COMBINATORIAL AND DISCRETE PROBLEMS IN CONVEX GEOMETRY

A dissertation submitted to
Kent State University in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy

by

Matthew R. Alexander

September, 2017

Dissertation written by

Matthew R. Alexander

B.S., Youngstown State University, 2011

M.A., Kent State University, 2013

Ph.D., Kent State University, 2017

Approved by

Dr. Artem Zvavitch, Co-Chair, Doctoral Dissertation Committee

Dr. Matthieu Fradelizi, Co-Chair

Dr. Dmitry Ryabogin, Members, Doctoral Dissertation Committee

Dr. Fedor Nazarov

Dr. Feodor F. Dragan

Dr. Jonathan Maletic

Accepted by

Dr. Andrew Tonge, Chair, Department of Mathematical Sciences

Dr. James L. Blank, Dean, College of Arts and Sciences

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGEMENTS

The list of those to whom I owe a debt of gratitude during my journey to this point is quite long, and if I were to write out every act of kindness or aid given I would quite likely exceed in length the content of this dissertation several times over. So instead I shall briefly give thanks to those without whom I would not have been able to achieve what I have, and beg forgiveness of those I may have unintentionally overlooked or omitted.

First, I would like to thank my advisers, Artem and Matthieu. I could not have asked for a more patient and kind adviser in Artem, and yet I found another in Matthieu. Always available to answer questions and provide help, and always encouraging me along the way, Artem showed me what a wonderful world research in mathematics can be, introducing me to many amazing individuals, including Matthieu who, along with his gracious family, made my time in Paris such an incredible experience.

I would also like to acknowledge the Chateaubriand Fellowship of the Office for Science & Technology of the Embassy of France in the United States and The Centre National de la Recherche Scientifique funding visiting research at Université Paris-Est Marne-la-Vallée for my time spent in Paris. My time in Paris was not only extremely educational and productive for me as a mathematician, but also personally a life changing experience. Further affording me the opportunity to meet many talented individuals, I would like to acknowledge the National Science Foundation and Kent State University for providing travel funding for many conferences, workshops, and meetings. I would also like to thank Dr. Tonge and the Department of Mathematical Sciences at Kent State for further support of my professional development.

My time at both Kent State and Youngstown State were made all the more enjoyable and fulfilling by the encouraging professors and supportive staff in each department. Moreover, the amazing friends and colleagues that I met made each department feel like a second home.

Finally, I'd like to thank my wife, Jessica, and the rest of my family for their support during some of the most difficult and stressful times of my life (so far). It was often difficult living two hours (or an ocean) apart, but we made it.

NOTATION

List of Notation

Let A, B be non-empty subsets of $\mathbb{R}^d$, $x, y, z \in \mathbb{R}^d$, $\alpha, \lambda \in \mathbb{R}$, $K, L \subset \mathbb{R}^d$ convex bodies, f, g real valued functions, Γ, Λ lattices. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product.

- $A + B = \{a + b : a \in A, b \in B\}$
- $\lambda A = \{\lambda a : a \in A\}$
- $\mathrm{conv}(A)$ is the minimal convex set containing A
- $\mathrm{span}\,(A)$ is the minimal linear subspace containing A
- $\mathrm{aff}(A)$ is the minimal affine subspace containing A
- $\mathrm{int}(A) = \{a : \left(a + \varepsilon B_2^d\right) \subset A \text{ for some } \varepsilon > 0\}$
- $\dim(A)$ is the dimension of $\mathrm{span}\,(A - A)$
- $\mathrm{diam}(A) = \sup\{|a| : a \in A - A\}$
- $|x| = \sqrt{\langle x, x \rangle}$
- $\lfloor \alpha \rfloor$ the integer part of $\alpha \in \mathbb{R}$
- $\{\alpha\}$ the fractional part of $\alpha \in \mathbb{R}$
- $|A|$ is the d-dimensional Lebesgue measure of A.
- $\mathrm{vol}_k(A)$ is the k-dimensional Lebesgue measure of A.
- $\#A$ is the cardinality of the set A
- ∂A denotes the boundary of A
- $A^{\perp} = \{x \in \mathbb{R}^d : \langle a, x \rangle = 0 \text{ for all } a \in A\}$

- $A^z = \{x \in \mathbb{R}^d | \langle a - z, x - z \rangle \leq 1 \text{ for all } a \in A\}$

- $\mathcal{P}(K) = \inf\{|K||K^z| : z \in \mathrm{int}(K)\}$

- $s(K)$ is the unique point such that $|K^{s(K)}| = \min_{z \in \mathrm{int}(K)} |K^z|$

- $f(x) = O(g(x))$ if $\limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty$

- $f(x) = o(g(x))$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

- $B_2^d = \{x \in \mathbb{R}^d : |x| \leq 1\}$

- $B_\infty^d = \{x \in \mathbb{R}^d : \langle x, e_i \rangle \leq 1 \text{ for } 1 \leq i \leq d\}$

- $B_1^d = \{x \in \mathbb{R}^d : \sum_{i=1}^n \langle x, e_i \rangle \leq 1\}$

- $B_p^d = \{x \in \mathbb{R}^d : \sum_{i=1}^n |\langle x, e_i \rangle|^p \leq 1\}$ for $0 < p < \infty$

- $\mathcal{K}^d$ the family of all convex bodies in $\mathbb{R}^d$

- $\mathcal{K}_0^d$ the family of all origin-symmetric bodies in $\mathbb{R}^d$

- $\Pi(\Gamma) = \{\sum_{i=1}^d \alpha_i u_i : 0 \leq \alpha_i < 1 \in \mathbb{R}^d \text{ for } 1 \leq i \leq d\}$ where $\{u_i\}_{i=1}^d$ is a basis of Γ

Part I

Introduction

CHAPTER 1

This Thesis

The main topics of this thesis are Convex and Discrete Geometry. These results come from attempts to find versions of classical facts from Convex Geometry and Geometric Tomography in discrete and non-linear settings. My interest in this area began during my time as an undergraduate in a summer REU where, under the supervision of Dimitry Ryabogin, we explored a discrete version of Aleksandrov's uniqueness theorem. Our exploration followed a call by Gardner, Gronchi, and Zong in [24] to begin bringing the theory of discrete tomography in line with the fuller theory of continuous geometric tomography.

Geometric Tomography concerns the reconstruction of objects from incomplete data such as projections or sections. The objects of interest are often taken to be convex bodies so that the tools of convex geometry may be applied. There is a rich theory for continuous convex bodies in $\mathbb{R}^d$. However, much less has been done for discrete convex lattice sets in $\mathbb{Z}^d$. Here, a convex body is a compact convex set with non-empty interior, and a convex lattice set is a set of points equal to the intersection of $\mathbb{Z}^d$ with a convex body.

A large reason for this current disparity is that many of the basic results for continuous convex bodies fail in the discrete setting. For example, for two convex bodies in $\mathbb{R}^d$ their Minkowski sum is convex. However, in the discrete case this does not remain true (see figure 1.1). Another example is Brunn's theorem which says that for any origin-symmetric convex body and given any unit vector the section of largest volume perpendicular to the unit vector passes through the origin (see figure 4.1). We can formulate this theorem by taking volume to be the cardinality of a discrete set, but we again find that the theorem does not hold.

In chapter 5 we will explore a discrete question that follows from questions relating to the isomorphic Busemann-Petty problem based on joint works with Artem Zvavitch and Martin Henk [3]. In particular we will find that for a convex origin symmetric body K the discrete volume of the largest slice of K is larger than the discrete volume of K up to a constant depending only on the dimension. We will also find the best possible bound in the case of unconditional bodies still depends on the dimension, and generalize the result to slices of higher co-dimension.

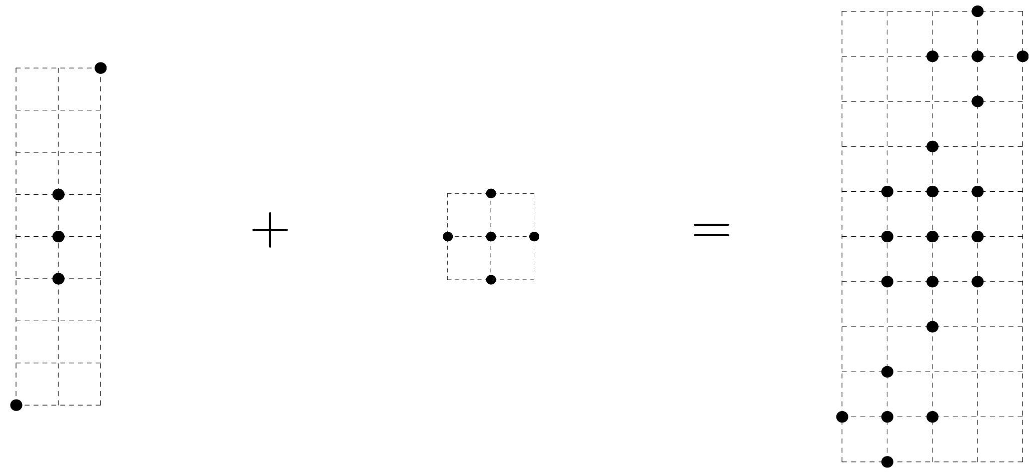


Figure 1.1: Minkowski sum of convex lattice sets need not be convex

Another well known open problem is the conjecture of Mahler related to the minimum value of the volume product of a convex body. In chapter 7 we investigate its maximum for several cases for the class of polytopes with less than a certain number of vertices. In particular, we give a new proof of a result from [64] that the regular N-gon is the polygon of maximal volume product for all polygons with N vertices. We will also study the maximal bodies in the class of convex polytopes with $d+2$ vertices, and symmetric polytopes with $2d+4$ vertices. In chapter 9 we explore a more discrete version of the volume product that comes from associating the space of Lipschitz functions over a metric space to a symmetric polytope with conditions on its vertices, called the unit ball of the Lipschitz-free space. We study the maximal body in this setting in dimension two, and the minimal body in dimension three. These sections are based on joint work with Artem Zvavitch and Matthieu Fradelizi [1, 2].

We will provide an overview of the pertinent background information and main theorems to be used throughout the paper in chapter 2. In chapters 3 and 4 we will discuss further results related to our exploration of the discrete slicing problem in chapter 5. In chapter 6 we will explore the main techniques for two of the following chapters, chapters 7 and 9, which explore questions surrounding volume product of polytopes.

CHAPTER 2

Preliminaries

2.1 Convex Geometry

We will begin with a review of classical theorems and notation in Convex Geometry. These can be found in [23, 45, 14, 79, 83]. We will work primarily in the usual Euclidean setting of finite dimension d, denoted $\mathbb{R}^d$, equipped with the usual inner product $\langle \cdot, \cdot \rangle$ with the standard basis vectors of $\mathbb{R}^d$ denoted by $e_1, \ldots, e_d$. By this we mean that every element $x \in \mathbb{R}^d$ can be expressed uniquely as the sum of scalar products of the basis vectors.

We start with the definition of a convex body which will be the primary object of study. The most direct definition traditionally used is thus: A set $K \subset \mathbb{R}^d$ is *convex* if for every two points $x, y \in K$ the line segment connecting x and y is contained in K. This can be given more precisely in the following way: Let $\lambda \in \mathbb{R}$, then K is convex if $\lambda x + (1-\lambda)y \in K$ for all $x, y \in K$ and $0 < \lambda < 1$. Moreover, for our purposes we will want the additional requirements that K is non-empty, compact, and equal to the closure of its interior. This we will call a *convex body.* If a set X is not convex, we may make it convex by taking the *convex hull* of X, i.e.

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{m} \lambda_i x_i : x_i \in X, \lambda_i > 0 \text{ for all } i, \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

The convex hull of X is the smallest convex set containing X.

K is *origin-symmetric* if $K = -K$, where $\lambda K = \{\lambda x : x \in K\}$, for $\lambda \in \mathbb{R}$, and *unconditional* if it is symmetric with respect to every coordinate hyperplane, that is if $x = (x_i)_{i=1}^d \in K$ then $(\epsilon_i x_i)_{i=1}^d \in K$ for any choice of signs $\epsilon_i \in \{-1, 1\}$. We denote $\mathcal{K}^d$ the family of all convex bodies in $\mathbb{R}^d$, and $\mathcal{K}_0^d$ the family of all origin-symmetric convex bodies. For a set K we denote by $\dim(K)$ its *dimension*, that is, the dimension of the affine hull of K. Where the *affine hull* is given by

$$\mathrm{aff}(X) = \left\{ \sum_{i=1}^{m} \alpha_i x_i : x_i \in X, \alpha_i \in \mathbb{R} \text{ for all } i, \text{ and } \sum_{i=1}^{m} \alpha_i = 1 \right\}.$$

We define $K + L = \{x + y : x \in K, y \in L\}$ to be the *Minkowski sum* of $K, L \subset \mathbb{R}^d$. We will also denote by vol_d the d-dimensional Hausdorff measure, and if the body K is d-dimensional we will call $\mathrm{vol}_d(K)$ the *volume* of K. We will often write $|K| = \text{vol}_d(K)$ or omit d when the context is clear. Let us denote by $\xi^\perp$ a hyperplane perpendicular to a unit vector ξ, i.e.

$$\xi^\perp = \{x \in \mathbb{R}^d : \langle x, \xi \rangle = 0\}.$$

We denote the set of all vectors of length one in dimension d by S^{d-1} the unit sphere.

A commonly studied family of unconditional convex bodies are the ℓ_p balls in dimension d given by

$$B_p^d = \left\{ x \in \mathbb{R}^d : \sum_{i=1}^d |\langle x, e_i \rangle| \leq 1 \right\} \quad \text{for } 1 \leq p < \infty.$$

In particular, we will regularly reference the euclidean sphere, B_2^d, and cross-polytope B_1^d. For $p = \infty$ we define the unit cube by

$$B_\infty^d = \left\{ x \in \mathbb{R}^d : \max_i \left(|\langle x, e_i \rangle| \leq 1 \right) \right\}.$$

Another commonly studied family of convex bodies are polytopes. One way to define a polytope is as the convex hull of a finite set of points. If these points are also *extreme points*, points that do not fall within any line segment inside the body, then they are called vertices. Equivalently, a polytope is the intersection of a finite number of half-spaces which is bounded. It is often easier and more convenient to work with polytopes when studying the properties of convex bodies due to their discrete nature.

2.2 Basic functions and their properties

We define the *Minkowski Functional* of K in $\mathbb{R}^d$ for a convex body $K \subset \mathbb{R}^d$ and for all $x \in \mathbb{R}^d$ as

$$||x||_K = \min\{\alpha \geq 0 : x \in \alpha K\}.$$

For any origin-symmetric convex body this functional is a norm on $\mathbb{R}^d$. The radial function of K on the sphere S^{d-1} is given by

$$\rho_K(u) = \max\{r > 0 : ru \in K\} = ||u||_K^{-1}$$

for all directional vectors $u \in S^{d-1}$. The support function of K on the sphere S^{d-1} is given by

$$h_K(u) = \max\{\langle u, y \rangle : y \in K\}$$

for all directional vectors $u \in S^{d-1}$. We call the hyperplane in the direction u that first "touches" the body K the *supporting hyperplane*. That is, $H(u) = \{x \in \mathbb{R}^d : \langle x, y \rangle = h_K(u)\}$. For polytopes we say that a *face* of the polytope is the intersection of the body and a supporting hyperplane. We can classify the surfaces that the supporting hyperplane intersect by their dimension. That is, we call a face a vertex if $\dim(K \cap H(u)) = 0$, an edge if the dimension is 1, and a facet if the dimension is $d-1$.

The projection of a body K in the direction ξ is given by

$$K|\xi^{\perp} = \{x \in \xi^{\perp} | x + \lambda\xi \in K \text{ for some } \lambda \in \mathbb{R}\}.$$

The section of a body K in the direction ξ is denoted $K \cap \xi^{\perp}$ and is exactly the set of points of K in the hyperplane with unit normal ξ.

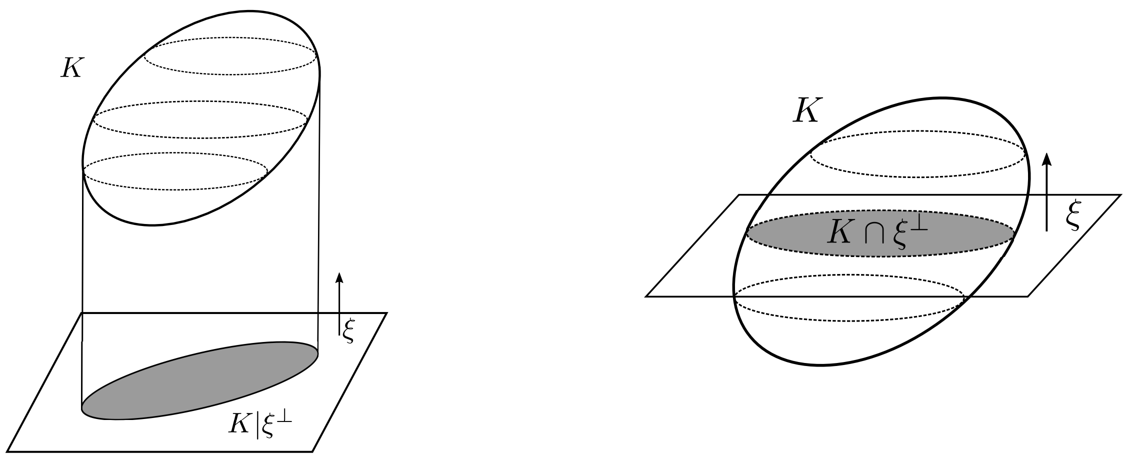


Figure 2.1: Sections and projections of convex bodies

2.3 Classical theorems

We can now begin to examine several theorems that provide the basis for much of the theory related to our explorations beginning with the Brunn-Minkowski inequality. Details and history can be found in, for example, [83, 34, 91]

Theorem 2.3.1. *Brunn-Minkowski Inequality: Let A, B be non-empty compact subsets of $\mathbb{R}^d$. Then*

$$|A+B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}},$$

or equivalently,

$$|\lambda A + (1-\lambda)B| \geq |A|^{\lambda}|B|^{1-\lambda},$$

for all $\lambda \in [0,1]$.

From the Brunn-Minkowski inequality we can derive the following theorem which tells us that the largest section of any origin-symmetric convex body must pass through the origin.

Theorem 2.3.2. *Brunn's Theorem: Let $K \subset \mathbb{R}^d$ be a convex body and*

$$A_{K,\xi}(t) = |K \cap \{\xi^\perp + t\xi\}|.$$

Then $A_{K,\xi}(t)^{\frac{1}{d-1}}$ is a concave function on its support. Moreover, if K is origin-symmetric then $A_{K,\xi}(0) \geq A_{K,\xi}(t)$ for all $t \in \mathbb{R}$.

An interesting question is how to compare when two bodies are "close" to each other, for a fuller discussion see [18]. One way to do this is to take the *Hausdorff distance* between two bodies K and L, given by

$$d_{\mathcal{H}}(K, L) = \min\left\{\lambda \geq 0 : K \subset L + \lambda B_2^d, L \subset K + \lambda B_2^d\right\}.$$

It can be equivalently defined by

$$d_{\mathcal{H}}(K, L) = \max\left(\max_{x\in K}\min_{y\in L}|x-y|, \max_{y\in L}\min_{x\in K}|x-y|\right).$$

It is well known that any convex body can be approximated by polytopes in the Hausdorff metric.

Let $GL(d)$ be the set of all invertible linear transformations on $\mathbb{R}^d$, then we define the Banach-Mazur distance between two origin-symmetric bodies by

$$d_{BM}(K, L) = \min\left\{r \geq 1 : L \subset TK \subset rL \text{ for some } T \in GL(d)\right\}.$$

For non-symmetric bodies we can define the Banach-Mazur distance using invertible affine transformations. That is if $AL(d)$ is the set of all invertible affine transformations on $\mathbb{R}^d$, and K and L are arbitrary bodies in $\mathbb{R}^d$ then

$$d_{BM}(K, L) = \min\left\{r \geq 1 : YL \subset TK \subset rYL \text{ for some } T, Y \in AL(d)\right\}.$$

The set of all symmetric convex bodies in $\mathbb{R}^d$ is compact with respect to the Banach-Mazur distance which can be seen as a result of the following classical theorem.

Theorem 2.3.3. *F. John's Theorem: For any convex origin-symmetric body $K \subset \mathbb{R}^d$,*

$$d_{BM}(K, B_2^d) \leq \sqrt{d}.$$

Equality holds only when K is an ellipsoid.

Note that for non-symmetric bodies the above theorem holds with constant d.

2.4 Polarity

Let K be a convex body in $\mathbb{R}^d$ containing the origin in its interior. Then we can define the *polar body* of K by

$$K^\circ = \left\{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1 \text{ for all } x \in K\right\}.$$

Note, as any origin-symmetric convex body is the unit ball of the norm space corresponding to the Minkowski functional, we can view also the polar body as the unit ball of the dual space. Thus, we may sometimes refer to the polar body of K as the dual body for origin-symmetric convex bodies, and write K^* in place of K°. Above, the center of polarity is taken to be the origin. However, one may chose any point z in the interior of K and define the *polar body of K with center of polarity z* by

$$K^z = \{y \in \mathbb{R}^d : (y - z) \cdot (x - z) \leq 1 \text{ for all } x \in K\}.$$

Note that $K^z = (K - z)^\circ + z$, and the bipolar theorem says that $(K^z)^z = K$, for $z \in \operatorname{int}(K)$ (see [35], p. 47). Polarity also reverses inclusions, that is, if $K \subset L$ then $L^z \subset K^z$. For K and L origin symmetric $(K \cap L)^z = \operatorname{conv}(K^z, L^z)$. There is also a correspondence between sections and projections of bodies and their polar body. Namely,

$$\left(K \cap \xi^\perp\right)^\circ = K^\circ | \xi^\perp$$

if we restrict the polar body to the hyperplane that the section lives in. Finally $(TK)^\circ = (T^*)^{-1} K^\circ$ where T^* is the adjoint matrix of T.

Further, as every vertex of a polytope is the intersection of adjacent supporting hyperplanes, there is an easy correspondence between the vertices of $K \subset \mathbb{R}^d$ and faces of K^z. In fact, if we denote the family of k dimensional faces of K by $\mathcal{F}_k(K)$ then there is a bijection between $\mathcal{F}_k(K)$ and $\mathcal{F}_{d-k-1}(K^z)$ (see [35]).

2.5 Volume Product

The volume of K^z is a strictly convex function for z in the interior of K, and tends to infinity as z approaches the boundary of K (see [43, 66]). A well known result of Santaló [81] (see also [83], p. 419) states that in every convex body K in $\mathbb{R}^d$, there exists a unique point $s(K)$, called the *Santaló point* of K, such that

$$|K^{s(K)}| = \min_{z \in \operatorname{int}(K)} |K^z|.$$

The *volume product* of K is defined by

$$\mathcal{P}(K) = \inf\{|K||K^z| : z \in \operatorname{int}(K)\} = |K|\ |K^{s(K)}|.$$

The volume product is affinely invariant, that is, $\mathcal{P}(A(K)) = \mathcal{P}(K)$ for every affine isomorphism $A : \mathbb{R}^d \to \mathbb{R}^d$. Observe that if we denote $L = K^{s(K)}$ then

$$\mathcal{P}(K^{s(K)}) = |L||L^{s(L)}| \leq |L||L^{s(K)}| = |K^{s(K)}||K| = \mathcal{P}(K).$$

So

$$\mathcal{P}(K^{s(K)}) \leq \mathcal{P}(K).$$

Since the set of all convex bodies in $\mathbb{R}^d$ is compact with respect to the Banach-Mazur distance and $K \mapsto \mathcal{P}(K)$ is continuous (see, for example, [19]), so it is natural to ask for maximal and minimal values of $\mathcal{P}(K)$.

The Blaschke-Santaló inequality states that

$$\mathcal{P}(K) \leq \mathcal{P}(B_2^d),$$

where B_2^d is the Euclidean unit ball. The equality in the above inequality is possible only for ellipsoids ([81], [72], see [67] or also [63] for a simple proof of both the inequality and the case of equality).

The minimal value of $\mathcal{P}(K)$ is an open question which is often called Mahler's conjecture [56, 57], which states that, for every convex body K in $\mathbb{R}^d$,

$$\mathcal{P}(K) \geq \mathcal{P}(\Delta^d) = \frac{(d+1)^{d+1}}{(d!)^2}, \tag{2.1}$$

where Δ^d is an d-dimensional simplex. It is also conjectured that equality in (2.1) is attained only if K is a simplex.

Note that the symmetric case of Mahler conjecture states that for every *origin-symmetric* convex body $K \subset \mathbb{R}^d$:

$$\mathcal{P}(K) \geq \mathcal{P}(B_1^d) = \mathcal{P}(B_\infty^d) = \frac{4^d}{d!}, \tag{2.2}$$

where B_1^d and B_∞^d are the cross-polytope and its polar body, the cube, respectively.

The inequalities (2.1) and (2.2) for $d = 2$ were proved by Mahler [56] with the case of equality proved by Meyer [62] in the general case and by Reisner [74] in the symmetric case, but the question is still open in many cases. A solution has recently been proposed for the symmetric case of dimension three in [39]. However, it is still unknown for the non-symmetric case in dimension three, and in general for dimension four and above. In the d-dimensional case, the conjecture has been verified for some special classes of bodies such as unconditional bodies [61, 75, 80], convex bodies having hyperplane symmetries which fix only one common point [8], zonoids [31, 74], and bodies of revolution [64]. It has also been shown that bodies with some positive curvature assumptions may not be minimizers of the volume product [32, 76, 81]. Bourgain and Milman, [13], proved the isomorphic version of the conjecture. That is, there exists an absolute constant c such that $\mathcal{P}(K) \geq c^d \mathcal{P}(Q)$, for all convex bodies K. Kuperberg [52] gave a new proof of this result with a better constant (see also [69], [25] for different proofs of the inequality and [5], [79] for more information). See [58, 79, 86, 85] for detailed discussions of the Mahler problem and properties of the dual volume.

An interesting note regarding the difficulty of Mahler's conjecture versus the solution of the Santaló inequality in the symmetric case is that there is only one maximizer for the volume product, though there are many minimizers if the conjecture holds. Indeed, there exist multiple bodies in dimension $d \geq 4$ that are the direct product of cubes and cross-polytopes in lower dimension. Recall that the direct product of two bodies $K \subset \mathbb{R}^{d_1}$ and $L \subset \mathbb{R}^{d_2}$ is given by

$$K \oplus L = \left\{(x, y) \in \mathbb{R}^{d_1+d_2} : x \in K, y \in L\right\}.$$

These figures are called Hanner Polytopes (see [42, 43]) and may also be constructed using their Minkowski functionals to define the L_1 sum of two bodies in the following way: The body $K \oplus_1 L$ is such that $||(x_1, x_2)||_{K\oplus_1 L} = ||x_1||_K + ||x_2||_L$ for $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$. The L_1 sum of bodies is dual to the direct (L_∞) sum of two bodies, that is, $(K \oplus L)^\circ = K^\circ \oplus_1 L^\circ$, and can be seen geometrically as the convex hull of two bodies in their complementary spaces, i.e. $K \oplus_1 L = \mathrm{conv}(K \times \{0\}, \{0\} \times L)$. Thus Hanner polytopes may also be described as the L_1 sums of cubes and cross-polytopes.

The equality of the volume product of Hanner polytopes to the cube or cross-polytope follows simply from the following lemma [79].

Lemma 2.5.1. *Given two origin symmetric bodies $K \subset \mathbb{R}^{d_1}$ and $L \subset \mathbb{R}^{d_2}$, then*

$$\mathcal{P}(K \oplus L) = \frac{d_1! d_2!}{(d_1 + d_2)!} \mathcal{P}(K)\mathcal{P}(L)$$

Thus, for $B_\infty^{d_1} \subset \mathbb{R}^{d_1}$ and $B_1^{d_2} \subset \mathbb{R}^{d_2}$ with $d = d_1 + d_2$,

$$\mathcal{P}\left(B_\infty^{d_1} \oplus B_1^{d_2}\right) = \frac{d_1! d_2!}{(d_1 + d_2)!} \cdot \frac{4^{d_1}}{d_1!} \cdot \frac{4^{d_2}}{d_2!} = \frac{4^d}{d!} = \mathcal{P}(B_1^d).$$

So there are examples of bodies whose volume product is the same as the conjectured minimum value that are neither a cube nor a cross-polytope.

Part II

The Discrete Slicing Problem

CHAPTER 3

The Geometry of Numbers

3.1 Introduction

The geometry of numbers is a field of mathematics formalized by Hermann Minkowski, bringing together earlier results surrounding packings, tilings, and continued fractions. Minkowski's study of when a body is sufficiently large enough in volume to contain a non-zero integer point gave new insight into Diophantine approximation and combinatorics. The geometry of numbers now occupies an area in the intersections of discrete geometry and number theory where modern results connect ideas in positive quadratic forms, algorithms, and crystallography. For a more detailed discussion of the history and introduction to the topic see [34, 54, 17, 33, 35].

3.2 Lattices

A lattice $\Gamma \subset V$ is a discrete, additive subgroup of V which spans V where V is a real vector space equipped with an inner product. Recall that the span of a set of vectors is given by

$$\operatorname{span}\{x_i\} = \left\{ \sum_i \alpha_i x_i : \alpha_i \in \mathbb{F} \right\}$$

where $\mathbb{F}$ is the underlying field of the vector space V. The rank of a lattice is the dimension of the vector space it spans. The standard example of a lattice is the set of all points with integer coordinates, denoted $\mathbb{Z}^d$. Here, $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^d$, and $\Gamma = \mathbb{Z}^d$. Another simple example is the lattice with integer coordinates whose sum of coordinates is even, denote this lattice by D. That is

$$D = \left\{ x \in \mathbb{Z}^d : \sum_{i=1}^{d} x_i \equiv 0 \mod 2 \right\}.$$

This second lattice is a subset of the first, and further, is a subgroup. Thus we say that D is sublattice of $\mathbb{Z}^d$. In general, a lattice $\Gamma' \subset \Gamma$ is a sublattice of Γ if it is algebraically a subgroup (see figure 3.1).

Every lattice is generated by a finite number of vectors equal to its rank. This set of generating vectors is called the basis of a lattice. Two lattices are said to be isomorphic if there exists an invertible linear transformation that carries basis vectors of one lattice to basis vectors of another. It is a standard fact that all lattices are isomorphic to the standard integer lattice of the same rank with the matrix that carries a lattice to the standard lattice composed of the basis vectors of the original lattice. Let us call the matrix composed of these basis vectors T. Then if the lattice is of full rank d we say that the determinant of the lattice $\det(\Gamma) = \det(T)$. This has the following geometric interpretations.

If $\{u_i\}$ is the set of basis vectors for the lattice Γ with rank d then

$$\Pi(\Gamma) = \left\{ \sum_{i=1}^{d} \alpha_i u_i : 0 \leq \alpha_i < 1 \text{ for } i = 1 \ldots d \right\}$$

is the fundamental parallelepiped of Γ with basis $\{u_i\}$ (see figure 3.1). Then it turns out that $\mathrm{vol}_d(\Pi) = \det(\Gamma)$. We can also define the determinant of a lattice $\det(\Gamma)$ by the formula

$$\frac{1}{\det(\Gamma)} = \lim_{\rho \to \infty} \frac{|\Gamma \cap \rho B_2^d|}{\mathrm{vol}(\rho B_2^d)}.$$

This definition links clearly with the previous when we consider that $\mathbb{R}^d$ may be tiled with copies of Π, and that the last equation calculates the number of points in $\mathbb{R}^d$ per lattice point. In particular, we can see from this definition and the previous definition that no matter the basis chosen for Γ the fundamental parallelepiped has the same volume. From this we see that any transformation between bases of the same lattice must be a unimodular tranformation, that is, from the set of square matrices whose determinant is ± 1. Note that $\det(\Gamma)$ is often written as $\mathrm{vol}(\mathbb{R}^d/\Gamma)$ in literature.

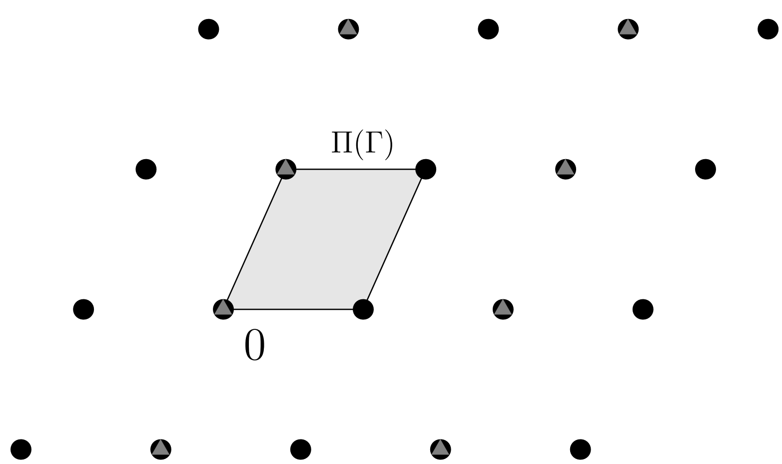


Figure 3.1: Example of a lattice, sublatice, and fundamental parallelepiped.

3.3 Minkowski's Theorems

One of the most well known results in the geometry of numbers is Minkowski's first theorem which gives a necessary condition for an origin-symmetric convex body to contain an integer point (see [88], Theorem 3.28 pg 134 and [9], Theorem 2.8 pg 38).

Theorem 3.3.1. *Minkowski's First Theorem: Let $K \subset \mathbb{R}^d$ be an origin-symmetric convex body such that $\mathrm{vol}_d(K) \geq 2^d$ then K contains at least one non-zero element of $\mathbb{Z}^d$.*

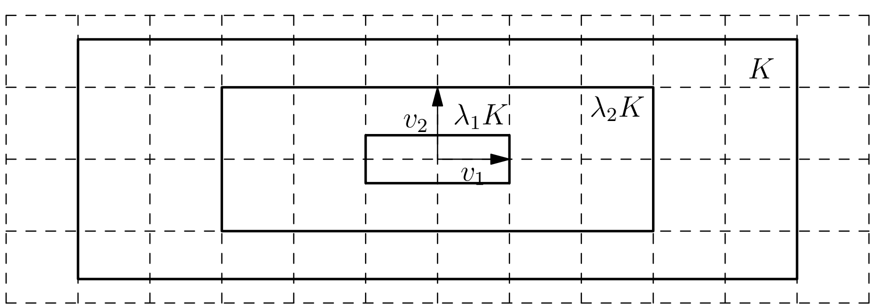
This theorem can be restated to work in the setting of a general lattice by simply incorporating the determinant of the lattice.

Theorem 3.3.2. *Minkowski's First Theorem (General): Let $K \subset \mathbb{R}^d$ be an origin-symmetric convex body, and $\Gamma \subset \mathbb{R}^d$ a lattice of rank d. If $\mathrm{vol}_d(K) \geq 2^d \det(\Gamma)$ then K contains at least one non-zero element from Γ.*

We can extend Minkowski's first theorem to better respect the shape of the body using successive minima which are defined in the following way.

Definition 3.3.3. Let Γ be a lattice in $\mathbb{R}^d$ of rank k, and let K be an origin-symmetric convex body in $\mathbb{R}^d$. For $1 \leq j \leq k$ define the successive minima to be

$$\lambda_j = \lambda_j(K, \Gamma) = \min\{\lambda > 0 : \lambda \cdot K \text{ contains } j \text{ linearly independent elements of } \Gamma\}.$$



$\lambda_1 = \frac{1}{5}$ $\qquad\qquad$ $\lambda_2 = \frac{3}{5}$

Figure 3.2: The successive minima and basis vectors

Notice that it follows directly from the definition that $\lambda_k \geq \lambda_{k-1} \geq ... \geq \lambda_1$. In addition, the assumption that K contains d linearly independent lattice points of Γ implies that Γ has rank d and that $\lambda_d \leq 1$.

Moreover, according to the definition of the successive minima there exists a set of linearly independent vectors from Γ, $v_1, \ldots, v_k$, such that v_i lies on the boundary of $\lambda_i \cdot K$ but the interior of $\lambda_i \cdot K$ does not contain any lattice vectors outside the span of $v_1, \ldots, v_{i-1}$. The vectors $v_1, \ldots, v_k$ are called a directional basis, and we note that they may not necessarily form a basis of Γ.

Theorem 3.3.4. *Minkowski's Second Theorem: Let Γ be a lattice in $\mathbb{R}^d$ of rank d, K be an origin-symmetric convex body with successive minima λ_i. Then,*

$$\frac{1}{d!}\prod_{i=1}^{d}\frac{2}{\lambda_i} \leq \frac{\mathrm{vol}(K)}{\det(\Gamma)} \leq \prod_{i=1}^{d}\frac{2}{\lambda_i}.$$

Notice that these inequalities are sharp. The cube (transformed by the same matrix as the lattice) gives the right hand side while the cross-polytope is equal on the left.

3.4 The Ehrhart Polynomial

Minkowski's first theorem gives us the necessary condition for a body to have at least one integer point. However, we are often interested in knowing precisely how many lattice points are contained in a given set compared to the volume. In $\mathbb{R}^2$ we can find this precisely using the following theorem

Theorem 3.4.1. *Pick's Theorem: Let P be an integral 2-dimensional convex polygon, then*

$$A = I + \frac{1}{2}B - 1$$

where $A = \mathrm{vol}_2(P)$ is the area of the polygon, I is the number of lattice points in the interior of P, and B is the number of lattice points on the boundary.

Here a polygon is called integral if it can be described as the convex hull of lattice points. In the below figure 3.3, we have 4 interior points and 8 boundary points, hence the area by Pick's theorem is 7.

Unfortunately, in higher dimensions we are unable to precisely compare the volume and lattice count. However, there is a strong theory regarding the number of lattice points in dilations of integral polytopes. If we call $L_t(P) = \#\left(tP \cap \mathbb{Z}^d\right)$ the lattice point enumerator of P a body whose centroid is at the origin and whose vertices are in $\mathbb{Z}^d$, then it turns out that this function can be extended to a rational polynomial in t of degree d. That is $L_t(P) = a_0 + \ldots + a_{d-1}t^{d-1} + a_d t^d$ where $a_i \in \mathbb{Q}$, and $L_t(P)$ coincides with the above definition at integer values of t. Clearly, $a_0 = 1$ as K will only contain the origin as it shrinks to one point. Using the idea that the volume and number of lattice points converge as the body dilates to infinity (see the

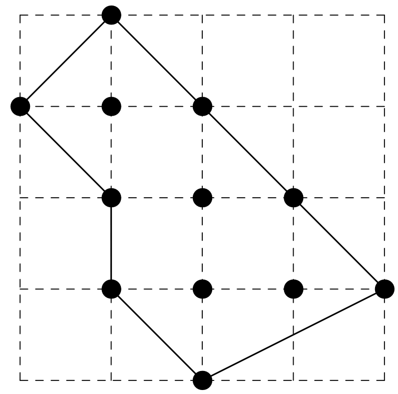


Figure 3.3: The integral polytope has an area of 7 by Pick's theorem

well known Gauss's circle problem [38]) we can see that $a_d = \mathrm{vol}_d(P)$. It can be shown further that a_{d-1} corresponds to the normalized surface area with respect to the lattice, however, there are no further known geometric correlations to the coefficients of the lattice point enumerator. See [9] for a full discussion on the lattice point enumerator and its applications.

CHAPTER 4

Geometric Tomography

4.1 Introduction

Geometric Tomography concerns the reconstruction of objects from incomplete data such as projections or sections. The objects of interest are often taken to be convex bodies so that the tools of convex geometry may be applied. There is a rich theory for continuous convex bodies in $\mathbb{R}^d$, however, much less has been done for discrete convex lattice sets in $\mathbb{Z}^d$. Here, a convex body is a compact convex set with non-empty interior, and a convex lattice set is a set of points equal to the intersection of $\mathbb{Z}^d$ with a convex body.

A large reason for this current disparity is that many of the basic results for continuous convex bodies fail in the discrete setting. For example, for two convex bodies in $\mathbb{R}^d$ their Minkowski sum is convex. However, in the discrete case this does not remain true as we saw in figure 1.1. Another example is Brunn's theorem which gives us that for any origin-symmetric convex body and given any unit vector the section of largest volume perpendicular to the unit vector passes through the origin. We can formulate this theorem by taking volume to be the cardinality of a discrete set, but we again find that the theorem does not hold.

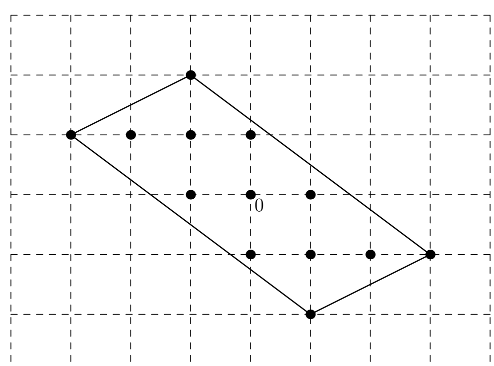


Figure 4.1: Central section may have fewer integer points.

The comprehensive book of Gardner [23] provides many of the details of the well studied continuous aspects of tomography. In [24] there was a call to study discrete analogues of known tomography results.

Interestingly, problems may be trivial in the discrete case, or even prove to be untrue! Below, we will list four related problems in Tomography and briefly discuss discrete results related to them.

4.2 Projections of sets

The first type of partial information that we will attempt to use is that of projections of sets. Recall, the projection of a body K in the direction ξ is given by $K|\xi^{\perp} = \{x \in \xi^{\perp} | x + \lambda\xi \in K \text{ for some } \lambda \in \mathbb{R}\}$

Theorem 4.2.1. *Aleksandrov's Theorem: Let K and L be origin symmetric convex bodies in $\mathbb{R}^d$. If $\mathrm{vol}_{d-1}(K|\xi^{\perp}) = \mathrm{vol}_{d-1}(L|\xi^{\perp})$ for every $\xi \in \mathbb{S}^{d-1}$ then $K = L$.*

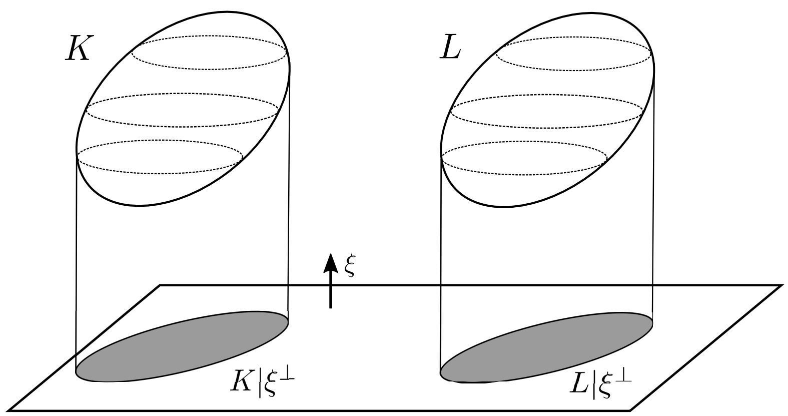


Figure 4.2: Aleksandrov's Theorem

A discrete analogue of Aleksandrov's was considered in [24], where discrete projections are taken to be the cardinality of the projection of the discrete set onto a hyperplane. It is interesting that counter-examples can be found for origin-symmetric bodies such as in figure 4.3. This leaves the question of how the theorem might be reconciled with the continuous version. Certainly as the size of the lattice set grows they approximate the continuous setting, so it is reasonable to propose additional conditions on the lattice sets such as minimal size. In [92] it was show to hold for bodies contained in a very narrow strip. In [94] the theorem was shown to be true for bodies whose projections were the same in all directions also for a dilate of the body. Finally, in [78] the question was studied for an analogue on the surface area of the projections. However, the question largely remains open.

Problem 4.2.2. *Shephard's Problem [84]: Let K and L be origin symmetric convex bodies in $\mathbb{R}^d$. If $\mathrm{vol}_{d-1}(K|\xi^{\perp}) \leq \mathrm{vol}_{d-1}(L|\xi^{\perp})$ for every $\xi \in \mathbb{S}^{d-1}$ then is it true that $\mathrm{vol}_d(K) \leq \mathrm{vol}_d(L)$?*

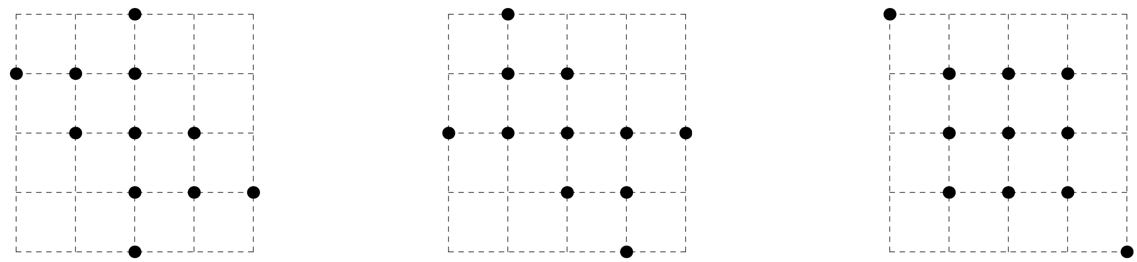


Figure 4.3: Counterexamples for a discrete analog of Aleksandrov's theorem

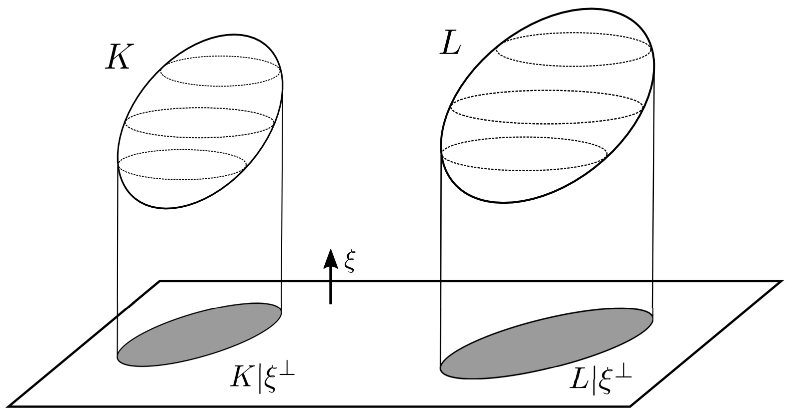


Figure 4.4: Shephard's Problem

It was shown by Petty [71] and Schneider [82], that the statement is, in fact, false in dimension 3 and greater. However, in the discrete case there is always a direction from which the projection counts all points in the body and so the problem is trivial for discrete measure.

4.3 Sections of sets

Dual to projections are sections of sets. Recall that the central section of K in the direction $\xi \in S^{d-1}$ is the intersection K with $\xi^\perp$. Denoted by $K \cap \xi^\perp$.

Theorem 4.3.1. *Funk's Theorem: Let K and L be origin symmetric convex bodies in $\mathbb{R}^d$. If $\mathrm{vol}_{d-1}(K \cap \xi^\perp) = \mathrm{vol}_{d-1}(L \cap \xi^\perp)$ for every $\xi \in \mathbb{S}^{d-1}$ then $K = L$.*

It is clear that in the discrete setting Funk's theorem is trivial. If we take any hyperplane intersecting a convex set of points, then we may move the hyperplane slightly so that it contains only lattice points in a line. The lattice count of this slice in every possible direction would yield the radial function for our discrete set, allowing us to recreate the set.

Problem 4.3.2. *Busemann-Petty Problem [15]: Let K and L be origin symmetric convex bodies in $\mathbb{R}^d$. If*

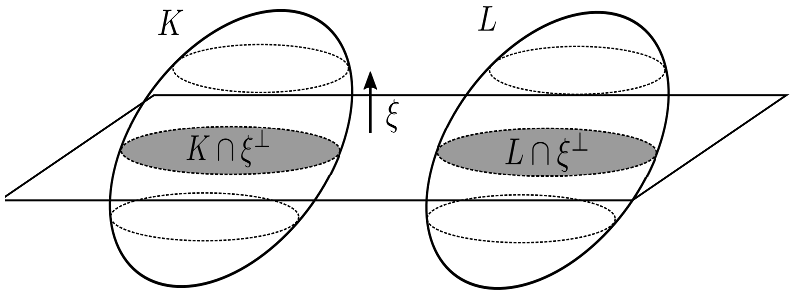


Figure 4.5: Funk's theorem

$\mathrm{vol}_{d-1}(K\cap\xi^{\perp}) \le \mathrm{vol}_{d-1}(L\cap\xi^{\perp})$ *for every* $\xi\in\mathbb{S}^{d-1}$ *then is it true that* $\mathrm{vol}_d(K)\le\mathrm{vol}_d(L)$*?*

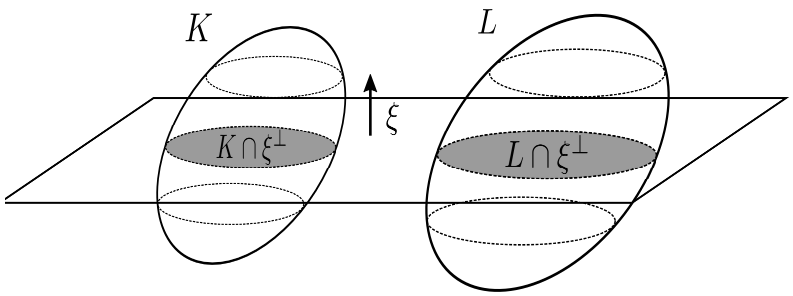


Figure 4.6: The Busemann-Petty problem

The answer to the Busemann-Petty problem is negative for $d\ge 5$ and affirmative for $d\le 4$. The solution appeared as the result of a sequence of papers: [53] for $d\ge 12$, [6] for $d\ge 10$, [26] and [10] for $d\ge 7$, [70] and [20] for $d\ge 5$, [21] for $d=3$, [93] and [22] for $d=4$. See [23] and [45] for historical remarks. It is somewhat natural to ask if the problem can be saved by adding a constant to the inequality, the following section will explore this.

4.4 The Isomorphic Busemann-Petty Problem

The isomorphic version of the Busemann-Petty problem is equivalent to the slicing problem of Bourgain [12, 11], which is, undoubtedly, one of the major open problems in convex geometry, which asks if an origin-symmetric convex body of volume one must have a large (in volume) hyperplane section. More precisely, it asks whether there exists an absolute constant $\mathcal{L}_1$ so that for any origin-symmetric convex body K in $\mathbb{R}^d$

$$\mathrm{vol}_d(K)^{\frac{d-1}{d}} \le \mathcal{L}_1 \max_{\xi\in\mathbb{S}^{d-1}} \mathrm{vol}_{d-1}(K\cap\xi^{\perp}). \tag{4.1}$$

The problem is still open, with the best-to-date estimate of $\mathcal{L}_1 \le O(d^{1/4})$ established by Klartag [41], who improved the previous estimate of Bourgain [11]. We refer to [67] and [14] for detailed information and history of the problem. Koldobsky recently proposed an interesting generalization of the slicing problem [46, 48, 50, 49, 47]: Does there exists an absolute constant $\mathcal{L}_2$ so that for every even measure μ on $\mathbb{R}^d$ with a positive density, and for every origin-symmetric convex body K in $\mathbb{R}^d$ such that

$$\mu(K) \le \mathcal{L}_2 \max_{\xi \in \mathbb{S}^{d-1}} \mu(K \cap \xi^{\perp}) \mathrm{vol}_d(K)^{\frac{1}{d}}? \tag{4.2}$$

Koldobsky was able to solve the above question for a number of special cases of the body K and provide a general estimate of $O(\sqrt{d})$. Recently [44] it was shown that the constant can be bounded from below with an estimate of $O\left(\sqrt{\frac{d}{\log\log d}}\right)$ for a particular measure. The most amazing fact here is that the constant $\mathcal{L}_2$ in (4.2) can be chosen independent of the measure μ under the assumption that μ has even positive density. In addition, Koldobsky and Zvavitch were able to prove in [51] that $\mathcal{L}_2$ is of order $O\left(d^{1/4}\right)$ if one assumes that the measure μ is s-concave. We note that the assumption of positive density is essential for the above results and (4.2) is simply not true if this condition is dropped. Indeed, to create a counterexample consider an even measure μ on $\mathbb{R}^2$ uniformly distributed over $2N$ points on the unit circle, then the constant $\mathcal{L}_2$ in (4.2) will depend on N. The following chapter will explore the question for the discrete counting measure.

CHAPTER 5

The Discrete Slicing Problem

5.1 Introduction

During the 2013 AIM workshop on "Sections of convex bodies" Koldobsky asked if it is possible to provide a discrete analog of inequality (4.2): Let $\mathbb{Z}^d$ be the standard integer lattice in $\mathbb{R}^d$, K be a convex, origin-symmetric body, define $\#K = \mathrm{card}(K \cap \mathbb{Z}^d)$, the number of points of $\mathbb{Z}^d$ in K.

Question: *Does there exist a constant $\mathcal{L}_3$ such that*

$$\#K \le \mathcal{L}_3 \max_{\xi \in \mathbb{S}^{d-1}} \left(\#(K \cap \xi^\perp)\right) \mathrm{vol}_d(K)^{\frac{1}{d}},$$

for all convex origin-symmetric bodies $K \subset \mathbb{R}^d$ containing d linearly independent lattice points?

We note here that we require that K contains d linearly independent lattice points, i.e., $\dim(K \cap \mathbb{Z}^d) = d$, in order to eliminate the degenerate case of a body whose maximal section contains all lattice points in the body, but whose volume may be taken to 0 by eliminating a dimension. For example, take a box $[-1/n, 1/n]^{d-1} \times [-20, 20]$.

The main goal of this chapter is to study Koldobsky's question. In section 5.2 we will show the solution for the 2-dimensional case. The solution is based on the classical Minkowski's First and Pick's theorems from the chapter 3 and gives a general idea of the approach to be used in Sections 5.3, 5.5, and 5.7. In Section 5.3 we apply a discrete version of the theorem of F. John due to T. Tao and V. Vu [87] to give a partial answer to Koldobsky's question and show that the constant $\mathcal{L}_3$ can be chosen independent of the body K and as small as $O(d)^{7d/2}$. We then make a minor improvement using known inequalities in section 5.4. We start section 5.5 with a case of unconditional bodies and present a simple proof that in this case $\mathcal{L}_3$ can be chosen of order $O(d)$ which is the best possible. After, we prove a discrete analog of Brunn's theorem and use it to show that the constant $\mathcal{L}_3$, for the general case, can be chosen as small as $O(1)^d$. In fact, we prove the slightly more general result that

$$\#K \le O(1)^d d^{d-m} \max\left(\#(K \cap H)\right) \ \mathrm{vol}_d(K)^{\frac{d-m}{d}},$$

where the maximum is taken over all m-dimensional linear subspaces $H \subset \mathbb{R}^d$. Finally, we also provide a short observation that $\mathcal{L}_1 \le \mathcal{L}_3$.

5.2 Solution in $\mathbb{Z}^2$

Now we will use Theorems 3.3.1 and 3.4.1 above to show that the constant $\mathcal{L}_3$ in Koldobsky's question can be chosen independently of the origin-symmetric convex body in $\mathbb{R}^2$.

Theorem 5.2.1. *Let K be a origin-symmetric convex body in $\mathbb{R}^2$, $\dim(K \cap \mathbb{Z}^2) = 2$, then*

$$\#K \le 5 \max_{\xi \in \mathbb{S}^1} \#(K \cap \xi^{\perp}) \operatorname{vol}_2(K)^{\frac{1}{2}}.$$

Proof: Let $s = \sqrt{\operatorname{vol}_2(K)/4}$, then by Minkowski's First Theorem, since $\operatorname{vol}_2(\frac{1}{s}K) = 4$, there exists a non-zero vector $u \in \mathbb{Z}^2 \cap \frac{1}{s}K$. Then $su \in K$ and

$$\#\left(L_u \cap K\right) \ge 2\lfloor s \rfloor + 1,$$

where $\lfloor s \rfloor$ is the integer part of s, and L_u is the line containing u and the origin. Next, consider $P = \operatorname{conv}(K \cap \mathbb{Z}^2)$, i.e., the convex hull of the integral points inside K. P is an integral 2-dimensional convex polytope, and so by Pick's theorem we get that

$$\operatorname{vol}_2(P) = I + \frac{1}{2}B - 1 \ge \frac{I+B}{2} - \frac{1}{2},$$

using that $I \ge 1$. Thus

$$\#P = I + B \le 2\operatorname{vol}_2(P) + 1 \le \frac{5}{2}\operatorname{vol}_2(P),$$

since the minimal volume of an origin-symmetric integral convex polygon is at least 2. We now have that

$$\begin{aligned}
\#K = \#P &\le \frac{5}{2}\operatorname{vol}_2(P) \le \frac{5}{2}\operatorname{vol}_2(K) \\
&= \frac{5}{2}\left(2\, s\right) \operatorname{vol}_2(K)^{\frac{1}{2}} < 5\left(2\lfloor s \rfloor + 1\right) \operatorname{vol}_2(K)^{\frac{1}{2}} \\
&\le 5 \max_{\xi \in \mathbb{S}^1} \#(K \cap \xi^{\perp}) \operatorname{vol}_2(K)^{\frac{1}{2}}.
\end{aligned}$$

$\square$

5.3 Approach via Discrete F. John Theorem

It is a standard technique to get a first estimate in slicing inequalities, i.e. $\mathcal{L}_1 \le O(\sqrt{d})$ from 4.1, by using the classical F. John theorem (theorem 2.3.3), [40], [68], or [14], which claims that for every convex origin-symmetric body $K \subset \mathbb{R}^d$ there exists an ellipsoid E such that $E \subset K \subset \sqrt{d}E$. A discrete analog of F. John's theorem was first proved by Bárány and Vershik (Theorem 3 in [7]). The theorem claims that a convex origin-symmetric body can be approximated by a lattice parallelepiped. We will use a recent version of this result, see Theorem 5.3.2 below, proved by T. Tao and V. Vu (see [87, 88]), to show that the constant $\mathcal{L}_3$ in Koldobsky's question can be chosen independent of the origin-symmetric convex body $K \subset \mathbb{R}^d$. We first recall the definition of a generalized arithmetic progression (see [87, 88] for more details):

Definition 5.3.1. Let G be an additive group, $N = (N_1, \dots, N_d)$ a d-tuple of non-negative integers and $v = (v_1, \dots, v_d) \in G^d$. Then a generalized symmetric arithmetic progression $\mathbf{P}$ is a triplet (N, v, d). In addition, define

$$\text{Image}(\mathbf{P}) = [-N, N] \cdot v = \{n_1 v_1 + \ldots + n_d v_d : n_j \in [-N_j, N_j] \cap \mathbb{Z} \text{ for all } 1 \le j \le d\}.$$

The progression is called proper if the map $n \mapsto n \cdot v$ is injective for $n = (n_1, \dots, n_d)$, then $v = (v_1, \dots, v_d)$ is called its basis vectors, and d its rank. Finally, for $t > 0$, let the dilate $\mathbf{P}_t$ of $\mathbf{P}$ be a generalized symmetric arithmetic progression defined by (tN, v, d).

Below is a version for $\mathbb{Z}^d$ of the Discrete John theorem from [87] (Theorem 1.6 there):

Theorem 5.3.2. *Let K be a convex origin-symmetric body in $\mathbb{R}^d$. Then there exists a symmetric, proper, generalized arithmetic progression $\mathbf{P}$ with* $\text{Image}(\mathbf{P}) \subset \mathbb{Z}^d$*, such that rank*$(\mathbf{P}) \le d$ *and*

$$(O(d)^{-3d/2} K) \cap \mathbb{Z}^d \subset \text{Image}(\mathbf{P}) \subset K \cap \mathbb{Z}^d \subset \text{Image}(\mathbf{P}_{O(d)^{3d/2}}), \tag{5.1}$$

in addition

$$O(d)^{-7d/2} \# K \le \# \text{Image}(\mathbf{P}). \tag{5.2}$$

Now we are ready to state and prove our first estimate in Koldobsky's question and prove that for any origin-symmetric convex body $K \subset \mathbb{R}^d$, $\dim(K \cap \mathbb{Z}^d) = d$,

$$\# K \le O(d)^{7d/2} \max_{\xi \in \mathbb{S}^{d-1}} \left(\#(K \cap \xi^\perp)\right) \text{vol}_d(K)^{\frac{1}{d}}. \tag{5.3}$$

To prove (5.3) we apply the discrete John's theorem to get a symmetric, proper, generalized arithmetic progression $\mathbf{P} = (N, v, d)$ as in Definition 5.3.1. We note that $\text{rank}(\mathbf{P}) = d$. As otherwise $\dim(K \cap \mathbb{Z}^d) = d$ but $\dim(\text{Image}(\mathbf{P}_{O(d)^{3d/2}})) < d$ with $K \cap \mathbb{Z}^d \subset \text{Image}(\mathbf{P}_{O(d)^{3d/2}})$. Without loss of generality, take $N_1 \ge N_2 \ge$

$\ldots \geq N_d \geq 1$, then define $\xi^{\perp} = \operatorname{span}\{v_1, \ldots, v_{d-1}\}$. Application of (5.2) gives

$$
\begin{aligned}
\#K \leq & O(d)^{7d/2} \#(\mathrm{Image}(\mathbf{P})) \\
\leq & O(d)^{7d/2} \prod_{i=1}^{d} (2N_i + 1) \\
= & O(d)^{7d/2} (2N_d + 1) \prod_{i=1}^{d-1} (2N_i + 1) \\
\leq & O(d)^{7d/2} \left(\prod_{i=1}^{d} (2N_i + 1) \right)^{\frac{1}{d}} \#(K \cap \xi^{\perp}).
\end{aligned}
$$

Where the last inequality follows from the minimality of N_d and we use (5.1) to claim that

$$
\#(K \cap \xi^{\perp}) \geq \prod_{i=1}^{d-1} (2N_i + 1).
$$

Now we consider the volume covered by our progression. Take a fundamental parallelepiped

$$
\Pi = \{a_1 v_1 + \ldots + a_d v_d, \text{ where } a_i \in [0, 1), \text{ for all } i = 1, \ldots, d\}.
$$

Let $X = [-N, N-1] \cdot v$, where $N - 1 = (N_1 - 1, \ldots, N_d - 1)$. We notice that

$$
K \supset \bigcup_{x \in X} (x + \Pi).
$$

Indeed from $\mathrm{Image}(\mathbf{P}) \subset K \cap \mathbb{Z}^d$ we get that the vertices of $x + \Pi$ belong to $K \cap \mathbb{Z}^d$ for all $x \in X$ and thus, by convexity, $x + \Pi \subset K$ for all $x \in X$. Next

$$
\mathrm{vol}_d(K) \geq \left(\prod_{i=1}^{d} 2N_i \right) \det(v_1, \ldots, v_d) \geq \prod_{i=1}^{d} 2N_i,
$$

where the last inequality follows from the fact that $v_1, \ldots, v_d$ are independent vectors in $\mathbb{Z}^d$ and thus $\det(v_1, \ldots, v_d) \geq \det(\mathbb{Z}^d) = 1$.

Finally,

$$
\begin{aligned}
\#K \leq & O(d)^{7d/2} \left(\prod_{i=1}^{d} (2N_i + 1) \right)^{\frac{1}{d}} \#(K \cap \xi^{\perp}) \\
\leq & O(d)^{7d/2} \left(\prod_{i=1}^{d} (2N_i) \right)^{\frac{1}{d}} \#(K \cap \xi^{\perp}) \\
\leq & O(d)^{7d/2} \#(K \cap \xi^{\perp}) \mathrm{vol}_d(K)^{\frac{1}{d}}.
\end{aligned}
$$

We comment here that it is mentioned in [88] that the authors would be interested to see if the constant $O(d)^{-3d/2}$ could be improved to $e^{O(d)}$ or $d^{O(1)}$ which would immediately improve our result here.

5.4 Solution using known inequalities

Here we wish to remark on several results of inequalities relating the above properties of convex bodies. Some of these results will provide insight into the motivation of the solutions for the following chapter. This first theorem, found in [37] allows us to establish an upper bound on the volume by the number of lattice points.

Theorem 5.4.1. *(Van der Corput) Let $K \in \mathcal{K}_0^d$. Then*

$$\#\left(K \cap \mathbb{Z}^d\right) \geq 2\left\lfloor \frac{\mathrm{vol}(K)}{2^d} \right\rfloor + 1$$

We may rewrite

$$\frac{\mathrm{vol}(K)}{2^d} - 1 \leq \left\lfloor \frac{\mathrm{vol}(K)}{2^d} \right\rfloor \leq \frac{\#K - 1}{2}.$$

So we see that $\mathrm{vol}(K) \leq 2^{d-1}\left(\#K + 1\right)$.

The next theorem provides an upper bound for the number of lattice points in terms of the successive minima from [36].

Theorem 5.4.2. *(M. Henk) Let $d \geq 2$, $K \in \mathcal{K}_0^d$ and $\Gamma \subset \mathbb{R}^d$ be a lattice, then*

$$\#\left(K \cap \Gamma\right) < 2^{d-1} \prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i(K,\Gamma)} + 1 \right\rfloor.$$

Now we may use the above inequalities as well as Minkowski's Second theorem to find a slight improvement to our last estimate for the discrete slicing problem. We will need to consider counting points intersecting a body with a general lattice, and so we will adapt our notation slightly. Given a lattice Λ we will take $\#(K \cap \Lambda) = \mathrm{card}(K \cap \Lambda)$ and, as before, if the lattice is omitted we will take the lattice to be the standard integer lattice of appropriate dimension. To simplify the presentation of the proof, we will denote by $C, C_1, C_2, \ldots$ absolute positive constants.

Let $\{\lambda_i\}_{i=1}^d$ be the successive minima of K with $\{v_i\}_{i=1}^d$ the directional basis. We may assume that the intersection of the body and lattice is full dimensional, that is $\lambda_i \leq 1$ for all successive minima. Thus $\frac{2}{\lambda_i} + 1 \leq \frac{4}{\lambda_i}$. Let $H = \mathrm{span}\{v_2, \ldots, v_d\}$. Then, starting with Theorem 5.4.2 and using a similar argument as above, we have

$$\#K \le 2^{d-1}\prod_{i=1}^{d}\left\lfloor \frac{2}{\lambda_i}+1\right\rfloor = 2^{d-1}\left(\prod_{i=1}^{d}\left\lfloor \frac{2}{\lambda_i}+1\right\rfloor\right)^{1-\frac{1}{d}}\cdot\left(\prod_{i=1}^{d}\left\lfloor \frac{2}{\lambda_i}+1\right\rfloor\right)^{\frac{1}{d}}$$
$$\le 2^{d-1}\left(\left(\prod_{i=2}^{d}\frac{4}{\lambda_i}\right)^{\frac{1}{d-1}}\cdot\prod_{i=2}^{d}\left(\frac{4}{\lambda_i}\right)\right)^{\frac{d-1}{d}}\cdot\left(\prod_{i=1}^{d}\frac{4}{\lambda_i}\right)^{\frac{1}{d}}$$
$$= 2^{d-1}\prod_{i=2}^{d}\left(\frac{4}{\lambda_i}\right)\cdot\left(\prod_{i=1}^{d}\frac{4}{\lambda_i}\right)^{\frac{1}{d}}.$$

Now applying Theorem 3.3.4 twice, once on the hyperplane H, we get that:

$$\left(\prod_{i=1}^{d}\frac{4}{\lambda_i}\right)^{\frac{1}{d}} \le \left(d!C_1^{d-1}\frac{\mathrm{vol}(K)}{\det(v_1,\ldots,v_d)}\right)^{\frac{1}{d}} \le dC_2^d\mathrm{vol}(K)^{\frac{1}{d}}$$

$$\prod_{i=2}^{d}\left(\frac{4}{\lambda_i}\right) \le (d-1)!C_3^d\frac{\mathrm{vol}(K\cap H)}{\det(v_2,\ldots,v_d)}.$$

Combining all of the above we have that

$$\#K \le d!C_4^d\frac{\mathrm{vol}(K\cap H)}{\det(v_2,\ldots,v_d)}\mathrm{vol}(K)^{\frac{1}{d}}. \tag{5.4}$$

Now we use Theorem 5.4.1 on the slice $K\cap H$ to complete our estimate. Let $T=(v_2,\ldots,v_d)$, the matrix whose columns are the vectors v_i, so that $\Lambda = T\cdot Z^{d-1}$ and $L=K\cap H$, then

$$\mathrm{vol}(T^{-1}L) \le 2^{d-2}(\#(T^{-1}L\cap Z^{d-1})+1) = 2^{d-2}(\#(L\cap\Lambda)+1).$$

Then we have $\det(T)=\det(v_2,\ldots,v_d)$ and $\#(K\cap H)=\#L\cap\Lambda$ then

$$\frac{\mathrm{vol}(K\cap H)}{\det(v_2,\ldots,v_d)} \le C_5^d\#(K\cap H).$$

Applying this final inequality to 5.4 we have that

$$\#K \le d!C^d\#(K\cap H)\mathrm{vol}(K)^{\frac{1}{d}}$$

which gives us an estimate of $\mathcal{L}_3 \sim O(d)^d$.

5.5 Solution for Unconditional Bodies

The goal of this section and the next is to improve the estimates provided in Section 5.3 and 5.4. We will first study the behavior of constant $\mathcal{L}_3$ in the case of unconditional convex bodies. A set $K\subset\mathbb{R}^d$ is said to be unconditional if it is symmetric with respect to any coordinate hyperplane, i.e., $(\pm x_1,\pm x_2,\ldots,\pm x_d)\in K$, for any $x\in K$ and any choice of $\pm$ signs.

Theorem 5.5.1. *Let $K \subset \mathbb{R}^d$ be an unconditional convex body with* $\dim(K \cap \mathbb{Z}^d) = d$. *Then*

$$\#K \le O(d) \max_{i=1,\dots,d} \left(\#(K \cap e_i^{\perp})\right) \operatorname{vol}_d(K)^{\frac{1}{d}},$$

where $e_1, \dots, e_d$ are the standard basis vectors in $\mathbb{R}^d$. Moreover, this bound is the best possible.

Proof: This result follows from the simple observation that the section of K by a coordinate hyperplane ${e_i}^{\perp}$ is maximal in cardinality among all parallel sections of K, i.e.

$$\#(K \cap (e_i^{\perp} + te_i)) \le \#(K \cap e_i^{\perp}), \text{ for all } t \in \mathbb{R}, \text{ and } i = 1, \dots, d. \tag{5.5}$$

We can see this by considering a point $x \in K \cap (e_i^{\perp} + te_i) \cap \mathbb{Z}^d$. Let $\bar{x}$ be the reflection of x over ${e_i}^{\perp}$, i.e., $\bar{x} = (x_1, \dots, -x_i, \dots, x_d)$. Using the unconditionality of K, we get that $\bar{x} \in K$ and convexity gives us $(x + \bar{x})/2 \in K \cap e_i^{\perp}$. Hence, the projection of a point in $K \cap (e_i^{\perp} + te_i)$ is associated to a point in $K \cap e_i^{\perp}$, which explains (5.5).

Let $\{\lambda_i\}_{i=1}^d$ be the successive minima of K with respect to $\mathbb{Z}^d$. Using an argument similar to the one above one can show that that the vectors $v_1, \dots, v_d \in \mathbb{Z}^d$ associated with $\{\lambda_i\}_{i=1}^d$ may be taken as a rearrangement of $e_1, \dots, e_d$. We may assume without loss of generality that λ_d corresponds to e_d. So $e_d \in \lambda_d K$ and $\frac{1}{\lambda_d} e_d \in K$. Thus $\#(K \cap L_{e_d}) \le 2\lfloor \frac{1}{\lambda_d} \rfloor + 1$, where, as before, L_{e_d} is a line containing e_d and the origin. Using (5.5), we get

$$\#K \le \left(2 \left\lfloor \frac{1}{\lambda_d} \right\rfloor + 1\right) \#(K \cap {e_d}^{\perp}).$$

By assumption we have $\lambda_d \le 1$ and, using $\lambda_d \ge \lambda_i$, for all $i = 1, \dots, d$, we get

$$2 \left\lfloor \frac{1}{\lambda_d} \right\rfloor + 1 \le \frac{3}{\lambda_d} \le O(d) \left(\frac{1}{d!} \prod_{i=1}^{d} \frac{2}{\lambda_i} \right)^{1/d}.$$

Finally we use Theorem 3.3.4 to finish the proof:

$$\#K \le O(d) \#(K \cap {e_d}^{\perp}) \operatorname{vol}_d(K)^{\frac{1}{d}}.$$

The cross-polytope $B_1^d = \operatorname{conv}\{\pm e_1, \dots, \pm e_d\}$ of $\operatorname{vol}(B_1^d) = 2^d/d!$ shows that the bound is optimal up to multiplication with constants. The maximal section through the origin is the B_1^{d-1} slice with $2^{d-1}+1$ points. So

$$\#B_1^d = 2^d + 1 \le C\left(2^{d-1} + 1\right) \left(\frac{2^d}{d!}\right)^{\frac{1}{d}}$$

gives that C must have growth of $O(d)$.

$\square$

5.6 Discrete Brunn's Theorem

The idea of the proof of the above theorem follows from the classical Brunn's theorem: the central hyperplane section of a convex origin-symmetric body is maximal in volume among all parallel sections (see [23], [45], [79]). One may notice that, in general, it may not be the case that the maximal hyperplane in cardinality for an origin-symmetric convex body passes through the origin. For example, see Figure 4.1 above, or consider an example of a cross-polytope $B_1^d = \{x \in \mathbb{R}^d : \sum |x_i| \leq 1\}$, then $\#(B_1^d \cap (1/\sqrt{d}, \ldots, 1/\sqrt{d})^{\perp}) = 1$ but a face of B_1^d contains d integer points. We can also find that the ratio can be of an exponential order if we consider a cube of dimension $d-1$ embedded in the following way. Let $Q = [0,1]^{d-1}$ be the cube of unit volume in $\mathbb{R}^{d-1}$, and let $K = \mathrm{conv}\,(Q + e_d, -Q - e_d)$. Then $K \cap e_d^{\perp} + e_d = 2^{d-1}$ while $K \cap e_d^{\perp} = 1$.

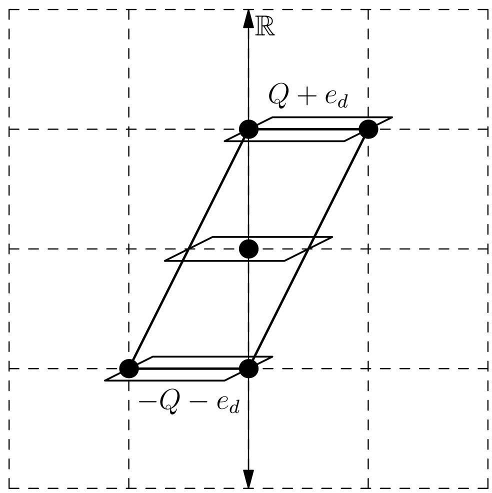


Figure 5.1: Exponential nature of the failure of a discrete analog of Brunn's theorem

While there is no equivalent of Brunn's theorem, still, we propose the following analog of Brunn's theorem in the discrete setting:

Theorem 5.6.1. *Consider a convex, origin-symmetric body $K \subset \mathbb{R}^d$ and a lattice $\Gamma \subset \mathbb{R}^d$ of rank d, then*

$$\#(K \cap \xi^{\perp} \cap \Gamma) \geq 9^{-(d-1)} \#(K \cap (\xi^{\perp} + t\xi) \cap \Gamma), \text{ for all } t \in \mathbb{R} \text{ and } \xi \in \mathbb{S}^{d-1}.$$

Before proving Theorem 5.6.1 we need to recall two nice packing estimates (see Lemma 3.21, [88]):

Lemma 5.6.2. *Let Λ be a lattice in $\mathbb{R}^d$. If $A \subset \mathbb{R}^d$ is an arbitrary bounded set and $P \subset \mathbb{R}^d$ is a finite non-empty set, then*

$$\#\left(A \cap (\Lambda + P)\right) \leq \#\left((A - A) \cap (\Lambda + P - P)\right). \tag{5.6}$$

If $B \subset \mathbb{R}^d$ is an origin-symmetric convex body, then

$$(kB) \cap \Lambda \textit{ can be covered by } (4k+1)^d \textit{ translates of } B \cap \Lambda. \tag{5.7}$$

Proof of Theorem 5.6.1: We first recall a standard observation, that the convexity of K gives us

$$K \cap \xi^\perp \supset \frac{1}{2}(K \cap (\xi^\perp + t\xi)) + \frac{1}{2}(K \cap (\xi^\perp - t\xi)).$$

Let $\Gamma' = \Gamma \cap \xi^\perp$ and assume that $\Gamma \cap (\xi^\perp + t\xi) \neq \emptyset$ (the statement of the theorem is trivial in the other case). Consider a point $\boldsymbol{\gamma} \in \Gamma \cap (\xi^\perp + t\xi)$ then

$$\Gamma \cap (\xi^\perp + t\xi) = \boldsymbol{\gamma} + \Gamma' \text{ and } \Gamma \cap (\xi^\perp - t\xi) = -\boldsymbol{\gamma} + \Gamma'.$$

Moreover,

$$K \cap \xi^\perp \supset \frac{1}{2}\left(\left[K \cap (\xi^\perp + t\xi)\right] - \boldsymbol{\gamma}\right) + \frac{1}{2}\left(\left[K \cap (\xi^\perp - t\xi)\right] + \boldsymbol{\gamma}\right).$$

Thus

$$\left(K \cap \xi^\perp\right) \cap \Gamma' \supset \left[\frac{1}{2}\left(\left[K \cap (\xi^\perp + t\xi)\right] - \boldsymbol{\gamma}\right) + \frac{1}{2}\left(\left[K \cap (\xi^\perp - t\xi)\right] + \boldsymbol{\gamma}\right)\right] \cap \Gamma'.$$

Our goal is to estimate the number of lattice points on the right hand side of the above inclusion. Let

$$B = \frac{1}{2}\left(\left[K \cap (\xi^\perp + t\xi)\right] - \boldsymbol{\gamma}\right)$$

then, using the symmetry of K, we get

$$-B = \frac{1}{2}\left(\left[K \cap (\xi^\perp - t\xi)\right] + \boldsymbol{\gamma}\right).$$

Thus $B - B$ is an origin-symmetric convex body in $\xi^\perp$. Next we use (5.7) from Lemma 5.6.2 to claim that

$$\#(2(B-B) \cap \Gamma') \leq 9^{d-1}\#((B-B) \cap \Gamma').$$

Notice that $2(B-B) = 2B - 2B$ thus we may use (5.6) from Lemma 5.6.2 with $P = \{\mathbf{0}\}$, $\xi^\perp$ associated with $\mathbb{R}^{d-1}$, and $\Lambda = \Gamma'$ to claim that

$$\#(2(B-B) \cap \Gamma') = \#((2B-2B) \cap \Gamma') \geq \#(2B \cap \Gamma') = \#(2B \cap \Gamma).$$

Thus we proved that

$$\begin{aligned}\#\Bigg(&\left[\frac{1}{2}(K \cap (\xi^\perp + t\xi) - \boldsymbol{\gamma}) + \frac{1}{2}(K \cap (\xi^\perp - t\xi) + \boldsymbol{\gamma})\right] \cap \Gamma\Bigg) \\ &\geq 9^{-(d-1)}\#\left(\left[K \cap (\xi^\perp + t\xi) - \boldsymbol{\gamma}\right] \cap \Gamma\right)\end{aligned}$$

but

$$\#\left(\left[K \cap (\xi^{\perp} + t\xi) - \gamma\right] \cap \Gamma\right) = \#\left(\left[K \cap (\xi^{\perp} + t\xi)\right] \cap \Gamma\right).$$

$\square$

Corollary 5.6.3. *Consider a convex, origin-symmetric body $M \subset \mathbb{R}^d$, lattice $\Lambda \subset \mathbb{R}^d$ and m-dimensional lattice subspace H, i.e., it contains m linearly independent points of Λ, then*

$$\#(M \cap H \cap \Lambda) \geq 9^{-m} \#(M \cap (H + z) \cap \Lambda), \text{ for all } z \in \mathbb{R}^d.$$

Proof: Let $z \in \mathbb{R}^d$. Then we may assume $z \in \Gamma \setminus \{H \cap \Gamma\}$ and let U be the linear space spanned by H and z. Then $\dim(U) = m + 1$ and the corollary follows from Theorem 5.6.1 with U associated with $\mathbb{R}^{m+1}$, $K = M \cap U$, and $\Gamma = \Lambda \cap U$.

$\square$

5.7 Solution in General

Let $G_{\mathbb{Z}}(i, d)$ be the set of all i-dimensional linear subspaces containing i-linearly independent lattice vectors of $\mathbb{Z}^d$, i.e., the set of all i-dimensional lattice hyperplanes. The next theorem gives a general bound on the number of integer points in co-dimensional slices.

We would like to estimate the number of points in $K \cap \mathbb{Z}^d$ using the number of points from $\mathbb{Z}^d$ in a central hyperplane section of K. Our goal is to find a direction for which the lattice width of K is small enough and use the discrete version of Brunn's theorem.

For a convex body $K \subset \mathbb{R}^d$ the usual definition of the width of the body K with respect to a vector $u \in \mathbb{R}^d$ is given by

$$w_K(u) = \max_{x \in K} u \cdot x - \min_{x \in K} u \cdot x.$$

For an origin symmetric body we may simply take twice the maximum. Often, u is taken to be a vector of unit length to give width of the body in a direction. However, we wish to consider the width with respect to our lattice, and so we restrict the vectors u to be in the lattice. For $u \in \mathbb{Z}^d \setminus \{0\}$ the *lattice width* of $K \in \mathcal{K}_0^d$ in the direction u is

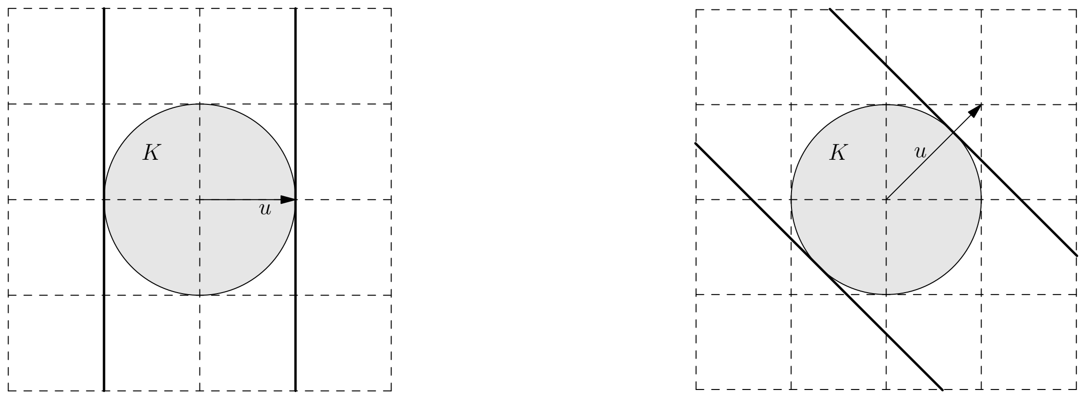
$$w_K(u) = 2 \max_{x \in K} x \cdot u.$$

Our interest is in the minimum width with respect to the lattice.

It is well known that the dual norm is the width of the body, i.e. $w_K(u) = ||u||^*_K$. Let $\lambda_i^* = \lambda_i(K^*, \mathbb{Z}^d)$ be the successive minima of the polar body of K. Then the minimal lattice width of our body K is $2\lambda_1^*$ since

$$||z||_{K^*} = \min\{t \in \mathbb{R} : z \in tK^*\} \geq \lambda_1^*,$$

for any $z \in \mathbb{Z}^d \setminus \{0\}$ and by the definition of the successive minima there exists a non-zero lattice point w with $||w||_{K^*} = \lambda_1^*$.



$w_K(u) = 2$ $\qquad$ $w_K(u) = 2\sqrt{2}$

Figure 5.2: Examples of width with respect to the lattice

Theorem 5.7.1. *Let $K \subset \mathbb{R}^d$ be an origin-symmetric convex body with $\dim(K \cap \mathbb{Z}^d) = d$. Then*

$$\#K \leq O(1)^d\, d^{d-m} \max\{\#(K \cap H) : H \in \mathrm{G}_{\mathbb{Z}}(m,d)\}\ \mathrm{vol}_d(K)^{\frac{d-m}{d}}. \tag{5.8}$$

Obviously, for $m = d-1$ we obtain the estimate for hyperplane slices

$$\#K \leq O(1)^d \max_{\xi \in \mathbb{S}^{d-1}} \left(\#(K \cap \xi^\perp)\right) \mathrm{vol}_d(K)^{\frac{1}{d}}. \tag{5.9}$$

Proof: Let $\{\lambda_i^*\}_{i=1}^d$ be the successive minima of the polar body

$$K^* = \{y \in \mathbb{R}^d : y \cdot x \leq 1, \text{ for all } x \in K\}$$

with respect to $\mathbb{Z}^d$ and let $v_1, \ldots, v_d \in \mathbb{Z}^d$ be the associated directional basis. These vectors are linearly independent and $v_i \in \lambda_i^*\, K^*$ for all i. Thus we have

$$K \subseteq \{x \in \mathbb{R}^d : |v_i \cdot x| \leq \lambda_i^*,\ 1 \leq i \leq d\}. \tag{5.10}$$

Let $U = \text{span}\{v_1, \ldots, v_{d-m}\}$ and let $\overline{H} = U^{\perp}$ be the orthogonal complement of U. Observe that $\overline{H} \in \mathrm{G}_{\mathbb{Z}}(m,d)$. Since for $z \in \mathbb{Z}^d$ we have $v_i \cdot z \in \mathbb{Z}$, $1 \leq i \leq d$, we also have $v_i \cdot (z\big|U) \in \mathbb{Z}$, $1 \leq i \leq d-m$, where $z\big|U$ is the orthogonal projection onto U. In view of (5.10) we obtain

$$(K \cap \mathbb{Z}^d)\big|U \subset \{y \in U : v_i \cdot y \in \mathbb{Z} \text{ and } |v_i \cdot y| \leq \lambda_i^*,\, 1 \leq i \leq d-m\}, \tag{5.11}$$

and thus

$$\#((K \cap \mathbb{Z}^d)\big|U) \leq \prod_{i=1}^{d-m} (2\lfloor \lambda_i^* \rfloor + 1)\,. \tag{5.12}$$

Due to our assumption that K contains d-linearly independent lattice points we have that $\lambda_1^* \geq 1$; otherwise (5.10) implies $v_1 \cdot z = 0$ for all $z \in K \cap \mathbb{Z}^d$. So we conclude by (5.11)

$$\begin{aligned}
\#K &\leq \#((K \cap \mathbb{Z}^d)|U)\, \max\{\#(K \cap (z + \overline{H})) : z \in \mathbb{Z}^d\} \\
&\leq \max\{\#(K \cap (z + \overline{H})) : z \in \mathbb{Z}^d\}\, 3^{d-m} \prod_{i=1}^{d-m} \lambda_i^* \\
&\leq 3^{d-m}\, O(1)^m \#(K \cap \overline{H}) \prod_{i=1}^{d-m} \lambda_i^* \leq O(1)^d\, \#(K \cap \overline{H}) \prod_{i=1}^{d-m} \lambda_i^*.
\end{aligned} \tag{5.13}$$

Here the last step follows from Corollary 5.6.3, the co-dimensional version of the discrete Brunn's Theorem.

Next Minkowski's Second Theorem (Theorem 3.3.4) gives the upper bound

$$\lambda_1^* \cdot \ldots \cdot \lambda_d^* \mathrm{vol}_d(K^*) \leq 2^d \tag{5.14}$$

and so we find

$$\left(\prod_{i=1}^{d-m} \lambda_i^*\right)^d \mathrm{vol}_d(K^*)^{d-m} \leq \left(\prod_{i=1}^{d} \lambda_i^*\right)^{d-m} \mathrm{vol}_d(K^*)^{d-m} \leq 2^{d(d-m)}. \tag{5.15}$$

Hence

$$\prod_{i=1}^{d-m} \lambda_i^* \leq 2^{d-m}\, \mathrm{vol}_d(K^*)^{\frac{m-d}{d}}. \tag{5.16}$$

By the Bourgain-Milman inequality (isomorphic version of reverse Santaló inequality, see [13, 25, 69, 52] or [79],) there exists an absolute constant $c > 0$ with

$$c^d \frac{4^d}{d!} \leq \mathrm{vol}_d(K)\mathrm{vol}_d(K^*)$$

and so we get

$$\mathrm{vol}_d(K^*)^{\frac{m-d}{d}} \leq O(d)^{d-m} \mathrm{vol}_d(K)^{\frac{d-m}{d}}. \tag{5.17}$$

Thus together with (5.16) and (5.13) we obtain

$$\#K \leq O(1)^d\, d^{d-m}\, \max\{\#(K \cap H) : H \in \mathrm{G}_{\mathbb{Z}}(m,d)\}\, \mathrm{vol}_d(K)^{\frac{d-m}{d}}. \tag{5.18}$$

$\square$

Remark 5.7.2. We notice that the methods used in Section 3, i.e. computation via discrete version of John's theorem (Theorem 5.3.2 from above), can also be used to provide a bound for general co-dimensional sections. But such computation gives the estimate of order $O(d)^{7d/2}$ which is worse than the one in the above theorem.

Remark 5.7.3. Observe that Theorem 5.7.1 can be restated for an arbitrary d-dimensional lattice Λ: Let Λ be a lattice in $\mathbb{R}^d$ and $K \subset \mathbb{R}^d$ be an origin-symmetric convex body with $\dim(K \cap \Lambda) = d$. Then

$$\#K \leq O(1)^d \, d^{d-m} \max\{\#(K \cap H) : H \in \mathrm{G}_\Lambda(m,d)\} \left(\frac{\mathrm{vol}_d(K)}{\det(\Lambda)}\right)^{\frac{d-m}{d}}. \tag{5.19}$$

We also notice that the methods used in the proofs of Theorem 5.5.1 and Theorem 5.7.1 can be used to provide an estimate for the co-dimensional slices of an unconditional convex body:

Theorem 5.7.4. *Let $K \subset \mathbb{R}^d$ be an unconditional convex body with $\dim(K \cap \mathbb{Z}^d) = d$. Then*

$$\#K \leq O(d)^{d-m} \max\{\#(K \cap H) : H \in \mathrm{G}_{\mathbb{Z}}(m,d)\} \; \mathrm{vol}_d(K)^{\frac{d-m}{d}}. \tag{5.20}$$

Proof: First we notice that if K is an unconditional body and H is a coordinate subspace of dimension m (i.e. it is spanned by m coordinate vectors) with $K \cap (H+z) \neq \emptyset$, then $K \cap (H+z)$ must be an unconditional convex body in $(H+z)$. Thus, using this property together with the proof of Theorem 5.5.1 we get that for any unconditional body K and for any coordinate subspace H

$$\#(K \cap H \cap \mathbb{Z}^d) \geq \#(K \cap (H+z) \cap \mathbb{Z}^d), \text{ for all } z \in \mathbb{R}^d.$$

Next we follow the steps of the proof of Theorem 5.7.1 and similarly to (5.13) get

$$\#K \leq 3^{d-m} \#(K \cap \overline{H}) \prod_{i=1}^{d-m} \lambda_i^*.$$

Finally, we finish the proof using Minkowski's Second Theorem and the Bourgain-Milman inequality.

$\square$

Remark 5.7.5. We also would like to test our estimates against two classical examples

(A) For the cube $B_\infty^d = \{x \in \mathbb{R}^d : |x|_\infty \le 1\}$ we have $\#B_\infty^d = 3^d$,

$\max\{\#(B_\infty^d \cap H) : H \in \mathrm{G}_{\mathbb{Z}}(m,d)\} = 3^m$, and $\mathrm{vol}_d(B_\infty^d)^{\frac{d-m}{d}} = 2^{d-m}$.

(B) For the cross polytope $B_1^d = \{x \in \mathbb{R}^d : |x|_1 \le 1\}$ we have $\#B_1^d = 2\,d+1$,

$\max\{\#(B_1^d \cap H) : H \in \mathrm{G}_{\mathbb{Z}}(m,d)\} = 2\,m+1$, and $\mathrm{vol}_d(B_1^d)^{\frac{d-m}{d}} \sim \frac{c^{d-m}}{d^{d-m}}$.

These examples show that we expect our constant to grow exponentially in the case of higher co-dimensional slices, though we do not expect our current estimates to be sharp.

We would like to add that Oded Regev [73] recently proved that for bodies of volume at most C^{d^2} one can improve our estimate for $\mathcal{L}_3$ and in fact get a tight bound of $O(d)$.

We finish this section with a remark about the relationship between the constant in the original slicing inequality, $\mathcal{L}_1$, and the constant in the discrete version, $\mathcal{L}_3$. Using the general idea from [24] and Gauss's Lemma on the intersection of a large convex body with a lattice we will show that $\mathcal{L}_1 \le \mathcal{L}_3$.

Consider a convex symmetric body K and let $\mathcal{L}_1(K) > 0$ be such that

$$\mathrm{vol}_d(K)^{\frac{d-1}{d}} = \mathcal{L}_1(K) \max_{\xi \in \mathbb{S}^{d-1}} \mathrm{vol}_{d-1}(K \cap \xi^\perp).$$

Thus

$$\mathcal{L}_1 = \max\{\mathcal{L}_1(K) : K \subset \mathbb{R}^d, K \text{ is convex, origin-symmetric body, } d \ge 1\}.$$

Then

$$\mathrm{vol}_d(K)^{\frac{d-1}{d}} \ge \mathcal{L}_1(K)\mathrm{vol}_{d-1}(K \cap \xi^\perp), \quad \text{for all } \xi \in \mathbb{S}^{d-1}.$$

Our goal is to study a central section of K with a maximal number of points from $\mathbb{Z}^d$, if $K \cap \xi^\perp$ is such a section, then, without loss of generality, we may assume that $\mathbb{Z}^d \cap \xi^\perp$ is a lattice of a full rank $d-1$. Indeed, if $\mathbb{Z}^d \cap \xi^\perp$ has a rank less then $d-1$ we may rotate ξ to catch $d-1$ linearly independent vectors in $\xi^\perp$, without decreasing the number of integer points in $K \cap \xi^\perp$. Now, we may use Gauss's Lemma (see for example Lemma 3.22 in [88]) to claim that for r large enough we have

$$\#(rK) = r^d \mathrm{vol}_d(K) + O\left(r^{d-1}\right) \text{ and}$$

$$\#(rK \cap \xi^\perp) = \frac{r^{d-1}\mathrm{vol}_{d-1}\left(K \cap \xi^\perp\right)}{\det\left(\mathbb{Z}^d \cap \xi^\perp\right)} + O\left(r^{d-2}\right),$$

which we can rearrange to get the following two equations

$$\mathrm{vol}_d\left(K\right) = \frac{1}{r^d}\#\left(rK\right) + O\left(\frac{1}{r}\right) \text{ and}$$

$$\mathrm{vol}_{d-1}\left(K\cap\xi^{\perp}\right)=\frac{1}{r^{d-1}}\#\left(rK\cap\xi^{\perp}\right)\det\left(\mathbb{Z}^d\cap\xi^{\perp}\right)+O\left(\frac{1}{r}\right).$$

Next, using that $\det\left(\mathbb{Z}^d\cap\xi^{\perp}\right)\geq 1$ and $\mathrm{vol}_d(K)\geq\mathcal{L}_1(K)\mathrm{vol}_{d-1}\left(K\cap\xi^{\perp}\right)\mathrm{vol}_d^{\frac{1}{d}}(K)$ we get

$$\begin{aligned}\frac{1}{r^d}\#\left(rK\right)\geq&\mathcal{L}_1(K)\left(\frac{1}{r^{d-1}}\right)\#\left(rK\cap\xi^{\perp}\right)\det\left(\mathbb{Z}^d\cap\xi^{\perp}\right)\mathrm{vol}_d^{\frac{1}{d}}(K)+O\left(\frac{1}{r}\right)\\ \geq&\mathcal{L}_1(K)\left(\frac{1}{r^d}\right)\#\left(rK\cap\xi^{\perp}\right)\mathrm{vol}_d^{\frac{1}{d}}(rK)+O\left(\frac{1}{r}\right).\end{aligned}$$

Then for $\epsilon>0$ there is a sufficiently large r_0 such that for all $r>r_0$

$$\#(rK)\geq\left(\mathcal{L}_1(K)-\epsilon\right)\#\left(rK\cap\xi^{\perp}\right)\mathrm{vol}_d^{\frac{1}{d}}(rK).$$

So then if $\#(rK)\leq\mathcal{L}_3\max_{\xi\in\mathbb{S}^{d-1}}\#\left(rK\cap\xi^{\perp}\right)\mathrm{vol}_d^{\frac{1}{d}}(rK)$ we have that $\mathcal{L}_1(K)-\epsilon\leq\mathcal{L}_3$ for all d, ϵ, and bodies K. Which leads us to conclude that $\mathcal{L}_1\leq\mathcal{L}_3$.

Part III

Discrete Considerations for the Volume Product

CHAPTER 6

Shadow Systems

6.1 Introduction

Shadow Systems were introduced by Rogers and Shephard [77] and further generalized by Shephard [84] to solve extremal problems. A *shadow system of convex sets* along a direction $\theta \in S^{d-1}$ is a family of convex sets $K_t \in \mathbb{R}^d$ which are defined by

$$K_t = \operatorname{conv}\{x + \alpha(x)t\theta : x \in B\},$$

where $B \subset \mathbb{R}^d$ is a bounded set, called the basis of the shadow system, $\alpha : B \to \mathbb{R}$ is a bounded function, called the speed of the shadow system, and t belongs to an open interval $J \subset \mathbb{R}$. We say that a shadow system is *non-degenerate*, if all the convex sets K_t have non-empty interior.

To better understand the origin of the name, we can explore the following more general approach. Let C be a closed convex set in $\mathbb{R}^{d+1}$. Let $(e_1, \ldots, e_{d+1})$ be an orthonormal basis of $\mathbb{R}^{d+1}$, write $\mathbb{R}^{d+1} = \mathbb{R}^d \oplus \mathbb{R}e_{d+1}$, so that $\mathbb{R}^d = e_{d+1}^{\perp}$. For every $u \in \mathbb{R}^d$ let P_u be the projection onto $\mathbb{R}^d$ parallel to $e_{d+1} - u$, that is, for $z \in \mathbb{R}^d$ and $s \in \mathbb{R}$,

$$P_u(z + se_{d+1}) = z + su.$$

We denote $K_u = P_u(C) \subset \mathbb{R}^d$. Let I be a convex subset of $\mathbb{R}^d$. Then the family $(K_u)_{u \in I}$ is a shadow system of convex sets.

We can reconcile the two definitions in the following way. Consider a base $B \subset \mathbb{R}^d$ (i.e. the set that we would like to move, from the first definition). Now take $C = \operatorname{conv}\{x + \alpha(x)e_{d+1} : x \in B\}$. Then

$$K_t = P_u(C) = \operatorname{conv}\{x + \alpha(x)u : x \in B\}$$

for t from some open interval $J \subset \mathbb{R}$ and $u = t\theta$.

A well known example of a shadow system is the Steiner Symmetral [91, 79]. For a convex body $K \subset \mathbb{R}^d$ and a direction $\xi \in S^{d-1}$, the Steiner Symmetral of K in the direction ξ is a body of the same volume

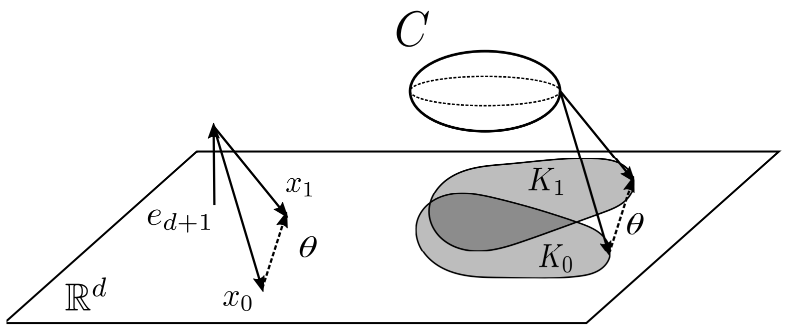


Figure 6.1: A shadow system

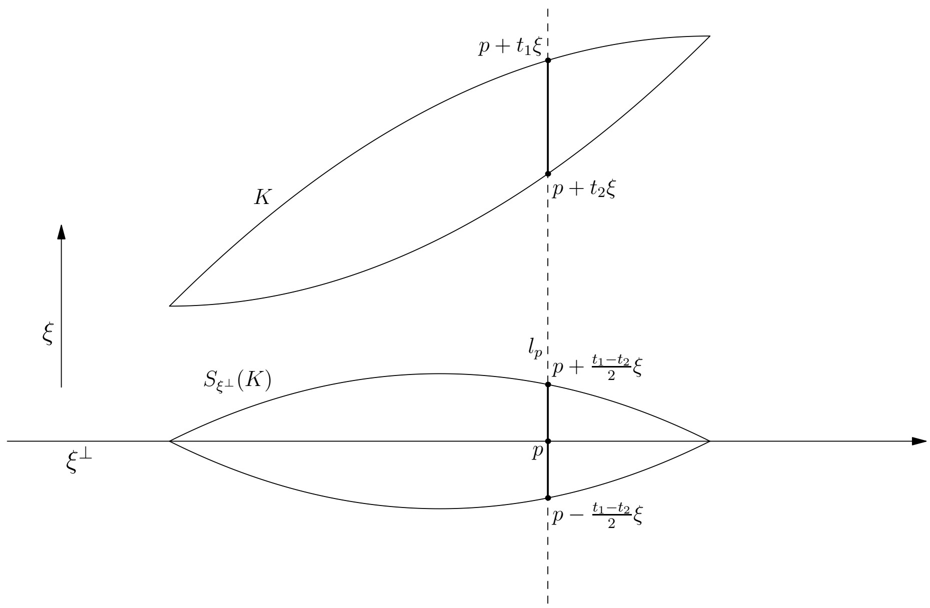


Figure 6.2: The Steiner Symmetral

as K made symmetric about the hyperplane $\xi^\perp$ in the following way. For each $p \in \xi^\perp$ let ℓ_p be the line perpendicular to $\xi^\perp$ through p. Then translate each segment $K \cap \ell_p$ so that the midpoint falls on the hyperplane $\xi^\perp$. The endpoints of each segment come from the basis B and are translated in the direction of ξ with speed appropriately determined.

By repeated application of the Steiner Symmetral it can be shown that for any convex body K there is a sequence of convex bodies which converge to the Euclidean ball of the same volume. This can be used to show, for example, the isoprimetric inequality and Brunn-Minkowski inequality.

6.2 Convexity of the volume of shadow systems

Our primary use for shadow systems will be for the following tools which allow us to compare the volume product of polytopes related by a shadow system.

Rogers and Shephard [77] proved the following theorem.

Theorem 6.2.1. *(Rogers, Shephard):For a shadow system $(K_x)_{x\in I}$, $x \mapsto |K_x|$ is convex on I.*

Campi and Gronchi [16] showed a similar result for polar bodies.

Theorem 6.2.2. *(Campi, Gronchi): Let I be a convex subset of $\mathbb{R}^d$ and $(K_x)_{x\in I}$ be a shadow system of origin symmetric convex bodies in $\mathbb{R}^d$, then $x \mapsto |K_x^\circ|^{-1}$ is convex.*

As a corollary, if the volume of K_x constant, then $x \mapsto \mathcal{P}(K_x)^{-1}$ is convex. Moreover if the function $x \mapsto |K_x|$ is affine then $x \mapsto \mathcal{P}(K_x)$ is the quotient of an affine function by a convex one. As it was noticed in [60] Lemma 12 and in [19] Corollary 2, it follows that it is quasi-concave: i.e. $\{x \in \mathbb{R}^d : \mathcal{P}(K_x) \geq s\}$ is convex, for every $s > 0$.

In [63], Meyer and Reisner generalized Theorem 6.2.2 to the non-symmetric case and studied the equality case. The following proposition is the key tool for us:

Theorem 6.2.3. *(Meyer, Reisner): Let I be a convex subset of $\mathbb{R}^d$ and $(K_x)_{x\in I}$, be a shadow system of convex bodies in $\mathbb{R}^d$ then $x \mapsto |K_x^{s(K_x)}|^{-1}$ is convex on I.*
If, moreover, $x \mapsto |K_x|$ is affine on I and $x \mapsto \mathcal{P}(K_x)$ is constant on I, then there exists $w \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$, such that for every $x, x_0 \in I$, one has $K_x = A_x(K_{x_0})$, where $A_x : \mathbb{R}^d \to \mathbb{R}^d$ is the affine map defined by

$$A_x(z) = z + (\langle w, z\rangle + \alpha)(x - x_0).$$

Finally, the following proposition is a combination of Propositions 1 and 2 of Kim and Reisner [43] which will help us to estimate the behaviour of $|L^z|$ when z is close enough to $s(L)$.

Theorem 6.2.4 ([43])**.** *Let K and L be two convex bodies in $\mathbb{R}^d$. Then there exists $\delta(K)$ such that, if $d_H(K, L) \leq \delta(K)$ then*

$$|L^{s(L)}| = |L^{s(K)}| + O(d_H(K, L)^2),$$

where O depends only on K. As a consequence, if the Santaló point of K is at the origin and $d_H(K, L) \leq \delta(K)$ then

$$\mathcal{P}(L) = |L||L^\circ| + O(d_H(K, L)^2).$$

A motivating example of the convexity of the volume product can be seen in Mahler's solution of his conjecture for $d = 2$ [79]. The idea of the proof is to translate vertices individually as a shadow system such that the volume of any polytope is preserved. In figure 6.3 below, this can be seen as the vertex z is translated in the direction $\theta = v_3 - v_1$. By theorem 6.2.2 we know that the volume product is minimal at the boundary of the domain of the function, which occurs when adjacent vertices disappear, when $t = \alpha$ or $t = \beta$ in the below figure. This process can be repeated until only four vertices remain.

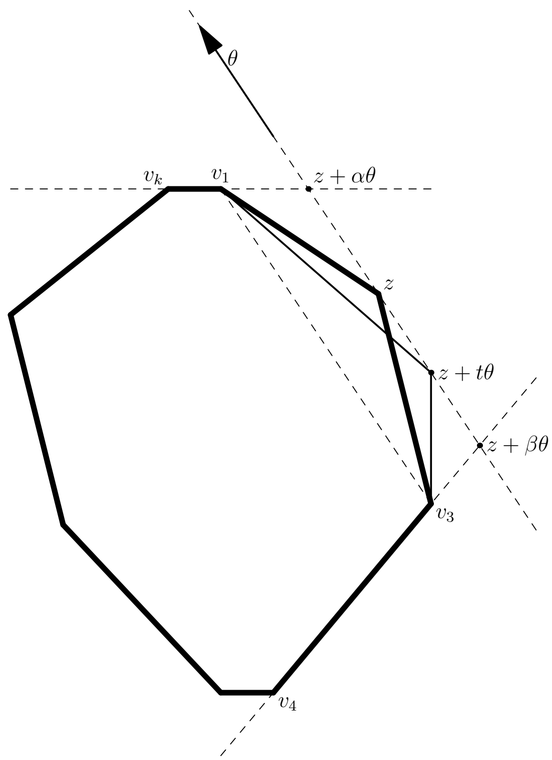


Figure 6.3: Sketch of the proof of Mahler's conjecture in dimension 2

CHAPTER 7

Polytopes of Maximal Volume Product

7.1 Introduction

We begin by recalling several definitions from chapter 2. The polar body K^z of K with the center of polarity z is defined by

$$K^z = \{y \in \mathbb{R}^d : \langle y - z, x - z\rangle \leq 1 \text{ for all } x \in K\}.$$

If the center of polarity is taken to be the origin, we denote the polar body of K by K° . Note that $K^z = (K - z)^\circ + z$, and the bipolar theorem says that $(K^z)^z = K$, for $z \in \text{int}(K)$ (see [35], p. 47).

The goal of this chapter is to study the maximal value of the volume product when we restrict ourselves to the class of polytopes with a bounded number of vertices. In Theorem 7.2.4 we show that the maximum value of the volume product among all convex polytopes in $\mathbb{R}^d$ with m vertices is increasing in m. Next, in Theorem 7.2.6 we prove that the polytopes of maximal volume product among polytopes with at most m vertices must satisfy some identities which imply in particular that it is simplicial.

In Section 7.3 we give a new proof of the result of Meyer and Reisner [64] showing that the regular N-gon is the only N-gon with maximal volume product among polygons with at most N vertices. Then, in section 7.4, we consider the case of convex polytopes with $d + 2$ vertices in $\mathbb{R}^d$ and in theorem 7.4.3 we find that the polytope with maximal volume product is the convex hull of two simplices living in supplementary affine subspaces of dimensions $\lceil \frac{d}{2} \rceil$ and $\lfloor \frac{d}{2} \rfloor$.

The following (classical) lemma is a key observation for us to treat the maximal cases of the volume product:

Lemma 7.1.1. *Let $K \in \mathbb{R}^d$ be a convex body, and F a concave continuous function $F : K \to \mathbb{R}$. Assume that K and F are invariant under linear isometries $T_1, ..., T_m$. Then there is $x_0 \in K$ such that $T_i(x_0) = x_0$, for all $i = 1, \dots, m$ and $F(x_0) \geq F(x)$ for all $x \in K$.*

Proof: Let us first assume that the function F is strictly concave, i.e. $F((x + y)/2) > (F(x) + F(y))/2$, for $x \neq y$. Then by continuity of F and compactness of K, the maximum of F is reached at $x_0 \in K$, moreover this point is unique by strict concavity, indeed if $x, y \in K$ are two distinct maximums, then $F((x+y)/2) > F(x)$ and $(x+y)/2 \in K$.

Moreover the function F is invariant under a map T_i so then $F(T_i x_0) = F(x_0)$. But because the maximum is reached at unique point we have $T_i x_0 = x_0$.

Now if F is only concave and not necessary strictly concave, we may approximate F by a sequence of strictly concave functions $F_k(x) = F(x) - |x|^2/k$. The maps T_i are isometries and thus $F_k(T_i x) = F_k(x)$ for all $i \in 1, \dots, m$ and $k \in \mathbb{N}$. By the previous argument applied to F_k, we deduce that for each k there is a unique $x_k \in K$ such that $\max_{x \in K} F_k(x) = F_k(x_k)$ and $T_i x_k = x_k$ for all $i \in 1, \dots, m$. Since K is compact we may select a convergent subsequence $\{x_{k_l}\}$ of $\{x_k\}$. Let $\lim x_{k_l} = x_0$, then $x_0 \in K$ and by continuity of T_i,we get $T_i x_0 = x_0$ for all i. Moreover, by continuity of F we get $\lim F(x_{k_l}) = F(x_0)$, therefore $\max_{x \in K} F(x) = F(x_0)$.

□

The preceding lemma also has an interesting classical consequence regarding the Santaló point of a convex body having some symmetries. Recall that the support function of a convex body K is defined by $h_K(u) = \sup_{x \in K} \langle x, u \rangle$, for all $u \in \mathbb{R}^d$. Thus for every $z \in \mathbb{R}^d$,

$$h_{K-z}(u) = \sup_{x \in K-z} \langle x, u \rangle = \sup_{y \in K} \langle y - z, u \rangle = h_K(u) - \langle z, u \rangle.$$

Hence, using the the usual polar integration volume formula, one has, for every $z \in \text{int}\,(K)$,

$$F_K(z) := |K^z| = |(K-z)^\circ| = \frac{1}{d} \int_{S^{d-1}} \frac{d\sigma(u)}{h_{K-z}(u)^d} = \frac{1}{d} \int_{S^{d-1}} \frac{d\sigma(u)}{(h_K(u) - \langle z, u \rangle)^d},$$

where σ is the Lebesque surface area measure on S^{d-1}. From this formula, it follows that the function F_K is strictly convex in $\text{int}\,(K)$ and tends to infinity when z tends to the boundary of K. This explains the existence and uniqueness of the minimizer of F_K, which is the Santaló point of K. If we assume that K is invariant with respect to a linear isometry T then, for every $z \in K$, we have

$$F_K(Tz) = F_{TK}(Tz) = |(T(K-z))^\circ| = |(K-z)^\circ| = F_K(z).$$

Thus F_K has the same symmetries as K and from the preceding lemma, we get that the Santaló point of K satisfies $T(s(K)) = s(T(K)) = s(K)$. Thus the Santaló point is invariant under linear transformations, and further, by the definition of the polar body, invariant under translation. We call such a point affine invariant. Affine invariant points are linked to the symmetry of a body. In general, the fewer unique affine invariant points that a body has the more symmetries it has. There is also a dual correspondence between certain affine invariant points such as the Santaló point and the centroid. For more information on affine invariant points see [65, 66, 89].

7.2 Properties of Polytopes of Maximal Volume Product

Definition 7.2.1. For $d \geq 1$ we denote by $\mathcal{K}^d$ the set of all convex bodies in $\mathbb{R}^d$ endowed with the Hausdorff distance. For $m \geq d+1$, we denote by $\mathbb{P}_m^d$ the subset of $\mathcal{K}^d$ consisting of the polytopes in $\mathbb{R}^d$ with non empty interior having at most m vertices and by $\mathbb{P}^d = \cup_{m\in\mathbb{N}}\mathbb{P}_m^d$, the dense subset of $\mathcal{K}^d$ consisting of all polytopes with non-empty interior. We denote by M_m^d the supremum of the volume product of polytopes with at most m vertices and non empty interior in $\mathbb{R}^d$

$$M_m^d := \sup_{K\in\mathbb{P}_m^d} \mathcal{P}(K).$$

Recall that from Blaschke-Santaló inequality one has $\sup_{K\in\mathcal{K}^d} \mathcal{P}(K) = \mathcal{P}(B_2^d)$. By the continuity of the function $K \mapsto \mathcal{P}(K)$ on $\mathcal{K}_d$ (see for example Lemma 3 in [19]) and the density of $\mathbb{P}^d$ in $\mathcal{K}^d$ we deduce that $\lim_{m\to+\infty} M_m^d = \mathcal{P}(B_2^d)$. Our aim is now to establish that the sequence M_m^d is strictly increasing. We start with a proposition that is of independent interest and gives a better understanding on the behavior of the volume product functional.

Lemma 7.2.2. *Let $d, m \in \mathbb{N}$ with $m \geq d+1$ and $K \in \mathbb{P}_m^d$. Let F be a facet of K with exterior normal $u \in S^{d-1}$, let x_F be in the relative interior of F and let $K_t = \mathrm{conv}(K, x_F + tu)$, for $t > 0$. Then for t small enough the volume product of K_t is strictly larger than the volume product of K:*

$$\mathcal{P}(K_t) > \mathcal{P}(K).$$

Notice that the polytope K_t defined in the above proposition has exactly $m+1$ vertices.

Proof: We may assume that the Santaló point of K is at the origin. Let $h > 0$ such that the affine hyperplane spanned by F is $H = \{x : \langle x, u\rangle = h\}$ and $K \subset H^-$, where $H^- = \{x : \langle x, u\rangle \leq h\}$. Let $F_1, \ldots, F_k$ be the facets of K which are adjacent to F and for $1 \leq i \leq k$, denote by u_i the exterior normal of F_i. Let $h_i > 0$ be such that $H_i = \{x : \langle x, u_i\rangle = h_i\}$ is the spanned affine hyperplane of F_i. Thus

$$K \subset \bigcap_{1\leq i\leq k} H_i^-.$$

We also denote by

$$R = \{x : \langle x, u\rangle \geq h,\ \langle x, u_i\rangle \leq h_i, \quad \text{for all } 1 \leq i \leq k\}$$

the polyhedral region bounded by F and the H_i, $i = 1, \ldots, k$. For every $x \in R$, let $K_x = \mathrm{conv}(K, x)$ then $(K_x)_{x \in R}$ is a shadow system and one has

$$|K_x| = |K| + \frac{1}{d}|F|(\langle x, u - h\rangle).$$

Hence $x \mapsto |K_x|$ is affine and thus $(K_x)_{x \in R}$ is an affine volume shadow system. It follows from Proposition 6.2.3 that the volume product $P(x) := \mathcal{P}(K_x)$ is quasi-concave on R. Let x_F be an interior point of F and let $x_t = x_F + tu$, then if $t > 0$ and small enough we get $x_t \in R$. Moreover, using that $x_F \in F$ and thus $\langle x_F, u\rangle = h$ we get $\langle x_t, u\rangle = h + t$ and

$$|K_{x_t}| = |K| + \frac{t}{d}|F|.$$

By polarity, the point u/h is a vertex of K°, the points u_i/h_i are its adjacent vertices and $K_x^\circ = \{y \in K^\circ; \langle y, x\rangle \le 1\}$ is the truncation of the polytope K° by the halfspace $\{y : \langle y, x\rangle = 1\}$. For every x in the interior of R this truncation cuts off the vertex u/h of K°. It also cuts the edges $[u/h; u_i/h_i]$ at some points $v_i = (1 - \lambda_i)u/h + \lambda_i u_i/h_i$, where $\lambda_i \in [0, 1]$ is determined by the fact that $\langle v_i, x\rangle = 1$. This gives

$$\lambda_i = \frac{(\langle x, u\rangle - h)\, h_i}{\langle x, (h_i u - h u_i)\rangle}.$$

Thus

$$v_i - \frac{u}{h} = -\lambda_i \left(\frac{u}{h} - \frac{u_i}{h_i}\right) = -\frac{\langle x, u\rangle - h}{h} \times \frac{h_i u - h u_i}{\langle x, h_i u - h u_i\rangle}.$$

Moreover one has

$$K^\circ \setminus K_x^\circ = \mathrm{conv}\left(\frac{u}{h}, v_1, \ldots, v_k\right) = \frac{u}{h} + \mathrm{conv}\left(0, v_1 - \frac{u}{h}, \ldots, v_k - \frac{u}{h}\right).$$

Hence

$$\begin{aligned} |K_x^\circ| &= |K^\circ| - \left|\mathrm{conv}\left(0, v_1 - \frac{u}{h}, \ldots, v_k - \frac{u}{h}\right)\right| \\ &= |K^\circ| - \left(\frac{\langle x, u\rangle - h}{h}\right)^d \left|\mathrm{conv}\left(0, \frac{h_1 u - h u_1}{\langle x, (h_1 u - h u_1)\rangle}, \ldots, \frac{h_k u - h u_k}{\langle x, (h_k u - h u_k)\rangle}\right)\right|. \end{aligned}$$

Applying this for $x = x_t$ and using that $\langle x_t, u\rangle = h + t$, we get

$$|K_{x_t}^\circ| = |K^\circ| - \left(\frac{t}{h}\right)^d \left|\mathrm{conv}\left(0, \frac{h_1 u - h u_1}{\langle x_t, (h_1 u - h u_1)\rangle}, \ldots, \frac{h_k u - h u_k}{\langle x_t, (h_k u - h u_k)\rangle}\right)\right|.$$

Thus for t small enough, we obtain

$$|K_{x_t}^\circ| = |K^\circ| + O(t^d).$$

Hence

$$|K_{x_t}||K_{x_t}^\circ| = (|K| + t|F|/d)\left(|K^\circ| + O(t^d)\right) = |K||K^\circ| + t|K^\circ||F|/d + o(t).$$

Moreover it follows from propositions 1 and 2 of Kim and Reisner [43] that if $d_H(K, L)$ is small enough then

$$|L^{s(L)}| \geq |L^{s(K)}| - c(K, L)d_H(K, L)^2,$$

where $c(K, L)$ is a positive constant depending on K and L. Applying this to $L = K_{x_t}$ and using that $d_H(K, K_{x_t}) \leq c(K)t$ for some constant $c(K)$ depending on K only, we get that for $t > 0$, small enough

$$\mathcal{P}(K_{x_t}) \geq \mathcal{P}(K) + t|K^\circ||F|/d + o(t) > \mathcal{P}(K).$$

□

Remark 7.2.3. It is tempting to state lemma 7.2.2 in a stronger form, saying that for any n-dimensional polytope $K \subset \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, such that $\mathrm{conv}(K, x)$ has more vertices than K we get $\mathcal{P}(\mathrm{conv}(K, x)) \geq \mathcal{P}(K)$. But such a statement is wrong. This can be seen by a direct computation, or from the following observation: consider $K = B_\infty^2$ and $x_\epsilon = (10, 1 - \epsilon)$. Then, continuity of the volume product gives us

$$\lim_{\epsilon \to 0} \mathcal{P}(\mathrm{conv}\{B_\infty^2, x_\epsilon\}) = \mathcal{P}(\mathrm{conv}\{(1, -1); (-1, -1); (-1, 1); (10, 1)\}) < \mathcal{P}(B_\infty^2),$$

where the last inequality follows from direct computation (see also Theorem 7.3.1, below).

Theorem 7.2.4. *Let $n \geq 1$ and $m \geq d + 1$. Then the supremum M_m^d is achieved at some polytope with exactly m vertices and the sequence M_m^d is strictly increasing in m.*

Proof: The fact that the supremum M_m^d is achieved follows the proof of the corresponding statement on the infimum established, for example, in Proposition 2 and Lemma 4 of [19]. By the affine invariance of $\mathcal{P}$ and John's theorem one has

$$M_m^d := \sup_{K \in \mathbb{P}_m^d} \mathcal{P}(K) = \sup\{\mathcal{P}(K) : K \in \mathbb{P}_m^d, B_2^d \subset K \subset dB_2^d\}.$$

Note that $\{K \in \mathbb{P}_m^d : B_2^d \subset K \subset dB_2^d\}$ is compact in the Hausdorff distance. Moreover the function $K \mapsto \mathcal{P}(K)$ is continuous on $\mathcal{K}_d$ (see for example Lemma 3 in [19]). Therefore as the supremum of a continuous function $\mathcal{P}$ on a compact set, we conclude that the supremum $M_m^d = \sup_{K \in \mathbb{P}_m^d} \mathcal{P}(K)$ is attained at some polytope K_m with at most m vertices.

Now let us prove that any polytope K_m achieving the supremum has exactly m vertices. The proof goes by induction on m. For $m = d + 1$, the result is clear. Let $m \geq d + 1$ be fixed and assume that the result is known for K_m. So K_m has exactly m vertices. From lemma 7.2.2 there exists x outside K_m such that $K_m(x) = \mathrm{conv}(K_m, x)$ has a volume product strictly larger than K. Since $K_m(x) \in \mathbb{P}^d_{m+1}$, it follows that

$$M^d_{m+1} = \mathcal{P}(K_{m+1}) \geq \mathcal{P}(K_m(x)) > \mathcal{P}(K_m) = M^d_m.$$

We conclude that K_{m+1} has exactly $m+1$ vertices and that the sequence $m \mapsto M^d_m$ is strictly increasing.

□

Remark 7.2.5. Notice that since the Euclidean ball is known to be the maximum in volume product among all bodies, then from this and the above theorem we can see that there is no polytope which is a local maximum of the volume product among all convex bodies.

Recall that one says that a polytope is simplicial if all its facets are simplices.

Theorem 7.2.6. *Let $n \geq 1$ and $m \geq d+1$. Let K be of maximal volume product among polytopes with at most m vertices. Then K is a simplicial polytope.*

For the proof, we need to introduce some more notation concerning polytopes. For any polytope K we denote by $\mathcal{E}(K)$ the set of its vertices and by $\mathcal{F}(K)$ the set of its facets.

Proof: Let K be a polytope with the origin in its interior. For any facet $F \in \mathcal{F}(K)$, we denote u_F its exterior normal and by h_F its distance to the origin. Let x be a vertex of K. Denote by $\mathcal{F}(x)$ the set of facets of K containing x. We denote by F_x the facet of K° corresponding to x:

$$F_x = \{y \in K^\circ; \langle y, x\rangle = 1\} = \left\{ y \in K^\circ; \left\langle y, \frac{x}{|x|} \right\rangle = \frac{1}{|x|} \right\}.$$

Notice that F_x has $\frac{x}{|x|}$ as exterior normal and its distance to the origin is $1/|x|$. Now we introduce a modification of K that was used by Meyer and Reisner in [64] in the plane: we define $K_t = \mathrm{conv}(K, (1+t)x)$, for small values of $t > 0$, so we extend K in the direction of x. We get the following modification for the volume of K:

$$|K_t| = |K| + \sum_{F \in \mathcal{F}(x)} |\mathrm{conv}(F, (1+t)x)|.$$

For any $F \in \mathcal{F}(x)$, one has $\langle u_F, x\rangle = h_F$, thus

$$|\mathrm{conv}(F, (1+t)x)| = \frac{1}{d}|F|(\langle u_F, (1+t)x\rangle - h_F) = \frac{t}{d}|F|h_F = t|\mathrm{conv}(F, 0)|.$$

Hence

$$|K_t| = |K| + t \sum_{F \in \mathcal{F}(x)} |\mathrm{conv}(F, 0)|.$$

The result of this change of K is a cutting of K° parallel to the facet F_x:

$$K_t^\circ = \left\{ y \in K^\circ; \langle y, x \rangle \le \frac{1}{1+t} \right\} = \left\{ y \in K^\circ; \left\langle y, \frac{x}{|x|} \right\rangle \le \frac{1}{(1+t)|x|} \right\}.$$

For sufficiently small $t > 0$ the distance between the facet F_x and the new parallel facet is

$$d_x = \frac{1}{|x|}\left(1 - \frac{1}{1+t}\right) = \frac{t}{(1+t)|x|} = \frac{t}{|x|} + o(t).$$

Thus it is not difficult to see that we get

$$|K_t^\circ| = |K^\circ| - t\frac{|F_x|}{|x|} + o(t) = |K^\circ| - dt|\mathrm{conv}(F_x, 0)| + o(t).$$

Together, we get

$$|K_t||K_t^\circ| = |K||K^\circ| + t\left(|K^\circ| \sum_{F \in \mathcal{F}(x)} |\mathrm{conv}(F, 0)| - d|K||\mathrm{conv}(F_x, 0)| \right) + o(t).$$

Now we assume that the Santaló point of K is at the origin. Then using the result of Kim and Reisner [43] similarly to lemma 7.2.2, since $d_H(K, K_t) = O(t)$ we get $\mathcal{P}(K_t) = |K_t||K_t^\circ| + O(t^2)$. Thus, for $t > 0$,

$$\mathcal{P}(K_t) = \mathcal{P}(K) + t\left(|K^\circ| \sum_{F \in \mathcal{F}(x)} |\mathrm{conv}(F, 0)| - d|K||\mathrm{conv}(F_x, 0)| \right) + o(t). \tag{7.1}$$

Now let us assume that K has maximal volume product among polytopes with at most m vertices. Since K_t has also m vertices, it follows that $\mathcal{P}(K_t) \le \mathcal{P}(K)$ and thus using (7.1) for any vertex x of K we have

$$|K^\circ| \sum_{F \in \mathcal{F}(x)} |\mathrm{conv}(F, 0)| \le d|K||\mathrm{conv}(F_x, 0)|. \tag{7.2}$$

Summing on all the vertices of K we get

$$\sum_{x \in \mathcal{E}(K)} |K^\circ| \sum_{F \in \mathcal{F}(x)} |\mathrm{conv}(F, 0)| \le \sum_{x \in \mathcal{E}(K)} d|K||\mathrm{conv}(F_x, 0)| = d|K||K^\circ|.$$

Simplifying by $|K^\circ|$ and inverting sums in the left hand side gives

$$\sum_{F \in \mathcal{F}(K)} \mathrm{card}(\mathcal{E}(F))|\mathrm{conv}(F, 0)| \le d|K|.$$

Since for any facet F, one has $\mathrm{card}(\mathcal{E}(F)) \ge d$, we get

$$d|K| \le \sum_{F \in \mathcal{F}(K)} \mathrm{card}(\mathcal{E}(F))|\mathrm{conv}(F, 0)| \le d|K|.$$

Thus we get equality in all previous inequalities, which implies that for any facet F one has $\mathrm{card}(\mathcal{E}(F)) = d$. Therefore every facet F is a simplex and so K is simplicial. We also get the following consequence, for any vertex $x \in \mathcal{E}(K)$ one has

$$|K^\circ| \sum_{F \in \mathcal{F}(x)} |\mathrm{conv}(F, 0)| = d|K||\mathrm{conv}(F_x, 0)|. \tag{7.3}$$

□

Remark 7.2.7. Let us notice that if a polytope K minimizes the volume product among polytopes with at most m vertices then the inequality (7.2) is reversed: for every vertex x of K one has

$$|K^\circ| \sum_{F \in \mathcal{F}(x)} |\mathrm{conv}(F, 0)| \geq d|K||\mathrm{conv}(F_x, 0)|.$$

It's easy to see that simplices and B_1^d satisfy the above inequality.

One may also establish the following lemma generalizing equation (7.3).

Lemma 7.2.8. *Let $d \geq 1$ and $m \geq d+1$. Let K be of maximal volume product among polytopes with at most m vertices. Assume that the Santaló point of K is at the origin. Let $x \in \mathcal{E}(K)$ be a vertex of K and denote by $\mathcal{F}(x)$ the facets of K containing x. Then one has*

$$|K^\circ| \sum_{F \in \mathcal{F}(x)} |\mathrm{conv}(F, 0)| y_F = d|K||\mathrm{conv}(F_x, 0)| g_{F_x}, \tag{7.4}$$

where g_{F_x} denotes the center of gravity of the facet F_x of K° corresponding to x and for every $F \in \mathcal{F}(x)$, y_F denotes the vertex of K° corresponding to F.

Proof: From Theorem 7.2.6, we know that K is simplicial. Using Lemma 5 of [19], we may apply a more general shadow system than the one used in the proof of Theorem 7.2.6. Let $Q = \mathrm{conv}(\mathcal{E}(K) \setminus \{x\})$ and for z in a neighborhood of x define $K(z) = \mathrm{conv}(Q, z)$. Then one has

$$|K(z)| = |K| + \frac{1}{d} \sum_{F \in \mathcal{F}(x)} |F| \langle z - x, u_F \rangle.$$

Hence, using that $y_F = u_F / h_F$,

$$\nabla |K(z)|_{z=x} = \frac{1}{d} \sum_{F \in \mathcal{F}(x)} |F| u_F = \sum_{F \in \mathcal{F}(x)} |\mathrm{conv}(F, 0)| y_F.$$

For the polar, one has $K(z)^\circ = \{y \in Q^\circ; \langle y, z \rangle \leq 1\}$. Indeed, using formula (3) on page 347 of [59] the derivative of this kind of function has been computed and one has

$$\nabla |K(z)^\circ|_{z=x} = -\frac{|F_x|}{|x|} g_{F_x} = -d|\mathrm{conv}(F_x, 0)| g_{F_x}.$$

Because all facets of K are simplices, z can move freely in a neighborhood of x and thus for K maximizing the volume product, we get that $\nabla\mathcal{P}(K(z))_{z=x} = 0$. Again, from Kim and Reisner [43] one has $\mathcal{P}(K(z)) = |K(z)||K(z)^\circ| + O(|z-x|^2)$ thus

$$\nabla(|K(z)||K(z)^\circ|)_{z=x} = \nabla\mathcal{P}(K(z))_{z=x} = 0 = |K|\nabla|K(z)^\circ|_{z=x} + |K^\circ|\nabla|K(z)|_{z=x}.$$

Hence we get that for every vertex $x \in \mathcal{E}(K)$

$$|K^\circ| \sum_{F\in\mathcal{F}(x)} |\mathrm{conv}(F,0)|y_F = d|K||\mathrm{conv}(F_x,0)|g_{F_x}.$$

□

Remark 7.2.9. Notice that if K has maximal volume product among symmetric polytopes with at most m vertices, then in the proof of Theorem 7.2.6 one can consider $K_t = \mathrm{conv}(K, \pm(1+t)x)$ and we get that K satisfies the inequality (7.2) and thus K must be simplicial.

Remark 7.2.10. We should note that a simple and simplicial polytope is either a polygon or a simplex (see, for example, [95], page 67). Thus if K has maximal volume product among the polytopes with at most $m > d+1$ vertices in dimension $d > 2$ then its polar is not of maximal volume product in its class and doesn't necessarily satisfy equation (7.4). Still, following [64] we may claim that, in $\mathbb{R}^2$, K° will satisfy the combinatorial properties of (7.4).

Indeed, let $K \subset \mathbb{R}^2$ be of maximal volume product among polygons with at most m vertices. Let $L = K^\circ$, y be a vertex of L and define $L(z)$ in the same way we defined $K(z)$ in the proof of Lemma 7.2.8, i.e. z is a small perturbation of the vertex y. Using that K is a polygon we get that $(L(z))^\circ$ has the same number of vertices as K.

We get that $\mathcal{P}((L(z))^\circ)$ is maximal when $z = y$ and that $\nabla\mathcal{P}((L(z))^\circ)|_{z=y} = 0$. Now, again, as in the proof of Lemma 7.2.8 we can use a stability result from [43], and since

$$d_H((L(z))^\circ, K) = d_H(((L(z))^\circ)^\circ, K^\circ) = d_H(L(z), L) = O(|z-y|)$$

we have

$$|((L(z))^\circ)^{S((L(z))^\circ)}| = |((L(z))^\circ)^{S(K)}| + O(|z-x|^2) = |L(z)| + O(|z-y|^2).$$

So then

$$|L(z)||L(z)^\circ| = |((L(z))^\circ)^{S((L(z))^\circ)}||L(z)^\circ| + O(|z-x|^2) = \mathcal{P}((L(z))^\circ) + O(|z-x|^2).$$

Thus $\nabla(|L(z)||L(z)^\circ|)_{z=y} = 0$ and we can conclude similarly as in proof of Lemma 7.2.8.

7.3 Maximality in $\mathbb{R}^2$

Let us fix some notation. For $\theta \in [0, 2\pi]$, we set R_θ to be the rotation about the origin of angle θ in the oriented plane $\mathbb{R}^2$. We denote by e_1, e_2 the canonical basis of $\mathbb{R}^2$. For $m \geq 3$ we consider the regular polygon with m vertices and unit circumcircle:

$$P_m := \operatorname{conv}\left\{ R^k_{\frac{2\pi}{m}}(e_1);\ k = 0, \dots, m-1 \right\}.$$

A simple calculation shows that $|P_m| = m \sin(\pi/m)\cos(\pi/m)$. Note that

$$P_m^\circ = \frac{1}{\cos(\pi/m)} R_{\frac{\pi}{m}}(P_m)$$

is also a regular polytope (obtained by rotating and dilating P_m). We deduce that $|P_m^\circ| = m\tan(\pi/m)$ and the volume product of P_m is thus

$$\mathcal{P}(P_m) = \big(m \sin(\pi/m)\big)^2. \tag{7.5}$$

Notice that $m \mapsto \mathcal{P}(P_m)$ is an increasing sequence. Indeed, the function $x \mapsto \sin(x)/x$ is positive and decreasing on $[0, \pi)$.

We shall give a new proof of the following result of Meyer and Reisner [64].

Theorem 7.3.1. *Let $m \geq 3$. The regular m-gon has maximal volume product among all polygons with at most m vertices, that is, polygons in $\mathbb{P}^2_m$. More precisely, for every polygon K with at most m vertices, one has*

$$\mathcal{P}(K) \leq \mathcal{P}(P_m),$$

with equality if and only if K is an affine image of P_m.

We proceed by induction on the number of vertices m. Denote

$$M_m = \sup_{K \in \mathbb{P}^2_m} \mathcal{P}(K).$$

Recall that by Proposition 7.2.4 the preceding supremum is attained. For $m = 3$, the result is clear. Let us for now assume that $m \geq 4$ and that the result holds for $m-1$. Then

$$M_{m-1} = \mathcal{P}(P_{m-1}) = \big((m-1)\sin(\pi/(m-1))\big)^2.$$

Notice that since this quantity is strictly increasing it follows that

$$M_m = \sup_{K \in \mathbb{P}^2_m} \mathcal{P}(K) \geq \mathcal{P}(P_m) > \mathcal{P}(P_{m-1}) = M_{m-1}.$$

We will now prove a lemma showing that if we apply a shadow system which moves any individual vertex of such a polytope K achieving the supremum but keeps all other vertices of K fixed, then the vertex must be on the line passing through the Santaló point of K and the middle of its two adjacent vertices.

Lemma 7.3.2. *Let $K \in \mathbb{P}_m^2$ having maximal volume product among polytopes in $\mathbb{P}_m^2$ and Santaló point at the origin. Then for any vertex x of K there exists a real number $\lambda = \lambda(x)$ such that $x = \lambda(x_1 + x_2)$, where x_1 and x_2 are the vertices of K adjacent to x.*

Proof: We assume that the Santaló point of K is at the origin. Let x be a vertex of K and denote by x_1 and x_2 its two adjacent vertices. Denote by y_1 and y_2 the vertices of K° corresponding to the edges $[x, x_1]$ and $[x, x_2]$ of K. We apply equation (7.4) of Lemma 7.2.8 to our situation, the center of gravity of the edge F_x^* of K° corresponding to the x is the center of gravity of the edge $[y_1, y_2]$, hence it is the middle of the segment $[y_1, y_2]$, thus $g_{F_x^*} = (y_1 + y_2)/2$. Hence equation (7.4) becomes:

$$|K||\mathrm{conv}(0, y_1, y_2)|(y_1 + y_2) = |K^\circ|(|\mathrm{conv}(0, x, x_1)|y_1 + |\mathrm{conv}(0, x, x_2)|y_2).$$

Because these quantities are equal and y_1 and y_2 are linearly independent, we may identify and get

$$|K||\mathrm{conv}(0, y_1, y_2)| = |K^\circ||\mathrm{conv}(0, x, x_1)| = |K^\circ||\mathrm{conv}(0, x, x_2)|.$$

Choosing an orientation, we deduce that

$$\det(x_1, x) = |\mathrm{conv}(0, x, x_1)| = |\mathrm{conv}(0, x, x_2)| = \det(x, x_2) = -\det(x_2, x).$$

Thus $\det(x_1 + x_2, x) = 0$. Hence there exists a real number $\lambda = \lambda(x)$ such that $x = \lambda(x_1 + x_2)$.

□

So we proved that for the polygon with maximal volume product, each vertex must have the property that it is a multiple of the sum of its two adjacent vertices. Since the dual of a polygon of N vertices is another polygon of N vertices then we can also conclude that this property holds in the dual as well. Now we will show that for any polygon with N vertices that has the property of Lemma 7.3.2 the constant λ is independent of the triple of vertices we consider:

Lemma 7.3.3. *Let K be a convex polygon with N vertices and maximal volume product with its Santaló point at the origin. Then there exists a real number, $\lambda > 1/2$ such that for any three adjacent vertices, v_1, v_2, and v_3, with v_2 adjacent to both v_1 and v_3, $v_2 = \lambda(v_1 + v_3)$.*

Proof: Let us order the vertices of the polygon counterclockwise as $x_1, \ldots, x_N$ and the vertices of the dual $y_1, \ldots, y_N$ with y_i such that $\langle x_i, y_i \rangle = \langle x_{i+1}, y_i \rangle = 1$. By Lemma 7.3.2 applied to P, there exists real numbers λ_i so that for all $1 \le i \le N-1$

$$x_i = \lambda_i (x_{i-1} + x_{i+1}).$$

Taking the scalar product with y_i and y_{i-1}, we get

$$\langle x_{i-1}, y_i \rangle = \langle x_{i+1}, y_{i-1} \rangle = \frac{1}{\lambda_i} - 1.$$

Now we can use Remark 7.2.10 to claim that P° will also satisfy the combinatorial conditions of Lemma 7.3.2. Thus, there exists μ_i such that for all $1 \le i \le N-1$

$$y_i = \mu_i (y_{i-1} + y_{i+1}).$$

Taking the scalar product with x_i and x_{i+1}, we get

$$\langle x_i, y_{i+1} \rangle = \langle x_{i+1}, y_{i-1} \rangle = \frac{1}{\mu_i} - 1.$$

Using the equations above, we deduce that $\lambda_i = \mu_i = \lambda_{i+1}$.

$\square$

Now using Lemma 7.3.3 and standard techniques to solve recurrence relations we can prove Theorem 7.3.1.

Proof of Theorem 7.3.1: By an affine transform, we may assume that the Santaló point of P is at the origin. Denote by $v_1, \ldots, v_m$ the vertices of P (counting clockwise). Again, applying a linear transformation we may assume $v_0 = v_m = e_1$, where, as before, e_1 is the first vector of the canonical basis (e_1, e_2) of $\mathbb{R}^2$. From Lemma 7.3.3 we have the recurrence relation for the vertices: $tv_k = v_{k+1} + v_{k-1}$. Then the recurrence holds also for the coordinates x_k and y_k of v_k. Since v_k is a vertex of P, one has $0 < t < 2$ thus the roots of the equation $y^2 - ty + 1 = 0$ are $\alpha = e^{i\theta}$ and $\beta = e^{-i\theta}$, with $\cos(\theta) = t/2$. Thus there exists $A, B \in \mathbb{R}$ such that for every k

$$y_k = A\cos(k\theta) + B\sin(k\theta),$$

with initial conditions $y_0 = y_m = 0$. Since $y_0 = 0$, we get $A = 0$. Notice that if $B = 0$ then all y_k are 0 and thus all vertices lie on the x-axis, so we discard this possibility. So by the initial conditions we have

$B\sin(m\theta) = 0$ hence $\sin(m\theta) = 0$. Thus there exist $j \in \mathbb{N}$ such that $\theta = \frac{j\pi}{N}$. The first coordinate x_k of v_k satisfies the same recurrence relation so there exists C and D such that for every k

$$x_k = C\cos\left(\frac{jk\pi}{m}\right) + D\sin\left(\frac{jk\pi}{m}\right),$$

with the initial conditions $x_0 = x_m = 1$. Since $x_0 = 1$, we get $C = 1$. Since $x_m = 1$, we get that j must be even. Taking in to account that K has exactly N vertices we get that $j = 2$. Finally

$$(x_k, y_k) = \left(\cos\left(\frac{2k\pi}{m}\right) + D\sin\left(\frac{2k\pi}{m}\right), B\sin\left(\frac{2k\pi}{m}\right)\right),$$

and thus K is the linear image of the regular m-gon P_m by the map

$$T = \begin{pmatrix} 1 & D \\ 0 & B \end{pmatrix}.$$

□

7.4 Convex hull of $d+2$ points in $\mathbb{R}^d$

For $K \subset \mathbb{R}^d$ being a convex body, we define $\mathcal{O}(K) = \{T \in O_d(\mathbb{R}); TK = K\}$ and $\mathrm{Fix}\,(K) = \{x \in \mathbb{R}^d; Tx = x, \text{ for all } T \in \mathcal{O}(K)\}$. We shall consider convex bodies K such that $\mathrm{Fix}\,(K)$ is one point, the origin. In this case, notice that all affine invariant points associated to K coincide with this point. In particular the Santaló point of K satisfies $s(K) = 0$.

Theorem 7.4.1. *Let $1 \le k \le d-1$ be integers and let E and F be two supplementary subspaces in $\mathbb{R}^d$ of dimensions k and $d-k$ respectively. Let $L \subset E$ and $M \subset F$ be convex bodies of the appropriate dimensions such that* $\mathrm{Fix}\,(L) = \mathrm{Fix}\,(M) = \{0\}$. *Then for every $x \in L$ and $y \in M$*

$$\mathcal{P}(\mathrm{conv}(L-x, M-y)) \le \mathcal{P}(\mathrm{conv}(L, M)) = \frac{\mathcal{P}(L)\mathcal{P}(M)}{\binom{d}{k}},$$

with equality if and only if $x = y = 0$.

Proof: Using the invariance of the volume product under linear transformations we may assume that E and F are perpendicular. Now consider the following shadow system $(x, y) \mapsto K_{x,y} = \mathrm{conv}(L-x, M-y)$, for $(x, y) \in K \times L$. Computing the volume, we get

$$|K_{x,y}| = \frac{|L-x||M-y|}{\binom{d}{k}} = \frac{|L||M|}{\binom{d}{k}}.$$

So $(x, y) \mapsto K_{x,y}$ is a volume constant shadow system. Thus, from Meyer-Reisner, the function $(x, y) \mapsto \mathcal{P}(K_{x,y})$ is quasi concave on $L \times M$. Moreover, for any $(S, T) \in \mathrm{O}(L) \times \mathrm{O}(M)$ one has $(S \times T)(L \times M) = L \times M$ and for any $(x, y) \in L \times M$

$$K_{S(x),T(y)} = \mathrm{conv}(L - S(x), M - T(y)) = \mathrm{conv}(S(L - x), T(M - y)) = (S \times T)(K_{x,y}).$$

Thus $\mathcal{P}(K_{S(x),T(y)}) = \mathcal{P}(K_{x,y})$. This means that the function $(x, y) \mapsto \mathcal{P}(K_{x,y})$ is invariant under the action of $\mathrm{O}(L) \times \mathrm{O}(M)$. By Lemma 7.1.1, we deduce that its maximum occurs at a fixed point of $\mathrm{O}(L) \times \mathrm{O}(M)$, which is reduced to the origin by the hypotheses. The equality case is clear.

$\square$

Corollary 7.4.2. *Let $L \subset \mathbb{R}^{d-1}$ be a convex body such that $\mathrm{Fix}\,(L)$ is one point. Then among all double pyramids $K = \mathrm{conv}(L, x, y)$ in $\mathbb{R}^d$ with base L separating apexes x and y, the volume product $\mathcal{P}(K)$ is maximal when x and y are symmetric with respect to the Santaló point of L.*

Theorem 7.4.3. *Let K be the convex hull of $d + 2$ points. Let $q = \lfloor \frac{d}{2} \rfloor$ and $p = \lceil \frac{d}{2} \rceil = d - q$. Then*

$$\mathcal{P}(K) \leq \frac{(p + 1)^{p+1}(q + 1)^{q+1}}{d!p!q!},$$

with equality if and only if K is the convex hull of two simplices Δ_q and Δ_p living in supplementary affine subspaces of dimensions q and p respectively.

Proof:

Let K be a body in $\mathbb{P}^d_{d+2}$. Then by Radon's theorem there exists $1 \leq k \leq d - 1$ such that one may split the $d + 2$ vertices of K into two subset I and J, with $\mathrm{card}(I) = k + 1$ and $\mathrm{card}(J) = d + 1 - k$ in such a way that if $L = \mathrm{conv}(I)$ and $M = \mathrm{conv}(J)$ then $L \cap M \neq \emptyset$. Since $K = \mathrm{conv}(I, J) = \mathrm{conv}(L, M)$ is full dimensional in $\mathbb{R}^d$, it follows that L and M are non-degenerate simplices in supplementary affine subspaces E and F of dimension k and $d - k$. By affine invariance, we may assume that $E \cap F = \{0\}$, E and F are orthogonal to each other and L and M are standard simplices of their respective dimensions so that one may write $L = \Delta_k + s(L)$, where $s(L)$ is the Santaló point of L and Δ_k is a regular simplex of dimension k with Santaló point at the origin. In the same way, one has $M = \Delta_{d-k} + s(M)$. Since $\mathrm{Fix}\,(\Delta_k) = \mathrm{Fix}\,(\Delta_{d-k}) = 0$, we may apply Theorem 7.4.1 to Δ_k and Δ_{d-k}. We get that

$$\mathcal{P}(K) \leq \mathcal{P}\left(\mathrm{conv}(\Delta_k, \Delta_{d-k})\right) = \frac{\mathcal{P}(\Delta_k)\mathcal{P}(\Delta_{d-k})}{\binom{d}{k}} := f_d(k).$$

Recall that the volume product of a non-degenerate simplex Δ_d in $\mathbb{R}^d$ is

$$\mathcal{P}(\Delta_d) = \frac{(d+1)^{d+1}}{(d!)^2}.$$

Hence after simplification, we get

$$f_d(k) = \frac{1}{d!} \times \frac{(k+1)^{k+1}}{k!} \times \frac{(d-k+1)^{d-k+1}}{(d-k)!} = \frac{g(k)g(d-k)}{d!},$$

where $g(x) = \frac{(x+1)^{x+1}}{\Gamma(x+1)}$, for $x \geq 0$. Then with the change of variable $t = xu$ we get, for $x > 0$,

$$\frac{\Gamma(x)}{x^x} = \frac{1}{x^x} \int_0^{+\infty} e^{-t} t^{x-1} dt = \int_0^{+\infty} (ue^{-u})^x \frac{du}{u}.$$

Hence from Hölder's inequality the function $x \mapsto \frac{\Gamma(x)}{x^x}$ is log-convex on $(0, +\infty)$. It follows that g is log-concave on $\mathbb{R}_+$. So f_d is log-concave as well, and since it satisfies $f_d(k) = f_d(d-k)$, for all $0 \leq k \leq n$, we deduce that

$$f_d(k) = \frac{g(k)g(d-k)}{d!} \leq \frac{g(\lfloor \frac{d}{2} \rfloor) g(\lceil \frac{d}{2} \rceil)}{d!},$$

with equality if and only if $K = \mathrm{conv}(\Delta_{\lfloor \frac{d}{2} \rfloor}, \Delta_{\lceil \frac{d}{2} \rceil})$.

Remark 7.4.4. One may conjecture that for $k \leq d$, among polytopes with at most $d+k$ vertices, the convex hull of k simplices living in orthogonal subspaces of dimensions $\lfloor \frac{d}{k} \rfloor$ or $\lceil \frac{d}{k} \rceil$ have maximal volume product (see Gluskin-Litvak [27] where such bodies where considered). Theorem 7.4.3 establishes this conjecture for $k = 2$.

7.5 Symmetric bodies in $\mathbb{R}^d$

Notice that Lemma 7.2.2, Theorem 7.2.4, and Theorem 7.2.6 also hold if we restrict the maximum to the case of symmetric polytopes. Indeed, consider $d \geq 1$ and $m \geq d$ and let K be of maximal volume product among symmetric polytopes with at most $2m$ vertices. Then K is a simplicial polytope which has exactly $2m$ vertices. Moreover the sequence of such maxima is strictly increasing in m.

CHAPTER 8

Graphs

A graph $G = (V, E)$ is composed of two sets, V the set of *vertices* (or *nodes*) which we will generally take to be points in $\mathbb{R}^d$, and E the set of *edges* which connect pairs of vertices. The vertices that an edge connects are said to be *incident* to the edge and *adjacent* to each other. Similarly, two edges are said to be *adjacent* if they share a vertex and the edges are said to be *incident* to the vertex. A graph may be either *directed*, specifying a head and tail for each edge, or *undirected* (sometimes called *bidirected*). A graph is called *simple* if no edge has the same vertex as its endpoints, that is, every edge is said to be *proper*. We will concern ourselves only with simple undirected graphs with no repeated edges which we will simply refer to as graphs.

The *degree* of a vertex is the number of proper edges incident to that vertex. The *complement* G^c of a graph G is a graph with the same vertex set but whose edge set consists of the edges not present in G. A graph $H = (V', E')$ is called a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. A subgraph I of a graph G is said to be *induced* if two vertices of I are connected by an edge of I if and only if they are connected by an edge in G.

A *walk* of a graph is a sequence of vertices $\{v_0, \ldots, v_k\}$ and associated edges such that for each $0 \leq i \leq k$ the vertices v_i and v_{i+1} are adjacent. A walk is said to be *closed* if the initial vertex (v_0) is the same as the final vertex (v_k), otherwise it is *open*. We call a walk a *trail* if no edge is repeated. We call a trail a *path* if no vertex is repeated. A graph is said to be connected if there is a path between every two vertices. A *cycle* is a closed path.

There are several special graphs which will be useful to our study. A graph is said to be *cyclic* or a cycle if it contains one cycle which traverses all edges and vertices. A graph is said to be a *tree* if any two vertices are connected by only one path. A graph is said to be *complete* if every pair of vertices are adjacent.

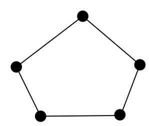


Figure 8.1: Example of a cycle

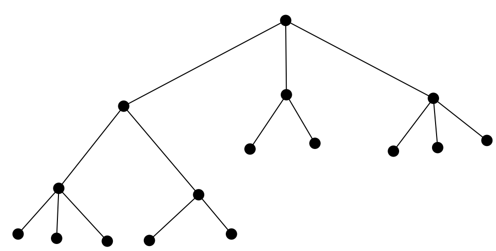


Figure 8.2: Example of a tree

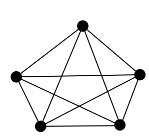


Figure 8.3: Example of a complete graph

CHAPTER 9

Volume Product and Lipschitz-free Banach Spaces

9.1 Introduction

Consider a metric space M containing a special designated point 0, such a pair is usually called a pointed metric space. To this metric space we can associate the space $\mathrm{Lip}_0(M)$ of Lipschitz functions $f : M \to \mathbb{R}$, with the special property $f(0) = 0$. We refer to [90, 29, 30] for many interesting facts about $\mathrm{Lip}_0(M)$ and its geometry. It turns out that $\mathrm{Lip}_0(M)$ is a Banach space with norm

$$\|f\|_{\mathrm{Lip}} = \sup\left\{\frac{f(x) - f(y)}{\rho(x,y)},\ \text{where } x, y \in X,\ \text{and } x \neq y\right\}. \tag{9.1}$$

This space is called the Lipschitz dual of M. The closed unit ball of the space $\mathrm{Lip}_0(M)$ is compact for the topology of pointwise convergence on M, and therefore this space has a canonical predual which is called the Lipschitz-free space over M and denoted by $\mathcal{F}(M)$. It is the closed subspace of $\mathrm{Lip}_0(M)^*$ generated by the evaluation functionals $\{\delta_x : x \in M\}$ defined by $\delta_x(f) = f(x)$, for every $f \in \mathrm{Lip}_0(M)$. The goal of this chapter is to study the geometry of the unit ball of $\mathcal{F}(M)$, when M is finite and in particular its volume product.

More precisely, assume our metric space space $M = \{a_0, a_1, \dots, a_d\}$, is finite with metric ρ. Consider $\mathrm{Lip}_0(M)$ the space of Lipschitz functions f on M with the restriction that $f(a_0) = 0$ and the norm

$$\|f\|_{\mathrm{Lip}} = \max_{a_i \neq a_j} \frac{f(a_i) - f(a_j)}{\rho(a_i, a_j)}.$$

Note that each function f on M is just a set of d values $f(a_1), \dots, f(a_d)$ and thus we can identify $\mathrm{Lip}_0(M)$ with $\mathbb{R}^d$, assigning to a function $f \in \mathrm{Lip}_0(M)$ a vector $f = (f(a_1), \dots, f(a_d)) \in \mathbb{R}^d$. Let us denote $\rho_{ij} = \rho(a_i, a_j)$, $K_M = B(\mathrm{Lip}_0(M))$ the unit ball of the $\mathrm{Lip}_0(M)$ and $L_M = K_M^\circ$, the unit ball of $\mathcal{F}(M)$.

Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{R}^d$ and let $e_0 = 0$. For $0 \le i \neq j \le d$ let

$$v_{ij} = \frac{e_i - e_j}{\rho_{i,j}} \quad \text{and} \quad V_M = \{v_{ij} : 0 \le i \neq j \le d\}.$$

Then $\|f\|_{\mathrm{Lip}} = \max_{v \in V_M} \langle f, v\rangle$ and $L_M = \mathrm{conv}(V_M)$.

We recall that the polar body K° of (a symmetric convex body) K is defined by

$$K^\circ = \{y \in \mathbb{R}^d : \langle y, x \rangle \leq 1 \text{ for all } x \in K\}. \tag{9.2}$$

Then the volume product of a symmetric convex body K is defined by

$$\mathcal{P}(K) = |K||K^\circ|,$$

and the volume product is invariant under invertible linear transformations on $\mathbb{R}^d$.

Our goal is to discuss the maximal and minimal properties of

$$\mathcal{P}(M) = |K_M||L_M|, \tag{9.3}$$

where M is a metric space of a fixed number of elements.

It is interesting to note that in this case the maximal case for $P(M)$ is also an extremely interesting and open problem. Indeed, $L_M \neq B_2^d$ for finite M. Thus the maximal case will not follow from Santaló inequality and we must look for other maximum(s) along with the possible conjectured minimizers.

We will begin with a discussion on the connection between graphs and the Lipschitz-free space of the associated metric space and our conjectured extremal metrics. Then, in Section 9.3, we will discuss known results concerning these spaces and the relationship between special graphs and metric spaces. Section 9.4 is dedicated to the connections of the structure of metric spaces and the corresponding volume product. In particular in Section 9.4.4 we compute the volume product corresponding to the compleate graph. In section 9.4.5 we will confirm the maximum for metrics on three points, and the minimum for metrics on four points in section 9.4.6.

9.2 Relationship to graphs and the main conjectures

There is a bijection between finite metric spaces and weighted undirected connected (finite) graphs. This bijection goes as follows: to any weighted undirected connected finite graph $G = (V, E, w)$, with vertices V, edges E and weight $w : E \to \mathbb{R}_+$ one can associate a finite metric space on its set of vertices V by using the shortest path distance. Reciprocally to any finite metric spaces (M, ρ), we can canonically associate a weighted undirected connected finite graph as follows: we first consider the complete weighted graph on M with the weight on the edge between two points being their distance. Then one erases the edge between two points if its weight is equal to the sum of the weights of the edges along a path joining the two points, i.e. if there is equality in the triangle inequality.

This representation is very well adapted to our study because the edges that appear in the graph model are exactly corresponding to the vertices of the unit ball of the Lipschitz-free space. This was recently proved by Aliaga and Guirao [4] in a more general setting. We give a simpler proof for finite sets in the Proposition 9.3.1 below.

Because of this bijection between pointed finite metric spaces (M, ρ) and weighted graphs, one may also see the Lipshitz free mapping as a way to attach to any weighted connected finite graph a finite dimensional Banach space (the Lipshitz-free space $\mathcal{F}(M)$ over M) or a centrally symmetric convex body ($L_M = \mathrm{conv}(v_{ij} : 0 \leq i \neq j \leq d)$). This mapping is no longer a bijection, as not every origin-symmetric convex body can be associated with a finite metric space. For example, these bodies have at most $d(d+1)$ vertices, but this is not the only constraint and it will be interesting to describe geometrically the class of convex bodies associated to with finite metric spaces via the above mapping. Since we are usually interested in isometric properties of Banach spaces it is also interesting to describe the class of graphs whose associated unit balls of their Lipschitz-free spaces are isometric.

There is a particular example of graphs which are very interesting for us, the graphs which are trees. It was proved by A. Godard in [28] that M is a tree if and only if L_M is an affine image of B_1^d. We give a simpler proof of this fact in Proposition 9.4.1 below.

We would like to make the following two conjectures:

Conjecture 9.2.1. Let $n \in \mathbb{N}$. Among pointed metric spaces (M, ρ) with $d+1$ elements, $\mathcal{P}(M)$ is minimal if M is a tree. Note that in this case L_M is the affine image of a cross polytope and $P(M) = 4^d/d!$.

Conjecture 9.2.2. Let $n \in \mathbb{N}$. Among pointed metric spaces (M, ρ) with $n+1$ elements, $P(M)$ is maximal when M is the complete graph.

9.3 General results on the Lipschitz-free spaces

The Lipschitz-free space $\mathcal{F}(X)$ associated to a metric space X is a Banach space associated to X and many of the properties of X may be studied at the Banach space level. In the case of finite metric spaces, the Lipschitz-free operation associates a finite dimensional Banach space to a graph, which amounts to a symmetric convex body. Thus it gives a correspondence between graphs and a subset of the convex bodies.

The Lipschitz-free operation gives a "linearization" of any Lipschitz map f between two metric spaces X and Y as the linear map $\tilde{f}$ between $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ defined by $\tilde{f}(\delta_x) = \delta_{f(\tilde{x})}$ and extended by linearity.

It implies that if you have a metric space (M,ρ) and you take a subset N of M, you can of course consider the restriction of ρ to N and it gives you another metric space (N,ρ). You get that $\mathcal{F}(N)$ is a subspace of $\mathcal{F}(M)$. Hence its unit ball, the convex body L_N, is a section of the convex body L_M.

Notice that the Lipschitz-free operation of a metric space (M,ρ) with two different chosen roots give two Banach spaces which are isometric such as in figures 9.1 and 9.2 below. In the general setting, the argument goes as follows. For $a \in M$ denote $\mathrm{Lip}_a(M)$ the set of Lipschitz functions $f : M \to \mathbb{R}$ such that $f(a) = 0$. For two different roots a and b in M, one constructs an isometry T between between $\mathrm{Lip}_a(M)$ and $\mathrm{Lip}_b(M)$ by defining $T(f)(x) = f(x) - f(b)$ for every $f \in \mathrm{Lip}_a(M)$. We have $T(f) \in \mathrm{Lip}_b(M)$ and $\|T(f)\|_{\mathrm{Lip}_b(M)} = \|f\|_{\mathrm{Lip}_a(M)}$. Thus the associated Lipschitz-free spaces are isometric. In the case of finite metric spaces this implies that the unit ball L_M^b of the Lipschitz-free space $\mathcal{F}_b(M)$ is an affine image of the unit ball L_M^a of the Lipschitz-free space $\mathcal{F}_a(M)$. This can also be seen directly in the following way. Assume that $M = \{a_1, \ldots, a_{d+1}\}$. Define

$$\mathcal{L}_M = \mathrm{conv}\left(\left\{\frac{e_i - e_j}{\rho_{ij}} : 1 \le i \ne j \le d+1\right\}\right) \subset \left\{x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i = 0\right\}.$$

Then $\mathcal{L}_M$ is an d-dimensional convex body living in an hyperplane of $\mathbb{R}^{d+1}$. For $1 \le i \le d+1$, denote by $L_M^{a_i}$ the unit ball of the Lipschitz-free space $\mathcal{F}_{a_i}(M)$ pointed at a_i. Then $L_M^{a_i} = \left(\mathcal{L}_M | e_i^\perp\right)$ is the orthogonal projection of $\mathcal{L}_M$ on $e_i^\perp$ and this projection is in fact bijective from $\{x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i = 0\}$ onto $\{x \in \mathbb{R}^{d+1} : x_i = 0\}$. Thus for different i and j, $L_M^{a_i}$ and $L_M^{a_j}$ are affine images of each other.

The edges that appear in the graph model are exactly corresponding to the vertices of the unit ball of the Lipschitz-free space associated to it. This was recently proved by Aliaga and Guirao [4] in a more general setting. We give a simpler proof for finite sets in Proposition 9.3.1 below.

Proposition 9.3.1 ([4]). *Let (M,ρ) be a pointed finite metric space, with $M = \{a_0, \ldots, a_d\}$. Let $G = (M,E,w)$ be the canonical weighted undirected connected finite graph associated to (M,ρ). Let $L_M = \mathrm{conv}(v_{ij} : 0 \le i \ne j \le d)$ be the unit ball of the Lipschitz-free Banach space $\mathcal{F}(M)$ associated to (M,ρ). Let $0 \le i \ne j \le d$. The following are equivalent.*

(i) The vector v_{ij} is not an extreme point of L_M.

(ii) There exists $k \notin \{i,j\}$ such that $\rho_{ij} = \rho_{ik} + \rho_{kj}$.

(iii) There exists $k \notin \{i,j\}$ such that $v_{ij} \in [v_{ik}, v_{kj}]$.

Proof: <u>*(ii)* $\Rightarrow$ *(iii)*</u>: let $k \notin \{i,j\}$ such that $\rho_{ij} = \rho_{ik} + \rho_{kj}$. Then one has

$$v_{ij} = \frac{e_i - e_j}{\rho_{ij}} = \frac{\rho_{ik}}{\rho_{ik} + \rho_{kj}} \times \frac{e_i - e_k}{\rho_{ik}} + \frac{\rho_{kj}}{\rho_{ik} + \rho_{kj}} \times \frac{e_k - e_j}{\rho_{kj}} = \frac{\rho_{ik}}{\rho_{ik} + \rho_{kj}} v_{ik} + \frac{\rho_{kj}}{\rho_{ik} + \rho_{kj}} v_{kj}.$$

Hence $v_{ij} \in [v_{ik}, v_{kj}]$.

<u>*(iii)* $\Rightarrow$ *(i)*</u>: let $k \notin \{i,j\}$ such that $v_{ij} \in [v_{ik}, v_{kj}]$. Since i,j,k are distinct it follows that $v_{ij} \neq v_{ik}$ and $v_{ij} \neq v_{kj}$ thus v_{ij} is not an extreme point of L_M.

<u>*(i)* $\Rightarrow$ *(ii)*</u>: if v_{ij} is not an extreme point of L_M then v_{ij} belongs to the relative interior of a face of L_M. Let $\mathcal{E} = \{(a_k, a_l) : 0 \leq k \neq l \leq n\}$. For $e = (a_k, a_l) \in \mathcal{E}$ we denote $\rho_e = \rho_{k,l}$ and $v_e = v_{k,l}$. By Carathéodory's theorem there exists $\Gamma \subset \mathcal{E}$, with $2 \leq \mathrm{card}(\Gamma) \leq d$ such that $v_{ij} \in \mathrm{conv}(v_e : e \in \Gamma)$. Let $\gamma \subset \mathcal{E}$ be the subset of $\mathcal{E}$ of smallest cardinality such that $v_{ij} \in \mathrm{conv}(v_e : e \in \gamma)$. Then $S_\gamma := \mathrm{conv}(v_e : e \in \gamma)$ is a simplex. Let $m = \mathrm{card}(\gamma)$. There exists $(\lambda_e)_{e \in \gamma}$ such that $\lambda_e > 0$, for all $e \in \gamma$, $\sum_{e \in \gamma} \lambda_e = 1$ and

$$v_{ij} = \frac{e_i - e_j}{\rho_{ij}} = \sum_{e \in \gamma} \lambda_e v_e.$$

Let $M_\gamma = \{a_k \in M : \exists e \in \gamma; k \in e\}$ be the vertices of M which belongs to some of the edges in γ. Let $\gamma' = \gamma \cup \{(i,j)\}$. We want to prove that the graph $C_\gamma := (M_\gamma, \gamma')$ is a cycle. First it is not difficult to see using the minimality of γ that the graph C_γ is connected. Since S_γ is an $(m-1)$-dimensional simplex whose affine hull doesn't contain the origin, its m vertices are linearly independent. Since these vertices are differences of basis vectors, one has at least $m+1$ basis vectors involved in the vertices of S_γ: $\mathrm{card}(M_\gamma) \geq m+1$. By linear independence, each $a_k \in M_\gamma$ belongs to at least two edges. By double counting one has

$$2(m+1) = 2\mathrm{card}(\gamma') = \sum_{a \in M_\gamma} \deg(a) \geq 2\mathrm{card}(M_\gamma) \geq 2(m+1).$$

Hence $\mathrm{card}(M_\gamma) = m+1$ and each vertex of C_γ has exactly degree two. Thus C_γ is a two-regular connected graph: it is the $(m+1)$-cycle graph. This implies that $\rho_{ij} = \sum_{e \in \gamma} \rho_e$. Since $m \geq 2$ there exists $k \notin \{i,j\}$ such that $k \in e$ for some $e \in \gamma$. Then one has $\rho_{ij} = \rho_{ik} + \rho_{kj}$.

$\square$

One deduces easily the following corollary.

Corollary 9.3.2. *Let (M, ρ) be a pointed finite metric space, with $M = \{a_0, \dots, a_d\}$. Let $G = (M, E, w)$ be the canonical weighted undirected connected finite graph associated to (M, ρ). Let $L_M = \mathrm{conv}(v_{ij} : 0 \leq i \neq j \leq d)$ be the unit ball of the Lipschitz-free Banach space $\mathcal{F}(M)$ associated to (M, ρ). Let $0 \leq i \neq j \leq d$.*

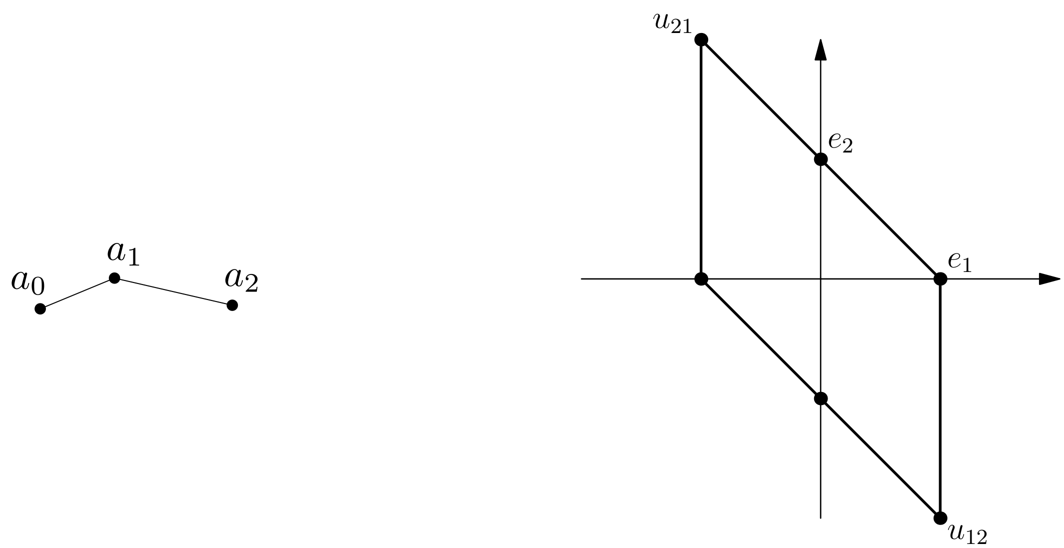


Figure 9.1: Three point tree and corresponding unit ball

Then v_{ij} is an extreme point of L_M if and only if $(a_i, a_j) \in E$. Moreover $L_M = \mathrm{conv}(v_{ij} : (a_i, a_j) \in E)$ and L_M has exactly $2card(E)$ vertices.

9.4 Extremal properties of the Volume product

9.4.1 Trees

The following nice relationship between trees and affine images of B_1^d was first proved by A. Godard in a more general setting in section 4 of [28]. We give here a simpler proof in the case of finite metric spaces.

Proposition 9.4.1 ([30])**.** *Let (M, ρ) be a pointed finite metric space, with $M = \{a_0, \ldots, a_d\}$. Let $G = (M, E, w)$ be the canonical weighted undirected connected finite graph associated to (M, ρ). Let $L_M = \mathrm{conv}(v_{ij} : 0 \leq i \neq j \leq d)$ be the unit ball of the Lipschitz-free Banach space $\mathcal{F}(M)$ associated to (M, ρ). Then G is a tree if and only if L_M is the linear image of B_1^d.*

Proof: Recall that the graph $G = (M, E, w)$ is a tree if and only it is connected and acyclic, equivalently it is connected and $\mathrm{card}(E) = \mathrm{card}(M) - 1 = d$. Since our graphs are all connected and using Corollary 9.3.2, we get that $G = (M, E, w)$ is a tree if and only if L_M has $2d$ vertices. But we know that L_M is full dimensional and is centrally symmetric so it has exactly $2d$ vertices if and only if it is an affine image of B_1^d.

□

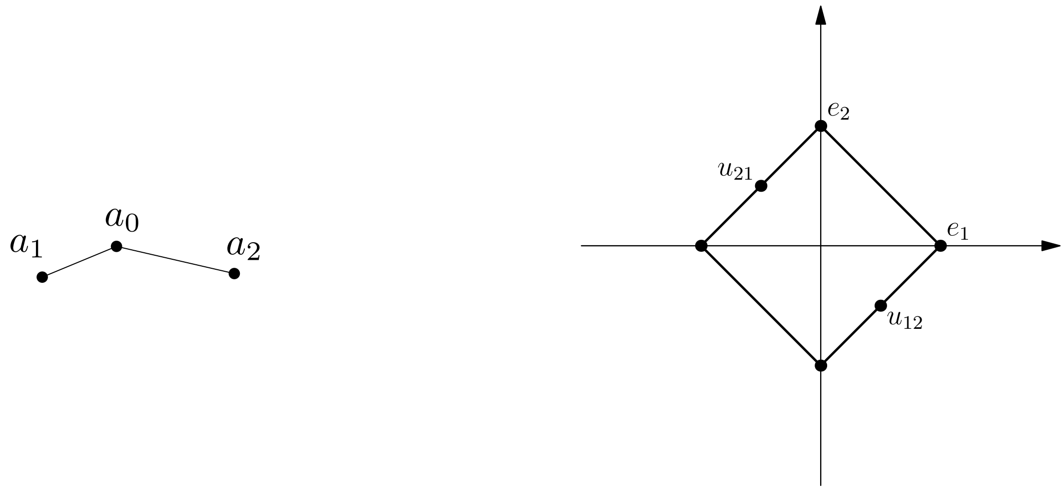


Figure 9.2: Alternative three point tree and corresponding unit ball

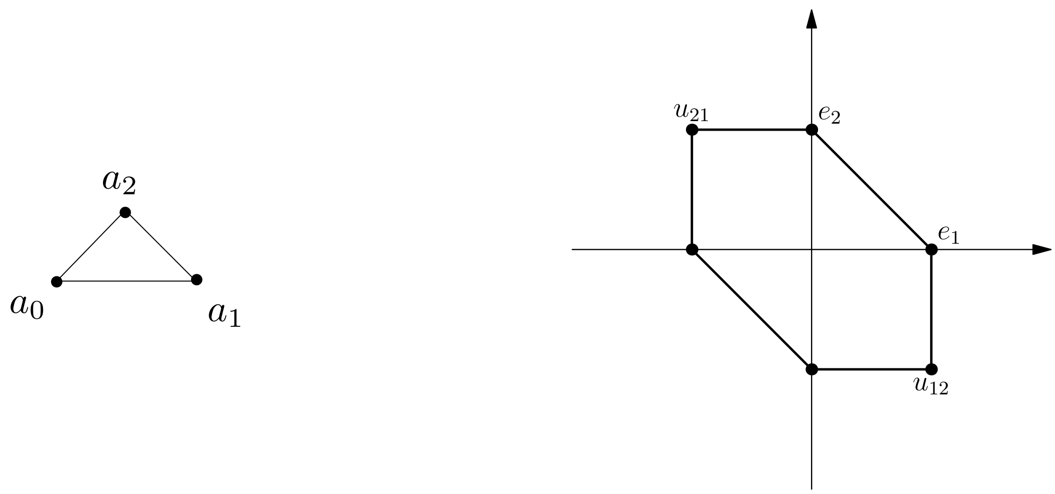


Figure 9.3: Complete graph of 3 points and corresponding unit ball

9.4.2 The cycle graph

We would like to consider another particular example of a graph: the cycle graph. Let (M, ρ) be a pointed finite metric space, with $M = \{a_0, \ldots, a_d\}$. Let $G = (M, E, w)$ be the canonical weighted undirected connected finite graph associated to (M, ρ). Since G is a cycle graph we have $\mathrm{card}(E) = d + 1$. We use Corollary 9.3.2, to deduce that the unit ball of the Lipschitz-free Banach space associated to a cycle graph (M, ρ) of $d + 1$ elements is exactly the convex hull of $2d + 2$ symmetric points. It was proved by Lopez and Reisner [53], that such convex bodies satisfy Mahler conjecture for dimension 8 and lower. Hence they satisfy the Conjecture 9.2.1 for metric spaces of nine points or fewer. We note that the limitation of [53] to dimension 8 is due to computational complexity and so the result may be extended further with additional computation.

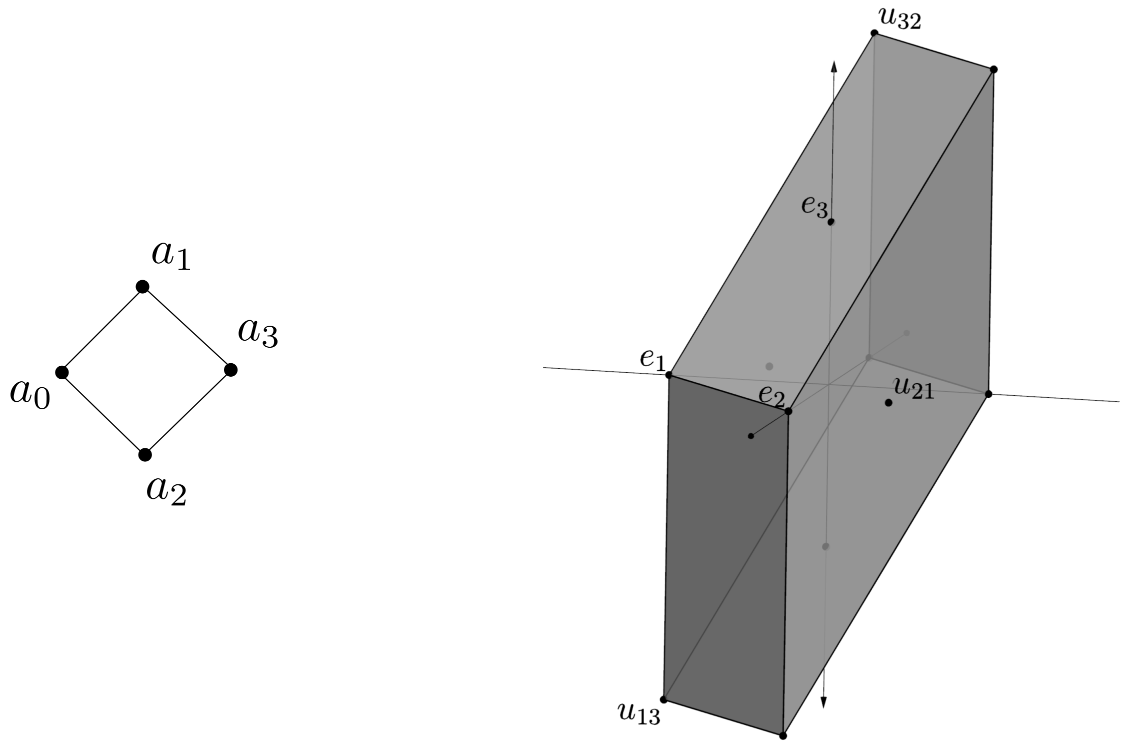


Figure 9.4: Cyclic graph of 4 points and corresponding unit ball

For Conjecture 9.2.2, i.e. the upper bound, one needs to prove that the volume product of the convex hull of $2d+2$ points is always smaller than the volume product of K_c.

9.4.3 Minimizers of the volume product

It is interesting to note that the Lipschitz-free Banach space associated to the series composition of two graphs is the L^1 sum of their Lipschitz-free Banach space, in particular, its unit ball is the convex hull of the two unit balls. More precisely, if $M = (\{a_0, \ldots, a_d\}, \rho)$ and $N = (\{b_0, \ldots, b_k\}, \rho)$ are two finite pointed metric spaces with $d+1$ and $k+1$ elements respectively then one defines the metric space $M \diamond N$ by taking the union of the sets, where a_0 and b_0 are identified. So we get a finite metric set of $d+k+1$ elements obtained by connecting the two graphs by their roots. Then we get that $L_{M \diamond N} = \mathrm{conv}(L_M \times \{0\}, \{0\} \times L_N) \subset \mathbb{R}^{d+k}$. By using Lemma 2.5.1 we may then calculate the volume product of the composition of the two graphs,

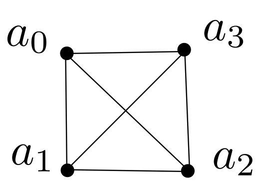
$$\mathcal{P}(M \diamond N) = \mathcal{P}(L_M \oplus_1 L_N) = \frac{d_1! d_2!}{(d_1+d_2)!} \mathcal{P}(L_M)\mathcal{P}(L_N).$$

We can then use this composition formula to find additional minimizers of the volume product, namely, the graphs corresponding to Hanner polytopes. For example, the unit ball of the Lipschitz free space of the cycle graph of four elements is the linear image of a cube in $\mathbb{R}^3$. Let us call this graph C_4. Then if we take any tree of $d-3$ elements, T, and compose the two using our diamond operation we have that

$$\mathcal{P}(C_4 \diamond T) = \frac{3!(d-3)!}{d!}\mathcal{P}(L_{C_4})P(L_T) = \frac{4^d}{d!}.$$

9.4.4 Complete graph

Let us consider the case of the complete graph K_{d+1} with all ρ_{ij} equal to 1. It corresponds to the discrete metric on $\{0,\dots,d\}$. The unit ball L_c of the Lipschitz-free space associated to K_{d+1} is $L_c = \mathrm{conv}\{\pm e_i, \pm(e_i - e_j); 1 \le i \ne j \le d\}$. It has exactly $d(d+1)$ vertices.



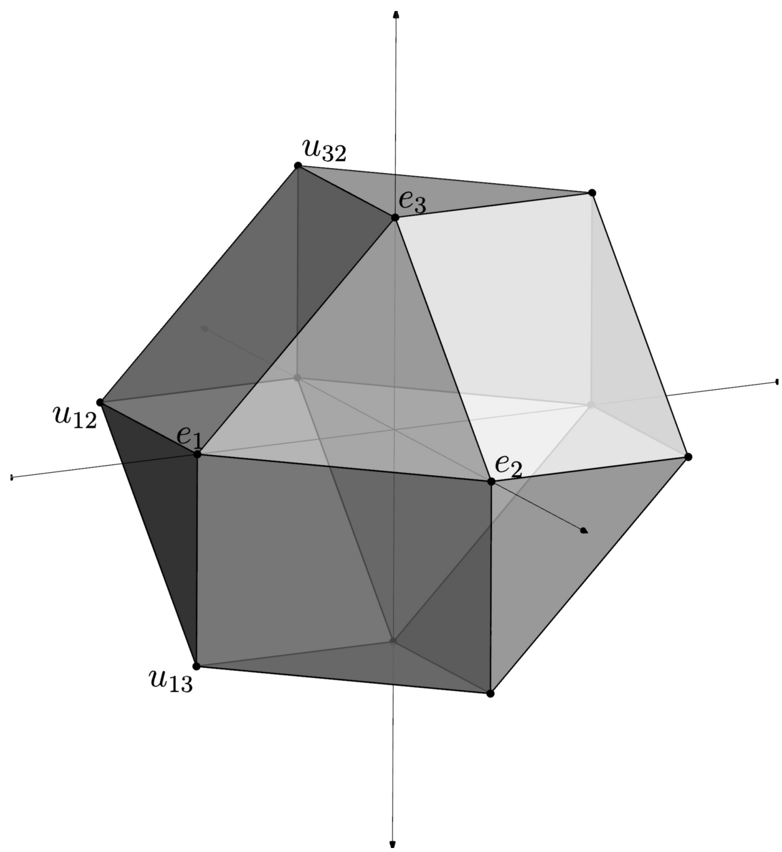


Figure 9.5: Complete graph of 4 points and corresponding unit ball

Let us describe more precisely the unit ball K_c of $\mathrm{Lip}_0(K_{d+1})$.

Claim 9.4.2. *The unit ball K_c of $Lip_0(K_{d+1})$ is*

$$K_c = \mathrm{conv}\left\{\pm\sum_{i\in I} e_i : I \subset \{1,\dots,d\}\right\}.$$

Proof: Denote by C the set on the right hand side. First let us prove that $C \subset K_c$. One has

$$K_c = L_c^\circ = \{x \in \mathbb{R}^d : |x_i| \leq 1, |x_i - x_j| \leq 1 \text{ for all } 1 \leq i \neq j \leq d\}.$$

For any $I \subset \{1, \dots, d\}$ let us denote $x(I) = \sum_{i \in I} e_i$. For any $k, l \in \{1, \dots, d\}$ one has $x(I)_k \in \{0; 1\}$ and $x(I)_k - x(I)_l \in \{-1; 0; 1\}$ hence $x(I) \in K_c$. Therefore $C \subset K_c$.

To show that $K_c \subset C$ we consider $x \in K_c$ then $|x_i| \leq 1$ and so we may assume, reordering our axes if necessary, that $-1 \leq x_1 \leq x_2 \leq \ldots \leq x_{d-1} \leq x_d \leq 1$. Further, since $|x_i - x_j| \leq 1$ we get $x_d - x_1 \leq 1$. Now let us consider the indices with positive entries and negative entries separately. That is, we let k be the last negative index, choosing it to be 0 if no entries are negative, and to be d if all entries are negative. Then

$$x = (x_2 - x_1)(-e_1) + (x_3 - x_2)(-e_1 - e_2) + \ldots + (x_k - x_{k-1}) \left(- \sum_{i \leq k-1} e_i \right)$$
$$+ x_{k+1} \sum_{i \geq k+1} e_i + (x_{k+2} - x_{k+1}) \sum_{i \geq k+2} e_i + \ldots + (x_d - x_{d-1}) e_d$$

Which is a convex combination of points in C. Therefore $K_c \subset C$.

□

Let us denote $\sum_{i=1}^d e_i = e$ then we make the following claim.

Claim 9.4.3.

$$K_c = \frac{1}{2} B_\infty^d + \frac{1}{2} [-e, e]$$

Proof: Denote by D the zonotope on the right hand side. First let us take an extreme point $x \in D$ then

$$x = \frac{1}{2} \sum_{i=1}^d \varepsilon_i e_i + \frac{1}{2} \varepsilon_{d+1} \sum_{i=1}^d e_i = \sum_{i=1}^d \frac{\varepsilon_i + \varepsilon_{d+1}}{2} e_i.$$

where the $\varepsilon_i \in \{-1; 1\}$. Let $I = \{i \in \{1, \dots, d\} | \varepsilon_i = \varepsilon_{d+1}\}$. Then

$$x = \varepsilon_{d+1} \sum_{i \in I} e_i \in K_c.$$

To see that $K_c \subset D$ we simply reverse our previous observation. So if we take $x \in C$ so $x = \varepsilon_{d+1} \sum_{i \in I} e_i$ where $I \subset \{1, \dots, d\}$. Then we define $\varepsilon_i = \varepsilon_{d+1}$ if $i \in I$ and $\varepsilon_i = -\varepsilon_{d+1}$ if $i \notin I$. Then by our choices of ε_i we have $x = \frac{1}{2} \sum_{i=1}^d \varepsilon_i e_i + \frac{1}{2} \varepsilon_{d+1} \sum_{i=1}^d e_i \in D$ as desired.

□

Let us compute the volume product of K_c.

Claim 9.4.4.

$$\mathcal{P}(K_c) = \frac{d+1}{d!}\binom{2d}{d}.$$

Proof: Let us first compute the volume of K_c. Recall, $e = \sum_{i=1}^d e_i$. Then K_c is a zonotope which is the following sum of $d+1$ segments

$$K_c = \frac{1}{2}\left(\sum_{i=1}^d [-e_i, e_i] + [-e, e]\right).$$

Thus one may use the following formula for the volume of zonotopes: if $Z = \sum_{i=1}^m [0, u_i]$ is a zonotope which is the sum of m vectors $u_1, \dots, u_m$ in $\mathbb{R}^d$, with $m \geq d$ then

$$|Z| = \sum_{\mathrm{card}(I)=d} |\det(u_i)_{i\in I}|.$$

Since for all $k \in \{1, \dots, d\}$ one has $|\det(e, (e_i)_{i\neq k})| = 1$ we get $|K_c| = d+1$.

Now let us compute the volume of $L_c = \mathrm{conv}\{\pm e_i, \pm(e_i - e_j); 1 \leq i \neq j \leq d\}$. We decompose L_c using the partition of $\mathbb{R}^d$ into 2^d parts defined according to the coordinate signs. For $I \subset \{1, \dots d\}$ let

$$C_I = \{x \in \mathbb{R}^d : x_i > 0 \text{ for all } i \in I \text{ and } x_j < 0 \text{ for all } j \notin I\}.$$

Then

$$K_c \cap C_I = \mathrm{conv}(0; (e_i)_{i\in I}; (-e_j)_{j\notin I}; (e_i - e_j)_{i\in I, j\notin I}) = \mathrm{conv}(0; (e_i)_{i\in I}) + \mathrm{conv}(0; (e_j)_{j\notin I}).$$

Hence if we denote $k = \mathrm{card}(I)$ then

$$|K_c| = |\mathrm{conv}(0; (e_i)_{i\in I})|_k |\mathrm{conv}(0; (e_j)_{j\notin I})|_{n-k} = \frac{1}{\mathrm{card}(I)!\mathrm{card}(I^c)|!} = \frac{1}{k!(n-k)!}.$$

Thus

$$|K_c| = \sum_{I\subset\{1,\dots,d\}} \frac{1}{\mathrm{card}(I)!\mathrm{card}(I^c)|!} = \sum_{k=0}^d \binom{d}{k} \frac{1}{k!(d-k)!} = \frac{1}{d!}\sum_{k=0}^d \binom{d}{k}^2 = \frac{1}{d!}\binom{2d}{d}.$$

□

9.4.5 Maximality for three points

Theorem 9.4.5. *For all metric spaces of three elements, $\mathcal{P}(M) \leq \mathcal{P}(K_C)$, where K_C is the complete metric space described above.*

Proof:

Let M be a metric space of three elements with distances between elements written as above (see Figures 9.1, 9.2, 9.3). Then after a linear transformation $K = B(M) = \text{conv}\{\pm e_1, \pm e_2, \pm \frac{\rho_1 e_1 - \rho_2 e_2}{\rho_{12}}\}$. We notice that we may exchange ρ_1 and ρ_2 without changing the volume product of our unit ball. Thus our volume product changes with ρ_{12}, and by symmetry the maximum must occur when $\rho_1 = \rho_2$. Without loss of generality let $\rho_1 = \rho_2 = 1$ and $t = \frac{1}{\rho_{12}}$. Then consider the volume of the shadow bodies $|K_t| = 1 + 2t$ and $|K_t^\circ| = \frac{4t-1}{t^2}$. By simple calculus, $\mathcal{P}(K_t) \leq \mathcal{P}(K_1) = 9$. Thus the maximum occurs in the complete graph when all distances are equal (Figure 9.3).

□

9.4.6 Minimality for four points

Theorem 9.4.6. *For all metric spaces of four elements, $\mathcal{P}(M) \geq \mathcal{P}(B_1^3)$*

Let M be a metric space of four elements with distances between elements written as above. Then after a linear transformation $K = B(M) = \text{conv}\{\pm e_1, \pm e_2, \pm e_3, \pm u_{12}, \pm u_{13}, \pm u_{23}\}$, where $u_{ij} = \frac{\rho_i e_i - \rho_j e_j}{\rho_{ij}}$. Notice that the points u_{ij} must lie on the two dimensional hyperplane spanned by e_i and e_j. Further, by the triangle inequalities constraining the ρ_{ij} we have that $\langle e_i - e_j, u_{ij}\rangle \geq 1$ and $|\langle e_i + e_j, u_{ij}\rangle| \leq 1$.

In order to calculate the volume of our body we first need to understand what facial structure is possible for our body, that is, when faces will not be triangular. For u_{ij} to be on a non-triangular face, it must be coplanar to three other vertices, one of which is another $u_{k\ell}$. This hyperplane intersects the hyperplane that u_{ij} lives on to form a line. For each u_{ij} we have two such lines which divide each coordinate hyperplane into four regions. We may determine when u_{ij} falls on this line, and is thus coplanar, by computing the determinants of matrices formed by the coordinates of u_{ij} and the adjacent vertices. By symmetry, we need only consider the following hyperplanes in three octants:

Octant 1: $(-,+,+)$ has points $u_{21} = \frac{\rho_2 e_2 - \rho_1 e_1}{\rho_{12}}$ and $u_{31} = \frac{\rho_3 e_3 - \rho_1 e_1}{\rho_{13}}$. Then we need to check only the plane $e_3 e_2 u_{21} u_{31}$.

Consider the determinant of the following matrix.

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -\frac{\rho_1}{\rho_{12}} & -\frac{\rho_1}{\rho_{13}} \\ 1 & 0 & \frac{\rho_2}{\rho_{12}} & 0 \\ 0 & 1 & 0 & \frac{\rho_3}{\rho_{13}} \end{vmatrix} = \frac{\rho_1(\rho_2 + \rho_{13} - \rho_3 - \rho_{12})}{\rho_{12}\rho_{13}}.$$

If $\Delta_1 > 0$ then $K \cap \mathcal{O}_{(-,+,+)} = \operatorname{conv}\{0, e_2, e_3, -u_{12}\} \cup \operatorname{conv}\{0, e_3, -e_1, -u_{13}, -u_{12}\}$ which gives the following volume

$$|K| = \frac{1}{6}\left(\frac{\rho_1}{\rho_{12}} + \frac{\rho_1\rho_2}{\rho_{12}\rho_{13}} + \frac{\rho_2\rho_3}{\rho_{12}\rho_{13}}\right).$$

If $\Delta_1 < 0$ then $K \cap \mathcal{O}_{(-,+,+)} = \operatorname{conv}\{e_2, -e_1, -u_{13}, -u_{12}\} \cup \operatorname{conv}\{0, e_3, e_2, -e_1, -u_{13}\}$ which gives the following volume

$$|K| = \frac{1}{6}\left(\frac{\rho_1}{\rho_{13}} + \frac{\rho_1\rho_3}{\rho_{12}\rho_{13}} + \frac{\rho_2\rho_3}{\rho_{12}\rho_{13}}\right).$$

Notice, if $\Delta_1 = 0$ then the two formulas coincide.

Octant 2: $(+,-,+)$ has points $u_{12} = \frac{\rho_1 e_1 - \rho_2 e_2}{\rho_{12}}$ and $u_{32} = \frac{\rho_3 e_3 - \rho_2 e_2}{\rho_{23}}$. Then we need to check only the plane $e_3 e_1 u_{12} u_{32}$.

Consider the determinant of the following matrix.

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & \frac{\rho_1}{\rho_{12}} & 0 \\ 0 & 0 & -\frac{\rho_2}{\rho_{12}} & -\frac{\rho_2}{\rho_{23}} \\ 0 & 1 & 0 & \frac{\rho_3}{\rho_{23}} \end{vmatrix} = \frac{\rho_2(\rho_3 - \rho_1 + \rho_{12} - \rho_{23})}{\rho_{12}\rho_{23}}.$$

If $\Delta_2 > 0$ then $K \cap \mathcal{O}_{(+,-,+)} = \operatorname{conv}\{0, e_1, e_3, u_{12}\} \cup \operatorname{conv}\{0, e_3, u_{12}, u_{32}\}$ which gives the following volume

$$|K| = \frac{1}{6}\left(\frac{\rho_2}{\rho_{12}} + \frac{\rho_1\rho_2}{\rho_{12}\rho_{23}} + \frac{\rho_1\rho_3}{\rho_{12}\rho_{23}}\right).$$

If $\Delta_2 < 0$ then $K \cap \mathcal{O}_{(+,-,+)} = \operatorname{conv}\{-e_2, e_1, u_{32}, u_{12}\} \cup \operatorname{conv}\{0, e_3, -e_2, e_1, u_{32}\}$ which gives the following volume

$$|K| = \frac{1}{6}\left(\frac{\rho_2}{\rho_{23}} + \frac{\rho_2\rho_3}{\rho_{12}\rho_{23}} + \frac{\rho_1\rho_3}{\rho_{12}\rho_{23}}\right).$$

Octant 3: $(-,-,+)$ has points $u_{32} = \frac{\rho_3 e_3 - \rho_2 e_2}{\rho_{23}}$ and $u_{31} = \frac{\rho_3 e_3 - \rho_1 e_1}{\rho_{13}}$. Then we need to check only the plane $-e_1 - e_2 u_{32} u_{31}$.

Consider the determinant of the following matrix.

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -\frac{\rho_1}{\rho_{13}} & 0 \\ 0 & -1 & 0 & -\frac{\rho_2}{\rho_{23}} \\ 0 & 0 & \frac{\rho_3}{\rho_{13}} & \frac{\rho_3}{\rho_{23}} \end{vmatrix} = \frac{\rho_3(\rho_2 - \rho_1 + \rho_{13} - \rho_{23})}{\rho_{13}\rho_{23}}.$$

If $\Delta_3 > 0$ then $K \cap \mathcal{O}_{(-,-,+)} = \mathrm{conv}\{0, -e_1, -e_2, e_3, u_{32}\} \cup \mathrm{conv}\{0, e_3, -e_1, u_{31}, u_{32}\}$ which gives the following volume

$$|K| = \frac{1}{6}\left(\frac{\rho_3}{\rho_{23}} + \frac{\rho_1\rho_2}{\rho_{13}\rho_{23}} + \frac{\rho_2\rho_3}{\rho_{13}\rho_{23}}\right).$$

If $\Delta_3 < 0$ then

$$|K| = \frac{1}{6}\left(\frac{\rho_3}{\rho_{13}} + \frac{\rho_1\rho_2}{\rho_{13}\rho_{23}} + \frac{\rho_1\rho_3}{\rho_{13}\rho_{23}}\right).$$

Now we proceed to create shadow systems by cases on the points u_{ij}. We begin by taking the points u_{ij} to be in a general position. We then begin to manipulate the points by changing individual distances. Without loss of generality we begin by changing ρ_{12} which will move u_{12} and its negative.

Case 1. u_{12} *hits the inner edge of its boundary in* span $\{e_1, e_2\}$

Then $u_{12} \in [e_1, -e_2]$ and so two vertices have vanished. We now may WLOG move ρ_{13} which yields three possibilities.

Case 1.1. u_{13} *hits the inner edge. Then* $u_{13} \in \mathrm{span}\{e_1, -e_3\}$*. Now there remains 8 vertices for which the minimal bodies are calculated via [55].*

Case 1.2. u_{13} *hits a hyperplane so that* $\Delta_3 = 0$.

Then $\rho_2 + \rho_{13} = \rho_1 + \rho_{23}$ We then proceed by moving ρ_3 which will give two cases.

Case 1.2.1. *We increase* ρ_3 *until we are left with 8 points when* $e_3 \in [e_2, u_{32}]$ *or* $e_3 \in [e_1, u_{31}]$

Case 1.2.2. *We decrease* ρ_3 *until we arrive at the* L_1 *ball.*

Case 1.3. u_{13} *hits an outer edge. Then there remains only 8 vertices as either* $e_1 \in [e_3, u_{13}]$ *or* $-e_3 \in [-e_1, u_{13}]$

Case 2. u_{12} *hits a hyperplane such that either* $\Delta_1 = 0$ *or* $\Delta_3 = 0$.

Without loss of generality, choose $\Delta_1 = 0$ then $\rho_2 + \rho_{13} = \rho_3 + \rho_{12}$. Then we may change ρ_{23}.

Case 2.1. *u_{23} hits the inner edge then this is the same as case 1.2*

Case 2.2. *u_{23} hits a hyperplane. Then u_{13} also hits a hyperplane and $\Delta_1 = \Delta_2 = \Delta_3 = 0$. However, this is the complete case, which we know is dual to a zonotope. So by [74, 31] we know that this is not minimal.*

$\square$

CHAPTER 10

Further Work

In chapter 5 we showed that for each $d \in \mathbb{N}$ there exists a constant $C(d)$ depending on d only, such that for any origin-symmetric convex body $K \subset \mathbb{R}^d$ containing d linearly independent lattice points

$$\#K \le C(d) \max(\#(K \cap H)) \operatorname{vol}_d(K)^{\frac{d-m}{d}},$$

where the maximum is taken over all m-dimensional subspaces of $\mathbb{R}^d$. We also proved that $C(d)$ can be chosen asymptotically of order $O(1)^d d^{d-m}$, and, in particular, we have order $O(1)^d$ for hyperplane slices. Additionally, we showed that if K is an unconditional convex body then $C(d)$ can be chosen asymptotically of order $O(d)^{d-m}$, which for hyperplane slices gives $O(d)^1$.

There are many future directions possible for work in this area. First, as stated above, is studying whether it is possible to improve the bound in Theorem 5.7.1. One likely area to begin improving is in the bound given in Theorem 5.6.1. Another question is if the continuous volume of the body, needed to preserve homogeneity, can be eliminated from the estimate. There are also questions about what can be said when restricting the size of the body to be either large or small. In [73] the constant is improved for bodies with volume less than C^{d^2}.

Further, there are many unanswered questions still open from [24]. In chapter 4 we mentioned several partial results for the discrete version of Aleksandrov's uniqueness theorem (Theorem 4.2.1. However, this largely remains open in that no "larger" counterexamples have been found in dimension 2, and no counterexamples are known in dimension 3 or above.

In chapter 7 we showed that the supremum of the volume product among all polytopes with at most m vertices is attained at a simplicial polytope with exactly m vertices. Further we provided a new proof of a result of Meyer and Reisner [64] showing that, in the plane, the regular polygon has maximal volume product among all polygons with at most m vertices. These results together would seem to suggest that the maximal simplicial polytope should have its vertices evenly distributed on a sphere, but we have no such result. Last, we treat the case of polytopes with $d+2$ vertices in $\mathbb{R}^d$ and symmetric polytopes with 8 vertices in $\mathbb{R}^3$. Both offer possible techniques for extending our results further.

Finally, in chapter 9 we studied the geometric and extremal properties of the convex body L_M, which is the unit ball Lipschitz-free Banach space associated a finite metric space M. We provided conjectures for the graphs which give the extremal volume products and show evidence for these conjectures. In particular we show that the maximum among all metric spaces of 3 points occurs when the corresponding graph is complete, and the minimum for spaces of 4 points must be a tree or the cycle graph. It is unclear which graphs correspond to cubes in dimensions 4 and 5, however, compositions of these graphs with trees are the conjectured minimizers whose unit balls, L_M, correspond to the minimizers of Mahler's conjecture. The conjecture is open starting from the case of 5 elements excluding some special graphs in low dimension and, clearly, the most interesting results correspond to the case of metric spaces with a large number of components (i.e. to large data structures).

BIBLIOGRAPHY

[1] M. Alexander, M. Fradelizi, and A. Zvavitch. Polytopes of maximal volume product. *preprint*, 2017.

[2] M. Alexander, M. Fradelizi, and A. Zvavitch. Volume product in lipshitz-free banach spaces. *preprint*, 2017.

[3] M. Alexander, M. Henk, and A. Zvavitch. A discrete version of koldobsky's slicing inequality. *Isreal Journal of Mathematics*, 2015. to appear.

[4] R. J. Aliaga and A. J. Guirao. On the preserved extremal structure of Lipschitz-free spaces. *ArXiv e-prints*, May 2017.

[5] Shiri Artstein-Avidan, Apostolos Giannopoulos, and Vitali D. Milman. *Asymptotic geometric analysis. Part I*, volume 202 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[6] Keith Ball. Cube slicing in $\mathbf{R}^n$. *Proc. Amer. Math. Soc.*, 97(3):465–473, 1986.

[7] I Bárány and AM Vershik. On the number of convex lattice polytopes. *Geometric and Functional Analysis*, 2(4):381–393, 1992.

[8] F. Barthe and M. Fradelizi. The volume product of convex bodies with many hyperplane symmetries. *Amer. J. Math.*, 135(2):311–347, 2013.

[9] Matthias Beck and Sinai Robins. Computing the continuous discretely: Integer-point enumeration in polyhedra. 2007. undergrad. texts in math., 2007.

[10] J. Bourgain. On the Busemann-Petty problem for perturbations of the ball. *Geom. Funct. Anal.*, 1(1):1–13, 1991.

[11] Jean Bourgain. Geometry of banach spaces and harmonic analysis. In *Proceedings of the International Congress of Mathematicians*, volume 1, page 2. Citeseer, 1986.

[12] Jean Bourgain. On high dimensional maximal functions associated to convex bodies. *American Journal of Mathematics*, 108(6):1467–1476, 1986.

[13] Jean Bourgain and Vitaly D Milman. New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$. *Inventiones Mathematicae*, 88(2):319–340, 1987.

[14] Silouanos Brazitikos, Apostolos Giannopoulos, Petros Valettas, and Beatrice-Helen Vritsiou. *Geometry of isotropic convex bodies*, volume 196 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014.

[15] H. Busemann and C. M. Petty. Problems on convex bodies. *Math. Scand.*, 4:88–94, 1956.

[16] Stefano Campi and Paolo Gronchi. On volume product inequalities for convex sets. *Proc. Amer. Math. Soc.*, 134(8):2393–2402, 2006.

[17] J.W.S. Cassels. *An Introduction to the Geometry of Numbers*. Classics in Mathematics. Springer Berlin Heidelberg, 2012.

[18] M. Fradelizi, M. Madiman, A. Marsiglietti, and A. Zvavitch. The convexification effect of Minkowski summation. *ArXiv e-prints*, April 2017.

[19] Matthieu Fradelizi, Mathieu Meyer, and Artem Zvavitch. An application of shadow systems to Mahler's conjecture. *Discrete Comput. Geom.*, 48(3):721–734, 2012.

[20] R. J. Gardner. Intersection bodies and the Busemann-Petty problem. *Trans. Amer. Math. Soc.*, 342(1):435–445, 1994.

[21] R. J. Gardner. A positive answer to the Busemann-Petty problem in three dimensions. *Ann. of Math. (2)*, 140(2):435–447, 1994.

[22] R. J. Gardner, A. Koldobsky, and T. Schlumprecht. An analytic solution to the Busemann-Petty problem on sections of convex bodies. *Ann. of Math. (2)*, 149(2):691–703, 1999.

[23] Richard J Gardner. *Geometric tomography; 2nd ed.* Encyclopedia of mathematics and its applications. Cambridge University Press, Cambridge, 2006.

[24] Richard J Gardner, Paolo Gronchi, and Chuanming Zong. Sums, projections, and sections of lattice sets, and the discrete covariogram. *Discrete & Computational Geometry*, 34(3):391–409, 2005.

[25] Apostolos Giannopoulos, Grigoris Paouris, and Beatrice-Helen Vritsiou. The isotropic position and the reverse Santaló inequality. *Israel J. Math.*, 203(1):1–22, 2014.

[26] Apostolos A. Giannopoulos. A note on a problem of H. Busemann and C. M. Petty concerning sections of symmetric convex bodies. *Mathematika*, 37(2):239–244, 1990.

[27] E. D. Gluskin and A. E. Litvak. On the asymmetry constant of a body with few vertices. *Geom. Dedicata*, 90:45–48, 2002.

[28] A. Godard. Tree metrics and their Lipschitz-free spaces. *Proc. Amer. Math. Soc.*, 138(12):4311–4320, 2010.

[29] G. Godefroy and N. J. Kalton. Lipschitz-free Banach spaces. *Studia Math.*, 159(1):121–141, 2003. Dedicated to Professor Aleksander Pelczyński on the occasion of his 70th birthday.

[30] Gilles Godefroy. A survey on Lipschitz-free Banach spaces. *Comment. Math.*, 55(2):89–118, 2015.

[31] Y. Gordon, M. Meyer, and S. Reisner. Zonoids with minimal volume-product—a new proof. *Proc. Amer. Math. Soc.*, 104(1):273–276, 1988.

[32] Yehoram Gordon and Mathieu Meyer. On the minima of the functional Mahler product. *Houston J. Math.*, 40(2):385–393, 2014.

[33] P. Gruber. *Convex and Discrete Geometry.* Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2007.

[34] P. M. Gruber and J. M. Wills, editors. *Handbook of convex geometry. Vol. A, B.* North-Holland Publishing Co., Amsterdam, 1993.

[35] Branko Grünbaum. *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

[36] Martin Henk. Successive minima and lattice points. *Rend. Circ. Mat. Palermo (2) Suppl.*, (70, part I):377–384, 2002. IV International Conference in "Stochastic Geometry, Convex Bodies, Empirical Measures & Applications to Engineering Science", Vol. I (Tropea, 2001).

[37] Martin Henk and Jörg M. Wills. Minkowski's successive minima. In *Number theory & discrete geometry*, volume 6 of *Ramanujan Math. Soc. Lect. Notes Ser.*, pages 129–142. Ramanujan Math. Soc., Mysore, 2008.

[38] D. Hilbert and S. Cohn-Vossen. *Geometry and the imagination.* Chelsea Publishing Company, New York, N. Y., 1952. Translated by P. Neményi.

[39] H. Iriyeh and M. Shibata. Symmetric Mahler's conjecture for the volume product in the three dimensional case. *ArXiv e-prints*, June 2017.

[40] Fritz John. Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*, pages 187–204. Interscience Publishers, Inc., New York, N. Y., 1948.

[41] Ravi Kannan and László Lovász. Covering minima and lattice-point-free convex bodies. *Ann. of Math. (2)*, 128(3):577–602, 1988.

[42] Jaegil Kim. Minimal volume product near Hanner polytopes. *J. Funct. Anal.*, 266(4):2360–2402, 2014.

[43] Jaegil Kim and Shlomo Reisner. Local minimality of the volume-product at the simplex. *Mathematika*, 57(1):121–134, 2011.

[44] B. Klartag and A. Koldobsky. An example related to the slicing inequality for general measures. *ArXiv e-prints*, June 2017.

[45] Alexander Koldobsky. *Fourier analysis in convex geometry*, volume 116 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[46] Alexander Koldobsky. A hyperplane inequality for measures of convex bodies in $\mathbb{R}^n$, $n \leq 4$. *Discrete Comput. Geom.*, 47(3):538–547, 2012.

[47] Alexander Koldobsky. A hyperplane inequality for measures of convex bodies in $\mathbb{R}^n$, $n \leq 4$. *Discrete Comput. Geom.*, 47(3):538–547, 2012.

[48] Alexander Koldobsky. A $\sqrt{n}$ estimate for measures of hyperplane sections of convex bodies. *Adv. Math.*, 254:33–40, 2014.

[49] Alexander Koldobsky. Slicing inequalities for measures of convex bodies. *Adv. Math.*, 283:473–488, 2015.

[50] Alexander Koldobsky. Slicing inequalities for subspaces of L_p. *Proc. Amer. Math. Soc.*, 144(2):787–795, 2016.

[51] Alexander Koldobsky and Artem Zvavitch. An isomorphic version of the Busemann-Petty problem for arbitrary measures. *Geom. Dedicata*, 174:261–277, 2015.

[52] Greg Kuperberg. From the Mahler conjecture to Gauss linking integrals. *Geom. Funct. Anal.*, 18(3):870–892, 2008.

[53] D. G. Larman and C. A. Rogers. The existence of a centrally symmetric convex body with central sections that are unexpectedly small. *Mathematika*, 22(2):164–175, 1975.

[54] C.G. Lekkerkerker and P. Gruber. *Geometry of Numbers.* North-Holland Mathematical Library. Elsevier Science, 1987.

[55] M. A. Lopez and S. Reisner. A special case of Mahler's conjecture. *Discrete Comput. Geom.*, 20(2):163–177, 1998.

[56] Kurt Mahler. Ein minimalproblem für konvexe polygone. *Mathematica (Zutphen)*, B7:118–127, 1939.

[57] Kurt Mahler. Ein übertragungsprinzip für konvexe Körper. *Časopis Pěst. Mat. Fys.*, 68:93–102, 1939.

[58] E Makai Jr. The recent status of the volume product problem. *arXiv preprint arXiv:1507.01473*, 2015.

[59] M. Meyer and S. Reisner. Characterizations of ellipsoids by section-centroid location. *Geom. Dedicata*, 31(3):345–355, 1989.

[60] M. Meyer and S. Reisner. Inequalities involving integrals of polar-conjugate concave functions. *Monatsh. Math.*, 125(3):219–227, 1998.

[61] Mathieu Meyer. Une caractérisation volumique de certains espaces normés de dimension finie. *Israel J. Math.*, 55(3):317–326, 1986.

[62] Mathieu Meyer. Convex bodies with minimal volume product in $\mathbf{R}^2$. *Monatsh. Math.*, 112(4):297–301, 1991.

[63] Mathieu Meyer and S. Reisner. Shadow systems and volumes of polar convex bodies. *Mathematika*, 53(1):129–148 (2007), 2006.

[64] Mathieu Meyer and S. Reisner. On the volume product of polygons. *Abh. Math. Semin. Univ. Hambg.*, 81(1):93–100, 2011.

[65] Mathieu Meyer, Carsten Schütt, and Elisabeth Werner. Dual affine invariant points. *Indiana Univ. Math. J.*, 64(3):735–768, 2015.

[66] Mathieu Meyer, Carsten Schütt, and Elisabeth M. Werner. Affine invariant points. *Israel J. Math.*, 208(1):163–192, 2015.

[67] V. D. Milman and A. Pajor. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.*, pages 64–104. Springer, Berlin, 1989.

[68] Vitali D. Milman and Gideon Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With an appendix by M. Gromov.

[69] Fedor Nazarov. The Hörmander proof of the Bourgain-Milman theorem. In *Geometric aspects of functional analysis*, volume 2050 of *Lecture Notes in Math.*, pages 335–343. Springer, Heidelberg, 2012.

[70] Michael Papadimitrakis. On the Busemann-Petty problem about convex, centrally symmetric bodies in $\mathbb{R}^n$. *Mathematika*, 39(2):258–266, 1992.

[71] C. M. Petty. Projection bodies. In *Proc. Colloquium on Convexity (Copenhagen, 1965)*, pages 234–241. Kobenhavns Univ. Mat. Inst., Copenhagen, 1967.

[72] C. M. Petty. Affine isoperimetric problems. In *Discrete geometry and convexity (New York, 1982)*, volume 440 of *Ann. New York Acad. Sci.*, pages 113–127. New York Acad. Sci., New York, 1985.

[73] O. Regev. A Note on Koldobsky's Lattice Slicing Inequality. *ArXiv e-prints*, August 2016.

[74] Shlomo Reisner. Zonoids with minimal volume-product. *Math. Z.*, 192(3):339–346, 1986.

[75] Shlomo Reisner. Minimal volume-product in Banach spaces with a 1-unconditional basis. *J. London Math. Soc. (2)*, 36(1):126–136, 1987.

[76] Shlomo Reisner, Carsten Schütt, and Elisabeth M. Werner. Mahler's conjecture and curvature. *Int. Math. Res. Not. IMRN*, (1):1–16, 2012.

[77] C. A. Rogers and G. C. Shephard. Some extremal problems for convex bodies. *Mathematika*, 5:93–102, 1958.

[78] Dmitry Ryabogin, Vladyslav Yaskin, and Ning Zhang. Unique determination of convex lattice sets. *Discrete Comput. Geom.*, 57(3):582–589, 2017.

[79] Dmitry Ryabogin and Artem Zvavitch. Analytic methods in convex geometry. In *Analytical and probabilistic methods in the geometry of convex bodies*, volume 2 of *IMPAN Lect. Notes*, pages 87–183. Polish Acad. Sci. Inst. Math., Warsaw, 2014.

[80] J. Saint-Raymond. Sur le volume des corps convexes symétriques. In *Initiation Seminar on Analysis: G. Choquet-M. Rogalski-J. Saint-Raymond, 20th Year: 1980/1981*, volume 46 of *Publ. Math. Univ. Pierre et Marie Curie*, pages Exp. No. 11, 25. Univ. Paris VI, Paris, 1981.

[81] L. A. Santaló. An affine invariant for convex bodies of n-dimensional space. *Portugaliae Math.*, 8:155–161, 1949.

[82] Rolf Schneider. Zur einem Problem von Shephard über die Projektionen konvexer Körper. *Math. Z.*, 101:71–82, 1967.

[83] Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.

[84] G. C. Shephard. Shadow systems of convex sets. *Israel J. Math.*, 2:229–236, 1964.

[85] Terence Tao. Santaló inequalities. http://www.math.ucla.edu/∼tao/preprints/Expository/santalo.dvi.

[86] Terence Tao. *Structure and randomness*. American Mathematical Society, Providence, RI, 2008. Pages from year one of a mathematical blog.

[87] Terence Tao and Van Vu. John-type theorems for generalized arithmetic progressions and iterated sumsets. *Adv. Math.*, 219(2):428–449, 2008.

[88] Terence Tao and Van H. Vu. *Additive combinatorics*, volume 105 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. Paperback edition [of MR2289012].

[89] G. Toth. *Measures of Symmetry for Convex Sets and Stability*. Universitext. Springer International Publishing, 2015.

[90] Nik Weaver. *Lipschitz algebras*. World Scientific Publishing Co., Inc., River Edge, NJ, 1999.

[91] Roger Webster. *Convexity*. Oxford University Press, 1994.

[92] Huan Xiong. On a discrete version of Alexandrov's projection theorem. *Acta Math. Sin. (Engl. Ser.)*, 29(8):1597–1606, 2013.

[93] Gaoyong Zhang. A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$. *Ann. of Math. (2)*, 149(2):535–543, 1999.

[94] Ning Zhang. An analogue of the Aleksandrov projection theorem for convexlattice polygons. *Proc. Amer. Math. Soc.*, 145(6):2305–2310, 2017.

[95] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.