}) : \psi$ an $\mathcal{L}^*_{(B_0,A_0)}$-formula and $(\mathfrak{B}_0, \mathfrak{A}_0) \vDash \psi(\bar{b})\}$, so $(\mathfrak{B}', \mathfrak{A}') \succeq (\mathfrak{B}_0, \mathfrak{A}_0)$ and $(\mathfrak{B}', \mathfrak{A}') \vDash \{\phi^U(\bar{x}, \bar{a}) : \phi(\bar{x}, \bar{a}) \in p\}$, so p is realized in $\mathfrak{A}'$. Thus, $(\mathfrak{B}', \mathfrak{A}')$ is a model as claimed to exist. ∎

Iterating claim 1, we build $(\mathfrak{B}^*, \mathfrak{A}^*) \succeq (\mathfrak{B}_0, \mathfrak{A}_0)$ countable and such that if $\bar{a} \in A_0$ and $p \in S_n(\bar{a})$ is realized by $\mathfrak{B}_0$, then p is realized in $\mathfrak{A}^*$.

Claim 2: If $\bar{b} \in B_0$ and $p \in S_n(\bar{b})$, then there is a countable $(\mathfrak{B}', \mathfrak{A}') \succeq (\mathfrak{B}_0, \mathfrak{A}_0)$ such that p is realized in $\mathfrak{B}'$.

proof of claim 2: Let $\Gamma(\bar{x}) = p \cup \{\psi(\bar{d}) : \psi$ an $\mathcal{L}^*_{(B_0,A_0)}$-formula and $(\mathfrak{B}_0, \mathfrak{A}_0) \models \psi(\bar{d})\}$. Let $\phi_1, \dots, \phi_m \in p$. Then $\mathfrak{B}_0 \models \exists \bar{x} \bigwedge_i \phi_i(\bar{x}, \bar{b})$ and therefore $(\mathfrak{B}_0, \mathfrak{A}_0) \models \exists \bar{x} \bigwedge_i \phi_i(\bar{x}, \bar{b})$. From here, we proceed just as in the proof of claim 1, and show that Γ is satisfiable and that there thus exists a countable pair of models $(\mathfrak{B}', \mathfrak{A}')$ as claimed. ■

We now build an elementary chain of countable models

$$(\mathfrak{B}_0, \mathfrak{A}_0) \preceq (\mathfrak{B}_1, \mathfrak{A}_1) \preceq \dots$$

such that

1. if $p \in S_n(T)$ is realized in $\mathfrak{B}_{3i}$, then p is realized in $\mathfrak{A}_{3i+1}$.
2. if $\bar{a}, \bar{b}, c \in A_{3i+1}$ and $tp^{\mathfrak{A}_{3i+1}}(\bar{a}) = tp^{\mathfrak{A}_{3i+1}}(\bar{b})$, then there is $d \in A_{3i+2}$ such that $tp^{\mathfrak{A}_{3i+2}}(\bar{a}, c) = tp^{\mathfrak{A}_{3i+2}}(\bar{b}, d)$.
3. if $\bar{a}, \bar{b}, c \in B_{3i+2}$ and $tp^{\mathfrak{B}_{3i+2}}(\bar{a}) = tp^{\mathfrak{B}_{3i+2}}(\bar{b})$, then there is $d \in B_{3i+3}$ such that $tp^{\mathfrak{B}_{3i+3}}(\bar{a}, c) = tp^{\mathfrak{B}_{3i+3}}(\bar{b}, d)$.

For (1), we use iterations of claim 1: as noted after claim 1, given countable $(\mathfrak{B}_k, \mathfrak{A}_k)$ we can iterate claim 1 to find $(\mathfrak{B}^*, \mathfrak{A}^*) \succ (\mathfrak{B}_k, \mathfrak{A}_k)$ countable and such that if $\bar{a} \in A_k$ and $p \in S_n(\bar{a})$ is realized by $\mathfrak{B}_k$, then p is realized in $\mathfrak{A}^*$. We let $(\mathfrak{B}_{k+1}, \mathfrak{A}_{k+1}) = (\mathfrak{B}^*, \mathfrak{A}^*)$.

We handle (2) similarly: given countable $(\mathfrak{B}_k, \mathfrak{A}_k)$ and $\bar{a}, \bar{b}, c \in A_k$ such that $tp^{\mathfrak{A}_k}(\bar{a}) = tp^{\mathfrak{A}_k}(\bar{b})$ we iterate claim 1 to find countable $(\mathfrak{B}^*, \mathfrak{A}^*)$ such that for any $c \in \mathfrak{A}_k$ we have $tp^{\mathfrak{A}_k}(\bar{a}, c)$ realized in $\mathfrak{A}^*$ by some $d \in A^*$ and then let $(\mathfrak{B}_{k+1}, \mathfrak{A}_{k+1}) = (\mathfrak{B}^*, \mathfrak{A}^*)$. (Note that to use claim 1 here we are making use of the fact that because $\mathfrak{A}_k \subseteq \mathfrak{B}_k$ we know that $tp^{\mathfrak{A}_k}(\bar{a})$ is realized in $\mathfrak{B}_k$.)

For (3), we use claim 2: given countable $(\mathfrak{B}_k, \mathfrak{A}_k)$ and $\bar{a}, \bar{b}, c \in B_k$ such that $tp^{\mathfrak{B}_k}(\bar{a}) = tp^{\mathfrak{B}_k}(\bar{b})$ we iterate claim 2 to find a countable $(\mathfrak{B}^*, \mathfrak{A}^*)$ such that for any $c \in \mathfrak{B}_k$ we have $tp^{\mathfrak{B}_k}(\bar{a}, c)$ realized in $\mathfrak{B}^*$ by some $d \in B^*$ and then let $(\mathfrak{B}_{k+1}, \mathfrak{A}_{k+1}) = (\mathfrak{B}^*, \mathfrak{A}^*)$.

Now, let $(\mathfrak{B}, \mathfrak{A}) = \bigcup_{i \in \omega} (\mathfrak{B}_i, \mathfrak{A}_i)$. Then by (1), $\mathfrak{A}$ and $\mathfrak{B}$ realize the same types and by (2) and (3) $\mathfrak{A}$ and $\mathfrak{B}$ are homogeneous, hence also isomorphic by theorem 2.21. Since $(\mathfrak{B}, \mathfrak{A}) \succeq (\mathfrak{B}_0, \mathfrak{A}_0)$ as $\mathcal{L}^*$-structures, that $(\mathfrak{B}_0, \mathfrak{A}_0)$ is Vaughtian pair gives that $(\mathfrak{B}, \mathfrak{A})$ is a Vaughtian pair. □

We are now ready to prove Vaught's Two-Cardinal Theorem.

Theorem 3.7 (Vaught's Two-Cardinal Theorem) *If T has a (κ, λ)-model where $\kappa > \lambda \geq \aleph_0$, then T has an $(\aleph_1, \aleph_0)$-model.*

proof: Say T has (κ, λ)-model. Then by lemma 3.4, T has a Vaughtian pair of models $(\mathfrak{B}, \mathfrak{A})$. By lemma 3.5, we may assume without loss of generality that $\mathfrak{B}, \mathfrak{A}$ are countable and by lemma 3.6 we may further assume without loss of generality that $\mathfrak{B}, \mathfrak{A}$ are homogeneous and realize the same types in $S_n(T)$, and hence that $\mathfrak{B} \cong \mathfrak{A}$ by theorem 2.21.

Since $(\mathfrak{B}, \mathfrak{A})$ is a Vaughtian pair, there is $\phi(\bar{x})$ an $\mathcal{L}_A$-formula with infinitely many realizations in A and none in $B \setminus A$. We build an elementary chain $(\mathfrak{B}_\alpha : \alpha < \omega_1)$, each $\mathfrak{B}_\alpha$ isomorphic to $\mathfrak{B}$ and $(\mathfrak{B}_{\alpha+1}, \mathfrak{B}_\alpha) \cong (\mathfrak{B}, \mathfrak{A})$ (as $\mathcal{L}^* = \mathcal{L} \cup \{U\}$-structures). In particular, $(\mathfrak{B}_{\alpha+1}, \mathfrak{B}_\alpha) \cong (\mathfrak{B}, \mathfrak{A})$ will mean that there are no realizations of ϕ in $B_{\alpha+1} \setminus B_\alpha$.

Let $\mathfrak{B}_0 = \mathfrak{B}$. Say we have completed the construction for all $\beta < \alpha$ and α is a limit ordinal. Then we let $\mathfrak{B}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{B}_\beta$. We use theorem 2.21 to show that $\mathfrak{B} \cong \mathfrak{B}_\alpha$. If p is a type realized by $\mathfrak{B}$, then as $\mathfrak{B} \subseteq \mathfrak{B}_\alpha$, $\mathfrak{B}_\alpha$ realizes p. Conversely, if $\mathfrak{B}_\alpha$ realizes a type p then the realization of p is contained in some $\mathfrak{B}_\beta$ for some $\beta < \alpha$. Thus $\mathfrak{B}_\beta$ realizes p and so, since $\mathfrak{B}_\beta \cong \mathfrak{B}$, we have p realized in $\mathfrak{B}$. So $\mathfrak{B}_\alpha$ and $\mathfrak{B}$ realize the same types.

Now say $X \subseteq B_\alpha$ is finite, $f : X \to B_\alpha$ is partial elementary and $b \in B_\alpha$. Then as X is finite $ran(f)$ is finite and so there is some $\beta < \alpha$ such that $A, ran(f) \subseteq B_\beta$ and $b \in B_\alpha$. Since $\mathfrak{B}_\beta \cong \mathfrak{B}$ and $\mathfrak{B}$ is homogeneous there is f^* extending f such that $f^* : X \cup \{b\} \to B_\beta$ is partial elementary. Thus $f^* : X \cup \{b\} \to B_\alpha$ is partial elementary and extends f, so $\mathfrak{B}_\alpha$ is homogeneous. Thus $\mathfrak{B}_\alpha$ is homogeneous and realizes the same types as $\mathfrak{B}$ so by theorem 2.21 $\mathfrak{B}_\alpha \cong \mathfrak{B}$.

For the case of a successor ordinal, say we have $\mathfrak{B}_\alpha \cong \mathfrak{B}$. Then because $\mathfrak{B} \cong \mathfrak{A}$ there is $\mathfrak{B}_{\alpha+1}$ an elementary extension of $\mathfrak{B}_\alpha$ such that $(\mathfrak{B}, \mathfrak{A}) \cong (\mathfrak{B}_{\alpha+1}, \mathfrak{B}_\alpha)$. Thus, $\mathfrak{B}_{\alpha+1} \succ \mathfrak{B}_\alpha$ and $\mathfrak{B}_{\alpha+1} \cong \mathfrak{B}_\alpha$.

Finally, let $\mathfrak{B}^* = \bigcup_{\alpha<\omega_1} \mathfrak{B}_\alpha$. Then as each $\mathfrak{B}_\alpha$ is countable, $|\mathfrak{B}^*| = \aleph_0 \times |\omega_1| = \aleph_0 \times \aleph_1 = \aleph_1$. If $\mathfrak{B}^* \models \phi(\bar{a})$, then by our construction $\bar{a} \in A$. Therefore $(\mathfrak{B}^*, \mathfrak{A})$ is an $(\aleph_1, \aleph_0)$-model of T. □

Corollary 3.8 *If T is $\aleph_1$-categorical, then T has no Vaughtian pairs and hence no (κ, λ)-models for $\kappa > \lambda \geq \aleph_0$.*

proof: In the proof of Vaught's theorem, the existence of a (κ, λ)-model was used only to give the existence of a Vaughtian pair of models of T. Thus, if

T has a Vaughtian pair of models, the rest of the argument in the last proof goes though and so T has an $(\aleph_1, \aleph_0)$-model. By the discussion just before lemma 3.2, having an $(\aleph_1, \aleph_0)$-model prevents T from being $\aleph_1$-categorical. Thus, if T *is* $\aleph_1$-categorical, it can have no Vaughtian pairs, and hence, by lemma 3.4, no (κ, λ)-models for $\kappa > \lambda \geq \aleph_0$. □

We now show that in the case of ω-stable theories, the existence of a (κ, λ)-model for some $\kappa > \lambda \geq \aleph_0$ is enough to deduce the existence of a $(\kappa, \aleph_0)$-model for any $\kappa > \aleph_1$. First, we need the following lemma.

Lemma 3.9 *Suppose that T is ω-stable, $\mathfrak{A} \models T$, and $|A| \geq \aleph_1$. There is a proper elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ such that if $\Gamma(\bar{v})$ is a countable type over A realized in $\mathfrak{B}$, then $\Gamma(\bar{v})$ is realized in $\mathfrak{A}$.*

proof:
Claim: There is an $\mathcal{L}_A$-formula $\phi(x)$ such that $|[\phi(x)]| \geq \aleph_1$ and for all $\mathcal{L}_A$-formulas $\psi(x)$ either $|[\phi(x) \wedge \psi(x)]| \leq \aleph_0$ or $|[\phi(x) \wedge \neg\psi(x)]| \leq \aleph_0$.

proof of claim: Toward a contradiction, assume otherwise. Then for any $\mathcal{L}_A$-formula $\phi(x)$ with $|[\phi(x)]| \geq \aleph_1$, there is an $\mathcal{L}_A$-formula $\psi(x)$ such that $[\phi(x) \wedge \psi(x)]$ and $[\phi(x) \wedge \neg\psi(x)]$ both have cardinality at least $\aleph_1$. Let $\phi_\varnothing$ be the formula $x = x$. Then for each $a \in A$ we have $\phi_\varnothing \in tp(a/A)$ and thus $|[\phi_\varnothing]| \geq |A| \geq \aleph_1$. Having the formula $\phi_\varnothing$ for which we know $|[\phi_\varnothing]| \geq \aleph_1$ we can use it as the base to build an infinite tree of formulas $(\phi_\sigma : \sigma \in 2^{<\omega})$ such that

1. $|[\phi_\sigma]| \geq \aleph_1$
2. $[\phi_{\sigma,0}] \cap [\phi_{\sigma,1}] = \varnothing$
3. if $\tau \supseteq \sigma$, then $\phi_\tau \models \phi_\sigma$

To do this, we start by letting $\phi_\varnothing$ be as above. Given ϕ_σ for $\sigma \in 2^{<\omega}$, there is by our assumption some $\mathcal{L}_A$-formula $\psi(x)$ such that $|[\phi_\sigma(x) \wedge \psi(x)]| \geq \aleph_1$ and $|[\phi_\sigma(x) \wedge \neg\psi(x)]| \geq \aleph_1$. We let $\phi_{\sigma,0}$ be $\phi_\sigma \wedge \psi$ and $\phi_{\sigma,1}$ be $\phi_\sigma \wedge \neg\psi$.

Now, let A_0 be the set of all parameters from A appearing in some ϕ_σ. Clearly, A_0 is countable. Let $f : \omega \to 2$. Then because we have $[\phi_{f|0}] \supseteq [\phi_{f|1}] \supseteq [\phi_{f|2}] \supseteq \ldots$ and $S_n^{\mathfrak{A}}(A_0)$ compact, there is $p_f \in \bigcap_n [\phi_{f|n}]$. Now, say $f, g : \omega \to 2$ and $f \neq g$. Then there is m such that $f|m = g|m$

but $f(m) \neq g(m)$. By constructions $\phi_{f|m+1} \models \neg\phi_{g|m+1}$ and therefore $p_f \neq p_g$. This means that $f \mapsto p_f$ is a one-to-one function from 2^ω into $S_n^{\mathfrak{A}}(A_0)$ and so $|S_n^{\mathfrak{A}}(A_0)| \geq \aleph_1$, contradicting the ω-stability of T. Therefore, a ϕ as claimed must exist. ■

Let $\phi(x)$ be as in the statement of the claim. Let $p = \{\psi(x) : \psi$ an $\mathcal{L}_A$-formula and $|[\phi(x) \wedge \psi(x)]| \geq \aleph_1\}$. Let $\psi_1, \ldots, \psi_m \in p$. Then $|[\phi(x) \wedge \bigvee \neg\psi_i(x)]| \leq \aleph_0$. This means that we must have $|[\phi(x) \wedge \bigwedge \psi_i(x)]| \geq \aleph_1$, and so $\bigwedge \psi_i(x) \in p$. Since $\bigwedge \psi_i(x) \in p$, there are uncountably many types in $S_n^{\mathfrak{A}}(A)$ including $\bigwedge \psi_i(x)$ as an element. Each of these types is realizable in some model of T and thus the subset $\{\psi_1, \ldots, \psi_m\}$ of p is realizable in some model of T, so p is finitely satisfiable, hence satisfiable. By our choice of $\phi(x)$, for any $\mathcal{L}_A$-formula $\psi(x)$ exactly one of $\psi(x)$ and $\neg\psi(x)$ is in p. Thus, p is a complete type over A.

By proposition 2.5 there is $\mathfrak{A}'$ an elementary extension of $\mathfrak{A}$ containing c a realization of $p(x)$. By theorem 2.15, there is $\mathfrak{B} \prec \mathfrak{A}'$ prime over $A \cup \{c\}$ such that every $\bar{a} \in B$ realizes an isolated type over $A \cup \{c\}$. Since $\mathfrak{B}$ is prime over $A \cup \{c\}$, $\mathfrak{A} \prec \mathfrak{B}$. (To ensure that $\mathfrak{B}$ is a proper extension of $\mathfrak{A}$ we may, if necessary, take $\mathfrak{A}'$ to be of cardinality larger than $\mathfrak{A}$ and replace A in the paragraph above with $A \cup \{\alpha\}$ where $\alpha \in A' \smallsetminus A$.)

Let $\Gamma(\bar{v})$ be a countable type over $\mathfrak{A}$ realized by $\bar{b} \in B$. By our choice of B, there is $\theta(\bar{v}, x)$ such that $\theta(\bar{v}, c)$ isolates $tp^{\mathfrak{B}}(\bar{b}/A \cup \{c\})$. Thus we have $\exists\bar{v}\theta(\bar{v}, x) \in p$. Further, for each $\gamma(\bar{v}) \in \Gamma(\bar{v})$ we have $\forall\bar{v}(\theta(\bar{v}, x) \to \gamma(\bar{v})) \in p$. Let $\Delta(x) = \{\exists\bar{v}\theta(\bar{v}, x)\} \cup \{\forall\bar{v}(\theta(\bar{v}, x) \to \gamma(\bar{v})) : \gamma \in \Gamma\}$. Then $\Delta(x) \subseteq p(x)$ is countable and, if c' realizes $\Delta(x)$ in $\mathfrak{A}$, then $\mathfrak{A} \models \exists\bar{v}\theta(\bar{v}, c')$, and so we would have $\bar{b}' \in A$ such that $\mathfrak{A} \models \theta(\bar{b}', c')$, in which case $\bar{b}'$ realizes Γ.

Enumerate $\Delta(x)$ as $\delta_0(x), \delta_1(x), \ldots$. By our choice of p, $|\{\alpha \in A : \phi(\alpha)\}| \geq \aleph_1$ and $|\{\alpha \in A : \phi(\alpha) \wedge \neg(\delta_0(\alpha) \wedge \ldots \wedge \delta_n(\alpha))\}| \leq \aleph_0$ for all $n \in \omega$. Thus $|\{\alpha \in A : \phi(\alpha)$ and α realizes $\Delta\}| \geq \aleph_1$. Thus there is $c' \in A$ realizing Δ and therefore, as in the last paragraph, a $\bar{b}'$ such that $\mathfrak{A} \models \theta(\bar{b}', c')$. This means that $\bar{b}'$ is a realization of Γ in $\mathfrak{A}$ and so Γ is realized in $\mathfrak{A}$, as desired. □

Having the lemma, we now proceed to the desired result.

Theorem 3.10 *Suppose that T is ω-stable and there is an $(\aleph_1, \aleph_0)$-model of T. If $\kappa > \aleph_1$, then there is a $(\kappa, \aleph_0)$-model of T.*

proof: Let $\mathfrak{A} \models T$ with $|\mathfrak{A}| \geq \aleph_1$ such that $|\phi(\mathfrak{A})| = \aleph_0$ and let $\mathfrak{B} \succ \mathfrak{A}$ be as in the last lemma. Let $\Gamma(v) = \{\phi(v)\} \cup \{v \neq a : a \in A$ and $\mathfrak{A} \models \phi(a)\}$. Then Γ

is countable and is not realized by $\mathfrak{A}$ and, by the last lemma, therefore not realized in $\mathfrak{B}$. Thus $\phi(\mathfrak{A}) = \phi(\mathfrak{B})$.

Iterating this construction, we build an elementary chain $(\mathfrak{A}_\alpha : \alpha < \kappa)$ such that $\mathfrak{A}_0 = \mathfrak{A}$ and $\mathfrak{A}_{\alpha+1} \neq \mathfrak{A}_\alpha$ but $\phi(\mathfrak{A}_\alpha) = \phi(\mathfrak{A})$. Letting $\mathfrak{B} = \bigcup_{\alpha<\kappa} \mathfrak{A}_\alpha$, we obtain $\mathfrak{B}$, a $(\kappa, \aleph_0)$-model of T. $\square$

Now that we have the theorem of Vaught that we need, we turn our attention to the theorem of Ramsey.

3.2 Ramsey's Theorem

Notation: Let X be a set and κ a cardinal (possibly finite). Then we let $[X]^\kappa$ be the collection of all subsets of X of size κ.

Definition 3.11 *For X a set, κ, λ cardinals (possibly finite), we call a function $f : [X]^\kappa \to \lambda$ a* **partition** *of $[X]^\kappa$.*

We say that $Y \subseteq X$ is **homogeneous** *for a partition f if there is $\alpha < \lambda$ such that $f(A) = \alpha$ for all $A \in [Y]^\kappa$ (i.e., if f is constant on $[Y]^\kappa$).*

Notation: If $\kappa, \eta, \mu, \lambda$ are cardinals (possibly) finite, we write $\kappa \to (\eta)^\mu_\lambda$ if whenever $|X| \geq \kappa$ and $f : [X]^\mu \to \lambda$, then there is $Y \subseteq X$ such that $|Y| \geq \eta$ and Y is homogeneous for f.

Theorem 3.12 (Ramsey's Theorem) *If $k, n < \omega$, then $\aleph_0 \to (\aleph_0)^n_k$.*

proof: By induction on n. If $n = 1$, then Ramsey's Theorem asserts that if X is an infinite set, $k \in \omega$ and $f : X \to k$ then there is some $i \in k$ such that $f^{-1}(i)$ is infinite. This is clearly true by the Pigeonhole Principle: we cannot split the infinite set X into k many piece without at least one piece being infinite.

Now suppose that the theorem is proved for $i < n$. Let $k < \omega$ and let X be infinite. Say X_0 is a countable subset of X and let $f : [X]^n \to k$ is a partition. If there is an infinite set $Y \subseteq X_0$ such that Y is homogeneous for $f|[X_0]^n$ then Y is infinite and homogeneous for f. Thus, without loss of generality, we may assume X is countable, in fact, we may (and do) assume that $X = \mathbb{N}$.

Let $f : [\mathbb{N}]^n \to k$. For $a \in \mathbb{N}$, let $f_a : [\mathbb{N} \setminus \{a\}]^{n-1} \to k$ be given by $f_a(A) = f(A \cup \{a\})$. Build the sequences $a_0 < a_1 < \ldots$ in $\mathbb{N}$ and $\mathbb{N} = X_0 \supset X_1 \supset X_2 \supset \ldots$, each X_i infinite, as follows: Given a_i, X_i, let $X_{i+1} \subset X_i \setminus \{0, 1, 2, \ldots, a_i\}$ be homogeneous for f_{a_i} (such an X_{i+1} exists by the inductive assumption) and let a_{i+1} be the least element of X_{i+1}.

Choose $c_i < k$ such that $f_{a_i}(A) = c_i$ for each $A \in [X_{i+1}]^{n-1}$. By the Pigeonhole Principle, there is $c < k$ such that $\{i : c_i = c\}$ is infinite. Let $X = \{a_i : c_i = c\}$. We show that X is homogeneous for f. Let $x_1 < x_2 < \ldots < x_n$ be elements of X. Then there is i such that $x_1 = a_i$. Since each x_j for $2 \leq j \leq n$ is greater than x_1, each such x_j is greater than a_i. Thus, if $2 \leq j \leq n$ and l is such that $x_j = a_l$, we have $l > i$. This means $X_l \subset X_i$ and so, because $a_l \in X_l$, $a_l \in X_i$. Thus, $x_2, \ldots, x_n \in X_i$. Therefore $f(\{x_1, \ldots, x_n\}) = f_{x_1}(\{x_2, \ldots, x_n\}) = c_i = c$.

Since $x_1 < \ldots < x_n$ was an arbitrary set of elements from X, this means that X is homogeneous for f, as desired. $\square$

Ramsey's Theorem is the starting point for a much larger theory of partitions. As this theory is not germane to our purposes, we do not develop it here. Nevertheless, we have placed Ramsey's Thoerem in its own section to emphasize its importance in this other area.

3.3 Order Indiscernibles

Definition 3.13 *Let I be an infinite set and suppose that $X = \{x_i : i \in I\}$ is a set of distinct elements of some $\mathcal{L}$-structure $\mathfrak{A}$. We say that X is an* **indiscernible set** *if whenever $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$ are two sequences of m distinct elements of I, then $\mathfrak{A} \models \phi(x_{i_1}, \ldots, x_{i_m}) \leftrightarrow \phi(x_{j_1}, \ldots, x_{j_m})$ for any $\mathcal{L}$-formula ϕ.*

Definition 3.14 *Let $\mathfrak{A}$ be an $\mathcal{L}$-theory. Let $(I, <)$ be an ordered set, and let $(x_i : i \in I)$ be a sequence of distinct elements of A. We say that $(x_i : i \in I)$ is a sequence of* **order indiscernibles** *if whenever $i_1 < i_2 < \ldots < i_n$ and $j_1 < j_2 < \ldots < j_n$ are two increasing sequences from I, then $\mathfrak{A} \models \phi(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \leftrightarrow \phi(x_{j_1}, x_{j_2}, \ldots, x_{j_n})$.*

In a theory with infinite models, it is always possible to find a sequence of order indiscernibles; indeed, we can even control what the ordering is like, so long as we make sure it is a linear ordering.

Theorem 3.15 *Let T be a theory with infinite models. For any infinite linear order $(I, <)$, there is $\mathfrak{A} \models T$ containing $(x_i : i \in I)$, a sequence of order indiscernibles.*

proof: Let $\mathcal{L}^* = \mathcal{L} \cup \{c_i : i \in I\}$, the c_i distinct new constant symbols. Let Γ be the union of

1. T
2. $c_i \neq c_j$ for $i, j \in I$ with $i \neq j$
3. $\phi(c_{i_1}, \ldots, c_{i_m}) \to \phi(c_{j_1}, \ldots, c_{j_m})$ for $\mathcal{L}$-formulas $\phi(\bar{x})$ where $i_1 < \ldots < i_m$ and $j_1 < \ldots < j_m$ are increasing sequences from I.

If $\mathfrak{A} \models \Gamma$, then $(c_i^{\mathfrak{A}} : i \in I)$ is an infinite sequence of order indiscernibles, so it suffices to show that there is an $\mathfrak{A} \models \Gamma$. We do this by showing that Γ is finitely satisfiable and applying Compactness 1.29. Let $\Delta \subset \Gamma$ be finite and let I_0 be the (necessarily finite) subset of I composed of all the $i \in I$ such that c_i appears in Δ. Now, say $\phi_1, \ldots, \phi_m$ are the formulas for which Δ asserts indiscernibility, i.e. the formulas such that a statement of the from of point (3) above appears in Δ. Assume that there are n free variables occurring in the ϕ_i's.

Let $\mathfrak{B} \models T$ be infinite and fix some linear order $<$ of B. We define a partition $F : [B]^n \to \mathcal{P}(\{1, 2, \ldots, m\})$. If $D = \{d_1, d_2, \ldots, d_n\}$ where $d_1 < \ldots < d_n$, then $F(D) = \{i : \mathfrak{B} \models \phi_i(d_1, \ldots, d_n)\}$. Then F partitions $[B]^n$ into a most 2^m subsets, so, by Ramsey's Theorem 3.12, we can find an infinite $X \subseteq B$ which is homogeneous for F. Let $\eta \subseteq \{1, 2, \ldots, m\}$ be such that $F(D) = \eta$ for all $D \in [B]^n$.

Choose $(x_i : i \in I_0)$ such that each $x_i \in X$ and such that $x_i < x_j$ if $i < j$. If $i_1 < i_2 < \ldots < i_n$ and $j_1 < j_2 < \ldots < j_n$, then $\mathfrak{B} \models \phi_k(x_{i_1}, \ldots, x_{i_n}) \iff k \in \eta \iff \mathfrak{B} \models \phi_k(x_{j_1}, \ldots, x_{j_n})$. Interpreting c_i as x_i for each $i \in I_0$, we make $\mathfrak{B}$ into a model of Δ. Thus, Γ is finitely satisfiable, hence satisfiable so there is $\mathfrak{A} \models \Gamma$, as desired. □

We will use order indiscernibles to build a specific model of a theory T. From the existence of this model, we will be able to draw a pair of corollaries which will give us one direction of Baldwin and Lachlan's characterization of uncountably categorical theories. Before doing this, however, we will need the idea of *skolemization*.

Definition 3.16 *We say that an $\mathcal{L}$-theory T has* **built-in Skolem functions** *if for all $\mathcal{L}$-formulas $\phi(w, \bar{x})$ there is a function symbol f such that $T \models \forall \bar{x}(\exists w \phi(w, \bar{x}) \to \phi(f(\bar{x}), \bar{x}))$.*

We call these f **Skolem functions**.

As the terminology suggests, "skolemization" is the process of "building in" Skolem functions to a theory that does not already have built-in Skolem functions. Formally, the result we want is:

Lemma 3.17 *Let T be an $\mathcal{L}$-theory. There are $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ an $\mathcal{L}^*$-theory such that T^* has built-in Skolem functions, and if $\mathfrak{A} \models T$, then we can expand $\mathfrak{A}$ to $\mathfrak{A}^* \models T^*$. What's more, we can choose $\mathcal{L}^*$ such that $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.*

proof: We build a sequence of languages $\mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subset \mathcal{L}_2 \subseteq \ldots$ and $\mathcal{L}_i$-theories T_i such that $T_0 = T \subseteq T_1 \subseteq T_2 \subseteq \ldots$.

Given $\mathcal{L}_i$, let $\mathcal{L}_{i+1} = \mathcal{L} \cup \{f_\phi : \phi(x, y_1, \ldots, y_n)$ is an $\mathcal{L}_i$-formula $n = 1, 2, \ldots\}$ where f_ϕ is an n-ary function symbol. For $\phi(x, \bar{y})$ an $\mathcal{L}$-formula, let Ψ_ϕ be the sentence $\forall \bar{y}(\exists x \phi(x, \bar{y}) \to \phi(f_\phi(\bar{y}), \bar{y}))$ and let $T_{i+1} = T_i \cup \{\Psi_\phi : \phi$ an $\mathcal{L}_i$-formula $\}$.

Claim: If $\mathfrak{A} \models T_i$, then we can interpret the functions symbols of $\mathcal{L}_{i+1} \setminus \mathcal{L}_i$ so that $\mathfrak{A} \models T_{i+1}$.

proof of claim: Let c be a fixed element of A. If $\phi(x, y_1, \ldots, y_n)$ is an $\mathcal{L}_i$-formula, choose some function $g : A^n \to A$ be such that if $\bar{a} \in A^n$ and $X_{\bar{a}} = \{b \in A : \mathfrak{A} \models \phi(b, \bar{a})\}$ is non-empty, then $g(\bar{a}) \in X_{\bar{a}}$ and if $X_{\bar{a}} = \emptyset$, then $g(\bar{a}) = c$. (The choice of c is irrelevant, we simply need some value for g to take when $X_{\bar{a}} = \emptyset$.) Thus, if $\mathfrak{A} \models \exists x \phi(x, \bar{a}))$, then $\mathfrak{A} \models \phi(g(\bar{a}), \bar{a}))$. Interpreting $f_\phi \in \mathcal{L}_{i+1} \setminus \mathcal{L}$ as g, we then get $\mathfrak{A} \models \Psi_\phi$. This proves the claim. ■

Let $\mathcal{L}^* = \bigcup_i \mathcal{L}_i$ and $T^* = \bigcup_i T_i$. If $\phi(x.\bar{y})$ is an $\mathcal{L}_i$-formula, then ϕ is an $\mathcal{L}_i$-formula for some i and $\Psi_\phi \in T_{i+1} \subseteq T^*$, so T^* has built-in Skolem functions. By iterating the claim, if $\mathfrak{A} \models T$, then we can interpret the symbols of $\mathcal{L}^* \setminus \mathcal{L}$ to make $\mathfrak{A} \models T^*$.

Finally, note that at each stage i we formed $\mathcal{L}_{i+1}$ by adding to $\mathcal{L}$ one new function symbol for each $\mathcal{L}_i$-formula. Since $\mathcal{L}_0 = \mathcal{L}$ is countable, it follows by induction that for each $\mathcal{L}_i$ the number of $\mathcal{L}_i$-formulas is countable. Thus, $|\mathcal{L}_{i+1}| = |\mathcal{L}_i| + \aleph_0$ for each i. Hence, each $\mathcal{L}_i$ has size $|\mathcal{L}| + \aleph_0$, so $\mathcal{L}^*$ is a countable union of sets of size $|\mathcal{L}| + \aleph_0$ hence is of size $\aleph_0 \times (|\mathcal{L}| + \aleph_0) = |\mathcal{L}| + \aleph_0$, as claimed. □

Definition 3.18 *If T^*, T are as above, then we call T^* a* **skolemization of** *T.*

Throughout the remainder of this sections, $\mathcal{L}^*$, T^* will be as above.

Definition 3.19 *Let T be an $\mathcal{L}$-theory, T^* a skolemization of T. If $\mathfrak{A} \models T^*$ and $X \subseteq A$, we let $\mathcal{H}(X)$ denote the $\mathcal{L}^*$-substructure of $\mathfrak{A}$ generated by X (i.e, the closure of X under the Skolem functions of $\mathcal{L}^*$). We call $\mathcal{H}(X)$ the* **Skolem hull** *of X.*

Note that since $\mathcal{L}^*$ has built-in Skolem functions, $\mathcal{H}(X) \preceq \mathfrak{A}$.

We are now ready to use order indiscernibles to build our desired model, and deduce our desired corollaries.

Theorem 3.20 *Let $\mathcal{L}$ be countable and T be an $\mathcal{L}$-theory with infinite models. For all $\kappa \geq \aleph_0$, there is $\mathfrak{A} \models T^*$ with $|A| = \kappa$ such that if $X \subseteq A$, then $\mathfrak{A}$ realizes at most $|X| + \aleph_0$ types in $S_n^{\mathfrak{A}}(X)$.*

proof: Let $\mathcal{L}^*, T^*$ be as above. Let $\mathfrak{A} \models T$ be the Skolem hull of a sequence of order indiscernibles I of order type $(\kappa, <)$ (such a sequence exists by theorem 3.15). Then we have $|A| = \kappa$.

Let $X \subseteq A$. For each $x \in X$, there is an $\mathcal{L}$-term t_x and $\bar{y}_x$ a sequence from I such that $x = t_x(\bar{y}_x)$. Let $Y = \{y \in I : y \text{ is in some } \bar{y}_x\}$. Then $|Y| = |X| + \aleph_0$ (our language $\mathcal{L}^*$ has $|X| + \aleph_0$ symbols, hence there are at most $|X| + \aleph_0$-many terms, and thus the number of y appearing in some $\bar{y}_x$ is at most $(|X| + \aleph_0) \times \aleph_0 = |X| + \aleph_0$).

Now, let $y_1 < y_2 < \ldots < y_m$ and $z_1 < z_2 < \ldots < z_m$ be two sequences from I. We say that $\bar{y} \sim_Y \bar{z}$ if for all $\eta \in Y$ we have:

$$y_i < \eta \text{ if and only if } z_i < \eta \text{ AND } y_i = \eta \text{ if and only if } z_i = \eta.$$

It is easy to see that $\sim_Y$ is an equivalence relation.

Claim: If $\bar{y}_1, \ldots \bar{y}_n$, $\bar{z}_1, \ldots, \bar{z}_n$ are sequences from I such that for each i we have $\bar{y}_i \sim_Y \bar{z}_i$ and $t_1, \ldots, t_n$ are Skolem terms (terms whose only functions symbols are Skolem functions), then $t_1(\bar{y}_1), \ldots, t_n(\bar{y}_n)$ and $t_1(\bar{z}_1), \ldots, t_n(\bar{z}_n)$ realize the same types in $S_n^{\mathfrak{A}}(X)$.

proof of claim: Let $x_1, \ldots, x_k \in X$. Because each $\bar{y}_i \sim_Y \bar{z}_i$ we know that each $\bar{y}_i$ and $\bar{z}_i$ are in the same position in the ordering relative to Y. Thus, by indiscernibility, we have

$$\begin{aligned}
&\mathfrak{A} \models \phi(t_1(\bar{y}_1), \ldots, t_n(\bar{y}_m), x_1, \ldots, x_k) \iff \\
&\mathfrak{A} \models \phi(t_1(\bar{y}_1), \ldots, t_n(\bar{y}_m), t_{x_1}(\bar{y}_{x_1}), \ldots, t_{x_k}(\bar{y}_{x_k})) \iff \\
&\mathfrak{A} \models \phi(t_1(\bar{z}_1), \ldots, t_n(\bar{z}_m), t_{x_1}(\bar{y}_{x_1}), \ldots, t_{x_k}(\bar{y}_{x_k})) \iff \\
&\mathfrak{A} \models \phi(t_1(\bar{z}_1), \ldots, t_n(\bar{z}_m), x_1, \ldots, x_k) \qquad \blacksquare
\end{aligned}$$

Since sequences of indiscernibles that are $\sim_Y$-equivalent realize the same types, it suffices show that $|I^m / \sim_Y| \leq |X| + \aleph_0$. Well, for $z \in I \setminus Y$, let $C_z = \{y \in Y : y < z\}$. Then $\bar{y} \sim_Y \bar{z}$ if and only if for each i:

1. if $y_i \in Y$, then $y_i = z_i$ and
2. if $y_i \notin Y$, then $z_i \notin Y$ and $C_{y_i} = C_{z_i}$.

Now, I is well ordered by $<$, so $C_y = C_z$ if and only if $C_y = C_z = \varnothing$ or $inf\{i \in I : i > C_y\} = inf\{i \in I : i > C_z\}$. There are at most $|Y| + 1$ possible cuts C_z in Y (one for each $z \in Y$ plus one for $\varnothing$) and so for a given $k \in \mathbb{N}$ there can be at most $k \times (|Y|+1)$ non-$\sim_Y$-equivalent tuples of length k from Y. Since $Y \subseteq X$, we have $k \times (|Y|+1) \leq |X| + \aleph_0$. Thus, the total number of $\sim_Y$-inequivalent tuples of any length from Y is at most $\aleph_0 \times (|X|+\aleph_0) = (|X|+\aleph_0)$. Therefore, $|I^m/\sim_Y| \leq |X| + \aleph_0$, and so $\mathfrak{A}$ realizes at most $|X| + \aleph_0$ types over X. □

Corollary 3.21 *Let T be a complete theory in a countable language with infinite models, and let $\kappa \geq \aleph_1$. If T is κ-categorical, then T is ω-stable.*

proof: Assume for a contradiction that T is not ω-stable. Then there is countable $\mathfrak{A} \models T$ with some $X \subseteq A$ such that $|S_n^{\mathfrak{A}}(X)| > \aleph_0$. A straightforward Compactness argument shows that we can find $\mathfrak{B}_0 \succ \mathfrak{A}$ of cardinality κ realizing uncountably many types in $S_n^{\mathfrak{A}}(X)$. Now, by theorem 3.20, we can find $\mathfrak{B}_1 \models T$ of cardinality κ such that for all $X \subset A$ if $|X| = \aleph_0$, then $\mathfrak{B}_1$ realizes at most $\aleph_0$ types over X. This gives us $\mathfrak{B}_0 \ncong \mathfrak{B}_1$, contradicting the κ-categoricity of T, and so T must be ω-stable. □

Corollary 3.22 *Let T be a complete theory in a countable language with infinite models. If $\kappa \geq \aleph_1$ and T is κ-categorical, then T has no Vaughtian pairs and hence no (κ, λ)-models for $\kappa > \lambda \geq \aleph_0$.*

proof: Since T is κ-categorical, the last corollary gives that T is ω-stable. If T has a Vaughtian pair, then by theorem 3.7 there is an $(\aleph_1, \aleph_0)$-model of T and hence, by theorem 3.10, a $(\kappa, \aleph_0)$-model of T. But now, by lemma 3.2, T also has a model in which every infinite set definable by an $\mathcal{L}$-formula has size κ, a contradiction to the κ-categoricity of T. Thus, T must not have any Vaughtian pairs. □

Corollaries 3.21 and 3.22 will form one direction of Baldwin and Lachlan's characterization of uncountably categorical theories, which we will give in section 3.5.

3.4 Strong Minimality and Algebraicity

It is possible to develop a notion of one element of a structure (i.e., of a structure's underlying set) being algebraic relative to other elements of the structure. Having developed this notion, one can then develop a notion of dimension for the subsets of a structure. This notion of dimension will be useful to us in proving Morley's Categoricity Theorem, and so we set about developing it now. Before getting to the actual, notions of algebraticity and dimension, however, we need to set up the context in which those definitions apply. For this, we need the notion of strongly minimal sets.

Definition 3.23 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and let $D \subseteq A^n$ be an infinite definable set. We say that D is* **minimal** *in $\mathfrak{A}$ if for any definable $Y \subseteq D$ either Y is finite or $D \setminus Y$ is finite. If $\phi(\bar{x}, \bar{a})$ is the formula that defines D, then we also say that $\phi(\bar{x}, \bar{a})$ is* **minimal**.

We say that D and ϕ are **strongly minimal** *if ϕ is minimal in any elementary extension $\mathfrak{B}$ of $\mathfrak{A}$.*

We say that a theory T is **strongly minimal** *if the formula $x = x$ is strongly minimal, i.e. if in every model $\mathfrak{A}$ of T, A is strongly minimal.*

We fix $\mathfrak{A} \vDash T$ and D a strongly minimal set in $\mathfrak{A}$.

Definition 3.24 *For $X \subseteq D$ and $y \in D$ we say that y is* **algebraic** *over D if there is a formula $\phi(x, \bar{a})$ with $\bar{a} \in X$ such that $\phi(\mathfrak{A}, \bar{a})$ is finite and $\mathfrak{A} \vDash \phi(y, \bar{a})$.*

For $X \subseteq D$ we let $acl(X)$, the **algebraic closure** *of X (in D), be $\{y \in D : y$ is algebraic over $X\}$.*

Nowhere in the preceding definition did we make use of the fact that D is strongly minimal. Thus, if we replaced D by A, the entire underlying set of $\mathfrak{A}$, or by any other set, in the above definition, it would still make sense. We have not done this because we will be interested almost exclusively in algebraticity in the context of elements or subsets of D. Nevertheless, there is one result we cover which is true in a more general context. We state and prove it now.

Lemma 3.25 *For any set D, strongly minimal or not, we have the following for $X, Y \subseteq D$:*

1. $acl(acl(X)) = acl(X) \supseteq X$.

2. *If* $X \subseteq Y$*, then* $acl(X) \subseteq acl(Y)$.

3. *If* $x \in acl(X)$*, then* $x \in acl(X_0)$ *for some finite* $X_0 \subseteq X$.

proof:

1. If $x \in X$ then the formula $v = x$ (where v is a variable symbol) witnesses that x is algebraic over X, so $x \in acl(X)$ and therefore $acl(X) \supseteq X$. Applying this to $acl(X)$, we immediately get $acl(X) \subseteq acl(acl(X))$. For the reverse inclusion, let $\xi \in acl(acl(X))$. Then there is a formula $\phi(v)$ with n (some $n \in \mathbb{N}$) realizations in $\mathfrak{A}$ and parameters from $acl(X)$ witnessing that ξ is algebraic over $acl(X)$. Say ϕ is the $\mathcal{L}$-formula $\psi(v, \bar{\xi}, \bar{x})$ where $\bar{\xi}$ are the parameters in ϕ from $acl(X) \setminus X$ and $\bar{x}$ are the parameters in ϕ from X.

   Since each ξ_i in $\bar{\xi}$ is in $acl(X)$ there is a formula $\chi_i(v)$ with parameters from X such that $\chi_i(\mathfrak{A})$ is finite and $\mathfrak{A} \vDash \chi_i(\xi_i)$. Let n_i be the number of realizations of χ_i in $\mathfrak{A}$. Let $\Phi(v)$ be the formula with parameters from X given by $\exists \bar{w}(\psi(v, \bar{w}, \bar{x}) \wedge \bigwedge_i \chi_i(w_i) \wedge |\{z \in A : \mathfrak{A} \vDash \psi(z, \bar{w}, \bar{x})\}| = n)$. Then we have $\mathfrak{A} \vDash \Phi(\xi)$ and Φ has at most $n(\Pi_i n_i)$ realizations. Thus, Φ witnesses that $\xi \in acl(X)$, so we have $acl(acl(X)) \subseteq acl(X)$ and therefore $acl(acl(X)) = acl(X)$, as we wished to show.

2. Let $\xi \in acl(X)$. Then there is a formula $\phi(v)$ with parameters from X and only finitely many realizations in $\mathfrak{A}$ such that $\mathfrak{A} \vDash \phi(\xi)$. But since $X \subseteq Y$, ϕ is a formula with parameters from Y and only finitely many realizations in $\mathfrak{A}$ such that $\mathfrak{A} \vDash \phi(\xi)$, so $\xi \in acl(Y)$. Thus $acl(X) \subseteq acl(Y)$.

3. Say $x \in acl(X)$ and $\phi(v)$ is a formula witnessing that x is algebraic over X. Let X_0 be the set of elements of X appearing in ϕ. Then X_0 is finite and ϕ witnesses that $x \in acl(X_0)$.

□

The following is an important property of the algebraic closure in strongly minimal sets.

Lemma 3.26 (Exchange Principle) *Suppose $\mathfrak{A}$ is an $\mathcal{L}$-structure and that $D \subset A$ is strongly minimal, $X \subseteq D$ and $x, y \in D$. If $x \in acl(X \cup \{y\}) \setminus acl(X)$, then $y \in acl(X \cup \{x\})$.*

proof: To simplify our notation, we write $acl(X, y)$ in place of $acl(X \cup \{y\})$. Let $x \in acl(X, y) \setminus acl(X)$. Say $\mathfrak{A} \models \phi(x, y)$ where ϕ is a formula with parameters in X and $\phi(u, w)$ has only finitely many realizations in D, say n realizations. Let $\psi(w)$ be the formula asserting that $|\{z \in D : \phi(z, w)\}| = n$. If $\psi(w)$ defines a finite subset of D, then we have $y \in acl(X)$. In this case, we have $x \in acl(X, y) \subseteq acl(acl(X))$. By lemma 3.25(1), $acl(acl(X)) = acl(X)$, so this gives $x \in acl(X)$, contrary to hypothesis. Thus, $\psi(w)$ must define a cofintie subset of D.

Now, if $\{z \in D : \phi(x, z) \wedge \psi(z)\}$ is finite, then we have $y \in acl(X, x)$ as desired. Thus, we assume for a contradiction that $\{z \in D : \phi(x, z) \wedge \psi(z)\}$ is infinite. Since D is strongly minimal, this means that $D \setminus \{z \in D : \phi(x, z) \wedge \psi(z)\}$ must be finite, say of size m. Let $\chi(v)$ be the formula asserting that $D \setminus \{z \in D : \phi(v, z) \wedge \psi(z)\}$ has size m. If $\chi(v)$ defines a finite subset of D, then, since $\mathfrak{A} \models \chi(x)$, we would have $x \in acl(X)$, again contrary to hypothesis. Therefore, $\chi(v)$ defines an infinite (hence cofinite) subset of D.

Choose $a_1, \ldots, a_{n+1}$ such that $\mathfrak{A} \models \chi(a_i)$ for each $1 \leq i \leq n+1$. Thus, for each i we have that the set $D \setminus \{z \in D : \phi(a_i, z) \wedge \psi(z)\}$ has size m, and so for each i the set $B_i = \{z \in D : \phi(a_i, z) \wedge \psi(z)\}$ is cofinite. Choose $\hat{b} \in \bigcap_i B_i$ (since each B_i is cofinite and D is infinite, such a $\hat{b}$ must exists). Then for $1 \leq i \leq n+1$ we have $\mathfrak{A} \models \phi(a_i, \hat{b})$, so $|\{z \in D : \phi(z, \hat{b})\}| \geq n+1$. But this contradicts the fact that $\psi(\hat{b})$! Thus, $\{z \in D : \phi(x, z) \wedge \psi(z)\}$ must be finite, so $y \in acl(X, x)$, as we wished to show. □

Not surprisingly, having a notion of algebraticity, we have a notion of independence as well.

Definition 3.27 *We say that $X \subseteq D$ is* **independent** *if $x \notin acl(X \setminus \{x\})$ for all $x \in X$. If $Y \subset D$, we say that X is* **independent over** *Y if $x \notin acl(Y \cup (X \setminus \{x\}))$ for all $x \in X$.*

A basic fact about independent sets is:

Lemma 3.28 *If $X \subseteq D$ is independent and $x \in D$ is such that $x \notin acl(X)$, then $X \cup \{x\}$ is independent.*

proof: Assume toward a contradiction that $X \cup \{x\}$ is not independent. Then there is $x_0 \in X \cup \{x\}$ such that $x_0 \in acl((X \cup \{x\}) \smallsetminus \{x_0\})$. Since $x \notin acl(X)$ by assumption, $x_0 \neq x$. Thus $(X \cup \{x\}) \smallsetminus \{x_0\} = (X \smallsetminus \{x_0\}) \cup \{x\}$, so we have $x_0 \in acl((X \smallsetminus \{x_0\}) \cup \{x\})$. As X is independent $x_0 \notin acl(X \smallsetminus \{x_0\})$, so applying the exchange principle 3.26 gives us that $x \in acl((X \smallsetminus \{x_0\}) \cup \{x_0\}) = acl(X)$. This is a contradiction, so we must have $X \cup \{x\}$ independent. □

Independence can be related to the types as follows.

Lemma 3.29 *Suppose that $\mathfrak{A}, \mathfrak{B} \models T$ and $\phi(x)$ is a strongly minimal formula with parameters from X, where either $X = \varnothing$ or $X \subseteq A_0$ where $\mathfrak{A}_0 \models T$, $\mathfrak{A}_0 \prec \mathfrak{A}, \mathfrak{B}$. If $\bar{a} \in \phi(\mathfrak{A})$ is independent over X and $\bar{b} \in \phi(\mathfrak{B})$ is independent over X, then $tp^{\mathfrak{A}}(\bar{a}/X) = tp^{\mathfrak{B}}(\bar{b}/X)$.*

proof: We first consider the case where $\phi(x)$ has parameters from $X \subseteq A_0$ for some $\mathfrak{A}_0 \prec \mathfrak{A}, \mathfrak{B}$. We proceed by induction on n, the length of $\bar{a}$ and $\bar{b}$.

Assume $n = 1$ and let $a \in \phi(\mathfrak{A}) \smallsetminus acl(X)$, $b \in \phi(\mathfrak{B}) \smallsetminus acl(X)$. Let $\psi(x)$ be any formula with parameters from X such that $\mathfrak{A} \models \psi(a)$. Since $a \notin acl(X)$ we must have $\phi(\mathfrak{A}) \cap \psi(\mathfrak{A})$ infinite (otherwise $\phi \wedge \psi$ would witness $a \in acl(X)$). Because ϕ is strongly minimal, $\phi(\mathfrak{A}) \smallsetminus \psi(\mathfrak{A})$ must be finite. Thus, there is some m such that $\mathfrak{A} \models |\{x : \phi(x) \wedge \neg\psi(x)\}| = m$. As $\mathfrak{A}_0 \prec \mathfrak{A}$, this means $\mathfrak{A}_0 \models |\{x : \phi(x) \wedge \neg\psi(x)\}| = m$. But now, $\mathfrak{A}_0 \prec \mathfrak{B}$, so $\mathfrak{B} \models |\{x : \phi(x) \wedge \neg\psi(x)\}| = m$. Since $b \notin acl(X)$, no formula with parameters from X and only finitely many realizations (in particular m realizations) can be realized by b. Thus, $\mathfrak{B} \not\models \phi(b) \wedge \neg\psi(b)$, so as $b \in \phi(\mathfrak{B})$ we must have $\mathfrak{B} \models \psi(b)$. So $tp^{\mathfrak{A}}(a/X) = tp^{\mathfrak{B}}(b/X)$, as desired.

Now suppose the claim is true for n and let $a_1, \ldots, a_{n+1} \in \phi(\mathfrak{A})$ and $b_1, \ldots, b_{n+1} \in \phi(\mathfrak{B})$ be independent sequences over X. Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$. By induction, $tp^{\mathfrak{A}}(\bar{a}/X) = tp^{\mathfrak{B}}(\bar{b}/X)$. Let $\psi(\bar{w}, x)$ be a formula with parameters from X such that $\mathfrak{A} \models \psi(\bar{a}, a_{n+1})$. Because $a_{n+1} \notin acl(X, \bar{a})$, it must be that, similarly to the previous case, $\phi(\mathfrak{A}) \cap \psi(\bar{a}, \mathfrak{A})$ is infinite and $\phi(\mathfrak{A}) \smallsetminus \psi(\bar{a}, \mathfrak{A})$ is finite. So there is some m such that $\mathfrak{A} \models |\{x : \phi(x) \wedge \neg\psi(\bar{a}, x)\}| = m$. Again, we have $\mathfrak{A}_0 \prec \mathfrak{A}$ so $\mathfrak{A}_0 \models |\{x : \phi(x) \wedge \neg\psi(\bar{a}, x)\}| = m$ and $\mathfrak{A}_0 \prec \mathfrak{B}$ so $\mathfrak{B} \models |\{x : \phi(x) \wedge \neg\psi(\bar{a}, x)\}| = m$. Now, since $tp^{\mathfrak{A}}(\bar{a}/X) = tp^{\mathfrak{B}}(\bar{b}/X)$ this gives us $\mathfrak{B} \models |\{x : \phi(x) \wedge \neg\psi(\bar{b}, x)\}| = m$ and thus, since $b_{n+1} \notin acl(A, \bar{b})$, we get $\mathfrak{B} \models \psi(\bar{b}, b_{n+1})$. Therefore, $tp^{\mathfrak{A}}(\bar{a}, a_{n+1}) = tp^{\mathfrak{B}}(\bar{b}, b_{n+1})$ and by induction this case is proved.

Now consider the case $X = \varnothing$. Again, we proceed by induction on n. Say $n = 1$ and let $a \in \phi(\mathfrak{A})$, $b \in \phi(\mathfrak{B})$. Let $\psi(x) \in tp^{\mathfrak{A}}(a))$. Then because ϕ is strongly minimal and a is not algebraic over $\varnothing$ we must have that $\phi(\mathfrak{A}) \cap \psi(\mathfrak{A})$ is infinite (else $\phi \wedge \psi$ witnesses that a is algebraic over $\varnothing$). Thus, by strong minimality of ϕ, we have $\phi(\mathfrak{A}) \cap \psi(\mathfrak{A})$ is finite. So there is m such that $\mathfrak{A} \models |\{x : \phi(x) \wedge \neg\psi(x)\}| = m$. Since T is a complete theory, this means that $T \models |\{x : \phi(x) \wedge \neg\psi(x)\}| = m$ and so $\mathfrak{B} \models T$ gives us $\mathfrak{B} \models |\{x : \phi(x) \wedge \neg\psi(x)\}| = m$. Now, since $b \in \phi(\mathfrak{B})$ is not algebraic over $\varnothing$, it must be that $\mathfrak{B} \not\models \phi(b) \wedge \neg\psi(b)$, i.e., must be that $\mathfrak{B} \models \psi(b)$. Therefore $tp^{\mathfrak{A}}(a) = tp^{\mathfrak{B}}(b)$.

Suppose now that the result is true for n and let $a_1, \ldots, a_{n+1} \in \phi(\mathfrak{A})$ and $b_1, \ldots, b_{n+1} \in \phi(\mathfrak{B})$ be independent sequences of elements. Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$. By induction, $tp^{\mathfrak{A}}(\bar{a}) = tp^{\mathfrak{B}}(\bar{b})$. Let $\psi(\bar{w}, x) \in tp^{\mathfrak{A}}(\bar{a}, a_{n+1})$. Then similarly to the previous cases, because a_{n+1} is independent over $\bar{a}$ we must have $\phi(\mathfrak{A}) \cap \psi(\bar{a}, \mathfrak{A})$ infinite. So there is m such that $\mathfrak{A} \models |\{x : \phi(x) \wedge \neg\psi(\bar{a}, x)\}| = m$. Now, since $tp^{\mathfrak{A}}(\bar{a}) = tp^{\mathfrak{B}}(\bar{b})$, this means that the formula $|\{x : \phi(x) \wedge \neg\psi(\bar{w}, x)\}| = m \in tp^{\mathfrak{B}}(\bar{b})$ and thus that $\mathfrak{B} \models |\{x : \phi(x) \wedge \neg\psi(\bar{b}, x)\}| = m$. Therefore, as previously, since b_{n+1} is independent of $\bar{b}$, we must have $\mathfrak{B} \models \psi(\bar{b}, b_{n+1})$, so $\psi \in tp^{\mathfrak{B}}(\bar{b}, b_{n+1})$ and thus $tp^{\mathfrak{A}}(\bar{a}, a_{n+1}) = tp^{\mathfrak{B}}(\bar{b}, b_{n+1})$ □

Definition 3.30 *Let $X = (x_i : i \in I)$ be a sequence of order indiscernibles in a structure $\mathfrak{A}$. We define $tp(X)$ to be*

$$\{\phi(\bar{x}) : \mathfrak{A} \models \phi(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), i_1 < i_2 < \ldots < i_n \in I, n \in \omega\}$$

We call $tp(X)$ the **type of the indiscernibles**

An immediate consequence of lemma 3.29 is:

Corollary 3.31 *If $\mathfrak{A}, \mathfrak{B} \models T$, X and $\phi(x)$ are as in theorem 3.29, Y is an infinite subset of $\phi(\mathfrak{A})$ independent over X and Z is an infinite subset of $\phi(\mathfrak{B})$ independent over X, then Y and Z are infinite sets of indiscernibles of the same type over X.*

Having defined a notion of independence, we can give a notion of a basis.

Definition 3.32 *We say that X is a* **basis** *for $Y \subseteq D$ if $X \subseteq Y$ is independent and $acl(X) = acl(Y)$.*

Lemma 3.33 *Let $X, Y \subseteq D$ be independent with $X \subseteq acl(Y)$.*

1. *Suppose that* $X_0 \subseteq X, Y_0 \subseteq Y, X_0 \cup Y_0$ *is a basis for* $acl(Y)$ *and* $x \in X \setminus X_0$. *Then there is* $y \in Y_0$ *such that* $X_0 \cup \{x\} \cup (Y_0 \setminus \{y\})$ *is a basis for* $acl(Y)$.

2. $|X| \leq |Y|$.

3. *If* X *and* Y *are bases for* $Z \subseteq D$, *then* $|X| = |Y|$.

proof:

1. Let $C \subseteq Y_0$ be of minimal cardinality such that $x \in acl(X_0 \cup C)$. Since X is independent, $x \notin acl(X_0)$, so we must have $C \neq \emptyset$, i.e. $|C| \geq 1$. Let $y \in C$. Then $x \in acl(X_0 \cup (C \setminus \{y\}) \cup \{y\})$ so by the exchange principle 3.26, $y \in acl(X_0 \cup \{x\} \cup (C \setminus \{y\}))$. This means that $acl(X_0 \cup \{x\} \cup (Y_0 \setminus \{y\}))$ contains y, so in particular $acl(X_0 \cup \{x\} \cup (Y_0 \setminus \{y\})) \supseteq acl(X_0 \cup Y_0) = acl(Y)$. Therefore, we get $acl(X_0 \cup \{x\} \cup (Y_0 \setminus \{y\}) = acl(Y)$. It remains, then, only to show that $X_0 \cup \{x\} \cup (Y_0 \setminus \{y\})$ is independent.

   Well, say $x \in acl(X_0 \cup (Y_0 \setminus \{y\}))$. Then $y \in acl(X_0 \cup (Y_0 \setminus \{y\}))$, contradicting the fact that $X_0 \cup Y_0$ is a basis. Thus, since $X_0 \cup (Y_0 \setminus \{y\})$, lemma 3.28 $X_0 \cup \{x\} \cup (Y_0 \setminus \{y\})$ is independent.

2. First consider the case Y finite. Toward a contradiction, suppose that $|Y| = n$ and $x_1, \ldots, x_{n+1}$ are distinct elements of X. Let $X_0 = \emptyset$ and $Y_0 = Y$. Applying (1) repeatedly, we can find distinct $y_1, \ldots, y_n \in Y$ such that $\{x_1, \ldots, x_i\} \cup (Y \setminus \{y_1, \ldots, y_i\})$ is basis for $acl(Y)$ for $i \leq n$. But this gives $acl(x_1, \ldots, x_n) = acl(Y)$. Since $x_{n+1} \in acl(Y)$, this contradicts the independence of X.

   Now consider the case Y infinite. Let $Y_0 \subseteq Y$ be finite. Then $X \cap acl(Y_0)$ is independent since it is contained in X and is clearly contained in $acl(Y_0)$. By the previous case, then, we have $|X \cap acl(Y_0)| \leq |Y_0|$, so $X \cap acl(Y_0)$ is finite. Now

   $$X \subseteq \bigcup_{Y_0 \subseteq Y \text{ finite}} acl(Y_0).$$

   Since there are $|Y|$-many finite subsets of Y, we thus have $|X| \leq |Y| \times \aleph_0 = |Y|$, where the last equality holds because Y is infinite.

3. This follows immediately from 2.

□

Having the notion of a basis, we can now obtain our desired notion of dimension.

Definition 3.34 *If $Y \subseteq D$, then the* **dimension** *of Y is the cardinality of a basis for Y. We let $dim(Y)$ denote the dimension of Y.*

An useful fact about dimension is the following:

Lemma 3.35 *If D is a strongly minimal set and $Y \subseteq D$ is uncountable, then $dim(Y) = |Y|$.*

proof: Say that $A \subset Y$ is such that $|A| < |Y|$. We will show that A is not a basis for Y. Since our language $\mathcal{L}$ is countable, there are only $|A|$-many formulas of the from $\phi(x, \bar{a})$ for $\bar{a} \in X$. Thus, there are at most $|A|$-many formulas of this from with finitely many realizations, say $\{\phi_\alpha : \alpha < \lambda\}$ for some cardinal $\lambda \leq |A|$. Then $acl(A)$ has size at most $\aleph_0 \times \lambda = max\{\aleph_0, \lambda\} \leq |A| < |Y|$. Thus $acl(A) \neq acl(Y)$, so A is not a basis for Y, so no set of size strictly less than $|Y|$ can be a basis for Y, so $dim(Y) = |Y|$. □

An important result for proving Morley's Categoricity theorem is the following.

Lemma 3.36 *Suppose T is a strongly minimal theory. If $\mathfrak{A}, \mathfrak{B} \models T$, then $\mathfrak{A} \cong \mathfrak{B}$ if and only if $dim(A) = dim(B)$.*

More generally, if $\mathfrak{A}, \mathfrak{B}$ and ϕ are as in lemma 3.29, and $dim(\phi(\mathfrak{A})) = dim(\phi(\mathfrak{B}))$, then there is a bijective partial elementary map $f : \phi(\mathfrak{A}) \to \phi(\mathfrak{B})$.

proof: Let X be a basis for $\phi(\mathfrak{A})$, Y a basis for $\phi(\mathfrak{B})$. Since $dim(\phi(\mathfrak{A})) = dim(\phi(\mathfrak{B}))$, we have $|X| = |Y|$ so there is a bijection $f : X \to Y$. By 3.31, X and Y are sets of indiscernibles of the same type, and thus f is elementary. Let $I = \{g : X' \to Y' : X \subseteq X' \subseteq \phi(\mathfrak{A}), Y \subseteq Y' \subseteq \phi(\mathfrak{B}), f \subseteq g$ partial elementary$\}$. By Zorn's Lemma, there is a maximal $g : X' \to Y'$. We will show that $\phi(\mathfrak{A}) = X'$. Toward a contradiction, suppose otherwise and let $\xi \in \phi(\mathfrak{A}) \setminus X'$.

Claim: There is a formula $\psi(x, \bar{d})$ isolating $tp^{\mathfrak{A}}(\xi/X')$.

proof of claim: Because ξ is algebraic over X' there is an $\mathcal{L}$ formula $\theta(x, \bar{c})$ for some $\bar{c} \in X'$ such that $\theta(\mathfrak{A}, \bar{c})$ is finite and $\mathfrak{A} \models \theta(\xi, \bar{c})$. Let $\{a_1, a_2, \ldots, a_k\}$ be the set of elements of $\theta(\mathfrak{A}, \bar{c})$ that do NOT realize the same type over X' as ξ. Then we have formulas $\alpha_1, \ldots, \alpha_k$ and $\bar{c}_1, \ldots, \bar{c}_k \in X'$ such that for $1 \leq i \leq k$ we have $\mathfrak{A} \models \alpha_i(a_i, \bar{c}_i)$ and $\mathfrak{A} \models \neg\alpha_i(\xi, \bar{c}_i)$. Let $\bar{d}$ be the tuple consisting of $\bar{c}$ followed by each $\bar{c}_i$ in order. Let $\psi(x, \bar{d})$ be the formula $\theta(x, \bar{c}) \wedge \neg\alpha_1(x, \bar{c}_1) \wedge \ldots \wedge \neg\alpha_k(x, \bar{c}_k)$. Then the only realizations of $\psi(x, \bar{d})$ in $\mathfrak{A}$ are realizations of $\theta(x, \bar{c})$ which have the same type over X' as ξ. Thus, if $\mu(x) \in tp^{\mathfrak{A}}(\xi/X')$ we have $T \models \forall x(\psi(x, \bar{d}) \to \mu(x))$, so $\psi(x, \bar{d})$ isolates $tp^{\mathfrak{A}}(\xi/X')$, as desired. ■

Now, $\mathfrak{A} \models \exists x(\psi(x, \bar{d}) \wedge \phi(x))$. Thus, because g is partial elementary, $\mathfrak{B} \models \exists x(\psi(x, g(\bar{d})) \wedge \phi(x))$ so there is some $\eta \in \phi(\mathfrak{B})$ such that $\mathfrak{B} \models \psi(\eta, g(\bar{d}))$. This means that $tp^{\mathfrak{A}}(\xi/X') = tp^{\mathfrak{B}}(\eta/Y')$ so we can extend g by mapping ξ to η. This contradicts the maximality of g, and so we must have $\phi(\mathfrak{A}) = X'$. A similar argument shows $Y' = \phi(\mathfrak{B})$. □

Thus far, we have been assuming the existence of a strongly minimal set in our results. We now show that in ω-stable theories strongly minimal sets do exist.

Lemma 3.37 *Let T be ω-stable.*

1. *If $\mathfrak{A} \models T$, then there is a minimal formula in $\mathfrak{A}$.*

2. *If $\mathfrak{A} \models T$ is $\aleph_0$-saturated and $\phi(\bar{x}, \bar{a})$ is a minimal formula in $\mathfrak{A}$, then $\phi(\bar{x}, \bar{a})$ is strongly minimal.*

proof: For (1), suppose for a contradiction that there is not a minimal formula in $\mathfrak{A}$. We build a tree of formulas $(\phi_\sigma : \sigma \in 2^{<\omega})$ such that:

- if $\sigma \subset \tau$, then $\phi_\sigma \models \phi_\tau$
- $\phi_{\sigma,i} \models \neg\phi_{\sigma,1-i}$
- $\phi_\sigma(\mathfrak{A})$ is infinite

Let $\phi_\varnothing$ be the formula $x = x$. Suppose we have ϕ_σ with $\phi_\sigma(\mathfrak{A})$is infinite. Since ϕ_σ is not minimal (by assumption), there is a formula ψ such that both

$(\phi_\sigma \wedge \psi)(\mathfrak{A})$ and $(\phi_\sigma \wedge \neg\psi)(\mathfrak{A})$ are infinite. Let $\phi_{\sigma,0}$ be $\phi_\sigma \wedge \psi$ and let $\phi_{\sigma,1}$ be $\phi_\sigma \wedge \neg\psi$.

Let A_0 be the set of all parameters from A occurring in any formula ϕ_σ. Clearly, A_0 is countable. Let $f : \omega \to 2$. Then we have $[\phi_{f|0}] \supseteq [\phi_{f|1}] \supseteq [\phi_{f|2}] \supseteq \ldots$ (where we recall that $[\phi] = \{p \in S_n^{\mathfrak{A}}(A_0) : \phi \in p\}$). Because $S_n^{\mathfrak{A}}(A_0)$ is compact, there is some $p_f \in \bigcap_{n=0}^{\infty}[\phi_{f|n}]$. We will show that the map $f \mapsto p_f$ is one-to-one.

Let $f, g : \omega \to 2$, $f \neq g$. Then there is some m such that $f(m) \neq g(m)$ but $f|m = g|m$. By our construction of the ϕ_σ, $\phi_{f|m+1} \vDash \neg\phi_{g|m+1}$ so we have $[\phi_{f|m+1}] \cap [\phi_{g|m+1}] = \varnothing$. Thus $p_f \neq p_g$, so $f \mapsto p_f$ is one-to-one, as desired. But this means that $f \mapsto p_f$ is a one-to-one map from 2^ω to $S_n^{\mathfrak{A}}(A_0)$, so $|S_n^{\mathfrak{A}}(A_0)| \geq 2^{\aleph_0}$, contradicting ω-stability. Thus, there must be a minimal formula.

Now, for (2) suppose otherwise and let $\mathfrak{B} \succ \mathfrak{A}$ witness that ϕ is not strongly minimal. Then there is some $\bar{b} \in B$ and some formula ψ such that $\psi(\mathfrak{B}, \bar{b})$ is an infinite, co-infinite subset of $\phi(\mathfrak{B}, \bar{a})$. Now, since $\mathfrak{A}$ is $\aleph_0$-saturated, there is $\bar{b}' \in A$ such that $tp^{\mathfrak{A}}(\bar{a}, \bar{b}') = tp^{\mathfrak{B}}(\bar{a}, \bar{b})$. Now, for each $n \in \mathbb{N}$, we have $\exists x_1 \ldots \exists x_n (\bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_i \phi(x_i, \bar{a}) \wedge \bigwedge_i \psi(x_i, \bar{b}))$ and $\exists x_1 \ldots \exists x_n (\bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_i \phi(x_i, \bar{a}) \wedge \bigwedge_i \neg\psi(x_i, \bar{b}))$ in $tp^{\mathfrak{A}}(\bar{a}, \bar{b}')$, so $\psi(\mathfrak{A}, \bar{b}')$ is an infinite, co-infinite subset of $\phi(\mathfrak{A}, \bar{a})$, a contradiction. □

A potential difficulty with the notions of minimal and strongly minimal sets is that we cannot, in general, express the idea that a formula has infinitely many realizations within a first-order language. However, if we are interested in subsets of a model of a theory with no Vaughtian pairs, we can get around this. The key is the following result, which shows that in this context, having infinitely many realizations is equivalent to having some finite number of realizations.

Theorem 3.38 *Suppose that T is an $\mathcal{L}$-theory with no Vaughtian pairs. Let $\mathfrak{A} \vDash T$, and let $\phi(\bar{x}, \bar{y})$ be a formula with parameters from A. There is a number n such that if $\bar{a} \in A$ and $|\phi(\mathfrak{A}, \bar{a})| > n$, then $\phi(\mathfrak{A}, \bar{a})$ is infinite.*

proof: Toward a contradiction, suppose otherwise. Then for each $n \in \mathbb{N}$ there is some $\bar{a}_n \in A$ such that $\phi(\mathfrak{A}, \bar{a}_n)$ is a finite set of size at least n. We work in the language $\mathcal{L}^* = \mathcal{L} \cup \{U\}$ for pairs of models introduced in the proof of lemma 3.5.

Let $\Gamma(\bar{v})$ be the $\mathcal{L}^*$-type containing T and asserting:

- U defines a proper $\mathcal{L}$-elementary submodel
- $\bigwedge_{i=1}^{m} U(v_i)$
- there are infinitely many elements $\bar{x}$ such that $\phi(\bar{x},\bar{v})$
- $\phi(\bar{x},\bar{v}) \to \bigwedge_{i=1}^{k} U(x_i)$

Let $\mathfrak{B}$ be a proper elementary extension of $\mathfrak{A}$ (such exists, for example, by the Upward Löwenheim-Skolem theorem). Since $\phi(\mathfrak{A},\bar{a}_n)$ is finite, there is some $N \in \mathbb{N}$ such that

$$\mathfrak{A} \models \exists \bar{x}_1 \ldots \exists \bar{x}_N \left(\left(\bigwedge_{1 \le i < j \le N} \bar{x}_i \neq \bar{x}_j \right) \wedge \forall \bar{x} \left(\phi(\bar{x},\bar{a}_n) \to \bigvee_{i=1}^{N} \bar{x} = \bar{x}_i \right) \right)$$

Thus, this sentence is also true in $\mathfrak{B}$, so, as $\mathfrak{A} \subseteq \mathfrak{B}$, we have $\phi(\mathfrak{B},\bar{a}_n) = \phi(\mathfrak{A},\bar{a}_n)$. Say $\Delta(\bar{v}) \subseteq \Gamma(\bar{v})$ is finite. Then for n sufficiently large, $\bar{a}_n$ realizes $\Delta(\bar{v})$ in $(\mathfrak{B},\mathfrak{A})$. Thus, $\Gamma(\bar{v})$ is finitely satisfiable and hence satisfiable.

Since $\Gamma(\bar{v})$ is satisfiable, there is an $\mathcal{L}^*$-structure $\mathfrak{C}^*$ such that some $\bar{c} \in C^*$ realizes $\Gamma(\bar{v})$. As in the proof of lemma 3.5 we build $\mathcal{L}$-structures $\mathfrak{A}_0 \subseteq \mathfrak{B}_0$ from $\mathfrak{C}^*$. Note that the realization $\bar{c}$ of $\Gamma(\bar{v})$ from $\mathfrak{C}^*$ is contained in the universe of $\mathfrak{B}_0$. Thus, from the assertions of Γ, it follows that $\mathfrak{B}_0 \succ \mathfrak{A}_0$, $\phi(\mathfrak{A}_0,\bar{c})$ is infinite and $\phi(\mathfrak{B}_0,\bar{c}) = \phi(\mathfrak{A}_0,\bar{c})$. This means $(\mathfrak{B}_0,\mathfrak{A}_0)$ is a Vaughtian pair of models of T, contrary to hypothesis. □

Corollary 3.39 *If T has no Vaughtian pairs, then any minimal formula is strongly minimal.*

proof: Let $\phi(\bar{x})$ be a minimal formula over some $\mathfrak{A} \models T$ (where ϕ may contain parameters). Suppose toward a contradiction that there is an elementary extension $\mathfrak{B} \succ \mathfrak{A}$, a $\bar{b} \in B$ and an $\mathcal{L}$-formula $\psi(\bar{x},\bar{y})$ such that $\psi(\mathfrak{B},\bar{b})$ is an infinite, co-infinite subset of $\phi(\mathfrak{B})$.

Then by lemma 3.38, there is a number n such that for any elementary extension $\mathfrak{C}$ of $\mathfrak{A}$ and $\bar{c} \in \mathfrak{C}$, $\psi(\mathfrak{C},\bar{c})$ is an infinite co-infinite subset of $\phi(\mathfrak{C})$ if and only if $|\psi(\mathfrak{C},\bar{c}) \cap \phi(\mathfrak{C})| > n$ and $|\neg\psi(\mathfrak{C},\bar{c}) \cap \phi(\mathfrak{C})| > n$. Now, however, since $\phi(\mathfrak{A})$ is a minimal set, we have that

$$\mathfrak{A} \models \forall \bar{v}(|\psi(\mathfrak{A},\bar{v}) \cap \phi(\mathfrak{A})| \le n \vee |\neg\psi(\mathfrak{A},\bar{v}) \cap \phi(\mathfrak{A})| \le n).$$

Because this statement is a first order statement and $\mathfrak{B} \succ \mathfrak{A}$ this statement must also be true in $\mathfrak{B}$, a contradiction. □

Putting the last several results together, we deduce the following corollary.

Corollary 3.40 *If T is ω-stable and has no Vaughtian pairs, then for any $\mathfrak{A} \models T$ there is a strongly minimal formula over $\mathfrak{A}$. In particular, there is a strongly minimal formula with parameters from $\mathfrak{A}_0$, the prime model of T.*

3.5 Morley's Categoricity Theorem

We establish Morley's Categoricity Theorem by establishing a characterization of uncountably categorical theories due to Baldwin and Lachlan. To start, we prove the following lemma.

Lemma 3.41 *If T has no Vaughtian pairs, $\mathfrak{A} \models T$ and $X \subseteq A^n$ is infinite and definable then no proper elementary substructure of $\mathfrak{A}$ contains X. If, in addition, T is ω-stable, then $\mathfrak{A}$ is prime over X.*

proof: Let $\phi(\bar{x})$ be a formula that defines X. Say $\mathfrak{B}$ is a proper elementary submodel of $\mathfrak{A}$ containing X. Then we have $X = \phi(\mathfrak{A}) = \phi(\mathfrak{B})$ and so $(\mathfrak{A}, \mathfrak{B})$ is a Vaughtian pair of models of T, contrary to hypothesis.

Now say T is ω-stable. In this case, theorem 2.15 gives that there exists a $\mathfrak{B} \prec \mathfrak{A}$ such that $\mathfrak{B}$ is prime over X. As T has no Vaughtian pairs, it must be that $\mathfrak{B} = \mathfrak{A}$, so $\mathfrak{A}$ is prime over X. □

We are now ready to prove Baldwin and Lachlan's characterization of uncountable categorical theories.

Theorem 3.42 *Let T be a complete theory in a countable language with infinite models, and let κ be an uncountable cardinal. Then T is κ-categorical if and only if T is ω-stable and has no Vaughtian pairs.*

proof: If T is κ-categorical for some uncountable κ, then by corollaries 3.21 and 3.22, T is ω-stable and has no Vaughtian pairs.

Conversely, assume that T is ω-stable and has no Vaughtian pairs. By theorem 2.13, since T is ω-stable, T has a prime model $\mathfrak{A}_0$. By lemma 3.37 and corollary 3.39 there is a strongly minimal formula $\phi(x)$ with parameters from $\mathfrak{A}_0$.

Now suppose that $\mathfrak{A}, \mathfrak{B}$ are models of T, each of size κ for some uncountable κ. We will show that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Since $\mathfrak{A}_0$ is prime, it embeds elementarily into both $\mathfrak{A}$ and $\mathfrak{B}$. Thus, we may assume without loss of generality that both $\mathfrak{A}$ and $\mathfrak{B}$ are elementary extensions of $\mathfrak{A}_0$. Consider $\phi(\mathfrak{A})$. If $|\phi(\mathfrak{A})| < \kappa$, then by the Downward Löwenheim-Skolem theorem we can find $\mathfrak{C} \prec \mathfrak{A}$ such that $\phi(\mathfrak{A})$ is contained in $\mathfrak{C}$ and $|\mathfrak{C}| = |\phi(\mathfrak{A})| < \kappa$. This contradicts the conclusion of lemma 3.41, so we conclude that $\phi(\mathfrak{A})$ must have cardinality κ. Similarly $|\phi(\mathfrak{B})| = \kappa$. By lemma 3.35, then, we must have $dim(\phi(\mathfrak{A})) = dim(\phi(\mathfrak{B})) = \kappa$. Thus, by theorem 3.36, there is a partial elementary bijection $f : \phi(\mathfrak{A}) \to \phi(\mathfrak{B})$.

Now, lemma 3.41 gives that $\mathfrak{A}$ is prime over $\phi(\mathfrak{A})$, so we can extend f to an elementary function $f^* : \mathfrak{A} \to \mathfrak{B}$. Since f^* is elementary, $f^*(\mathfrak{A})$ is an elementary substructure of $\mathfrak{B}$. Further, because $f^*(\phi(\mathfrak{A})) = \phi(\mathfrak{B})$, $\phi(\mathfrak{B}) \subseteq f^*(\mathfrak{A})$. By lemma 3.41, $\mathfrak{B}$ has no proper elementary substructure containing $\phi(\mathfrak{B})$ and so we must have $f^*(\mathfrak{A}) = \mathfrak{B}$. Thus, f^* is an isomorphism, so $\mathfrak{A} \cong \mathfrak{B}$. Since $\mathfrak{A}, \mathfrak{B}$ were arbitrary models of T of size κ, this means that T is κ-categorical, as desired. □

From this characterization we easily deduce Morely's Categoricity Theorem

Theorem 3.43 (Morley's Categoricity Theorem) *If T is a complete theory in a countable language then T is κ-categorical for some uncountable cardinal κ if and only if T is λ-categorical for every uncountable cardinal λ.*

proof: Say T is complete and κ-categorical for some $\kappa \geq \aleph_1$. Then by the preceding theorem, T is ω-stable and has no Vaughtian pairs. But now, for any uncountable λ, the preceding gives that because T is ω-stable and has no Vaughtian pairs, T is λ-categorical. Thus, T κ-categorical for some $\kappa \geq \aleph_1$ implies that T is λ-categorical for all $\lambda \geq \aleph_1$. The converse is obvious. □

Chapter 4

Further Directions

In this chapter, we try to give a brief sketch of where one might go next, now that we have proved Morley's Categoricity Theorem. Proofs will be deferred. Throughout the chapter, T will be a complete theory with infinite models in a countable first-order language $\mathcal{L}$. The discussion is based on Marker [2], except for the discussion of Shelah's Main Gap theorem, which is based on a paper of L. Harrington and M. Makkai [1].

A powerful notion for studying ω-stable theories is the concept of *forking*. Before we can give a definition of this concept, we will need several other definitions.

Definition 4.1 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and $\phi(\bar{x})$ an $\mathcal{L}_A$-formula. We define $RM^{\mathfrak{A}}(\phi)$, the* **Morley rank** *of ϕ in $\mathfrak{A}$, as follows:*
First, for α an ordinal, we define $RM^{\mathfrak{A}}(\phi) \geq \alpha$ if:

1. *$RM^{\mathfrak{A}}(\phi) \geq 0$ if and only if $\phi(\mathfrak{A}) \neq \varnothing$*

2. *if α is a limit ordinal, $RM^{\mathfrak{A}}(\phi) \geq \alpha$ if and only if $RM^{\mathfrak{A}}(\phi) \geq \beta$ for all $\beta < \alpha$*

3. *if α is any ordinal $RM^{\mathfrak{A}}(\phi) \geq \alpha + 1$ if and only if there are $\mathcal{L}_A$-formulas $\psi_i(\bar{x})$ for $i \in \mathbb{N}$ such that $\psi_1(\mathfrak{A}), \psi_2(\mathfrak{A}), \ldots$ is an infinite family of pairwise disjoint subsets of $\phi(\mathfrak{A})$ and $RM^{\mathfrak{A}}(\psi) \geq \alpha$ for all i*

Now, if $\phi(\mathfrak{A}) = \varnothing$ then $RM^{\mathfrak{A}}(\phi) = -1$; if $RM^{\mathfrak{A}}(\phi) \geq \alpha$ and $RM^{\mathfrak{A}}(\phi) \ngeq \alpha + 1$, then $RM^{\mathfrak{A}}(\phi) = \alpha$; if $RM^{\mathfrak{A}}(\phi) \geq \alpha$ for all ordinals α, then $RM^{\mathfrak{A}}(\phi) = \infty$.

An unsatisfactory feature of this definition is its dependence on a particular model $\mathfrak{A}$ of T. It can be shown, however, that if $\mathfrak{B}, \mathfrak{C} \geq \mathfrak{A}$ are $\aleph_0$-saturated models of T, then for any $\mathcal{L}_A$-formula $RM^{\mathfrak{B}}(\phi) = RM^{\mathfrak{C}}(\phi)$. Thus, we define the *Morley rank* $RM(\phi)$ to be the Morley rank of ϕ in an arbitrary $\aleph_0$-saturated extension of T.

Having defined Morley rank for a formula, it is a simple matter to extend the concept of Morley rank to definable sets:

Definition 4.2 *If $\mathfrak{A} \models T$ and $X \subseteq A^n$ is definable by the $\mathcal{L}_A$-formula $\phi(\bar{x})$ we define $RM(X) = RM(\phi)$.*

A few basic facts about the Morley ranks of definable sets are as follows:

Lemma 4.3 *Let X, Y be definable subsets of A^n for some $\mathfrak{A} \models T$. Then*

1. *if $X \subseteq Y$, then $RM(X) \leq RM(Y)$*
2. *$RM(X \cup Y) = max\{RM(X), RM(Y)\}$*
3. *if $X \neq \varnothing$, then $RM(X) = 0$ if and only if X is finite*

When concepts such as Morley rank, which *a priori* depend on the particular model being considered, are involved one can often become bogged down in uninteresting and unenlightening details of proving that the (apparent) dependence on the model can be avoided (such as by passing to an appropriately saturated elementary extension). To avoid this, it is a common practice in model theory to fix a so-called *Monster model* $\mathbb{M}$, such that $\mathbb{M} \models T$, $\mathbb{M}$ is saturated and $\mathbb{M}$ is "very large," i.e. of some cardinality large enough that any model we consider can be considered as an elementary submodel of $\mathbb{M}$ (while we have not discussed it in this work, it can be shown that if a model $\mathbb{M}$ is κ-saturated for $\kappa \geq \aleph_0$, any model of size less than κ can be elementarily embedded in $\mathbb{M}$).

This approach suffers from the problem that an arbitrary theory T may or may not have arbitrarily large saturated models, hence may not have a model large enough to server as our "Monster model." For us, this will not be an obstacle since we will be interested in ω-stable theories and it can be shown that if T is ω-stable then T has arbitrarily large saturated models. Throughout the rest of the chapter, we let $\mathbb{M}$ be our Monster model of T. From now on, we think of $RM(\phi)$ as being $RM^{\mathbb{M}}(\phi)$.

A notion related to that of Morley rank is Morley degree.

Proposition 4.4 *Let ϕ be an $\mathcal{L}_{\mathbb{M}}$-formula with $RM(\phi) = \alpha$ for some ordinal α. Then there is $d \in \mathbb{N}$ such that if $\psi_1(\mathbb{M}), \ldots, \psi_n(\mathbb{M})$ are $\mathcal{L}_{\mathbb{M}}$-formulas such that each $\psi_i(\mathbb{M}) \subseteq \phi(\mathbb{M})$ and $RM(\psi_i) = \alpha$, then $n \leq d$*

Definition 4.5 *For ϕ, d as above, we call d the* **Morley degree** *of ϕ and write $deg_M(\phi) = d$.*

The notions of Morley rank and degree can be extended from formulas to types.

Definition 4.6 *If $p \in S_n^{\mathbb{M}}(A)$ then the* **Morley rank** *of p is $RM(p) = \inf\{RM(\phi) : \phi \in p\}$. If $RM(p)$ is an ordinal, then the* **Morley degree** *of p is $deg_M(p) = \inf\{deg_M(\phi) : \phi \in p \text{ and } RM(\phi) = RM(p)\}$.*

Having defined the Morley rank of a type, we are in a position to define forking.

Definition 4.7 *Let T be a complete ω-stable theory. Let $A \subseteq B \subset \mathbb{M}$, $p \in S_n^{\mathbb{M}}(A)$, $q \in S_n^{\mathbb{M}}(B)$ be such that $p \subseteq q$. If $RM(p) < RM(q)$, then we say that q is a* **forking** *extension of p and that p forks over B. If $RM(p) = RM(q)$ we say that q is a* **nonforking** *extension of p.*

We have the following existence theorem:

Theorem 4.8 *Suppose that $p \in S_n^{\mathbb{M}}(A)$ and $A \subseteq B$. Then*

1. *There is $q \in S_n^{\mathbb{M}}(B)$ a nonforking extension of p.*
2. *There are at most $deg_M(p)$ nonforking extensions of p in $S_n^{\mathbb{M}}(B)$ and, if $\mathfrak{M}$ is an $\aleph_0$-saturated model with $A \subseteq M$, there are exactly $deg_M(p)$ nonforking extensions of p in $S_n^{\mathbb{M}}(M)$.*
3. *There is at most one $q \in S_n^{\mathbb{M}}(B)$, a nonforking extension of p with $deg_M(p) = deg_M(q)$. In particular, if $deg_M(p) = 1$, then there is a unique nonforking extension of p in $S_n^{\mathbb{M}}(B)$.*

Forking allows us to define a notion of independence for elements of models of ω-stable theories.

Definition 4.9 *Let $A, B \subset \mathbb{M}$ and let $\bar{a} \in \mathbb{M}$. We say that $\bar{a}$ is* **independent** *from B over A if $tp^{\mathbb{M}}(\bar{a}/A)$ does not fork over $A \cup B$, i.e., if $RM(tp^{\mathbb{M}}(\bar{a}/A)) = RM(tp^{\mathbb{M}}(\bar{a}/A \cup B))$. In this case we write $\bar{a} \downarrow_A B$.*

This notion of independence has several nice properties:

Lemma 4.10 *Let $\bar{a}, \bar{b}, \bar{c} \in \mathbb{M}$, $A, B, C \subset \mathbb{M}$, $B \subseteq C$. Then we have*

- ***Monotnicity*** *If $\bar{a} \downarrow_A B$, then $\bar{a} \downarrow_A C$*
- ***Transitivity*** *$\bar{a} \downarrow_A \bar{b}, \bar{c}$ if and only if $\bar{a} \downarrow_A \bar{b}$ and $\bar{a} \downarrow_{A,\bar{b}} \bar{c}$*
- ***Finite Basis*** *$\bar{a} \downarrow_A B$ if and only if $\bar{a} \downarrow_A B_0$ for all finite $B_0 \subseteq B$*
- ***Symmetry*** *If $\bar{a} \downarrow_A \bar{b}$, then $\bar{b} \downarrow_A \bar{a}$*

These notions of independence and forking are useful for analyzing the so called *spectrum* of a theory T: the number $I(T, \kappa)$ of non-isomorphic models of T of size κ. This analysis, however, is beyond the scope of the present work. To give a flavor of this area, we close by stating a theorem of Shelah concerning $I(T, \kappa)$.

Theorem 4.11 (Shelah's Main Gap Theorem) *For a countable complete theory T, the spectrum function $I(T, -)$ is either $I(T, \aleph_\alpha) = 2^{\aleph_\alpha}$ or it satisfies the inequality $I(T, \aleph_\alpha) < \beth_{\omega_1}(|\omega + \alpha|)$ for all $\alpha \geq 1$.*

Bibliography

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