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ON SOME PROPERTIES OF SKEW MCCOY RINGS (50 pages)

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In this dissertation, we study the McCoy properties. The first part of this dissertation concerns the effect on the condition of σ-compatibility or σ-semicompatibility to skew McCoy rings. We first show that for any semicommutative right (or left) artinian ring which is σ-semicompatible with an epimorphism σ, its Jacobson radical is σ-skew McCoy. From this fact, we see that the Jacobson radical of semicommutative artinian rings is always right McCoy. Also, we provide a general sufficient condition for σ-compatible semicommutative rings to be σ-skew McCoy. We also prove that σ-compatible right duo rings are σ-skew McCoy. Moreover, we show that for any σ-compatible regular ring, the properties of the σ-skew McCoy and the right McCoy are equivalent.

In the second part of this dissertation, we consider skew Camillo rings. We prove that every σ-compatible 2-primal ring is σ-skew Camillo. As a corollary, we get that σ-compatible semicommutative rings are σ-skew Camillo. Also, we investigate the relationships between semicommutative, matrix rings and linearly σ-skew Camillo rings.

In the third part of this dissertation, we focus on the McCoy property in the setting of Ore extensions. We show that the Jacobson radical of (σ, δ)-compatible right duo, local left (or right) artinian rings is (σ, δ)-skew McCoy. We also prove that if the group ring $K[Q_8]$ (where K is a field of characteristic 0 and Q_8 is the quaternion group with order 8) is (σ, δ)-compatible with a tracial σ-derivation δ, then $K[Q_8]$ is (σ, δ)-skew McCoy if and only if there exists no solution in K for the equation $1 + x^2 + y^2 = 0$.

ON SOME PROPERTIES OF SKEW MCCOY RINGS

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CHAPTER 1

Introduction

In 1942, N. H. McCoy showed that if a ring R is commutative and a polynomial $f(x) \in R[x]$ is a zero-divisor, then it has a nonzero annihilator in R [32, Theorem 2]. But, this result may fail without the assumption of commutativity on R. For example [20], let $R := M_2(\mathbb{C})$ be the ring of 2×2 matrices over the field of complex numbers $\mathbb{C}$ and take

$$f(x) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x.$$

Then we can see that for any $r \in R$, $f(x)r = 0$ implies $r = 0$. However, if we take

$$g(x) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x,$$

then $f(x)g(x) = 0$. We note that $M_2(\mathbb{C})$ is not commutative. This example says that the condition of commutativity in the McCoy Theorem is essential. Up to now, the McCoy theorem has been generalized in many different ways. We first review those attempts.

1.1 Definitions and preliminaries

Throughout this dissertation, we regard R as an associative ring with identity.

Following Nielsen [34], a ring R is said to be *right McCoy* (resp., *left McCoy*) if for any two polynomials $f(x)$, $g(x) \in R[x]\backslash\{0\}$ with $f(x)g(x) = 0$,

there exists a nonzero element $r \in R$ such that $f(x)r$=0 (resp., $rg(x) = 0$).

If a ring R satisfies both right and left McCoy, then R is called McCoy. The McCoy rings have been widely researched and the relationships with various rings have been discovered. We introduce several classes of rings closely related to McCoy ring:

A ring R is called *Armendariz* [37] if for any two polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfying $f(x)g(x) = 0$, we have $a_i b_j = 0$ for any $0 \le i \le m$ and $0 \le j \le n$;

reduced if there exists no nonzero nilpotent element in R; *reversible* if $ab = 0$ implies $ba = 0$ for every a, $b \in R$; *semicommutative* if $ab = 0$ implies $aRb = 0$ for every a, $b \in R$; *polynomial semicommutative* if $R[x]$ is semicommutative.

Then we can see the following well-known implications between these classes of rings (see for example [8]):

$$\text{reduced} \Longrightarrow \begin{cases} \text{reversible} \Longrightarrow \begin{cases} \text{McCoy;} \\ \text{semicommutative.} \end{cases} \\ \\ \text{polynomial semicommutative} \Longrightarrow \begin{cases} \text{McCoy;} \\ \text{semicommutative.} \end{cases} \\ \\ \text{Armendariz} \Longrightarrow \text{McCoy.} \end{cases}$$

We denote an endomorphism of R by σ and a σ-derivation of R by δ, which is an additive map δ of R satisfying $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$. We also denote a skew polynomial ring with the usual addition and multiplication defined by $xr = \sigma(r)x$ for all $r \in R$ as $R[x; \sigma]$. According to [10], a ring R is called *σ-skew McCoy* if for any two polynomials $f(x)$, $g(x) \in R[x; \sigma]\backslash\{0\}$ with $f(x)g(x) = 0$,

$$\text{there exists a nonzero element } r \in R \text{ such that } f(x)r = 0. \tag{1.1}$$

Also, a ring R is said to be *linearly σ-skew McCoy* if we restrict degrees of both $f(x)$ and $g(x)$ not to be greater than 1 in the equation (1.1). The study of σ-skew McCoy rings has been extensively done by many authors (for example, [1], [2], [3], [10], [12], [35], [36], [38], [40], [41], [42]).

We also denote an Ore extension with the usual addition and multiplication defined by $xr = \sigma(r)x + \delta(r)$ for all $r \in R$ as $R[x; \sigma; \delta]$. Following [16], a ring R is called *(σ, δ)-skew McCoy* if for any two polynomials

$$f(x) = \sum_{i=0}^{m} a_i x^i \quad \text{and} \quad g(x) = \sum_{j=0}^{n} b_j x^j \ \in R[x; \sigma; \delta]\backslash\{0\}$$

with $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ such that $a_i x^i r = 0$ for all $0 \leq i \leq m$. In the property of (σ, δ)-skew McCoy, if $\delta = 0_R$ (i.e., the zero mapping), then (σ, δ)-skew McCoy becomes σ-skew McCoy, and further if both $\delta = 0_R$ and $\sigma = I_R$ (i.e., the identity mapping), then it becomes right McCoy.

On the other hand, many authors have considered an endomorphism σ with certain conditions. We here review these endomorphisms: an endomorphism σ of a ring R is said to be *rigid* [26] if

$$a\sigma(a) = 0 \implies a = 0 \quad \text{for any } a \in R,$$

and a ring R is called *σ-rigid* if there exists a rigid endomorphism σ of R. Also, a ring R is called *right* (resp., *left*) *σ-reversible* [9] if

$$ab = 0 \implies b\sigma(a) = 0 \;\; (\text{resp., } \sigma(b)a = 0) \quad \text{for any } a,\, b \in R,$$

and R is called *σ-reversible* if it satisfies both right and left σ-reversible. A ring R is called *σ-compatible* ([5], [17]) if

$$ab = 0 \iff a\sigma(b) = 0 \quad \text{for any } a,\, b \in R,$$

and is called *δ-compatible* if

$$ab = 0 \implies a\delta(b) = 0 \quad \text{for any } a,\, b \in R.$$

Furthermore, R is called *(σ, δ)-compatible* if it satisfies both σ-compatible and δ-compatible. Considering the property of a module homomorphism, i.e., $\sigma(ab) = a\sigma(b)$, in general, this property does not satisfy the condition that $a\sigma(b) = 0 \Rightarrow ab = 0$. From the fact, we would like to give a weaker condition than σ-compatibility: Let σ be an endomorphism of a ring R. Then we say that R is *σ-semicompatible* if

$$ab = 0 \implies a\sigma(b) = 0 \quad \text{for any } a, b \in R.$$

The following relations among these several endomorphism conditions are useful.

Proposition 1.1.1. Let σ be an endomorphism of a ring R.

(a) If R is σ-semicompatible reversible and σ is a monomorphism, then R is σ-compatible.

(b) If R is σ-semicompatible reduced and σ is a monomorphism, then R is σ-rigid.

(c) If R is reversible, then R is right σ-reversible if and only if R is σ-semicompatible.

Proof. (a) Suppose $a\sigma(b) = 0$ for a, $b \in R$. But since R is reversible and σ-semicompatible, we have $\sigma(b)a = 0$ and $\sigma(ba) = 0$. Then by a monomorphism σ, we get $ba = 0$, which implies $ab = 0$.

(b) Let $a\sigma(a) = 0$ for $a \in R$. Since R is reduced, R is reversible. Thus $\sigma(a)a = 0$. By σ-semicompatibility and a monomorphism σ, $\sigma(a^2) = 0$, and hence $a^2 = 0$. But since R is reduced, $a = 0$.

(c) Assume R is right σ-reversible and let $ab = 0$ for a, $b \in R$. Then by reversibility, $ba = 0$. Since R is right σ-reversible, $a\sigma(b) = 0$. Conversely, suppose R is σ-semicompatible and let $ab = 0$ for a, $b \in R$. By reversibility, $ba = 0$, which implies $b\sigma(a) = 0$ by σ-semicompatibility.

□

However, we cannot guarantee that σ-semicompatibility implies σ-compatibility. It may fail even in the case of commutative rings. For example, we can see that there exists a σ-semicompatible commutative ring which is not σ-compatible.

Example 1.1.2. Take $R := \mathbb{Z}_6[x]$ and define an endomorphism σ of R by

$$\sigma\colon f(x) \mapsto f(0). \tag{1.2}$$

Clearly, R is commutative. But since $\mathbb{Z}_6$ is reduced, it is Armendariz. Then we have

$$f(x)g(x) = 0 \implies f(x)g(0) = 0 \implies f(x)\sigma(g(x)) = 0,$$

which implies that R is σ-semicompatible. However, R is not σ-compatible. To see this, let

$$f(x) = 2 \quad \text{and} \quad g(x) = 3 + x.$$

Then $f(x)\sigma(g(x)) = 0$, but $f(x)g(x) = 2x \neq 0$.

In this example, we note that Z_6 is cyclic. Thus, we may ask whether there exists a non-cyclic example. In fact, the answer to this question is affirmative. To see this, take

$$R := (\mathbb{Z}_2 \oplus \mathbb{Z}_2)[x].$$

Note that $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is non-cyclic commutative and reduced. If we define an endomorphism σ of R as in (1.2), we can see that R is σ-semicompatible by the same method in Example 1.1.2. However, R is not σ-compatible. Indeed, take

$$f(x) = (0,\,1) + (0,\,1)x \quad \text{and} \quad g(x) = (1,\,0) + (0,\,1)x.$$

Then

$$f(x)\sigma(g(x)) = f(x)g(0) = ((0,\,1) + (0,\,1)x)(1,\,0) = (0,\,0),$$

but

$$f(x)g(x) = (0,\,1)x + (0,\,1)x^2 \neq (0,\,0).$$

Further, we can also see that there exists a ring which is σ-semicompatible for some nontrivial endomorphism σ, but not σ-compatible for any endomorphism σ other than the identity map.

Example 1.1.3. Let $R := \mathbb{Z}_2[x]$ and I be the ideal generated by x^2. Take

$$S := R/I.$$

If we regard the elements of $\mathbb{Z}_2[x]$ as their images in the quotient ring S, then we have

$$S = \{0,\,1,\,x,\,x+1\}.$$

For an endomorphism σ of S, if we take

$$\sigma(x) := 1 \;\text{ or }\; x+1,$$

then

$$0 = \sigma(x^2) = \sigma(x)\sigma(x) = \begin{cases} 1 & (\text{if } \sigma(x) := 1) \\ (x+1)^2 = 1 & (\text{if } \sigma(x) := x+1), \end{cases}$$

which is a contradiction. Therefore, for an endomorphism σ to be well-defined, we must assign $\sigma(x)$ to one of the elements 0 and x. If $\sigma(x) = x$, then σ becomes the identity map. For the case of

$$\sigma(x) = 0, \tag{1.3}$$

the endomorphism σ is clearly not injective, which implies that there exists no monomorphism of S other than the identity map. But since all endomorphisms satisfying the compatible condition must be injective, we can say that there does not exist any endomorphism σ of S, except for the identity map, satisfying that S is σ-compatible. However, if we take (1.3), we can see that S is σ-semicompatible. To see this, we must check that for any s_1, $s_2 \in S$,

$$s_1 s_2 = 0 \implies s_1 \sigma(s_2) = 0.$$

But the only case that $s_1 s_2 = 0$ in S is $s_1 = s_2 = x$. Thus, it suffices to show that $x\sigma(x) = 0$, which is indeed true by our choice (1.3), Therefore, S is σ-semicompatible.

From these examples above, the condition of σ-semicompatibility may be a slightly more useful than that of σ-compatibility in the ring-theoretic aspect. Throughout this dissertation, we place a emphasis on the roles of both σ-semicompatibility and σ-compatibility in the McCoy property.

1.2 Overview

The structure of this dissertation is as follows:

In Chapter 2, we focus on the relations between the σ-skew McCoy condition and several types of rings, for example, semicommutative, duo, reversible, and regular rings. We first note that the Jacobson radical of a ring does not need to be right McCoy (see Example 2.1.1). However, we get a positive result for semicommutative artinian rings. The first main theorem of Chapter 2 is a generalization of this result to the σ-skew McCoy property: more concretely, for a σ-semicompatible semicommutative right (or left) artinian ring R with an epimorphism σ, the Jacobson radical $J(R)$ is σ-skew McCoy [28]. Therefore, it follows that the Jacobson radical of semicommutative artinian rings is right McCoy. On the other hand, it was shown [11, Theorem 8.2] that right duo rings are right McCoy. By contrast, right duo rings need not be σ-skew McCoy. However, as the second main theorem of Chapter 2, we show that if R is σ-compatible right duo, then R is σ-skew McCoy [28]. The third main theorem of Chapter 2 concerns the connection between (von Neumann) regular rings and σ-skew McCoy rings. We here prove that for a σ-compatible regular ring R, R is right McCoy if and only if R is σ-skew McCoy [28].

In Chapter 3, we study the property of (linearly) σ-skew Camillo rings. We recall [8] that a ring R is called *right Camillo* (resp., *left Camillo*) if for any two polynomials $f(x)$, $g(x) \in R[x]\backslash\{0\}$ with $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ such that $f(x)r = 0$ or $g(x)r = 0$ (resp., $rf(x) = 0$ or $rg(x) = 0$), and R is called *Camillo* if it is both left and right Camillo. We say that a ring R is *σ-skew Camillo* if for any two polynomials $f(x)$, $g(x) \in R[x;\sigma]\backslash\{0\}$ with $f(x)g(x) = 0$,

$$\text{there exists a nonzero element } r \in R \text{ such that } f(x)r = 0 \text{ or } g(x)r = 0. \tag{1.4}$$

Moreover, R is called *linearly σ-skew Camillo* if we restrict degrees of both $f(x)$ and $g(x)$ not to be greater than 1 in the equation (1.4). We recall that a ring R is called *2-primal* if the prime radical of R is the same as the set of nilpotent elements in R. Then it is well-known (see [8]) that

$$\text{semicommutative} \Longrightarrow \text{2-primal} \Longrightarrow \text{Camillo}. \tag{1.5}$$

The second implication was proven by V. Camillo and P. Nielsen [11, Theorem 9.2]. The main theorem of Chapter 3 extends this theorem: σ-compatible 2-primal rings are σ-skew Camillo [29]. Also, we prove that σ-semicompatible semicommutative rings are linearly σ-skew Camillo [28] and that matrix rings over a division ring are linearly σ-skew Camillo for every endomorphism σ [28].

Basically, Chapter 4 is based on [30]. In this chapter, we introduce a (σ, δ)-skew nil-McCoy property, and then show that (σ, δ)-compatible right duo rings are (σ, δ)-skew nil-McCoy (see Theorem 4.1.3). As an application of this result, the main theorem of Chapter 4 is that for any (σ, δ)-compatible right duo, local left (or right) artinian ring R, the Jacobson radical $J(R)$ is (σ, δ)-skew McCoy (see Theorem 4.2.2). Also, we introduce a notion of tracial σ-derivation, and then prove that if R is a (σ, δ)-compatible regular ring with a tracial σ-derivation δ, then R is reversible if and only if R is (σ, δ)-skew McCoy (see Theorem 4.3.2). We also give an interesting corollary about a group ring: if K is a field of characteristic 0 and the group ring $K[Q_8]$ (where Q_8 is the quaternion group) is (σ, δ)-compatible with a tracial σ-derivation δ, then $K[Q_8]$ is (σ, δ)-skew McCoy if and only if there exists no solution in K for the equation $1 + x^2 + y^2 = 0$ (see Corollary 4.4.1).

1.3 Notations

In this dissertation, we write:

R	an associative ring with identity
σ	an endomorphism of R
δ	a σ-derivation of R
$R[x]$	the ring of polynomials over R
$R[x;\sigma]$	the ring of skew polynomials over R
$R[x;\sigma;\delta]$	an Ore extension over R
$P(R)$	the prime radical of R
$J(R)$	the Jacobson radical of R
$N(R)$	the set of nilpotent elements in R
$Nil^*(R)$	the upper nilradical of R
$\mathit{coeff}(f(x))$	the set of all coefficients in $f(x)$
$\deg(f(x))$	the degree of $f(x)$
$M_n(R)$	the ring of n×n matrices over R
$R_1 \oplus R_2$	the direct sum of two rings R_1 and R_2
Z_n	the ring of residue classes modulo n

CHAPTER 2

Skew McCoy rings

In this chapter, we study the σ-skew McCoy property for semicommutative, duo, reversible, and regular rings. Chapter 2 is basically based on [28], while Example 2.1.1 and Problem 2.1.8, Lemma 2.1.9, Corollary 2.1.10 can be found in [30] and [29], respectively. First, we show that if R is a σ-semicompatible semicommutative right (or left) artinian ring with an epimorphism σ, then the Jacobson radical $J(R)$ is σ-skew McCoy (see Theorem 2.1.6). From this result, we get that the Jacobson radical of semicommutative artinian rings is right McCoy. Second, we prove that every σ-compatible right duo ring is σ-skew McCoy (see Theorem 2.2.2). Lastly, we show that if R is a σ-compatible regular ring, then R is σ-skew McCoy if and only if R is right McCoy (see Theorem 2.4.1).

2.1 Semicommutative rings

We recall that a ring R is said to be *semicommutative* if $ab = 0$ implies $aRb = 0$ for every a, $b \in R$. It was proven [18, Theorem 2.2] that if R is polynomial semicommutative, then R is right McCoy. But in general, semicommutative rings are not right McCoy [34, Section 3], and hence we do not expect it to be σ-skew McCoy. However, it was shown [16, Theorem 3.11] that σ-compatible semicommutative rings are linearly σ-skew McCoy. We first observe the semicommutative rings under the condition of σ-semicompatibility.

On the other hand, we note that the Jacobson radical of a ring does not need to be right McCoy as we can see in the following example.

Example 2.1.1. Let $R := K[[x]]$ be the ring of formal power series over a field K. Note that the Jacobson radical $J(R) =$ the set of formal power series with zero constant term. If we take

$$S := \begin{pmatrix} R & 0 \\ R & R \end{pmatrix},$$

then we note that

$$J(S) = \begin{pmatrix} J(R) & 0 \\ R & J(R) \end{pmatrix}.$$

Now, let

$$f(X) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -x \end{pmatrix} X \quad \text{and} \quad g(X) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} x & 0 \\ 1 & 0 \end{pmatrix} X \ \in J(S)[X].$$

Then

$$f(X)g(X) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -x \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] X$$
$$+ \left[\begin{pmatrix} 0 & 0 \\ 1 & -x \end{pmatrix}\begin{pmatrix} x & 0 \\ 1 & 0 \end{pmatrix}\right] X^2 = \left[\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -x & 0 \end{pmatrix}\right] X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

but we can see that if $f(X)r = 0$, where $r := \begin{pmatrix} a(x) & 0 \\ b(x) & c(x) \end{pmatrix} \in J(S)$, then

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a(x) & 0 \\ b(x) & c(x) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a(x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 \\ 1 & -x \end{pmatrix}\begin{pmatrix} a(x) & 0 \\ b(x) & c(x) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a(x) - xb(x) & -xc(x) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which implies $a(x) = b(x) = c(x) = 0$, i.e., there does not exist any nonzero element $r \in J(S)$ such that $f(X)r = 0$. Therefore, $J(S)$ is not right McCoy.

Thus, we do not expect that the Jacobson radical of a ring is McCoy. However, the result is affirmative for semicommutative artinian rings. To see this, we need the following lemmas.

Lemma 2.1.2. *Let R be a σ-semicompatible semicommutative ring and let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma]$ satisfying $f(x)g(x) = 0$. Then*

$$a_i \sigma^i(b_0^{i+1}) = 0 \quad \textit{for all } 0 \le i \le m. \tag{2.1}$$

Proof. We apply the technique of the proofs in [34, Lemma 1] and [10, Lemma 8(2)]. When $i = 0$, clearly $a_0 b_0 = 0$. Now, suppose by induction that $a_i \sigma^i(b_0^{i+1}) = 0$ for all $i < k$. From $f(x)g(x) = 0$, the coefficient of the term with degree k is 0, i.e.,

$$\sum_{j=0}^{k} a_j \sigma^j(b_{k-j}) = 0. \tag{2.2}$$

Also, we have $a_i \sigma^i(b_0^k) = 0$ for all $i < k$, and then it follows from the condition of σ-semicompatibility that $a_i \sigma^k(b_0^k) = 0$ for all $i < k$. But since R is semicommutative, we get

$$a_i \sigma^i(b_{k-i}) \sigma^k(b_0^k) = 0 \text{ for all } i < k.$$

Multiplying both sides of (2.2) by $\sigma^k(b_0^k)$ on the right, we get

$$\sum_{j=0}^{k} a_j \sigma^j(b_{k-j}) \sigma^k(b_0^k) = a_k \sigma^k(b_0) \sigma^k(b_0^k) = a_k \sigma^k(b_0^{k+1}) = 0.$$

This completes the induction step, and hence we are done. □

Note that Lemma 2.1.2 may be false without the condition of σ-semicompatibility. To see this, take

$$R := \mathbb{Z}_2 \oplus \mathbb{Z}_2 \tag{2.3}$$

with the usual addition and multiplication, and take an endomorphism σ of R defined by

$$\sigma\colon (a, b) \mapsto (b, a) \quad \text{for each } (a, b) \in R. \tag{2.4}$$

Then R is clearly commutative, and hence it is semicommutative. Now, if we let

$$f(x) = (1, 0) + (1, 0)x \quad \text{and} \quad g(x) = (0, 1) + (1, 0)x \ \in R[x; \sigma], \tag{2.5}$$

then $f(x)g(x) = 0$, but

$$a_1 \sigma(b_0^2) = (1, 0) \neq (0, 0).$$

This contradicts (2.1).

Lemma 2.1.3. *Let R be a σ-semicompatible semicommutative ring. If $f(x) = \sum_{i=0}^{m} a_i x^i$ and $0 \neq g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma]$ such that $f(x)g(x) = 0$ and $b_0^{m+1} \neq 0$, then there exists a nonzero element $r \in R$ such that $f(x)r = 0$.*

Proof. From Lemma 2.1.2, we have $a_i\sigma^i(b_0^{i+1}) = 0$ for all $0 \leq i \leq m$. If we multiply by $\sigma^i(b_0^{m-i})$ for each i, then we obtain $a_i\sigma^i(b_0^{m+1}) = 0$ for all $0 \leq i \leq m$. Therefore,

$$f(x)(b_0^{m+1}) = \sum_{i=0}^{m} a_i\sigma^i(b_0^{m+1})x^i = 0.$$

By the assumption $b_0^{m+1} \neq 0$, we are done. □

Note that $N(R) :=$ the set of nilpotent elements in R and *coeff*$(f(x)) :=$ the set of all coefficients of $f(x)$.

Lemma 2.1.4. *Let R be a σ-semicompatible semicommutative ring. If $f(x) \in R[x;\sigma]$ is a left zero-divisor with coeff$(f(x)) \in N(R)$, then there exists a nonzero element $r \in R$ such that $f(x)r = 0$.*

Proof. Suppose $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma]\backslash\{0\}$ satisfying that $f(x)g(x) = 0$ and $a_i \in N(R)$ for every $0 \leq i \leq m$. Without loss of generality, we assume $b_0 \neq 0$. If $b_0^{m+1} \neq 0$, then from Lemma 2.1.3, the result follows.

Instead, we suppose $b_0^{m+1} = 0$. Then there exists $k \in \mathbb{N}$ satisfying $b_0^k \neq 0 = b_0^{k+1}$. Since $b_0^k \neq 0$, if $f(x)b_0^k = 0$ then we are done. Let $f(x)b_0^k \neq 0$. If j_1 is the smallest integer such that $a_{j_1}\sigma^{j_1}(b_0^k) \neq 0$, then we get $a_{j_1}b_0^k \neq 0$ by σ-semicompatibility. By assumption, $a_{j_1} \in N(R)$, which implies that there exists the smallest integer $n_1 \in \mathbb{N}$ satisfying $a_{j_1}^{n_1}b_0^k \neq 0 = a_{j_1}^{n_1+1}b_0^k$. Since R is semicommutative, we obtain $a_i\sigma^i(a_{j_1}^{n_1}b_0^k) = a_i\sigma^i(a_{j_1}^{n_1})\sigma^i(b_0^k) = 0$ for all $i < j_1$. Then we see

$$f(x)(a_{j_1}^{n_1}b_0^k) = \sum_{i=j_1}^{m} a_i\sigma^i(a_{j_1}^{n_1}b_0^k)x^i.$$

If we assume $f(x)(a_{j_1}^{n_1}b_0^k) = 0$, we are done. Thus, assume $f(x)(a_{j_1}^{n_1}b_0^k) \neq 0$ and let $j_2 \in \mathbb{N}$ be the smallest integer such that $a_{j_2}\sigma^{j_2}(a_{j_1}^{n_1}b_0^k) \neq 0$. Note that $j_2 > j_1$. Then we obtain $a_{j_2}a_{j_1}^{n_1}b_0^k \neq 0$. Again, since $a_{j_2} \in N(R)$, we see that there exists the smallest integer $n_2 \in \mathbb{N}$ satisfying $a_{j_2}^{n_2}a_{j_1}^{n_1}b_0^k \neq 0 = a_{j_2}^{n_2+1}a_{j_1}^{n_1}b_0^k$. We also get $a_i\sigma^i(a_{j_2}^{n_2}a_{j_1}^{n_1}b_0^k) = a_i\sigma^i(a_{j_2}^{n_2})\sigma^i(a_{j_1}^{n_1}b_0^k) = 0$ for all $i < j_2$, which implies

$$f(x)(a_{j_2}^{n_2}a_{j_1}^{n_1}b_0^k) = \sum_{i=j_2}^{m} a_i\sigma^i(a_{j_2}^{n_2}a_{j_1}^{n_1}b_0^k)x^i.$$

If $f(x)(a_{j_2}^{n_2}a_{j_1}^{n_1}b_0^k) = 0$, we are done. Thus, we assume $f(x)(a_{j_2}^{n_2}a_{j_1}^{n_1}b_0^k) \neq 0$. Repeating the above procedure a finite number of times, we obtain

$$f(x)(a_{j_t}^{n_t}a_{j_{t-1}}^{n_{t-1}}\cdots a_{j_1}^{n_1}b_0^k) = 0, \text{ where } a_{j_t}^{n_t}a_{j_{t-1}}^{n_{t-1}}\cdots a_{j_1}^{n_1}b_0^k \neq 0 \text{ for some } t \in \mathbb{N}.$$

This concludes the proof. □

Corollary 2.1.5. *Let R be a semicommutative ring. Then for any two polynomials $f(x)$, $g(x) \in R[x]\backslash\{0\}$ satisfying that $f(x)g(x) = 0$ and coeff$(f(x)) \in N(R)$, there exists a nonzero element $r \in R$ such that $f(x)r = 0$.*

Proof. It follows from Lemma 2.1.4 with σ to be the identity map. □

We now prove:

Theorem 2.1.6. *Let R be a σ-semicompatible semicommutative right (or left) artinian ring where σ is an epimorphism of R. Then the Jacobson radical $J(R)$ is σ-skew McCoy. In particular, the Jacobson radical of semicommutative artinian rings is right McCoy.*

Proof. Note that the Jacobson radical of right or left artinian rings is nilpotent and that every surjective endomorphism preserves the Jacobson radical, i.e., $\sigma(J(R)) \subseteq J(R)$. Thus, we can see that σ is also an endomorphism of $J(R)$. By the hereditary property of both semicommutativity and σ-semicompatibility, $J(R)$ is also σ-semicompatible and semicommutative. Then since all elements of $J(R)$ are nilpotent, $J(R)$ is σ-skew McCoy by Lemma 2.1.4. The second statement comes from the first statement with the identity map σ. □

Note that $Nil^*(R) :=$ the upper nilradical of a ring R. Then we shall say that a ring R is *J-McCoy* (resp., *N-McCoy*) if for any nonzero polynomials $f(x)$, $g(x) \in J(R[x])$ (resp., $Nil^*(R)[x]$) with $f(x)g(x) = 0$,

there exists a nonzero $r \in R$ such that $f(x)r = 0$.

On the other hand, it is known [4, Theorem 1] that there exists a nil ideal I of R such that $J(R[x]) = I[x]$, which implies

$$J(R[x]) \subseteq Nil^*(R)[x]. \tag{2.6}$$

Thus, we can see that

$$\text{right McCoy} \Longrightarrow N\text{-McCoy} \Longrightarrow J\text{-McCoy}.$$

Further, it is well-known that for any ring R satisfying the Köthe conjecture, $J(R[x]) = Nil^*(R)[x]$ (see [4], [13], [25]). Hence, when R satisfies the Köthe conjecture, we have

$$R \text{ is } J\text{-McCoy} \Longleftrightarrow R \text{ is } N\text{-McCoy}.$$

We then have:

Corollary 2.1.7. *Every semicommutative ring is J-McCoy.*

Proof. It is immediate from (2.6) and Corollary 2.1.5. □

Next, we recall that in general, semicommutative rings are not right McCoy (see [34, Section 3]). Thus, we do not expect that σ-compatible semicommutative rings are σ-skew McCoy. Then we may ask the following:

Problem 2.1.8. *Find a general sufficient condition for σ-compatible semicommutative rings to be σ-skew McCoy.*

For an answer to Problem 2.1.8, we first observe:

Lemma 2.1.9. *Let R be a σ-compatible semicommutative ring and let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma]\backslash\{0\}$ with $f(x)g(x) = 0$. Suppose k is a positive integer less than or equal to $n+1$. If for each $j = 0, 1, \ldots, k-1$, $b_j \in N(R)$ and $b_j^{p_j} \neq 0 = b_j^{p_j+1}$ for some $p_j \geq 0$, then*

$$a_i b_k^{i+1} b_{k-1}^{p_{k-1}} b_{k-2}^{p_{k-2}} \cdots b_0^{p_0} = 0 \quad \text{for all } 0 \leq i \leq m. \tag{2.7}$$

Proof. We proceed by induction on i. If $i = 0$, then there are two cases to consider: in the case of $p_0 \neq 0$, the equation (2.7) is clear from $a_0 b_0 = 0$ and the semicommutative condition of R, and for the case of $p_0 = 0$, let $l \in \mathbb{N}$ be the smallest integer less than or equal to k satisfying $b_l \neq 0$. Then we have $a_0 b_l = 0$ from $f(x)g(x) = 0$. Thus, the equation (2.7) holds true because R is semicommutative. Now, suppose that the equation (2.7) holds for all $i < u$. From $f(x)g(x) = 0$, we can see that the coefficient of the term with degree $u + k$ is 0, i.e.,

$$\sum_{i=0}^{u+k} a_i \sigma^i(b_{u+k-i}) = 0.$$

If we multiply both sides by $b_k^u b_{k-1}^{p_{k-1}} b_{k-2}^{p_{k-2}} \cdots b_0^{p_0}$, then we have

$$0 = \sum_{i=0}^{u+k} a_i \sigma^i(b_{u+k-i}) b_k^u b_{k-1}^{p_{k-1}} b_{k-2}^{p_{k-2}} \cdots b_0^{p_0}. \tag{2.8}$$

On the other hand, by the inductive assumption together with the semicommutative condition, we get

$$a_i b_k^u b_{k-1}^{p_{k-1}} b_{k-2}^{p_{k-2}} \cdots b_0^{p_0} = 0 \text{ for all } i < u.$$

By semicommutativity of R again,

$$a_i\sigma^i(b_{u+k-i})b_k^u b_{k-1}^{p_{k-1}} b_{k-2}^{p_{k-2}} \cdots b_0^{p_0} = 0 \text{ for all } i < u. \tag{2.9}$$

Furthermore, from the assumption that $b_j^{p_j+1} = 0$ for all $0 \leq j \leq k-1$, we also have

$$a_i\sigma^i(b_{u+k-i})b_k^u b_{k-1}^{p_{k-1}} b_{k-2}^{p_{k-2}} \cdots b_0^{p_0} = 0 \text{ for all } u+1 \leq i \leq u+k. \tag{2.10}$$

Thus, by (2.9) and (2.10), we rewrite the equation (2.8):

$$a_u\sigma^u(b_k)b_k^u b_{k-1}^{p_{k-1}} b_{k-2}^{p_{k-2}} \cdots b_0^{p_0} = 0,$$

which implies from σ-compatibility of R that

$$a_u b_k^{u+1} b_{k-1}^{p_{k-1}} b_{k-2}^{p_{k-2}} \cdots b_0^{p_0} = 0.$$

This completes the proof. □

On the other hand, in Lemma 2.1.3, we showed that whenever R is σ-semicompatible semicommutative, if $f(x) = \sum_{i=0}^m a_i x^i$ and $g(x) = \sum_{j=0}^n b_j x^j \in R[x;\sigma]\backslash\{0\}$ with $f(x)g(x) = 0$ and $b_0^{m+1} \neq 0$, then there exists a nonzero element $r \in R$ such that $f(x)r = 0$. We now generalize it under the σ-compatible condition, which gives a general sufficient condition for σ-compatible semicommutative rings to be σ-skew McCoy.

Corollary 2.1.10. *Let R be a σ-compatible semicommutative ring and let $f(x) = \sum_{i=0}^m a_i x^i$ and $g(x) = \sum_{j=0}^n b_j x^j \in R[x;\sigma]\backslash\{0\}$ with $f(x)g(x) = 0$. Suppose that $b_0^{m+1} \neq 0$ or*

$$b_k^{m+1} b_{k-1}^{p_{k-1}} b_{k-2}^{p_{k-2}} \cdots b_0^{p_0} \neq 0 \quad \text{for some } k \geq 1 \tag{2.11}$$

with $b_j \in N(R)$ for each $0 \leq j \leq k-1$ and $b_j^{p_j} \neq 0 = b_j^{p_j+1}$ for some $p_j \geq 0$. Then there exists a nonzero element $r \in R$ such that $f(x)r = 0$.

Proof. If $b_0^{m+1} \neq 0$, then it is obvious from Lemma 2.1.3. Assume that $b_0^{m+1} = 0$ and (2.11) is satisfied. If we take $r := b_k^{m+1} b_{k-1}^{p_{k-1}} b_{k-2}^{p_{k-2}} \cdots b_0^{p_0}$, then by Lemma 2.1.9 together with the condition of semicommutativity, we have $f(x)r = 0$. □

2.2 Duo rings

We recall that a ring R is called *right* (resp., *left*) duo if every right (resp., left) ideal is two-sided. This definition is equivalent to the property that for any r, $s \in R$, there exists an element $r' \in R$ (resp., $s' \in R$) such that $rs = sr'$ (resp., $rs = s'r$). Note that right duo rings are right McCoy (see [11, Theorem 8.2]). But, they need not to be σ-skew McCoy. To see this, we put R, σ, $f(x)$, and $g(x)$ as in (2.3), (2.4), and (2.5), respectively. Then $f(x)g(x) = 0$, but there does not exist any nonzero element $r \in R$ such that $f(x)r = 0$. However, under the condition of σ-compatibility, right duo rings are σ-skew McCoy (see Theorem 2.2.2).

We first extend the result of [11, Lemma 5.4].

Lemma 2.2.1. *Let R be a σ-semicompatible semicommutative ring and let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma]$ such that $f(x)g(x) = 0$. Then*

$$a_0^{i+1} b_i = 0 \quad \textit{for each } 0 \leq i \leq n.$$

Proof. It is similar to the proof in Lemma 2.1.2. □

We now prove:

Theorem 2.2.2. *σ-compatible right duo rings are σ-skew McCoy.*

Proof. We apply the method of the proofs in [11, Theorem 8.2] and [34, Theorem 2]. Let R be a σ-compatible right duo ring and let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $0 \neq g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma]$ satisfying $f(x)g(x) = 0$ and $a_m \neq 0$. Without loss of generality, assume $b_0 \neq 0$. We denote

$$I_{g(x)} := \text{the right ideal generated by the set } \{a,\ \sigma^n(a) : a \in \textit{coeff}(g(x)) \text{ and } n \in \mathbb{N}\}.$$

We proceed by induction on the degree of $f(x)$ to show that there exists a nonzero element $r \in I_{g(x)}$ such that $f(x)r = 0$. If deg($f(x)$)=0, then we have $f(x)b_0 = 0$. We now suppose by induction that for any polynomial $p(x) \in R[x;\sigma]$ with $\deg(p(x)) = k$ for all $k < m$, if there exists a nonzero polynomial $q(x) \in R[x;\sigma]$ such that $p(x)q(x) = 0$, then there exists a nonzero element $r \in I_{q(x)}$ such that $p(x)r = 0$. We divide the proof into two cases:

Case 1 ($a_0 g(x) = 0$): We have $a_0 I_{g(x)} = 0$ because R is σ-semicompatible. Take

$$f_1(x) := \sum_{i=1}^{m} a_i x^{i-1}$$

and

$$g_1(x) := \sigma(b_0) + \sigma(b_1)x + \sigma(b_2)x^2 + \cdots + \sigma(b_n)x^n.$$

Note that $g_1(x) \neq 0$ by a monomorphism of σ. We observe

$$0 = f(x)g(x) = (a_1x + a_2x^2 + \cdots + a_mx^m)g(x) = f_1(x)xg(x) = f_1(x)g_1(x)x,$$

which implies $f_1(x)g_1(x) = 0$. Since we also have $\deg(f_1(x)) < \deg(f(x))$, by induction, there exists a nonzero element $r \in I_{g_1(x)}$ such that $f_1(x)r = 0$. But since $I_{g_1(x)} \subseteq I_{g(x)}$, it follows that $r \in I_{g(x)}$ and $f(x)r = 0$ by σ-semicompatibility.

Case 2 ($a_0g(x) \neq 0$): Let j be the smallest integer satisfying $a_0b_j \neq 0$. Then from Lemma 2.2.1, there exists $k \geq 1$ such that $a_0^kb_j \neq 0 = a_0^{k+1}b_j$. But since R is right duo, there exists $s_1 \in R$ such that $a_0^kb_j = b_js_1$. We take $g_1(x) := g(x)s_1$. Then we can see that $g_1(x) \neq 0$ because R is σ-compatible and that $f(x)g_1(x) = 0$. Similarly, if we assume $a_0g_1(x) = 0$, then we see $a_0I_{g_1(x)} = 0$. Take

$$f_1(x) := \sum_{i=1}^{m} a_ix^{i-1}$$

and

$$g_1'(x) := \sigma(b_0s_1) + \sigma(b_1\sigma(s_1))x + \sigma(b_2\sigma^2(s_1))x^2 + \cdots + \sigma(b_n\sigma^n(s_1))x^n.$$

We observe

$$0 = f(x)g_1(x) = f_1(x)xg_1(x) = f_1(x)g_1'(x)x,$$

which implies $f_1(x)g_1'(x) = 0$. From the fact that $\deg(f_1) < \deg(f)$ and $g_1' \neq 0$, by induction, there exists a nonzero element $r_1 \in I_{g_1'(x)}$ such that $f_1(x)r_1 = 0$. We also have $I_{g_1'(x)} \subseteq I_{g(x)}$, so that $r_1 \in I_{g(x)}$ and $f(x)r_1 = 0$, and hence the result follows. Instead, let $a_0g_1(x) \neq 0$ and let j_1 be the smallest integer such that $a_0(b_{j_1}\sigma^{j_1}(s_1)) \neq 0$. Note that $j < j_1$ because we have $a_0(b_js_1) = 0$. From Lemma 2.2.1, we again obtain a nonzero integer $n_1 \geq 1$ such that $a_0^{n_1}(b_{j_1}\sigma^{j_1}(s_1)) \neq 0 = a_0^{n_1+1}(b_{j_1}\sigma^{j_1}(s_1))$, and then it follows from the right duo property that there exists $s_2 \in R$ such that $a_0^{n_1}(b_{j_1}\sigma^{j_1}(s_1)) = (b_{j_1}\sigma^{j_1}(s_1))s_2$. Put $g_2(x) := g_1(x)s_2$, and then $fg_2 = 0$. If we repeat the above procedure a finite number of times, we obtain $a_0g_t(x) = 0$, where $g_t(x) := g(x)(s_1s_2\cdots s_t)$ for some $t \in \mathbb{N}$, which implies that there exists a nonzero element $r_t \in I_{g(x)}$ such that $f(x)r_t = 0$. This completes the proof. □

2.3 Reversible rings

Now, we focus on the σ-skew McCoy property for reversible rings. We first recall [16, Theorem 3.6] that

$$\text{every } \sigma\text{-compatible reversible ring is } \sigma\text{-skew McCoy.} \tag{2.12}$$

Then we can strengthen (2.12) using σ-semicompatibility with a monomorphism σ in place of σ-compatibility.

Lemma 2.3.1. *Let R be a σ-semicompatible reversible ring where σ is a monomorphism of R. Then R is σ-skew McCoy.*

Proof. We claim that such rings are σ-compatible. To see this, let $a, b \in R$ such that $a\sigma(b) = 0$. Since R is reversible, $\sigma(b)a = 0$, which implies from σ-semicompatibility that $\sigma(ba) = 0$. Also, since σ is a monomorphism, we have ba=0, and hence $ab = 0$. Therefore, by (2.12), R is σ-skew McCoy. $\square$

Example 2.3.2. (a) Let R be a domain and σ be a monomorphism of R. Then we can see that R is σ-semicompatible reversible. Therefore, by Lemma 2.3.1, R is σ-skew McCoy.

(b) Let R be a reduced ring and take

$$S := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R \right\}.$$

Define an endomorphism σ of S by

$$\sigma\colon \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Then S is reversible by [24, Proposition 1.6]). Also, we see that σ is a monomorphism of S. Indeed,

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = 0 \implies \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} = 0,$$

which implies $a = b = 0$. Further, we have S is σ-semicompatible. To see this, if we let

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = 0, \text{ then } \begin{cases} ac = 0, \\ ad + bc = 0. \end{cases} \tag{2.13}$$

Then from (2.13), $a(ad+bc)=0$. But since R is reduced, it is semicommutative. Thus, $abc=0$ by $ac=0$. Then $aad=0$, which implies $(ad)^2=0$, and hence $ad=0$ because R is reduced. Also, we have $bc=0$ by (2.13). Therefore, we obtain

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \sigma\left(\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & -d \\ 0 & c \end{pmatrix} = 0.$$

Thus, by Lemma 2.3.1, S is σ-skew McCoy.

(c) Lemma 2.3.1 may fail if the condition of σ-semicompatibility is omitted. To see this, we take R and σ as in (2.3) and (2.4), respectively. Then we can see that σ is an automorphism and R is reduced, but not σ-skew McCoy.

Then we can recapture the result of [10, Theorem 9].

Corollary 2.3.3. *Let R be a right σ-reversible and reversible ring where σ is a monomorphism of R. Then R is σ-skew McCoy.*

Proof. It is enough to show that if R is right σ-reversible and reversible, then R is σ-semicompatible. Indeed, assume R is right σ-reversible and reversible. Suppose $a, b \in R$ such that $ab=0$. Since R is reversible, $ba=0$. Then we have $a\sigma(b)=0$ by right σ-reversibility, which implies that R is σ-semicompatible. Thus, from Lemma 2.3.1, R is σ-skew McCoy. □

2.4 Regular rings

We recall that a ring R is called (*von Neumann*) *regular* [21] if for any element $a \in R$, there exists an element $b \in R$ satisfying $a = aba$. Then it was shown [27, Theorem 20] that if R is a regular ring, then

$$R \text{ is right McCoy} \Longleftrightarrow R \text{ is reduced.} \tag{2.14}$$

We now generalize (2.14) with the σ-skew McCoy property and show that whenever a ring is σ-compatible regular, the notions of the σ-skew McCoy and the right McCoy are equivalent.

Theorem 2.4.1. *If R is a σ-compatible regular ring, then R is right McCoy if and only if R is σ-skew McCoy.*

Proof. From (2.14), it is enough to prove that

$$R \text{ is reduced} \Longleftrightarrow R \text{ is } \sigma\text{-skew McCoy.}$$

Let R be reduced, which implies R is reversible. From the fact that σ-compatibility implies σ-semicompatibility with a monomorphism σ, we have that R is σ-skew McCoy by Lemma 2.3.1.

Conversely, suppose R is σ-skew McCoy. Assume to the contrary that R is not reduced. Let $a \in R \backslash \{0\}$ satisfying $a^2 = 0$. Since R is regular, there exists an element $b \in R$ such that $a = aba$. Put $c := bab$, then $aca = a$, $cac = c$, and $c \neq 0$. If we let

$$f(x) = \sigma(a) - (1 - ca)x \quad \text{and} \quad g(x) = ac + \sigma(c)x,$$

then we have $f(x) \neq 0 \neq g(x)$ and

$$f(x)g(x) = \sigma(a)ac + (\sigma(a)\sigma(c) - (1 - ca)\sigma(ac))x - (1 - ca)\sigma^2(c)x^2.$$

We observe the following with the condition of σ-compatibility:

(i) $\sigma(a)ac = 0$ from the fact that $\sigma(a^2) = 0$ implies $\sigma(a)a = 0$.

(ii) $\sigma(a)\sigma(c) - (1 - ca)\sigma(ac) = \sigma(ac) - \sigma(ac) + ca\sigma(ac) = c(a\sigma(a))\sigma(c) = 0$ from the fact that $a^2 = 0$ implies $a\sigma(a) = 0$.

(iii) $(1 - ca)\sigma^2(c) = 0$ from the fact that $(1 - ca)c = 0$.

Therefore, we get $f(x)g(x) = 0$. But since R is σ-skew McCoy, there exists a nonzero element $r \in R$ satisfying $f(x)r = 0$. Then

$$f(x)r = (\sigma(a) - (1 - ca)x)r = \sigma(a)r - (1 - ca)\sigma(r)x = 0,$$

which implies

$$\begin{cases} \sigma(a)r = 0, \\ (1 - ca)\sigma(r) = 0. \end{cases} \tag{2.15}$$

Then by σ-compatibility, we get $a\sigma(r) = 0$ from $\sigma(a)r = 0$. Thus, from (2.15), $\sigma(r) = ca\sigma(r) = 0$, and hence we obtain $r = 0$, which is a contradiction. Therefore, R is reduced. □

Example 2.4.2. Let $S = \mathbb{Z}_p(t)$ be the field of fractions of the polynomial ring $\mathbb{Z}_p[t]$, where p is a prime, and let ϕ be the Frobenius endomorphism of S defined by $\phi(s) = s^p$. Take

$$R := S \oplus S$$

and define an endomorphism σ of R by

$$\sigma\colon (a,\, b) \mapsto (\phi(a),\, \phi(b)).$$

Clearly, R is commutative, which implies R is right McCoy. Also, we can easily see that R is σ-compatible regular. Therefore, by Theorem 2.4.1, R is σ-skew McCoy.

On the other hand, we observe the right linearly McCoy property. It was shown [23, Proposition 2.14] that if R is a regular ring, then R is right linearly McCoy if and only if R is reduced. We show that if R is a σ-compatible regular ring, then R is linearly σ-skew McCoy if and only if R is reduced. To do so, we first extend [22, Lemma 3.4], which asserts that if R is a right linearly McCoy ring and $a \in R$ with $a^2 = 0$, then $1 - ra$ cannot be a left zero-divisor for all $r \in R$.

Lemma 2.4.3. *Let R be a σ-compatible linearly σ-skew McCoy ring. Then for any $a \in R$ with $a^2 = 0$, $1 - ra$ cannot be a left zero-divisor for all $r \in R$.*

Proof. We apply the method of the proof in [22, Lemma 3.4]. Assume to the contrary that $1 - ra$ is a left zero-divisor for some $r \in R$. Then there exists a nonzero $k \in R$ such that $(1 - ra)k = 0$. Take

$$f(x) = \sigma(a) + (1 - ra)x \quad \text{and} \quad g(x) = ak - \sigma(k)x.$$

Then $f(x) \neq 0 \neq g(x)$ and

$$f(x)g(x) = \sigma(a)ak + (-\sigma(a)\sigma(k) + (1 - ra)\sigma(ak))x + (-(1 - ra)\sigma^2(k))x^2.$$

Since R is σ-compatible, we have

(i) $\sigma(a)ak = 0$ because $0 = \sigma(a^2k) = \sigma(a)\sigma(ak)$, and hence $\sigma(a)ak = 0$.

(ii) $-\sigma(a)\sigma(k) + (1 - ra)\sigma(ak) = -ra\sigma(ak) = 0$ because $a\sigma(ak) = 0$.

(iii) $(1 - ra)\sigma^2(k) = 0$ because $(1 - ra)k = 0$.

Thus, we have $f(x)g(x) = 0$. Since R is linearly σ-skew McCoy, there exists $0 \neq s \in R$ such that $f(x)s = 0$. Then

$$f(x)s = (\sigma(a) + (1 - ra)x)s = \sigma(a)s + (1 - ra)\sigma(s)x = 0,$$

which implies $\sigma(s) = 0$, and thus $s = 0$, which is a contradiction. Therefore, the result follows. □

We then have:

Corollary 2.4.4. *Let R be a σ-compatible regular ring. Then R is linearly σ-skew McCoy if and only if R is reduced.*

Proof. Suppose R is linearly σ-skew McCoy. Assume to the contrary that there exists $0 \neq a \in R$ such that $a^2 = 0$. Since R is regular, there exists $b \in R$ such that $aba = a$. If we take $c := bab$, then we have $aca = a$ and $cac = c$. Thus, we get $c \neq 0$ and $(1 - ca)c = 0$, which contradicts Lemma 2.4.3. Conversely, it is immediate from Theorem 2.4.1 via (2.14). □

Corollary 2.4.5. *Let R be a σ-compatible regular ring. Then the following are equivalent:*

(i) R is right McCoy;

(ii) R is linearly σ-skew McCoy;

(iii) R is σ-skew McCoy.

Proof. It is immediate from Theorem 2.4.1 and Corollary 2.4.4 together with (2.14). □

CHAPTER 3

Skew Camillo rings

In this chapter, we consider the skew Camillo property. To do so, we first remind of the following definitions: a ring R is called *right Camillo* (resp., *left Camillo*) [8] if for any two polynomials $f(x)$, $g(x) \in R[x]\backslash\{0\}$ with $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ such that $f(x)r = 0$ or $g(x)r = 0$ (resp., $rf(x) = 0$ or $rg(x) = 0$).

We say that a ring R is *σ-skew Camillo* if for any two polynomials $f(x)$, $g(x) \in R[x;\sigma]\backslash\{0\}$ with $f(x)g(x) = 0$,

$$\text{there exists a nonzero element } r \in R \text{ such that } f(x)r = 0 \text{ or } g(x)r = 0. \tag{3.1}$$

Moreover, R is said to be *linearly σ-skew Camillo* if we restrict degrees of both $f(x)$ and $g(x)$ not to be greater than 1 in the equation (3.1).

We also recall that a ring R is called 2-primal if $P(R) = N(R)$. Then it was shown [11, Theorem 9.2] that

$$R \text{ is 2-primal} \Longrightarrow R \text{ is Camillo.} \tag{3.2}$$

Chapter 3 is essentially based on [29], while Section 3.2 can be found in [28]. First, we show that if R is σ-compatible 2-primal, then R is σ-skew Camillo (see Theorem 3.1.3), which extends the result of (3.2). By the result, we get that σ-compatible semicommutative rings are σ-skew Camillo, which generalizes a result by P. Nielsen [34, Theorem 4]. Second, we prove that if R is σ-semicompatible semicommutative, then R is linearly σ-skew Camillo (see Theorem 3.2.1) and that if R is a matrix ring over a division ring, then R is linearly σ-skew Camillo for any endomorphism σ (see Theorem 3.2.3).

3.1 2-primal rings

We first recall that a ring R is called *σ-skew Armendariz* [19] if for any two polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma]$ satisfying $f(x)g(x) = 0$,

$$a_i \sigma^i(b_j) = 0 \text{ for all } 0 \leq i \leq m \text{ and } 0 \leq j \leq n.$$

Then in order to prove that every σ-compatible 2-primal ring is σ-skew Camillo, we need the following lemmas.

Lemma 3.1.1. *Let R be a σ-semicompatible reduced ring with a monomorphism σ. Then R is σ-skew Armendariz.*

Proof. We use an idea of the proof in [7, Lemma 1]. Since $f(x)g(x) = 0$, we have

$$\begin{cases} a_0 b_0 = 0, \\ a_0 b_1 + a_1 \sigma(b_0) = 0, \\ a_0 b_2 + a_1 \sigma(b_1) + a_2 \sigma^2(b_0) = 0, \\ \qquad \vdots \\ a_0 b_{m+n} + a_1 \sigma(b_{m+n-1}) + a_2 \sigma^2(b_{m+n-2}) + \cdots + a_{m+n} \sigma^{m+n}(b_0) = 0, \end{cases} \tag{3.3}$$

where $a_k = 0$ for all $k > m$ and $b_k = 0$ for all $k > n$. Since R is reduced, R is reversible, which implies from $a_0 b_0 = 0$ that $b_0 a_0 = 0$. If we multiply the second equation of (3.3) by b_0 on the left, we obtain $b_0 a_1 \sigma(b_0) = 0$, so that $b_0 a_1 \sigma(b_0 a_1) = 0$. From the reversible condition, $\sigma(b_0 a_1) b_0 a_1 = 0$. Then $(b_0 a_1)^2 = 0$ by σ-semicompatibility with a monomorphism σ. Since R is reduced, $b_0 a_1 = 0$, and thus $a_1 b_0 = 0$. Then by σ-semicompatibility, we get $a_1 \sigma(b_0) = 0$. Again, multiplying the third equation of (3.3) by b_0 on the left, we also get $b_0 a_2 \sigma^2(b_0) = 0$, and hence $a_2 \sigma^2(b_0) = 0$ by the same way. If we repeat this process, we have $a_i \sigma^i(b_0) = 0$ for all $0 \leq i \leq m+n$. Then the equation (3.3) reduces to:

$$\begin{cases} a_0 b_1 = 0, \\ a_0 b_2 + a_1 \sigma(b_1) = 0, \\ a_0 b_3 + a_1 \sigma(b_2) + a_2 \sigma^2(b_1) = 0, \\ \qquad \vdots \\ a_0 b_{m+n} + a_1 \sigma(b_{m+n-1}) + \cdots + a_{m+n-1} \sigma^{m+n-1}(b_1) = 0. \end{cases} \tag{3.4}$$

By repeating the procedure from the equation (3.3) in the equation (3.4) with b_1 instead of b_0, we also have $a_i\sigma^i(b_1) = 0$ for all $0 \le i \le m+n-1$. Continuing this argument, we obtain $a_i\sigma^i(b_j) = 0$ for all $0 \le i \le m$ and $0 \le j \le n$. □

In [6, Proposition 2.1], it was shown that if R is a ring and $N(R)$ is an ideal, then $f(x)g(x) \in N(R)[x]$ implies $ab \in N(R)$ for all $a \in coeff(f(x))$ and $b \in coeff(g(x))$. We now apply the ideas of this result to extend it.

Lemma 3.1.2. *Let R be a σ-compatible ring and $N(R)$ be an ideal. Then for any $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma]$,*

$$f(x)g(x) \in N(R)[x;\sigma] \Longrightarrow a_i\sigma^i(b_j) \in N(R) \text{ for all } 0 \le i \le m \text{ and } 0 \le j \le n.$$

Proof. Suppose $f(x)g(x) \in N(R)[x;\sigma]$. Let $\bar{R} := R/N(R)$ and $\bar{\sigma}$ be an endomorphism of $\bar{R}$ defined by

$$\bar{\sigma} \colon r + N(R) \mapsto \sigma(r) + N(R).$$

Clearly, $\bar{R}$ is reduced and $\bar{f}(x)\bar{g}(x) = 0$, where $\bar{f}$ and $\bar{g}$ are the corresponding polynomials in $\bar{R}[x;\bar{\sigma}]$. We claim that $\bar{\sigma}$ is a monomorphism and $\bar{R}$ is $\bar{\sigma}$-semicompatible. To see this, let $\bar{\sigma}(r + N(R)) = N(R)$. Then $\sigma(r) \in N(R)$, which implies $r \in N(R)$ by a monomorphism σ. Hence, $\bar{\sigma}$ is a monomorphism. To show $\bar{\sigma}$-semicompatibility, we first claim that

$$ab \in N(R) \Longrightarrow a\sigma(b) \in N(R). \tag{3.5}$$

Indeed, if $(ab)^t = 0$ for some $t \ge 1$, then by σ-compatibility, we have $a\sigma(b(ab)^{t-1}) = 0$ and $a\sigma(b)\sigma((ab)^{t-1}) = 0$, and thus $a\sigma(b)(ab)^{t-1} = 0$. Continuing this process, we obtain $(a\sigma(b))^t = 0$, which shows (3.5). We now assume $(a+N(R))(b+N(R)) = N(R)$. Then since $ab \in N(R)$, we have $a\sigma(b) \in N(R)$ by (3.5). Then $(a + N(R))(\sigma(b) + N(R)) = N(R)$, which implies $(a + N(R))\bar{\sigma}(b + N(R)) = N(R)$. Therefore, $\bar{R}$ is $\bar{\sigma}$-semicompatible.

Now, if we apply Lemma 3.1.1 with $\bar{R}$ and $\bar{\sigma}$, then we see $a_i\sigma^i(b_j) \in N(R)$ for all $0 \le i \le m$ and $0 \le j \le n$. □

We are ready to prove the following:

Theorem 3.1.3. *σ-compatible 2-primal rings are σ-skew Camillo.*

Proof. Let R be a σ-compatible 2-primal ring. Suppose

$$f(x) = \sum_{i=0}^{m} a_i x^i \text{ and } g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma]\backslash\{0\} \text{ with } f(x)g(x) = 0.$$

From the 2-primal condition, we have $P(R) = N(R)$, so that $N(R)$ is an ideal. Note that $f(x)g(x) = 0 \in N(R)[x;\sigma]$. Thus, by Lemma 3.1.2, we get $a_i\sigma^i(b_j) \in N(R)$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. We claim that

$$a_i\sigma(b_j) \in N(R) \text{ for all } i,\ j. \tag{3.6}$$

To see this, since $(a_i\sigma^i(b_j))^t = 0$ for some $t \geq 1$, by σ-compatibility, we see

$$\begin{aligned}(a_i\sigma^i(b_j))^t = 0 &\Longrightarrow a_i\sigma^i(b_j(a_i\sigma^i(b_j))^{t-1}) = 0\\ &\Longrightarrow a_i\sigma(b_j)\sigma((a_i\sigma^i(b_j))^{t-1}) = 0\\ &\Longrightarrow a_i\sigma(b_j)(a_i\sigma^i(b_j))^{t-1} = 0.\end{aligned}$$

Repeating this argument, we obtain $(a_i\sigma(b_j))^t = 0$, which proves (3.6). Therefore, $a_i\sigma(b_j) \in P(R)$ for all i, j. Take

$$S := \{a_i\sigma(b_j) \mid 0 \leq i \leq m \text{ and } 0 \leq j \leq n\} \subseteq P(R). \tag{3.7}$$

Case 1 ($S = \{0\}$): Choose a nonzero coefficient b_s of $g(x)$ for some $0 \leq s \leq n$. Then by a monomorphism σ, $\sigma(b_s) \neq 0$. From the fact that $S = \{0\}$ and R is σ-compatible, we get $a_i\sigma^i(b_s) = 0$ for all $0 \leq i \leq m$, which implies

$$f(x)b_s = \sum_{i=0}^{m} a_i\sigma^i(b_s)x^i = 0.$$

Case 2 ($S \neq \{0\}$): Note that $P(R)$ is locally nilpotent, which implies S is nilpotent. Thus, there exists $k \geq 1$ satisfying $S^k \neq \{0\} = S^{k+1}$. If we take $r \in S^k\backslash\{0\}$ satisfying $b_j r = 0$ for all $0 \leq j \leq n$, then by σ-compatibility, we have $b_j\sigma^j(r) = 0$ for each j. Thus,

$$g(x)r = \sum_{j=0}^{n} b_j\sigma^j(r)x^j = 0.$$

Instead, suppose $b_c r \neq 0$ for some $0 \leq c \leq n$. Since $S^{k+1} = \{0\}$, $a_i\sigma(b_c)r = 0$ for all $0 \leq i \leq m$. Then from σ-compatibility, we get $a_i\sigma^i(b_c r) = 0$ for each i. Hence,

$$f(x)(b_c r) = \sum_{i=0}^{m} a_i\sigma^i(b_c r)x^i = 0.$$

We completes the proof. □

We recall that

an ideal P is *prime* if for any elements a, $b \in R$, $aRb \subseteq P$ implies $a \in P$ or $b \in P$,

a prime ideal P is *completely prime* if for any elements a, $b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$.

Then we would like to remark the following:

Remark 3.1.4. We claim that if R is a σ-compatible 2-primal ring and $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma]\backslash\{0\}$ with $f(x)g(x) = 0$, then for any minimal prime ideal P, we get

$$\{a_0, \cdots, a_m\} \subseteq P \text{ or } \{\sigma(b_0), \cdots, \sigma(b_n)\} \subseteq P. \tag{3.8}$$

Indeed, assume to the contrary that there exists a minimal prime ideal P satisfying $a_t \notin P$ and $\sigma(b_s) \notin P$ for some $0 \leq t \leq m$ and $0 \leq s \leq n$. From (3.6), $a_i\sigma(b_j) \in N(R)$ for all i, j. Then we have $a_t\sigma(b_s) \in N(R) = P(R)$, and hence $a_t\sigma(b_s) \in P$, which is a contradiction to the fact that all minimal prime ideals of R are completely prime if and only if R is 2-primal [39, Proposition 1.11]. Therefore, the result of (3.8) follows (see [11, the proof of Theorem 9.2]).

However, we do not guarantee that (3.8) holds for $P(R)$ in place of P. If (3.8) holds for $P(R)$, then we can prove Theorem 3.1.3 in a little different way. The outline of the different proof is as follows. If we suppose that $S := \{a_0, \cdots, a_m\} \subseteq P(R)$, then clearly $S \neq \{0\}$. But since S is nilpotent, there exists $t \geq 1$ such that $S^t \neq 0 = S^{t+1}$. Choose $s \in S^t\backslash\{0\}$. Then from the fact that $S^{t+1} = 0$ and R is σ-compatible, $a_i\sigma^i(s) = 0$ for all i. Therefore, $f(x)s = \sum_{i=0}^{m} a_i\sigma^i(s)x^i = 0$. Instead, suppose $\{\sigma(b_0), \cdots, \sigma(b_n)\} \subseteq P(R)$. Then similarly, we have $g(x)s' = 0$ for some $s' \in R\backslash\{0\}$.

On the other hand, by Theorem 3.1.3 and (1.5), we see that every σ-compatible semicommutative ring is σ-skew Camillo, which generalizes [34, Theorem 4]. However, since semicommutative rings need not to be right McCoy (see [34, Section 3]), we can conclude that in general, σ-compatible semicommutative rings are not σ-skew McCoy.

3.2 Linearly skew Camillo rings

According to [16, Theorem 3.11], σ-compatible semicommutative rings are linearly σ-skew McCoy. But if the condition of σ-compatibility is weakened to that of σ-semicompatibility, it is not obvious whether this result is true.

However, we can see that such rings are linearly σ-skew Camillo as shown in the following theorem.

Theorem 3.2.1. *If R is a σ-semicompatible semicommutative ring, then R is linearly σ-skew Camillo.*

Proof. Let $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x \in R[x;\sigma]\backslash\{0\}$ with $f(x)g(x) = 0$. Suppose $b_0 = 0$. Then $0 = f(x)g(x) = f(x)b_1x$, which implies $f(x)b_1 = 0$. But from $g(x) \neq 0$, we have $b_1 \neq 0$, and hence we are done. Now, let $b_0 \neq 0$. In this case, if $b_0^2 \neq 0$, then by Lemma 2.1.3, there exists $r \in R\backslash\{0\}$ such that $f(x)r = 0$. Instead, we assume $b_0^2 = 0$.

Case 1 ($b_1\sigma(b_0) = 0$): Then we have $g(x)b_0 = 0$.

Case 2 ($b_1\sigma(b_0) \neq 0$): Put $g_1(x) := g(x)b_0 = b_1\sigma(b_0)x$. Then since $f(x)g_1(x) = 0$, we have $f(x)(b_1\sigma(b_0)) = 0$. Therefore, the result follows. □

However, it is not clear whether Theorem 3.2.1 can be strengthened to "σ-skew Camillo" instead of "linearly σ-skew Camillo". But we can see a weakened version. To do that, we give the following definition: a ring R is *weakly σ-skew Camillo* if for any left zero-divisor $f(x) \in R[x;\sigma]\backslash\{0\}$, there exists a nonzero $g(x) \in \mathrm{r.ann}_{R[x;\sigma]}(f(x))$ satisfying $f(x)r = 0$ or $g(x)r = 0$ for some $r \in R\backslash\{0\}$, where $\mathrm{r.ann}_{R[x;\sigma]}(f(x)) :=$ the set of right annihilators of $f(x)$ in $R[x;\sigma]$.

We then have:

Theorem 3.2.2. *Every σ-semicompatible semicommutative ring is weakly σ-skew Camillo.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x;\sigma]\backslash\{0\}$ with $f(x)g(x) = 0$. Without loss of generality, assume that $g(x)$ is a nonzero polynomial of minimal degree such that $f(x)g(x) = 0$. Note that $b_0 \neq 0$. If we suppose $b_0^{m+1} \neq 0$, then from Lemma 2.1.3, there exists a nonzero $r \in R$ such that $f(x)r = 0$. Now, let $b_0^{m+1} = 0$. Then there exists $k \geq 1$ satisfying $b_0^k \neq 0 = b_0^{k+1}$. If $\deg(g(x)) = 0$, then $g(x)b_0^k = 0$. Thus, let $\deg(g(x)) \geq 1$. If we take

$$g_1(x) = g(x)b_0^k \quad \text{and} \quad g_2(x) = g_1(x)/x,$$

then clearly we have $0 = f(x)g_1(x) = f(x)g_2(x)x$, and hence $f(x)g_2(x) = 0$. But the case of $g_2(x) \neq 0$ leads to a contradiction by minimality of g. Thus, we can see $g_2(x) = 0$, which implies $g(x)b_0^k = 0$. This completes the proof. □

On the other hand, it was proven [8, Corollary 3.7] that if R is a matrix ring over a division ring, then R is linearly Camillo. We recall ([8], [14]) that a ring R is called *right eversible* if every right zero-divisor of R is a left zero-divisor. Then the proof of [8, Corollary 3.7] is basically proving that right eversible rings are linearly right Camillo (see [8, Proposition 3.6]). We can further show by adopting the method of the proof in [8, Proposition 3.6].

Theorem 3.2.3. *Every matrix ring over a division ring is linearly σ-skew Camillo for all endomorphism σ.*

Proof. Since matrix rings over a division ring are right eversible, it is enough to show that right eversible rings are linearly σ-skew Camillo. To see this, we assume R is right eversible. Let $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x \in R[x;\sigma]\backslash\{0\}$ satisfying $f(x)g(x) = 0$. We observe the following two cases.

Case 1 ($a_0 = 0$): From $f(x)g(x) = 0$, we have $a_1\sigma(b_0) = 0$ and $a_1\sigma(b_1) = 0$, which implies that $f(x)b_0 = 0$ and $f(x)b_1 = 0$. But since $g(x) \neq 0$, we see $b_0 \neq 0$ or $b_1 \neq 0$. Therefore, the result follows.

Case 2 ($a_0 \neq 0$): Note that b_0 is a right zero-divisor from $a_0b_0 = 0$. Then by assumption, there exists a nonzero $r \in R$ such that $b_0r = 0$. If $b_1\sigma(r) = 0$, then $g(x)r = (b_0+b_1x)r = b_0r+b_1\sigma(r)x = 0$. Instead, let $b_1\sigma(r) \neq 0$. Observe

$$0 = f(x)g(x)r = f(x)(b_0 + b_1x)r = f(x)(b_0r + b_1\sigma(r)x) = f(x)(b_1\sigma(r)x),$$

which implies $f(x)(b_1\sigma(r)) = 0$. This completes the proof. □

CHAPTER 4

McCoy property on Ore extensions

In this chapter, we consider the McCoy property from the aspect of the Ore extensions. Basically, Chapter 4 is based on [30]. First, we provide a notion of (σ, δ)-skew nil-McCoy property, and then show that if R is a (σ, δ)-compatible right duo ring, then R is (σ, δ)-skew nil-McCoy (see Theorem 4.1.3). Second, we prove that for a (σ, δ)-compatible right duo, local left (or right) artinian ring R, the Jacobson radical $J(R)$ is (σ, δ)-skew McCoy (see Theorem 4.2.2). Third, we show that whenever R is a (σ, δ)-compatible regular ring with a tracial σ-derivation δ, R is reversible if and only if R is (σ, δ)-skew McCoy (see Theorem 4.3.2). Lastly, we prove that when K is a field of characteristic 0 and the group ring $K[Q_8]$ is (σ, δ)-compatible with a tracial σ-derivation δ, $K[Q_8]$ is (σ, δ)-skew McCoy if and only if K contains no solution to the equation $1 + x^2 + y^2 = 0$ (see Corollary 4.4.1).

4.1 (σ, δ)-skew nil-McCoy rings

We recall the result of ([16, Theorem 3.6]): for a (σ, δ)-compatible ring R,

$$R \text{ is reversible } \Longrightarrow R \text{ is } (\sigma, \delta)\text{-skew McCoy.} \tag{4.1}$$

We do not know whether the implication (4.1) holds for the "right duo condition" instead of the "reversible condition". But we can give a weakened version of this. To do that, we first introduce a new notion.

Definition 4.1.1. *A ring R is called (σ, δ)-skew nil-McCoy if for any nonzero two polynomials*

$$f(x) = \sum_{i=0}^{m} a_i x^i \text{ and } g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma;\delta] \text{ with } a_i \in N(R) \text{ for all } 0 \le i \le m,$$

$f(x)g(x) = 0$ implies there exists an element $r \in R \backslash \{0\}$ such that $a_i x^i r = 0$ for all i.

Lemma 4.1.2. *If R is a (σ, δ)-compatible ring, then for every $r_1, r_2 \in R$,*

$$r_1 x^i r_2 = 0 \Longrightarrow \begin{cases} r_1 x^i \sigma(r_2) = 0, \\ r_1 x^i \delta(r_2) = 0. \end{cases} \qquad \textit{for each } i = 0, 1, 2, \ldots.$$

Proof. We proceed by induction on i. If $i = 0$, it is clear by (σ, δ)-compatibility. Now, suppose $r_1 x^k r_2 = 0$ for some $k \in \mathbb{N}$. Observe

$$0 = r_1 x^{k-1}(x r_2) = r_1 x^{k-1}(\sigma(r_2)x + \delta(r_2)),$$

which implies $r_1 x^{k-1}\sigma(r_2)$=0 and $r_1 x^{k-1}\delta(r_2)$=0. Then from $r_1 x^{k-1}\sigma(r_2)$=0, we have by induction that

$$r_1 x^{k-1}\sigma^2(r_2) = 0$$

and

$$r_1 x^{k-1}\delta(\sigma(r_2)) = 0.$$

Therefore,

$$r_1 x^k \sigma(r_2) = r_1 x^{k-1}(x\sigma(r_2)) = r_1 x^{k-1}(\sigma^2(r_2)x + \delta(\sigma(r_2))) = 0.$$

Similarly, we can see $r_1 x^k \delta(r_2) = 0$ by $r_1 x^{k-1}\delta(r_2)$=0. This completes the proof. □

We now prove:

Theorem 4.1.3. *(σ, δ)-compatible right duo rings are (σ, δ)-skew nil-McCoy.*

Proof. We adopt an idea of the proofs in [11, Theorem 8.2] (or [34, Theorem 2] and [28, Theorem 2.8]). Let

$$f(x) = \sum_{i=0}^{m} a_i x^i \quad \text{and} \quad g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma;\delta]\backslash\{0\}$$

with $f(x)g(x) = 0$, $a_m \neq 0$, and $a_i \in N(R)$ for all $i = 0, \cdots, m$. Without loss of generality, assume $b_n \neq 0$. We write $I_{g(x)}$:= the right ideal generated by the set of all coefficients in $g(x)$ and the values of any finite number of σ and δ at each coefficient of $g(x)$, i.e.,

$$\Big\{b,\ \sigma^{t_1}(\delta^{t_2}(\sigma^{t_3}\cdots(\delta^{t_l}(b))\cdots)) :\ b \in \textit{coeff}(g),\ t_i \in \mathbb{N}\cup\{0\},\ l \in \mathbb{N}\Big\}.$$

We will use induction on deg(f) to get the result. If deg$\,(f) = 0$, then clearly, $f(x)b_n = 0$, and hence we are done.

Now, we suppose that if $u(x) = \sum_{i=0}^{l} u_i x^i \in R[x;\sigma;\delta]\backslash\{0\}$ with deg$(u(x))$=k for any $k < m$ and all u_i's are nilpotent, then for any nonzero polynomial $v(x) \in R[x;\sigma;\delta]$,

$$u(x)v(x) = 0 \Longrightarrow \exists\, r \in I_{v(x)}\backslash\{0\} \text{ such that } u_i x^i r = 0 \text{ for all } i.$$

We split the proof into two cases:

Case 1 ($a_0 g(x) = 0$): If we take

$$f_1(x) := a_1 + a_2 x + \cdots + a_m x^{m-1} \quad \text{and}$$

$$g_1(x) := \delta(b_0) + (\sigma(b_0) + \delta(b_1))x + (\sigma(b_1) + \delta(b_2))x^2 + \cdots$$
$$\cdots + (\sigma(b_{n-1}) + \delta(b_n))x^n + \sigma(b_n)x^{n+1},$$

then we have $f_1(x)g_1(x) = 0$ by the fact that

$$0 = f(x)g(x) = (a_1 x + \cdots + a_m x^m)g(x) = (a_1 + a_2 x + \cdots + a_m x^{m-1})xg(x)$$
$$= f_1(x)xg(x) = f_1(x)g_1(x).$$

Note that $f_1(x) \neq 0 \neq g_1(x)$ because $b_n \neq 0$ implies $\sigma(b_n) \neq 0$, and that $\deg(f_1) < \deg(f)$ and *coeff*$(f_1) \subseteq N(R)$. Thus, it follows from the inductive assumption that there exists a nonzero $r \in I_{g_1(x)}$ such that $a_i x^{i-1} r = 0$ for all $i = 1, \ldots, m$. Therefore, by Lemma 4.1.2, we obtain $a_i x^{i-1}\sigma(r) = 0$ and $a_i x^{i-1}\delta(r) = 0$ for all i, which implies that

$$a_i x^i r = a_i x^{i-1}(xr) = a_i x^{i-1}(\sigma(r)x + \delta(r)) = 0 \quad \text{for all } i = 1, \ldots, m.$$

But from $a_0 g(x) = 0$, we obtain $a_i x^i r = 0$ for all $i = 0, \cdots, m$. Also, we see $r \in I_{g_1(x)} \subseteq I_{g(x)}$, and therefore we are done.

Case 2 ($a_0 g(x) \neq 0$): Let j be the minimal index satisfying $a_0 b_j \neq 0$. By $a_0 \in N(R)$, there exists $t > 1$ such that $a_0^t = 0$, so that $a_0^t b_j = 0$. Choose $k \geq 1$ such that $a_0^k b_j \neq 0 = a_0^{k+1} b_j$. Then since R is right duo, we can see that there exists $s_1 \in R$ such that $a_0^k b_j = b_j s_1$ $(\neq 0)$. Let k_1 be the largest index satisfying $b_{k_1} s_1 \neq 0$. Then we see $0 = b_{k_1+1}s_1 = b_{k_1+2}s_1 = \cdots = b_n s_1$, and hence we have $0 = b_{k_1+1}x^{k_1+1}s_1 = b_{k_1+2}x^{k_1+2}s_1 = \cdots = b_n x^n s_1$ by (σ, δ)-compatibility. Set

$$g_2(x) := (b_0 + b_1 x + \cdots + b_{k_1} x^{k_1})s_1.$$

We observe that $f(x)g_2(x) = 0$ and the leading coefficient of $g_2(x)$ is $b_{k_1}\sigma^{k_1}(s_1)$. But since R is σ-compatible and $b_{k_1}s_1 \neq 0$, we have $b_{k_1}\sigma^{k_1}(s_1) \neq 0$, which implies $g_2(x) \neq 0$. Again, we consider two cases.

We first suppose $a_0 g_2(x) = 0$. If we take $g_2'(x) := x g_2(x)$, then we have $f_1(x) g_2'(x) = 0$ and $g_2'(x) \neq 0$ since $g_2'(x) = x(b_0 s_1 + \cdots + b_{k_1} \sigma^{k_1}(s_1) x^{k_1})$, so that the leading coefficient of $g_2'(x)$ is $\sigma(b_{k_1} \sigma^{k_1}(s_1)) \neq 0$ by σ-compatibility. But since $\deg(f_1) < \deg(f)$ and $\mathit{coeff}(f_1) \subseteq N(R)$, by the inductive assumption, there exists a nonzero $r_1 \in I_{g_2'(x)}$ satisfying $a_i x^{i-1} r_1 = 0$ for all $i = 1, \ldots, m$. Using the same method as in Case 1, we can get $a_i x^i r_1 = 0$ for all $i = 0, \ldots, m$. Further, we observe $r_1 \in I_{g_2'(x)} \subseteq I_{g_2(x)} \subseteq I_{g(x)}$, and hence we are done.

Next, we suppose $a_0 g_2(x) \neq 0$. If j_1 is the minimal index satisfying $a_0 b_{j_1} x^{j_1} s_1 \neq 0$, then we have $j_1 > j$ since $a_0 b_j s_1 = a_0^{k+1} b_j = 0$, so that $a_0 b_j x^j s_1 = 0$ by (σ, δ)-compatibility. Note that $a_0 b_{j_1} s_1 \neq 0$ and $a_0^t b_{j_1} s_1 = 0$. Thus, there exists $k_1' \geq 1$ such that $a_0^{k_1'} b_{j_1} s_1 \neq 0 = a_0^{k_1'+1} b_{j_1} s_1$. By the right duo condition, there exists $s_2 \in R$ satisfying $a_0^{k_1'}(b_{j_1} s_1) = (b_{j_1} s_1) s_2$ $(\neq 0)$. Choose the largest index k_2 such that $b_{k_2} s_1 s_2 \neq 0$. Thus, $0 = b_{k_2+1} s_1 s_2 = b_{k_2+2} s_1 s_2 = \cdots = b_{k_1} s_1 s_2$. Since R is (σ, δ)-compatible, we see $0 = b_{k_2+1} x^{k_2+1} s_1 s_2 = b_{k_2+2} x^{k_2+2} s_1 s_2 = \cdots = b_{k_1} x^{k_1} s_1 s_2$. Take

$$g_3(x) := (b_0 + b_1 x + \cdots + b_{k_2} x^{k_2}) s_1 s_2.$$

Similarly, we have $b_{k_2} \sigma^{k_2}(s_1 s_2) \neq 0$ from $b_{k_2} s_1 s_2 \neq 0$, and hence $g_3(x) \neq 0$ and $f(x) g_3(x) = 0$. Continuing this process a finite number of times, we obtain $p \in \mathbb{N}$ satisfying

$$a_0 g_p(x) = 0, \quad \text{where} \ \ g_p(x) := g(x) s_1 s_2 \cdots s_{p-1}.$$

Hence, we can see that there exists a nonzero $r_{p-1} \in I_{g(x)}$ such that $a_i x^i r_{p-1} = 0$ for all $i = 0, \ldots, m$. This concludes the proof. □

Note that the condition of Theorem 4.1.3 is essential as we can see in the following examples.

Example 4.1.4. (a) Let R be an integral domain. Take

$$S := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R \right\}.$$

Then S is right duo because S is commutative. Define an endomorphism σ of S by

$$\sigma : \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$$

and define a σ-derivation δ of S by

$$\delta\colon \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} -b & 0 \\ 0 & -b \end{pmatrix}.$$

Then we have S is σ-compatible. Indeed, let

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = 0, \text{ and then } \begin{cases} ac = 0, \\ ad + bc = 0. \end{cases} \tag{4.2}$$

Since R is a domain and $ac = 0$, $a = 0$ or $c = 0$. If $a = 0$, we have $bc = 0$ by (4.2), and hence $-ad + bc = 0$. For the case of $c = 0$, we have $ad = 0$ from (4.2), and thus $-ad + bc = 0$. Therefore, we get

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \sigma \left(\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & -d \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The converse of the σ-compatible condition is similar. However, S is not δ-compatible. To see this, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \delta \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, S is not (σ, δ)-compatible. Now, let

$$f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \quad \text{and} \quad g(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \ \in S[x; \sigma; \delta].$$

We observe that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(S)$ and

$$
\begin{aligned}
f(x)g(x) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\delta\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \\
&+ \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\sigma\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\delta\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\right] x \\
&+ \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\sigma\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\right] x^2 \\
&= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
&+ \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right] x + \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right] x^2 \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}
$$

but there does not exist any nonzero element $s \in S$ such that $\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x\right) s = 0$. Therefore, S is not (σ, δ)-skew nil-McCoy.

(b) Let R be the free algebra generated by the two noncommuting indeterminates a, b and I be the ideal generated by a^2 and b^2. Take

$$S := R/I \oplus R/I.$$

Note that the right ideal generated by (a, b) of S contains no element $(b, a)(a, b) = (ba, ab)$, which implies that S is not right duo. Let σ be an endomorphism of S defined by

$$\sigma\colon (a, b) \mapsto (b, a)$$

and let $\delta = 0_S$. Then S is not σ-compatible. Indeed, $(a, b)(a, b) = (0, 0)$, but

$$(a, b)\sigma\left((a, b)\right) = (a, b)(b, a) \neq (0, 0).$$

On the other hand, if we take

$$f(x) = (a,\, b) + (a,\, b)x \quad \text{and} \quad g(x) = (-a,\, -b) + (b,\, a)x \;\in S[x;\sigma;\delta],$$

then we have

$$\begin{aligned} f(x)g(x) &= (a,\, b)(-a,\, -b) + (a,\, b)\delta((-a,\, -b)) \\ &\quad + [(a,\, b)(b,\, a) + (a,\, b)\sigma((-a,\, -b)) + (a,\, b)\delta((b,\, a))]\, x + [(a,\, b)\sigma((b,\, a))]\, x^2 \\ &= [(ab,\, ba) + (a,\, b)(-b,\, -a)]\, x + [(a,\, b)(a,\, b)]\, x^2 \\ &= (0,\, 0), \end{aligned}$$

but there exists no nonzero element $s \in S$ such that $(a,\, b)s = 0 = ((a,\, b)x)\, s$, which implies that S is not $(\sigma,\, \delta)$-skew nil-McCoy.

4.2 $(\sigma,\, \delta)$-skew McCoy property on Jacobson radicals

Next, we focus upon the $(\sigma,\, \delta)$-skew McCoy property of the Jacobson radical. We first note that a σ-derivation δ of a ring R does not need to preserve $N(R)$. To see this, if we take S and δ as in Example 4.1.4-(a), then we can see that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(S)$, but $\delta\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \notin N(S)$. However, the result is positive under the $(\sigma,\, \delta)$-compatible condition:

Lemma 4.2.1. *Let R be a $(\sigma,\, \delta)$-compatible ring. Then*

$$\delta(N(R)) \subseteq N(R).$$

Proof. Let $a \in N(R)$. Then $a^n = 0$ for some $n \in \mathbb{N}$. We claim that $\delta(a)^n = 0$. Indeed, since $a^n = 0$, we see

$$0 = \delta(a^n) = \delta(a)a^{n-1} + \sigma(a)\delta(a^{n-1}). \tag{4.3}$$

But by the fact that R is $(\sigma,\, \delta)$-compatible and $0 = \sigma(a^n) = \sigma(a)\sigma(a^{n-1})$, we have $\sigma(a)a^{n-1} = 0$, which implies $\sigma(a)\delta(a^{n-1}) = 0$. Then from (4.3), we get

$$\delta(a)a^{n-1} = 0.$$

Again, since R is $(\sigma,\, \delta)$-compatible, we observe

$$0 = \delta(a)\delta(a^{n-1}) = \delta(a)\Big(\delta(a)a^{n-2} + \sigma(a)\delta(a^{n-2})\Big) \tag{4.4}$$

and

$$\delta(a)a^{n-1} = 0 \;\Rightarrow\; \delta(a)\sigma(a^{n-1}) = 0 \;\Rightarrow\; \delta(a)\sigma(a)\sigma(a^{n-2}) = 0 \;\Rightarrow\; \delta(a)\sigma(a)\delta(a^{n-2}) = 0.$$

Therefore, we obtain $\delta(a)^2a^{n-2} = 0$ from (4.4). Repeating this argument, we can see $\delta(a)^n = 0$, and hence we are done. □

We then prove:

Theorem 4.2.2. *Let R be a right duo, local left (or right) artinian ring which is (σ, δ)-compatible. Then the Jacobson radical $J(R)$ is (σ, δ)-skew McCoy.*

Proof. We recall that for any left or right artinian ring R, the Jacobson radical $J(R)$ is nilpotent. Now, we first show that

$$\sigma(J(R)) \subseteq J(R) \quad \text{and} \quad \delta(J(R)) \subseteq J(R). \tag{4.5}$$

To see this, we note that right duo rings are 2-primal, i.e., $P(R) = N(R)$ (see [31]). Since, in general, $P(R) \subseteq J(R)$ and $J(R) \subseteq N(R)$ from the fact that $J(R)$ is nilpotent, we have $J(R) = N(R)$. But since σ preserves nilpotent elements, we get $\sigma(J(R)) \subseteq J(R)$. Also, by Lemma 4.2.1, the second inclusion of (4.5) is obvious. This shows (4.5). Note that

$$J(R) \text{ is } (\sigma, \delta)\text{-compatible} \tag{4.6}$$

from the heredity of (σ, δ)-compatibility. Next, we show that

$$J(R) \text{ is right duo.} \tag{4.7}$$

To see this, let $a, b \in J(R)\backslash\{0\}$. Since R is right duo, we have $ab = bd$ for some $d \in R$. It is enough to prove that $d \in J(R)$. Then we assume to the contrary that $d \notin J(R)$. But since R is local, we see that $J(R)$ is the set of non-units of R, which implies d is a unit. On the other hand, since $J(R)$ is nilpotent, there exists $n > 1$ such that $a^n = 0$. Then from $ab = bd$, we observe $0 = a^nb = a^{n-1}(bd) = \cdots = bd^n$. But since d is a unit, we obtain $b = 0$, which is a contradiction. This shows (4.7). Therefore, since every element of $J(R)$ is nilpotent, we conclude that $J(R)$ is (σ, δ)-skew McCoy by Theorem 4.1.3 together with (4.5), (4.6) and (4.7). □

Corollary 4.2.3. *Let R be a (σ, δ)-compatible reversible left (or right) artinian ring. Then $J(R)$ is (σ, δ)-skew McCoy.*

Proof. Since the reversible condition is hereditary, $J(R)$ is also reversible. Note that every reversible ring is 2-primal. Then by the same argument as in the proof of Theorem 4.2.2 together with the assertion (4.1), the result follows. □

With a ring R and a derivation δ of R, we denote a differential polynomial ring with the usual addition and multiplication defined by $xr = rx + \delta(r)$ for every $r \in R$ by $R[x;\delta]$. We shall say that a ring R is called *δ-differential McCoy* if for any nonzero two polynomials

$$f(x) = \sum_{i=0}^{m} a_i x^i \quad \text{and} \quad g(x) = \sum_{j=0}^{n} b_j x^j \ \in R[x;\delta],$$

$f(x)g(x) = 0$ implies there exists a nonzero element $r \in R$ such that $a_i x^i r = 0$ for all $0 \leq i \leq m$.

We then have:

Corollary 4.2.4. *Let R be a δ-compatible local left (or right) artinian ring with a derivation δ of R. Then $J(R)$ is δ-differential McCoy whenever R is right duo or reversible.*

Proof. It is immediate from Theorem 4.2.2 and Corollary 4.2.3. □

4.3 Tracial σ-derivations

By Theorem 2.4.1 via [27, Theorem 20], it is known that if for a σ-compatible regular ring R,

$$R \text{ is reversible} \iff R \text{ is } \sigma\text{-skew McCoy.} \tag{4.8}$$

In spite of that, we cannot guarantee that when R is a (σ, δ)-compatible regular ring, R is reversible if and only if R is (σ, δ)-skew McCoy. But we may get the result under some constraints. To do so, we introduce a new notion: a σ-derivation δ of R is called *tracial* if δ sends commutators to zero, i.e.,

$$\delta(ab) = \delta(ba) \quad \text{for all } a, b \in R.$$

Clearly, if R is commutative, then every σ-derivation of R is a tracial σ-derivation. However, it is not trivial whether there exists a tracial σ-derivation on a noncommutative ring. The following example shows that there exists such a derivation.

Example 4.3.1. Let R be a commutative ring and $T_2(R)$ be an upper triangular matrix ring whose elements are in R, i.e.,

$$T_2(R) := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a,\, b,\, c \in R \right\}.$$

Then $T_2(R)$ is clearly a noncommutative ring. Let σ be an endomorphism of $T_2(R)$ defined by

$$\sigma\colon \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

and let δ be a σ-derivation of $T_2(R)$ defined by

$$\delta\colon \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} 0 & c-a \\ 0 & 0 \end{pmatrix}.$$

We claim that δ is tracial. To see this,

$$\delta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\right) = \delta\left(\begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}\right) = \begin{pmatrix} 0 & cf-ad \\ 0 & 0 \end{pmatrix}$$

and

$$\delta\left(\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \delta\left(\begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}\right) = \begin{pmatrix} 0 & fc-da \\ 0 & 0 \end{pmatrix}.$$

But since R is commutative, the result follows.

We now generalize the result of (4.8) with a tracial σ-derivation δ:

Theorem 4.3.2. *Let R be a $(\sigma,\, \delta)$-compatible regular ring with a tracial σ-derivation δ. Then*

$$R \text{ is reversible} \iff R \text{ is } (\sigma,\, \delta)\text{-skew McCoy.}$$

Proof. ($\Rightarrow$): By (4.1), this holds without the regularity condition.

($\Leftarrow$): It is similar to the proof of Theorem 2.4.1. Suppose R is $(\sigma,\, \delta)$-skew McCoy. Assume to the contrary that R is not reversible, which implies R is not reduced. Thus, there exists a nonzero $a \in R$ such that $a^2 = 0$. Since R is regular, there exists a nonzero $c \in R$ such that $a = aca$ and $c = cac$. Then by the same method as in Theorem 2.4.1, we take

$$f(x) = \sigma(a) - (1-ca)x \quad \text{and} \quad g(x) = ac + \sigma(c)x.$$

Then a direct calculation gives $f(x) \neq 0 \neq g(x)$ and

$$\begin{aligned} f(x)g(x) = & \Big(\sigma(a)ac - (1-ca)\delta(ac)\Big) \\ & + \Big(\sigma(a)\sigma(c) - (1-ca)\sigma(ac) - (1-ca)\delta(\sigma(c))\Big)x \\ & - (1-ca)\sigma^2(c)x^2. \end{aligned} \tag{4.9}$$

Observe:

(i) Since $a^2 = 0$, and hence $\sigma(a)^2 = 0$, it follows from σ-compatibility that $\sigma(a)a = 0$ and $a\sigma(a) = 0$. Then $\sigma(a)ac = 0$ and $ca\sigma(ac) = 0$;

(ii) Note that $(1-ca)ca = 0$. Since R is (σ, δ)-compatible, we have $(1-ca)\delta(ca) = 0$. But since δ is tracial, we have $(1-ca)\delta(ac) = 0$;

(iii) Since $(1-ca)c = 0$, we have $(1-ca)\sigma^i(c) = 0$ $(i = 0, 1, \dots)$ by σ-compatibility, and hence $(1-ca)\delta(\sigma(c)) = 0$ by δ-compatibility.

Combining (4.9) with the above arguments (i), (ii), and (iii), we get $f(x)g(x) = 0$. Since R is (σ, δ)-skew McCoy, there exists $0 \neq r \in R$ such that

$$\begin{cases} \sigma(a)r = 0, \\ (1-ca)xr = 0. \end{cases} \tag{4.10}$$

Since $\sigma(a)r = 0$, we have $a\sigma(r) = 0$ by σ-compatibility. Also, from the second equation of (4.10), $(1-ca)\sigma(r) = 0$. Thus, $\sigma(r) = ca\sigma(r) = 0$, so that $r = 0$, a contradiction. Therefore, R is reversible. This completes the proof. □

4.4 An application to group rings

Now, we conclude with an application of Theorem 4.3.2 to the group ring. Then we consider the group ring with a field and the quaternion group Q_8. To do so, we write the group ring with a group G and a ring R as $R[G]$. On the other hand, it was proven [15, Theorem 3.1] that when K is a field of characteristic different from 2,

$$K[Q_8] \text{ is reversible} \iff \text{the equation } 1 + x^2 + y^2 = 0 \text{ contains no solution in } K. \tag{4.11}$$

Then we may wonder about the following question: when does the group ring $K[Q_8]$ have the McCoy property, more concretely, the (σ, δ)-skew McCoy property?

With the aid of Theorem 4.3.2, we can give an answer to this question:

Corollary 4.4.1. *Let K be a field of characteristic 0. If $K[Q_8]$ is (σ, δ)-compatible with a tracial σ-derivation δ, then the following are equivalent:*

(i) *$K[Q_8]$ is (σ, δ)-skew McCoy;*

(ii) *the equation $1 + x^2 + y^2 = 0$ contains no solution in K.*

Proof. Note that if $a \in Q_8$, then the order of $a \in \{1, 2, 4\}$. Since K is a field of characteristic 0, we have $1 \cdot 1_K$, $2 \cdot 1_K$, and $4 \cdot 1_K$ are nonzero, and thus they are units in K. Hence, there exist $s_1, s_2, s_4 \in K$ satisfying

$$(1 \cdot 1_K)s_1 = 1_K, \quad (2 \cdot 1_K)s_2 = 1_K, \quad (4 \cdot 1_K)s_4 = 1_K.$$

Then for any $r \in K$, we have

$$\begin{cases} 1 \cdot (s_1 r) = (1 \cdot 1_K)s_1 r = r, \\ 2 \cdot (s_2 r) = (2 \cdot 1_K)s_2 r = r, \\ 4 \cdot (s_4 r) = (4 \cdot 1_K)s_4 r = r. \end{cases}$$

But $s_i r$ $(i = 1, 2, 4)$ is the unique element such that $i \cdot (s_i r) = r$ since the equation $i \cdot t = r$ for some $t \in K$ implies

$$t = 1_K t = (i \cdot s_i)t = s_i(i \cdot t) = s_i r.$$

Therefore, we conclude that

$$K \text{ is uniquely divisible by the orders of elements in } Q_8. \tag{4.12}$$

On the other hand, it is known ([33, Theorem 2]) that for a locally finite group G and a commutative regular ring R which is uniquely divisible by the order of each element in G, the group ring $R[G]$ is regular. Thus, since Q_8 is clearly a locally finite group and we have the assertion (4.12), we obtain $K[Q_8]$ is regular. Therefore, the result follows from Theorem 4.3.2 together with (4.11). □

Corollary 4.4.2. *If K is a field of characteristic 0, then $K[Q_8]$ is right McCoy if and only if the equation $1 + x^2 + y^2 = 0$ contains no solution in K.*

Proof. It follows from Corollary 4.4.1 with $\sigma = I_K$ and $\delta = 0_K$. □

Example 4.4.3. Let $\mathbb{Q}$ be the field of rational numbers. Then the characteristic of $\mathbb{Q}$ is clearly 0. Moreover, there does not exist any solution in $\mathbb{Q}$ for the equation $1 + x^2 + y^2 = 0$. Thus, it follows from Corollary 4.4.2 that $Q[Q_8]$ is right McCoy. However, if $\mathbb{C}$ is the field of complex numbers, then $\mathbb{C}[Q_8]$ is not right McCoy because the equation $1 + x^2 + y^2 = 0$ has a solution $(x = 0,\ y = i)$ in $\mathbb{C}$.

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