ADULT LEARNERS' KNOWLEDGE OF FRACTION ADDITION AND SUBTRACTION

A dissertation submitted to the
Kent State University College
of Education, Health, and Human Services
in partial fulfillment for the requirements
for the degree of Doctor of Philosophy

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December 2017

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MUCKRIDGE, NICOLE, A., Ph.D., December 2017 TEACHING, LEARNING, AND CURRICULUM STUDIES

ADULT LEARNERS' KNOWLEDGE OF FRACTION ADDITION AND SUBTRACTION (168 pp.)

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The purpose of this study was to examine adult developmental mathematics (ADM) students' knowledge of fraction addition and subtraction as it relates to their demonstrated fraction schemes and ability to disembed in multiplicative contexts with whole numbers. The study was conducted using a mixed methods sequential explanatory design. In the first phase, 72 developmental mathematics students took a written assessment containing disembedding, fraction scheme, and fraction addition/subtraction items. Based upon the results of the assessment, three individuals from the first phase were selected to participate in one-on-one clinical interviews. These interviews were aimed at identifying and describing the cognitive processes underlying the participants' performance on the written assessment items.

Results from the quantitative phase indicated statistically significant moderate correlations between disembedding in multiplicative contexts, demonstrated fraction schemes, and fraction addition/subtraction. Moreover, regression analysis revealed that age, fraction schemes score, disembedding score, and number of repeated mathematics courses were all significant predictors of a participant's fraction addition/subtraction score. Analysis of the clinical interviews revealed that norming and the equi-partitioning scheme play an important role in ADM learners' conceptions of fractions.

This study quantitatively measured the relationship between disembedding, fraction schemes, and fraction addition/subtraction, which has been hypothesized in prior qualitative research. The results also have important instructional implications. Instructors of ADM courses should use the results of this study as an indication of the importance of determining their students' existing schemes and providing them with opportunities to engage in actions associated with higher-level schemes.

ACKNOWLEDGMENTS

"Give thanks in all circumstances; for this is the will of God in Christ Jesus for you."

1 Thessalonians 5:18 (*ESV*)

This Ph.D. journey has taught me much more than you will find in the results chapter, and I have many people to thank for their support and encouragement along the way. Thank you to my committee, Dr. Karl Kosko, Dr. Joanne Caniglia, and Dr. Jay Jahangiri. I am extremely grateful for your time and the various perspectives you each provided to my research. I would like to extend a special appreciation to my advisor and committee co-chair, Dr. Karl Kosko. Your guidance throughout this entire dissertation process has been invaluable. I cannot thank you enough for your quick and thoughtful feedback, as well as for reassuring me at several difficult points in time.

To my critical friends, Linda and Md, I am so grateful to have met you. Thank you both for providing personal support as well as feedback on my writing throughout the coursework and dissertation phases of this degree. This journey would not have been the same without the two of you! Md, thank you for always offering to read my work and for the thoughtful comments you provided. But more importantly, thank you for not letting me take myself too seriously! Linda, I could never fully express my gratitude to you for your support (especially your sense of humor!) throughout this entire journey. It was truly invaluable to have someone alongside me with whom I shared so many aspects of my life. I have been blessed by your friendship well beyond the scope of this degree.

Additionally, thank you to all of the students who participated in this study. I appreciate your willingness to do even more mathematics outside of your regular math classes in order to help further the knowledge base of adult mathematics education through this study.

Most importantly, a tremendous thank you goes to my family. Thank you to my husband, John, for your support throughout this long and often tiring journey. You never discouraged me from finishing no matter how busy our lives got. Rather, you made ways for me to have plenty of quiet time alone to read and write. You are an amazing partner and wonderful father. To my sister, Krista, sharing this experience with you kept me sane! I am so thankful that I was able to encourage you and be encouraged by you. To my mom and dad, you have supported me and believed in me from the very beginning. My work ethic is undoubtedly a result of the wonderful example you both set for me. Thank you also for providing me with “office space” at your kitchen table when I needed a close and quiet place to write (especially after my actual office turned into a toy room!). Mom, “thank you” doesn’t seem sufficient to express the deep gratitude I have for all that you did to help me while I was completing this degree. Thank you for watching the kids countless times when I needed to study or write, as well as for the meals you made us during the especially busy times. I couldn’t have done this without you!

Finally, this dissertation is dedicated to my three sons, Jack, Luke, and Evan, who always knew how to make me smile on a bad day. They were a constant reminder of life beyond these pages and motivated me to keep going. I know you were not able comprehend the scope of this Ph.D. journey at the time, but I hope that in some way it has taught you what it means to persevere and to use the gifts that God has given to you in order to bring Him glory.

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CHAPTER I

INTRODUCTION

Statement of the Problem

Research on adult teaching and learning of mathematics is relatively new, with the earliest publications originating in the 1960s. Although adult learning of mathematics is a relatively new area for research, it is an exceptionally complex one. Just as research in mathematics education becomes intertwined with research in psychology, sociology, mathematics, and other fields, research in adult mathematics education is inevitably linked to these disciplines as well as to the study of general adult education. As Coben, FitzSimons, and O'Donoghue (2000) pointed out, adult learning of mathematics is an "under-theorised domain which needs to draw upon as many relevant disciplines as possible in order to develop" (p. 6). To add to the complexity of defining adult learning of mathematics as a research field, adults learn mathematics in several different settings. The most common of these settings include postsecondary schools (two- and four-year colleges/universities, vocational and trade schools), the home, the workplace, and adult basic education (ABE) courses. Moreover, the types of subjects offered to adults studying mathematics within these settings can vary tremendously, ranging from arithmetic to graduate level abstract algebra, analysis, and topology.

Although adult learning occurs in a variety of settings and across many subjects, prior research on adult mathematics education is overwhelmingly affective in nature. For example, studies tend to focus on adults' perceptions of mathematics (Crawford, Gordon, Nicholas, & Prosser, 1998; FitzSimons & Godden, 2000) and mathematics anxiety

(Higbee & Thomas, 1999; S. Tobias, 1978). Additionally, a large body of research is centered around curriculum, focusing on what subjects and concepts should be taught to adult learners of mathematics and why (Safford-Ramus, 2008; Tout & Schmitt, 2002). Lastly, research on adult mathematics education is comprised of instruction based studies, which focus on how mathematics should be taught and which "strategies" work with adult learners (Hodara, 2011; Mireles, Offer, Ward, & Dochen, 2011; Spradlin & Ackerman, 2010). Studies involving adult learners' cognitive processes of mathematics are few in number. For example, algebra is a huge subject area of mathematics and one that nearly all adults are required to take in post-secondary settings. Yet, as Manly and Ginsburg (2010) pointed out, algebraic thinking in adults has no body of research comparable to the work that has been conducted with children. Moreover, the study of how to teach algebra to adult learners is more robust than the study of how adults think algebraically.

This lack of theory pertaining to how adults learn mathematics can also be seen through a review of the *Adults Learning Mathematics International Journal*, or *ALMIJ*. Although the *ALMIJ* calls for "research and theoretical perspectives in the area of adults learning mathematics/numeracy" (Hector-Mason & Díez-Palomar, 2012, p. 2), theories pertaining to how adults learn mathematics have been missing from the journal since its inception. Instead of theorizing adult mathematics learning, articles published in the *ALMIJ* over the past decade have been focused primarily on the relationship between school mathematics and mathematics in everyday life, namely the workplace. Whereas three of the past 14 issues were dedicated to affective factors involved in adult mathematics learning, such as anxiety and perceptions of mathematics, only one issue was devoted to "how adult learners grow to comprehend mathematical concepts" (Hector-Mason & Díez-Palomar, 2012, p. 5). This examination of the literature indicates that further theorizing on adult mathematics learning is necessary.

In particular, theorizing on adult learners' knowledge of fractions is an area of needed research. According to Lamon (2007), fractions, ratios, and proportions have arguably been both cognitively challenging for students as well as most difficult to teach. Yet, at the same time, these concepts are extremely critical to student success in higher level mathematics and science. The National Mathematics Advisory Panel (2008) surveyed a large, nationally representative panel of high school Algebra I teachers and reported that these teachers rated students as having very poor preparation in "rational numbers and operations involving fractions and decimals" (p. 28). Furthermore, a poor understanding of fractions was ranked by these teachers as one of the two most important weaknesses in students' preparation for Algebra I. Therefore, because students continue to struggle with fractions even after studying them throughout middle and high school, fractions are a very important field of research within adult mathematics education.

Significance of the Study

Despite the enormous emphasis on developing students' mathematical abilities for postsecondary education (American Diploma Project, 2004; American Mathematical Association of Two Year Colleges, 1995, 2002; Conley & Bodone, 2002), current research shows that this goal is not being met. Students who are not prepared for postsecondary school coursework take what are most commonly known as developmental courses. Developmental mathematics courses prepare students with the mathematical content knowledge and skills necessary to take college-level mathematics courses at a given postsecondary institution (Parsad & Lewis, 2003). Successful completion of these types of courses does not result in credit toward a student's degree. In contrast, college-level mathematics courses are those in which students are awarded credit towards their degree upon successful completion of the course (Bonham & Boylan, 2011).

Wirt et al. (2004) reported that almost 30% of incoming freshman students entering a postsecondary institution in the fall of 2000 needed developmental coursework due to a lack of preparedness for standard credit-earning courses. The majority of these developmental courses were in the area of mathematics. In addition to an increase in the number of students needing developmental mathematics courses, these courses usually have the highest failure and incompletion rates among all developmental subjects (Bonham & Boylan, 2011). Analyzing data from the National Educational Longitudinal Study, Attewell, Lavin, Domina, and Levey (2006) found that only 30% of students pass all of the developmental mathematics courses in which they enroll, as compared to 68% in writing and 71% in reading. Failure and incompletion of developmental mathematics courses impacts both the time and money students spend on their education. These factors often discourage students and can become a barrier for students to complete their degrees. For example, only 42% of students taking two or fewer developmental mathematics courses during the fall of 2000 went on to complete their certification or degree program (Wirt et al., 2004). Thus it is important to identify the course topics with which students struggle.

Fractions are cited as a difficult topic for students of all ages (Moss & Case, 1999; Vinner, Hershkowitz, & Bruckheimer, 1981). At the same time, fraction knowledge is an essential topic for moving on to college-level mathematics courses. For example, fraction operations, such as addition and subtraction, are necessary for students to understand in order to operate on rational functions and trigonometric functions in college algebra and trigonometry. Understanding how developmental mathematics students think about fraction addition and subtraction will help practitioners to reach students who are struggling with these notoriously difficult topics in addition to contributing to a limited body of existing research on adult learning.

Research Goals and Questions

The overarching focus of this study is to examine adult developmental mathematics students' knowledge of fraction addition and subtraction. Prior research conducted with children has studied the relationship between a learner's whole number knowledge and his or her fractional knowledge (Behr, Wachsmuth, Post, & Lesh, 1984; Steffe & Olive, 1990). Research on children's fractional knowledge has also found a hierarchy of fraction schemes inferred from analyzing children's engagement with fraction tasks (Steffe & Olive, 2010). Thus, the first and second research questions in this study examine the relationship between these aspects of knowledge and students' performance and understanding on fraction addition/subtraction tasks. Furthermore, this study focused on adult developmental mathematics learners, an under-studied population in existing literature. As Wedege (1998) pointed out, "the situation for learning mathematics depends on the experience of the individual adult with mathematics in school and everyday practice and their individual perspectives for learning" (p. 214). Thus, the goal of the third research question is to understand the ways in which these experiences have informed adult learners' fractional knowledge. Specifically, the following research questions were addressed:

1. Does adult developmental mathematics students' performance on whole number disembedding tasks inform how they add/subtract fractions? If so, what is the nature of the relationship?
2. Do fraction schemes demonstrated by adult developmental mathematics students inform how they add/subtract fractions? If so, what is the nature of the relationship?
3. In what ways do adult developmental mathematics students' prior mathematics experiences inform their fraction conceptions, particularly regarding addition/subtraction of fractions?

The remainder of this paper is organized as follows. Chapter 2 provides a review of literature pertaining to children's and adults' understandings of fractions. Specifically, it focuses on the different types of fraction conceptions as well as the role of the unit in such conceptions. Because very little literature exists regarding how adult developmental mathematics students conceptualize fractions, much of the theory draws on research with children and preservice teachers (PSTs). Chapter 3 discusses the specific methodology and procedures used to answer the research questions. A mixed methods sequential explanatory design was used, whereby the researcher first collected and analyzed quantitative data followed by the collection and analysis of qualitative data. Chapter 4 presents the findings from both the quantitative and qualitative phases of the study. Finally, Chapter 5 provides implications of this study for educators and future research.

CHAPTER II

LITERATURE REVIEW

This chapter reviews literature associated with the learning and conceptualizations of fractions. Specifically, it is concerned with the role of the unit in students' understandings of whole numbers and fractions, as well as with the addition and subtraction of fractions. Reviewing the literature on the mathematical thinking of adult learners revealed that adult learners' conceptions of fractions have not been widely researched. Thus, it was necessary to use research conducted with other populations to create a model for describing adults' knowledge of fractions. Because developmental mathematics students are studying mathematical concepts, such as fractions, first introduced at the elementary and middle school levels, this chapter begins by synthesizing relevant literature in K–12 mathematics education. Additionally, preservice teachers (PSTs) are a subset of adult learners whose mathematical content knowledge regarding fractions has been studied in prior research. Therefore, a summary of the literature on PSTs' knowledge of fractions is also presented. This chapter concludes with a proposed framework for describing adults' fraction knowledge. In addition to research on the fraction knowledge of children and PSTs, this framework is informed by the limited studies regarding adults' knowledge of fractions as well as mathematical thinking of adult learners in general.

Children's Knowledge of Fractions

Research on the teaching and learning of fractions has constituted a large body of the literature in mathematics education since the last quarter of the 20th century (Lamon, 2007; Olive, 1999). In an effort to better understand the cognitive demands associated with teaching and learning fractions, researchers have worked to classify children's conceptions of fractions. The work done by Kieren beginning in the late 1970s is considered seminal in identifying subconstructs of rational numbers necessary to build a full understanding of rational numbers (Behr, Harel, Post, & Lesh, 1992). The work done through the Rational Number Project (RNP) was largely based on Kieren's subconstructs. Additionally, Kieren's subconstructs remain the foundation of many of the studies pertaining to children's conceptions of fractions (Olive, 1999). It was not until the work of Steffe and Olive (1990, 2010) that a second approach to classifying children's conceptions of fractions was established.

One of the key differences between the two approaches to classifying children's conceptions of fractions involves how the conceptions are initially determined. For example, some researchers (Behr, Lesh, Post, & Silver, 1983; Kieren, 1980) use rational task analysis, which examines mathematical content, such as fractions, from a "mature mathematical perspective, making assumptions about the ways of thinking that are necessary to solve problems" (Lamon, 2007, p. 641). Other researchers (Norton & Wilkins, 2009; Steffe & Olive, 1990) take a more *emergent* approach, focusing on the mathematics of children in order to analyze the processes involved in understanding complex mathematical concepts such as fractions. One of the results of working from the viewpoint of how children construct fraction knowledge as opposed to a more top-down mathematical view of fractions is a disagreement over the role that natural numbers play in students' understanding of rational numbers.

Some prior research on children's fractional knowledge assumes that a child's natural number knowledge interferes with his or her fractional knowledge (Behr et al., 1984). However, Steffe and Olive's hypothesis, known as the reorganization hypothesis, claims that "children's fractional knowing can emerge as accommodations in their natural number knowing" (2010, p. vii). Specifically, children reorganize their previous way of knowing (within natural numbers) to construct a new way of knowing (within fractions). As opposed to using their own mature constructs to interpret their experience of children's mathematics, Steffe and Olive focused on the learning trajectories of children. Therefore, they did not make the assumption that a child's natural number knowledge interferes with his or her fractional knowledge. Rather, Steffe and Olive's teaching experiments have demonstrated that children's fraction knowledge can emerge by way of a reorganization of their numerical counting schemes, provided their instructional and material environment supports their thinking.

As a result of these differing epistemological assumptions concerning how natural number knowledge relates to fraction knowledge, researchers have adopted different ways of classifying children's conceptions of fractions. These two approaches to classifying children's conceptions of fractions are the focus of the remainder of this section. Although the current study primarily is concerned with the fraction schemes stemming from the work of Steffe and Olive (2010), the subconstructs of rational numbers established by Kieren also are summarized, as they are the basis for a large number of studies within this review.

Subconstructs of Rational Numbers

Kieren (1980; 1988) indicated that he believes a fully developed rational number concept consists of four subconstructs: measure, quotient, ratio, and multiplicative operator. In the measure subconstruct, a fraction is considered a number and is also associated with the measure of a quantity relative to one unit of that quantity. For example, 2/3 corresponds to the distance of two one-third units from zero. Under this subconstruct, students should be able to locate a given fraction on a number line as well as identify the fraction represented by a given point on a number line. Under the quotient subconstruct, rational numbers are understood in terms of division and/or equal sharing. For example, the fraction 2/5 could be viewed as 2 ÷ 5 or as sharing two candy bars equally among five friends. The notion of equal sharing is one that is rooted in the experiences of many children. The ratio subconstruct involves interpreting rational numbers as comparisons between two quantities (of the same or different type). For example, the fraction 4/6 could represent a comparison of the number of cups of sugar and the number of cups of flour in a recipe (4 cups of sugar to every 6 cups of flour). Alternatively, it could represent the number of cookies that can be purchased for a certain amount of money (4 cookies for $6). Lastly, the multiplicative operator subconstruct involves using fractions as operators to stretch or shrink another number. The most common example of this subconstruct would be taking 1/2 of 6 or 3/4 of 8.

Behr et al. (1983) further developed Kieren's model by adding the part-whole interpretation as a subconstruct. Although Kieren (1988) acknowledged this interpretation of rational numbers, he, along with researchers such as Lamon (2007), considered the part-whole interpretation to be a manifestation of the measure subconstruct. The part-whole interpretation of a fraction involves partitioning a continuous quantity (e.g., a candy bar) or a discrete number of objects (e.g., five pizzas) into parts of equal size. The fraction is thus a comparison between a number of parts of the partitioned unit and the total number of parts into which the whole (unit) is partitioned. From this perspective, the numerator should be less than or equal to the denominator, which limits students' understanding to proper fractions and does not easily translate to improper fractions. This has traditionally been the most commonly encountered approach to fractions and teaching fractions in school mathematics (Lamon, 2007; Streefland, 1991). Kieren (1980) pointed out that although various subconstructs appear to have similar mathematical properties, situations related to different subconstructs of the same rational numbers can draw different responses from students. Additionally, the subconstructs each allow for different connections to other mathematical ideas.

Emergent Fraction Schemes

Steffe and colleagues' approach to examining children's conceptions of fractions adopts the perspective that schemes are attributed to students by researchers and teachers as a way of explaining and predicting the students' actions (Norton & McCloskey, 2008). According to von Glasersfeld (1980), a scheme consists of three components: (a) a recognition template for recognizing situations in which the scheme applies; (b) an action or operation associated with the situation; and (c) an expected result of the action or operation. For example, consider a problem such as: "Luke has 4 toy cars. Jack gives him 3 more toy cars. How many toy cars does Luke have now?" A child given this problem might recognize the situation as involving addition and thus one where the "counting on" scheme applies. The action of counting on would be invoked by the student with the expected result of this action to be the value achieved after starting at 4 and "counting on" 3.

Steffe and Olive's (2010) fraction schemes are models of children's fractional knowing and are based on partitioning schemes. Partitioning involves separating a whole (or unit) into equal parts. Steffe and Olive identified three partitioning schemes: the *simultaneous partitioning* scheme, the *equi-partitioning* scheme, and the *splitting* scheme. In the *simultaneous partitioning* scheme, the student mentally projects a composite unit into a continuous whole and uses the composite unit as a "partitioning template" with the goal of establishing equal parts. For example, given a rectangular bar, a student who is asked to shade 1/7 of the bar would mentally project a composite unit of 7 parts into the whole bar. Within the *equi-partitioning* scheme, the student would know and use the simultaneous partitioning scheme with the additional understanding that the partitioned parts are identical and that any one of them can be iterated a certain number of times to reproduce the original whole. Thus, a student who is given a rectangular bar and asked to shade 1/7 of the bar would not only produce 7 equally spaced units within the bar but would also iterate any one of the parts to reproduce the whole bar. Lastly, under the *splitting* scheme, a whole is first partitioned into equally sized pieces and then one of those pieces is iterated to produce a part of the whole. Additionally, when invoking the splitting scheme, the student establishes a multiplicative relation between the part produced and the given whole. For example, if a student is told to make a stick such that a given stick is 5 times as long, the student would partition the given stick into 5 pieces and iterate one of the pieces to produce a partial stick. Furthermore, this student would recognize the whole stick to be 5 times as long as the partial stick. In the case of the splitting scheme, the student may or may not recognize the partial stick to be 1/5 as long as the given stick. In other words, this scheme is independent of any fraction scheme.

In addition to the partitioning schemes, Steffe and Olive (2010) have identified fraction schemes by which to describe the situations, actions, and results of students' fractional knowing. Five of these schemes examined in this study are the *part-whole* fraction scheme, the *partitive unit* fraction scheme, the *partitive* fraction scheme, the *reversible partitive* fraction scheme, and the *iterative* fraction scheme (see Table 1).

In the *part-whole* fraction scheme (PWFS), the student unitizes a whole, partitions the whole into equal pieces, and disembeds some number of pieces within the partitioned whole, while leaving the whole intact. Disembedding refers to an operation that involves "taking a part out of a whole unit without mentally destroying the whole" (Hackenberg, 2010, p. 385). In the part-whole scheme, a student conceives of a fraction as a certain number of pieces of a partitioned whole out of the total number of partitioned pieces of the whole (see Figure 1). For example, the fraction 3/4 is understood to represent 3 pieces out of 4 equal pieces of a whole. Students who have constructed this scheme do not yet recognize 3/4 as 3 iterations of one of the fourths, and so they do not unitize the fraction (Boyce & Norton, 2016). Additionally, since this scheme involves identifying fractions as parts out of a whole, it does not provide an adequate way for students to conceive of improper fractions.

Table 1

Fraction Schemes and their Associated Actions and Example Tasks

Fraction Scheme	Actions	Example Task
Part-whole fraction scheme (PWFS)	Unitizing a whole, partitioning the whole into equal pieces, and disembedding some number of pieces within the partitioned whole	Shade ¾ of the bar. (See Figure 1.)
Partitive unit fraction scheme (PUFS)	Given a whole and a unit fractional part of the whole, iterating the fractional part an appropriate number of times to reproduce the whole, understanding that the number of iterations determines the size of the fraction relative to the whole	If the longer bar is one whole, what fraction is the shorter bar? (See Figure 2.)
Partitive fraction scheme (PFS)	Given a whole and a proper fractional part of the whole, partitioning the fractional part to produce a unit fractional part, iterating the unit fractional part to produce the proper fraction and the whole, while maintaining the relation between the unit fraction and the whole	If the longer bar is one whole, what fraction is the shorter bar? (See Figure 3.)
Reversible partitive fraction scheme (RPFS)	Given a proper fractional part of a whole, partitioning the proper fractional part to produce a unit fractional part and iterating the unit fractional part an appropriate number of times to produce the whole	If the bar shown is 4/5 as long as a whole bar, draw the whole bar. (See Figure 4.)
Iterative fraction scheme (IFS)	Given any fractional part of a whole (including improper fractions), partitioning the fractional part to produce a unit fractional part and iterating the unit fractional part an appropriate number of times to produce the whole	If the bar shown is 4/3 as long as a whole bar, draw the whole bar. (See Figure 5.)

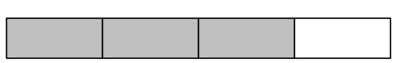


Figure 1. Part-whole fraction scheme.

In the *partitive unit* fraction scheme (PUFS), a student understands that any unit fractional part can be iterated a certain number of times to reproduce the whole and that the number of iterations determines the size of the fraction relative to the whole. For example, a student under this scheme would understand that a unit fraction such as 1/3 could be iterated three times to produce a whole (see Figure 2). Furthermore, the student would also understand that three iterations indicate that the fraction 1/3 is one-third of the whole. One limitation of this scheme is that it cannot be used to determine the fractional size of a non-unit fraction where no number of iterations of the fraction will reproduce the whole (e.g., 2/5).

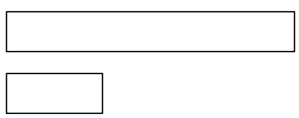


Figure 2. Partitive unit fraction scheme. The shorter bar is 1/3.

The *partitive* fraction scheme (PFS) is a generalization of the partitive unit fraction scheme. Thus, a student working in this scheme can produce composite fractions from unit fractions through iteration, while maintaining the relation between the unit fraction and the whole. For example, a student would be able to produce the fraction 3/5 by iterating 1/5 of the whole three times (see Figure 3). This scheme involves units coordination at two levels (units of units) because the student views 3/5 as a unit of three units (fifths).

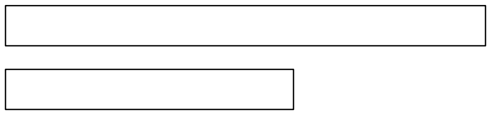


Figure 3. Partitive fraction scheme. The shorter bar is 3/5.

The *reversible partitive* fraction scheme (RPFS) involves producing an implicit whole from a proper fractional part of it by partitioning the fractional part to produce a unit fraction and iterating the unit fraction an appropriate number of times to produce the whole. For example, given a bar that is 4/5 as long as another bar, a student in the reversible partitive fraction scheme would understand 4/5 as 4 one-fifths, partition the given bar into four pieces, and iterate a piece five times to produce the whole (see Figure 4). This scheme also involves units coordination at two levels (units of units), as the student would establish 4/5 as a unit of four units (fifths).

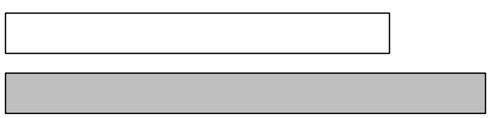


Figure 4. Reversible partitive fraction scheme. The given bar is unshaded, and the whole bar is shaded.

Under the *iterative* fraction scheme (IFS), a student produces an implicit whole from any fraction (including improper fractions) by partitioning a given fraction into equal parts with the understanding that one of the parts could be iterated a certain number of times to produce the whole. For example, given a bar that represents 4/3, a student in the iterative fraction scheme would be able to produce a 3/3 bar by partitioning the 4/3 bar into four equal pieces and iterating one of those pieces three times to produce a whole (see Figure 5). This scheme involves units coordination at three levels (units of units of units) because the student would view 4/3 as a unit of four units (thirds), any of which imply a unit (whole) of three units (thirds). Here, fractions are freed from relying on the whole for meaning even though they stand in relation to the whole (Hackenberg, 2007).

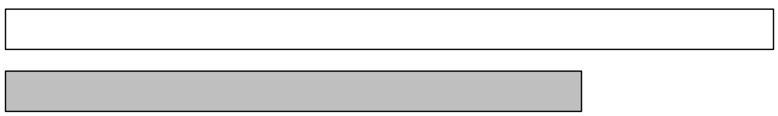


Figure 5. Iterative fraction scheme. The given bar is unshaded, and the whole bar is shaded.

The aforementioned fraction schemes, identified in Steffe and Olive's (2010) teaching experiments, have been further studied by other researchers (Hackenberg, 2007; Nabors, 2003; Norton, 2008; Saenz-Ludlow, 1994). Through use of teaching experiment methodology, these researchers have found a theoretical hierarchy to fraction schemes. In this hierarchy, each preceding scheme is reorganized to form the succeeding scheme. Students tend to develop a part-whole fraction scheme early and might not ever move beyond it. If they do move beyond this first scheme, students generally progress through some or all of the partitive fraction schemes, with which fractions become comparisons of relative size within the whole. After reaching the reversible partitive fraction scheme, a student who keeps progressing would then reach the iterative fraction scheme in which fractions are still understood in relation to the whole but are additionally "freed from the whole."

The Role of the Unit

Both content analyses (Behr & Harel, 1990; Harel & Behr, 1990) and research on children's thinking (Lamon, 1989; Steffe, 1994) point to unitizing as a mechanism for the growth of mathematical thinking. As noted earlier, a unit is commonly defined as "an entity that is treated as a whole" (Confrey & Harel, 1994, p. xvii). Thus, unitizing refers to the process of treating a collection of units as a whole. Furthermore, research on the development of language hierarchies also suggests that more sophisticated thinking results when one reframes a situation in terms of a more collective unit because this allows the student to think about both the aggregate and the individual items that compose it (Lamon, 1996). Steffe (1988) pointed out that curriculum on whole number arithmetic gives problem situations in which children deal with quantities essentially expressed only in single units, rather than providing situations in which they represent quantity in various unit types. There are some situations in which whole number arithmetic involving a variety of unit types and units of units will provide them with a more adequate foundation for learning and understanding whole number arithmetic as well as serve as a bridge to developing rational number understanding.

Lamon (1996) used partitioning tasks in clinical interviews in order to study children's unitizing processes. She found that "the reunitization of a quantity in terms of more composite units appears to be a complex process that happens in stages" (Lamon, 1996, p. 188). Namely, children's ability to form composite units involves their prior understanding of number sense and counting processes with whole numbers. For example, a child's whole number sense might help him or her to make the decision to use more composite units as opposed to decomposing into more pieces than necessary when completing a fair sharing problem.

The role of the unit is extremely important when it comes to students' development of fraction concepts (Lamon, 1994, 2007). How many or how much is meant by a fraction symbol is ambiguous unless we know to which whole it refers, also known as the *referent unit*. The whole could be continuous or discrete, as well as a single unit or a composite unit. For example, the numerical symbol 2/5 can represent different amounts depending on what the unit whole happens to be. If the whole is a cake cut into five pieces, then 2/5 of the cake represents 2 pieces. If the whole is one dollar, then 2/5 of one dollar represents $0.40. Additionally, the importance of creating "units of units" (two levels) and "units of units of units" (three levels) has been found in research on fractions and proportional reasoning. For example, as discussed in the previous section, the ability to coordinate three levels of units as opposed to two levels of units is what distinguishes a student operating under the iterative fraction scheme from one operating under a partitive fraction scheme.

Several studies have examined the role that units and unitizing plays in children's conceptions of fractions (Boyce & Norton, 2016; Hackenberg, 2007; Hui-Chuan, 2014). In one such study, Izsák, Tillema, and Tunç-Pekkan (2008) conducted semi-structured interviews with sixth grade students. The tasks used in the interviews pertained to fraction addition and subtraction involving number lines and fraction strips. One of the three pairs of students interviewed possessed a part-whole interpretation of fractions (also referred to by the authors as the *n-out-of-m structure*). As a result of this interpretation, these two students had difficulty maintaining a fixed unit whole when reasoning with the number line and fraction strips (Izsák et al., 2008). However, once one of these students attended to a fixed unit, she was able to represent fractions appropriately by taking partitions of partitions. This study is one that illustrates the important role that maintaining a fixed unit has in children's knowledge of fractions.

Maintaining a fixed unit whole is a key component of the disembedding operation. Recall that disembedding refers to "taking a part out of a whole unit without mentally destroying the whole" (Hackenberg, 2010, p. 385). Beginning with the part-whole scheme, disembedding is an important operation necessary in each of the fraction schemes. Students who disembed parts from a whole unit can then operate on these parts while maintaining the understanding that they are part of another whole. Specifically with respect to fraction addition and subtraction, maintaining a fixed unit whole lays the foundation for students to understand fractions as numbers. As Fazio and Siegler (2010) pointed out, "the common error of trying to add fractions by first adding the numerators and then adding the denominators stems in part from not understanding that fractions are numbers with magnitudes" (p. 10). When students do not understand fractions as numbers, they often overlook such fraction addition and subtraction errors (Brown & Quinn, 2006). For example, when adding 2/3 + 1/5, students arriving at an answer of 3/8 are neglecting the magnitude of the addends and sum. Students who understand fractions as numbers know that 2/3 is greater than 3/8 and so 3/8 does not make sense given the magnitude of the two addends. Several studies discussed in this review, including both children and PSTs, report that students have difficulty treating whole units when working with fractions. Therefore, one focus of this study is to examine the relationship between the disembedding operation and fraction addition/subtraction performance among adult developmental students.

Preservice Teachers' Knowledge of Fractions

There is very limited research on adult developmental learners' knowledge of fractions. However, there have been a number of studies conducted with preservice teachers regarding fractions. Because PSTs are one type of adult learner, these studies can help to provide insight into the potential conceptions adult developmental learners might have of fractions. Through a review of literature on the mathematical content knowledge of preservice teachers across three time periods, Thanheiser, Browning, Edson, and Whitacre (2014) reported that the content area of fractions had the highest number of publications during two of the time periods. Within this larger review, Olanoff, Lo, and Tobias (2014) summarized the research that had been done on preservice elementary teachers' knowledge of fractions. They found that the research indicated: (a) preservice teachers' common content knowledge is strong when performing procedures but lacking in "fraction number sense;" (b) preservice teachers struggle when it comes to understanding why procedures work; and (c) preservice teachers tend to prefer using the part-whole representation of fractions and have difficulty representing fractions using operator and number line models. Additionally, studies involving preservice teachers have investigated the role that units and unitizing play in the PSTs' knowledge of fractions. Some of the common misconceptions revealed through Olanoff, Lo, and Tobias' (2014) review are summarized in Tables 2 and 3 and discussed in paragraphs that follow.

Table 2

Preservice Teachers' Misconceptions Regarding Fraction Operations

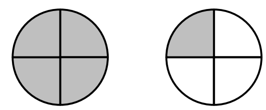
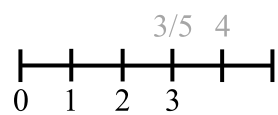
Misconception	Example	Example Task
Solution method is dependent upon the nature of the denominator as opposed to the given operation (Newton, 2008).	In an addition problem, keep the denominator the same if the addends have like denominators but add the denominators together if the addends have unlike denominators.	$\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$, while $\frac{2}{3}+\frac{1}{5}=\frac{3}{8}$
Absent, ambiguous, or inconsistent references to unit whole (McAllister & Beaver, 2012; Tobias, 2015; Toluk-Uçar, 2009)	When representing an addition problem with a story, using unequal whole units for the addends.	Q: Write a story problem that could be solved by 2/3 + 4/5. A: "Two thirds of the kindergarten class and four fifths of the eighth-grade class mixed together. What fraction of the two classes was mixed?" (McAllister & Beaver, 2012, p. 93)
Treating fractions as number of pieces rather than as quantities (Tobias, Roy, & Safi, 2015; Toluk-Uçar, 2009)	When given a fair sharing problem, answering in the form of a whole number rather than a fraction.	Q: Determine how much of a pizza each person gets given that three pizzas are shared equally among four people. A: 3 pieces (as opposed to 3/4 of a pizza, the correct response)

Note. Although the term *misconception* has been contested in some literature (see Watanabe, 1991, p. 30), it is being used here to describe knowledge that results in mathematical errors.

Newton (2008) found that preservice teachers' errors on fraction computation problems were a result of misapplying algorithms, such as adding the denominators of fractions with unlike denominators. Overall, the most common error at the beginning of the study was "to keep the denominator the same when it was not appropriate to do so"

Table 3

Preservice Teachers' Misconceptions Regarding Fractions

Misconception	Example	Example Task
Unable to recognize composite wholes (Rosli, Gonzalez, & Capraro, 2011)	Given two or more whole objects of the same shape and size, inability to interpret the group of objects as one whole unit.	Q: Determine how much of the pizza was eaten (as denoted by the shaded regions).  A: 1-1/4 of the pizza (as opposed to 5/8 of the pizza, the correct response)
Do not understand fractions as numbers (Domoney, 2001; Toluk-Uçar, 2009)	On a given number line (labeled from 0 to 3), correctly marking 4 but incorrectly marking 3/5.	

Note. Although the term *misconception* has been contested in some literature (see Watanabe, 1991, p. 30), it is being used here to describe knowledge that results in mathematical errors.

(Newton, 2008, p. 1104). Such errors indicated that the preservice teachers' misconceptions stemmed from their prior knowledge of fractions rather than their prior knowledge of whole numbers. Newton also measured the flexibility, or "inclination *to choose* alternate procedures," of the preservice teachers on the same fraction computation problems. She found the preservice teachers' flexibility to be low. For example, only seven out of 70 preservice teachers recognized that 2/4 – 3/6 = 0 because each of the fractions was equivalent to 1/2, whereas the remaining preservice teachers needed to find a common denominator and subtract the numerators before concluding the answer to be zero. Lastly, Newton (2008) found that 40% of the preservice teachers in her study did not understand the importance of using equal wholes when adding fractions.

Toluk-Uçar (2009) gave a 10 question, open-ended fraction assessment to 95 preservice elementary teachers. The questions covered a range of fraction concepts, including addition, subtraction, multiplication, division, comparing fractions, and equivalent fractions. In contrast to the Newton (2008) study, the results indicated that the preservice teachers were very capable of applying procedures to solve the questions. However, when asked to generate appropriate word problems for given fraction situations, most of the preservice teachers had difficulty doing so because "they were unable to identify the unit to which each fraction in an operation referred" (Toluk-Uçar, 2009, p. 170). For example, when representing 3/4 – 1/2, many of the participants generated word problems that actually represented 3/4 – 3/8 due to creating unequal wholes. Furthermore, Toluk-Uçar found that many preservice teachers treated fractions as number of pieces rather than as quantities, resulting in whole number answers rather than fractions.

Similar to Toluk-Uçar (2009), McAllister and Beaver (2012) explored the error types made by preservice teachers when asked to write word problems for given fraction operations. One of the most common errors was "an apparent lack of understanding of the concept of unit whole" (McAllister & Beaver, 2012, p. 95). This was evident through the preservice teachers' ambiguity with respect to units when writing word problems for all types of operations. For example, the participants in this study did not identify the unit whole to which the fractions were referring. When writing a scenario for 3 1/6 ÷ 4/5, one participant wrote, "Sarah had 3 1/6 cups of pretzels and she wanted to divide them in groups of 4/5. How much in each group?" (McAllister & Beaver, 2012, p. 93). This participant neglected to specify whether the 4/5 referred to cups or pretzels.

Several other studies examined preservice teachers' understandings of units within the content area of fractions. Rosli, Gonzalez, and Capraro (2011) used naturalistic inquiry to gain an understanding of three preservice teachers' knowledge of unit and unitizing fractions. Through semi-structured, open-ended interviews based on tasked-based fraction problems, the researchers found that these preservice teachers did not understand that a group of pizzas could represent a whole. In other words, these preservice teachers were unable to recognize composite units.

Tobias, Roy, and Safi (2015) conducted a teaching experiment with 33 preservice teachers in an elementary mathematics content course centered on unitizing and operations involving whole numbers and fractions. One of the tasks given to these preservice teachers was to determine how much of a pizza each person would get given that four pizzas were to be shared equally among five people. Their results indicated that only eight of the preservice teachers were able to correctly identify the whole to be a/one pizza. Nearly half of the participants gave answers in terms of pieces/slices. This is similar to one of the findings in the Toluk-Uçar (2009) study. Additionally, Tobias et al. (2015) found that the preservice teachers understood the idea of iteration with unit and composite fractions and were also "able to start making connections with unitizing across whole numbers and fractions" (p. 26).

Domoney (2001) investigated preservice elementary teachers' understandings of fractions, focusing on the concept of fractions as numbers. Through clinical interviews with four such preservice teachers, he found the part-whole subconstruct to be the dominant way in which fractions were interpreted. In addition, when given a number line labeled from zero to three and asked to mark the number four and the number 3/5, all participants correctly marked the whole number four. However, three of the four participants marked the whole number three instead of 3/5, indicating that they saw the number line as one whole and three as three-fifths of the whole line. This illustrates that these preservice teachers did not have the understanding of a fraction as a number.

Adult Developmental Learners' Knowledge of Fractions

As a result of incomplete understandings about fractions during the elementary, middle, and secondary school years, many adults pursuing postsecondary degrees continue to have difficulties understanding and performing operations on fractions. Therefore, adults' knowledge of fractions is a needed area of research within adult mathematics education. Existing literature in the field of adult mathematics education is limited in scope, particularly in the area of fractions. Thus, it becomes necessary to use literature from research on children and PSTs to construct a framework for organizing and interpreting adult learners' knowledge of fractions. Based on such literature, as well as the limited literature on adult mathematics learners (see Muckridge, 2015), I have theorized two types of knowledge that inform adults' fractional knowledge: *unorganized procedural knowledge* and *contextualized knowledge*. *Unorganized procedural knowledge* consists of rules and procedures for operating with fractions, which have been acquired by the learner throughout his or her formal mathematics instruction. This type of knowledge is unorganized in the sense that the learner does not always know when and/or why to use these rules and procedures[1]. *Contextualized knowledge* refers to more organic, learner generated approaches that are grounded in the real world experiences of the learner. The learner may or may not identify these experiences as mathematical (Coben, 2000). Models, or representations, of fractions also inform adults' fractional knowledge. Here, a model is referring to a written or mental pictorial representation of a fraction. For example, models might include fraction bars, fraction circles, or number lines.

The theorized relationships between unorganized procedural knowledge, contextualized knowledge, models of fractions, and adults' fractional knowledge are depicted in a diagram in Figure 6. In this diagram, there is a dashed line between unorganized procedural knowledge and contextualized knowledge, representing a possible relationship between the two types of knowledge. For this relationship to exist, for example, the learner would make a connection or connections between a procedure and a real world situation in which the procedure would be used. Similarly, a dashed line between models of fractions, contextualized knowledge, and unorganized procedural knowledge indicates a possible relationship. In some cases, a learner will create a model to represent fractions in a real world situation in an effort to better understand the situation itself. On the other hand, fraction models might be used in a more procedural sense whereby the learner goes through the process of creating a model despite not using the model to help solve the problem.

[1] It is possible for an adult student to demonstrate *organized procedural knowledge*, whereby he or she knows when and why to use rules and procedures but is unable to transfer this knowledge to a real world context or situation. This type of knowledge is not the focus here since students demonstrating this type of knowledge would likely not be placed into developmental mathematics courses.

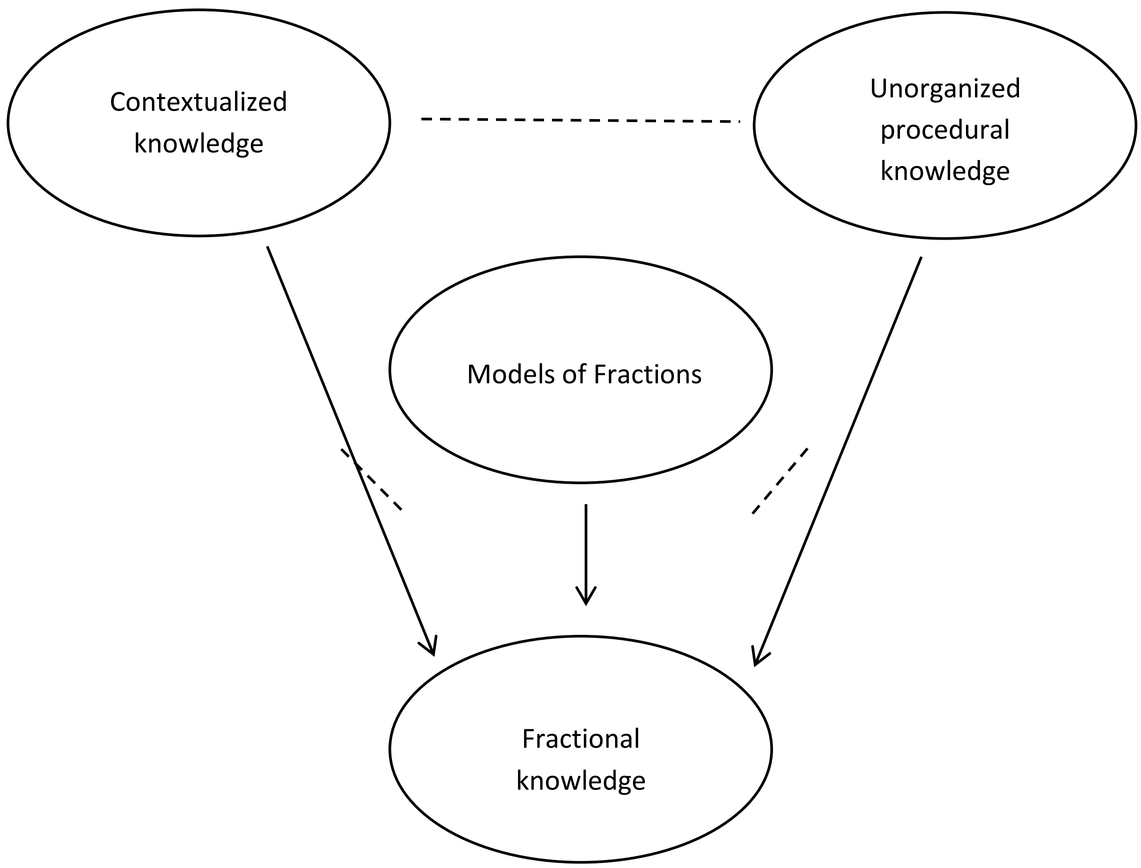


Figure 6. Framework of adult learners' knowledge of fractions.

Existing research indicates that adult mathematics learners tend to rely on an unorganized procedural knowledge of fractions based on their prior instruction (Alexander, 2013; Muckridge, 2015; Silver, 1983). Prior formal instruction in mathematics impacts both cognitive and affective aspects of an adult learner's mathematical experience. Namely, students learn concepts and procedures in tandem with acquiring perceptions of the subject of mathematics and their own ability to do mathematics. Thus, an adult learner's unorganized procedural knowledge is comprised of these acquired perceptions as well as the learned procedures that the learner does not always know when and/or why to apply.

Research has shown that adults often report having had negative past experiences in school mathematics (FitzSimons & Godden, 2000; Sierpinska, Bobos, & Knipping, 2007). Such experiences have resulted in lower self-confidence towards the subject (Lawrence, 1988; Lehmann, 1987). Students then come to expect failure, attributing it to a lack of mathematical ability (Tobias, 1978). Additionally, Buerk concluded that adults perceive mathematics to be a static and rigid subject that has a definitive set of answers which are separate from his or her own thought processes (as cited in Lawrence, 1988, p. 5). Rote memorization, drill and repeated practice divorced from meaning, and standardized testing are just some of the classroom components that innately place emphasis on finding answers, rather than conceptual understanding. FitzSimons and Godden (2000) noted that "for many adult learners mathematics in school was characterized instead by low-level activity and rule-following" (p. 15).

As a result, adult students tend to try to recall previously learned procedures, rules, and algorithms when attempting problems, including those involving fractions. The trouble with this approach is that if the adult learner forgets how and/or when to apply such devices, he or she might not use reasoning to pursue the problem. As Givvin, Stigler, and Thompson (2011) pointed out:

> When our interview questions asked students to solve problems, students would quickly choose a procedure they remembered from school, and then set about applying it to the situation . . . By itself, that approach might not have been

> problematic. The problem lies in the fact that the procedures called upon were often either inappropriate for the situation or were executed with critical errors—errors that would surely have been caught had students understood the concept underlying the procedure or noticed that the magnitude of the answer was not reasonable. Without a conceptual understanding of the procedures in their toolbox, students were left to rely solely on a memory of which to use and how to use it. It appeared that over the years and with an increasing collection of procedures from which to draw, that memory had eroded. (p. 7)

With respect to fractions, this finding is also consistent with prior research on adult developmental learners (Alexander, 2013; Stigler et al., 2010; Muckridge, 2015). Namely, due to a lack of conceptual understanding on fraction tasks, adult learners neglect to use the magnitude of fractions to check their work and identify errors. At the high school level, Brown and Quinn (2006) also found that elementary algebra students overlooked unreasonable answers on problems involving fraction operations due to overreliance on algorithms.

Another error found in literature on children, PSTs, and adult learners is the tendency to add numerators and denominators on fraction addition problems, namely those involving unlike denominators (Hui-Chuan, 2014; Muckridge, 2015; Newton, 2008; Silver, 1983; Vinner et al., 1981). Silver (1983) discovered that while fraction comparison and ordering errors were fairly easy to correct, fraction addition and subtraction errors were very resistant to instruction. He goes on to emphasize:

> The fraction addition error [adding across numerators and denominators] may be so robust because it has considerable conceptual support in the thinking of the subjects who make the error. It is fairly common and reasonable to attribute the error, when it is made by young children, to an incorrect generalization of whole number addition. Such an explanation, however, may not be sufficient to explain the error for the subjects in this study. For an error to persist for as many years as this error did for many of the subjects, despite numerous instructional attempts to eradicate it, the error must have considerable conceptual support in subjects' thinking about rational numbers. (p. 115)

Here, it is important to note a possible key distinction between children's conceptions of fractions and adults' conceptions of fractions. Whereas some previous literature on children's conceptions of fractions suggests that whole number knowledge interferes with fraction knowledge (Behr et al., 1984), research on the fraction knowledge of PSTs and adults (Muckridge, 2015; Newton, 2008; Silver, 1983) suggests that adults' misconceptions stem from prior knowledge of fractions as opposed to prior knowledge of whole numbers. This is evident from the types of errors that children make versus the types of errors that adults make when operating with fractions.

As a result, Silver (1983) acknowledged the need to take a more emergent approach to understanding adult students' conceptions of fractions. He articulates that "in order to effect any substantial improvement in performance it is necessary that the underlying conceptual support system be carefully examined" (Silver, 1983, p. 117). In other words, in order for educators and researchers to effectively assess student understanding, we must begin by gaining insight into the conceptions that adult learners have regarding fractions. Because adult learners have received prior fraction instruction, often at several different times in their educational experience, they, unlike children learning fractions for the first time, have the added difficulty of overcoming prior misunderstandings. This prior instruction also makes it difficult to assess an adult learner's conceptions of fractions. Namely, in order to more deeply understand the learner's cognitive processes, researchers must develop tasks in which the learner cannot apply a procedure in order to arrive at the answer (Busi et al., 2015). Stigler, Givvin, and Thompson (2010) found that by giving students permission to reason, as well as asking them questions that could be answered by reasoning, students were both interested in and willing to think hard about fundamental mathematics concepts including fractions.

Although existing research shows adult learners' reliance on unorganized procedural knowledge, it is also true that adults have many real world experiences that shape their contextualized knowledge (Knowles, 1990; Lawrence, 1988). For example, adults often have a large gap in the time between their formal secondary and post-secondary mathematics education. As a result of this gap in their learning history, adults are often exposed to real-life mathematics and further separated from school mathematics. Therefore adult learners do not see the connection between the mathematics they do in their daily life, perhaps at their job, and how it relates to what is taught in formal education. Many adults perceive certain mathematics as being strictly common sense and not mathematics. Through interviews with adults, some of whom were students, Coben (2000) found that many adults were doing mathematics in their daily lives but did not recognize such knowledge and skills as being mathematical. She referred to this category of mathematics as "invisible mathematics" (p. 55). One woman in Coben's study said that many household jobs, such as hanging shelves, involve performing mathematical measurements and calculations that she considered to be common sense. Perhaps the opposite of invisible mathematics is the idea that adult students tend to ignore what they know to be true in order to solve what they perceive to be "real math" (Kerka, 1995). They abandon common sense and their innate problem-solving abilities instead of applying them to mathematical thinking. Ginsburg and Gal (1995) interviewed adult students to gain insight into their knowledge and use of percentages. They found that adult learners had at least a small amount of informal knowledge of percentages which they could apply to some situations but not others. One explanation for their inability to apply knowledge across various situations was that the students' knowledge of percent was restricted to specific contexts and was not a part of a more comprehensive mathematical structure. This contextualized knowledge further supports the need for future research to take an emergent approach when considering adult learners' conceptions of fractions.

Finally, various types of fraction models play an important role in adult learners' fractional knowledge. Research with both children and adults has investigated how and the degree to which learners use models when working with fractions. Silver (1983) examined both correct and incorrect solution methods among community college and college students on fraction and ratio tasks. He found that most of the participants' fraction thinking was based on a circular model or no model at all. Brown and Quinn

(2006) found that high school algebra students also tended to apply algorithms as opposed to using pictorial models of fractions. Further, many of the students in Silver's (1983) study who correctly compared and added fractions were unable to relate their procedures to any fraction model. A similar finding was discovered among students in the middle grades. Hui-Chuan (2014) found that even when students were able to perform well on fraction operation tasks, they often lacked the ability to recognize pictorial representations of these tasks. In the unlikely event that a student uses some type of fraction model to make sense of a fraction task, research indicates that students do not experience any dissonance when arriving at a certain answer using procedures and a different answer using models. For example, Silver (1983) found that "one characteristic of the subjects who failed to improve is that they had no apparent expectation that the answer one obtained when working with written symbols should be the same as the answer obtained with concrete materials" (p. 114). This finding supports the theorized connection between models of fractions and unorganized procedural knowledge, as it is an example of using models in a procedural sense without relying on the models to obtain an answer.

Conclusion

This literature review highlights research associated with the learning and conceptualizations of fractions. Most of the studies have been conducted with children and preservice teachers, as these are two populations of learners who traditionally encounter fractions in formal education. The growing number of adult developmental mathematics learners, along with limited studies involving these learners, indicates a need for more research to be conducted among this population. Because developmental mathematics students are studying fraction concepts that are first introduced at the elementary and middle school levels, adult developmental mathematics learners' conceptions of fractions can be informed by prior research conducted with K–12 students and PSTs.

Prior research involving children and PSTs reveals several common errors encountered when operating on fractions, including: (a) solution methods that are dependent upon the nature of the denominator rather than the given operation; (b) absent, ambiguous, or inconsistent references to a unit whole; and (c) not treating fractions as numbers. The limited research with adult developmental learners indicates similar errors are made by this population as well. Like research on PSTs' knowledge of fractions, research on adult learners' knowledge of fractions is overwhelmingly limited to studies that analyze the learners' incorrect solution methods, sometimes referred to as error analysis. As Newton (2008) pointed out with respect to PSTs, "more studies should examine knowledge from multiple perspectives, including an examination of *correct* solution methods" (p. 1105). The same is true with respect to the research on adult developmental learners' knowledge of fraction concepts. Studies involving adult learners' cognitive processes of mathematics in general are very limited. As discussed in this chapter, the disembedding operation and its attention to maintaining a fixed unit whole has been emphasized in studies with children and PSTs. Thus the purpose of this study is to examine adult developmental learners' knowledge of fraction addition and subtraction, as well as the role that the unit plays in this knowledge.

The next chapter describes the methodology and procedures for this study. This includes a rationale and description of a mixed methods sequential explanatory research design. Because this design involves collecting both quantitative and qualitative data, the procedures for each of these two phases are discussed. The written assessment used to collect quantitative data draws on the literature discussed in the current chapter, particularly Steffe and Olive's (2010) fraction schemes, the role of the unit via disembedding, and fraction addition and subtraction. Qualitative data were collected to gain a more in-depth understanding of participants' responses on the written assessment. The procedures for analyzing both types of data also are addressed in Chapter 3.

CHAPTER III

METHODOLOGY AND PROCEDURES

The goal of this study was to examine adult developmental mathematics (ADM) students' knowledge of fraction addition and subtraction, focusing on the role of the unit. To accomplish this goal, a mixed methods sequential explanatory approach was taken. Quantitative data were collected using a written assessment containing questions related to whole number disembedding, fraction schemes, and adding and subtracting fractions. Qualitative data were then collected through one-on-one clinical interviews, which were informed by the results of the quantitative data. Because research on ADM students' knowledge of fraction addition and subtraction (especially as it relates to their understanding of the role of the unit) is extremely limited, mixing both quantitative and qualitative methods aims to capture a comprehensive view of this complex area of adult mathematics education. This chapter focuses on the specific methodology used in this study. The procedures for sampling, data collection, and data analysis for each of the two phases of this study are described.

Research Questions

The overarching focus of this study is to examine adult developmental mathematics students' knowledge of fraction addition and subtraction. Specifically, the following research questions are addressed:

1. Does adult developmental mathematics students' performance on whole number disembedding tasks inform how they add/subtract fractions? If so, what is the nature of the relationship?

2. Do fraction schemes demonstrated by adult developmental mathematics students inform how they add/subtract fractions? If so, what is the nature of the relationship?
3. In what ways do adult developmental mathematics students' prior mathematics experiences inform their fraction conceptions, particularly regarding addition/subtraction of fractions?

Research Design

The present study used a mixed methods sequential explanatory design. Mixed methods research is generally defined as an approach to research where the investigator gathers both qualitative and quantitative data, integrates these two types of data, and then draws interpretations based on the combined strengths of each set of data to understand a research problem or problems (Creswell, 2015). This can be done within a single study or through a series of studies that investigate a single underlying phenomenon (Onwuegbuzie & Leech, 2006). A core assumption of mixed methods research is that combining, or mixing, the two types of data will provide a better understanding of the research problem(s) than either form of data alone. As a result of this assumption, integrating the data sets and playing upon the strength(s) that each type of data brings is key.

This study's sequential explanatory design was a two-phase design in which the quantitative data collection and analysis came first, followed by the qualitative data collection and analysis. Because the quantitative phase was completed first, the researcher was able to use the findings from this phase to determine what to explore further in the qualitative phase. Namely, the quantitative phase provided the researcher with insight for the selection of participants and interview questions for the qualitative phase. Furthermore, this design allowed the researcher to use the qualitative results to explain the quantitative findings. Figure 7 provides a visual model for this study's mixed methods sequential explanatory design. The key ideas presented by Ivankova, Creswell, and Stick (2006) were used to guide the creation of this diagram. Specifically, these authors suggested a one-page, three-column diagram that identifies each stage of a mixed methods study, as well as the procedures and products, or outcomes, associated with each stage.

As is depicted in Figure 7, the first phase of this study involved collecting quantitative data from a written assessment containing questions related to whole number disembedding, fraction schemes, and adding and subtracting fractions. In the second phase, qualitative data were collected through one-on-one clinical interviews. The rationale for this approach was that the quantitative data and results would provide a general picture of how ADM students perform on whole number disembedding and fraction tasks, while the qualitative data and its analysis would help refine and explain the statistical results by exploring participants' knowledge in more depth.

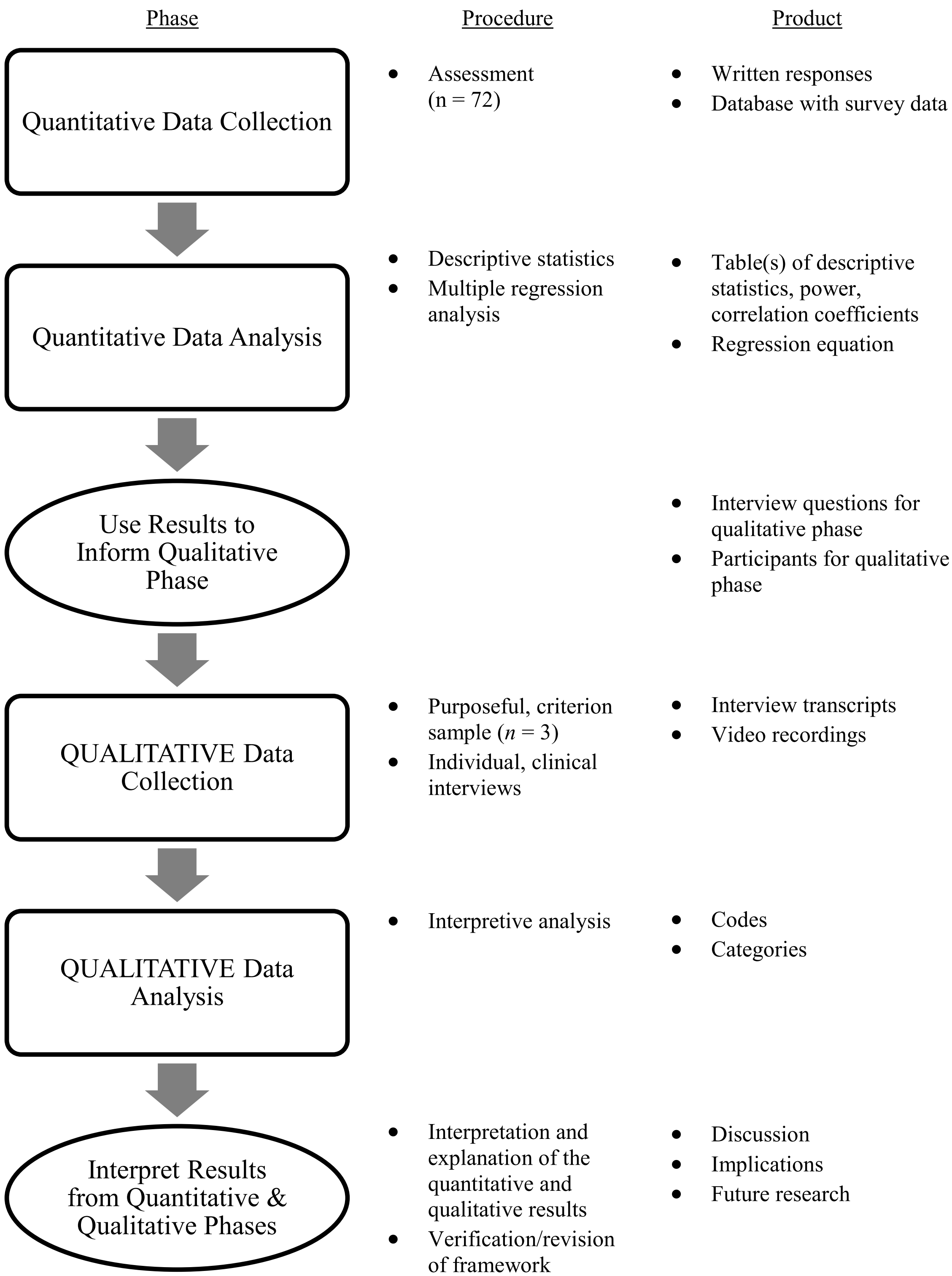


Figure 7. Visual model for mixed methods sequential explanatory design.

Recall from the review of literature in Chapter 2 that prior research calls for a more emergent approach to understanding ADM students' conceptions of fractions, whereby students are given the opportunity to reason and explain their thinking (Silver, 1983; Stigler et al., 2010). Clinical interviews provide a useful method for doing so because their flexible method of questioning explores the richness of students' thoughts and cognitive processes (Ginsburg, 1981). Because adult developmental mathematics learners have diverse backgrounds in both formal and informal mathematical knowledge, quantitative data were extremely helpful in identifying a purposeful sample of students to interview for further insight regarding their conceptions of fraction addition/subtraction. In addition to identifying a purposeful sample for the qualitative phase of the study, the quantitative component addressed the first two relational research questions:

1. Does adult developmental mathematics students' performance on whole number disembedding tasks inform how they add/subtract fractions? If so, what is the nature of the relationship?
2. Do fraction schemes demonstrated by adult developmental mathematics students inform how they add/subtract fractions? If so, what is the nature of the relationship?

Because these questions sought to describe the relationship between multiple variables (demographics, whole number disembedding, demonstrated fractions schemes, and performance on fraction addition/subtraction problems), multiple regression analysis was conducted. The qualitative component was used to explain the results of the quantitative phase of the study in greater detail and also addressed the third research question:

3. In what ways do adult developmental mathematics students' prior mathematics experiences inform their fraction conceptions, particularly regarding addition/subtraction of fractions?

The priority in this design was given to the qualitative method, because the qualitative research represents the major aspect of data collection and analysis in the study, focusing on in-depth explanations of quantitative results. In the diagram in Figure 7, capitalized letters are used to denote the priority given to the qualitative component. A quantitative component goes first in the sequence and was used to reveal students' performance on whole number disembedding, fraction schemes, and fraction addition/subtraction tasks. The quantitative and qualitative data were integrated at multiple points in this study. As opposed to "mixing" the data in the sense that one type of data dissolves into the other, integration refers to connecting the two types of data while keeping each type intact (Creswell, 2015). Integration of data occurred at the beginning of the qualitative phase while selecting the participants for clinical interviews and developing the interview questions based on the results of the statistical tests. Additionally, the results of the two phases were integrated during the discussion of the outcomes of the whole study.

The goals of each phase of this sequential explanatory study aligned with the purposes of development and complementarity in mixed methods research, outlined by Greene, Caracelli, and Graham (1989). Complementarity involves seeking elaboration, illustration, enhancement, and/or clarification of the results from one method with the results from the other method. Development uses the results from one method to help inform the other method in one or more ways. There were three goals for the quantitative phase of this study. The first goal was to determine how adult developmental mathematics students respond to questions pertaining to whole number disembedding, fraction schemes, and adding and subtracting fractions. The second goal was to allow for purposefully selecting informants for the second phase. Lastly, the third goal of the quantitative phase was to develop interview questions for the second phase. Both the second and third goals of the quantitative phase allowed for the development of the second (qualitative) phase of this study. The primary goal for the qualitative phase was to gain deeper insight into the participants' knowledge of disembedding and fraction schemes as it relates to how they solve fraction addition and subtraction problems. Thus, the results from the qualitative component of the study allowed for both elaboration and clarification of the results from the quantitative phase.

Setting

The research site for both phases of this study was a regional campus of a Midwestern public university. The total campus enrollment during the spring 2016 semester was 2,462 students. Of these students, 914 were enrolled in a mathematics course and 531 were enrolled in a developmental mathematics course. Thus, approximately 58.1% of the students enrolled in a mathematics course were enrolled in a developmental mathematics course. This compares similarly to prior large scale national studies reporting that nearly 60% of all incoming students take at least one developmental mathematics course (Attewell et al., 2006; Bailey, Jeong, & Cho, 2010). It also compares similarly to the other regional campuses in this study's university system.

Phase 1: Quantitative

This section of the chapter describes in detail the methods of the quantitative phase of this sequential explanatory study. Information pertaining to the participants is given first, followed by a description of the written assessment and the procedures for implementing it. Also discussed are the measures for this phase of the study. This section concludes with a description of the multiple regression analysis that was conducted using the data collected from the assessment.

Participants

Participants for the first phase of this study consisted of adult mathematics students enrolled in one of five developmental mathematics courses (Course A, Course B, Course C, Course D, or Course E) during either the fall 2016 or spring 2017 semester. All of these courses were considered to be developmental mathematics courses by the university, as their aim was to prepare students with the mathematical knowledge necessary to take college-level mathematics courses. Therefore, they served as prerequisites for college-level courses. Table 4 gives a description of how the fraction addition and subtraction content was approached in each of the five courses, as well as the number of study participants in each course.

Table 4

Descriptions of Courses and Number of Participants in Each Course

	Fraction Addition/Subtraction with Like/Unlike Denominators	**Number of Participants**
Course A	New topic	5
Course B	Review topic	34
Course C	Review topic	29
Course D	Prerequisite topic	3
Course E	Prerequisite topic	1

All of the courses in this study used computer-aided instruction. Specifically, the program "Assessment and LEarning in Knowledge Spaces" (ALEKS) was the primary means for course instruction and assessment. ALEKS is a web-based, artificially-intelligent, software program based on Knowledge Space Theory (www.aleks.com). All ALEKS assessments use adaptive questioning to determine which topics a student knows and doesn't know in a course. Students are then instructed by the software on the topics they are ready to learn. Because of this individual approach, students progress through the topics in each course at their own pace. Additionally, students are placed into one of the five courses based on an ALEKS placement assessment numerical score. Therefore, for example, one student could enter Course B having already mastered fraction addition and subtraction whereas another student entering the same course might need instruction on this topic. This is why fraction addition and subtraction are considered "review topics" in Courses B and C. However, if and when a student takes Courses D or E, fraction addition and subtraction are considered to be prerequisite topics since students are given no ALEKS instruction on these topics and are expected to apply them to new topics of instruction. Note that students would not typically take all five of these developmental mathematics courses. Rather, a student would stop at the point in the sequence when the prerequisite for his or her college-level mathematics course has been met.

Sample and Procedures

An IRB was issued from the research site's institute in September 2016, allowing the researcher to begin data collection in October 2016. The researcher asked instructors teaching one or more sections of any of the five developmental mathematics courses for permission to recruit students to participate in the quantitative phase of the study. Once the researcher obtained consent from willing participants, the researcher administered the written assessment to the participants during their regularly scheduled class time. Data were collected from two classes during the fall 2016 semester and from nine classes during the spring 2017 semester. Because it was not necessary for the participants to have received formal instruction pertaining to fraction addition and subtraction, the assessment was administered near the beginning of the term. Students were informed that their participation was strictly voluntary and that they could withdraw from the study at any time without consequence. Additionally, they were informed that their choice to participate, along with their choice to withdraw, would not have any direct impact on their course grade. Names and identities of study participants have been protected using pseudonyms.

Instrument

The written assessment contained 16 problems related to whole number disembedding, fraction schemes, and adding and subtracting fractions (see Table 5 and Appendix A). The first page of the assessment contained five demographic questions pertaining to the participant's age, gender, major, campus, current course, number of years since last math course, and number of repeated courses. Items #6–12 consisted of whole number disembedding tasks adapted from Kosko and Singh (in press) that required participants to disembed either ones or non-ones from a composite to find another whole. Students were given two rectangular rods, one shaded with a given length and one unshaded with an unknown length, and were asked to find the length of the unshaded rod based on the given length of the shaded rod. Items #13–17 assessed the participants' ability to construct each of the fraction schemes described in Chapter 2, namely the part-whole fraction scheme, the partitive unit fraction scheme, the partitive fraction scheme, the reversible partitive fraction scheme, and the iterative fraction scheme. These items were taken from Norton and Wilkins (2009), who confirmed the validity of the item types in measuring the underlying schemes. Lastly, items #18–21 assessed the participants' knowledge of fraction addition and subtraction with like and unlike denominators. These problems included both symbolic and non-symbolic tasks. The primary reason for including both symbolic and non-symbolic tasks was due to findings in prior research indicating that students might respond differently on fraction addition and subtraction tasks presented using only symbols as compared to presented as word problems (Muckridge, 2015). The written assessment was first administered to a group of nine students not contained in the study's sample. The purpose was to ensure readability of the items and estimate the time it might take participants to complete the assessment.

Table 5

Quantitative Assessment Item Descriptions

Items	Description	Source
7, 9, 12	Disembedding 1s from a composite to find another whole	Kosko & Singh, in press
6, 8, 10, 11	Disembedding non-1s from a composite to find another whole	
13	Parts-within-wholes OR Part-whole fraction scheme	Norton & Wilkins, 2009*
14	Partitive unit fraction scheme	Norton & Wilkins, 2009*
15	Partitive fraction scheme	Norton & Wilkins, 2009*
16	Reversible partitive fraction scheme	Norton & Wilkins, 2009
17	Iterative fraction scheme	Norton & Wilkins, 2009
18a, 18b, 18c, 18d	Fractions with like denominators – symbolic	original
18e, 18f, 18g, 18h, 18i, 18j	Fractions with unlike denominators – symbolic	original
19	Fractions with like denominators – non-symbolic	Illustrative Mathematics (4.NF)*
20, 21	Fractions with unlike denominators – non-symbolic	Illustrative Mathematics (5.NF)*

Note. *Similar to items used in test or research

Measures

The written assessment was designed to measure several variables, which is discussed in the paragraphs that follow. The dependent and independent variables, along with the covariates for the study, are identified in this section. For each variable, there is an explanation as to which items were used to measure the variable, along with a description of how each item was scored.

Dependent variable. The dependent variable for all three research questions was performance on fraction addition/subtraction tasks (*AddSubScore*). This variable was measured using scores on items #18–21 of the assessment. Item #18 contained 10 parts, labeled (a) through (j), and each part contained a symbolic fraction addition or subtraction task. Items #19–21 contained a single non-symbolic (word problem) fraction addition or subtraction task. For each task, correct solutions were assigned 1 point and incorrect solutions were assigned 0 points. Thus, there were 13 possible points for items #18–21, and the variable *AddSubScore* represents the total number of points out of 13 possible.

Independent variables. The independent variable for the first research question was performance on disembedding tasks. This variable was measured using scores on items #6–12 of the assessment. All of these items required a numerical response. Correct solutions were assigned 1 point and incorrect solutions were assigned 0 points. The variable *DisembedScore* represents the total number of points out of 7 possible.

The independent variables for the second research question were demonstrated fraction schemes. These variables were measured using items #13–17 on the assessment.

Each item was used as an indicator of the existence of a particular fraction scheme (see Table 5). Thus, four dummy coded variables (*dPUFS, dPFS, dRPFS, dIFS*) were used with the part-whole fraction scheme as the reference group. Responses to each of the items #13–17 were scored using a 0 or a 1, based on whether or not there was an indication that the participant had operated in a way that was compatible with the particular fraction scheme (Norton & Wilkins, 2009). Thus, a "0" score might include an incorrect response and/or markings that were incompatible with actions that fit the scheme. A "1" score might include a correct response and partitions of the given figure. Items #14 and #15 required a numerical response, whereas items #13, #16, and #17 required a drawing. In order to determine whether or not a participant's drawing demonstrated the participant's construction of a given fraction scheme, the researcher used guidelines established by Norton and Wilkins (2009) and Wilkins, Norton, and Boyce (2013). The variable *FSScore* represents the total number of fraction schemes attained out of five possible.

The independent variables for the third research question were number of years (*NumYears*) since last math course and number of repeated courses (*NumRepeated*). Item #4 on the assessment asked students, "When did you take this [most recent successfully completed] course (mm/yyyy)?" The variable *NumYears* was created by calculating the number of years and months between the participant's current course and his or her response to item #4. Then the number of years and months was converted to a decimal which represented the number of years, rounded to the nearest hundredth, since the participant's last successfully completed math course. Items #2 and #5 asked students the number of times they had attempted their current course and their last successfully completed math course, respectively. Participants' responses to these two items were summed to create the variable *NumRepeated*.

Covariates. The present study focuses on how disembedding, demonstrated fraction schemes, and prior mathematics experiences affect adult developmental mathematics students' performance on fraction addition/subtraction tasks. However, other factors have been found in prior research to affect mathematics performance. Though some of the results are conflicting, age, gender, and current course have each been associated with differences in mathematics performance. Walker and Plata (2000) found a significant relationship between age and pass-fail frequencies among students in developmental mathematics courses. Namely, older students were more successful than younger students in lower-level developmental mathematics courses. On the other hand, Elliott (1990) found mathematics achievement of nontraditional students (age 25 years or older) in a college algebra class to be as high as that of traditional students (less than 25 years old) in the class. Thus the continuous variable *Age* was also included as a covariate in the present study.

A meta-analysis conducted by Frost, Hyde, and Fennema (1994) revealed gender differences in mathematics performance to be small, at most, in the elementary and middle school years but moderate in the high school and college years. Furthermore, Walker and Plata (2000) also found there to be a significant relationship between gender and pass-fail frequencies of students in developmental mathematics classes. Therefore, a dummy coded variable (*dFemale*) was used for gender so that the effect would associate with being female. Thus, a value of 0 was assigned to "male" responses and a value of 1 was assigned to "female" responses.

When analyzing the results of a fraction assessment given to developmental algebra students, Alexander (2013) found scores to be significantly different based on course level. Students in two higher-level developmental mathematics courses scored significantly higher than students in two lower-level developmental mathematics courses. Walker and Plata (2000) also found significant relationships between student performance and current course. To represent students' current course in this study, four dummy coded variables (*dCourseB, dCourseC, dCourseD, dCourseE*) were used with Course A as the reference group. Thus, a score of 1 indicated that the student was enrolled in a particular course whereas a score of 0 indicated that he or she was not enrolled in a course.

Instrument Validity and Reliability

The reliability and internal consistency of the assessment was evaluated using classical item analysis, which analyzes the participants' responses to individual assessment items (Crocker & Algina, 2006). The analysis included the computation of Cronbach's alpha for item-total correlations, item difficulty, and item discrimination. Computing Cronbach's alpha for item-total correlations indicates whether or not a correct response to an item has a correlation with the overall score on the assessment. According to Nunnally and Bernstein (1994), items with high item-total correlations "have more variance relating to what the items have in common and add more to the test's reliability than items with low *r* values" (p. 305). An item-total correlation of 0.30 or higher indicates a meaningful correlation between an item and the overall test score (Nunnally & Bernstein, 1994). Item difficulty represents the proportion, as a percentage, of participants who answered a given item correctly. Higher percentages indicate that more participants answered a given item correctly, implying that the item is less difficult. Conversely, a lower percentage implies that an item is more difficult. Lastly, item discrimination determines the correlation between the average item score and the average total score. This allows the researcher to determine how well each assessment item discriminates, or differentiates, between lower and higher scoring participants. Because each item was dichotomously scored and the total score was continuous, a point biserial correlation was used to determine the item discrimination correlations for the fraction addition/subtraction items and the disembedding items. An item discrimination correlation of at least 0.30 is considered satisfactory, whereas 0.40 or higher is considered to have very good discriminatory power (Crocker & Algina, 2006). These item statistics are reported in the sections and tables that follow.

Fraction scheme items. A Cronbach's alpha coefficient of 0.642 was calculated for the overall response score for the five fraction scheme items (M = 3.08, SD = 1.319, *Range* = 0–5). Item-total correlations were calculated for each of the fraction scheme items and can be found in Table 6. All items had sufficient item-total correlations of 0.30 or higher, with the exception of *dPFS*. This item had an item-total correlation of 0.196. Additionally, if this item were deleted, the Cronbach's alpha would increase from 0.642 to 0.676. Item difficulty was also calculated for each of the fraction scheme items and can be found in Table 6. Because of the hierarchy found among the fraction schemes

(Norton & Wilkins, 2009), it was expected that the difficulty of the schemes would increase from part-whole to PUFS and from PUFS to PFS, and so forth. However, as can be seen in Table 6, the item difficulty increases from PUFS to PFS but then decreases from PFS to RPFS. Lastly, the discriminatory power of each of the fraction scheme items was computed and is reported in Table 6. The part-whole item had satisfactory discriminatory power, whereas the PUFS, RPFS, and the IFS items had very good discriminatory power. The PFS item had a discrimination correlation below 0.30, indicating that this item does not do a satisfactory job at differentiating between lower and higher scoring participants.

Table 6

Item Reliability Statistics for Fraction Scheme Items (n = 72)

Item	**No. Correct Answers**	**Item-Total Correlation**	**% Correct**	**Discrimination Correlation**
Q13 (Part-whole)	67	0.404	93.1%	0.371**
Q14 (PUFS)	61	0.398	84.7%	0.463**
Q15 (PFS)	14	0.196	19.4%	0.260*
Q16 (RPFS)	45	0.572	62.5%	0.571**
Q17 (IFS)	35	0.477	48.6%	0.522**

Note. **Correlation is significant at the 0.01 level (2-tailed).
*Correlation is significant at the 0.05 level (2-tailed).

Disembedding items. A Cronbach's alpha coefficient of 0.803 was calculated for the overall response score for the seven disembedding items (*M* = 5.21, *SD* = 2.076, *Range* = 0–7). This reliability estimate indicates a high level of internal consistency for the disembedding portion of the assessment (Wiersma & Jurs, 2009). Item-total correlations were calculated for each of the disembedding items and can be found in Table 7. All items had sufficient item-total correlations of 0.30 or higher, indicating that a correct response on a disembedding item had a meaningful correlation with a higher overall score and vice versa. Table 7 also reports the item difficulty for each of the disembedding items. Lastly, the discrimination correlations for each of the disembedding items can be found in Table 7 as well. Because all of the disembedding items had a correlation of 0.40 or higher, it can be concluded that these questions had very good discriminatory power and differentiate between lower and higher scoring participants.

Fraction addition/subtraction items. The fraction addition/subtraction items also had a high level of internal consistency, as determined by a Cronbach's alpha of 0.896 (*M* = 9.57, *SD* = 3.745, Range = 0–13). Item-total correlations were calculated for each of the fraction addition/subtraction items and can be found in Table 8. All items had sufficient item-total correlations of 0.40 or higher, indicating that a correct response on a fraction addition/subtraction item had a meaningful correlation with a higher overall score and vice versa. Item difficulty was also calculated for each of the fraction addition/subtraction items and can be found in Table 8. Items 18j and 18d had the lowest item difficulty percentages, indicating that only 52.8% (*n* = 38) and 61.1% (*n* = 44) of the participants answered these items correctly, respectively. Both items were mixed number

Table 7

Item Reliability Statistics for Disembedding Items (n = 72)

Item	No. Correct Answers	Item-Total Correlation	% Correct	Discrimination Correlation
Q6	52	0.513	72.2%	0.478**
Q7	57	0.490	79.2%	0.407**
Q8	55	0.641	76.4%	0.461**
Q9	57	0.707	79.2%	0.513**
Q10	50	0.418	69.4%	0.557**
Q11	54	0.501	75.0%	0.473**
Q12	50	0.511	69.4%	0.567**

Note. **Correlation is significant at the 0.01 level (2-tailed).

subtraction problems in which the participants had to borrow from the whole part and regroup to the fraction part. The mixed numbers in item 18j had unlike denominators, whereas the mixed numbers in 18d had like denominators. Lastly, the discriminatory power of each of the fraction addition/subtraction items was computed and is reported in Table 8. All of the fraction addition/subtraction items had significant discrimination correlations of 0.40 or higher, indicating that these questions differentiate between lower and higher scoring participants.

Table 8

Item Reliability Statistics for Fraction Addition/Subtraction Items (n = 72)

Item	No. Correct Answers	Item-Total Correlation	% Correct	Discrimination Correlation
Q18a	60	0.620	83.3%	0.658**
Q18b	64	0.452	88.9%	0.403**
Q18c	61	0.526	84.7%	0.543**
Q18d	44	0.485	61.1%	0.532**
Q18e	56	0.648	77.8%	0.691**
Q18f	56	0.680	77.8%	0.656**
Q18g	58	0.792	80.6%	0.800**
Q18h	49	0.701	68.1%	0.642**
Q18i	45	0.593	62.5%	0.551**
Q18j	38	0.592	52.8%	0.635**
Q19	55	0.604	76.4%	0.596**
Q20	54	0.664	75.0%	0.700**
Q21	49	0.448	68.1%	0.523**

Note. **Correlation is significant at the 0.01 level (2-tailed).

Data Analysis

Multiple regression analysis was conducted to examine relationships between the dependent variable (performance on fraction addition/subtraction tasks) and various independent variables, including performance on disembedding tasks, demonstrated fraction schemes, age, gender, number of years since last math course, and number of repeated courses. An important condition when conducting a multiple regression analysis is that the predictor, or independent, variables be correlated with the dependent variable but not highly correlated with other predictor variables (Gliner, Morgan, & Leech, 2009). When independent variables are highly correlated with each other, it is referred to as multicollinearity and can make it more difficult to reliably estimate the coefficients of the collinear variables (Allison, 1999). Prior research (Hackenberg & Tillema, 2009; Steffe, 2002) indicates a strong relationship between fraction schemes and units coordination (which includes disembedding). Thus, bivariate correlational analysis was used to determine if relationships were present between the variables. Because some moderate correlations were found, collinearity statistics were also run before conducting the multiple regression analysis. Findings from these tests, along with the testing of additional assumptions for multiple regression, are discussed in the next chapter.

Recall the first research question examined the relationship between performance on fraction addition/subtraction tasks and performance on disembedding tasks. The second research question examined the relationship between performance on fraction addition/subtraction tasks and demonstrated fraction schemes. Lastly, the third research question examined the relationship between performance on fraction addition/subtraction tasks and prior mathematics experiences. After bivariate correlations were run and the assumptions of multiple regression were found to be satisfied, one regression equation was used to model the relationship between the dependent variable (*AddSubScore*) and six independent variables (*Age, dFemale, Disembedscore, FSScore, AddSubScore, NumYears,* and *NumRepeated*). This model is as follows:

$$AddSubScore = B_0 + B_1 \cdot (Age) + B_2 \cdot (dFemale) + B_3 \cdot (FSScore) + B_4 \cdot (DisembedScore) + B_5 \cdot (NumYears) + B_6 \cdot (NumRepeated) + \varepsilon.$$

The term B_0 is the *AddSubScore* intercept, which represents the value of the variable *AddSubScore* when all other independent variables are equal to 0. Each coefficient B_k, for $k = 1, 2, \ldots 6$, of an independent variable represents the change in *AddSubScore* relative to a unit change in the respective independent variable when all other independent variables are constant. Additionally, ε is the error of prediction of the model. Finally, the dummy coded variables representing the participant's current course were excluded from the regression model because of the small number of participants in some of the courses (see Table 4). These small numbers did not allow for an accurate measure of the variance due to each course. Since it was possible that these variables would mistakenly account (or not account) for variance, they were excluded from the regression model.

Phase 2: Qualitative

This section of the chapter describes in detail the methods pertaining to the qualitative phase of this study. Information concerning the participants is given first, followed by a description of the questions and protocol used for the one-on-one clinical interviews. The section concludes with an explanation of the methods used to analyze the interview data.

Participants

For the second phase of this study, a purposeful sample of three developmental mathematics students was drawn from those who participated in the first phase. Note that the qualitative sample was a subset of the quantitative sample for several reasons. First, one intent of the design of this study was to use the qualitative data to explain the quantitative results. Thus it was important that the qualitative sample contain the same participants as the quantitative sample to increase the validity of the qualitative results. Additionally, the results of the quantitative phase were used in the development of questions for the qualitative interviews. Therefore, participants from the qualitative sample needed to be capable of answering the qualitative questions as they pertained to the quantitative tasks.

The intent of purposeful sampling is to select participants who can best help the investigator to understand the central phenomenon of the research study (Creswell, 2015). In addition to noting that the qualitative sample was a subset of the quantitative sample, it is also important to describe the strategy used for obtaining a purposeful sample in this study. Several purposeful sampling strategies exist, such as sampling to achieve representativeness or comparability, sampling special or unique cases, and sequential sampling (Teddlie & Yu, 2007). This phase of the study used criterion sampling, a type of sampling unique cases, whereby the researcher selects participants who meet a special criterion. Specifically, the participants were selected based on extreme differences in scores on each of the types of tasks. Participants who scored high on fraction addition/subtraction tasks and low on disembedding tasks were of interest. Also of interest were participants who scored high on fraction addition/subtraction tasks and low on demonstrated fraction schemes. Because prior research indicates adult mathematics learners draw on contextualized knowledge and/or unorganized procedural knowledge, following up with such participants would allow the researcher to ask clarifying questions and note actions or steps taken that were not indicated on the written assessment.

Interview Design and Protocol

During the second phase of data collection for this study, one-on-one clinical interviews were conducted with the purposeful sample of participants regarding their responses on the written assessment. The aim of the clinical interviews was to identify and describe the cognitive processes underlying the participants' performance on the assessment tasks (Ginsburg, 1981). Because the clinical interviews in this study were aimed at identification of the participants' cognitive processes, the clinical interviews were designed: (a) to employ tasks which channeled the participants' activity into disembedding, fraction schemes, and fraction addition/subtraction; (b) to demand reflection; and (c) with the interviewer's questions contingent upon the participant's response (Ginsburg, 1981, p. 7).

Each participant in the sample was interviewed once during the qualitative phase. The interviews lasted approximately 30–40 minutes each. All meetings were audio and video recorded. The participants' task during the clinical interview was to explain their solution to selected problems on the written assessment. Because the researcher's questions were contingent upon the participant's response, the interview protocol was flexible. The researcher asked clarifying questions until the participant was unable to provide any new insight into his or her solution method (see Appendix B). In addition to asking clarifying questions during the interview, the researcher also presented participants with some new tasks based upon the results of the quantitative assessment (see Appendix C). For example, as reported in the item analysis results, item 15 (measuring the indication of the PFS) had a low item-total correlation, low item difficulty percentage, and low discriminatory power. Therefore, a similar task using different numbers but also measuring for evidence of the PFS was presented to interviewees. The purpose of doing so was to provoke participant responses that might indicate evidence of the partitive fraction scheme. Notes were taken by the researcher to document participants' actions, and participants' drawings were collected as an additional artifact to support video analysis.

Data Analysis

Qualitative data analysis was guided by the procedures established by Creswell and Plano Clark (2011). Audio recordings were transcribed verbatim, and these transcriptions were also checked against the video's audio to ensure accuracy. The participants were given the option to review the transcripts and make corrections prior to the data being analyzed. Once the transcripts were finalized, the researcher read through each one and wrote short memos in the margins to capture her initial thoughts. After the initial memos were written, the researcher then watched the video interviews and took notes on the transcript pertaining to the actions made by each participant as well as by the researcher. Such actions include drawing or making marks on the problems, gestures, and so forth. These actions are extremely important to note, as they provide clarity for interpreting and understanding the dialogue in the transcript. Many times the participant would point to a certain part of the problem while only referring to it orally as, for example, “this one” or “those.” By watching the video, the researcher was able to have a much deeper understanding of the participants’ thought processes that could not have been captured through audio alone.

Because of the generative purpose of this phase of the study, interpretive analysis was used to analyze the qualitative interview data and clarify the results of the quantitative phase. *Interpretive analysis* involves a researcher openly interpreting large episodes of interview data without observation categories being fixed ahead of time. According to Clement (2000), generative purposes “can deal with behaviors that are quite unfamiliar, for which there is very little in the way of existing theory” (p. 557). As discussed in Chapter 2, little theory exists surrounding adult mathematics learners’ cognitive processes. Thus, interpretive analysis allowed the researcher to revise the framework for adult learners’ knowledge of fractions theorized in Chapter 2 as well as to explain as much of the interview data as possible.

Recall from Chapter 2 that the theorized framework for adult learners’ knowledge of fractions consists of three aspects: contextualized knowledge, unorganized procedural knowledge, and models of fractions. These three aspects guided the qualitative data analysis. Contextualized knowledge is grounded in real-life experiences of the adult and may or may not be recognized as mathematical. Some examples of contextualized fraction knowledge would be using fractions to measure ingredients while cooking or to measure lengths of materials during building. Unorganized procedural knowledge consists of rules, procedures, prior formal learning void of meaning, and perceptions of mathematics. Models of fractions include visual representations such as fraction bars, fraction circles, and number lines. In addition to coding the interview transcripts using these three aspects as categories and their corresponding examples as codes, new categories and codes were generated by the researcher's interpretation of large excerpts of interview data.

Conclusion

In order to understand adult developmental mathematics students' knowledge of fraction addition/subtraction, a mixed methods sequential explanatory study was designed. The first, quantitative phase consisted of creating and implementing an assessment measuring participants' performance on fraction addition/subtraction tasks, demonstrated fraction schemes, and performance on disembedding tasks. The second, qualitative phase involved conducting one-on-one clinical interviews with a purposeful sample identified based on the results of the first phase. This chapter described the procedures for collecting and analyzing data for each phase of the study. The details of both the quantitative and qualitative analyses are described in Chapter 4.

CHAPTER IV

ANALYSIS OF THE FINDINGS

The purpose of this study was to examine adult developmental learners' knowledge of fraction addition and subtraction, as well as the role that the unit plays in this knowledge. In order to accomplish this purpose, quantitative data were collected via a written fraction and whole number assessment, and qualitative data were collected through one-on-one clinical interviews. This chapter presents findings for the following research questions:

1. Does adult developmental mathematics students' performance on whole number disembedding tasks inform how they add/subtract fractions? If so, what is the nature of the relationship?
2. Do fraction schemes demonstrated by adult developmental mathematics students inform how they add/subtract fractions? If so, what is the nature of the relationship?
3. In what ways do adult developmental mathematics students' prior mathematics experiences inform their fraction conceptions, particularly regarding addition/subtraction of fractions?

The discussion of this study's findings first includes the results of the quantitative analysis to be followed by the results of the qualitative analysis. The quantitative findings report correlations found between the variables, as well as how these correlations informed the multiple regression model. The regression equation with its coefficients is presented in this section as well. The qualitative findings section of this chapter includes a summary of the themes found for each of the interview participants. Also included is a cross-case analysis comparing and contrasting these themes across all three participants. Lastly, the findings from the quantitative and qualitative phases are integrated at the end of this chapter.

Phase 1: Quantitative Findings

This section of the chapter discusses the quantitative findings of the study. It begins by presenting the descriptive statistics pertaining to the 72 participants who completed the written assessment. Next, the findings from the correlational analyses of the assessment data are reported. This section of the paper concludes with a description of the multiple regression analysis assumptions and results, including the regression equation.

Descriptive Statistics

The participants ranged in age from 18 to 64 years, with a mean age of 26 (SD = 9.7), median of 23, and mode of 19. Gender distribution was 65% females (n = 47) and 35% males (n = 25). Participants were primarily enrolled in Courses B and C, with the overall distribution as follows: Course A (n = 5, 6.9%), Course B (n = 34, 47.2%), Course C (n = 29, 40.3%), Course D (n = 3, 4.2%), and Course E (n = 1, 1.4%). Descriptive statistics for the independent and dependent variables were also calculated and are presented in Table 9.

Table 9

Descriptive Statistics for the Independent and Dependent Variables

Variable	*n*	M	Median	SD	Range
DisembedScore	72	5.21	6.00	2.076	0 – 7
FSScore	72	3.08	3.00	1.32	0 – 5
AddSubScore	72	9.57	11.00	3.75	0 – 13
NumYears	64	3.65	0.33	7.50	0.083 – 33
NumRepeated	72	0.75	0.00	1.16	0 – 6

Preliminary Examination of Variables

Bivariate correlational analysis was used to determine if relationships were present between the variables as well as to check for possible multicollinearity before conducting the multiple regression analysis. As discussed in Chapter 3, it is important that multicollinearity, or strong linear relationships among the independent variables, not be present when conducting a multiple regression analysis in order to allow for a reliable estimate of the coefficients. The Pearson *r* product moment correlation coefficient was calculated pairwise for the following variables: *Age*, *dFemale*, *Disembedscore*, *FSScore*, *AddSubScore*, *NumYears*, and *NumRepeated*. Tables 10, 11, and 12 present these correlation results. As can be seen in Table 10, neither of the covariates (age and gender) were significantly correlated with the dependent variable (*AddSubScore*).

Table 10

Correlations Between Covariates and Dependent Variable

	Age	dFemale	AddSubScore
Age	1	—	—
dFemale	-0.074	1	—
AddSubScore	0.190	-0.116	1

Meaningful and statistically significant correlations were found between *DisembedScore, FSScore,* and *AddSubScore*. As is reported in Table 11, *AddSubScore* was positively correlated with both *DisembedScore* ($r = 0.441$, $p < 0.001$) and *FSScore* ($r = 0.495$, $p < 0.001$), indicating that those with higher scores on the disembedding and fraction scheme items tended to have higher scores on the fraction addition and subtraction items. There was also a moderate correlation between two of the independent variables, *DisembedScore* and *FSScore* ($r = 0.405$, $p < 0.001$). This suggests that, in general, participants who scored high on the disembedding items also scored high on the fraction scheme items, and vice versa. This correlation was anticipated based on prior research, which had indicated a strong relationship between fraction schemes and units coordination (Hackenberg & Tillema, 2009; Steffe, 2002).

Table 11

Correlations Between Independent and Dependent Variables

	DisembedScore	FSScore	AddSubScore
DisembedScore	1	—	—
FSScore	0.405**	1	—
AddSubScore	0.441**	0.495**	1

Note. **Correlation is significant at the 0.01 level (2-tailed).

As can be seen in Table 12, there was a small negative correlation between *NumYears* and *NumRepeated* (r = -0.265, p = 0.034), which indicates that the more years a student has been out of school, the less mathematics courses they are likely to have repeated. Table 12 also displays a moderate positive correlation between Age and *NumYears* (r = 0.383, p = 0.002). This, not surprisingly, suggests that the higher a participant's age, the more years it has been since his or her previous mathematics course was taken. This is not a very meaningful finding in and of itself but this statistically significant and moderate correlation indicates a need to check for multicollinearity in the regression model.

Table 12

Correlations Between Covariates, Independent, and Dependent Variables

	Age	dFemale	NumYears	NumRepeated	AddSubScore
Age	1	—	—	—	—
dFemale	-0.074	1	—	—	—
NumYears	0.383**	0.004	1	—	—
NumRepeated	0.047	0.044	-0.265*	1	—
AddSubScore	0.190	-0.116	0.077	-0.152	1

Note. **Correlation is significant at the 0.01 level (2-tailed).
*Correlation is significant at the 0.05 level (2-tailed).

As discussed above, some of the independent variables were found to be significantly correlated. Therefore, two additional tests were run to check for multicollinearity. Table 13 displays the tolerance levels and the Variance Inflation Factors (VIF) for these variables. The tolerance is found by taking 1 minus the R^2 for each of the independent variables. Thus, it is important to look for low tolerances as a possible indication of multicollinearity. Moreover, the VIF is simply the reciprocal of the tolerance, and its square root shows “how much larger the standard error is, compared with what it would be if that variable were uncorrelated with the other [independent] variables in the equation” (Allison, 1999, p. 142). As displayed in Table 13, the tolerance levels for the independent variables in this study are not below 0.1, and their VIF scores are beneath 2.5. Thus, it was concluded that multicollinearity was not a concern in this model (Allison, 1999).

Table 13

Regression Model Collinearity Statistics

	Tolerance	VIF
Age	0.731	1.368
dFemale	0.920	1.087
FSScore	0.762	1.312
DisembedScore	0.856	1.168
NumYears	0.715	1.398
NumRepeated	0.895	1.118

Multiple Regression

Multiple linear regression analysis was used to develop a model for predicting participants' fraction addition/subtraction scores from their age, gender, disembedding score, fraction scheme score, number of mathematics courses repeated, and number of years since their previous mathematics course. Prior to conducting the multiple regression, however, it was necessary to verify that the data satisfied certain assumptions. These assumptions, along with results from the post hoc power analysis, are presented in the paragraphs that follow. Also presented are the results from the multiple regression analysis.

Statistical assumptions. The assumption of linearity was checked using scatterplots. Partial regression plots indicated linear relationships between the dependent variable (*AddSubScore*) and each of the independent, non-categorical variables (*Age*, *NumYears*, *NumRepeated*, *DisembedScore*, and *FSScore*). In addition, plotting the studentized residuals against the predicted values indicated that the independent variables were (collectively) linearly related to the dependent variable. Visual inspection of the aforementioned plot also verified the assumption of homoscedasticity. The assumption of normality of the standardized residuals was met, as assessed by a P-P Plot. Finally, there was independence of residuals, as assessed by a Durbin-Watson statistic of 1.595.

A post hoc power analysis was conducted using the G*power software package (Faul, Erdfelder, Buchner, & Lang, 2009). Statistical power is defined as $1 - \beta$, where β is the probability of making a Type II error (Cohen, 1992). In other words, the power of a statistical test is the likelihood that the test will detect an effect when there is in fact one to detect. The sample size of 72, along with a six predictor variable equation, was used for this study's power analysis. The alpha level used for the analysis was 0.05. The post hoc power analysis revealed that the statistical power for this study was 0.99 for a large effect size of $f^2 = 0.845$ (Cohen, 1992). Effect size, or the degree to which the null hypothesis is false, assesses how strong the relationship is between the variables (J. Cohen, 1988)

Multiple regression model. Multiple linear regression analysis was used to develop a model for predicting participants' fraction addition/subtraction scores from their age, gender, disembedding score, fraction scheme score, number of mathematics courses repeated, and number of years since their previous mathematics course. The overall model was significant, $F(6, 56) = 7.886$, $p < 0.001$, and accounted for 45.8% of the variance in the dependent variable (*AddSubScore*). The results indicated that age, fraction schemes score, disembedding score, and number of repeated mathematics courses were significant predictors of the fraction addition and subtraction score (see Table 14). Gender and number of years since last mathematics course were not significant predictors of the fraction addition and subtraction score (B_2 = 0.868, p = 0.290 and B_5 = -0.094, p = 0.157, respectively). The constant term, B_0 = -0.849, represents the value of a participant's fraction addition/subtraction score when all other independent variables are equal to 0. This value is not very meaningful in the context of this model, as it would represent the fraction addition/subtraction score of a male with an age of 0, having 0 years since his last mathematics course, 0 repeated mathematics courses, and a score of 0 on all assessment items. Positive regression weights were calculated for *FSScore* (B_3 = 1.388, p < 0.001) and *DisembedScore* (B_4 = 0.528, p = 0.008) and can be seen in Table 14. This finding indicates that participants with higher scores on these items were expected to have higher scores on the fraction addition/subtraction items, after controlling for the other variables in the model. For example, the regression coefficient B_4 = 0.528 shows that an increase of two points in the disembedding score is associated with an increase of one point in the fraction addition/subtraction score, when all other independent variables are held constant. *Age* also had a positive regression weight (B_1 = 0.146, p = 0.001), indicating that older students were expected to have higher fraction addition/subtraction scores. The number of repeated mathematics courses had a statistically significant negative effect (B_6 = -0.769, p = 0.019), indicating that after accounting for all other variables, the higher the number of repeated mathematics courses, the lower a participant's fraction addition/subtraction score. Thus the regression equation for prediction scores on fraction addition/subtraction items was:

$$AddSubScore = B_0 + B_1 \cdot (Age) + B_2 \cdot (dFemale) + B_3 \cdot (FSScore) + B_4 \cdot (DisembedScore) + B_5 \cdot (NumYears) + B_6 \cdot (NumRepeated) + \varepsilon.$$

Table 14

Regression Model With Coefficients and Significance Tests

	Unstandardized B	Coefficients Std. Error	Standardized β	t	Sig.
(Constant)	-0.849	1.925		-0.441	0.661
Age	0.146	0.043	0.389	3.380	0.001
dFemale	0.868	0.813	0.110	1.069	0.290
FSScore	1.388	0.317	0.494	4.386	0.000
DisembedScore	0.528	0.192	0.292	2.749	0.008
NumYears	-0.094	0.066	-0.167	-1.434	0.157
NumRepeated	-0.769	0.319	-0.251	-2.411	0.019

Phase 2: Qualitative Findings

This section of the chapter discusses the qualitative findings of this study. The section begins with a description of the sampling procedures used to select participants based on the results of the first phase. Following this description, the findings from the one-on-one clinical interviews are presented. The themes that emerged for each participant are presenting in cases, concluding with results from the cross case analysis.

Selection of Interview Participants

As discussed in Chapter 3, participants for the qualitative phase of this study were selected based on extreme differences in scores on each of the types of assessment items presented during the quantitative phase. These item types include disembedding, fraction schemes, and fraction addition and subtraction. Once the assessments were scored and entered into Excel, scores on each item type (*DisembedScore*, *FSScore*, *and AddSubScore*) were custom sorted so the researcher could identify participants who scored high on one item type and low on one or both of the other two item types. Eighteen participants were identified as having extreme differences in their individual item type scores. Also identified were five participants who scored high on all three item types. These 23 individuals were then recruited via email to take part in the qualitative phase. Of these 23, three students agreed to complete a follow-up interview. The scores of each interview participant on each of the three item types can be found in Table 15. As can be seen in the table, Anna scored high on all three item types, missing only one of the fraction addition/subtraction items. Chris scored high on the disembedding and fraction addition/subtraction items but low on the fraction scheme items. Lastly, Steve's disembedding and fraction addition/subtraction scores were low while his fraction scheme score was high.

Table 15

Item Type Scores of Interview Participants

	DisembedScore (Max = 7)	*FSScore* (Max = 5)	*AddSubScore* (Max = 13)
Chris	6	2	13
Anna	7	5	12
Steve	0	4	6

The next section of this chapter presents the findings from the clinical interviews conducted with Chris, Anna, and Steve. Each section begins with a general overview of the participant, followed by an examination of his or her conceptions related to disembedding, fraction schemes, and fraction addition/subtraction. Also discussed is evidence of the three aspects of this study's theorized framework for adult learners' knowledge of fractions. Recall that these three aspects include contextualized knowledge, unorganized procedural knowledge, and models of fractions. Following the individual findings, there is a cross-case analysis, which examines the three cases as a collective study. This cross-case analysis presents both the similarities and differences between the three individual cases.

Chris

Chris was a 23-year-old male student majoring in criminology and justice studies. He was enrolled in Course B during the spring 2017 semester. Prior to his enrollment in Course B, the last mathematics course he successfully completed was high school Algebra I five years earlier. On the written assessment, Chris answered six out of seven of the disembedding items correctly as well as all 13 of the fraction addition/subtraction items. His score on the fraction scheme items was two correct out of five, with his correct responses being on the part-whole and the PUFS tasks. In analyzing Chris' responses during the one-on-one interviews, some common themes regarding Chris' cognitive processes emerged within and across the various item types.

Estimation with benchmarking. One theme that was evident in both the disembedding items and the fraction scheme items was estimation with benchmarking. When he completed the written assessment during the quantitative phase of the study, Chris did not make any markings on the disembedding items and marked only one of the fraction scheme items. Thus, it was important to ask him to explain how he arrived at his answers. Chris' clinical interview revealed that he used familiar units (i.e., *benchmarks*) such as halves and fourths to arrive at most of his answers, as opposed to engaging in the expected actions associated with each item based on prior research. This allowed him to reasonably answer, very often correctly, these items without having made any markings. Evidence of Chris' use of estimation and benchmarking was found in his interview responses to items 7, 9, 12, 13, and 15. For example, when asked to explain how he arrived at an answer of seven on item 7, he responded:

> I didn't know if there was an actual formula on how to figure it out so I kind of just . . . (pause) . . . I figured this was five right here (pointing to second partition of the scale at the bottom of the problem) and just kind of estimated that it was seven. Maybe eight would be like right here (pointing close to the middle of the second partition of the scale), so I just said seven. (Chris, Interview, April 24, 2017)

When asked how he knew each of the intervals on the scale was five, he replied:

> I just kind of like figured, because if it [the shaded bar] equals ten so . . . I figured this would be five (making an imaginary line with his finger from the midline on

> the scale up to the unshaded bar) and you just add a couple so that's kind of just like what I, kind of like guessed on that. (Chris, Interview, April 24, 2017)

Chris' explanation does not show any evidence of disembedding 1s from the composite unit (shaded bar of length 10) as one might expect on this item type. Rather, he uses estimation with his benchmark being the midpoint line of the shaded bar, or the value five. As he described, he concluded that the value of the unshaded bar would have to be between five and 10. His "guess" of seven was a result of adding "a couple" to five.

Chris' response to item 13 on the assessment can be found in Figure 8. As is shown, Chris made a vertical line at what appears to be three-fourths of the whole bar. He also shaded between the beginning of the bar and this vertical line.

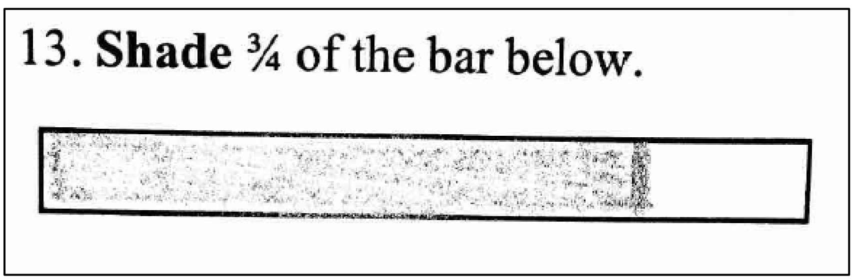


Figure 8. Chris' response to item 13 on the written assessment.

Although Chris had a correct response on this item, it was not clear if he partitioned the bar (mentally or otherwise) into four equal pieces since there were no markings made to indicate so. Therefore, during his one-on-one interview, the researcher asked him to describe how he decided where to shade (see Figure 9). He replied:

> Okay so, I just kind of estimated that . . . this roughly was the middle of it so this would be like half (pointing and marking near the middle of the bar). And I was

right in the middle of half and full, so I figured this would be like three-fourths (drawing another vertical line above his original vertical line on the bar). (Chris, Interview, April 24, 2017)

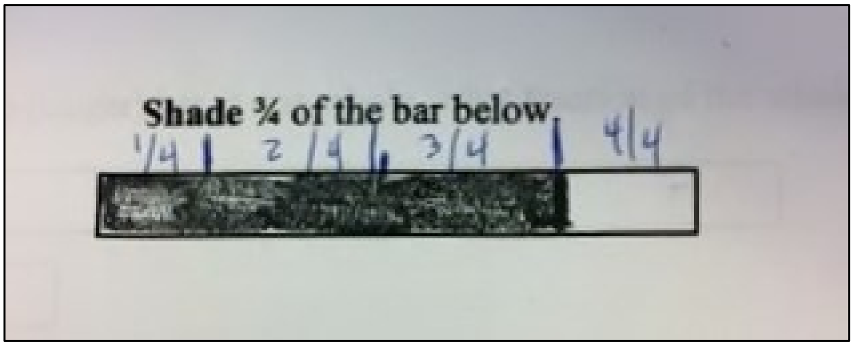


Figure 9. Chris' markings (in blue) on item 13 during the interview.

Once again, Chris' interview reveals the use of estimation by way of finding the middle of the bar to be one-half and then the middle of the second half of the bar to be three-fourths. From his markings on the written assessment and then his explanation during the one-on-one interview, it does not appear that he partitioned the bar into four equal pieces and disembedded three of them, which are two actions associated with the PWFS. However, the researcher then asked Chris if he could show where one-fourth would be on the bar. Chris proceeded to draw a vertical line above the bar at approximately one-fourth and continued to label and describe them verbally as "1/4, 2/4, 3/4, and the whole" (see Figure 9). This indicates that although he relied on estimation with benchmarking to arrive at his answer on the written assessment, he still constructed the PWFS since he conceived of the fractions one-fourth, two-fourths, and three-fourths as "so many pieces in the partitioned fraction out of so many pieces in the partitioned whole" (Norton & Wilkins, 2009, p. 151).

Chris' interview response to item 15 again revealed his use of estimation to arrive at an answer of three-fourths instead of the correct answer of three-fifths (see Figure 10). When the researcher asked him to explain how he came up with his answer, Chris replied:

> With this one I kind of just tried to guess on where the middle would be (drawing a vertical line near the middle of the top bar) . . . so it was a little more than half but not quite a whole so I just said three fourths . . . I guess it might even be less than three-fourths . . . three-fourths is right here (drawing a vertical line near the middle of the second half of the whole bar) . . . I'm not exactly sure how to break down three-fourths to make it in between a half and three fourths. (Chris, Interview, April 27, 2017)

Chris' response indicates that he knew the shorter bar was between one-half and three-fourths of the whole bar, which provides evidence of estimation. However, he was unsure of how to determine the exact value of the shorter bar as a fraction of the whole bar. This illustrates that Chris did not engage in the actions of partitioning and disembedding as we would expect to see under the PFS.

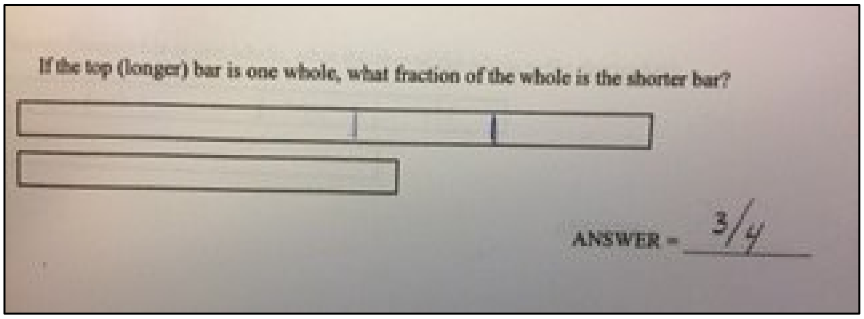


Figure 10. Chris' markings (in blue) on item 15 during the interview.

As mentioned earlier, item 15 had low reliability statistics. Therefore, during the interview, the researcher asked Chris to complete two additional tasks designed to trigger the PFS. Chris' responses on these items are discussed in the next few paragraphs, and the unmarked items, named 15a and 15b, can be found in Appendix C. The first task, 15a, was presented in the same format as the original PFS assessment item. The only change made was to the size of the longer bar so that the shorter bar was three-fourths of the longer bar, rather than three-fifths of the longer bar. This change was made to see if the size of the associated partition (denominator) had an impact on the participants' responses. Chris was given this new task and asked to read it and then "think through" aloud how to solve it. The dialogue between Chris and the researcher is below, and Chris' written response to item 15a can be found in Figure 11.

Chris: Okay so this is one whole, so I would say right here would be half (drawing a vertical line near the middle of the whole bar). Yeah (drawing another vertical line near the middle of the second half of the whole bar). So I would say, so this would be one fourth, two fourths, three fourths, four fourths, one whole (writing each of these fractions above the respective pieces of the whole bar). So I would say that's three fourths.

I: Okay, good.

Chris: Now I'm not, there's no way of telling that this is broken down into four sections or if they wanted like I don't know, you know, different sections or whatever, so that's just what I would say.

I: Okay.

Chris: Do you understand what I'm saying? Like if they wanted it broken down to like ten sections or whatnot.

I: That's a good point. How are you deciding on the fourth and not some other sized piece?

Chris: I would say that it's probably more, I don't know if I would say like standard but more just kind of like what I usually see on the fractions.

The conversation above, along with Chris' actions and markings on the whole bar (see Figure 11), displays evidence of the PFS. However, rather than identify the unit fraction (one-fourth) first, Chris again estimated one-half of the whole bar first. He also marked the vertical line for three-fourths before he marked the one-fourth line. The order of his approach displays more of an understanding of halves and fourths as fraction numbers than it does disembedding. Moreover, it is interesting that Chris mentioned being unsure of how the bar was to be "broken down." Even when asked how he knew to use fourths and not another unit, he reiterated that fourths are more common and familiar to him.

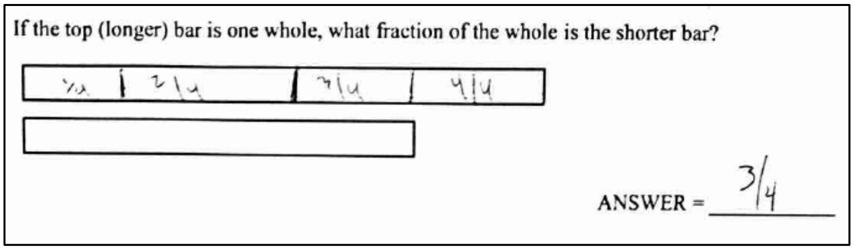


Figure 11. Chris' interview response to item 15a.

The second task, 15b, eliciting use of the PFS was designed using language (e.g., "as long as") intended to trigger schemes relating to size (Wilkins et al., 2013). As a result, someone working through this task might be more inclined to use the PFS, if they are able to, rather than the PWFS. Chris' response to item 15b illustrates how the PFS was triggered (see Figure 12).

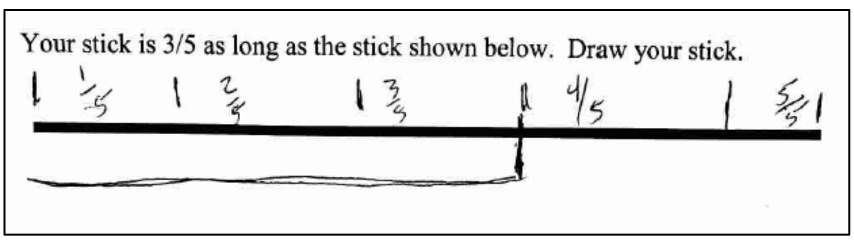


Figure 12. Chris' interview response to item 15b.

Unlike most of Chris' prior responses, he did not begin by finding the middle of the stick. Rather, he immediately partitioned the given stick into fifths, noting that the pieces should be of equal size even though his marks are a little unevenly spaced. After partitioning, Chris correctly drew his stick beneath the given stick, using the vertical mark between three-fifths and four-fifths as his guide for where his stick should end. Thus Chris was able to correctly complete item 15b while demonstrating evidence of the

PFS, even though he could not complete the original item 15. This finding is consistent with prior research, as Wilkins et al. (2013) noted, "students seem to have an established way of operating that includes the ability to solve some (but not all) tasks for which construction of a PFS is theoretically both necessary and sufficient" (p. 41).

Unorganized procedural knowledge. Another theme that emerged through the analysis of Chris' clinical interview was unorganized procedural knowledge. Recall from Chapter 2 that unorganized procedural knowledge consists of rules and procedures that have been acquired by the learner throughout his or her formal mathematics instruction. This type of knowledge is unorganized in the sense that the learner does not always know when and/or why to use these rules and procedures. At the beginning of Chris' explanation for item 7, he stated, "I didn't know if there was an actual formula on how to figure it out" (Chris, Interview, April 24, 2017). Thus, Chris' initial approach was to attempt to recall and rely on a procedure to solve an unfamiliar problem, indicating unorganized procedural knowledge. Prior research has revealed that students with five or more years of traditional mathematics instruction tend to employ rules and algorithms as opposed to reasoning strategies (Lamon, 2007). Considering Chris has had considerably more than five years of traditional mathematics instruction, this finding is consistent with Lamon's prior research. Another example of Chris' evidence of unorganized procedural knowledge was revealed on the fraction addition/subtraction items and interview. Recall that Chris correctly answered all 13 of the fraction addition/subtraction items on the written assessment. He also showed detailed steps regarding how he arrived at his final answers in this section. Based on his work, it was clear that he knew when it was necessary to convert the fractions to equivalent fractions with like denominators. He also confirmed this during his one-on-one interview when asked how he arrived at his answers for items 18a and 18e.

I: So for this first one [part a], three-fifths plus one-fifth equals four-fifths . . . How did you approach part (a) differently than part (e)?

Chris: Well I know that when adding and subtracting fractions you need a common denominator . . . This one [part (a)] already had it so I didn't worry about that, so I made the denominator five and just added three and one to get four-fifths.

I: Good. And so then how is this one [part (e)] different than part (a)?

Chris: This [part (e)] didn't have a common denominator so I had to multiply by three for the one-half and then the one-thirds I had to multiply by two to get three over six and two over six which is where I got five-sixths.

I: Okay good. So how did you know . . . whether you need a common denominator?

Chris: I don't know when I learned it but I just remembered that adding and subtracting you need one [a common denominator], and when you multiply and divide you don't.

I: Okay, great. Could you explain why you need one when you add and subtract?

Chris: I don't know. I just . . . I learned it . . . I remember it from high school but it's just one of those things where I learned it and it just stuck with me but I don't remember where.

The dialogue above clearly indicates that Chris knew how to rewrite equivalent fractions with common denominators as well as the idea that fractions need to have a common denominator before adding or subtracting. What was missing from his understanding was why a common denominator is necessary when adding/subtracting. This is an example of unorganized procedural knowledge, as Chris knew how and when to apply the procedure of finding and converting to a common denominator but he did not know why he was applying this procedure.

Use of the disembedding scale. A third theme was found in Chris' responses to the disembedding items. This theme involves his use of the scale given below the shaded and unshaded bars in each of the disembedding items. Figure 13 shows Chris' response to item 9 on the written assessment.

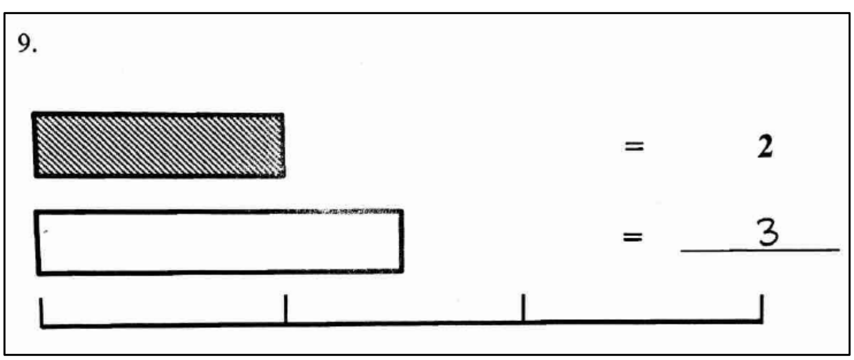


Figure 13. Chris' response to item 9.

Since there were no markings on his paper, the researcher again began by asking Chris to explain how he arrived at his answer of "3" for this question. Below is his initial response and the dialogue that followed between the researcher and Chris.

Chris: I don't remember this one but now that I see three, I don't know why I wrote that because I feel like three would be out to here (pointing to the end of the scale). So this [unshaded bar] would, I think, probably be one and a half. I don't know why I put three unless I was rushing.

I: Well, so what size is this shaded bar?

Chris: I think that would be one third.

I: Well if you remember, this is given to be two though, so if this shaded bar is two –

Chris: Okay, so the whole thing would be six.

I: What whole thing?

Chris: Like if it was like this horizontally to here (pointing to the end of the scale), it would be six.

I: Okay good, good.

Chris: So then that (pointing to the end of the unshaded bar) should be like two point five or three. Oh okay . . . I remember where I went now.

I: So can you walk me through what you were thinking then with the six?

Chris: So one of these (pointing to the shaded bar) equals two and then if you have three of these [shaded bars] and this part right here (pointing to the end of the scale) would be six. So then if this (pointing to the unshaded

> bar) ends right in the middle [of the scale], then it would be three. It took me a minute to realize what the problem is again.

The dialogue above also shows no clear evidence of Chris disembedding 1s from the composite unit (shaded bar of length two). Mathematically, Chris multiplied two by three and then divided by two, which is equivalent to multiplying one by three. Therefore, it could be that he disembedded 1s but his explanation does not explicitly indicate this. What is clear, however, is that Chris relied on the scale to reason through to his final answer. Specifically, he had to go out to six by iterating the shaded bar of length two three times and then take half of it to convince himself that the unshaded bar was three units long. Moreover, when Chris initially looked at this item, he ignored that the shaded bar was given to be two units long. He paid more attention to the scale being partitioned into three pieces, causing him to assume the shaded bar was one unit long and thus that the unshaded bar was one and a half units long.

Anna

Anna was a 45-year-old female student majoring in nursing. She was enrolled in Course B for the first time during the spring 2017 semester. Prior to this course, Anna successfully completed Algebra I in high school 29 years earlier. On the written assessment, Anna correctly answered all seven disembedding items and all five fraction scheme items. Additionally, her score on the fraction addition/subtraction items was 12 correct out of 13. The only fraction addition/subtraction item she answered incorrectly was 18j due to a computational error made when rewriting a mixed number with a new denominator. In analyzing Anna's responses during the one-on-one interviews, some common themes regarding her cognitive processes emerged within and across the various item types.

Benchmarking. One theme that was evident in Anna's responses to the fraction scheme items in particular was benchmarking. During Anna's clinical interview, her responses revealed her reliance on common units (e.g., halves, fourths, tenths) to help her arrive at her final answers. This can be seen, for example, in her response to item 15a. Anna was asked to read this item and then "think through" aloud how to solve it. Her written work for this item can be found in Figure 14.

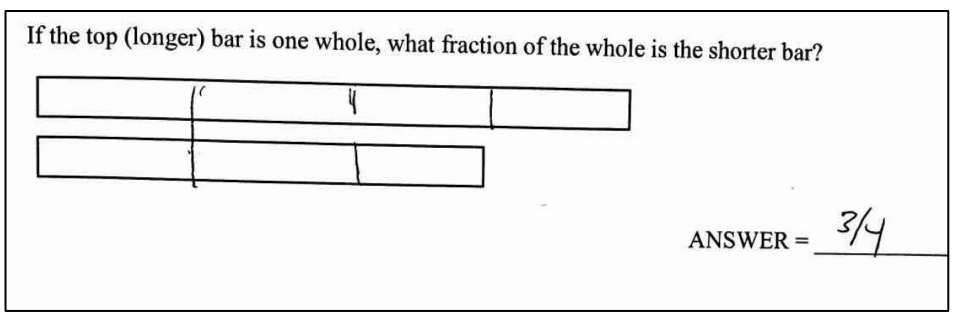


Figure 14. Anna's interview response to item 15a.

Anna began working through item 15a by stating, "Well I'm going to look at it in halves. I try to look at it in halves" (Anna, Interview, April, 25, 2017). As she said this, she used her pen to partition the whole bar into two equal pieces (see Figure 15).

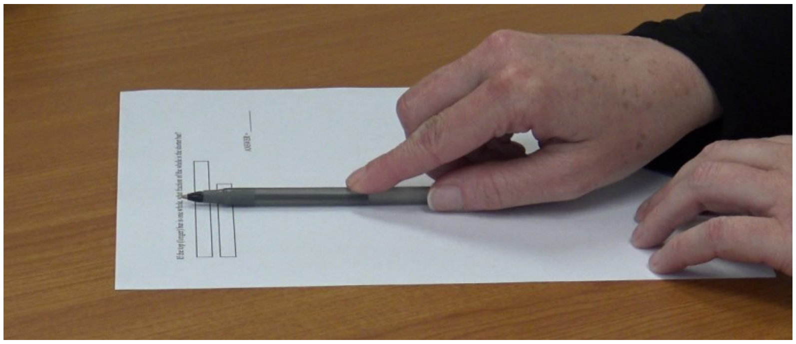


Figure 15. Anna demonstrating with her pen how she partitioned the whole bar.

She continued to describe how this benchmarking helped her:

> And if I can divide it [the whole bar] in half and it look even on both sides, which to me if you divide this in half (see vertical lines drawn in Figure 16) . . . this (pointing to the piece as seen Figure 16) and the blank space (drawing an imaginary rectangle from the end of the shorter bar to the end of the whole bar) are even. (Anna, Interview, April 25, 2017)

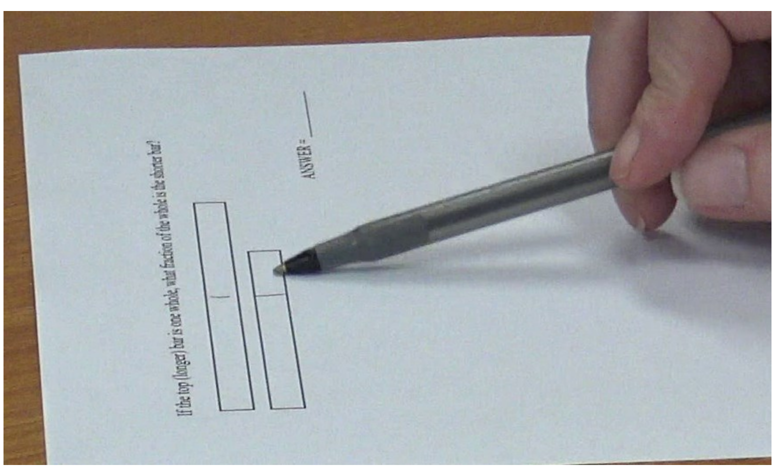


Figure 16. Anna pointing to a one-fourth sized piece of the shorter bar.

Anna finished working through this task, explaining:

> If you put your marking there (drawing a vertical line from the whole bar down through the shorter bar to create a unit of one-fourth) and a marking there (drawing a vertical line at three-fourths of the whole bar) you're going to have three fourths. This bar is three fourths of the whole bar. (Anna, Interview, April 25, 2017)

This response shows Anna's use of one-half as a benchmark that facilitated her ability to partition the smaller bar to produce a unit fractional part (one-fourth).

Benchmarking can also be seen in Anna's response to item 15b in Figure 17, where she used tenths as opposed to the given unit of fifths:

> Anna: Okay. So I'm just making lines (drawing a line that partitioned the stick in half).
>
> I: Okay. Why are you making that marking?
>
> Anna: Well because at this point, three-fifths is uh, kind of wild. It's easier to put it in tenths (partitioning each half into five pieces as she talks).
>
> I: Good and so the markings that you made then, you said something about tenths. Where would the tenths be represented on –
>
> Anna: I, so I have one, two, three, four, five, six, seven, eight, nine, ten (pointing to each of the partitions she created). So now . . . my stick is now going to be six tenths as long as this.
>
> I: Okay, great. So then can you draw your stick?

Anna: Okay, so this is the fifth mark (pointing) and I'm just going to "boop" down to this part (moving down to the sixth mark). And I'm just going to, one, two, three, four, five, six (drawing her stick from right to left as she counts).

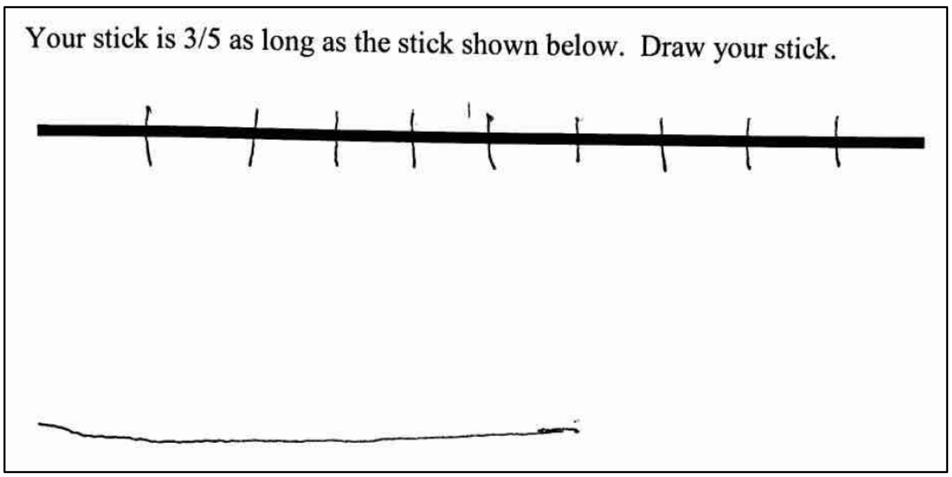


Figure 17. Anna's interview response to item 15b.

It is interesting that Anna initially chose to partition the stick into tenths to avoid making fifths, describing them as "wild." However, she eventually had to make fifths in order to partition each half of the given stick. Anna seemed very comfortable going back and forth between tenths and fifths, immediately noting that three-fifths is equal to six-tenths and that her stick was going to be "six tenths as long as this [given stick]."

In addition to using benchmarking to complete the fraction scheme items, Anna's interview also revealed the use of benchmarking with halves to facilitate disembedding. That is, even if the unshaded bar was not one-half or twice the shaded bar, Anna still used halves (e.g., finding the midpoint of the scale) to determine a unit to disembed. For example, in her response to item 7 (see Figure 18), Anna stated, “The half of—the bar (pointing to the vertical mark in the middle of the scale) below the halfway point [of the shaded bar] is five, which is half of the ten” (Anna, Interview, April 25, 2017). Thus, Anna’s process for determining the unit to disembed was to first find the size of one half of the scale and then partition the other half based on knowing that size.

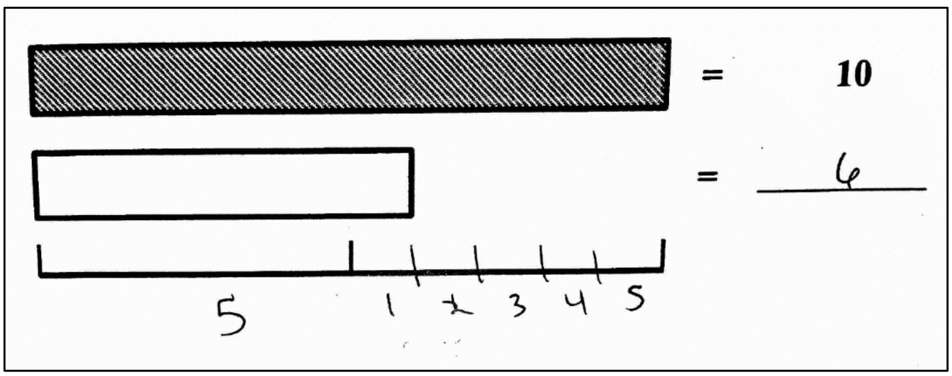


Figure 18. Anna’s response to item 7 on the written assessment.

Use of the disembedding scale. Another theme was found in Anna’s responses to the disembedding items. This theme involves her use of the scale given below the shaded and unshaded bars in each of the disembedding items. On the written assessment, all of Anna’s markings on the disembedding items were made on the scales as opposed to the bars themselves. Furthermore, she used the scale, along with the given size of the shaded bar, to arrive at most of her answers. One example of this reasoning can be seen in Anna’s response to item 6. Her written response to this item can be found in Figure 19.

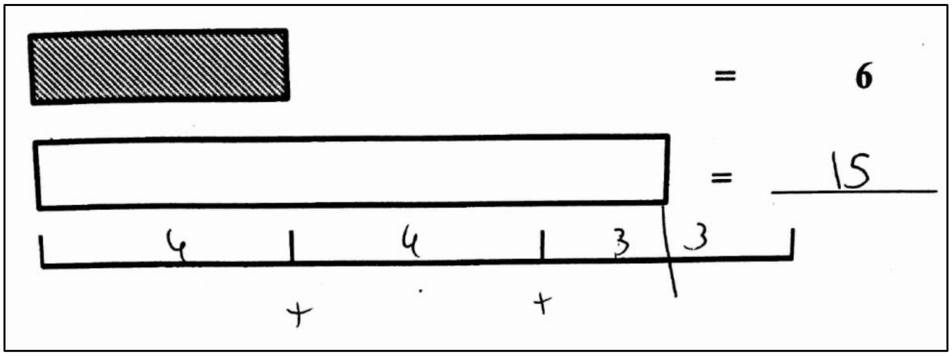


Figure 19. Anna's response to item 6 on the written assessment.

The markings made show evidence of disembedding, as Anna wrote "6" in the first two intervals and drew a vertical line at the end of the unshaded bar and through the scale below. In the third interval of the scale, she wrote "3" in each of the partitions created by the vertical line she drew. During her one-on-one interview, Anna's verbal explanation confirmed her ability to disembed 3s from the composite unit of six using the partitions of the scale:

> I put six because these (pointing to the three partitions of the scale) are already divvied up to a third. So I put six and six and six because this (pointing at the shaded bar) is six. So there's a six, there's a six and there's a six (pointing at the three partitions of size six on the scale) and that is a half of the six (pointing at the first piece of the third partition of the scale). So, half of six is three . . . so you add the six and the six and the three which equal fifteen. (Anna, Interview, April 25, 2017)

Notice that in the very first sentence of her explanation, she refers to the scale being "already divvied up to a third." Using that information, along with the size of the given bar, Anna partitioned the third interval of the scale in half to determine that each piece of the third interval would be three units long. She then iterated six twice and three once, summing those values together to arrive at her final answer of 15.

Anna's written response to item 11 on the assessment can be found in Figure 20. In her written work, Anna labeled the first two intervals of the scale with the number "8" and also appeared to have partitioned the first interval into four equal pieces of size two. This suggests that Anna disembedded non-1s, specifically twos, from the composite unit of length eight.

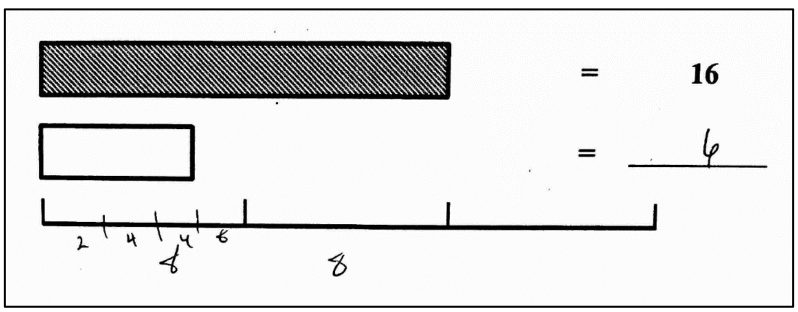


Figure 20. Anna's response to item 11 on the written assessment.

During her interview, Anna was asked to explain how she arrived at an answer of "6" for the unshaded bar.

Anna: Okay so I didn't count that, so –

I: You didn't count what?

Anna: This one (pointing to the last interval on the scale) at all even though it was there . . . I just counted this part (covering up the third interval on the scale). So it's already dashed here (pointing at the vertical bar partitioning the first and second intervals of the scale). So half of sixteen is eight, and

> then there's the other half is eight. So then you take here (pointing to the first interval) and you mock up here (pointing to the vertical lines she drew to partition the first interval into four pieces) and make sure that you even the lines and figure out how many go exactly in there and then add it.

As can be seen in her verbal explanation, Anna's very first consideration was that the last interval of the scale extended beyond the shaded bar. She then continued to explain how she partitioned the first two intervals of the scale in order to determine the size of the unshaded bar.

Attention to the unit. A common theme throughout Anna's responses on all of the item types was her careful attention to the unit. Specifically, she made a clear distinction between whole numbers and fractions in both her written work and verbal explanations. On the disembedding items, Anna confirmed her ability to disembed units of 1 and non-1 through her verbal explanations of her written work. For example, when asked to explain her written markings on item 9 (see Figure 21), Anna stated, "There are three of the twos and then there's half of a two. So you're at two and then that's . . . two plus a half of two is a one, [which] gives you three" (Anna, Interview, April 25, 2017).

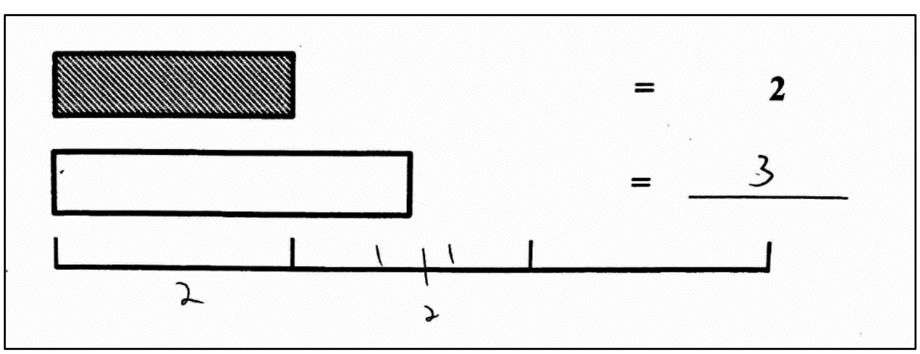


Figure 21. Anna's response to item 9 on the written assessment.

Anna's verbal response confirms that she coordinated between the composite unit of length two and the unit of length one. She did not mentally destroy the whole (shaded bar), as is evident in her last statement. Rather, she is able to maintain the whole composite unit of length two while using one part of this whole to operate on when she adds "two plus half of a two is a one, [which] gives you three."

Anna's response to item 17 is another example of her careful attention to the unit. Her written response to this item can be found in Figure 22. During the one-on-one interview, the researcher asked Anna to explain what she did to arrive at the correct answer.

Anna: Because the bar that I needed was just one whole and the other bar was four thirds as long which is, like one and one third. So if I needed one . . . third extra as the top bar . . . I just divided the bar into, well, fours and then take one of those off just to have—does that make sense?

I: Yes definitely . . . Tell me in your words, why did you divide this [given bar] into four? You mentioned something about one and one third.

Anna: Well because three equals one.

I: Three what?

Anna: Three parts equals the one whole.

I: Okay.

Anna: So and then I needed to have one left over (pointing at the last partition of the given bar).

I: One of the parts left over?

Anna: Yes. So and I needed my bottom [bar] to have three parts to equal the whole.

I: Good.

Anna: So that's why they're all kind of that way.

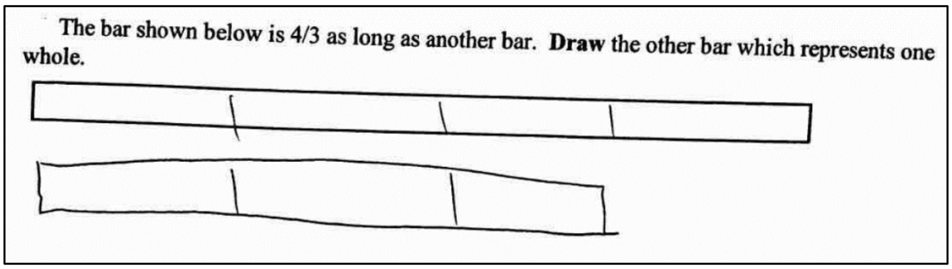


Figure 22. Anna's written response to item 17 (IFS) on the assessment.

It is clear from Anna's written and verbal response that she did not redefine the whole as the larger stick, as is done by students operating with a PFS and not an IFS. Rather, she maintained the size of the given bar and knew that the whole bar would be smaller than the given bar. Consistent with the IFS, Anna partitioned the improper fractional part (i.e., the given bar) into four pieces to produce unit fractional parts (one-thirds). Although she didn't specify iterating the one-third part three times to produce the whole, she does display a solid understanding that the whole is comprised of three one-third parts (e.g., "three parts equals the one whole"). Thus, it is evident that Anna is coordinating three levels of units, as she viewed four-thirds as a unit of four units (thirds), any of which imply a unit (whole) of three units (thirds).

Contextualized knowledge. Lastly, a theme that surfaced in Anna's interview pertaining to the fraction addition/subtraction items was her contextualized knowledge of fractions. Recall that Anna correctly answered all 13 of the fraction addition/subtraction items on the written assessment. She also showed detailed steps regarding how she arrived at her final answers in this section. Based on her work, it was clear that she knew when it was necessary to convert the fractions to equivalent fractions with like denominators. She confirmed this, along with contextualized knowledge, during her one-on-one interview when asked how she arrived at her answers for items 18a and 18g.

I: Could you explain [for part a] what you did and why you did it?

Anna: Okay because if you have three-fifths of something and you're adding one more fifth, then you would have a total of four-fifths. So even if you took out the fifth part, you add three plus one . . . you're going to get four, and they just happen to be fifths.

I: Okay . . . Why do they have to be fifths? Why does that matter? Why did you mention that they had to be fifths?

Anna: Well because they can't be thirds because a third is a different measurement completely.

I: Okay.

Anna: It all has to . . . that bottom number has to be the same when you're adding and subtracting because one-half of something, one-third of something, two-thirds of something . . . those are different measurements.

I: Different measurements. Okay.

Anna: That's how I know with cooking. One-half cup and one-third cup are two different things, and if you would add them, it doesn't equal two of the same thing.

In her explanation of steps for part a, Anna was careful to refer to the fractions as numbers (e.g., "three-fifths" and "four-fifths") and not as whole numbers. When she mentioned, " . . . even if you took out the fifth part . . . you're going to get four, and they just happen to be fifths," it might initially appear that she was operating only on the numerator of the problem. However, she went on to clarify the importance of paying attention to the denominator, stating that "they can't be thirds because a third is a different measurement completely" and "the bottom number has to be the same when you're adding and subtracting because one-half of something, one-third of something . . . those are completely different measurements." These comments are evidence of Anna's understanding of fractions as numbers.

The researcher also asked Anna how she approached 18g differently than 18a.

Anna: Well it's the bottom number so you need to make sure that your bottom numbers are the same. Since you can't reduce five-eighths any further, you need to increase the three-fourths to . . . they need to have the same common denominator. So you need to make that [three-fourths], and it's easy, you just double it – six eighths. And then you add it and it's eleven-eighths and then eight goes into eleven one time, leaving three over. So it's actually just one whole and three eighths.

I: Okay, great. How would you explain to someone else why you need a common denominator?

Anna: Why . . . I refer to it as cooking. You can't, you know. If the recipe calls for two-thirds cup and . . . you just can't substitute you know . . . four-halves of a cup or something . . . it all has to match or it's not going to . . . come out right. Probably because it's part of a whole of one thing.

This last excerpt from Anna's interview pertaining to the fraction addition/subtraction problems highlights multiple facets of her conceptions about fractions. She was able to explain how items 18a and 18g needed solved differently as a result of having unlike denominators. Thus, she demonstrated organized procedural knowledge because she knew when and how to find a common denominator, explaining how to convert three-fourths to an equivalent fraction with a denominator of eight. Furthermore, Anna indicated contextualized knowledge in that she related fraction addition to measurements made while cooking.

Steve

Steve was a 28-year-old male student majoring in communications. He was enrolled in Course B for the second time during the spring 2017 semester. On the written assessment, Steve did very well on the fraction scheme items, answering all but the PFS item correctly. He was unable to correctly answer any of the disembedding items and correctly answered six out of 13 on the fraction addition/subtraction items. In analyzing Steve's responses during the one-on-one interviews, some common themes regarding his cognitive processes emerged within and across the various item types.

Use of the disembedding scale. Like Chris and Anna, Steve also made use of the given scale of each of the disembedding items. However, Steve did not use the scale in the same manner as the other two participants. When Steve completed the disembedding items on the written assessment, he seemed to have ignored the values given for the shaded bars. Rather, he took the scale to be a composite unit and its intervals to be units of size one, or 1s. During his follow up interview, the researcher brought this to Steve's attention and asked him to take into consideration the size of the shaded bar. Once Steve understood the shaded bar as the composite unit with a given size, he was able to correctly complete three out of four disembedding items with a bit of guiding questions from the researcher. However, this conception of the disembedding items was so strong that even after completing three of them correctly, Steve reverted to his previous conception on the fourth interview item, item 9. Figure 23 shows Steve's response to item 9 on the written assessment. As is seen in this figure, Steve's original answer of "1½" depicts his assumption that the length of the first interval of the scale, and thus the shaded bar, was one.

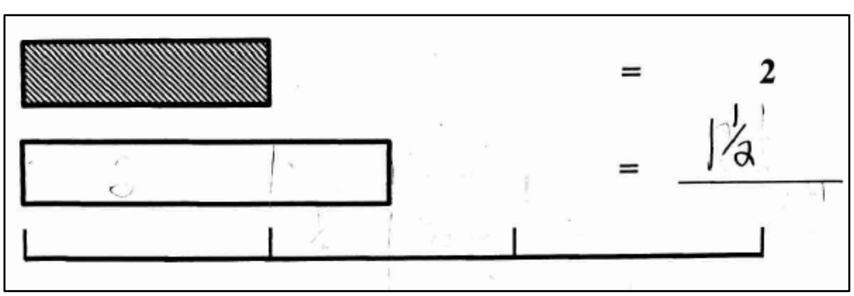


Figure 23. Steve's written response to item 9.

During his interview, the researcher asked Steve to revise his answer based on the given shaded bar being length two. Steve replied:

> I would say two and one-half . . . because if this whole thing (pointing to the second interval of the scale) is two, you'd have to have one and one (partitioning the second interval into two parts) because it [unshaded bar] extends to one. So two and one-half. (Steve, Interview, May 3, 2017)

Steve's partition of the second interval into pieces of "one and one" shows some indication of his conception of the interval's length as two. Additionally, the whole part of his final answer ("two and one-half") is consistent with the length two interval. However, Steve neglected the partitions of length one that he created and thus arrived at a mixed number rather than a whole number for his final answer.

Conception of the whole. Stemming from the previous theme, a second theme that emerged from Steve's interview, particularly on the fraction scheme items, was his inconsistent conception of the whole. When Steve was given a value for the whole (as in the disembedding problems) or given an image of the whole (as in the fraction scheme items), he had a much harder time maintaining what was given as the whole while he worked through the task. Similar to Steve's use of the scale on the disembedding items (see Figure 23), his answer to item 15 indicated that he took the partitions he made to be representative of whole units. Doing so caused him to arrive at an answer larger than one whole, namely "2½" (see Figure 24).

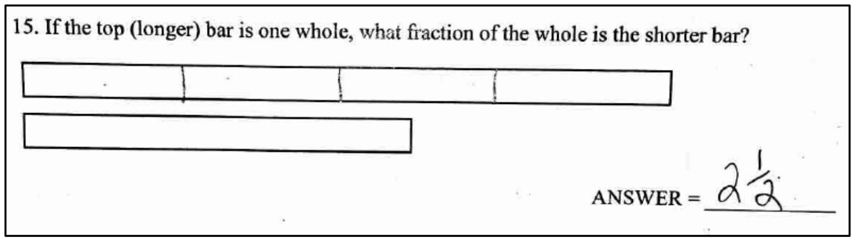


Figure 24. Steve's written response to item 15 (PFS) on the assessment.

Steve's inconsistency with the whole was further displayed in his interview explanation of his written work, whereby he took the whole to be both the given bar and the one-fourth partition he had made. When asked to explain his markings on item 15, Steve replied:

> Making it into fourths or four equal parts and then looking at this I see that . . . if I flip this [shorter bar] over (gesturing vertically as though flipping the shorter bar on top of the longer bar) more than two [partitions] would be filled in or covered and so you logically have to have two. And one whole—and my thought was that it would be a fraction of the one whole so if I have two of those [partitions] and it extends to, what seems to be the middle of one [partition], so two and one-half, or no. And that would be two and – yeah but I think that I kept wanting to say two and one-half of a fourth . . . I don't know how to write that. (Steve, Interview, May 3, 2017)

As his written work and interview indicate, Steve partitioned the whole bar into fourths. Yet, when he was trying to determine the size of the shorter bar, he neglected the established fourths and treated each partition as one whole in order to arrive at his answer of 2½. At the end of his explanation, he expressed dissonance between his two different choices for the whole but he was unable to resolve this conflict and express "one-half of a fourth."

What is especially interesting about Steve's conception of the whole is that he seemed to have an idea of the size of an object relative to the given whole. For example, when asked to explain how he arrived at his answer to item 7 (see Figure 25), Steve displayed his understanding that one and one-fifth is larger than one.

Steve: If I see this as one whole, I notice that there's two—and now that I'm looking at it, there's no way that that can be one and one-fifth, it would—

I: Why don't you think it could be one and one-fifth?

Steve: Well you don't—this [unshaded] bar doesn't extend past this [shaded bar] and it [unshaded bar] would actually have to extend past . . . that [shaded bar] to be one and one-fifth.

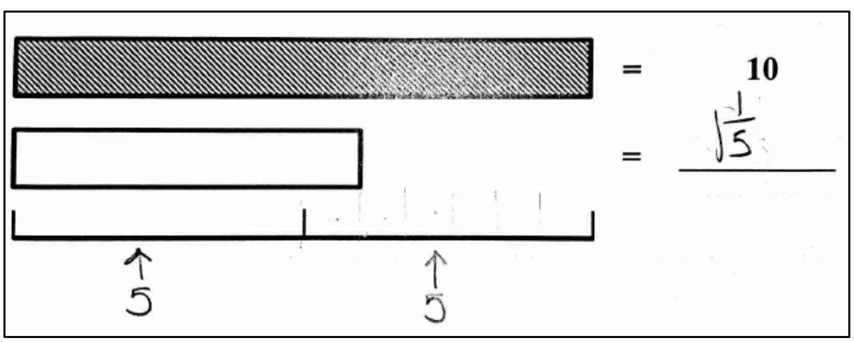


Figure 25. Steve's written response to item 7 on the assessment.

Thus, we see Steve question the reasonableness of his original response to item 7 based on the size of his choice of the whole. Although Steve did not correctly take the given shaded bar to be 10, his answer was correct within his assumption that the shaded bar was one whole. Similarly, Steve's responses to the other disembedding items on the written assessment indicated his understanding of the size of the unshaded bar relative to his assumption that the shaded bar was one whole. Figure 26 shows two examples of such understanding in Steve's written responses to item 8 and item 12.

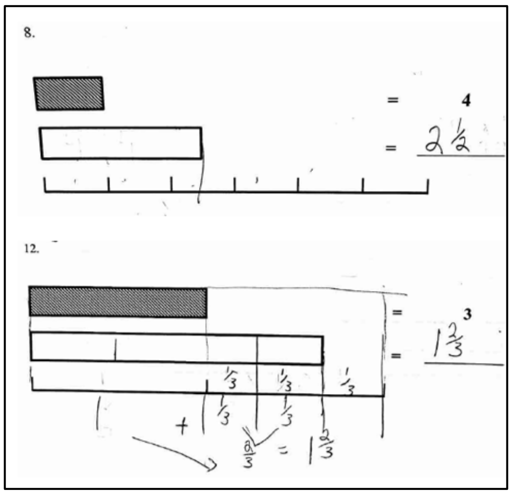


Figure 26. Steve's written responses to item 8 and item 12 on the assessment.

As is seen in Steve's responses in Figure 26, he took the given shaded bar as one whole and then partitioned the intervals based on the size of the unshaded bar relative to the whole. Steve also displayed this understanding of the size of fraction relative to the whole in item 16 and item 17. Figure 27 shows Steve's written response to item 16 on the assessment, which assessed the RPFS.

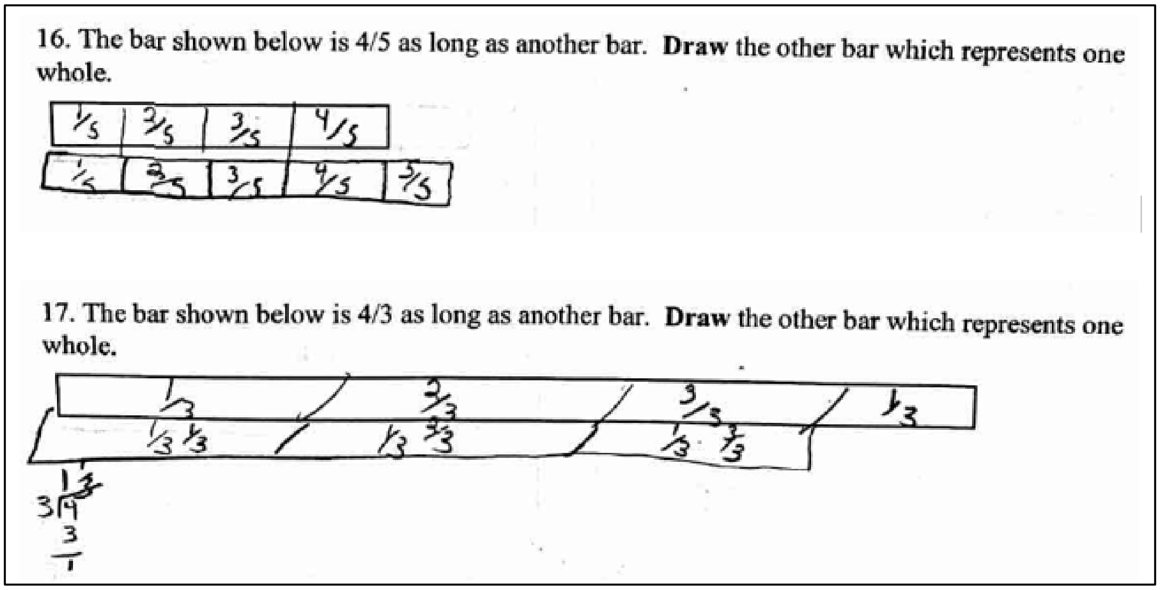


Figure 27. Steve's written response to items 16 (RPFS) and 17 (IFS) on the assessment.

When asked by the researcher to explain his markings on item 16, Steve replied:

> Yeah, if it's four fifths, this one [given bar] has to only have four sections, the one below it has to have five . . . and if you split it up evenly, it's just going to extend one more imaginary section (pointing at the piece he labeled five-fifths). (Steve, Interview, May 3, 2017)

Thus, on item 16, Steve understood that if the given bar was four-fifths as long as the whole, then the whole bar would be one-fifth larger than the given bar. Likewise, on item 17, Steve understood that if the given bar was four-thirds as long as the whole, then the whole bar would have to be one-third smaller than the given bar.

Unorganized procedural knowledge. Another theme that emerged through the analysis of Steve's clinical interview was unorganized procedural knowledge. This was revealed primarily through his responses on the fraction addition/subtraction items. On the written assessment, Steve correctly answered six out of the 13 fraction addition/subtraction items. However, his one-on-one interview revealed he knew more than what the assessment indicated. For example, consider the dialogue between Steve and the researcher regarding item 18a:

I: Can you tell me how you came up with your answer of four-tenths?

Steve: Not at all . . . because I believe that should be four-fifths.

I: Okay. Why?

Steve: In adding . . . the denominator doesn't change . . . you just straight add the numerators, which is four-fifths.

I: Great and so any idea how you know that? How do you know that you're not supposed to change the denominator?

Steve: Irritating repetitive ALEKS courses telling me I was wrong.

Thus, Steve immediately recognized his response as incorrect. However, when asked by the researcher how he knew that he should have gotten a common denominator, Steve did not explain why. Rather, he cited his current ALEKS course as the reason he knew what to do and when to do it, evidence of unorganized procedural knowledge. Nearly 10 minutes later in the interview, when asked by the researcher how he could explain to someone why it is necessary to get a common denominator, Steve replied, "you can't jump from Fahrenheit to Celsius, just like 32 Fahrenheit can't just be 32 Celsius and it be the same feeling" (Interview, May 3, 2017). Here, Steve is referencing units of measure in terms of temperature, alluding to the importance of the unit when trying to compare two values. Although Steve was still unable to precisely explain why a common denominator is necessary when adding/subtracting unlike fractions, we see a glimpse into what might be the beginning of a deeper understanding.

Fraction models. The final theme found in Steve's fraction addition/subtraction responses was his use of fraction models. Recall from Chapter 2 that fraction models refer to written or mental pictorial representations of fractions, such as fraction bars, fraction circles, or number lines. Steve was the only participant in the study to make use of fraction models to help him solve the addition/subtraction items. Figure 28 shows Steve's written response to item 18d.

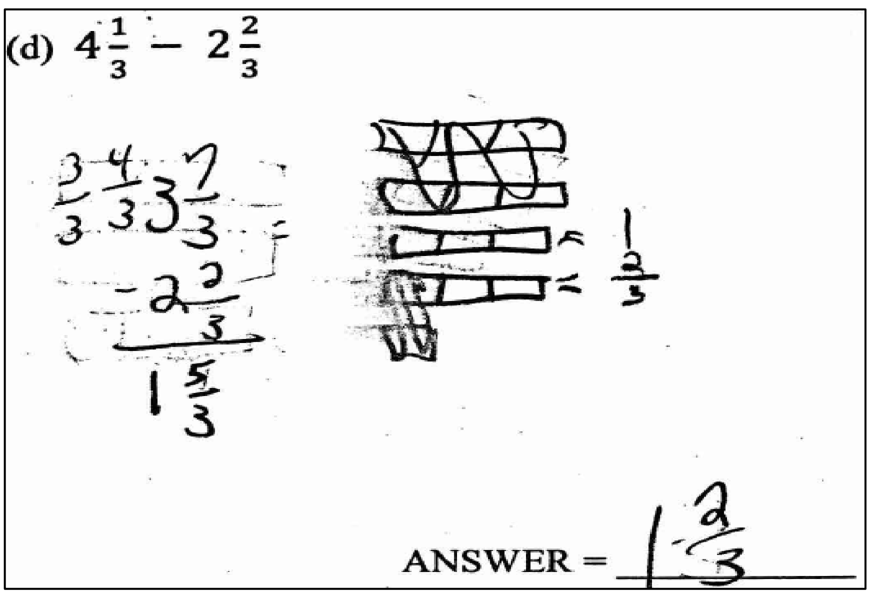


Figure 28. Steve's written response to item 18d using fraction models.

In the interview, the researcher asked Steve to explain the picture and how it helped him arrive at his final answer:

Steve: Yeah, I had my starting value [four and one-third] here so I knew that I had to have four bars, and within those four bars it had three sections.

I: Great.

Steve: And then I just took away two whole bars and two sections within one of those.

I: Great, and then that left you with what?

Steve: One whole . . . one and two-thirds . . . my one whole here and then two-thirds (pointing at the respective bars in his drawing).

Steve's verbal explanation of how he used his fraction bar model depicted at least a part-whole understanding of fractions, as he described the fractions by comparing the number of partitioned parts to the total number of parts in the bar. Furthermore, his drawing indicated that he knew the importance of maintaining a fixed whole when subtracting fractions since he kept the size of the bars the same. What is also interesting to note on this item is that Steve first attempted this problem using only procedures (borrowing and regrouping), but ultimately it was his model and not the procedures that led to the discovery of his correct final answer.

Cross Case Analysis

The findings from the qualitative phase of this study revealed multiple themes pertaining to each participant's conceptions of disembedding and fractions. Some of these themes were shared by two or more of the participants, whereas other themes were unique to a specific individual. One theme that emerged from all three of the participant interviews was the use of the scale on the disembedding items. Although, they each used the scale in different ways and with different outcomes. Both Chris and Steve ignored that the size of the shaded bar was given and instead took each interval of the scale to be a one-unit, resulting in mixed number answers. Steve did so when he initially took the written assessment and still reverted back to this conception once after the researcher pointed out the given value of the shaded bar. Chris, on the other hand, did not ignore the scale during the written assessment but did so once when presented with the second disembedding question of the interview (item 9). Unlike Steve and Chris, Anna relied on the disembedding scale, along with the size of the shaded bar, to determine how to partition the scale in order to find the size of the unshaded bar. On the written assessment, all of Anna's markings on the disembedding items were made on the scales as opposed to the bars themselves.

Another theme that emerged from the qualitative findings was benchmarking, or using common units. Both Chris and Anna relied on units that they were more familiar with, especially halves, to complete various assessment items. For Chris, benchmarking was a way for him to estimate his answers to the disembedding and fraction scheme problems. On nearly all of these items, he explained that he found the middle of the given bar first and then estimated his answers from that point. Anna also repeatedly used benchmarks on the disembedding and fraction scheme items. In addition to halves, Anna used fourths and tenths, which she described as easier to partition. The main difference between Anna and Chris' use of common units was that Anna was much more precise than Chris in her attention to the unit.

A third theme that was revealed through the qualitative analysis was the type of knowledge displayed by the participants. Both Chris and Steve showed evidence of unorganized procedural knowledge of fractions, whereas Anna possessed a contextualized knowledge of fractions. Chris' immediate response on the disembedding items was to try to recall a "formula" to complete them. Additionally, Chris had a perfect score on the fraction addition/subtraction items. Yet he was unable to explain why it was necessary to get a common denominator when adding/subtracting fractions despite being able to perform all of the steps. Steve, like Chris, was also aware of when and how to find a common denominator when adding/subtracting fractions but unable to connect this concept with a real-life experience to give it meaning. Anna's contextualized knowledge of fractions was seen through her cooking example. Because Anna knew halves and thirds, for example, were different measurements, she was able to explain why fractions with unlike denominators cannot be added/subtracted.

Integration of Quantitative and Qualitative Findings

This study's sequential explanatory design was a two-phase design in which the quantitative data collection and analysis came first, followed by the qualitative data collection and analysis. The rationale for this approach was that the quantitative data and results would provide a general picture of how adult developmental mathematics students perform on whole number disembedding and fraction tasks, while the qualitative data and its analysis would help refine and explain the statistical results by exploring participants' knowledge in more depth. Thus, a key aspect of this design is the integration of the results of each phase, whereby the researcher must determine whether and how the qualitative results help to explain the quantitative results. This section of the chapter summarizes and integrates the results of the two phases of this study.

The data collected through the written assessment revealed statistically significant positive correlations between a participant's disembedding score, fraction scheme score, and fraction addition/subtraction score. Specifically, participants with high scores on the disembedding and fraction scheme items tended to have high scores on the fraction addition/subtraction items. Additionally, participants who scored high on the disembedding items also scored high on the fraction scheme items, and vice versa. The results of the multiple linear regression analysis indicated that age, fraction schemes score, disembedding score, and number of repeated mathematics courses were significant predictors of the fraction addition and subtraction score (see Table 14). Specifically, it was found that an increase of two points in the disembedding score was associated with an increase of one point in the fraction addition/subtraction score, when all other independent variables were held constant. Additionally, an increase of three points in the fraction scheme score was associated with an increase of four points in the fraction addition/subtraction score, when all other independent variables were held constant. Thus, participants whose scores did not follow these findings were of interest during the qualitative phase of the study. Chris and Steve represented two such participants, whereas Anna's scores were consistent with the findings of the quantitative phase.

The qualitative interviews revealed multiple themes for each of the three participants. Chris, who scored high on the disembedding and fraction addition/subtraction items and low on the fraction scheme items, demonstrated unorganized procedural knowledge during his interview. He also relied on benchmarks to arrive at most of his disembedding and fraction scheme answers, as opposed to engaging in the expected actions associated with each item. Anna's interview revealed her contextualized knowledge, along with careful attention paid to the unit. While she also used benchmarks to arrive at many of her answers, the use of benchmarking facilitated the actions of disembedding and partitioning. Lastly, Steve, who scored high on the fraction scheme items and low on both the disembedding and fraction addition/subtraction items, also demonstrated unorganized procedural knowledge during his interview. His conception of the whole was inconsistent within and across the various items.

In line with the aim of this study, the qualitative results did help to explain the quantitative results of the interview participants by giving participants the opportunity to clarify and elaborate on their written responses. The researcher was able to understand the actions and cognitive processes that led Chris, Anna, and Steve to their final answers on the written assessment. The clinical interviews of the qualitative phase allowed the researcher to see the order in which students made their markings as well as how and why they determined to make such markings. Benchmarking, for example, was not something that could have been found through the collection and analysis of quantitative data alone. It was only revealed through in-depth interviews with the participants in which they were given the opportunity to explain their thinking to the researcher. For example, Chris correctly answered six out of seven of the disembedding items correctly. Looking at only this score, one might assume that Chris engaged in the actions of disembedding. However, the clinical interview revealed his estimation strategy for arriving at correct answers as opposed to his engagement in the actions of disembedding. Thus, this helps to explain Chris' quantitative results. Specifically, it illustrates how he could have scored high on the disembedding items and low on the fraction scheme items despite their correlation with one another. Similarly, Steve scored high on the fraction scheme items but low on the disembedding and fraction addition/subtraction items. At first glance, it appeared that Steve had constructed the RPFS and the IFS based on his quantitative results. However, once Steve was interviewed about his responses on the disembedding items in particular, it became clear that he had an inconsistent conception of the whole. While he was able to correctly complete the RPFS and IFS items on the written assessment, his shaky conception of the whole provided insight concerning his high fraction scheme score and low disembedding score despite their correlation with one another.

Conclusion

This chapter discussed the findings of both the quantitative and qualitative phases of the study. The multiple regression analysis of the written assessment data indicated that age, fraction schemes score, disembedding score, and number of repeated mathematics courses were all significant predictors of a participant's fraction addition/subtraction score. Analysis of the clinical interviews revealed several themes concerning participants' knowledge of disembedding, fraction schemes, and fraction addition/subtraction, as well as insights that helped to explain the quantitative results. These findings have implications for educators and researchers, which are discussed in Chapter 5.

CHAPTER V

DISCUSSION, IMPLICATIONS, AND RECOMMENDATIONS

The purpose of this study was to examine adult developmental mathematics students' knowledge of fraction addition and subtraction as it relates to their demonstrated fraction schemes and ability to disembed. The study was conducted using a mixed methods sequential explanatory design. In the first phase, 72 developmental mathematics students took a written assessment containing disembedding, fraction scheme, and fraction addition/subtraction items. Based upon the results of the assessment, three individuals from the first phase were selected to participate in one-on-one clinical interviews. These interviews were aimed at identifying and describing the cognitive processes underlying the participants' performance on the written assessment items. Results from the quantitative phase indicated statistically significant moderate correlations between disembedding in multiplicative contexts, demonstrated fraction schemes, and fraction addition/subtraction. Moreover, regression analysis revealed that age, fraction schemes score, disembedding score, and number of repeated mathematics courses were all significant predictors of a participant's fraction addition/subtraction score. Analysis of the clinical interviews revealed several themes concerning participants' knowledge of disembedding, fraction schemes, and fraction addition/subtraction, as well as insights that helped to explain the quantitative results. This chapter begins by examining the study's results in relation to relevant literature. Following the discussion of the findings, limitations of this study are presented. Lastly, the chapter concludes with a discussion of implications for both educators and researchers.

Discussion of Findings

The review of the literature in Chapter 2 was concerned with the learning and conceptualizations of fractions, with a specific focus on fraction schemes, disembedding, and conceptions of the whole. The literature review developed a framework to study adult mathematics learners' knowledge of fractions. Because the research on adult thinking and learning of mathematics, namely fractions, was largely missing from prior literature, the framework developed was also informed by research on the fraction knowledge of children and preservice teachers. Thus, it is important to discuss the findings of this study within the context of prior research on the fraction knowledge of children and PSTs as well as the proposed framework for adult mathematics learners' knowledge of fractions.

Quantitative Findings

Several important findings emerged from the quantitative phase of this study. First, the correlation statistics reported in Chapter 4 indicated a moderate and statistically significant relationship between ADM students' performance on tasks involving: (a) disembedding and fraction addition/subtraction, (b) fraction schemes and fraction addition/subtraction, and (c) disembedding and fraction schemes. These relationships have been hypothesized in prior qualitative studies but not measured quantitatively until the present study. Through the use of teaching experiments, several studies have examined the relationship between disembedding in multiplicative contexts and fraction schemes (Boyce & Norton, 2016; Hackenberg, 2010; Hackenberg & Tillema, 2009; Olive & Vomvoridi, 2006; Steffe, 2004; Steffe & Olive, 2010). These studies have suggested that a child's ability to disembed in multiplicative contexts plays an important role in his or her construction of fraction schemes. For example, the teaching experiments conducted by Steffe and Olive (2010) with fourth- and fifth-grade students suggested that the lack of construction of the disembedding operation severely constrained the construction of the splitting operation and thus the PFS. Additionally, Olive and Vomvoridi's (2006) teaching episodes with a sixth-grade student revealed that the student's lack of the disembedding operation hindered him from developing a PWFS. Therefore, the correlation statistics found in the present study provide confirmation of the hypothesized relationship between disembedding in multiplicative contexts and fraction schemes that has been developed through qualitative research.

Not only is there a quantitatively measured relationship between disembedding in multiplicative contexts and students' demonstrated fraction schemes, this study's correlation statistics and multiple regression analysis indicated that both of these factors are also directly related to how ADM students solve fraction addition and subtraction tasks. These findings confirm and extend prior qualitative research on children's and PSTs' knowledge of fraction addition and subtraction. As discussed in Chapter 2, the common errors made by both children and PSTs suggest a necessary connection between disembedding, fraction schemes, and fraction addition/subtraction. For example, the sixth-grade student in Olive and Vomvoridi's (2006) study who had not yet constructed the disembedding operation or the PWFS was unable to correctly add ½ + ¼.

Specifically, this student did not display an understanding of a fixed unit whole (Olive & Vomvoridi, 2006). Recall that maintaining a fixed unit whole is a key component of the disembedding operation. Students who disembed parts from a whole unit are able to operate on the parts with the understanding that they are part of another whole. In addition to Olive and Vomvoridi, other prior qualitative studies conducted with children and PSTs have reported that fraction addition and subtraction errors occurred when students did not maintain a fixed unit whole (Izsák et al., 2008; McAllister & Beaver, 2012; Toluk-Uçar, 2009).

Prior qualitative research with children and PSTs also suggests that these learners have a deeper understanding of rational numbers and are better able to operate with fractions as they move beyond the part-whole fraction scheme (Steffe & Olive, 2010; Tobias et al., 2015). Recall from Chapter 2 that since the PWFS involves identifying fractions as parts out of a whole without the understanding of the fraction as a unit, it does not provide an adequate way for children to conceive of improper fractions (Hackenberg, 2007; Tzur, 1999). Moreover, the PUFS is the first fraction scheme in which a child is able to unitize a fraction, which is a necessary skill to acquire the remaining fraction schemes (Boyce & Norton, 2016; Steffe & Olive, 2010). Thus, the positive correlation found in this study between the ADM learners' demonstrated fraction schemes and performance on fraction addition/subtraction tasks quantitatively confirms the aforementioned findings. Namely, adult learners who are farther along in the fraction scheme hierarchy are better able to perform on addition/subtraction tasks involving proper fractions, improper fractions, and mixed numbers.

The results from the multiple regression analysis provide additional insight into the confirmed relationship between disembedding in multiplicative contexts, demonstrated fraction schemes, and fraction addition/subtraction. Because of the correlations between disembedding, fraction schemes, and fraction addition/subtraction, conducting the multiple regression analysis allowed the researcher to examine the impact of each of these factors while taking into account the remaining factors. Findings from the multiple regression analysis indicated that age, fraction schemes score, disembedding score, and number of repeated mathematics courses were significant predictors of an adult learner's performance on fraction addition/subtraction tasks. Fraction schemes score and disembedding score had positive weights in the regression model, indicating that participants who scored higher on the fraction scheme and disembedding items tended to have higher scores on the fraction addition/subtraction items. Age also had a positive weight in the regression model, which implies that older students tended to have higher scores on the fraction addition and subtraction items. Recall from the review of the literature that the relationship between an adult student's age and their performance in college mathematics courses is inconsistent. The findings from the present study are consistent with Walker and Plata (2000) who reported that older students were more successful than younger students in lower-level developmental mathematics courses. Although all of the aforementioned variables contributed to the explanation of the variance in the fraction addition/subtraction score, the most powerful predictor was fraction scheme score, followed by age and then disembedding score. The model created to predict performance on fraction addition/subtraction tasks accounted for 45.8% of the variance in this score. The ability of the regression model to account for variance in participants' fraction addition/subtraction score extends the findings of the correlational analysis.

Qualitative Findings

During the qualitative phase of this study, multiple themes emerged from the analysis of the one-on-one clinical interviews. These themes provided additional insight into the cognitive processes of the ADM students who engaged in completing the assessment tasks. As a result, the qualitative findings helped the researcher to explain some of the quantitative results. One of the qualitative themes was participants' use of the disembedding scale. Recall that on the disembedding items, a shaded bar and its length were given. Below the shaded bar was an unshaded bar of unknown length as well as a scale. Students were expected to disembed 1s or composite units from the shaded bar in order to determine the value of the unshaded bar. During their interview, Steve and Chris had difficulty taking the length of the shaded bar as it was given. Furthermore, once the researcher pointed out the value of the shaded bar and asked them to reconsider their answer in light of this information, Steve and Chris both struggled with reconceptualizing the problem in relation to the given length. This process of reconceptualizing a given situation in relation to some fixed unit is known as *norming* (Lamon, 1994). Although Steve and Chris were able to temporarily work with the length of the given bar, they did not completely engage in norming, as they jumped back and forth between the lengths within a single task. Lamon emphasized the importance of norming, along with unitizing, in the development of children's fraction knowledge.

Specifically, she concluded that it is essential for students working with fractions to be able to reinterpret information in terms of different whole units (Lamon, 1994). Research conducted with PSTs has also reported similar findings on the ability of students to engage in the norming process (McAllister & Beaver, 2012; Rosli et al, 2011; Toluk-Uçar, 2009).

Another theme found was benchmarking, or using familiar units, to complete the disembedding and fraction scheme items. This action is associated with the equi-partitioning scheme found through teaching experiments with children (Steffe & Olive, 2010). As Steffe (2004) described,

> The purpose of the equi-partitioning scheme is to estimate one of several equal parts of some quantity and to iterate the part in a test to find whether a sufficient number of iterations produce a quantity equal to the original. (p. 132)

In the present study, both Chris and Anna appeared to engage in the equi-partitioning scheme. Although it was not clear from the written responses, one-on-one interviews with Chris and Anna revealed that fractions such as fourths and halves triggered this scheme. During their interview, both participants described estimating part of a whole and iterating that part in order to test whether or not it would produce the given whole. Similar to the findings of Steffe and Olive (2010), the actions of this scheme emerged without any concentrated attempts by the researcher to provoke the scheme.

A third theme found through the qualitative phase of this study was the type of knowledge displayed by the participants, as it relates to the theoretical framework for adult learners' knowledge of fractions. All three aspects of the framework (contextualized knowledge, unorganized procedural knowledge, and models of fractions) were revealed through the qualitative findings, specifically with respect to the participants' conceptions of fraction addition and subtraction. Chris demonstrated unorganized procedural knowledge when it came to understanding the importance of getting a common denominator when adding/subtracting fractions. He knew it was necessary to do and was able to perform the corresponding operations but he was unaware of why it was necessary. Anna, on the other hand, demonstrated contextualized knowledge in her understanding of fraction addition/subtraction. She had generated her own explanation, related to her everyday life, as to why getting a common denominator was necessary when adding/subtracting fractions. Lastly, Steve made use of fraction bar models in his attempt at solving fraction addition/subtraction tasks. These models helped him to correctly solve problems that he was unable to solve through direct computation. Although the purpose of this study was not to formally test the theoretical framework, the study's results do verify the role that its aspects play in the fraction knowledge of adult mathematics learners.

Limitations

Along with the findings of this study, some limitations also must be addressed. One limitation of this study is the time (as much as two weeks) that passed between the participants' completion of the assessment and their follow-up interviews. In some cases, because of this lapse in time, participants had difficulty remembering what they were thinking when they initially solved the problem on the assessment. Thus, although their interview responses did in fact portray their cognitive processes, it is possible that the processes they described in the interview were not the same as the ones elicited at the time of the written assessment.

A second limitation, related in part to the first limitation, was the inability to see students interact with the assessment items at the time the assessment was administered. For example, Steve in particular was very sporadic with his approaches to problems of the same type. On the fraction addition/subtraction items, he would sometimes get a common denominator and other times would not. Similarly, he would draw a picture for some of the items but not for others. Without having seen him complete the written assessment and the order in which he attempted the tasks, it was not always clear how or why he chose the strategies that he did since this was something he could not recall during the interview.

Another limitation of this study was the sample size of the qualitative phase. Although there is no clear standard for an “acceptable” number of participants in the qualitative phase of a sequential explanatory design, the present study was missing certain viewpoints. In particular, students who scored high on disembedding and fraction scheme items but low on fraction addition/subtraction items were not interviewed. Also missing from the interviews were students who scored low on disembedding and fraction scheme items but high on fraction addition/subtraction items. Having these viewpoints, along with the viewpoints of other students whose scores countered the trend, could have given a more in-depth explanation to the quantitative results.

Implications and Future Research

The findings from this study have implications for adult developmental mathematics educators as well as for researchers in the field of ADM education. Instructors of ADM courses can use the results of this study in several ways in order to understand the cognitive processes of their students. By doing so, these instructors will be better equipped to help their students understand fraction addition and subtraction. First, it is important for ADM instructors to be aware that their students' demonstrated fraction schemes and ability to disembed predicts their performance on fraction addition/subtraction tasks. Because of this, instructors should take time to determine which fraction scheme(s) their students have attained as well as if and how their students can disembed. Knowing the fraction scheme(s) their students have demonstrated provides instructors with the opportunity to engage their students in actions associated with higher-level fraction schemes. Using tools, such as JavaBars (Biddlecomb & Olive, 2000), instructors have the ability to help their students construct operations such as partitioning, disembedding, and iterating that are essential for constructing higher-level fraction schemes. Additionally, ADM instructors should also be aware of the aspects of the framework for adult learners' knowledge of fractions. Recognizing that students possess different aspects of fraction knowledge in light of their prior mathematical and life experience is critical for determining an adult learner's needs and abilities.

In some instances, qualitative results revealed that a participant did not engage in the expected actions associated with a task despite having correctly answered the task on the assessment. One example of this was Chris, whose ability to benchmark and estimate on the disembedding items caused him to get most of these items correct, despite not having shown evidence of disembedding through his markings or interview. Another example of this was Steve's ability to successfully complete the RPFS and the IFS items without explicitly having demonstrated the actions associated with these schemes. Thus, future studies in adult developmental mathematics education should address this phenomenon in both the disembedding and fraction scheme items.

As reported earlier in this chapter, all three participants made use of the disembedding scale in different ways and to different degrees. Steve struggled with norming more than Chris, and Anna seemed to have the most complete notion of norming out of the three participants. Research on children's multiplicative concepts, which distinguish between anticipatory schemes versus those constructed in activity, could be a way to explain these results (Kosko, in press). Specifically, future studies might use interviewing as a way to understand the point at which different types of students engage in the norming process. Namely, is the norming operation constructed by a student while working on a task or has it already been constructed prior to engaging in the task? Schemes that are constructed in activity versus prior to (anticipatory) tend to limit a student's flexibility when completing a task, as the student is focused on constructing the scheme (Kosko, in press).

Because the field of research on ADM learners' knowledge of fractions is so limited, future research has many opportunities. The present study was aimed at describing the cognitive processes of ADM learners. Future research with this same goal in mind should consider using an in-depth case study or teaching experiment to examine learners' cognitive processes for disembedding and fraction schemes separately. Perhaps such studies might also want to examine how participants approach disembedding and fraction scheme items differently when given alternative models (e.g., Cuisenaire rods, sticks, etc.). Recall that both Chris and Steve were able to complete a PFS task when given a stick (as in item 15b) but not when given a length (bar) model. This is consistent with Wilkins et al. (2013) who reported that some students "seem to have an established way of operating that includes the ability to solve some (but not all) tasks for which construction of a PFS is theoretically both necessary and sufficient" (p. 41). Therefore, future research should continue to address if and how different models elicit ADM learners' disembedding and fraction schemes.

Conclusion

The motivation behind this study was to learn more about how adult developmental mathematics students think about fractions. Specifically, it sought to examine the hypothesized relationship between ADM learners' ability to disembed, their demonstrated fraction schemes, and their performance on fraction addition and subtraction tasks that has been reported in prior qualitative studies. The present study has contributed to this body of literature by quantitatively measuring these hypothesized relationships through correlational and multiple regression analyses. The findings from this study found correlation statistics that confirm a statistically significant relationship between: (a) disembedding and fraction addition/subtraction, (b) fraction schemes and fraction addition/subtraction, and (c) disembedding and fraction schemes. Moreover, the multiple regression analysis revealed ADM learners' performance on disembedding and fraction scheme tasks to be statistically significant predictors of their performance on fraction addition and subtraction tasks. Furthermore, the qualitative findings of this study clarified and elaborated on the quantitative results. Specifically, interviews showed that norming and the equi-partitioning scheme play an important role in ADM learners' conceptions of fractions. Instructors of ADM courses should use the results presented in this study as indication of the importance of determining their students' existing schemes and providing them with opportunities to engage in actions associated with higher level schemes.

APPENDICES

APPENDIX A

ASSESSMENT

Appendix A

Assessment

Whole Number and Fraction Assessment

Introduction: This assessment is comprised of three parts.

Part I contains demographic questions.
Part II contains problems involving whole numbers.
Part III contains problems involving fractions, including fraction addition and subtraction.

Please read the directions for each part carefully, and show all of your work for each problem in Parts II and III. Your effort on this assessment is greatly appreciated!

Part I: Demographics

Name: ____

Email Address: ____

Major: ________________________________ Campus: ____________________

Age: ____________________ Gender: ____________________

Current Math Course Information

1. What math course are you currently enrolled in?

____

2. Not including this attempt, how many times have you attempted this course in the past?

____

Previous Math Course Information

3. Prior to your current course, what is the last math course you successfully completed (C or higher grade)?

____

4. When did you take this course (mm/yyyy)? ________ /_____________

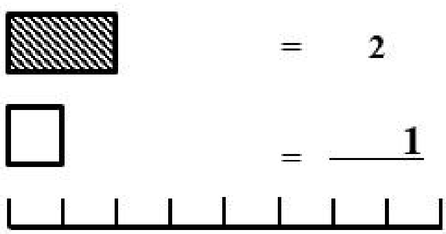
5. How many times did you attempt this course? _________________________________

Part II: Whole Number Problems

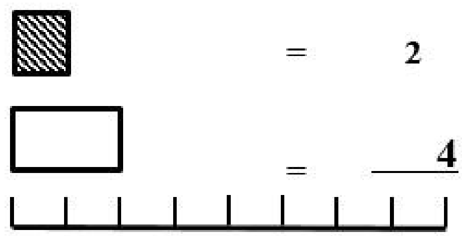
For each problem #6-12, you are asked to find how long the white rod is based on how long the shaded rod is. The problems on the same page may be very different, and you will need to pay attention to one problem at a time. The sample items below show correct answers to two different kinds of problems you might see in the next pages. For each problem, **show your work and write your answer** in the blank for the white rod.

Sample Items

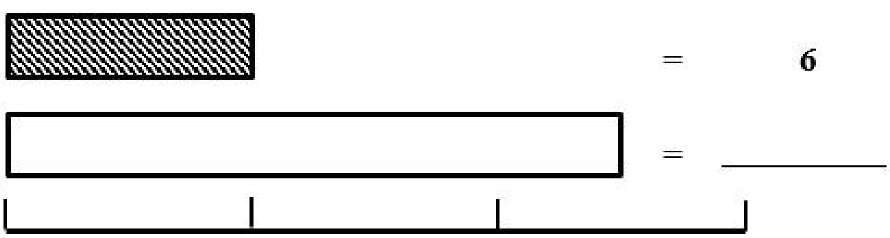
Sample Item 1



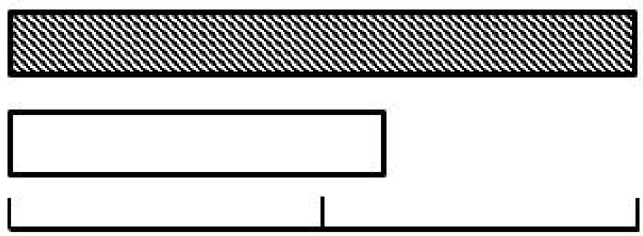
Sample Item 2



6.



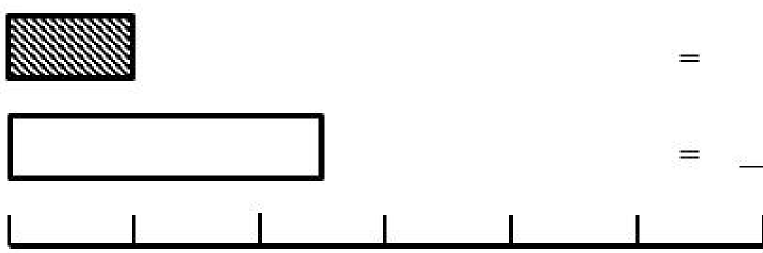
7.



= **10**

= ________

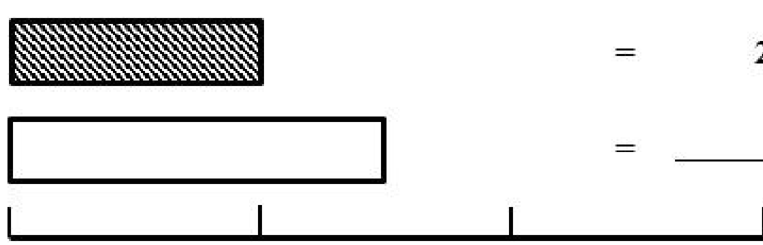
8.



= **4**

= ________

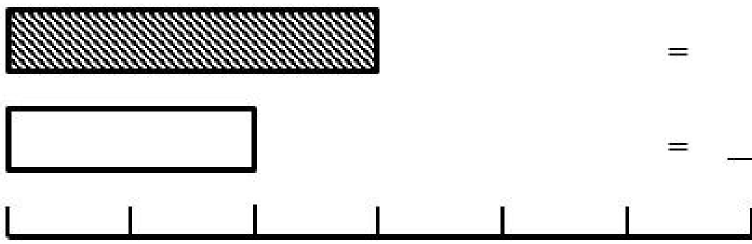
9.



= **2**

= ________

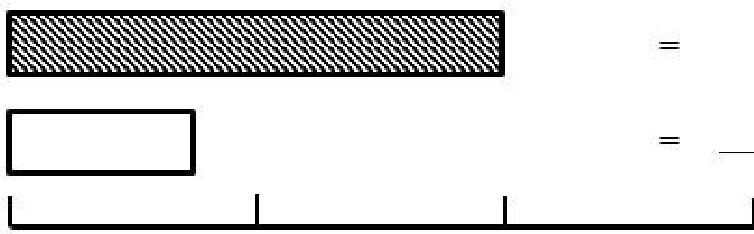
10.



= **24**

= ________

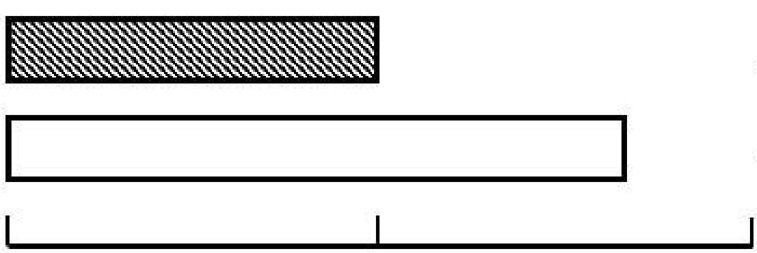
11.



= **16**

= ________

12.



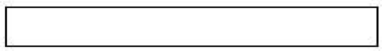
= **3**

= ________

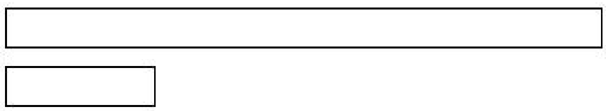
Part III: Fraction Problems

For each problem #13-21, the questions pertain to fractions, including fraction addition and subtraction. For each problem, **show your work and write your answer** in the blank where applicable.

13. **Shade** ¾ of the bar below.

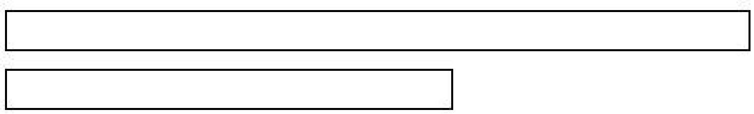


14. If the top (longer) bar is one whole, what fraction of the whole is the shorter bar?



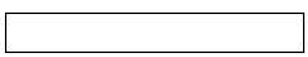
ANSWER = ___________

15. If the top (longer) bar is one whole, what fraction of the whole is the shorter bar?

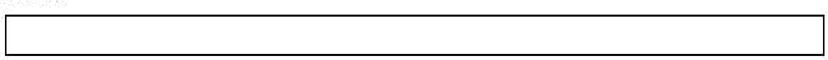


ANSWER = ___________

16. The bar shown below is 4/5 as long as another bar. **Draw** the other bar which represents one whole.



17. The bar shown below is 4/3 as long as another bar. **Draw** the other bar which represents one whole.



18. Use any method to solve the following. Show all of your work.

(a) $\frac{3}{5} + \frac{1}{5}$

ANSWER = ___________

(b) $2\frac{3}{8} + 3\frac{1}{8}$

ANSWER = ___________

(c) $7\frac{5}{6} - 1\frac{1}{6}$

ANSWER = ___________

(d) $4\frac{1}{3} - 2\frac{2}{3}$

ANSWER = ___________

(e) $\frac{1}{2} + \frac{1}{3}$

ANSWER = ___________

(f) $\frac{3}{4} - \frac{1}{6}$

ANSWER = ___________

(g) $\frac{3}{4} + \frac{5}{8}$

ANSWER = ___________

(h) $1\frac{2}{7} + 3\frac{1}{2}$

ANSWER = ___________

(i) $3\frac{7}{9} - 2\frac{1}{2}$

ANSWER = ___________

(j) $5\frac{2}{15} - 2\frac{3}{5}$

ANSWER = ___________

19. Cynthia is making her famous "Perfect Punch" for a party. After looking through the recipe, Cynthia knows that she needs to mix 5/8 gallons of fruit juice concentrate with 7/8 gallons of sparkling water. Just as she is about to get started, she realizes that she only has a 1-gallon container to use for mixing. Will this container be big enough to hold all the ingredients? Explain your answer using words and/or pictures.

20. Alex is training for a race and needs to run at least 1 mile per day. If Alex runs to his grandma's house, which is 5/8 of a mile away, and then to his friend Justin's house, which is 1/2 of a mile away, will he have trained enough for the day? Explain your answer using words and/or pictures.

21. The sum of 1/12 and 7/8 is closest to which of the following numbers? (Circle one choice.)

A. 20

B. 8

C. ½

D. 1

APPENDIX B

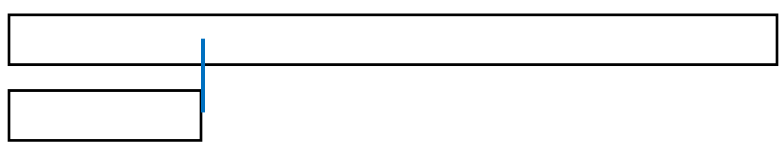
CLINICAL INTERVIEW PROTOCOL

Appendix B

Clinical Interview Protocol

Interviews can help to clarify students' thoughts and articulate what cannot be captured through markings on a drawing alone. For example, consider a student's response to item #14 shown below. Here the student has drawn one vertical line between the longer bar and the shorter bar and has given an answer of ¼. With only this single marking, it does not clearly indicate how she used this mark to arrive at the correct answer of ¼. For example, did the student iterate the shorter piece four times to conclude it was ¼ of the whole? Or, alternatively, did the student partition the whole into various sized pieces until the size of the partitioned piece matched the size of the shorter bar?

14. If the top (longer) bar is one whole, what fraction of the whole is the shorter bar?



ANSWER = ¼

Interview:

Provide a copy of the participant's assessment response on a single sheet of paper.

(a) How did you decide where to make your mark?
(b) Can you tell me how you determined your answer to be ¼?
(c) How could you check that the shorter bar is ¼ of the whole?

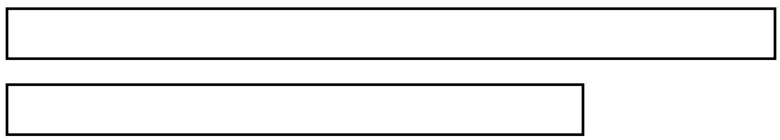
APPENDIX C

ADDITIONAL FRACTION SCHEME INTERVIEW ITEMS

Appendix C

Additional Fraction Scheme Interview Items

15a. If the top (longer) bar is one whole, what fraction of the whole is the shorter bar?



ANSWER = ____________

15b. Your stick is 3/5 as long as the stick shown below. Draw your stick.

APPENDIX D

CONSENT FORM

Appendix D

Consent Form

KENT STATE UNIVERSITY

Informed Consent to Participate in a Research Study

Study Title: *Adult Learners' Knowledge of Fraction Addition and Subtraction*

Principal Investigator: *Karl Kosko*
Co-Investigator: *Nicole Muckridge*

You are being invited to participate in a research study. This consent form will provide you with information on the research project, what you will need to do, and the associated risks and benefits of the research. Your participation is voluntary. Please read this form carefully. It is important that you ask questions and fully understand the research in order to make an informed decision. You will receive a copy of this document to take with you.

Purpose
The purpose of this study is to understand how adult developmental mathematics students think about fraction addition and subtraction. Specifically, this study will examine the relationship between students' knowledge of whole numbers and students' performance and understanding on fraction addition/subtraction tasks. It will also examine the relationship between students' demonstrated fraction schemes and students' performance and understanding on fraction addition/subtraction tasks. Finally, this study aims to understand the ways in which students' prior experiences have informed their fractional knowledge.

Procedures
If you choose to participate in this study, you will be asked to complete a 16-question assessment related to whole numbers and fraction addition/subtraction. This will last no more than 30 minutes and will be conducted during your regularly scheduled mathematics class.

After participating students have completed the assessment, a small sample of students will be contacted (via email) to seek participation in an interview. Interviews are a separate part of the study, with only a few interviews planned (less than 10). Those receiving the email will be asked to participate in an interview scheduled outside of their regular mathematics class and lasting no more than 45 minutes.

Interviews will be audio and video recorded, and the interviewer will take notes.

Audio and Video Recording and Photography
The interview will be audio and video recorded, and this raw data will not be accessible to anyone besides the researchers. If you desire to watch or listen to the recordings, you are welcome to do so. The audio recordings will be transcribed by the researchers, and these transcriptions will then be used for analysis. Likewise, the researchers will use to video recordings to make observations and take notes. The audio and video files will not be shared outside of the research group.

KENT STATE UNIVERSITY

No identifying information will be placed on any artifacts, interviews, or transcripts; pseudonyms will be used throughout to assure confidentiality. It is our intention that data from this study may be presented at an educational research conference or included as part of a journal article. If you consent to take part in this study, you are indicating agreement that your data can be used in each of these situations – I would once again like to assure you that no identifying information will be used in the write-up of the study, and there will be no follow-up requirements on your part unless you wish to do so.

Benefits
No compensation is being offered for participation in this study. The potential benefits of participating in this study may include a deeper understanding of fractions and/or fraction addition/subtraction. Your participation in this study will also help us to better understand adults' conceptions of fraction addition/subtraction, especially as they related to whole number understanding.

Risks and Discomforts
There are no anticipated risks beyond those encountered in everyday life.

Privacy and Confidentiality
Your study related information will be kept confidential within the limits of the law. Any identifying information, including audio and video tapes, will be kept in a secure location and only the researchers will have access to the data. Research participants will not be identified in any publication or presentation of research results.

Your research information may, in certain circumstances, be disclosed to the Institutional Review Board (IRB), which oversees research at Kent State University, or to certain federal agencies. Confidentiality may not be maintained if you indicate that you may do harm to yourself or others.

Voluntary Participation
Taking part in this research study is entirely up to you. You may choose not to participate or you may discontinue your participation at any time without penalty or loss of benefits to which you are otherwise entitled. Deciding to participate or not will not impact your grades/class standing/relationship to the institution. You will be informed of any new, relevant information that may affect your health, welfare, or willingness to continue your study participation.

Participant Contact Information
If you agree to participate in this research project, your name and university email address will be collected and retained for possible participation in interviews for the second phase of this study. By signing this section of the consent form, you are authorizing the study investigators to retain your name and university email address, as well as to possibly contact you regarding participation in a follow-up interview.

____________________________ ____________________
Participant Signature **Date**

KENT STATE
UNIVERSITY

Investigator Contact Information

If you have any questions or concerns about this research, you may contact Nicole Muckridge at 330-675-8910 or Karl Kosko at 330-672-0660. This project has been approved by the Kent State University Institutional Review Board. If you have any questions about your rights as a research participant or complaints about the research, you may call the IRB at 330.672.2704.

Consent Statement and Signature

I have read this consent form and have had the opportunity to have my questions answered to my satisfaction. I voluntarily agree to participate in this study. By signing this consent form, I affirm that I am 18 years or older. I understand that a copy of this consent will be provided to me for future reference.

______________________________ ____________________

Participant Signature **Date**

APPENDIX E

AUDIO/VIDEO CONSENT FORM

Appendix E

Audio/Video Consent Form

KENT STATE UNIVERSITY

AUDIOTAPE/VIDEO CONSENT FORM

ADULT LEARNERS' KNOWLEDGE OF FRACTION ADDITION AND SUBTRACTION
KARL KOSKO, PRINCIPAL INVESTIGATOR
NICOLE MUCKRIDGE, CO-INVESTIGATOR

I agree to participate in an audiotaped/videotaped interview about whole numbers and fractions as part of this project and for the purposes of data analysis. I agree that Karl Kosko and/or Nicole Muckridge may audiotape/videotape this interview. The date, time, and place of the interview will be mutually agreed upon.

____

Signature Date

I have been told that I have the right to listen to the recording of the interview before it is used. I have decided that I:

____want to listen to the recording ____do not want to listen to the recording

Sign now below if you do not want to listen to the recording. If you want to listen to the recording, you will be asked to sign after listening to them.

____

Signature Date

Teaching, Learning and Curriculum Studies
P.O. Box 5190 • Kent, Ohio 44242-0001
Phone: (330) 672-2580 • Fax: (330) 672-3246 • http://www.ehhs.kent.edu/tlc/

APPENDIX F

ASSESSMENT RECRUITMENT SCRIPT

Appendix F

Assessment Recruitment Script

Recruiting Script – Assessment

Study Title: *Adult Learners' Knowledge of Fraction Addition and Subtraction*

Hi. I am conducting research on adult students' understanding of fraction addition and subtraction. You were selected to participate in this study because you are enrolled in a developmental mathematics course. Participants must be age 18 or older. Please indicate if you do not meet this criterion.

If you choose to participate in this study, you will be asked to complete a 16-question assessment related to whole numbers and fraction addition/subtraction. This will last approximately 30 minutes or less and will be conducted during your regularly scheduled mathematics class. You might be selected to participate in one additional follow-up interview regarding your responses on the assessment. This interview will last approximately 45 minutes or less and will be conducted outside of your regularly scheduled mathematics class.

Your participation in this study is greatly appreciated. No compensation is being offered for participation in this study. The potential benefits of participating in this study may include a deeper understanding of fractions and/or fraction addition/subtraction. Your participation in this study will also help us to better understand adults' conceptions of fraction addition and subtraction, especially as they relate to whole number understanding. There are no anticipated risks beyond those encountered in everyday life.

Participation is strictly voluntary, and you may refuse to participate at any time. Deciding to participate or not will not impact your grades/class standing/relationship to the institution. If you are interested in participating, please read and sign the consent form I will provide.

The information you provide us will be kept strictly confidential and will not be shared with anyone outside of the research group. If you have any questions regarding the nature of this study, or your participation in it, please feel free to contact me at (330) 675-8910 or the Principal Investigator, Asst. Prof. Karl Kosko, at (330) 672-0660. Thank you.

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APPENDIX G

INTERVIEW RECRUITMENT SCRIPT

Appendix G

Interview Recruitment Script

Recruitment Email Text – Interviews

Study Title: *Adult Learners' Knowledge of Fraction Addition and Subtraction*

Hello,

Thank you again for your recent completion of the Whole Number and Fraction Assessment. Based on your responses on this assessment, you have been selected to participate in a follow-up interview regarding these responses. This interview will last approximately 45 minutes or less and will be conducted outside of your regularly scheduled mathematics class.

Once again, your participation in this study is greatly appreciated. The potential benefits of participating in this study may include a deeper understanding of fractions and/or fraction addition/subtraction. Your participation in this study will also help us to better understand adults' conceptions of fraction addition and subtraction, especially as they relate to whole number understanding. There are no anticipated risks beyond those encountered in everyday life. All participants who complete this follow-up interview will receive a $20 Barnes and Noble gift card.

Participation is strictly voluntary, and you may refuse to participate at any time. Deciding to participate or not will not impact your grades/class standing/relationship to the institution. If you are interested in participating, please **reply to this email** indicating so. Also, please read and sign the attached audiotape/video tape consent form.

The information you provide us will be kept strictly confidential and will not be shared with anyone outside of the research group. If you have any questions regarding the nature of this study, or your participation in it, please feel free to contact me at (330) 675-8910 or the Principal Investigator, Asst. Prof. Karl Kosko, at (330) 672-0660.

Thank you,
Nicole Muckridge

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