ELEMENTARY GRADE STUDENTS' DEMONSTRATED FRAGMENTING WITH VISUAL STATIC MODELS

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By

Maryam Zolfaghari

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A dissertation written by

Maryam Zolfaghari,

B. S., Azad University, 2007

M. S., Alzahra University, 2013

Ph.D., Kent State University, 2023

Approved by

________________________________, Chair, Doctoral Dissertation Committee
Karl W. Kosko

________________________________, Member, Doctoral Dissertation Committee
Joanne Caniglia

________________________________, Member, Doctoral Dissertation Committee
Clarissa Thompson

Accepted by

________________________________, School of Teaching, Learning and Curriculum Development
George Kamberelis

________________________________, Dean, College of Education, Health and Human Services
James C. Hannon

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The purpose of this study was to understand children's operations on fragmenting schemes while they are engaged with three different visual models, including circle, rectangle, and length model. This study was a sequential explanatory mixed method, which included two phases that happened sequentially with the dominant use of the quantitative approach. A fragmenting survey was given to 132 first and second grade students from three different schools located in the Midwestern U.S. Out of 132 participants who completed the test, 11 participants were purposefully selected for one-on-one clinical interviews.

Results from the descriptive statistics showed that almost half of the students are at Level 1 of fragmenting. It implies that students are able to partition a model into a given number of parts but are not able to make the parts equal or exhaust the whole. Analysis of the Friedman test and clinical interviews showed that students performed better when partitioning models into even numbers than into odd numbers. Further, findings from comparing the circle, rectangle, and length models and their relation to children's fragmenting scheme suggest that children's fragmenting knowledge is consistent across the three visual models. However, the type of model operated on children's successful partitioning when they work with odd numbers of parts. The findings of the study suggested circle models can add an extra layer of difficulty for equal partitioning into third and fifth.

Keywords: Elementary mathematics education, early knowledge of fractions, rational numbers

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CHAPTER I

INTRODUCTION

Statement of the Problem

Fractions are considered a foundation for success for many mathematics concepts. However, teaching and learning fractions remains challenging and difficult for many students and teachers (Anderson-Pence et al., 2014; Tunç-Pekkan, 2015; Son, 2012; Wilkins & Norton, 2018; Yang & Reys, 2008). To ease the difficulty of teaching and learning fractions, scholars have crafted learning trajectories for fractions based on observed fractions schemes (Davis & Hunting, 1990; Hackenberg et al., 2016; Nunes & Bryant, 2009, Steffe & Olives, 2010; Wilkins & Norton, 2018). An initial milestone in learning fractions is the part-whole scheme. In this scheme, children understand fractions as the relationship between the part(s) and whole. Children see the fraction as partitioning a whole into n equal units and taking out m parts of it, while the m parts can be seen as parts within the continuous whole or out of the continuous whole. Although the part-whole scheme is considered an initial level of fraction knowledge, studies suggested the necessity of developing some pre-requisite schemes to comprehend the part-whole scheme and the fractions' concept in general (Empson, 1995; Hackenberg et al., 2016; Nunes et al., 2009; Steffe & Olive, 2010).

The fragmenting scheme is considered a foundational scheme that is needed to learn fractions (Hackenberg et al., 2016). Fragmenting begins at an early age when a child attempts to fracture objects, such as a cookie (Empson, 1995). It then develops gradually through different detailed stages (Nunes et al., 2009; Steffe & Olive, 2010). The stages include the ability to perceive the whole and divide it into given parts, the relation between the number of cuts and the number of parts, making sure the parts are equal while exhausting a whole, and being able to see the parts nested in a whole and iterating parts make the whole. Successful fragmenting facilitates the transition to learning fractions. As Hackenberg et al. (2016) stated, “successful equal sharing of single items can allow students to develop fraction language and fraction notation that are linked to fragmenting they do to equally share” (p. 42).

Research on understating children’s knowledge of fractions is always associated with using different representations. Cramer et al. (2008) used circular models to understand children’s knowledge of addition and subtraction fractions. Hackenberg et al. (2016) used length models to understand children’s understating in unit fractions. Other studies compared the impact of using different models on children’s performance of different fractions problems (Tunç-Pekkan, 2015; Sidney et al., 2019). Results suggested little to no difference in students’ performance using circle and rectangle models for solving fractions tasks. However, most students have demonstrated difficulty solving fraction problems using a number line.

Significance of the Study

Given the sophistication of fractions, most researchers have suggested the necessity of developing prior schemes such as multiplicative reasoning (Hackenberg, 2007; Kosko, 2019) and fragmenting schemes (Empson, 1995; Steffe & Olive, 2010) before certain fractional schemes can emerge. Despite the importance of developing these pre-requisite schemes, most elementary students and even secondary school students are not developed these schemes yet. Examining data from 326 middle school students, Zwanch & Wilkins (2021) found that only 39.9% of students indicated different levels of multiplicative reasoning. This percentage increased slightly in the study to 46.3 % for fourth graders and 62.5% for fifth graders in demonstrating some advanced level of multiplicative reasoning (Kosko, 2019).

Similar issues have been observed for fragmenting schemes. In a study of a seventh-grade student's unit faction knowledge, Hackenberg (2013) observed one seventh-grade student's lack of knowledge of the equal sharing concept. The student had difficulty in drawing 1/3 of a given bar and instead showed 4/3 of it, indicating the absence of fragmenting schemes or equal sharing concept. Although children intuitively learn to fragment an object at an early age by fracturing a cookie or sharing candies with their friends (Empson, 1995), improving and developing fragmenting skills is necessary for comprehending different topics of fractions, such as unit fractions (Hackenberg, 2013).

To scaffold children's fractions knowledge, visual models play an essential role. Two common visual models that are used in teaching and learning early fractions concepts are circular and rectangular length models (Ni, 2001). Several scholars have found that most upper elementary graders rely on circle models in solving problems related to fractions (Cramer and Henry, 2002; Kaminski, 2018; Hamdan & Gunderson, 2017). For instance, in a study of fourth and fifth graders, Cramer and Henry (2002) found that children mostly rely on circle models to solve problems related to fractions additions and subtractions. Kaminski (2018) suggested children's reliance on circle models might be due to the easier visualization of circle models. Yet, several scholars have found rectangular length models beneficial in teaching and learning fractions (Hackenburg et al., 2016; Larson, 1980; Tunç-Pekkan, 2015). Due to the similarity of rectangular length model to linear models, the fractions' learning transition from length models to the number line should, hypothetically, be easier for children. While there has been a long debate on the privilege of using each of these models, there is no consensus on prioritizing one of the models in fraction instructions. There are some studies that compare upper elementary graders' knowledge of fractions while using circle and rectangle (Tunç-Pekkan, 2015; Sidney et al., 2019). All the studies found no difference in the upper graders' performance in circle and rectangle fraction questions.

Acknowledging that both models each have their affordances in the teaching and learning of fractions, there is a need to examine the appropriateness of models from the perspectives of early elementary children. Stated differently, teachers may use a specific model to introduce a topic from their own viewpoint, but the children may not interpret the same meaning as the teacher from that model (Tunç-Pekkan, 2015). Moreover, the studies that compared students' fraction knowledge mostly focused on fourth and fifth graders. Children begin to develop fragmenting skills much earlier than fourth and fifth grades (Empson, 1995). Studying children who are at the early stages of establishing fractional knowledge, such as partitioning and fragmenting, is essential for understanding which model would be the most useful for teaching and learning fraction. Thus, this study focused on the first and grade students to examine how their interactions with fragmenting schemes correspond with visual models.

Research Goals and Questions

The overarching goal of this study was to understand how children's interaction with fragmenting schemes correlated to three different visual models, including circle, rectangle, and length models. Prior studies found fragmenting scheme as a foundation for developing fraction knowledge (Empson, 1995; Steffe & Olive, 2010). Also, the necessity of using visual models to help children to develop different mathematical topics, including fraction concepts, has been emphasized by many researchers (Bialystok & Codd, 2000; Bruner, 1964, NCTM, 2000, Lesh et al., 1987). Thus, the first research question in this study investigates what types of fragmenting schemes children in the early elementary grades demonstrate. Due to the importance of using visual models to scaffold fractions' knowledge, the second research question looks at the interaction between observed schemes and children's use of different visual models (i.e., circle, rectangle, and length model). Finally, the last research question seeks to understand if there is any difference in children's partitioning due to the number of partitions. Studies suggest that as children develop a partitioning scheme, they can cut different shapes into different numbers of partitions (Steffe & Olive, 2010). However, some evidence suggests that children are more comfortable in cutting even numbers such as cutting into halve or fourth as compared to odd numbers (Piaget, et al., 1960). Nunaz and Bryant (2009) stated "a major strategy in carrying out successful partitioning was the use of successive division in two: so children are able to succeed in dividing a whole into fourth before they can succeed with third" (p. 18). This study focuses on the following three research questions:

1. What fragmenting schemes do early elementary students demonstrate?
2. How are fragmenting schemes manifested when children engage in three different visual models, including circle, rectangle, and length model?
3. How do children enact their partitioning schemes when working with odd and even numbers?

Key Terms

Disembedding: involves the coordination of units to specify a new unit while keeping track of prior units (Kosko, 2018).

Distributing: Distributing is forming a unit into other composite units. Usually, in the context of multiplication, this action is represented as repeated addition or forming groups of groups (Hackenberg et al., 2016)

Equal partitioning: partitioning a continuous whole into an equal number of parts and finishing the whole.

Fragmenting: Fracture or break a continuous whole into different parts (Hackenberg et al., 2016).

Iterating: the action of making a larger unit by using smaller units and repeating it.

Part-whole: The relation between m parts and n whole (m/n). Partitioning a whole into n equal units and taking out m parts of it, while the m parts can be seen as parts within the continuous whole or out of the continuous whole (Nunes et al., 2009).

Scheme: The process of assimilation and accommodation are controlled by the human mind and then represented as an action, which is referred to as a scheme (von Glasersfeld, 1998).

Unitizing: is a mental action of unifying the numbers of discrete and countable objects into a new convenient and familiar group/set to operate with that numbers (Hackenberg, et al., 2016).

CHAPTER II

LITERATURE REVIEW

This study aims to understand the effect of using different static models on children's early knowledge of fractions. I specifically focus on children's demonstration of fragmenting schemes as a primary set of schemes for developing fractions' knowledge and explore the effect of using three different static models, including circle, rectangle, and length model in constructing fragmenting schemes. This chapter first reviews the literature related to children's knowledge of fractions from different perspectives. Then, it focuses on the scheme theory that is used to ground the current study. Finally, I use scheme theory to discuss literature that studied children's use of various visual models for fractions.

Different Theoretical Lenses for Fractions

Rational numbers are one of the fundamental topics for students to develop other mathematical concepts such as rates, percentages, slopes, and many other topics in secondary school mathematics (Son, 2012; Yang & Reys, 2008). Despite such fundamental importance, most students, and many adults, find fractions a difficult topic to learn. For instance, in responding to find the closest whole number to $\frac{11}{12}+\frac{7}{8}$ only 24% of almost 20,000 eighth graders chose the correct answer (Carpenter et al., 1983, as cited in Siegler & Lortie-Forgues, 2017). To ease the difficulty of fraction concepts and to understand how students learn and conceptualize this concept, many researchers have studied rational numbers from various perspectives/theories.

One of the most well-known theories in teaching and learning rational numbers was established by Kieren (1976). Kieren (1976) viewed teaching and learning rational numbers as a distinct topic of math rather than an extension of the whole number. He identified different interpretations for rational numbers and called them subconstructs. The subconstructs include *quotient*, *ratio*, *operators*, and *measures*. He argued that a complete understanding of rational numbers depends on a thorough understanding of subconstructs and their interrelation. Building on his work, many researchers used the subconstructs to examine how children conceptualize rational numbers. Rational Number Project (RNP) (Behr et al., 1983) and different studies done by Nunes & Bryant (2008) and Mamede & Nunes (2008) are two examples among many studies that used Keiren's subconstructs as the foundation of their studies.

Kieren (1993) believed that understanding rational number is "not a simple extension of knowing whole numbers" (p.56). Rather, children's prior conceptions of whole numbers can lead to difficulty in understand rational number. However, others suggest that particular conceptions of whole numbers facilitate learning fractions (Lamon, 1993; Steffe & Olive, 2010) and that there are other factors that contribute to the difficulty of teaching and learning fractions (Siegler & Lortie-Forgues, 2017). According to Siegler and Lortie-Forgues (2017), there are two factors for the difficulty of learning and understanding fractions: cultural and inherent elements. The cultural element relates to the teacher's lack of knowledge of teaching fractions, the textbooks' content delivery, and the language used for fractions. The inherent element relates to the complex nature of fractions, the relation between the rational and whole numbers, and the relation between the rational numbers and operations. Siegler and Lortie-Forgues (2017) suggested both whole numbers and rational numbers should be comprehended as a type of number (i.e., magnitudes). They stated knowing the magnitude aspect of the whole numbers, which is the first type of number children learn, can facilitate children's learning of other types of numbers, such as rational numbers.

The emphasis on learning fractions as a type of number/magnitude has been highlighted by many researchers as well (Hamdan, & Gunderson, 2017; Siegler et al., 2011; Soni &

Okamoto, 2020). In an integrated theory of numerical development, Siegler et al. (2011) proposed learning numerical magnitudes as the central conceptual structure for developing all types of numbers. Relying on children's prior numerical knowledge can promote their understanding of fractions. Although, such numerical knowledge should be in the format of magnitude.

The role of children's prior knowledge of whole numbers and real numbers to learn rational numbers has also been investigated from the lens of other learning theories. In Scheme theory (which is also explained in the context of reorganization hypothesis), Steffe & Olive (2010) highlighted the role of the unit in all types of numbers as an important key to understanding rational numbers (Steffe, 2001; Steffe & Olive, 2010). They stated that children use their understanding of a unit in real numbers (i.e., called a numerical scheme) and reorganize it to accommodate the new knowledge of fractions. Since the focus of this study is children's developmental trajectory of learning fractions, the remainder of this chapter reviews the literature focused on how children construct fractions knowledge based on their prior schemes. Despite the focus of the study on schemes, it is worthwhile to first review Kieren's rational number subconstructs as the initial foundation of all works that have been done in teaching and learning fractions.

Emerging Fractions Schemes: A History

Rational numbers can be expressed as an integer or the quotient of an integer divided by a nonzero integer (i.e., $\frac{a}{b}$ which $b \neq 0$ and $bx=a$) (Merriam-Webster2022). To have a complete comprehension of rational numbers, Kieren (1993) defined rational numbers using four different subconstructs, including *quotients, measures, operators,* and *ratios*. In the quotients, subconstruct a rational number considered as dividing or equal partitioning, for example, dividing/sharing three cookies among seven people (3÷7). In the measure subconstruct, a fraction is considered as a magnitude. A child interprets fractions using a number line and understands that the unit in the number line can be divided into any number of congruent parts. Operators or Mapping subconstruct involves" the rational number $\frac{p}{q}$ as a transformer or operator changing a geometric figure into a figure $\frac{p}{q}$ time as big." (Kieren, 1976, p. 120). Perhaps such a definition can conceptualize profoundly in relation to scaling and mapping problems. For instance, in real life example provided by Kieren (1976), students were asked to use segment paring and map the small house based on Mr. Jones' big house. Ratio subconstruct involves understanding rational numbers as pairing a number in relation with other presented numbers (e.g., if we know (2, 8) $\approx$ (3, 12) then what is X in (X, 20)). Although presenting the numbers seems simple algorithmically, it is complex in concept (Kieren, 1976). The ratio conception is associated with a size and whole numbers, and many children refer to those concepts to solve the ratio problems (Neuman, 1993).

Later, in the Rational Number Project, Behr et al. (1983) developed Kieren's subconstruct and added part-whole knowledge as the fifth component of rational numbers. The part-whole subconstruct is considered as the ability to partition either a continuous quantity or a set of discrete objects into equal-sized subparts (Behr et al., 1983). It should be noted that part-whole reasoning is identified as a fundamental element of perceiving and interpreting rational numbers for Kieren (Kieren, 1976). However, he believed that such reasoning could be constructed under the quotient or measure subconstruct as "the dynamic comparison of a quantity to a dividable unit that allows for the generation of rational numbers as extensive quantities" (p.57).

Acknowledging that children's knowledge of rational numbers is constructed through these five subconstructs, Mack (1993) argued for the important role of informal knowledge in constructing rational numbers. She suggested using informal knowledge as the viable alternative to developing rational numbers understanding. For instance, by using a bread loaf as a representation in adding $\frac{3}{8} + \frac{2}{8}$, the students overcome the common mistake of adding numerators across and denominators across. The students partitioned each loaf into eight parts and then took out three parts from one bread and two from the other to find the sum. In this manner, they could see that the denominator did not change in the problem. Mack suggested the role of informal knowledge as a bridge to connect the abstract concept to real-life problems.

Along with Mack's idea of using informal knowledge of children to rational numbers knowledge, Lamon (1993) examined the influence of mathematical structures in facilitating children's knowledge of fractions. She suggested that basic mathematical structures such as arithmetic knowledge (which develops informally at early ages and construct formally at early grades) can facilitate children's knowledge of fractions. The construct of units and composition of units are the basic elements of interpreting ratios and fractions. Children develop unitizing as they hold a finger to coordinate it to a number (e.g., number five). Then, by decomposing or composing numbers into different units through multiplication and division they practice multiple compositions. For instance, in finding $\frac{3}{4}$ of 16 objects, the solution involves the construction of three collections of units:16 one-units, 4 composite units each consisting of 4 one-units, and 1 three-unit consisting of 3 of the 4 four-units (Lamon, 1993, p.133).

Lamon's (1993) suggestion of using number knowledge to define ratios is in contradiction with Kieren's (1993) idea of whole numbers for learning fractions. Kieren (1993) believed that understanding rational numbers as the extension of the whole number impedes learning rational numbers. However, Lamon suggested using former knowledge such as the whole number can help students to develop fractional knowledge. Such a view is aligned with Steffe's (2001) scheme theory (also known as the reorganization hypothesis), in which he stipulates that the ways students build and work with units in whole number contexts influence the ways they coordinate units in fractions contexts.

Following Lamon (1993), Steffe (2001) believed that mathematical structures such as whole number reasoning could help children conceptualize fractions. However, Steffe's (2001) tool to understand children's knowledge was different from his peers. In his investigation to understand how children understand fractions' knowledge, he went beyond observing children's mathematical activities and investigated the mental actions involved in children's fractions' activities. He believed that describing children's activities is not sufficient in terms of understanding children's cognitive developments. Rather, one might understand their conceptual operations by going beyond the description of children's work and understanding their repeatable and generalized actions while they do the problem. Thus, he took the scheme as a conceptual tool to analyze children's language and actions.

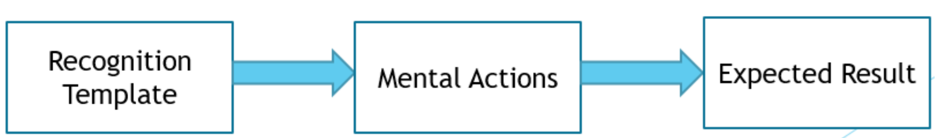
In the following section, I focus on the studies that use the scheme as their tool to understand children's fractional knowledge. I introduce the scheme from Piaget's perspective and continue with von Glasersfeld's scheme theory. Then, I explain how Steffe (2001) recognized different schemes that are contributed to learning fractions. Lastly, I review the studies that used Steffe's scheme (2001) for teaching and learning fractions.

Rational Number Knowledge in the Context of Scheme Theory

According to Piaget (1980), cognitive growth occurs through a series of stages, and it happens over time. Such growth is the result of the interaction of two processes of modifying the new information to adapt to prior information (*assimilation*) and reorganizing the prior knowledge to accommodate the new knowledge (*accommodation*). The process of assimilation and accommodation are controlled by the human mind and then represented as an action, which is referred to as a *scheme*. Building on Piaget's definition of the scheme, von Glasersfeld (1998) defined *scheme* as the objective-based activities which consist of three parts, including "perceptual situation (perceptual template), an activity associated with it, and the result the activity is thought likely to obtain" (p. 8). For von Glasersfeld (1998), a scheme is a tool that helps researchers elaborate the prior knowledge based on observable actions and then accommodate new situations based on the results. In other words, in forming a scheme, children perceive the subject, which triggers them to recognize a situation. Such recognition is the same as what Piaget named assimilation. That means children fit the new information/subject to their prior knowledge through the activity to get the actual result. "The actual result of the action must be such that it can be assimilated to expected one" (p. 6). In case of failing the expected result, the researcher must focus his attention on the initial situation and the trigger subject. Figure 1 demonstrates these three parts of the schemes.

Figure 1

The structure and pattern of the scheme

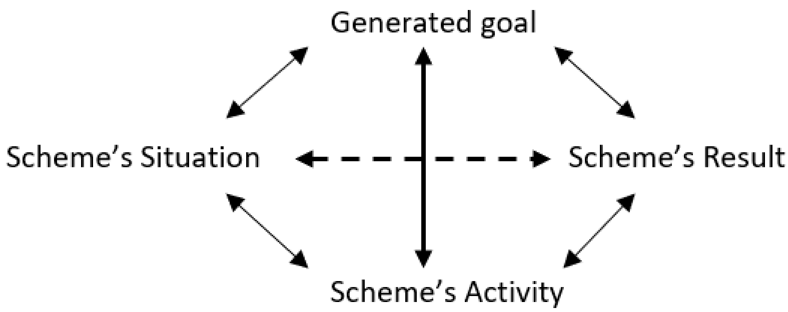


Note: (von Glasersfeld, 1998, p.6)

Borrowings from von Glasersfeld's scheme theory, Steffe (2001) postulated the scheme's structure using four interactive elements. For Steffe (2001), a scheme is a concept or knowledge that gain through children's mathematical language and activities. He pictured it as a tetrahedron (See Figure 2). The apex of the tetrahedron is the generated goal, and the three vertices (schemes' situation, schemes' activity, and scheme' result) are elements of a scheme. The double arrows indicate the comparability and relation of all elements in vertices. Moreover, the dashed arrow shows the expected result from the scheme. The diagram shows that the schemes can be reversible as well as one-way. Depending on the activity and situation that children may be involved with, they may need to start from the result and rebuild the situation using inverse operations. For instance, in order to find the fractions that represent $\frac{1}{2}$ of $\frac{1}{4}$ Children need to re-establish the whole by "using two equal pieces of half each partitioned into four equal parts" (p.23). On the other hand, some schemes are one-way schemes, and they proceed from subject-activity straight to the result.

Figure 2

A diagram for the structure of a scheme



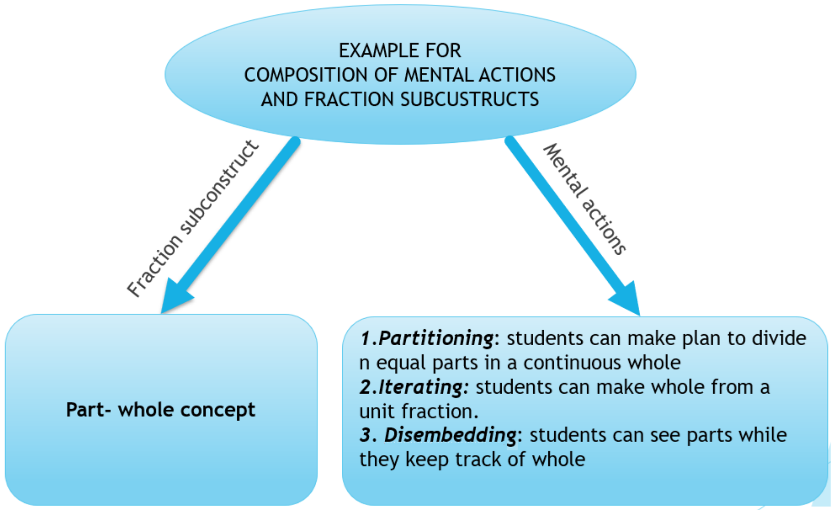
Note: (Steffe & Olive, 2010, p.23)

Rational Number Knowledge as Scheme

According to Steffe (2001), children's construction of fractional schemes emerged through their prior numerical counting schemes. Meaning that children use other schemes to build new schemes by reorganizing them. Steffe (2001) referred to the idea of establishing schemes as the reorganization hypothesis. However, to generate all these schemes, one must identify the mental actions contributed to the content of the activity first. For instance, the three mental actions associated with part-whole activity problems are partitioning, iterating, and disembedding (see Figure 3).

Figure 3

The Part whole fraction scheme



Originating from Steffe's (Steffe & Olive, 2010) reorganization hypothesis, Hackenberg et al. (2016) identified five mental actions including *unitizing*, *fragmenting*, *partitioning*, *iterating*, *disembedding*, and *distributing* (Hackenberg et al., 2016). *Unitizing* is a mental action of unifying the numbers of discrete and countable objects into a new convenient and familiar group/set to operate with that numbers (Lamon, 2007). Within the fraction context, the mental action can follow the same structures (Hackenberg et al., 2016). This means the whole can be identified as a unit, and by partitioning a whole into smaller parts, one can group those smaller parts into convenient units to operate with it. For instance, $\frac{1}{4}$ can be conceptualized as a unit itself which four of it create one whole ($\frac{1}{4}$ of a unit: ($\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$)). Alternatively, it can perceive as a whole that consists four of unit ($\frac{1}{4}$ of four one unit: (($\frac{1}{4}$), ($\frac{1}{4}$), ($\frac{1}{4}$), ($\frac{1}{4}$)) which means ($\frac{1}{4}$ ((1-unit) s of the unit).

Fragmenting and partitioning are cognitive processes that are developed through children's intuitive understanding of fractions (i.e., fracturing and equal sharing) and involve dividing a whole (either discrete or continuous) into smaller parts/units. In *fragmenting*, children fracture a continuous whole or share a certain number of objects among certain people with or without exhausting them. However, in *partitioning*, children require more complicated mental actions, including unitizing. For instance, in sharing ten cookies between two people, a child with fragmenting mental actions would share the cookies one after another till they finish the ten cookies. However, with partitioning, a child would anticipate that they need two units of five in their minds before they start sharing cookies. Thus, a child unitizes the cookies before doing the partitioning. I will explain more details about this mental action in the part-whole section.

Iterating is the cognitive process of making a larger unit by using smaller units. Anticipating a whole and using the appropriate number of units to create the whole is a reversible action that occurs in iterating. For instance, in order to know the number of times $\frac{1}{4}$ needs to be iterated to create a four, children need to imagine a whole and then project how many of $\frac{1}{4}$ can fit the whole. Thus, they need to anticipate a whole, then project $\frac{1}{4}$ as a unit into the whole to estimate the number of times that need to iterate $\frac{1}{4}$ to exhaust the whole. The iteration and partitioning actions in this activity can be considered as the inverse of one another to some extent.

Disembedding involves the coordination of units to specify a new unit while keeping track of prior units (Kosko, 2018). Within the fraction context, disembedding involves identifying parts and separating them from the whole while keeping track of the whole. For example, in the representation of $\frac{2}{5}$, children may be able to show $\frac{2}{5}$ in the context of two parts within the whole. But, if we present $\frac{2}{5}$ separated from the whole (i.e., two parts out of the whole), children who do not perceive disembedding may fail to connect the disembedded part to the whole and see the $\frac{2}{5}$ as a new and separate whole.

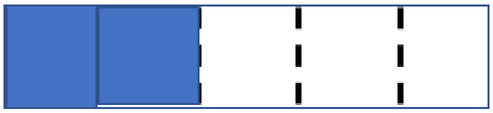
Distributing is forming a unit into other composite units. Usually, in the context of multiplication, this action is represented as repeated addition or forming groups of groups. However, this heuristic does not work very well for fractions. Instead, students can think of fraction multiplication as "taking a fraction of another fraction – composing two fractions". (Hackenberg et al., 2016, p.20). For instance, in making $\frac{1}{7}$ of $\frac{4}{3}$, given that the thirds marks are maintained, the distributive action/reasoning involves dividing the whole into three parts and taking $\frac{1}{7}$ of each part four times (Hackenberg &Tillema, 2009). As shown in other mental actions' examples, the distributing also engages other mental actions such as recursive partitioning operation.

The above mental actions have been identified in various children's mathematical activities and topics including the numerical counting scheme (Steffe, 2001), multiplicative and divisional scheme (Steffe & Cobb, 1998), and fraction scheme (Steffe & Olive, 2010). Acknowledging all these schemes intertwined and knowing one can facilitate learning the other, in the following, I only focus on fraction's scheme as of the interest of this paper.

The fractional schemes are categorized into five schemes as described in the following (Hackenberg, 2013). 1. The *part-within-whole fraction scheme* involves unitizing and partitioning as the mental actions. In this scheme, children are able to unitize the whole and partition the continuous whole. For instance, to represent $\frac{2}{5}$ of a continuous whole, the students partition the whole into five one units and then shade 2 of them. This partitioning may be of equal sizes or not (Punç-Pekkan, 2015). This scheme is also named a simultaneous partitioning scheme by Norton and McCloskey (2008). Figure 4 shows the example for a part-whole fraction scheme.

Figure 4

Part-within-the whole fraction scheme (showing 2/5)



2. The *part-whole fraction scheme* is where the children view a partitioned whole in terms of parts within and out of the whole. Unitizing, partitioning, and disembedding are three mental actions involved in the recognition of this scheme. In this scheme, children can identify units, partition a whole into units, and shade the parts that they have been asked to. However, the main step in this scheme is that children can track the whole while they do these actions. So, they successfully can represent the shaded parts out of the whole. Figure 5 shows an example of a successful *part*-whole fraction scheme.

Figure 5

Part-whole fraction scheme (showing 1/5)

The above schemes are associated with Kieren's part-whole subconstruct. However, such understanding is limited, and students in those levels of schemes "cannot identify the size of a given fractional part by iterating it within an unpartitioned whole" (Norton & McCloskey, 2008, p. 52). Thus, to go further with part-whole reasoning and conceptualize fraction as a measure, the child needs to develop the third scheme, which is partitive unit fraction schemes. *3. partitive unit fraction schemes are involved with iterating as a new mental action. For instance, in asking to prove how the child knows this is 1/4 of the whole, they would use* unitizing, partitioning, and disembedding to show 1/4 and then iterate ¼ to make a whole (see Figure 6).

Figure 6

Partitive unit fraction schemes (the shorter rectangle represents 1/4)

Conceptualizing that iterating a unit can make a whole is a fundamental element for children to understand non-unit fractions (Tzur,1999). The construction of this scheme also can be reversible. Hackenberg (2013) stated that children who develop this scheme could draw a whole based on a given part. For instance, "Given this segment that represents 1/6 of a whole, can you draw the whole?" (p.542). These children iterate a given part until they exhaust the whole.

4. Partitive fraction scheme involves unitizing, partitioning, disembedding, and iterating mental actions. However, it is at a higher level than partitive unit fraction schemes. That means children at this level can actually construct non-unit fractions out of the whole by iterating a unit from a given non-unit segment/fraction. More explanation, if the children were asked to show $\frac{2}{4}$ of a given whole, the children partition a whole into four equal parts, then take $\frac{1}{4}$ of it out and iterate it to show the $\frac{2}{4}$ (see Figure 7).

Figure 7

Partitive fraction schemes (1st rectangle represents 2/4, and the child takes 1/4 of it to make a whole)

5. *Iterative fraction scheme* is the last scheme, and children in this scheme can construct any proper or improper fractions through their three levels of unitizing (e.g., units of units of units), mental action, and the other actions. For instance, in constructing $\frac{12}{11}$, children iterate a unit fraction while keeping the unit in mind to construct the larger number (improper fraction). So, they have to identify $\frac{1}{11}$ as a unit and keep track of a whole (which is the other unit) until they get a $\frac{12}{11}$ (which can be considered a new unit). See Figure 8.

Figure 8

Iterative fraction scheme (the 1st rectangle [the given one] represents 12/11, the second is the whole [11/11])

It is to be noted that there has been another study that identified five fractional schemes from Steffe's work (Norton & McCloskey, 2008). In this study, Norton and McCloskey (2008) identified five schemes that are slightly different from Hackenburg's schemes. They identified *the Equi-partitioning scheme (i.e.,* unitizing, partitioning, and then mentally iterating any parts to determine their identity with the other parts) as the third fraction's scheme. They did not include the *Iterative fraction* in their identification of fractions' schemes.

Later, Wilkins & Norton (2018) synthesized all descriptions for fractions schemes that built upon Steffe and Olive (2010), Hackenberg (2013), and Norton & McCloskey (2008) descriptions. They basically synthesized all different descriptions of fractional operational schemes in the context of measurement (Wilkins & Norton, 2018). Their measurement schemes for fractions started with a *part-whole scheme* (PWS) as the indicator of children's understanding of the relation between part and whole. To construct this scheme children, utilize the two mental actions of partitioning and disembedding. The next fraction scheme is known as the *measurement scheme for unit fractions* (MSUF), in which children needed to contribute the iterating actions in constructing it. The *measurement scheme for proper fractions* (MSPF) is the scheme that deals with non-unit fractions. In this scheme children need to "iterate the unit fraction, as if it were a unit of 1, to measure both the whole and the non-unit fraction" (Wilkins & Norton, 2018, p. 4). The action that contributes to MSPF is called splitting, which is the combination of using partitioning and iterating at the same time. If students can produce any type of fractions including proper and improper fractions by the iterating unit of $\frac{1}{n}$, they successfully construct the *generalized measurement scheme for fractions* (GMSF).

As noted earlier, the first fraction scheme (i.e., *part-whole scheme)* is the beginning of constructing fractional knowledge, and it is generally considered to be constructed in elementary grades (as this advanced concept is introduced in elementary grades). However, studies show that many students do not construct the pre-requisite schemes (such as multiplicative reasoning) for developing fractional knowledge not only by the end of elementary grades (Hackenberg, 2007; Kosko, 2019), but also in middle school (Zwanch & Wilkins, 2021). For instance, Kosko (2019) found that only 46.3 % of fourth-graders demonstrated some advanced level of multiplicative concepts by the end of the school year. This percentage increased to 62.5% for fifth graders. In a similar study on 326 middle school students, Zwanch & Wilkins (2021) found that only 39.9% of whole students constructed different levels of multiplicative reasoning.

Given the sophistication of fractions, researchers suggest the necessity of developing prior schemes such as multiplicative reasoning (Hackenberg, 2007; Kosko, 2019), fragmenting schemes (Empson, 1995; Steffe & Olive, 2010), and numerical schemes (Steffe & Olive, 2010) before fractional schemes will develop. However, as noted above, most students are not demonstrating such pre-requisite schemes for understanding fractions even in middle school. Thus, it is necessary to further study those pre-requisite schemes and examine to what extent children are prepared to learn fractions or to construct fraction schemes. Across all those different pre-requisite schemes, fragmenting is the one that begins to develop at early ages. "For young students, the origin for fragmenting can come accidentally, like dropping a plate on the floor, or intentionally, such as breaking a cookie into parts to share with a parent or sibling." (Hackenberg et al., 2016, p.41). Despite the emergence of this scheme at early ages, developing fragmenting has a fundamental role to support part-whole scheme and other fractional schemes in general (Hackenberg et al., 2016; Steffe & Olive, 2010). The following chapter is allocated to studies focused on fragmenting scheme and how it is constructed and developed in children.

Fragmenting Schemes

Kieren (1993) suggested that part-whole reasoning has a fundamental role in understanding the other four subcontracts, including ratio, operator, quotient, and measure. Part-whole reasoning occurs when children view a partitioned whole in terms of parts within and out of the whole (Hackenberg et al., 2016). For instance, a child can see "a sandwich cut into fourths as one whole sandwich and also as a grouping of four 1/4-size pieces of a sandwich because each scenario describes the same amount of sandwich" (Empson et al., 2020, p. 279). The two mental actions contributed to this subconstruct discussed before. Although part-whole schemes have been identified as the first subconstruct of learning and developing fractional knowledge (Behr et al.,1983; Hackenberg et al., 2016; Kieren, 1993), researchers suggested an important pre-requisite scheme that needs to be developed before the part-whole scheme (Empson, 1995; Hackenberg et al., 2016; Piaget, 1960; Steffe & Olive, 2010). They called it a fragmenting scheme, which developed intuitively and informally through children's interaction and engagement with real-life problems (Empson, 1995). The following section focuses on how children develop fragmenting schemes aligned with the stages of developing such schemes.

Stages of Fragmenting

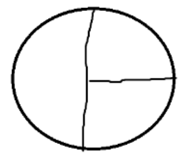
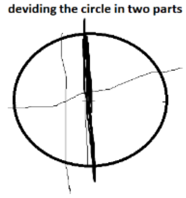
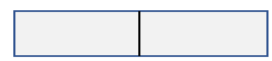
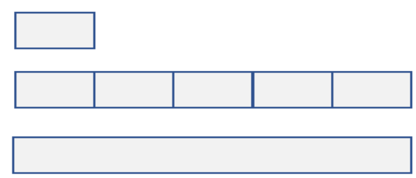
Children use their intuitive knowledge of sharing to do the basic stages of equal sharing (Empson, 1995). They apply real-life methods such as one-to-one correspondence, fragmenting, and partitioning to share discrete or continuous models. Examining children aged 3 to 4 years old, Hunting and Sharpley (1988) found that most children successfully shared 12 crackers among three dolls using one-to-one correspondence, or the 'dealing out' method. Using such methods correspond with their initial understanding of number sequence (Steffe & Olives, 2010). In comparing the above discrete model sharing task to its similar one but with a continuous model, scholars found the latter more complicated and challenging for children.

In the study conducted by Piaget (1960), he found that children aged four showed difficulties dividing a circle cake into half and sharing it with two dolls. In other words, children at age four usually divide the object not equally or divide part of an object without exhausting the whole. However, they were able to correspond each of the "non-equal" fragments to the dolls. So, regardless of the types of models, children are able to use their numerical sequence knowledge to correspond each figurative quantity to the number. The ability occurs at four different levels of fragmenting schemes that were founded by Steffe & Olive (2010) and later developed into five levels by Hackenberg et al. (2016).

At *the first level* of fragmenting, children are able to share a whole into two or three parts. However, these parts may not be equal. Steffe and Olives (2010) suggested more development of the operations regarding share into three parts. However, they hypothesized that children who establish figurative quantity are able to share a whole into either two or three parts (Steffe & Olives, 2010). In *the second level* of fragmenting, children are constructing equal sharing within the activity. They usually use trial and error to share a whole into two or three parts and exhaust the whole. It indicates that children cannot coordinate two goals of sharing the whole equally and exhausting the whole simultaneously. Moreover, focusing on one goal usually disturbs another one. *The third level* of fragmenting is that children accomplish two goals of sharing parts while exhausting the whole at the same time. So, at this level, children can anticipate the equal parts, project them into the whole in their minds, and then apply it to the task. At *the Fourth* level of fragmenting, children know how to coordinate the two goals for any given number of parts. They also know repeating one part can reproduce the whole "equi-partitioning". In other words, at this level, children can see the parts out of the whole. At *the fifth level* of fragmenting, children can do the same thing at level four but with multiple wholes into multiple parts. "For instance, sharing four bars among six people." (Hackenberg et al.,2016, p. 43). Table 1 presents the fragmenting levels aligned with the illustration for each level.

Table 1

Five levels of fragmenting

Levels of fragmenting	Examples of indicators of each level
Level 1: Children can mark a length into two/three parts. • Children are sensitive to make the parts equally, but it doesn't mean they would be successful)	Deviding cricle into three parts 
Level 2: Children are challenged to coordinate the two goals of making parts and exhausting the material. They are constructing the scheme within the activity (Pre coordinating level) • Trial and error can help them to achieve the goal.	deviding the circle in two parts 
Level 3: partitioning (Anticipatory scheme) • Children can coordinate parts and wholes	Share a continuous whole into two parts 
Level 4: equi-partitioning (connecting parts and wholes by iterating parts to construct the whole). • Partitioning a continuous whole into equally sized pieces, understanding that any one of those pieces could be iterated to reproduce the whole.	Iterating parts to construct the whole 
Level 5: involves multiple items sharing among multiple people. • Using partitioning, disembedding, and iterating at the same time (can see the chunk ($\frac{3}{5}$) of parts out of whole)	Sharing two pieces and three pieces of a whole between two people. 

Note. (Hackenberg, et al., 2016).

Developing fractions schemes is usually associated with the use of different representations. Many researchers employ different representations to observe children's cognitive developments and whether they have developed a specific scheme. However, they believe the types of representations should not impede children's success in doing the operation successfully. As Tunç-Pekkan (2015) mentioned:

> From the Piagetian perspective, if students have the necessary operations and schemes (partitive unit, partitive fraction, and iterative fraction schemes), then it should not matter

what material students work with, they should successfully produce results in the given situations (p.425)

Competence in using any representation as children's developed schemes is ideal. However, different studies indicate that children demonstrate different degrees of success in constructing fractions as the representation changes. For instance, studying fourth and fifth graders, Tunç-Pekkan (2015) found students' performance was significantly lower when using the number line compared to area models. In another study conducted by Cramer et al., (2008), they found that children's use of circle models helped them to understand the meaning of relative sizes for fractions.

The above studies indicate the necessity of taking a deeper look at how children with different fraction schemes use different representations. Thus, the following section reviewed the literature on different types of representations used in mathematics in general from different perspectives. Then it narrows down the focus on the specific representations that have been used to observe children's fractional knowledge through schemes.

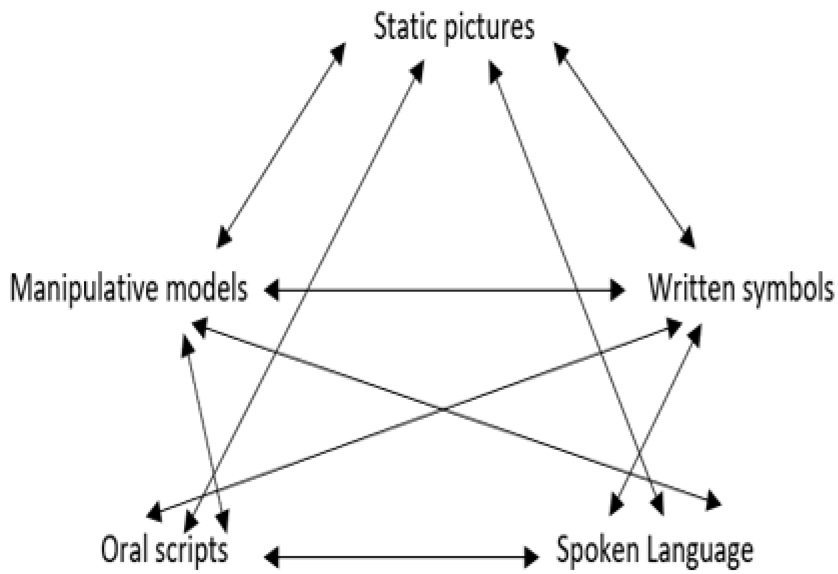
Mathematical Visual Representations

There are three types of representation defined by Bruner (1964): *enactive*, *iconic*, and *symbolic*, which proceed from the concrete to the abstract. In order to learn mathematics, children need to experience each of these levels respectively, and each level is a pre-requisite for the next level. *Enactive* representation is when children learn through acting and doing a math problem with concrete objects. *Iconic* representation is the image that illustrates the concept and conveys its meaning from the enactive experiences. The last representation is *symbolic*, which is abstract. "A word neither points directly to its referent here and now, nor does it resemble it as a picture" (Bruner, 1964, p.2). So, at this level, the concrete concept is transferred to conventional or symbolic representation. Children develop the concept through these three representations hierarchically. The big challenge is the transition from iconic to symbolic representation. In their experiment involving ordering an array of cups, Bruner and Kenney (1965) asked children aged 5 to 7 to rearrange nine differently scaled plastic glasses in three different orders. These cups were initially located in 3-by-3 grids in a cupboard in order of their size and height. Then Bruner and Kenny (1965) asked children to arrange the cups in two different ways after they were scrambled. First, they asked them to arrange the scrambled cups in the way they were initially introduced to them and then explain the order. Second, they left one of the cups on the grid and asked the children to arrange them just like the first time but kept this one cup here. The findings suggested that all children were easily able to do the first task, but only seven-year-old children were able to use mathematical language to explain the order of the cups (e.g., using higher and shorter to explain their dimensions as well as bigger or smaller to relate this to magnitude). Interestingly, only these children, those who could use mathematical language, could do the second transformed task. Bruner and Kenny (1965) suggested that children stuck at the iconic levels lack mathematical language and, as these features develop, they can transfer to symbolic representation.

Inspired by Bruner (1965)'s representations, Lesh et al. (1987) developed mathematical representation into five different elements: 1) Manipulative models, 2) Static pictures or diagram-static figural models, 3) Spoken language, 4) Written symbols, and 5) Real script (real experience) models. Each of these representations has a significant role in learning mathematics, but the transformations among these representations are also necessary (e.g., how children transform their knowledge using concrete models to written symbols). To indicate the importance of transitions among all these representations, Lesh et al. (1987) mapped them using double arrows as below (See Figure 9).

Figure 9

Five types of mathematical representations and the transformations among them



Note: (Lesh, et al., 1987, pp.33-40)

Later, Goldin (2003) presented a comprehensive definition of representation in mathematics and then proposed a theory of representation. Based on his definition, mathematical representation is the configuration of signs, characters, icons, or objects that can somehow stand for or "represent" something else (Goldin, 2003, p.276). Similar to Lesh et al. (1987), Goldin viewed mathematical representation as a system in which all characters/signs, configurations, and structures are meaningfully intertwined. He categorized mathematical representation into two forms of *internal* and *external* representation and attributed specific elements to each of them.

The internal representation is defined as "the natural language of individuals; their visual imagery and spatial, tactile, and kinesthetic representation; their problem-solving heuristics and strategies; their personal capabilities, including conceptions and misconceptions, in relation to conventional mathematical notations and configurations; their personal symbolization constructs and assignments of meaning to all these; and their effect especially in relation to mathematics" (Goldin, 2003, p.277).

External representation is defined as "conventional graphical, diagrammatic, and formal notational systems of mathematics; structured learning environments that may include concrete manipulative materials or computer-based microworlds, and sociocultural structures, such as those of kinship, economic relationships, political hierarchies, or school systems" (Goldin p. 277). For Goldin, external representation serves as a means to foster the development of internal representation (Goldin & Shteingold, 2001). His perspective aligned with what Zhang (1997) suggested as seeing external representation more than a simple input and stimulus to the internal mind. Thus, internal and external representations are intertwined (Behr, Lesh, post & Silver 1983; Goldin & Shteingold, 2001; Zhang, 1997) and, in terms of understanding children's knowledge, one should consider the interactions between both representations of problem situations (Behr et al., 1983).

As mentioned above, both representations are considered essential in children's mathematical education. However, because internal representation is not easy to observe, most teaching instructions and textbooks refer to external representations as their primary representation in math education (Mainali, 2021). The external representations have been classified into different elements/modes based on the nature of representation employed in teaching and learning mathematics. For instance, Janvier (1987) suggested four modes of external representation including: verbal (i.e., text, symbols, and sentences), table, graph (i.e., drawings, pictures), and formulae (equations). Lesh et al. (1987) classified modes of external representation into five elements: manipulative models, visual representation (i.e., visual static model, diagrams, and figural models), spoken language, written/symbols, and real scripts (real experience) models. All these five external representations have been employed in different mathematical topics, including a visual representation (e.g., array models and group in) for multiplication and division (Kosko, 2019), a figural model for teaching functions (Yerushalmy and Shternberg, 2001), the graphical representation for algebra (Friedlander and Tabach, 2001), written symbols and pictorial representation for number sequence (Bialystok and Codd, 2000), and manipulatives for adding and subtracting fractions (Suh, 2005).

Acknowledging the importance of all different modes of mathematical representations, one of the popular modes that reinforce teaching and learning mathematics is visual static representation (Arcavi, 2003; NCTM, 2000). Visual static representation is "a specific type of visual mathematical representation that include fixed pictorial images of mathematical concepts" (Anderson-Pence et al., 2014, p. 2). Children often use these pictorial images to show their understanding of the math concept before they use symbolic representation (Bruner,1964). For instance, Allardice (1997) found children aged three to four years old dominantly use visual static representations such as pictures, tallies, or circles to indicate the number of toy mice and buttons on the table. The tendency to use visual static models decreased in older children (six years old) as they exposed more to symbolic representation (e.g., writing numbers). The essential role of drawing and using visual models as initial steps to scaffold student's thinking has been stressed for many math topics including fractions (Allardice, 1997; Bialystok & Codd, 2000; Bruner, 1964, NCTM, 2000, Lesh et al., 1987). The following section is allocated to the research that used different visual representations to understand students' reasoning of fractions.

Static Representations of Rational Numbers

There are several visual representations of teaching and learning fractions in elementary grades. For instance, Behr et al. (1983) categorized fractional visual representation into three main groups: geometric regions, sets of discrete objects, and the number line. They found these three groups as the most common representations of fractions that have been used in elementary and junior high school. Later, Cooper et al. (2012) used a more generic classification for fractions representations and classified them into continuous and discrete visual representations (Cooper et al., 2012). Continuous visual representation is the model that "can be subdivided into smaller pieces" (p.7) and can be classified into three types of area models, length models, and number lines. On the other hand, discrete visual representation is the model "consists of a group of undividable objects." Set models are usually classified in this category.

Studies show that variance of cognitive demands is needed depending on using a discrete or continuous model. However, both groups of models scaffold children's cognitive development of fractions (Hunting & Sharpley, 1998; Piaget, 1960; Behr et al.,1983). For instance, Hunting & Sharpley (1998) found that most children successfully shared 12 crackers (i.e., discrete model) among three dolls using one to one corresponding (dealing out) method. However, they showed some difficulty in sharing one whole cookie (i.e., continuous model) among three dolls. Piaget et al. (1960) suggested that such difficulty is due to less cognitive demands in sharing discrete models. It is rarely needed to track the whole in sharing discrete models. However, the coordination between sharing equally while exhausting the whole is necessary using continuous models. Despite the variation of cognitive demands between these two models, the interaction of both models is necessary to have a robust understanding of rational numbers. As mentioned by Behr et al. (1983): "It seems plausible that the part-whole subconstruct, based both on continuous and discrete quantities, represents a fundamental construct for rational-number concept development" (P.11). However, the present study focuses on a continuous model as it needs more cognitive process, and it uses more often to teach and learn fractions. Specifically, the study focuses on the area model, including the circle, rectangle, and length model as the most common continuous representation that is used in fraction education and specifically with the part-wholes concept (Gould, 2013; Ni, 2000; Tunç-Pekkan, 2015; Watanabe, 2002).

Circle models

The circle model is one of the popular area models for teaching fractions, specifically part-whole concepts (Tunç-Pekkan, 2015). Referring to cutting a circular pizza makes the circle model as one of the popular real-life examples of presenting the part-whole concept. In the rational number project (RNP), Cramer and Henry (2002) found that fourth and fifth-grade students dominantly used circles to represent the fractional models (Cramer & Henry, 2002). Circle models support students' mental images of fractions and help them to conceptualize some difficult aspects of adding and subtracting fractions, such as finding common denominators (Cramer et al., 2008).

The third graders' outperformance in showing fractions using circle models was also found in Kaminski (2018)'s study. However, Kaminski (2018) surmised the children's outperformance on the circle model is due to the easier processing of the visual proportion than in other types of models. Kaminski (2018) suggested circle models "may be suitable for early introduction of basic fraction labeling, but they appear to provide little or no benefit for instruction on fraction arithmetic and fractions greater than 1." (P.6). Similarly, Hamdan & Gunderson (2017) found that despite most second and third grade's students performed well showing fractions using circle models; they often failed to transfer such knowledge to fraction magnitude comparisons (Hamdan & Gunderson, 2017).

Rectangle models

On a similar note, textbooks and teachers also heavily rely on using rectangular models to represent the concept of part-whole. Since rectangular models are considered as a linear model (Tunç-Pekkan, 2015), and there is a link between the linear model and the number line (Larson, 1980), researchers recommend that using rectangular fractions can facilitate students' transition from area moles to number lines (Hackenburg et al., 2016; Larson, 1980).

While there has been a long debate on the privilege of using each of these area models, there is no consensus on prioritizing one of the models (i.e., circle or rectangle) in fraction teaching and learning. Prior studies on comparing the static representations found different results on students' representation performance. In the study conducted by Tunç-Pekkan (2015), she used three pictorial representations (circle, rectangle, number line) to compare 656 fourth and fifth students' solving tasks related to fraction representation. She found that fourth and fifth-grade students performed similarly on the circle and rectangle models; however, they showed difficulty using number lines. Consistent with this result, Sidney et al. (2019) found no difference in students' responses using circles and rectangles for solving fraction divisions problem. However, Sidney et al. (2019) noticed students' who used number lines were more accurate in solving fraction divisions problem. Despite the similarity of students' performance on both static models (circle and rectangle), scholars have a different tendency to use each model. Cramer and Henry (2002) found Fraction circles as a "powerful model for fraction addition and subtraction." (p. 490). Yet, a rectangular model can use as a facilitator to move knowledge from the area model to length model, and then the number line.

Number Lines

Researchers and practitioners may have different ideas on prioritizing circle and rectangle models, but they all agree on the importance of using number lines in teaching and learning fractions (Behr et al., 1983; Hackenberg et al., 2016). It is probably not exaggerating if we assume the number line is the most challenging/difficult representation and the most important representation for conceptualizing fractions. In studying 77 fourth-grade students, Behr et al. (1983) found the number line is the most challenging representation among all other types of representation (e.g., set models and area models). Children's understanding of fractions is mostly held in the context of area models. And they do not conceptualize fraction as a magnitude itself to show it in the number line.

In exploring students' understanding of fractions using a number line, Bright et al. (1988) identified three main differences in number line representation as to the main reasons that number line requires a higher level of cognition. The three differences include:

> "First, a length represents the unit, and the number line model suggests not only iteration unit but also simultaneous subdivisions of all iterated units. That number line can be treated as a ruler. Second, on a number line there are visual separation between consecutive units. That is, the model is continuous. Both sets and regions as models possess visual discreteness. When regions are used, for example, space is typically left between of the unit. Third, the number line requires the use of symbols to convey part intended meaning" (p.215).

Such differences are the main reasons children cannot spontaneously make the transition from area representation of fractions to number line representation of fractions. In their examination of five fourth graders' understanding fractions using a number line, Bright et al.

(1988) found that children are more successful in dealing with number lines showing only whole numbers of ticks.

However, they showed considerable difficulty when they had to repartition number lines to indicate the reduced fraction symbols. Bright et al. (1988) suggested that partitioning and repartitioning play a fundamental role in understanding fractions in the context of the number line. They added, "as long as partitioning and un partitioning are difficult for children, number line representation of fractions may not be easily taught" P. 229. Similarly, Saxe et al. (2007) found that children in 3rd to 5th grades struggled to put a fractional quantity when the number line included the partitioning ticks and the ticks that indicate the whole numbers. Most children count the ticks without considering the space between whole numbers. Likewise, Cramer et al. (2017) observed students' errors in locating a unit's fractions on a number line. According to Cramer et al. (2017), to relieve such errors, the number line model should be introduced after students have practiced with other models that allow them to form strong mental pictures of fractions as numbers.

It appears that prioritizing models in teaching and learning fractions and using them in logical hierarchical orders would benefit children's comprehension of fractions. However, it seems such consideration (prioritization of models) needs to be in the direction of children's scheme development. Children often being exposed to different visual models simultaneously as scaffolding tools to build and develop their fractions' scheme at different stages. As children construct schemes stepwise, it might be beneficial for them to interact with a model that help them the most in developing a certain scheme. In the next chapter, I discuss the conjectures about the connection between visual models and children's fragmenting schemes.

Overview of Study

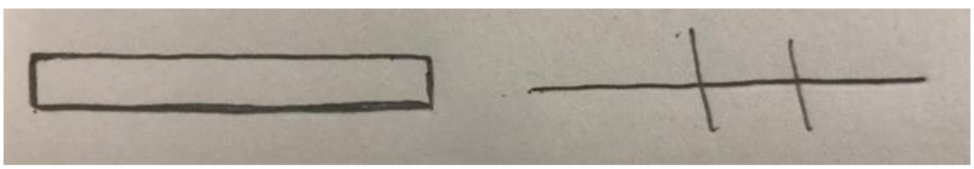
The purpose of this study is to investigate children's operations on fragmenting schemes while they are engaged with different visual models. Previous literature discussed the important role of fragmenting schemes to develop other fractional schemes (Hackenberg et al., 2016; Steffe & Olive, 2010). They also discuss the essential role of visual models in scaffolding students' thinking for fractions as one of the advanced mathematical topics (Allardice, 1997; Bialystok & Codd, 2000; Bruner, 1964, NCTM, 2000, Lesh et al., 1987). Several studies focused on students' fractional thinking and its interaction with visual models focusing on fourth and fifth graders (Cramer and Henry, 2002; Sidney et al., 2019; Tunç-Pekkan, 2015). However, evidence from the literature shows that children at early ages intuitively develop the concept of fragmenting and equal partitioning by fracturing cookies, sharing amongst friends, and so forth (Behr et al., 1983; Hackenberg et al., 2016; Empson, 1995). To understand how visual models interact with children's knowledge of fragmenting it is important to focus on students in their early school years. This study focuses on first and second grade students to answer the three following questions:

1. What fragmenting schemes do early elementary students demonstrate?
2. How are fragmenting schemes manifested when children engage in three different visual models, including circle, rectangle, and length model?
3. How do children enact their partitioning schemes when working with odd and even numbers?

Upon the investigation of the above research questions, there are two different conjectures about the connection between children's formation of fragmenting schemes and visual models; dimensionality of a shape and numerality of partitions. The first conjecture deals with the dimensionality of the visual models and how conceiving the dimension of the model affects children's reasoning via fragmenting. In cutting two-dimensional shapes, children need to coordinate the two goals of cutting parts and exhausting the whole (Hackenberg, et al., 2016). It should be noted that I am not considering if the parts are equal at this stage, because itneeds more development schemes. However, I conjecture that even though both circles and rectangles are two-dimensional shapes, the cognitive demands of coordinating two goals for circle models might be different from rectangular models that can be conceptualized as length models. For instance, in cutting a length model into three parts, children might consider a 2-dimensional length model as a 1-dimension line, and then begin cutting the line three times (See Figure 10).

Figure 10

The illustration of mentally transforming a 2-dimensional length model to a 1-dimensional line for cutting into 3 parts



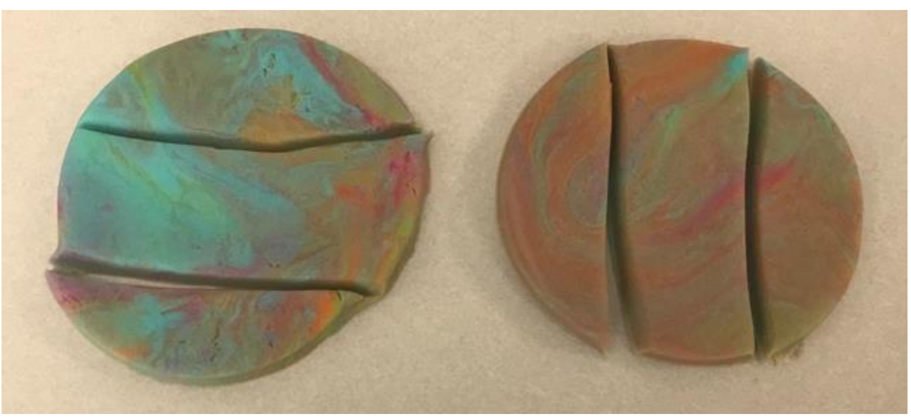
Considering a length model as a one-dimensional line is a less cognitively demanding task. In cutting a one-dimensional line into three parts, children deal with coordinating one goal of cutting into a given part rather than coordinating two goals of cutting into three parts while considering the nature of the whole.

In contrast to the length model, reconceptualizing a 2-dimensional circle shape into one dimension shape is difficult and even impossible for children at early school age. So, in cutting a circle into three parts, children deal with two actions of cutting a 2-dimensional circle into three parts and using up the whole circle. The necessity of coordinating these two actions/goals at the same time for circle shapes makes this shape more challenging for young children as compared to the length model. I conjecture that children may need more cognitive development for cutting a circle as compared to the length model.

The act of coordinating two goals is not the only thing that increases cognitive demand of partitioning a circle. Children might be able to coordinate these two goals and cut the circle into three parts while finishing up the whole circle, but that doesn't guarantee they can successfully partition the circle into equal parts. For instance, a child who constructs the coordination of two goals in their mind may cut the circle horizontally or vertically (see Figure 11). However, for successful equal partitioning, children need to identify the center of the circle which requires some explicit instruction. In all, the characteristic of the shape adds extra layers of complexity to the development of fraction schemes and in particular to the fragmenting schemes.

Figure 11

Illustration of cutting a circle into three parts vertically and horizontally without considering a center



The second conjecture focuses on children's partitioning shapes into even or odd numbers. Equi-partitioning schemes are considered a fundamental scheme for developing other fraction schemes (Steffe & Olives,2010). It is expected that children who developed equi-partitioning schemes are able to cut different shapes into any type of number (i.e., even and odd).

However, children are more comfortable cutting even numbered portions (i.e., half or fourth) than an odd number of portions. One possible reason is the amount of experience that a child has with cutting different types of number. Children often partition a shape into halves intuitively as early as preschool (Empson et al., 2006). However, such experience rarely happens in cutting a shape into three or five parts. The experience of cutting into even numbers may positively affect their performance in using different shapes.

The two conjectures described above served as motivation for answering the aforementioned research questions. The following chapter describes the methodology and procedures for this study and how the method helps to address the research questions. The chapter involves providing explanation and rationale of using sequential explanatory mixed-methods study design for this project, with the description of the process of collecting and analyzing both quantitative and qualitative data.

CHAPTER III

METHODS AND PROCEDURES

The goal of the study was to examine first and second graders' early fraction knowledge within three different visual static models, including circular, rectangular, and length models. Notably, the study used the variation of early fractions' tasks aligned with fragmenting levels created by Hackenberg et al. (2016) to understand children's fragmenting knowledge. A sequential explanatory mixed-methods study design was taken to achieve the purpose of the study. In which the quantitative data were first collected using the designed tasks containing the four levels of fragmenting questions. The goal of collecting quantitative data was to understand the children's knowledge of early fractions and the effect of different static models on their achievements. After collecting and analyzing the quantitative data, the qualitative data were collected subsequently. The data was collected through the one-to-one clinical interview using purposeful participants based on their responses to the tasks. Collecting data through clinical interviews elaborated the underlying reasoning of children's responses to each task and backed up/confirmed the why and which of the quantitative results. Thus, in this study, the sequential explanatory mixed-methods design is utilized in which a quantitative method is conducted to answer the research questions, and then the qualitative analysis is applied to explain and reflect the results obtained from the quantitative phase (Cresswell & Clark, 2011).

Research Questions

The general purpose of the study is to examine children's early knowledge of fractions and to examine the relationship between children's fragmenting schemes and their use of different static models. Particularly, the study addressed the three following questions:

1. What fragmenting schemes do early elementary students demonstrate?
2. How are fragmenting schemes manifested when children engage in three different visual models, including circle, rectangle, and length model?
3. How do children enact their partitioning schemes when working with odd and even numbers?

The main objective of research question one is to understand children's early fraction knowledge level. The question was answered using a quantitative method of collecting children's responses to the tasks. The data were first coded/graded by two experts, and then the interrater reliability was examined. Following that, the descriptive statistical analysis was used to identify children's early fractions knowledge level.

In order to answer research question two, I used ordinal logistic regression. The regression modeled the effect of using three different visual static models (i.e., circular, rectangular, and length models) and students' grade levels on early elementary students' fragmenting levels. The non-parametric Friedman tests aligned with descriptive statistical analysis were also used to address research question three and to know if children's interaction with fragmenting schemes relates to the type of number (i.e., odds and even numbers). Additionally, clinical interview data were collected and interpreted qualitatively to identify the underlaid cognitive reasonings of children's responses to the tasks and the results from quantitative data.

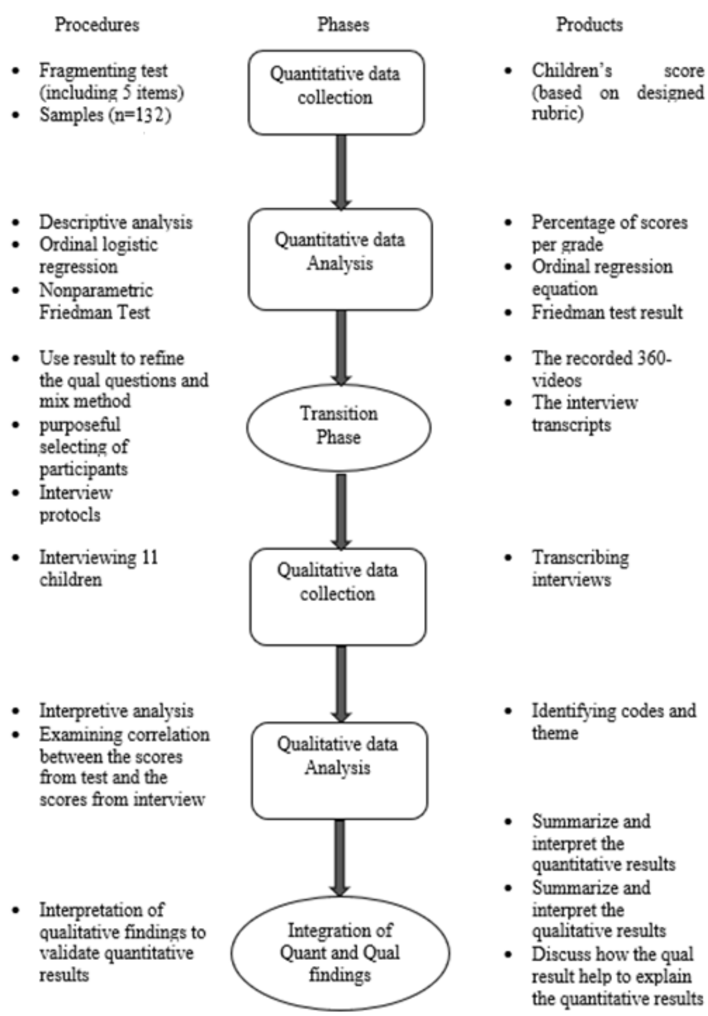
Research Design

Mixed-method analysis is an approach that associates both qualitative and quantitative forms. In a mixed-methods study, the researcher collects and analyzes qualitative and quantitative data, then integrates or links them either later or concurrently while framing these procedures within theoretical perspectives (Cresswell & Clark, 2011). To conduct a mixed-method study, a researcher needs to have a comprehensive knowledge of the worldviews. Worldviews or paradigms are the "general orientations/beliefs about the worlds and the nature of research that a researcher holds ... the types of beliefs held by individual researchers will often lead to embracing a qualitative, quantitative, or mixed-method approach in their research" (Cresswell, 2009, chapter 1, p.6). Among different types of worldviews, four inform different methods, including Post positivism, Constructivism, Advocacy/participatory, and pragmatism (Cresswell, 2009; Cresswell & Clark, 2011). Each of these worldviews has different assumptions/stances about the nature of reality (ontology), knowledge (epistemology), value, and valuation (Axiology) and has different perspectives about the method (methodology). For instance, a postpositive worldview deals with the approximation of existing reality and, thus, is associated with the quantitative approach (Hatch, 2002). Whereas for constructivism, the reality is known through actions, and it is more dialectical and inductive; thus, this worldview is more associated with qualitative methodology (Guba and Lincoln, 2005). In doing mixed-method research, a researcher believed in singular and multiple realities (Cresswell & Clark, 2011). Thus, they require to integrate other paradigms and all insights that provide quantitative and qualitative in a practical way.

For the purpose of this study, a sequential explanatory mixed-method study design was taken to understand children's levels of early fractional knowledge using different static models. This method has two phases that happen sequentially with the dominant use of the quantitative approach. In the quantitative phase, the data was collected through three sets of tasks (i.e., each set uses different circular, rectangular, and length) containing four fragmenting questions. Then, in the qualitative phase, the purposeful sample data was collected quantitively through a one-to-one clinical interview in the second phase. The purposeful selecting of children and interviewing and observing them provided insights into their understanding of early fractions concepts. Namely, it elaborated on children's interactions with these three visual models, the types of number for partitioning, and their fragmenting scheme. Figure 12 illustrates the visual model of the sequential explanatory design that is utilized in this study.

Figure 12

Summary flowchart for the explanatory sequential design



Participants and Setting

The participants for this mixed-method study were the first and second grade students attending from 3 different schools located in the Midwestern U.S. The total first and second grade students enrolled in all three schools were 132 students, where 50 % of the students' population self-identified as female, and 50% of them self-identified as male (will add description of race). The children were in the age range of seven years old to nine years old, with the mean age of $\bar{X}$ = 8.

Because the study used a mixed-method sequential design primarily focused on the quantitative phase and followed the quantitative method, the appropriateness and justification for selecting participants for each phase need to be described separately. Thus, the following section is devoted to the description of the sample for each phase, along with the utilized tasks and rubric.

Participants for the Quantitative Phase

Participants included 132 elementary students from three distinct schools located in the Midwestern United States. Across all 132 students, 49.2 % (n= 65) were first graders and 50.8% (n=67) were second graders. Descriptive statistics of participants' major can be found in Table 2.

Table 2

Descriptive statistics for students' grade level and gender

	Gender					
	Male		Female		Total	
Grade	Frequency	Percent	Frequency	Percent	Frequency	Percent
First	35	26.5%	30	22.7%	65	49.2%
Second	36	27.3%	31	23.5%	67	50.8%
Total	71	53.8%	61	46.2%	132	100%

Two criteria were considered regarding choosing this sample. First, children start to develop an understanding of fractions as a number in third grade (Common Core State Standards, 2010). However, most of them intuitively develop the concept of fractions by fracturing cookies or sharing candies among their friends at a much earlier age (Behr et al., 1983; Hackenberg et al., 2016; Empson, 1995). Even many textbooks such as McGraw-Hill My Math (2017) and Everyday Mathematics (2016) practice early fraction concepts such as partitioning and use some fractional language such as half, third and fourth in first and second grades. Thus, selecting children from these grade levels was ideal for examining children's fragmenting.

The second criterion was related to selecting the appropriate sample size to conduct the study. Since the study is quantitative dominant and used an ordinal logistic regression model, the appropriateness of the size of the sample should be considered to have the valuable statistics that represent the variables in the targeted population. Acknowledging different recommendations for the size of the sample in doing logistic regression (Peduzzi et al.,1996; Long,1997), this study followed Hosmer et al. (2013) suggestion of using the rule of 10 (or 5–9) events per parameter. As Hosmer et al., "the ten events per parameter rule may be a good conservative working strategy for models with continuous covariates and discrete covariates with a balanced distribution over its categories" (Hosmer et al., 2013; p.313). Since the study had nine parameters, selecting 132 participants allowed a more 'conservative' (i.e., having a larger sample size is always recommended) and an appropriate sample size to generate statistics that reflect the intended population.

Participants for the Qualitative Phase

Different types of purposeful sampling techniques were suggested as using mixed-method study. The achieve representativeness or comparability sampling was used for this sequential explanatory design. According to Teddlie & Yo (2007), achieve representativeness or comparability sampling is used "when the researcher wants to (a) select a purposive sample that represents a broader group of cases as closely as possible or (b) set up comparisons among different types of cases" (Teddlie & Yu, 2007, p. 80). The qualitative phase of the study aims to understand children's cognitive reasoning at different stages of fragmenting and compare the difference across them. Thus, the comparability sampling helped to have a wide range of children cover all levels of fragmenting.

To find the participants for the clinical interview, at the end of the test the participants were asked if they would also be willing to participate in a clinical interview. Of the total of 132 children, 90.1% (*n*= 119) of children were agreed to participate in one-to-one clinical interviews. After careful reviewing of the responses from those children who agreed to interview, 11 were selected to participate in one-to-one clinical interviews to embrace the different range of fragmenting abilities. Among all 11 students, five were first graders, and six were second graders—four of the participants self-identified as boys, while seven self-identified as girls. Two criteria were considered to select these students: first, to make sure that their answers to the survey were as varied as possible from each other. The wide range of responses allowed us to have all possible illustrated fragmenting schemes. In doing so, I hypothesized different levels of fragmenting based on students' responses, and then I chose them from the consent pool.

The second criterion was related to the model of the survey. Recall that students used different sets of a survey with different models of circle, rectangle, or length. To ensure that we have all three types of shapes in our clinical interviews, we picked participants based on the survey's model. Table 3 presents students' pseudonyms, demographic descriptions, and the hypothesized levels of fragmenting for clinical interview.

Table 3

Description of all case studies of students chosen for clinical reviews

Student Pseudonym	Grade level	Gender	Hypothesized level of fragmenting	Model of the survey
1.Sam	1st	Male	One	Length
2.Bella	1st	Female	One	Circle
3.Dina	2nd	Female	One	Length
4.Judy	2nd	Female	Three	Rectangle
5.Jack	1st	Male	Two	Circle
6.Hannah	2nd	Female	Two	Circle
7. Zara	2nd	Female	Three	Length
8. Adam	1st	Male	Three	Circle
9. Erika	2nd	Female	Two	Rectangle
10. Liam	1st	Male	Three	Length
11.Gabby	2nd	Female	Two	Rectangle

Measure and Protocols

In this section I described the measure and protocols for both quantitative and qualitative phases. First I described the measure that is used for the study and the rubric for coding the data. Then I explained the protocols and steps for collecting qualitative data.

Quantitative Measures

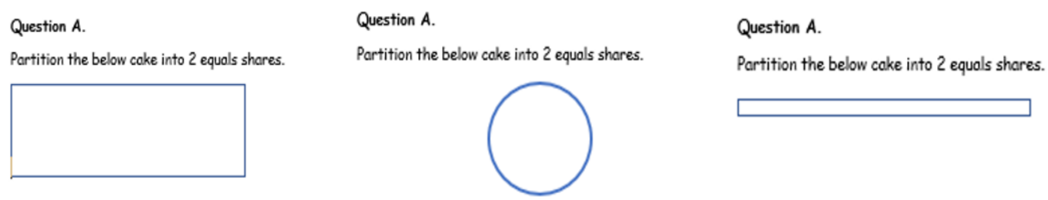
The fragmenting test was designed to examine if there is a difference in children's performance on their levels of fragmenting within three statistical models. In doing so, the open-ended items inspired by the list of activities and assessments described by Hackenberg et al.

(2016) were created. The fragmenting test started with one demographic question concerning participants' gender. The fragmenting test consisted of four items plus a bonus item designed in three different forms circle, rectangle, and length model. The overall purpose of these open-ended items was to examine if students can share the continuous whole equally into less than five parts while exhausting the whole. The selection of parts less than five was due to the focus of the study on the first four-level of fragmenting (Hackenberg et al., 2016; Steffe & Olive, 2010). It is expected that children at these four levels can share the whole into less than five parts. "There would be no a priori necessity to share the whole of the continuous unit into so many equal parts" at these four levels (Steffe & Olive, 2010, p. 69).

The first item tasked participants with fragmenting or partitioning the whole into two parts. It particularly asked to partition a cake (i.e., circular, rectangular, and length) into two equal parts. The second, third, and fourth items followed the same construct but asked to partition a whole into three, four, and five equal shares, respectively. The bonus item is asked children to draw one-fourth of a cake. Since children are not expected to know the fractional language at this age (Common Core State Standards, 2010), this item was designed as a bonus, so children could easily pass the item without answering it. The rationale for including this item is to assess students' knowledge of fractions' notation and its connection to fragmenting. Prior studies showed a gap between children's knowledge of fragmenting and fraction language (Olive & Vomvoridi, 2006), and this item allowed me to experiment if children with the upper level of fragmenting were able to connect their knowledge to the fraction's language. Figure 13 demonstrates the first item of the fragmenting test for all three models of rectangle, circle, and length. The illustration of all items can be found in appendix A.

Figure 13

The five designed items use rectangle, circle, and length of representation



The intended purpose of fragmenting test is to assess children's level of fragmenting schemes and their interaction with different representations. To measure it, we coded the fragmenting test using Hackenberg et al.'s (2016) fragmenting levels as the main resource. Initially, ten samples of students' responses for fragmenting test (from all three forms of representations) were selected, and then two researchers used the rubric independently to code them. The data was coded using the modified fragmenting framework (see Table 4).

Table 4

Modified rubric for designed questions

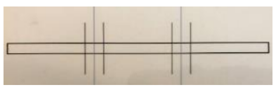
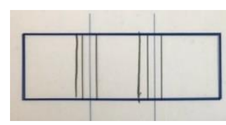
Levels of Fragmenting	Examples of indicators of each level per question			
	Q1: dividing into 2 parts	Q2: dividing into 3 parts	Q3: dividing into 4 parts	Q4: dividing into 5 parts
Level 0: No attempt to segmenting				
Level 1: Making a whole into up to three				
Level 2: Coordinating the two goals of making parts (up to five) and exhausting whole				

Level 3: Coordinating the two goals of making "equal" parts (up to five) and exhausting whole (constructed in the activity)	
Level 4: Equal partitioning a whole up to five parts (anticipatory level)	

It should be noted that, coding whether the students' responses were "equally" partitioned a shape or note not is challenging. There are sometimes disagreements on how teachers and math education scholars assess a child's partitioning in terms of equality of parts– and so while some students' cutting shape may seem obviously partitioned equally and correct (or correct enough) to one teacher, it may seem more obvious to be incorrect to another. Thus, I designed a survey of students' different ways of cutting shapes and shared it with ten math educators and teachers to find the "acceptable range" among all answers. Building on their responses, I created a transparent template of the acceptable range for all three shapes and all of the parts. Finally, the twelve templates were used to assess whether the children's partitioning shapes fit into the acceptable equal range. The template for partitioning into four equal parts with all three models is shown in Figure 14. The templates for other parts are provided in Appendix B.

Figure 14

Transparent templates of acceptable range for each model into 3 parts

Transparent templates for length model	Transparent templates for circle model	Transparent templates for rectangle model
		
Student's ***correct*** partitioning a length into three parts is proven by ttransparent templates	Student's ***correct*** partitioning a circle into three parts is proven by transparent templates	Student's ***incorrect*** partitioning a rectangle into three parts is proven by transparent templates
		

All four items were checked simultaneously and were coded from level 0 to level 4 based on responses to all items. Thus, for each student, only one level of fragmenting was designated. The bonus item didn't include for measuring the fragmenting levels and was coded separately as 0 or 1, in which 0 indicated not successful, and 1 indicated a successful performance on that item. This step allowed us to examine if the initial rubric could identify all possible students' strategies in the data. After initial coding, the two researchers met and discussed the applicability of the rubric for data of the study and the possible ways of improving and modifying it by using examples and explanations. First, we discussed if there is any variance in terms of coding between two of the raters (i.e., the inter-rater reliability). Second, if there is any strategy or theme emerged across children's responses that cannot be explained by the initial rubric. After finalizing the rubric, two researchers used it independently to code the data received from fragmenting test. It should be noted that the reliability and trustworthiness of the collecting data will be explained later in the reliability and trustworthiness section.

Qualitative Protocol

Following administration and coding of the fragmenting test data for the quantitative phase, participants were identified for clinical interviews in the qualitative phase. Out of 132 participants who completed the test, 11 participants were selected for one-on-one clinical interviews. The purpose of clinical interview was to 1) examine children's responses to each task and explore their underlying rationales for their actions, and 2) to validate whether scores on the fragmenting test aligned with children's responses in the interview. The interview lasted between 20 to 30 minutes and was videotaped and recorded. The type of video used for recording was 360 video. This specific type of video records in a spherical direction (omnidirectionally), allowing the viewer, in this case the researcher, to select what is perceivable from the physical position of the camera (Kosko et al., 2021). Thus, the recorded 360 video captured a wider range of what occurred during the interview, which allowed for moments that could be missed with standard video to be included for analysis.

For the interview, each child was asked to use pre-cut circles, rectangles, and length shapes made from playdough to explain their reasoning. Each child used a safe plastic knife to divide each premade shape into different parts and explained why they think the parts are equal. The steps and details of the interview protocol are described in Table 5.

Table 5

The interview protocol

Interview Questions	Materials provided
Ask a child to explain what they did for each question.	The prior test
Ask a child how we can tell whether the shares are equal? Can they prove it? The child can use manipulative provided for them to show their work.	Play-dough (all three shapes were cut out using playdough; thus, the shapes were made with accuracy)
Ask the child to use the other type of model (the model that did not already use for the test) and try to solve them.	The other two test with different models
Ask the child which one they think was easier to make equal? And why?	Play-dough (all three models were prepared using playdough; thus, the shapes were made with accuracy)

Reliability and Trustworthiness

The process of reliability and trustworthiness were applied separately for both quantitative phase and qualitative phase. Two pieces of evidence of expert judge and inter-rater reliability were utilized to assess the reliability of the fragmenting test in the quantitative phase. Also, three different pieces of evidence, including peer debriefing, self-awareness of subjectivity, and audit trail were used as trustworthiness evidence in the qualitative phase. The description of each piece of evidence is described below.

Reliability Evidence for the Quantitative Phase

Two pieces of evidence were considered to assess the validity and reliability of the fragmenting test. First, the judgment of two experts in the mathematics education field was applied to examine the relationship between fragmenting test and the designed rubric. Particularly, I asked the experts to judge if the rubric could cover all possible responses from the test. Second, the measure of agreement was used to analyze scores' reliability (inter-rater reliability). The measure of the agreement allowed researchers to realize to what extent they agree on scoring the same test and to what extent this scoring is consistent between them. Considering the scoring data as ordinal, weight was accounted to have an accurate representation of interrater reliability agreement (Landis and Koch, 1977). For instance, on the ordinal scale, if one coder scored two and the other scored three for a participant's response, the disagreement is counted as less than if that coder scores one and the other coder scores four for that response. Thus, I used weighted Cohen Kappa to measure the level of agreement for these six samples. According to Landis & Koch (1977) Kappa less than .0 is considered poor,.0 to 0.20 is slight, 0.21 to 0.40 is fair, .041 to 0.60 is moderate, 0.061 to 0.80 is substantial and 0.81 to 1.00 is almost perfect agreement.

Trustworthiness Evidence for the Qualitative Phase

Niesz (n.d.) mentioned that "producing high-quality, rigorous, and trustworthy research is a high priority for interpretive researchers" (p. 3). To achieve valid outcomes for my qualitative interpretive study, I need to be self-aware of my subjectivity in all aspects of doing the interview, taking notes, writing a research journal, and analyzing data. Being aware of the relationship between the researcher and what and who are being studied is one technique that can validate the study (Merriam & Grenier, 2019). This technique calls reflexivity. Because the interviewees are children, being aware of my influence while interviewing is crucial. Thus, to keep subjectivity out of my judgments and make sure that I've had a neutral perspective while I interpret data, I need to be self-critical. Besides reflexivity, I used other techniques such as peer debriefing and an audit trail to build trustworthiness for this study.

Peer debriefing is one of the techniques that I used to indicate the rigor and quality of my study. Peer debriefing is "a process of exposing oneself to a disinterested peer in a manner paralleling an analytic session and for the purpose of exploring aspects of the inquiry that might otherwise remain only implicit within the inquirer's mind" (Lincoln and Guba,1985, p.308). Using this technique helped me to uncover my biases and assumptions in this research.

Finally, I made sure to describe the process of the study entirely and with all details. This description should be in a way that "the independent readers can authenticate the finding of the study by following the trail of the researcher" (Merriam & Grenier,2019, p. 28). So, by using the audit trail technique and 360 video recording, I made sure to explain the process of collecting data as well as the method used to analyze and interpret data.

Analysis

The data analysis in a sequential explanatory mixed methods research design involves two separate phases. In the first step, the collected data from the quantitative phase analyzed and interpreted. Following that, the findings from qualitative data were used in the subsequent interpretation and deeper insight of the result from the analysis of quantitative data. For the purpose of this study, different strategies were used to analyze data in each phase.

For the quantitative phase, descriptive statistics were used to answer the first research question, how do early elementary students understand fragmenting levels. Then the ordinal regression analysis was conducted to understand the effect of using visual statical models (independent variable) and their grades (independent variable) on children's fragmenting knowledge (dependent variable). In other words, the ordinal logistic regression was used to predict to what degrees different statistical models and children's grade levels can vary children's levels of fragmenting. Modeling such equations allowed us to answer the second research question.

$$\text{Y}\ (0,1,2,3) = e^{\beta_0+\beta_1 \text{ static models (circle,rectangle,length)} + \beta_2\ Grades\ (\ first,second)}$$

Or

$$c_k(x) = \ln\ [\beta_0(x) + \beta_1(Static\ model\ )_1 + B_2(Grade\ bands)_2\ ]$$

Before conducting ordinal logistic regression, the assumption of proportional odds, as well as the goodness of fit of the model, needed to be checked (Hosmer et al., 2013). The assumption of proportional odds is used to examine if the slopes from independent variables to the logits are the same. In other words, the proportional odds assumption indicates if the relationship between the independent variables across all logits for the dependent variable is consistent (they all have the same slope). The result from chi-square needs to be not statistically significant ($\chi 2(df{=}12) = 7.417$, $p = 0.829$) indicating the assumption was met and the slopes did not differ across all logits. So, the ordinal logistic regression can be conducted.

Finally, the non-parametric Friedman tests were utilized to examine the performance of children across the items for each representation with specific consideration to odd and even numbers. It should be noted that due to the ordinal nature of the data, the Friedman test was used as the most appropriate test (Siegel & Castellan, 1988).

After analyzing all data in the quantitative phase, the data collected from the qualitative phase needed to be analyzed as the backup to support the result that was achieved in phase one (Creswell and Clark, 2011). Analyzing data began as I started to collect data in the qualitative phase. According to Niesz (n.d.), "Interpretive researchers do not wait until after all of their sources of data have been collected before they begin to review and analyze them. Data analysis must begin with the earliest data collection such that their analysis can guide further data collection". Using the method of interpreting data while collecting them direct data in a way to answer the research question. Specifically, because I conducted clinical interviews with each participant separately at different times, reviewing data and analyzing the first interview before I conducted the second one allowed me to be more cautious of my role as an interviewer. It helped me understand where and in what item I needed to shift my role and "getting nosy" and where I needed to be quiet and observe participants (Schram, 2006). Besides that, the process of analyzing data in a qualitative study consists of noticing, collecting, and thinking about the data. This process is not linear, and it is recursive and iterative (Seidel, 1998). So, while you are thinking about your data, you might also notice new things and ask better follow-up questions and generate more data that could answer my research question.

Transcribing the interview was performed after collecting all interview data and related field notes. This method was more formal than the previous one. First, the transcribed data were reviewed and categorized. Additionally, field notes aided me better understand the information gathered through the transcription. For instance, some of the sentences in transcript data had no meaning by themselves; using research journals identified their meanings and helped to understand the data better. Second, the line-by-line coding method was utilized to find categories. I used some descriptive words or the exact term of the sentences to describe each line. Then categorized them by the words that were created. Finally, the generated theme emerged through a cluster of words (after completing the study, I need to add a few sentences related to the research that connect the generic method to the study).

Conclusion

The study investigated children's knowledge of early fractions using three different visual static models, including circular, rectangular, and length models. A mixed-method sequential explanatory study was utilized to fulfill the purpose of the study. This chapter explained the method of designing the fragmenting test and collecting data and the validation process for each phase, qualitative and quantitative. In the next chapter, the description of analysis for both qualitative and quantitative phases was discussed.

CHAPTER IV

ANALYSIS OF THE FINDINGS AND RESULTS

The purpose of this study was to investigate children's operations on fragmenting schemes while they are engaged with different visual models. This chapter presents findings for the following research questions using a sequential explanatory mixed-methods study design:

1. What fragmenting schemes do early elementary students demonstrate?
2. How are fragmenting schemes manifested when children engage in three different visual models, including circle, rectangle, and length model?
3. How do children enact their partitioning schemes when working with odd and even numbers?

In order to accomplish the purpose of this study and address the research questions, quantitative data were collected by designing a fragmenting survey including five questions. Following the quantitative data, 11 participants were purposefully selected for one-on-one clinical interviews. Clinical interviews were used to corroborate and explain the quantitative results and to get insight into the children's thinking processes. The quantitative results, followed by the qualitative findings, served to address all three research questions.

Following the collection of all data, the initial framework designed by Hackenberg et al. (2016) and developed by the author (see Chapter 3, Table 2) was used to identify each student at a level of fragmenting. First, a subset of the students' answers were chosen at random identify a level of fragmenting (n = 10). Two mathematics educators coded these students' responses for reliability of the framework. After several attempts to place students' written works at particular fragmenting levels, both coders concluded that the framework did not align with over half of the students' responses and thus did not have enough reliability to be used for coding the rest of the data. Thus, a new revised framework was constructed, based on data from this study, to code all students' responses and address the study's purpose.

The first part of this chapter explains how the new fragmenting framework was made. Descriptions for each level of the revised fragmenting framework as well as the test for reliability of framework was provided. Following proposing the revised framework, the fragmenting framework was used to address the purpose of this study, which concerned with students' fragmenting knowledge and its relation to three statical models.

Following discussion of the revised framework, this chapter begins with the quantitative results for the first research question, which was about students' knowledge of fragmenting. Descriptive analysis was the primary tool for reporting students' knowledge of fragmenting. Following that, ordinal regression was used for the second research question to understand the relation between children's fragmenting knowledge and their use of different visual models as well as their grade bands. After reporting the results from ordinal regression, findings from clinical interviews are presented to explain the underlying reasons and rationales for such results.

Finally, the third research question of this study was addressed. This question concerns with differences in students' performance in partitioning shapes into different types of numbers. Both quantitative and qualitative analysis were conducted to address this research question. Specifically, first the Friedman test was used to examine for differences across different numbers of partitioning. Then, cross-case analysis was used to compare and contrast interviewees' knowledge and actions regarding partitioning and to see if these findings were consistent with quantitative results. This section concluded with a summary of the findings from this sequential mixed-methods explanatory study.

Summary of Study Demographics

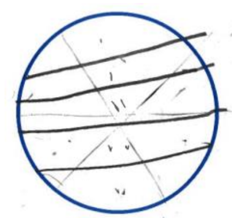
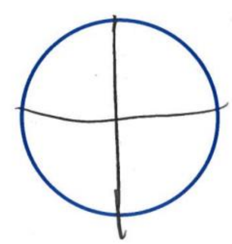
To address the purpose of study in both quantitative and qualitative phases, two sets of data were collected. For the quantitative analysis, 132 elementary students participated consisted of 49.2 % (*n*= 65) first graders and 50.8% (*n*=67) second graders were used. Of the 132 students, 11 were purposefully selected for the qualitative phase including five first graders and six second graders.

Description of Developing the New Fragmenting Framework

In order to achieve the purpose of this study and answer the research questions, I first needed to develop a new fragmenting framework. The new framework can identify one specific level of fragmenting for each student based on their overall responses to the five questions of the survey. Recall that the original fragmenting framework did not align with students' written work. Students' responses to each question of survey placed them in different level of fragmenting, which made the final decision of putting them in a one specific level impossible. To illustrate this, an example of how one student's responses were coded with the original framework is shown in Table 6. In this example the student's overall responses were inconsistent with the student being at Levels 4, 0, 4, and 2. More explanation, the student successfully answered the first question of the survey and partitioned the model into two equal parts in their first try. This answer led him to be placed at level 4 of framework which is equal partitioning a whole up to five parts. However, the student then was not able to answer the second question which is cutting the model into three different parts which place him into level 0 of fragmenting. For the next question, the student equally cut the shape into four parts. His work placed him at level 4 because of his indicating erasures. And finally for the last question, the student cut the shape into five unequal parts, indicating level 2 of fragmenting.

Table 6

Example of coding student written response based on initial fragmenting framework

Levels of fragmenting	dividing into 2 parts (Q1)	dividing into 3 parts (Q2)	dividing into 4 parts(Q3)	dividing into 5 parts(Q4)
Level 0: No attempt in segmenting				
Level 1: Marking a whole into up to three parts				
Level 2: Coordinating the two goals of making parts (up to five) and exhausting whole				
Level 3: Coordinating the two goals of making "equal" parts (up to five) and exhausting whole)/ (constructed in the activity)				
Level 4: equal partitioning a whole up to five parts (anticipatory level)				

The wide discrepancy in students' fragmentation levels, as shown in Table 6, made the final decision of assigning one specific level to each student difficult. Thus, there was a need to revise the original framework, which aligns with students' responses and identifies a level for each student based on their overall responses to all survey questions.

In order to develop the new framework, prior research-based categorizations on children's ability to fracture, partition, or equi-partition a model into given parts was used (Hackenberg et al., 2016; Nunes & Bryant, 2008; Steffe & Olives, 2010). The categories included 1: Did not partition a shape into halves, thirds, fourths, and fifths; 2: Partitioned a shape into halves, thirds, fourths, and fifths; 3: Equally partitioned a shape into halves, thirds, fourths, and fifths. After coding all data, descriptive analysis was conducted to show possible patterns across students' responses. Table 7 summarizes students' partitioning performance per question.

Table 7

Descriptive results of students' partitioning for each question

Partitioning	Divide into halve	Divide into third	Divide into fourth	Divide into fifth
Not partitioned	3.8%	30.3%	11.4%	43.9%
	n=5	*n*=40	*n*=15	*n*=58
Partitioned	5.3%	42.4%	47.7%	50%
	n=7	*n*=56	*n*=63	*n*=66
Equally	90.9%	27.3%	40.9%	6.1%
partitioned	*n*=120	*n*=36	*n*=54	*n*=8

Observable in Table 7, more students are successful partitioning halves than fourths, followed by thirds and fifths. This led to revise the original framework and design a new one which is presented in Table 8. The four-level framework is created to characterize the differences among students demonstrated fragmenting schemes.

Table 8

Levels of revised fragmenting knowledge of children

Levels	Sublevel	Description of Levels
Level 0	N/A	**Students are not attempting to fragment shapes**
Level 1		**Students are able to partition models into at least one of the numbers 5 parts or fewer (the parts are not necessarily equal).**
	1.1	Students are able to partition even numbers
	1.2	Students are able to partition even and odd numbers
Level 2		**Students are able to equally partition even numbers**
	2.1	Students are able to equally partition even numbers (2 & 4)
	2.2	Students are able to equally partition even numbers (2 & 4) and partitioning odd numbers, *but it is not internalized yet. Practicing the partitioning into odd numbers prepare them to construct this scheme*
Level 3		**Students are able to equally partition odd and even numbers**
	3.1	Students are able to equally partition at least one odd and one even number. *Students internalized the equal portioning scheme, however, at this level there is a chance of error due to practicing/constructing in activity*
	3.2	Successful equal partitioning for all even and odd numbers

No attempt to cut a shape is considered level zero for developing the fragmenting scheme, in which students are not incorporated in cutting the shapes. It should also be noted that all students (expect the four students who did not respond to any of the questions) who contributed to the activity of fragmenting were able to accurately divide models into two parts. Therefore, the evidence for equitably slicing in half is not seen as strong or dependable for a higher level of fragmenting knowledge. Students demonstrating these actions were placed at different levels of fragmenting.

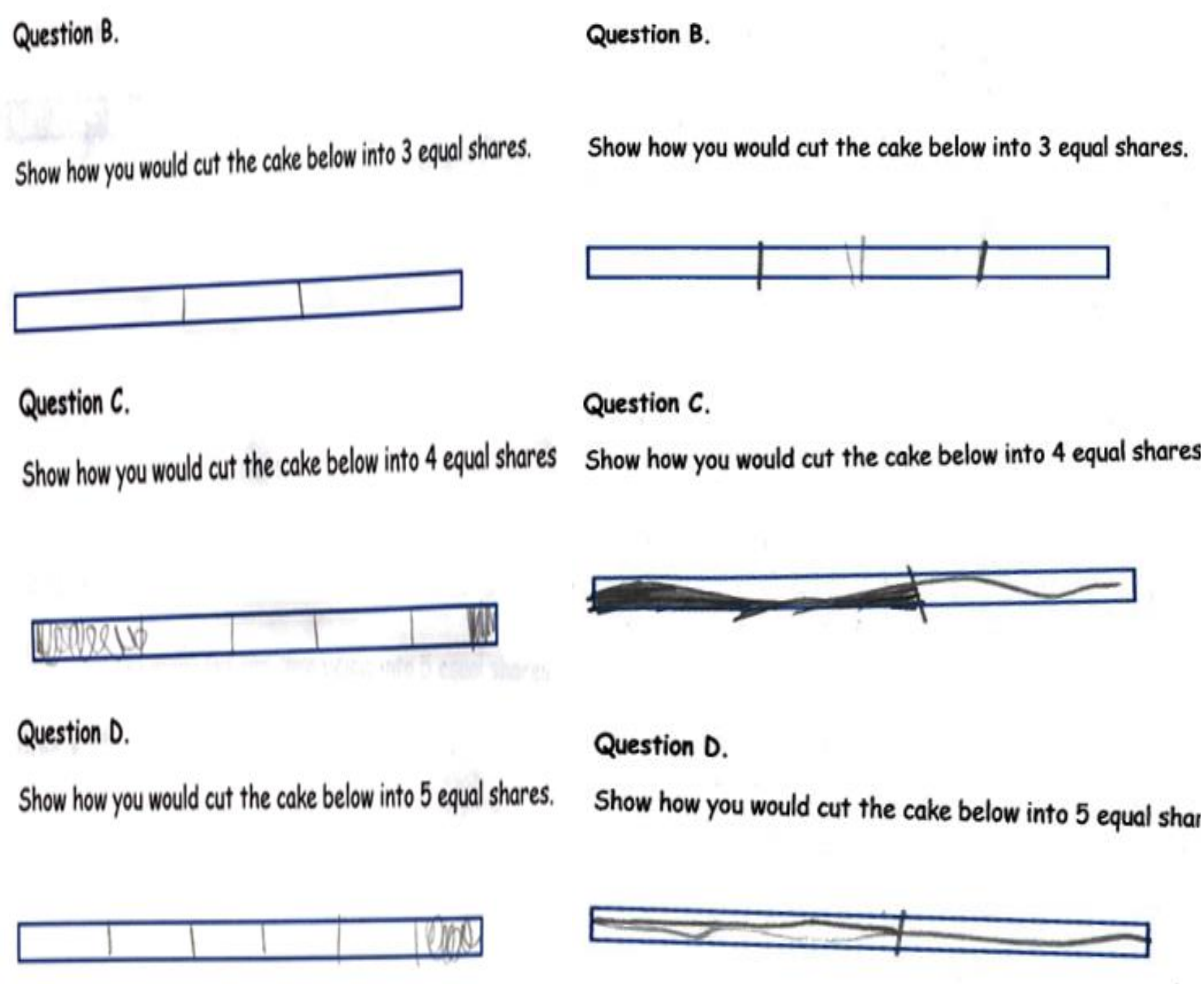
Partitioning less than five is Level 1 of fragmenting. At this level, students are able to partition a shape into at least one of the numbers less than five parts. This level mainly incorporates students' knowledge of counting numbers. Students usually start to create a unit by cutting the shape and then continue to create other unlike units until they achieve the goal of finishing the number of parts. Thus, the main goal for students at this level is to show the "number" of parts without considering a whole or making parts equal. The goal of making equal parts is often ignored at this level by either showing unlike parts or not exhausting the whole.

Figure 15 shows two examples of student's work that are placed at Level 1 of fragmenting. Student A was able to cut the length model into any given parts. But the parts are not equal, or the whole is not exhausted. This student indicates the ability to use their numerical knowledge of counting and apply it to achieve the goal of partitioning the length model into given parts. However, they were not able to coordinate the two goals of making parts into a given number (see Figure 15; student A: questions B, D) and using up all length (see Figure 15; student A: questions C and D). Focusing on one of these goals often results in failure in another. On the other hand, Student B shows the knowledge of counting parts and applies it to the length model for some questions (see Figure 15; student B: questions B, C), but not all of them (see Figure 15; student B: question D). Indicating students at this level might be able to partition all numbers less than five or partition at least one of the numbers less than five. However, they do not demonstrate the ability to make parts equally unless they fail to exhaust the whole (see Figure 15; student B: questions B, C).

Figure 15

Two students' demonstrated work which is placed at first level of fragmenting

Student A **Student B**



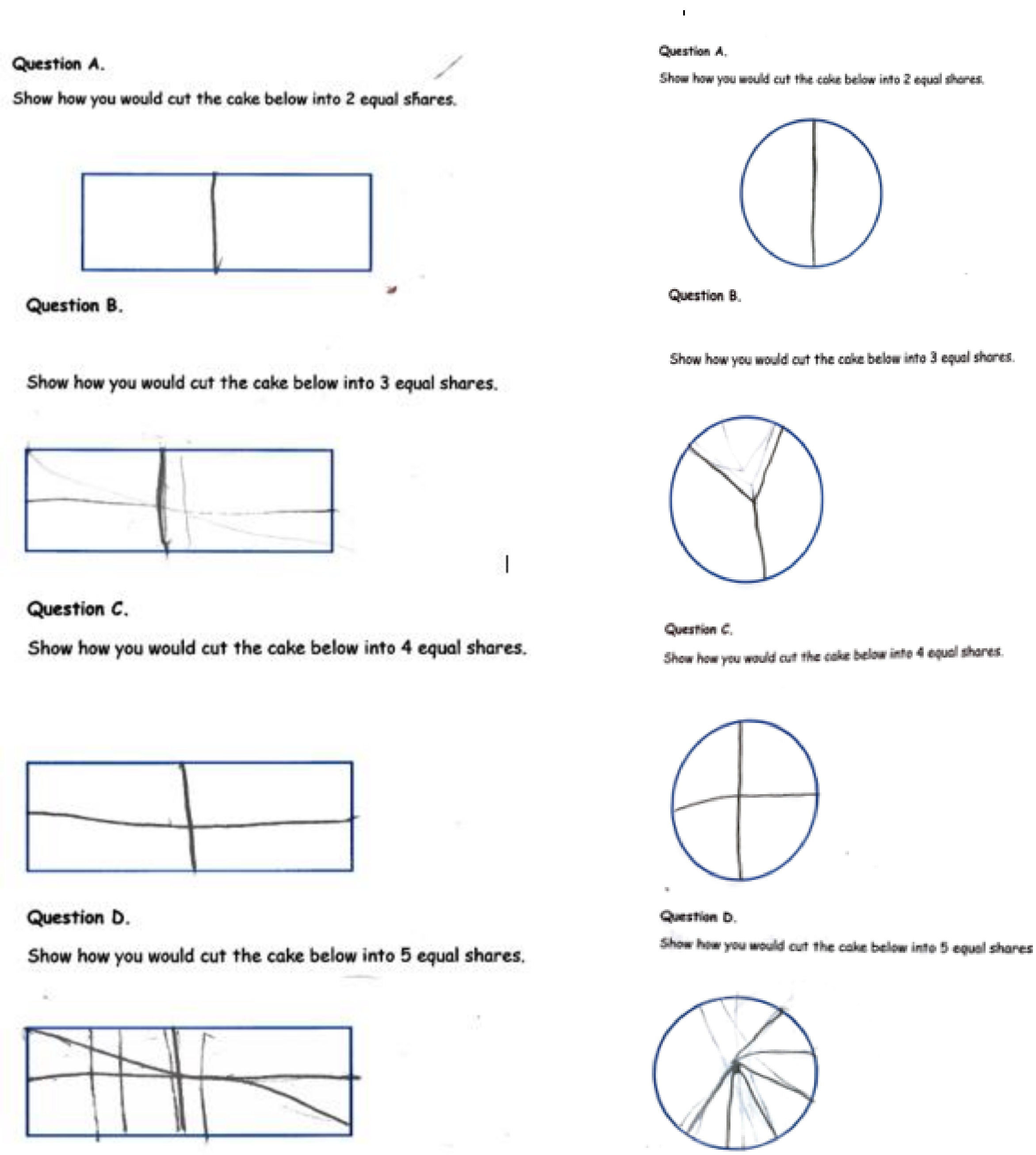
Successful equal partitioning for even numbers is considered Level 2 of fragmenting. At this level, students can coordinate the three goals of partitioning a shape, making parts equal, and exhausting the whole, but only for two and four parts. Students cannot illustrate these three actions/goals when partitioning models into three or five parts. At this level, students are aware of coordinating the given parts and dividing them equally while exhausting the whole, but they do not have enough experience or knowledge to achieve those goals with odd numbers. Students at Level 2 of fragmenting often indicate a lot of trial-and-error efforts in achieving those goals. However, most of their *within activities* efforts failed by making unequal or wrong parts.

It is important to distinguish students' indication of cutting shapes into incorrect or unequal parts from not knowing how to cut the shapes into three or five parts. Students at Level 2 are aware of the need to cut shapes equally into odd parts. However, their actions do not necessarily demonstrate accurate equal partitioning. As presented in Figure 16, student A's indication of erasures suggests they attempted to cut the rectangles into three and five parts. The student's constructed this in activity (i.e., they did not have the model internalized a priori) did not have a successful partitioning. In the same scenario with student B (Figure 16), this student also used trial and error to partition the circle into three and then five parts. Their several attempts to solve a task led them to successful partitioning for three parts, but not for five parts. Such within activity actions are only observed for cutting into odd numbers for students at Level 2 of fragmenting, but not even numbers. It appears that students' real-world experience in cutting pizza or cake into even numbers, the natural tendency to see everything as symmetry (Lipka et al., 2019), and the number of formal and informal STEM topics indirectly related to the concept of even, such as folding geometric shapes and drawing symmetrical shapes, help students to know at least one strategy for equally partitioning shapes into even numbers.

Figure 16

Two students' demonstrated work which is placed at second level of fragmenting

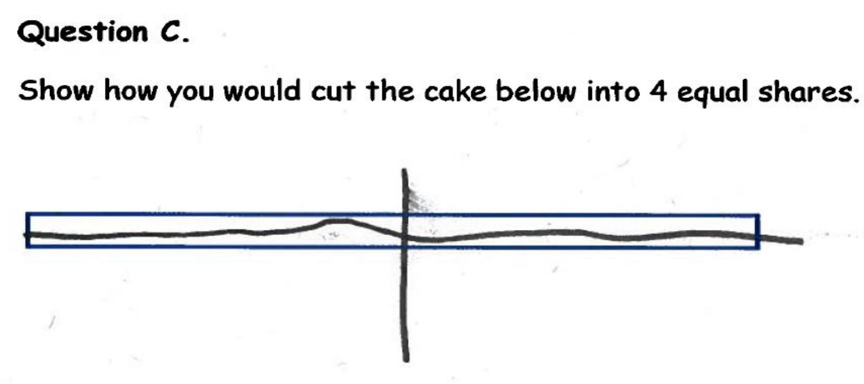
Student A **Student B**



Some of the strategies that usually apply for cutting shapes into even numbers were evident in some of students' responses. For instance, despite the difficulty of using the cross vector cutting method to cut a length into four parts (i.e., the standard or common method in cutting circles and rectangles into four parts), some students used that strategy to cut that model. Figure 17 provides an example of one student's effort to implement cross-cutting strategy to cut a length model into four parts.

Figure 17

Applying the common cross-cur strategy for partitioning length into four equal parts



At Level 3 of fragmenting, students are able to equally partition numbers less than five parts regardless of the type of the number. Students at this level have some strategies for equally partitioning both even and odd numbers. Employing these strategies guarantee their successful actions in cutting the model into given parts (no matter the type of number) and finishing up the whole.

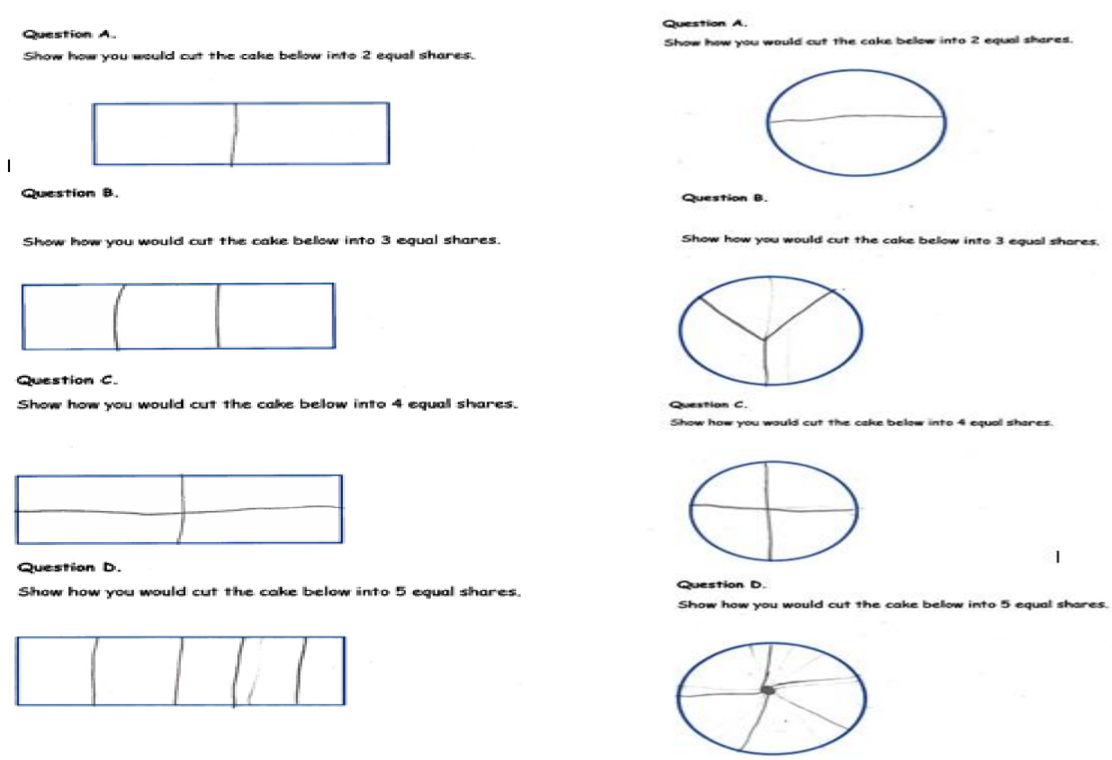
It is expected that students at this level can equally cut shapes into any given number less than five. However, in reaching this goal, there is a risk of not making those parts equally for larger numbers such as four or five. The issue is not associated with their knowledge of equal partitioning but is related to their rush to finish the survey or any sort of distractions.

The difference between this level and Level 2 is that the students at Level 2 have not developed any techniques or strategies to partition odd numbers. And their knowledge of equally cutting shapes into even numbers largely comes from their exposure to real-life observations and practicing those observations. Enough practice in cutting shapes into three parts helped them internalize the concept of equal partitioning for any numbers, placing them into level three. Figure 18 presents two students' work which was placed at Level 3 of fragmenting. As shown here, student A demonstrated successful equal partitioning for all rectangles except for partitioning into five parts. Furthermore, student B demonstrated the equal partitioning scheme by equally cutting all the shapes into the correct parts.

Figure 18

Two students' demonstrated work which is placed at third level of fragmenting

Student A **Student B**



Testing Reliability of the Proposed Framework

The reliability tests were conducted for the new fragmenting framework. The test was related to the inclusiveness of the framework. In order to do that, I used the framework to place each student at one level of fragmenting framework. Across all 132 students' responses, 97.0 % (*n*=128) of students were placed into a different level of fragmenting. The four remained participants did not partition a shape into half, but they partitioned other shapes into given parts. Given the low proportion of the students with this issue (i.e., $\frac{4}{128}$), this was assumed to be due to test error. Thus, I placed these four students based on their responses to the remaining questions. The resulting framework with the number of students at different levels is presented in Table 9

Table 9

The number of students being placed in each level of constructed fragmenting framework

Levels	Description Levels	Number of cases
Level 0	No attempt on cutting models into the given parts	*N*=4
Level 1	Students are able to cut models into at parts at five parts or fewer (the parts are not necessarily equal)	*N*=55
Level 2	Students are able to equally partition even numbers	*N*=34
Level 3	Students are able to equally partition odd and even numbers	*N*=35

Guided by prior research and the patterns observed in collected data, the revised framework was prepared to address the purpose of the study and its research questions. The remainder of this chapter provides the results for three research questions, including:

1. What fragmenting schemes do early elementary students demonstrate?

2. How are fragmenting schemes manifested when children engage in three different visual models including circle, rectangle, and length model?

3. How do children enact their partitioning schemes when working with odd and even numbers?

Findings for Children's Fragmenting Knowledge (RQ1)

The main goal of the first research question (RQ1) is to understand what fragmenting schemes children demonstrate. The question was answered using a quantitative method of collecting children's responses to the survey. The revised fragmenting framework (see Table 9) was used to code the data. Next, descriptive statistics were used to identify children's early fractions knowledge level.

Results from the descriptive statistics showed that across 132 first and second graders, 4% (*N*=4) were at level 0 of fragmenting, 44.7% (*N*=59) at level 1 of fragmenting, 33.3% (*N*=44) at level 2 of fragmenting, and 25% (*N*=25) were at level 3 of fragmenting. These results suggested that most students could partition a whole into less than five parts (first level of fragmenting) or equally partition a whole into even numbers but not odd numbers (second level of fragmenting). Interestingly, only 25% of children were placed at the third level of fragmenting, in which they can equally partition any number less than five. These results indicate that most children are not ready to begin developing part-whole reasoning (such as showing $\frac{1}{5}$ of a shape) as they move up to third grade.

Further analysis between grade levels showed the difference between first and second graders performance on fragmenting schemes was not statistically significant ($\chi 2$ (df=3) = 6.853, p = .077). Table 10 illustrates the frequency of students' performance on each fragmenting level based on their grade level.

Table 10

Frequency of students' performance on each fragmenting level based on their grade level

	level 0		Level 1		Level 2		Level 3		Total	
	n	%	*n*	%	*n*	%	*n*	%	*n*	%
First Grade	0 *2*	0%	33 *29.3*	25%	23 *21.3*	17%	9 *12.4*	7%	65	49%
Second Grade	4 *2*	3%	26 *29.7*	19%	21 *21.7*	16%	16 *12.6*	12%	67	51%
Total	4	3%	59	44%	44	33%	25	19%	132	100%

Given that fragmenting knowledge is a prerequisite to learn fractions (Hackenberg et al., 2016; Steffe & Olives, 2010), one would anticipate that second graders would perform better in partitioning in compared to first graders. However, results of this study show no statistically significant difference in the level of fragmenting for both grades.

Findings For the Relationship Between Children' Fragmenting Knowledge and Three Different Visual Models (RQ2)

Ordinal regression analysis in the quantitative phase, followed by cross-case analysis for the qualitative phase, was used to understand the relationship between children's fragmenting knowledge and circle, rectangle, and length model. In the following section, the findings for both analyses are presented.

Quantitative Results

Ordinal regression analysis was used to understand the effect of using visual statical models, including circle, rectangle, and length and grade levels (first and second graders), on children's fragmenting knowledge. The statical models and the grade levels were the independent variables, and the fragmenting levels were considered the dependent variable. Before running the regression model, the assumptions for proportional odds were examined. The result from chi-square indicated no statistically significant difference ($\chi2(df=12) = 7.417$, p = 0.829), suggesting that the assumption was met, and the slopes did not differ across all logits.

Next, the regression model presented in Equation Y below was used to understand if the model fits the data. The overall model fit was found not to be statistically significant from the intercept-only model ($\chi2(df=3) = 4.507$, $p = 0.212$), suggesting the model is not a good fit for the data (Hosmer et al., 2013). Thus, there was no statistically significant relationship between children's fragmenting knowledge and the statical models or their grade bands. This indicates that children's level of fragmenting is consistent across all visual models and both grade levels.

$$Y(0,1,2,3) = e^{\beta_0+\beta_1\, static\ models\ (\ circle, rectangel, length)+\beta_2 Grades(\ first, second)}$$

Or

$$ck(x) = Ln\ [\ \beta_0(\text{x}) + \beta_1(\text{Static models}) +\ \beta_2(\text{Grade bands})]$$

Qualitative Findings

In order to better understand the result of the regression model, children's clinical interview was investigated. Accordingly, students' fragmenting actions on three models, including rectangle, length, and circle, were observed. A cross-case analysis of students' fragmenting operations using three models revealed an acceptable performance in cutting all models into two and four equal parts. However, students at the clinical interview showed some difficulties in cutting shapes into three and five equal parts. Specifically, the struggle was more evident when attempting to cut circle models into three and five equal parts. Cutting shapes into three and five parts is challenging for students but using the circle model added an extra layer of difficulty. Almost all of the students in clinical interviews demonstrated significant effort to figure out how to cut a circle into three or five sections, but they frequently ended up cutting it vertically or skipping the task (see Figure 19).

Figure 19

Students' efforts to cut the circle into three equal parts

During one interview, Adam was very confident when cutting length and rectangle models into three and five parts, but he did not show the same confidence in cutting the circle into three parts. Adam quickly partitioned rectangles and length models into all numbers less than five and used a strong rationale for his actions, saying "the parts are equal because they are the same length and sizes."

Adam also demonstrated the same reasonings while he cut the circle model into half and fourths. However, he struggled cutting the circle into three and five parts. After several seconds of thinking quietly to partition the circle into three parts, he asked "did you say equal or not equal?" By asking this question and attempting several tracings to find a strategy, Adam stated that if he drew vertical or horizontal lines on the shape, he would not have equal parts, so he did not try those methods for cutting a circle into three and skipped the questions. His action plus his reasonings for how parts are equal provide strong evidence of his fragmenting scheme. However, he did not know a practical strategy to partition the circle into three or five equal parts.

In the same situation with Judy, when I asked her to cut the circle into three parts, she cut it vertically into half and then said "if I cut it in another way [indicating horizontally], it is going to be fourth." But after she moved to the question about cutting the circle into five parts, she found that if she cut the circle into diagonal lines, she could make five pieces equally (see Figure 20).

Figure 20

Judy made all three models into five equal parts

What became apparent during the clinical interview was students' consistent difficulty in cutting the circle into three and five parts. The clinical interview findings raised the question of why the regression model did not detect this difference in students' actions when partitioning the circle into odd numbers. One possible reason for this issue might be due to the fact that the regression model was very inclusive in examining the fragmenting levels rather than particular cases of partitioning. Thus, a more parsimonious model was needed to examine for such differences in models using odd numbers. In the following section, a supplemental analysis (i.e., Kruskal-Walli's test) was conducted to examine the difference between models when partitioning for odd numbers of parts.

Supplemental Analysis: Kruskal-Walli's test

Another set of analyses was conducted to reexamine the findings from the clinical interview and check the possibility of finding any difference across models while partitioning them into three and five parts. Given the ordinal nature of dependent variables (not partitioning, partitioning, equally partitioning) and to not violate the normality assumption, the Kruskal-Wallis test was used to examine the differences across models for partitioning into three and five parts.

Results from the test suggested no statistically significant differences across three models for partitioning into five ($H = .047, df = 2, p = 0.132$) and for partitioning into three ($H = .807, df = 2, p = 0.09$). Despite observing no statistically significant discrepancy across all three models, post-hoc pairwise comparison showed differences between the circle and rectangle models for cutting into three and five parts. For partitioning into three, a pairwise analysis showed a statistically significant difference between the rectangle and circle (p = 0.034). Moreover, for partitioning into five parts, the p-value was p= 0.057, suggesting a difference between the two models (see Table 11).

Table 11

Students' partitioning Post Hoc comparisons across circle, length, and rectangle models

Models	Partitioning into three			Partitioning into five		
	H test	P-value	Standard error	H test	P-value	Standard error
Circle vs Length	-12.216	.118	7.812	-11.368	.126	7.421
Circle vs Rectangle	-15.915	.034	7.515	-13.569	.057	7.138
Length vs Rectangle	-3.699	.627	7.611	-2.202	.761	7.230

Pairwise comparison indicated that students were more successful in partitioning into thirds and fifths using rectangles compared to circles. The mean rank for partitioning into three for the rectangle is 72.83, which is higher than the mean rank for the circle, which is 56.92. Similarly, the mean rank for partitioning into five for the rectangle is 71.60, which is higher than the mean rank for the circle, which was 58.03.

Findings For Children's Partitioning Schemes When Working with Odd and Even Numbers (RQ3)

To answer the third research question (RQ3), non-parametric Friedman tests were initially conducted. Following that, analysis for the clinical interviews was used to explain the results from the quantitative phase. The following section presents findings for both analyses.

Quantitative Results

The non-parametric Friedman tests were conducted to understand students' performance on partitioning half, third, fourth, and fifth. This test was mainly utilized due to the ordinal and not normally distributed data (Siegel & Castellan, 1988). As indicated in Table 12, the result shows a significant difference across all four types of partitioning ($\chi 2$ (df=3) =205.375, p<.001). Post-hoc analysis with Wilcoxon signed-rank test was conducted with a Bonferroni correction applied to determine which partitions were among these differences. Test results showed a statistically significant difference across all sets of comparisons. Particularly, there was a statistically significant difference between partitioning into three and two (z = -8.266, p<.001), partitioning into four and two (z=- 7.368, p<.001), partitioning into five and two (z = -9.919 , p<.001), partitioning into four and three (z = -4.022 , p<.001), partitioning into five and three (z = -5.402 , p<.001), and partitioning into five and four (z = -7.422 , p<.001). The mean rank for partitioning into two parts (M=2.87) and for partitioning into four parts (M=2.29) is higher than the mean rank for partitioning into three parts (M=1.97) and for partitioning into five parts (M=1.63). These results suggest that children might need to construct a higher level of fragmenting scheme when they partition into odd numbers compared to even numbers.

Table 12

Students Mean rank for each partitioning

	N	Mean Rank	Chi-Square	Degree of freedom	Significant Value
Partitioning into two parts	132	2.87	205.373	3	<0.001
Partitioning into four parts	132	2.29			
Partitioning into five parts	132	1.93			
Partitioning into three parts	132	1.97			

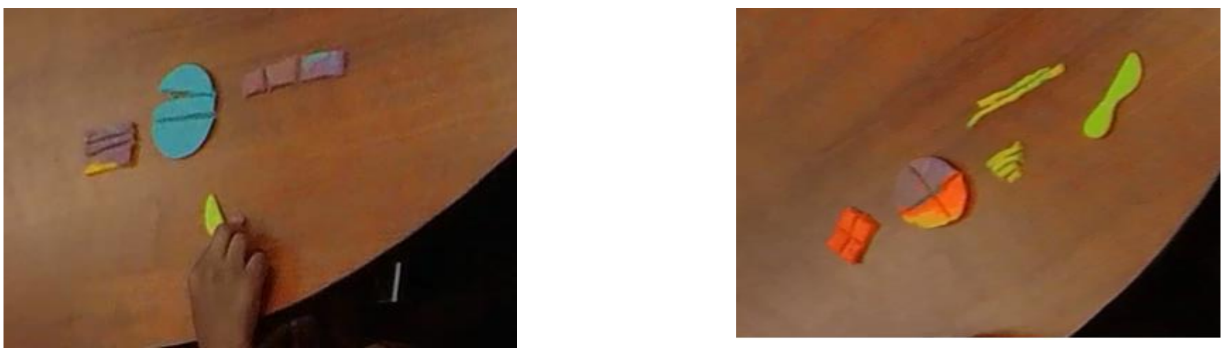
Qualitative Findings

Following the quantitative results, I analyzed students' recordings from clinical interviews to ensure that the findings from the qualitative analysis offered the same insights about students' different performances in cutting odd and even numbers. Interestingly, among the most common themes that emerged across all interviews were students' reactions, pauses, and ways of cutting odd numbers compared to cutting even numbers. Students could *successfully partition two and four into equal parts.* However, their strategy for coordinating the goals of making equal parts while exhausting the whole was often ineffective for odd numbers. One example of this reasoning can be seen in Hannah's two different responses and actions to cut the shapes into three versus four parts. In a question about cutting shapes into three equal parts, Hannah cut three horizontal lines for the circle and rectangle model and then changed her strategy for the length model. She used vertical lines for length, but then she said none of the parts she created were equal. When I asked how she knew, for each shape she put the parts underneath each other and said each shape has one big part (see Figure 21).

Hannah's response was very different for making four equal parts. When I put the pre-made rectangle and circle in front of her, she quickly made four equal parts by making cross-cut lines and saying it was very easy. She used the same strategy of cross-cutting for the length model. She first made a horizontal line on pre-made length but could not successfully make it. She then did another horizontal cut. I conjecture this led her to realize the impracticality of cross cutting strategy and plan a new strategy. So, she changed her strategy, picked one of the parts she had already created from the length model, and drew four vertical lines on it. Then she put all pieces under each other and said they were not equal. Figure 21 compares Hannah's work while cutting into the third and fourth.

Figure 21

Hanna's work while cutting all three models into three equal parts (the left shape) vs. four equal parts (the right shape)



Like Hannah, Jack was confident that he could divide two and four into two equal parts. However, he demonstrated difficulty when he tried to cut shapes into three and five pieces. His focus on dividing the length into three and five parts kept him away from finishing the whole (see the right picture in Figure 22) or making the parts the same size (see the left picture in Figure 22).

Figure 22

Jacks work while cutting length model into three and five equal parts

Jack cut length into three.

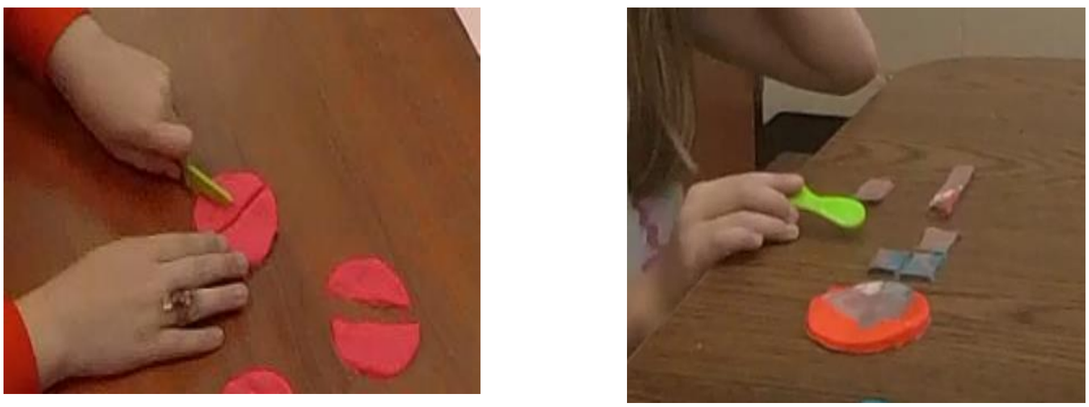
Jack cut length into five parts.

As noted above, Jack expressed comfort in cutting shapes into even numbers. He referenced the strategy of cutting into half as a way to cut into fourth, stating "I am going to put a line on the playdough just like this one [points to the pre-made circle that he cut in half], but then I cut like that way [cuts the circle vertically to have the four parts]." The excerpt of his work is shown in Figure 23 on the left side of the table.

Almost every student interviewed expressed more confidence in cutting shapes into two and four parts. This was particularly evident when cutting circle and rectangle models. Whereas in cutting shapes into odd numbers, students seemed unsure about the cutting strategy. They often spent more time to come up with a practical strategy and sometimes skip the work. For instance, when I asked Gabby to cut the circle into three parts, she smiled and said, "that is hard," and when I asked why she responded, "because I did not do any circles, but I did squares [indicating the rectangle].” When I told her to try a rectangle first, she failed to cut it into three equal parts. She crosscut the rectangle, as she did for cutting into four parts, and then took away one of the parts to have three equal parts. When I asked her how she knew the pieces were “three” equal parts, she said “When I count them, they are three, and all of them are the same, and if you add the four, that makes the *whole* shape, but you take away to make it a third” (see the right-hand side of Figure 23).

Figure 23

The excerpt of students' work (Jack and Gabby) in cutting shapes into even numbers (left picture) and odd numbers (right picture)



Chapter Summary

The revised fragmenting framework proposed four different levels to characterize the differences among students’ demonstrated fragmenting schemes. Level 0 is the lowest level of the framework, in which students do not attempt to segment. And Level 3 is the highest level of fragmenting in which students show the ability to equi-partition different models. Analyzing 132 students’ demonstrated fragmenting scheme indicated almost half (44%) of the students are at Level 1. This suggests that students are able to partition a model into a given number of parts but are not able to make the parts equal or exhaust the whole. Further, no statistically significant or meaningful differences were observed in children’s level of fragmenting based on their grade levels, and both first and second grade students showed the same level of fragmenting schemes.

Findings from comparing the circle, rectangle, and length models and their relation to children’s fragmenting scheme suggest that children’s fragmenting knowledge is consistent across the three visual models. However, the type of model operated on children’s successful partitioning when they work with odd number parts. The study’s findings suggest that children have more difficulty cutting shapes into odd numbers compared to even numbers. Also, circles add an extra layer of difficulty for partitioning into odd numbers when compared to rectangles. Students at different levels of fragmenting demonstrated this difficulty. However, this challenge was more procedural than conceptual for students at level 3 of fragmenting. These students internalized the concept of equal partitioning and can both demonstrate and explain the steps of partitioning while using rectangles and length models. However, they were not able to enact the same process using circle models.

CHAPTER V

DISCUSSION AND IMPLICATIONS

Given the complexity attributed to learning fractions in upper elementary grades, most researchers have suggested the need for establishing prior schemes, including but not limited to multiplicative reasoning, numerical schemes, and fragmenting schemes (Empson, 1995; Hackenberg, 2007; Kosko, 2019; Steffe & Olive, 2010). Fragmenting schemes usually begins to develop at an early age such as when a child breaks a cookie and tries to share it with their friends (Empson, 1995; Hackenberg et al., 2016). This scheme then develops gradually by equally partitioning a continuous and discrete model into given shares. Developing a fragmenting scheme prior to learning fractions, and especially the concept of part-whole gives children a higher chance of comprehending fractions (Hackenberg et al., 2016).

In addition to having a good foundation for fragmenting schemes, using different visual models facilitates fraction learning. However, whether there is a difference between visual representations (circle, rectangle, length) in children's early awareness of fractions has been debated (Cramer & Henry, 2002; Hamdan & Gunderson, 2017; Larson, 1980; Tunç-Pekkan, 2015). This study examined children's fragmenting schemes and its relation to three different visual models, including the rectangle, circle, and length models. Following with three research questions:

1. What fragmenting schemes do early elementary students demonstrate?
2. How are fragmenting schemes manifested when children engage in three different visual models, including circle, rectangle, and length model?
3. How do children enact their partitioning schemes when working with odd and even numbers?

Summary of the Study

Mathematics educators generally agree that fractions are one of the most challenging topics to learn and teach (Anderson-Pence et al., 2014; Tunç-Pekkan, 2015; Son, 2012; Wilkins & Norton, 2018). One theoretical approach to studying fraction is through scheme theory (Steffe & Olives, 2010; Wilkins & Norton, 2018). Scheme involves “a student’s way of working with, or conceiving of, a mathematical object, such as fraction” (Hackenberg et al., 2016, p.187). Accordingly, different learning trajectories were built based on empirically observed schemes (Hackenberg et al.,2016; Steffe & Olives, 2010; Wilkins & Norton, 2018), with each of these beginning with children developing a part-whole scheme. The part-whole scheme is about children’s demonstrating of fractions as the relationship between the part(s) and the whole. Although the part-whole scheme is considered an initial scheme to learn fractions, scholars have suggested the necessity of developing pre-requisite schemes such as numerical scheme (Steffe & Olives, 2010), multiplicative reasoning (Hackenberg, 2007; Kosko, 2019), and fragmenting scheme (Empson, 1995; Steffe & Olive, 2010).

This study focuses on fragmenting schemes, which involves students’ learning to partition shapes into equal parts (Empson, 1995; Hackenberg et al., 2016). Children often know how to fracture a continuous model at very early ages (Empson, 1995; Nunes et al., 2009), then they develop such knowledge through different developmental stages, or fragmenting schemes (Hackenberg et al., 2016). Fragmenting is a set of schemes a child must have before learning the part-whole scheme. Based on the Common Core State Standards, students begin to learn fractions by the end of third grade (Common Core State Standards, 2010). Thus, students should develop a fragmenting scheme prior to third grade. However, Hackenberg (2013) found a lack of fragmenting schemes for many students as late as secondary school. In order to understand if children demonstrate fragmenting knowledge before learning fractions, this study focused on understanding first and second graders fragmenting knowledge.

Fractions education is highly associated with visual models (Tunç-Pekkan, 2015). Children are able to visually demonstrate their knowledge of fragmenting or fractions by using different models. The three common models used in teaching and learning fractions are the rectangle, length, and circle models (Gould, 2013; Ni, 2000; Tunç-Pekkan, 2015). While each of these models is beneficial in scaffolding fragmenting knowledge, this study investigated the effect of each model on students' fragmenting knowledge to prioritize the use of the models in pedagogy.

For the purpose of this study, a sequential explanatory mixed-methods study design was used. 132 students responded to a survey about partitioning shapes into different equal parts. Of those participants, 11 were purposefully selected for the clinical interview. Quantitative data were collected and analyzed using descriptive analysis, Friedman test, ordinal regression, and Kruskal Wallis test. Then, qualitative data were gathered through clinical interviews and analyzed using a cross-case approach. The following sections discuss the findings of the study.

A Revised Fragmenting Framework

This study provided a revised constructed framework to understand children's knowledge of fragmenting (see Table 13). Prior research on designing fragmenting frameworks has often included upper elementary, secondary and high school students who may already be familiar with fractions (Hackenberg et al., 2016; Wilkins & Norton, 2018). Targeting the age group of students who already have experience with fractions helped prior researchers to design and develop a broader range (specifically higher levels) of fragmenting schemes. However, it may have prevented observation of initial steps of fragmenting scheme formation that occur in children and who are not as familiar with fractions. Fragmenting schemes usually develop within early elementary grades and earlier (Empson, 1995). By targeting children in first and second grades, I can observe further details steps on their development of fragmenting. For example, level one of Hackenberg et al. (2016)'s fragmenting scheme consists of the ability to coordinate two goals of making equal parts and exhausting the whole for numbers less than three. However, in observing children's fragmenting schemes, I found that it was harder for children to make equal parts and use up the whole when they cut shapes into three than when they cut them into two. In fact, twos and fours are often the first numbers that children can successfully partition.

Table 13

Three levels of fragmenting

Levels	Description of Levels
Level 0	No attempt on cutting models into the given parts
Level 1	Students are able to partition models into any number of 5 parts or fewer (the parts are not necessarily equal)
Level 2	Students are able to equally partition even numbers
Level 3	Students are able to equally partition odd and even numbers

A revised fragmenting framework allowed me to examine the difference between first and second graders' demonstrated fragmenting schemes. Findings indicated that 44% of students could achieve the two goals of making parts while exhausting the whole for most numbers less than five. Interestingly, students' knowledge of fragmenting did not differ across their grade bands. Considering that the data was collected at the end of the academic year, it is expected second graders should be more prepared for learning fractions as they move to third grade compared to first graders. However, the findings suggested their equal demonstrated fragmenting knowledge. This result support prior studies that showed older students' lack of prerequisite schemes for understanding fractions (Hackenberg, 2013; Kosko, 2019; Zwanch & Wilkins, 2021). Prior studies indicated that secondary and high school students lack pre-requisite fractions' schemes, including fragmenting schemes. The finding of this study showed the same trends. Moving on to the next grade level does not guarantee students' development in the fragmenting scheme.

Another finding in this study is that children generally performed better at cutting shapes into halves and fourths than cutting them into thirds and fifths. To the author's understanding, there are no prior empirical studies about this finding. However, I found several reasons related to prior studies that may help to explain why children performed better in cutting shapes into even numbers. The reasons discussed in prior literature including children's real-world experience in cutting cakes into even numbers (Nunes et al., 2009), the natural human tendency to perceive everything as symmetrical (Lipka et al., 2019), and indirect mathematics and science practices involving folding geometric shapes, drawing the symmetrical shape, etc., (Common Core State Standards, 2010). Such informal and formal experiences of cutting objects into even numbers help children to unintentionally or intentionally know how to cut shapes into two or four equal parts. The phrase unintentionally was employed here because children's knowledge of cutting a shape into fourths alone does not guarantee that they internalized the concept of equal partitioning. More explanations, children benefit from the practice of cutting real-life objects into four equal parts. They use this experience to cut shapes into four equal parts. However, there is always the possibility that some students do not comprehend the concept of equal dividing and simply copy the activities they witnessed. Teachers can determine if a kid has internalized the concept of equal partitioning by observing a child's action when they divide a shape into three equal sections. Future research is needed to investigate potential reasons for the difference in children's partitioning dealing with different types of numbers.

Fragmenting Schemes and Visual Models

The ordinal regression analysis did not indicate any statistically significant relationship between visual models and children’s fragmenting levels. However, analysis of the clinical interviews revealed that cutting circles into three and five parts are difficult for students. Recall that the analysis from the Friedman test already suggested students had more difficulty with odd numbers compared to even numbers. Through clinical interviews, I found circles add an extra layer of difficulty for partitioning into odd numbers in compared to rectangles. This difficulty was observed across all students at different levels of fragmenting. In particular, for children with the highest level of fragmenting, the challenge of cutting circles into odd numbers presented a more procedural than conceptual. Children at the highest level of fragmenting (Level 3) demonstrated the equal partitioning scheme. However, they stated they did not know the process of making circles into three or five equal parts while exhausting the whole. This is an important finding regarding the instruction aspect of different representations for teaching and learning fractions. Using multiple representations has always received considerable support in learning fractions (Rau & Matthews, 2017). However, this study showed that some of the representations, specifically circles (as they are the focus of this study), are complex. And despite students' knowledge of concepts (fragmenting), using that representation (circle) to show their knowledge is difficult. The findings of this study suggest explicit instructions for partitioning circle models into third and fifth parts. Additional research is required to find effective ways of teaching how to partition this model into odd numbers.

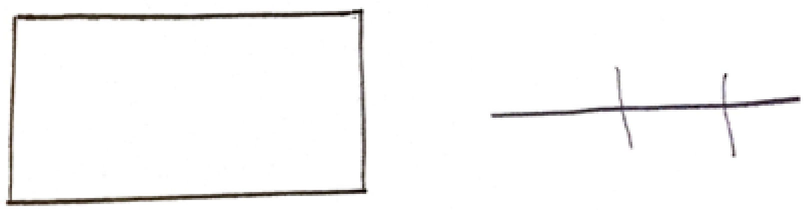
Considering that the regression model was an inclusive model that was not meant to detect the underlying relationships between particular sorts of numbers and shapes, I needed a more parsimonious model to validate the clinical interview findings. The supplementary analysis (i.e., Kruskal Wallis) was undertaken to determine if there is a relationship between the third and fifth partitioning of children and the visual models. According to the results, students performed differently while cutting circles into three and five pieces compared to when cutting rectangles.

Results of both quantitative and qualitative analysis suggest circle models can add an extra layer of difficulty for equal partitioning into third and fifth. There are several potential reasons for why circles are more difficult than rectangles. One of the reasons for such difficulty may be rooted in the nature of the circle itself. Circles are non-edged shapes with no starting or end points. Thus, it is difficult for students to pick a *starting point* to cut pieces of a circle (i.e., to create a unit) and then repeat that piece two times to make a circle into three equal parts. Rather, a child needs to know to start at the center of a circle.

In contrast to a circle, a rectangle has four sides and edges, giving students a visual clue on where to start and end. By having these reference points, students can create and estimate a piece (i.e., create a unit). Then by looking at the end, they can anticipate how big that piece or unit should be, so iterating that unit two more times would give them a rectangle with three equal parts. Another conjecture relates to the dimensionality of the visual models. Both circle and rectangle models are two-dimensional shapes. But it is possible that some students may imagine the rectangle model as a 1-dimension line that can be folded equally into given parts (see Figure 24). Although the above conjectures for the difficulty of circles appear reasonable, they must be empirically investigated in the future.

Figure 24

Visualizing a two-dimension rectangle model as a one-dimension line



Implications

This study provides insights into the nature of fragmenting schemes and how it develops in first and second grade students. The levels described in fragmenting framework are particularly designed to show researchers and educators how children demonstrate different forms of partitioning. Results from this study can also be used by mathematics educators in scaffolding children's fragmenting knowledge with different models. The following section describes several implications that inform both research and practice.

Implications for Research

The revised fragmenting framework has significant potential to help scholars better comprehend the nature and development of fragmenting in children. The framework consists of three primary levels, which each have some sub-levels (see Table 8). The sublevels provide more detail on how children develop fragmenting schemes (Battista, 2012). Despite the practicality of the revised framework, this study represents its initial development. Future work is needed to better understand students' actions at each level and sub-level. This can be done in a number of ways including conducting longitudinal studies or constructivist teaching experiments for students at different ages. Especially future studies are needed to focus on earlier ages to reveal more about how students learn fractions and, in particular, fragmenting schemes.

The same fragmenting knowledge of first- and second-grade students confirms and extends the findings of other studies regarding the absence of prerequisite schemes for learning fractions in students (Empson, 1995; Hackenberg, 2007; Kosko, 2019; Steffe & Olive, 2010). Prior research on secondary and upper elementary students showed that students lacked some prerequisite knowledge for fractions, including multiplicative reasoning (Kosko, 2019; Zwanch & Wilkins, 2021) and fragmenting knowledge (Hackenberg, 2007). The findings of this study extend those results about a lack of fragmenting knowledge for older children (i.e., secondary school students). The common assumption is that children are more prepared to learn fractions as they get closer to third grade. However, the current study suggests this is not so. That is, growing older does not ensure acquisition of fragmenting. Such results may be partially attributed to the fact that second graders receive fewer lessons and topics relating to partitioning and early fractions concepts (Common Core State Standards, 2010). However, future research is needed to examine for curricular factors, and any other potentially related factors.

Implications for Future Practice

The revised fragmenting framework is a useful aid for practitioners and teachers to understand how children develop fragmenting schemes. Teachers can utilize the three-level fragmenting framework for students in the same grade to determine if their students are prepared to learn fractions. Partitioning a shape is something a child is exposed to at an early age (Hackenberg et al., 2016; Piaget et al., 1960). However, knowing how to make the parts equal while exhausting the whole requires explicit instruction. This skill is essential for students to acquire before learning fractions. For instance, to show 1/3 of a shape, a child must first know how to make the shape into three equal parts. Thus, this framework gives insight to teachers on children’s preparation for learning fractions.

No difference between first graders and second graders in terms of developing fragmenting schemes informing teachers about the importance of having more fragmenting activities in second grades or even lower grades. Purposeful partitioning activities at the lower grades build the foundations for understanding the part-whole concept. Further, designing explicit instructions for equal partitioning in third grade can be a good preparation topic before teaching fractions. Findings of my study showed that partitioning into three and five parts are more difficult for children. Children need more explicit and purposeful exposure to such opportunities.

The similarities across all visual models suggest that teachers and practitioners can and should use a variety of visual models to scaffold children's fragmenting knowledge. In this study, I used drawing on paper and cutting playdough shapes to understand children's demonstrated fragmenting. Other materials, such as folded paper shapes, can be another useful way of teaching partitioning. Teachers can use stepwise techniques of folding or cutting shapes to help children visualize the steps of cutting shapes into equal parts. Given the observed difficulties with equally cutting circles, even for children at the highest level of fragmenting, it is also important to drill and practice the techniques for equal cutting of this shape. Despite all above suggestions, future research is needed to study the effective ways to teach how to partition circles into different parts.

Limitations

While the study's findings shed light on the understanding of children's knowledge of fragmenting, several limitations of the study are addressed. The first limitation of the study is related to the range of participants. In this study, the participants were from the first and second grades. However, children start to develop fragmenting at early ages (Empson, 1995;

Hackenberg et al., 2016). A wide range of participants, including kindergarteners and preschoolers, gave more insights into how children develop fragmenting schemes. Since first graders showed equal knowledge of fragmenting schemes, it is really important to know how kindergarteners and preschoolers demonstrate such schemes.

Further, in this study, I used a cross-case analysis to understand children's demonstrated scheme. Although cross-case analysis provide useful information in terms of children's different levels of fragmenting, there is a need to collect more longitudinal data to understand how these schemes develop. The fragmenting framework provided in this study includes three main levels with different sublevels. Given the sophistication of children's developmental schemes, and the steps that they take to move on from one level to the next, a constructivist teaching experiment may be a useful next step in better understanding how students build early fraction knowledge. Teaching experimental design allows researchers to work with students stepwise and design more in-depth activities based on students' prior work (Steffe & Olive, 2010). Thus, it can add more details on how children develop fragmenting schemes.

Three sets of surveys with different visual models were designed for this study, each including five same questions. While the surveys served well in understanding children's demonstrated fragmenting knowledge, I think an inclusive survey that includes all three shapes together would be beneficial for understanding more details about the effect of each model on children's fragmenting knowledge. In addition, an inclusive survey allows the researcher to add more evidence for the reliability of the survey and use the test broadly.

Conclusion

Fragmenting schemes are one a pre-requisite to learning fractions (Empson, 1995; Hackenberg, 2007; Steffe & Olive, 2010). Prior research has shown the lack of development of fragmenting schemes for most elementary and secondary school students (Hackenberg, 2007; Steffe & Olive, 2010). This study took action to investigate how first and second graders demonstrated fragmenting schemes while working with visual models.

The results and findings of this study indicate that most first and second graders demonstrated only early levels of fragmenting schemes. This result supported prior studies that stated students lack of preparation in learning fractions (Empson, 1995; Hackenberg, 2007; Kosko, 2019; Steffe & Olive, 2010). And if the trend that observed in this study is typical of U.S. students, there is a significant need to improve pedagogy and curriculum for fragmenting shapes. The finding of the study also suggested that children's fragmenting reasonings are consistent across different models, including circle, rectangle, and length. However, children are more likely to partition successfully when using rectangle models. More explanation, children with a higher level of fragmentation had a more difficult time dividing circles into three or five equal parts compared to dividing rectangles into three or five equal sections. Although the finding could not put an end to the long debates among scholars for no consensus on prioritizing one of the models in fraction instructions (Cramer et al., 2008; Hackenberg et al., 2016; Kaminski, 2018; Tunç-Pekkan, 2015), it shed light on how different models can facilitate children's learning of fragmenting at certain level.

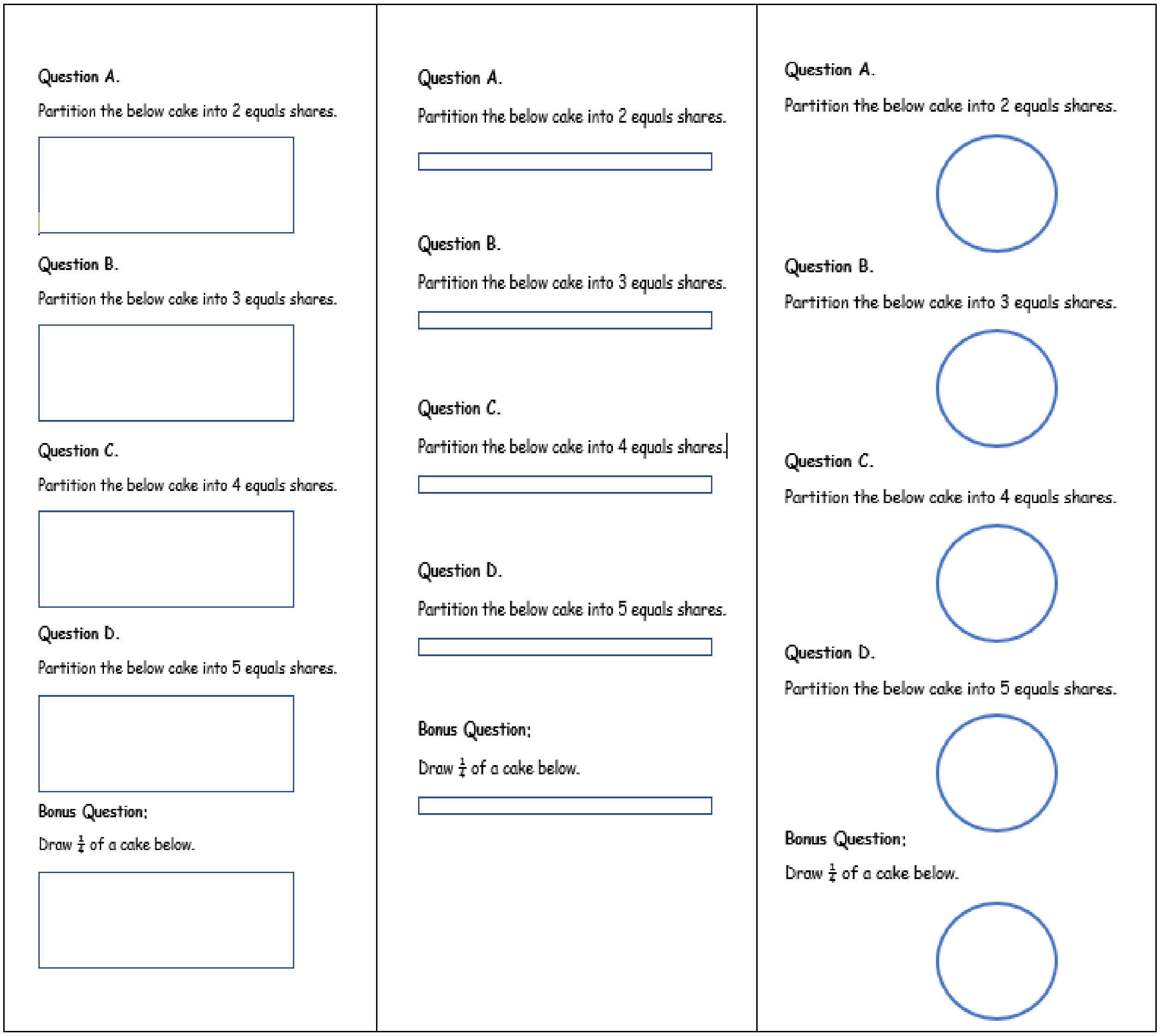
APPENDICES

APPENDIX A

RECTANGLE, LENGTH, AND CIRCLE FRAGMENTING SURVEY

Appendix A

Rectangle, Length, and Circle Fragmenting Survey



Question A.

Partition the below cake into 2 equals shares.

Question B.

Partition the below cake into 3 equals shares.

Question C.

Partition the below cake into 4 equals shares.

Question D.

Partition the below cake into 5 equals shares.

Bonus Question:

Draw $\frac{1}{2}$ of a cake below.

Question A.

Partition the below cake into 2 equals shares.

Question B.

Partition the below cake into 3 equals shares.

Question C.

Partition the below cake into 4 equals shares.

Question D.

Partition the below cake into 5 equals shares.

Bonus Question:

Draw $\frac{1}{2}$ of a cake below.

Question A.

Partition the below cake into 2 equals shares.

Question B.

Partition the below cake into 3 equals shares.

Question C.

Partition the below cake into 4 equals shares.

Question D.

Partition the below cake into 5 equals shares.

Bonus Question:

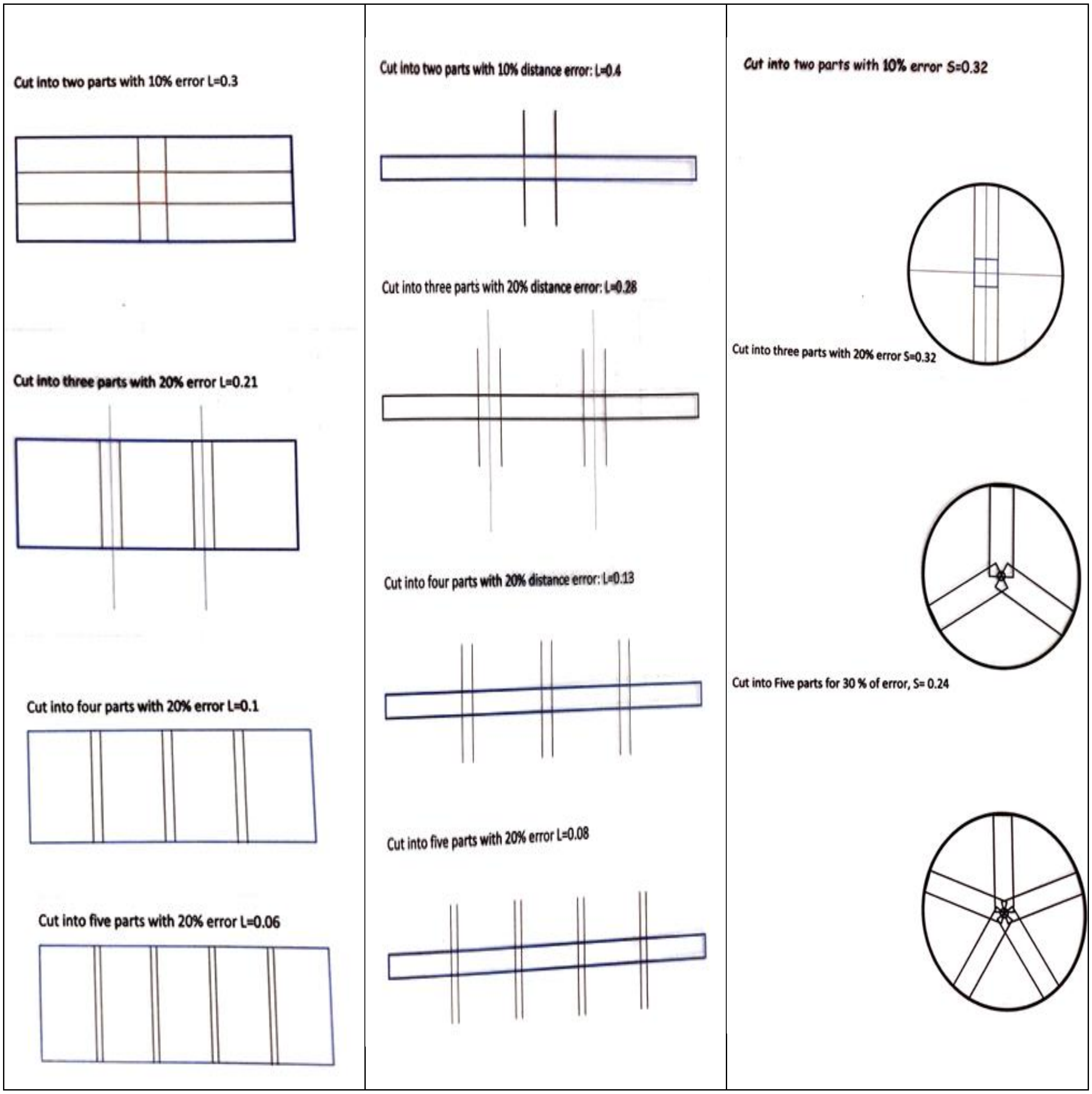
Draw $\frac{1}{2}$ of a cake below.

APPENDIX B

EQUAL PARTITIONING TEMPLATE

Appendix B

Equal Partitioning Template



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