The Upper Bound of Finite Additive 2-Bases

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§1. Introduction

In this thesis, we are interested in analytic upper bounds for the so called “finite additive 2-bases of integers”. This problem first was introduced by Rohrbach [8] in which he discussed the finite additive bases of order h, and introduced the extremal problem of determining the largest integer n for which there exists a set A consisting of at most k nonnegative integers such that A is a basis of order 2 for n.

Let n be a positive integer, and let A be a set of nonnegative integers such that $A \coloneqq \{a_i \in [0, n] \cap \mathbb{Z}: 0 = a_0 < a_1 < \cdots < a_{k-1}\}$. The set A is called a basis of order h for n if every integer $m \in [0, n]$ can be represented as the sum of h elements of the set A, where $h \geq 2$.

Rohrbach’s problem is to estimate the smallest number of elements $k_h(n)$ that can form an h-basis for n. The next table is an example for some of $k_h(n)$ and its basis A.

$k_h(n) = k$	$A = \{a_i: i = 0, 1, \dots, k-1\}$
$k_2(1) = 2$	$\{0,1\}$
$k_2(4) = 3$	$\{0,1,3\}$
$k_2(6) = 4$	$\{0,1,3,5\}$
$k_2(8) = 5$	$\{0,1,3,5,7\}$
$k_2(16) = 6$	$\{0,1,3,5,7,8\}$
$k_3(27) = 7$	$\{0,1,3,5,7,8,11\}$
$k_4(70) = 8$	$\{0,1,3,5,7,8,11,27\}$

Notice that $k_h(n)$ doesn’t necessarily have unique minimal 2-basis. For example, the unique minimal basis of $k_2(1)$ is $\{0,1\}$, but we can have $\{0,1,2\}$ as another minimal 2-basis for $k_2(4)$.

Rohrbach [8] observed that if A is a minimal additive 2-basis for n such that $k = k_2(n)$, then there are exactly $\binom{k+1}{2}$ ordered pairs of the form (a_i, a_j) where $a_i, a_j \in A$ and $a_i \leq a_j$. This gives the upper bound

$$n \leq \binom{k+1}{2} = \frac{k^2+k}{2} = \frac{k^2}{2} + O(k),$$

and hence

$$\lim_{n\to\infty} sup \frac{n}{k^2} \leq \frac{1}{2}.$$

It is an open problem to compute the upper bound of this upper limit. (There are also such upper bounds for the more general additive h-bases, c.f. [5,6,7,10]).

Define σ by

$$\sigma = \lim_{n\to\infty} sup \frac{n}{k^2}.$$

The upper bound of σ has been improved using various approaches as follows:

$\sigma \leq 0.4992$	Rohrbach [8]	1937
$\sigma \leq 0.4903$	Moser [5]	1960
$\sigma \leq 0.4867$	Riddell [7]	1960
$\sigma \leq 0.4847$	Moser, Pounder and Riddell [6]	1969
$\sigma \leq 0.4802$	Klotz [4]	1969
$\sigma \leq 0.4789$	Güntürk and Nathanson [1]	2006
$\sigma \leq 0.4778$	Horváth [3]	2007
$\sigma \leq 0.4698$	Yu [9]	2009
$\sigma \leq 0.4691$	Habsieger [2]	2014
$\sigma \leq 0.4585$	Yu [10]	2015

These bounds were obtained by either combinatorial arguments or methods of harmonic analysis, or a combination of both. In particular, Rohrbach's result relies on a combinatorial argument only. All other results more or less are related to applications of Fourier series.

In this thesis, we will review some of the more efficient approaches used by previous authors. Moreover, we will improve the hitherto sharpest bound by the approach of Yu [10].

Theorem 1.1. we have

$$\sigma \leq 0.45504.$$

In the next chapters, we will mainly explore the methods of Moser [5], Güntürk and Nathanson [1], Yu [9], and Yu [10]. In particular, we will prove our theorem by using the approach of [10].

Remark. It should be mentioned that, in an unpublished note, Kevin Ford conjectured that

$$\sigma = \frac{\pi}{8}.$$

Ford made the conjecture based on a continuous analog of additive 2-bases.

§2. Preliminaries

For a finite set of integers A, let $q_A(z)$ denote the generating function

$$q_A(z) = \sum_{m\in A} z^m.$$

Also, let $f_A(t)$ denote the complex generating function

$$f_A(t) = q_A\left(e^{2\pi it}\right) = \sum_{m\in A} e^{2\pi imt}.$$

We define the representation function $r_A(m)$ and the difference function $d_A(m)$ as follows:

$$r_A(m) \; = \#\{(a_1, a_2) \in A^2 : \; a_1 + a_2 = m\},$$

and

$$d_A(m) = \#\{(a_1, a_2) \in A^2 : \; a_1 - a_2 = m\}.$$

Lemma 2.1. Suppose $n \in \mathbb{N}$ and $a \in \mathbb{Z}$, then we have

$$\sum_{m=0}^{n-1} e^{\frac{2ma\pi i}{n}} = \begin{cases} n & if\ n|a, \\ 0 & if\ n \nmid a. \end{cases}$$

Proof. When $n|a$, the sum is clearly equal to n.

Suppose $n \nmid a$. Note that

$$\sum_{m=1}^{n} e^{\frac{2ma\pi i}{n}} = \sum_{m=0}^{n-1} e^{\frac{2ma\pi i}{n}}, \qquad (2.1)$$

because $e^{\frac{2(n)a\pi i}{n}} = e^{\frac{2(0)a\pi i}{n}} = 1$.

On the other hand, we have

$$e^{\frac{2a\pi i}{n}} . \sum_{m=0}^{n-1} e^{\frac{2ma\pi i}{n}} = \sum_{m=1}^{n} e^{\frac{2ma\pi i}{n}},$$

and since $e^{\frac{2a\pi i}{n}} \neq 1$, then by (2.1) we get

$$\sum_{m=0}^{n-1} e^{\frac{2ma\pi i}{n}} = 0.$$

Lemma 2.2. If A is a 2-basis for n, then for any integer a not divisible by n, we have

$$\left|f_A\left(\frac{a}{n}\right)\right|^2 \lesssim k^2 - 2n.$$

Proof. By Lemma 2.1, for $n \nmid a$ we have

$$\sum_{m=0}^{n-1} e^{\frac{2ma\pi i}{n}} = 0.$$

Hence we get

$$\left|f_A\left(\frac{a}{n}\right)\right|^2 = \sum_{m=0}^{2n} r_A(m)\, e^{\frac{2ma\pi i}{n}} = \sum_{m=0}^{2n} r^*{}_A(m)\, e^{\frac{2ma\pi i}{n}},$$

where

$$r^*{}_A(m) = \begin{cases} r_A(m) - 2 & if\ 0 \le m \le n-1, \\ r_A(m) & otherwise. \end{cases}$$

Thus

$$\left|f_A\left(\frac{a}{n}\right)\right|^2 = \left|\sum_{m=0}^{2n} r^*{}_A(m)\, e^{\frac{2ma\pi i}{n}}\right| \le \sum_{m=0}^{2n} r^*{}_A(m) + O(k)$$

$$= \sum_{m=0}^{2n} r_A(m) - 2n + O(k) = k^2 - 2n + O(k).$$

In this thesis, Fourier series is the main tool of the studies. Let $\varphi(t) \in C^1(-\infty,\infty)$ be a periodic function on the period $2L$, then the Fourier series of the function φ can be given as

$$\varphi(t) = a_0 + \sum_{m=1}^{\infty}\left[a_m \cos\left(\frac{2\pi mt}{L}\right) + b_m \sin\left(\frac{2\pi mt}{L}\right)\right],$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \varphi(t)dt,$$

and for $m \ge 1$

$$a_m = \frac{1}{L}\int_{-L}^{L} \varphi(t)\cos\left(\frac{2\pi mt}{L}\right)dt, \text{ and } b_m = \frac{1}{L}\int_{-L}^{L} \varphi(t)\sin\left(\frac{2\pi mt}{L}\right)dt.$$

In Chapter 4, we also need Fourier series of function or two variables. Let $\varphi(t_1, t_2)$ be a function with period $2L$ on each variable, then its Fourier series

$$\varphi(t_1, t_2) = \sum_{m_1 \epsilon \mathbb{Z}} \sum_{m_2 \epsilon \mathbb{Z}} \hat{\varphi}(m_1, m_2) e^{\frac{2\pi i m_1 t_1}{L}} e^{\frac{2\pi i m_2 t_2}{L}},$$

where

$$\hat{\varphi}(m_1, m_2) = \frac{1}{4L^2} \int_{-L}^{L} \int_{-L}^{L} \varphi(t_1, t_2) e^{\frac{2\pi i m_1 t_1}{L}} e^{\frac{2\pi i m_2 t_2}{L}} dt_1 dt_2.$$

§3. Moser's approach

In this approach, Moser [5] used harmonic analysis to obtain an upper bound of additive 2-basis for n. Suppose the finite set A is a 2-basis for n with $|A| = k$. Let

$$g(z) = \frac{q_A(z)^2 + q_A(z^2)}{2}. \tag{3.1}$$

defined $\delta(m)$ such that

$$g(z) = 1 + z + z^2 + \cdots + z^{n-1} + \sum_{m \epsilon 2A} \delta(m) z^m, \tag{3.2}$$

where

$$\delta(m) = \begin{cases} r_A(m) - 1 & if\ m \epsilon \{0,1,\dots,n-1\}, \\ r_A(m) & otherwise. \end{cases}$$

Then put $z = 1$ in (3.1) and (3.2), then we get the estimate

$$n = \frac{k^2 + k}{2} - \sum_{m \epsilon 2A} \delta(m). \tag{3.3}$$

Now our goal is to find a lower bound for $\sum_{m \epsilon 2A} \delta(m)$. The first estimate for $\sum_{m \epsilon 2A} \delta(m)$ can be found if we let $l = \#\{a \epsilon A : a \geq n/2\}$. Notice that for any two elements $a_i, a_j \in l$, then $a_i + a_j \geq n$. Hence we have

$$\sum_{m \epsilon 2A} \delta(m) \geq \sum_{m \geq n} \delta(m) = \sum_{m \geq n} r_A(m) \geq \frac{l(l+1)}{2} \geq \frac{l^2}{2}. \tag{3.4}$$

We obtain the second estimate by using the complex generating function. Let

$z = e^{\frac{2a\pi i}{n}}$. Then we get

$$q_A\left(e^{\frac{2a\pi i}{n}}\right) = f_A\left(\frac{a}{n}\right) = \sum_{m \in A} e^{\frac{2ma\pi i}{n}}.$$

By Lemma 2.1 for every $a \in A$, if a be an integer not divisible by n. Then

$$\sum_{m=0}^{n-1} e^{\frac{2ma\pi i}{n}} = 0,$$

hence by (3.1) and (3.2) we obtain

$$\frac{f_A\left(\frac{a}{n}\right)^2+f_A\left(\left(\frac{a}{n}\right)^2\right)}{2}=\sum_{m\epsilon 2A}\delta(m)e^{\frac{2ma\pi i}{n}},$$

and we have

$$\sum_{m\epsilon 2A}\delta(m)\geq\left|\sum_{m\epsilon 2A}\delta(m)e^{2\pi imt}\right|\geq\frac{\left|f_A\left(\frac{a}{n}\right)\right|^2-k}{2}.$$

Since we are dealing with a finite set, then the maximum exists. Let

$$M=\max\{\left|f_A\left(\frac{a}{n}\right)\right|:a\not\equiv 0\ mod\ n\},$$

then

$$\sum_{m\epsilon 2A}\delta(m)\geq\frac{M^2-k}{2}. \tag{3.5}$$

Let $\varphi(t)$ be a function with period 1 and its Fourier series

$$\varphi(t)=a_0+\sum_{m=1}^{\infty}[a_m\cos(2\pi mt)+b_m\sin(2\pi mt)],$$

with convergence coefficients. Then

$$\sum_{a\epsilon A}\varphi\left(\frac{a}{n}\right)=\sum_{a\epsilon A}[a_0+\sum_{m=1}^{\infty}[a_m\cos(2\pi am/n)+b_m\sin(2\pi am/n)]].$$

Notice that $M\geq|\sum_{a\epsilon A}\cos(2\pi am/n)|$ and $M\geq|\sum_{a\epsilon A}\sin(2\pi am/n)|$. Therefore, we get

$$\left|\sum_{a\epsilon A}\varphi\left(\frac{a}{n}\right)\right|\leq M\sum_{\substack{m=1\\ n\nmid m}}^{\infty}[|a_m|+|b_m|]+k\sum_{\substack{m=0\\ n|m}}^{\infty}|a_m|.$$

For simplicity let $C_0=\sum_{\substack{m=0\\ n|m}}^{\infty}|a_m|$ and $C=\sum_{\substack{m=1\\ n\nmid m}}^{\infty}[|a_m|+|b_m|]$ to get the formula

$$\left|\sum_{a\epsilon A}\varphi\left(\frac{a}{n}\right)\right|\leq MC+kC_0, \tag{3.6}$$

and choose

$$\alpha_1=\min_{[0,\frac{1}{2}]}\varphi \quad \text{and} \quad \alpha_2=\min_{[\frac{1}{2},1]}\varphi.$$

Then we also have

$$\sum_{a\epsilon A}\varphi\left(\frac{a}{n}\right)\geq\ \alpha_1(k-l)+\alpha_2 l=\alpha_1 k+(\alpha_2-\alpha_1)l. \qquad (3.7)$$

Compare (3.6) and (3.7) to we get

$$\alpha_1 k+(\alpha_2-\alpha_1)l\ \leq MC+kC_0,$$

then

$$M\geq\frac{(\alpha_1-C_0)k+(\alpha_2-\alpha_1)l}{C}.$$

Hence we have now another lower bound for $\sum_{m\epsilon 2A}\delta(m)$, which is

$$\sum_{m\epsilon 2A}\delta(m)\geq\frac{\left[\frac{(\alpha_1-C_0)k+(\alpha_2-\alpha_1)l}{C}\right]^2-k}{2}. \qquad (3.8)$$

Now all we need is to find the maximum of the lower bound for $\sum_{m\epsilon 2A}\delta(m)$ by using (3.4) and (3.8) to get

$$\max\left[l,\frac{(\alpha_1-C_0)k+(\alpha_2-\alpha_1)l}{C}\right]\geq\frac{(\alpha_1-C_0)k}{(\alpha_1-\alpha_2)+C}.$$

Now by (3.5), we have the lower bound

$$\sum_{m\epsilon 2A}\delta(m)\geq\frac{1}{2}\left(\frac{(\alpha_1-C_0)k}{(\alpha_1-\alpha_2)+C}\right)^2-\frac{k}{2}.$$

By inserting this lower bound our estimate for n in (3.3), we get

$$n\leq\frac{1}{2}\left(1-\left(\frac{(\alpha_1-C_0)}{(\alpha_1-\alpha_2)+C}\right)^2\right)k^2+k.$$

Then the value of upper bound of σ is

$$\sigma\leq\frac{1}{2}\left(1-\left(\frac{(\alpha_1-C_0)}{(\alpha_1-\alpha_2)+C}\right)^2\right). \qquad (3.9)$$

We can use the same example of Moser [5]. Let

$$\varphi(t)=\frac{1}{2}\cos(4\pi t)+\sin(2\pi t)=\begin{cases}\frac{1}{2} & for\ 0\leq t\leq\frac{1}{2},\\ \frac{-3}{2} & for\ \frac{1}{2}\leq t\leq 1.\end{cases}$$

Then function $\varphi(t)$ has $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{-3}{2}$. Also we have $C = \frac{1}{2} + 1 = \frac{3}{2}$ and $C_0 = 0$. By using (3.9), we get the upper bound

$$\sigma \leq \frac{1}{2}\left(1 - \left(\frac{1}{7}\right)^2\right) = 0.4898.$$

§4. Double Fourier series approach

This method was introduced by Güntürk and Nathanson [1]. Let $\varphi(t_1, t_2)$ be a function with period 1 in each variable t_1 and t_2, then use the Fourier series

$$\varphi(t_1, t_2) = \sum_{m_1 \epsilon \mathbb{Z}} \sum_{m_2 \epsilon \mathbb{Z}} \hat{\varphi}(m_1, m_2) e^{2\pi i m_1 t_1} e^{2\pi i m_2 t_2},$$

where its coefficients converge absolutely. Let's choose $\hat{\varphi}(m_1, m_2) = 0$ when $m_1 = m_2 = 0$, and also we define α_1 and α_2 such that

$$\alpha_1 = \min_{t_1 + t_2 < 1} \varphi(t_1, t_2) \text{ and } \alpha_2 = \min_{t_1 + t_2 \geq 1} \varphi(t_1, t_2),$$

and choose our function $\varphi(t_1, t_2)$ such that $\alpha_1 > \alpha_2$. Define L such that

$$L = \#\{(a_1, a_2) \in A \times A: a_1 + a_2 \geq n\}.$$

Notice that $L \geq l^2$, then $\sum_{m \epsilon 2A} \delta(m)$ has a lower bound

$$\sum_{m \epsilon 2A} \delta(m) \geq \sum_{m \geq n} \delta(m) = \sum_{m \geq n} r_A(m) = \frac{L + l}{2} \geq \frac{L}{2}, \tag{4.1}$$

and since $k^2 - L = \#\{(a_1, a_2) \in A \times A: a_1 + a_2 < n\}$. Then we have

$$\sum_{a_1 \epsilon A} \sum_{a_2 \epsilon A} \varphi\left(\frac{a_1}{n}, \frac{a_2}{n}\right) \geq \alpha_1(k^2 - L) + \alpha_2 L = \alpha_1 k^2 - (\alpha_1 - \alpha_2)L, \tag{4.2}$$

also we have

$$\sum_{a_1 \epsilon A} \sum_{a_2 \epsilon A} \varphi\left(\frac{a_1}{n}, \frac{a_2}{n}\right) = \sum_{a_1 \epsilon A} \sum_{a_2 \epsilon A} \sum_{m_1 \epsilon \mathbb{Z}} \sum_{m_2 \epsilon \mathbb{Z}} \hat{\varphi}(m_1, m_2) e^{\frac{2\pi i m_1 a_1}{n}} e^{\frac{2\pi i m_1 a_2}{n}}$$

$$= \sum_{m_1 \epsilon \mathbb{Z}} \sum_{m_2 \epsilon \mathbb{Z}} \hat{\varphi}(m_1, m_2) \sum_{a_1 \epsilon A} \sum_{a_2 \epsilon A} e^{\frac{2\pi i m_1 a_1}{n}} e^{\frac{2\pi i m_1 a_2}{n}},$$

and sine $f_A(t) = \sum_{a \in A} e^{2\pi i a t}$. Then we can have

$$\sum_{a_1 \epsilon A} \sum_{a_2 \epsilon A} \varphi\left(\frac{a_1}{n}, \frac{a_2}{n}\right) = \sum_{m_1 \epsilon \mathbb{Z}} \sum_{m_2 \epsilon \mathbb{Z}} \hat{\varphi}(m_1, m_2) f_A\left(\frac{a_1}{n}\right) f_A\left(\frac{a_2}{n}\right).$$

where $\hat{\varphi}(m_1, m_2)$ is the Fourier series coefficients. Let $\left|f_A\left(\frac{a_i}{n}\right)\right| \leq M$ for $i = 1, 2$ then we have

$$\left|f_A\left(\frac{a_i}{n}\right)\right| \leq \begin{cases} M & if\ n \nmid m_i, \\ k & if\ \ n | m_i. \end{cases}$$

Hence

$$\left|\sum_{a_1 \epsilon A} \sum_{a_2 \epsilon A} \varphi\left(\frac{a_1}{n}, \frac{a_2}{n}\right)\right| \leq C_0 k^2 + C_1 kM + C_2 M^2, \quad (4.3)$$

where $C_0 = \sum_{\substack{m_1 \epsilon \mathbb{Z} \\ n|m_1}} \sum_{\substack{m_2 \epsilon \mathbb{Z} \\ n|m_2}} |\hat{\varphi}(m_1, m_2)|$, $C_2 = \sum_{\substack{m_1 \epsilon \mathbb{Z} \\ n\nmid m_1}} \sum_{\substack{m_2 \epsilon \mathbb{Z} \\ n\nmid m_2}} |\hat{\varphi}(m_1, m_2)|$

and $C_1 = \sum_{\substack{m_1 \epsilon \mathbb{Z} \\ n|m_1}} \sum_{\substack{m_2 \epsilon \mathbb{Z} \\ n\nmid m_2}} |\hat{\varphi}(m_1, m_2)| + \sum_{\substack{m_1 \epsilon \mathbb{Z} \\ n\nmid m_1}} \sum_{\substack{m_2 \epsilon \mathbb{Z} \\ n|m_2}} |\hat{\varphi}(m_1, m_2)|$.

Comparing the inequalities (4.2) and (4.3), we get

$$\alpha_1 k^2 - (\alpha_1 - \alpha_2)L \leq C_0 k^2 + C_1 kM + C_2 M^2.$$

Hence we have

$$L \geq \frac{(\alpha_1 - C_0)k^2 - C_1 kM - C_2 M^2}{(\alpha_1 - \alpha_2)}.$$

Güntürk and Nathanson [1] used the following quantities

$$C_{axial} = \sum_{\substack{m \epsilon \mathbb{Z} \\ m \neq 0}} (|\hat{\varphi}(m, 0)| + |\hat{\varphi}(0, m)|) \text{ and } C_{main} = \sum_{\substack{m_1 \epsilon \mathbb{Z} \\ m_1 \neq 0}} \sum_{\substack{m_2 \epsilon \mathbb{Z} \\ m_2 \neq 0}} |\hat{\varphi}(m_1, m_2)|.$$

In [1], Güntürk and Nathanson proved that

$$|[\frac{(\alpha_1 - C_0)k^2 - C_1 kM - C_2 M^2}{(\alpha_1 - \alpha_2)}] - [\frac{\alpha_1 k^2 - C_{axial} kM - C_{main} M^2}{(\alpha_1 - \alpha_2)}]| < \varepsilon.$$

Hence we get

$$L \geq \frac{\alpha_1 k^2 - C_{axial} kM - C_{main} M^2}{(\alpha_1 - \alpha_2)}. \quad (4.4)$$

Hence, we can use (4.4) and (4.1) to get the maximum lower bound for $\sum_{m \epsilon 2A} \delta(m)$.

$$\sum_{m \epsilon 2A} \delta(m) \geq \frac{L}{2} \geq \frac{1}{2}\left(\frac{\alpha_1 k^2 - C_{axial} kM - C_{main} M^2}{(\alpha_1 - \alpha_2)}\right). \quad (4.5)$$

Now all we need is to find the maximum of the lower bound for $\sum_{m \epsilon 2A} \delta(m)$ by using (3.5) and (4.5) to get

$$\max\left[M^2, \frac{\alpha_1 k^2 - C_{axial} kM - C_{main} M^2}{(\alpha_1 - \alpha_2)}\right].$$

Snice $M^2 \geq L$, we have the inequality

$$M^2 \geq \frac{\alpha_1 k^2 - C_{axial}kM - C_{main}M^2}{(\alpha_1 - \alpha_2)},$$

then we get

$$-\alpha_1 k^2 + C_{axial}kM + (C_{main} + \alpha_1 - \alpha_2)M^2 \geq 0.$$

This is a polynomial of a second degree. Therefore

$$[{C_{axial}}^2 + 4\alpha_1(C_{main} + \alpha_1 - \alpha_2)]k^2 - C_{axial}kM - 2(C_{main} + \alpha_1 - \alpha_2)M^2 \geq 0,$$

Hence we have

$$M \geq \frac{[{C_{axial}}^2 + 4\alpha_1(C_{main} + \alpha_1 - \alpha_2)]^{\frac{1}{2}} - C_{axial}}{2(C_{main} + \alpha_1 - \alpha_2)}k. \tag{4.6}$$

By inserting (4.6) in (3.3) we get

$$\sigma = \frac{n}{k^2} \leq \frac{1}{2}\left(1 - \left(\frac{[{C_{axial}}^2 + 4\alpha_1(C_{main} + \alpha_1 - \alpha_2)]^{\frac{1}{2}} - C_{axial}}{2(C_{main} + \alpha_1 - \alpha_2)}\right)^2\right). \tag{4.7}$$

In [1], Güntürk and Nathanson used the function

$$\varphi(t_1, t_2) = \begin{cases} 1, & if\ t_1 + t_2 \leq 1, \\ 1 - 40(1 - t_1)(1 - t_2)(1 - t_1 - t_2), & if\ t_1 + t_2 \geq 1. \end{cases}$$

They computed $\alpha_1 = 1$ and $\alpha_2 = -3.7247$. Also, they gave the estimates $2.90278 \leq C_{axial} \leq 2.90289$ and $4.75145 \leq C_{main} \leq 4.76146$. Hence, we have

$$\sigma \leq 0.4789.$$

§5. Using periods greater than one

Yu [9] introduced a special weight function. Let $u(x)$ be nonnegative function on $[0,1]$, with piecewise continuous derivative, such that $\int_0^1 u(t)dt = 1$. Let

$$W(x) \coloneqq \int_0^{1-|x|} u(t)u(t+|x|)dt,$$

where $W(x)$ is an even function on $[-1,1]$. Now, let $\omega_{p,\delta}(x)$ be a periodic function on $[-p/2\,,p/2]$, which is defined by

$$\omega_{p,\delta}(x) = \begin{cases} W(x) & if\ |x| \le \delta, \\ 0 \quad if\ \delta \le |x| \le p/2 \end{cases} \tag{5.1}$$

where p and δ are real numbers, and $0 < \delta < \frac{p}{2}$.

Note that the formal Fourier expansion into the cosine series is

$$\omega_{p,\delta}(x) = \frac{a_{p,\delta}(0)}{2} + \sum_{r=1}^{\infty} a_{p,\delta}(r)\cos(\frac{2r\pi x}{p}).$$

Therefore, we have

$$a_{p,\delta}(r) = \frac{2}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} \omega_{p,\delta}(x)\cos(\frac{2r\pi x}{p})dx.$$

Take $u(t) = 1$ on [0,1]. Then we have

$$\omega_{p,\delta}(x) = \begin{cases} 1 - \frac{|x|}{\delta} & if\ |x| \le \delta, \\ 0 \quad if\ \delta \le |x| \le p/2. \end{cases} \tag{5.2}$$

Hence we have

$$a_{p,\delta}(r) = \frac{p}{(\pi r)^2\delta}[1 - \cos\left(\frac{2r\pi\delta}{p}\right)] \ge 0,$$

for $r \ge 1$ and we have

$$\sum_{r=1}^{\infty} a_{p,\delta}(r) = 1 - \frac{\delta}{p} \quad \text{and} \quad a_{p,\delta}(0) = \frac{2\delta}{p}. \tag{5.3}$$

Let

$$D_{p,\delta}(A) = \sum_{m=-n}^{n} \omega_{p,\delta}\left(\frac{m}{n}\right) d_A(m),$$

and

$$R_{p,\delta,\alpha}(A) = \sum_{m=0}^{2n} \omega_{p,\delta}\left(\frac{m}{n} + \alpha\right) r_A(m).$$

Lemma 5.1. Let $A \subset [0, n] \cap \mathbb{Z}$, and $\omega_{p,\delta}(x)$ is given in (5.1) then we have

$$D_{p,\delta}(A) + R_{p,\delta,\alpha}(A) \geq \frac{2\delta}{p} k^2 \text{ and } D_{p,\delta}(A) \geq R_{p,\delta,\alpha}(A).$$

Proof. Notice that

$$D_{p,\delta}(A) = \sum_{m=-n}^{n} d_A(m)\omega_{p,\delta}\left(\frac{m}{n}\right)$$

$$= \sum_{m=-n}^{n} d_A(m)\left(\frac{a_{p,\delta}(0)}{2} + \sum_{r=1}^{\infty} a_{p,\delta}(r)\cos\left(\frac{2r\pi m}{pn}\right)\right)$$

$$= \frac{\delta}{p}\sum_{m=-n}^{n} d_A(m) + \sum_{r=1}^{\infty} a_{p,\delta}(r)\sum_{m=-n}^{n} d_A(m)\cos\left(\frac{2r\pi m}{pn}\right)$$

$$= \frac{\delta}{p}k^2 + \sum_{r=1}^{\infty} a_{p,\delta}(r)\sum_{m=-n}^{n} d_A(m)\cos(\frac{2r\pi m}{pn}), \tag{5.4}$$

and

$$R_{p,\delta,\alpha}(A) = \sum_{m=0}^{2n} r_A(m)\omega_{p,\delta}\left(\frac{m}{n} + \alpha\right)$$

$$= \sum_{m=1}^{\infty} r_A(m)\left(\frac{a_{p,\delta}(0)}{2} + \sum_{r=1}^{\infty} a_{p,\delta}(r)\cos\left(\frac{2r\pi(m+\alpha n)}{pn}\right)\right)$$

$$= \frac{\delta}{p}\sum_{m=-n}^{n} r_A(m) + \sum_{r=1}^{\infty} a_{p,\delta}(r)\sum_{m=-n}^{n} r_A(m)\cos\left(\frac{2r\pi(m+\alpha n)}{pn}\right)$$

$$= \frac{\delta}{p}k^2 + \sum_{r=1}^{\infty} a_{p,\delta}(r)\sum_{m=-n}^{n} r_A(m)\cos\left(\frac{2r\pi(m+\alpha n)}{pn}\right). \tag{5.5}$$

By (5.4) and (5.5) we can get

$$D_{p,\delta}(A) + R_{p,\delta,\alpha}(A) \geq \frac{2\delta}{p} k^2 \text{ and } D_{p,\delta}(A) \geq R_{p,\delta,\alpha}(A).$$

Lemma 5.2. Suppose A is 2-basis for n, and $0 < \delta < \frac{1}{2}$. If we have $p \geq 1$ then

$$D_{p,\delta}(A) \leq D_{1,\delta}(A) \leq \delta k^2 + (1-\delta)(k^2 - 2n).$$

Proof. for $p \geq 1$ we have

$$\omega_{p,\delta}(x) \leq \omega_{1,\delta}(x) \text{ for } x\epsilon[-1,1].$$

Hence

$$D_{p,\delta}(A) \leq D_{1,\delta}(A).$$

Now we have

$$D_{1,\delta}(A) = \delta k^2 + \sum_{r=1}^{\infty} a_{1,\delta}(r) \sum_{m=-n}^{n} d_A(m) \cos\left(\frac{2r\pi m}{n}\right)$$

$$\leq \delta k^2 + (1-\delta)(k^2 - 2n).$$

Yu [9] approach is:

Let δ and ε be real numbers where $0 < \varepsilon < \delta < \frac{1}{2}$. Then by Lemma 5.1. we have

$$D_{1+2(\delta-\varepsilon),\delta}(A) + R_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(A) \geq \frac{2\delta}{1+2(\delta-\varepsilon)} k^2,$$

and

$$D_{1+3\delta-\varepsilon,\delta}(A) + R_{1+3\delta-\varepsilon,\delta,\delta}(A) \geq \frac{2\delta}{1+3\delta-\varepsilon} k^2.$$

Now combining the two inequalities

$$R_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(A) + R_{1+3\delta-\varepsilon,\delta,\delta}(A) \geq \left(\frac{2\delta}{1+2(\delta-\varepsilon)} + \frac{2\delta}{1+3\delta-\varepsilon}\right) k^2 - 2D_{1,\delta}(A).$$

Hence by Lemma 5.2.

$$R_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(A) + R_{1+3\delta-\varepsilon,\delta,\delta}(A) \geq \left(\frac{2\delta}{1+2(\delta-\varepsilon)} + \frac{2\delta}{1+3\delta-\varepsilon} - 2\right) k^2 + 4(1-\delta)n. \quad (5.6)$$

In Yu [9], we have the estimation

$$R_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(A) + R_{1+3\delta-\varepsilon,\delta,\delta}(A) \leq k^2 - 2n(1 - \frac{\varepsilon^2}{\delta}). \quad (5.7)$$

By comparing (5.6) and (5.7) we get

$$k^2 - 2n + \frac{2\varepsilon^2}{\delta} n \geq \left(\frac{2\delta}{1+2(\delta-\varepsilon)} + \frac{2\delta}{1+3\delta-\varepsilon} - 2\right) k^2 + 4(1-\delta)n,$$

hence

$$\frac{n}{k^2} \leq \frac{3-\frac{2\delta}{1+2(\delta-\varepsilon)}-\frac{2\delta}{1+3\delta-\varepsilon}}{6-4\delta-\frac{2\varepsilon^2}{\delta}}.$$

In [9], Yu used $\delta = 0.2257$ and $\varepsilon = 0.0882$, and he got the upper bound

$$\sigma = \frac{n}{k^2} \leq 0.46972.$$

§6. A new approach of Yu: proof of Theorem 1.1.

This new method was introduced by Yu [10], which involves a non-negative periodic function $\omega(x)$ of period 2. Suppose $\omega(x)$ is even and has cosine series expansion

$$\omega(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} a_r \cos(rx\pi) \ ,$$

where $a_r \geq 0$ for all $r \epsilon \mathbb{N}$. Now consider the sum

$$S = \sum_{|m|\leq n} d_A(m)\, \omega\left(\frac{m-n}{n}\right).$$

Then we have

$$S = \sum_{|m|\leq n} d_A(m) \left(\frac{a_0}{2} + \sum_{r=1}^{\infty} (-1)^r a_r \cos\left(\frac{mr\pi}{n}\right)\right)$$

$$= \frac{a_0}{2} \sum_{|m|\leq n} d_A(m) + \sum_{r \text{ even}} a_r \sum_{|m|\leq n} d_A(m) \cos\left(\frac{mr\pi}{n}\right)$$

$$- \sum_{r \text{ odd}} a_r \sum_{|m|\leq n} d_A(m) \cos\left(\frac{mr\pi}{n}\right)$$

$$= \frac{a_0}{2} k^2 + \sum_{r \text{ even}} a_r \left|f_A\left(\frac{m}{2n}\right)\right|^2 - \sum_{r \text{ odd}} a_r \left|f_A\left(\frac{m}{2n}\right)\right|^2.$$

Since $\omega(x)$ is non-negative function then we have $S \geq 0$, and hence

$$\frac{a_0}{2} k^2 + \sum_{r\ even} a_r \left|f_A\left(\frac{m}{2n}\right)\right|^2 \geq \sum_{r\ odd} a_r \left|f_A\left(\frac{m}{2n}\right)\right|^2.$$

Let

$$S_1 = \frac{a_0}{2} k^2 + \sum_{r\ even} a_r \left|f_A\left(\frac{m}{2n}\right)\right|^2,$$

and

$$S_2 = \sum_{r\ odd} a_r \left|f_A\left(\frac{m}{2n}\right)\right|^2,$$

hence we have

$$S_1 \geq S_2 \ . \qquad (6.1)$$

Now we need an upper bound for S_1 and a lower bound for S_2.

By using Lemma 2.2, we have

$$S_1 = \frac{a_0}{2}k^2 + \sum_{r \text{ even}} a_r \left|f_A\left(\frac{m}{2n}\right)\right|^2,$$

$$\leq \frac{a_0}{2}k^2 + \sum_{r \text{ even}} a_r\,(k^2 - 2n).$$

By using the fact that

$$\frac{a_0}{2} + \sum_{r \text{ even}} a_r = \frac{1}{2}\left[\left(\frac{a_0}{2} + \sum_{r=0}^{\infty} a_r\right) + \left(\frac{a_0}{2} + \sum_{r=0}^{\infty} (-1)^r a_r\right)\right] = \frac{1}{2}\big(\omega(0) + \omega(1)\big)\ ,$$

we have

$$S_1 \leq \frac{a_0}{2}k^2 + \sum_{r \text{ even}} a_r\,(k^2 - 2n)$$

$$\leq \frac{1}{2}\big(\omega(0) + \omega(1)\big)k^2 - (\omega(0) + \omega(1) - a_0)2n. \tag{6.2}$$

On the other hand, let $a_r \geq |b_r|$

$$S_2 = \sum_{r \text{ odd}} a_r \sum_{|m|\leq n} d_A(m) \cos\left(\frac{mr\pi}{n}\right)$$

$$= \sum_{r=0}^{\infty} b_{2r+1} \sum_{|m|\leq n} d_A(m) \cos\left(\frac{m(2r+1)\pi}{n}\right)$$

$$\geq \sum_{r=0}^{\infty} b_{2r+1} \sum_{0\leq m\leq 2n} r_A(m) \sin\left(\frac{m(2r+1)\pi}{n}\right)$$

$$= \sum_{0\leq m\leq 2n} r_A(m) \sum_{r=0}^{\infty} b_{2r+1} \sin\left(\frac{m(2r+1)\pi}{n}\right).$$

Let

$$M = \max_{\frac{m}{n}\epsilon[0,2]} \left\{\sum_{r=0}^{\infty} b_{2r+1} \sin\left(\frac{m(2r+1)\pi}{n}\right)\right\}.$$

Then we have

$$S_2 \geq \sum_{0\leq m\leq 2n} r_A(m) \left(\sum_{r=0}^{\infty} b_{2r+1} \sin\left(\frac{m(2r+1)\pi}{n}\right) + M\right) - M \sum_{0\leq m\leq 2n} r_A(m)$$

$$\geq 2\sum_{m\leq n} \left(\sum_{r=0}^{\infty} b_{2r+1} \sin\left(\frac{m(2r+1)\pi}{n}\right) + M\right) - Mk^2. \tag{6.3}$$

It is easy to see that for a positive integer r, we have

$$\sum_{m\le n} \sin\left(\frac{m(2r+1)\pi}{n}\right) = \int_0^n \sin\left(\frac{(2r+1)\pi x}{n}\right) dx + O(1)$$

$$= n\int_0^1 \sin\big((2r+1)\pi x\big)\, dx + O(2r+1)$$

$$= \frac{2n}{(2r+1)\pi} + O(2r+1).$$

By this we have now

$$\sum_{m\le n}\sum_{r=0}^{\infty} b_{2r+1} \sin\left(\frac{(2r+1)\pi}{n}\right) = \sum_{r=0}^{\infty} b_{2r+1}\left(\frac{2n}{(2r+1)\pi} + O(2r+1)\right)$$

$$= \sum_{r=0}^{n^{\frac{1}{2}}} b_{2r+1}\left(\frac{2n}{(2r+1)\pi} + O(2r+1)\right) + O(\sqrt{n})$$

$$= \frac{Bn}{2} + O(\sqrt{n}), \qquad (6.4)$$

where

$$B = \sum_{r=0}^{\infty} \frac{4b_{2r+1}}{(2r+1)\pi}\ .$$

By using (6.3) and (6.4), we get

$$S_2 \ge (B+2M)n - Mk^2\ . \qquad (6.5)$$

Then using (6.1), (6.2) and (6.5), we get

$$\frac{1}{2}\big(\omega(0)+\omega(1)\big)k^2 - (\omega(0)+\omega(1)-a_0)n \ge (B+2M)n - Mk^2\ ,$$

and thus

$$(B+2M)n + (\omega(0)+\omega(1)-a_0)n \le \frac{1}{2}\big(\omega(0)+\omega(1)\big)k^2 + Mk^2.$$

Hence we obtain

$$\sigma = \frac{n}{k^2} \le \frac{1}{2}\left(\frac{\omega(0)+\omega(1)+2M}{B+2M+\omega(0)+\omega(1)-a_0}\right). \qquad (6.6)$$

Proof of Theorem 1.1. Let

$$u(t) = \frac{9}{4} - 100\left(t - \frac{1}{2}\right)^4 ,$$

on [0,1], and $a_0 = \frac{1}{4}$. Calculating with Maple, we get

$$\omega(0) = \frac{34}{9} \text{ and } \omega(1) = 0.$$

We further take

$$b_{2r+1} = 0 \text{ for } r > 15 \text{ and } b_{2r+1} = a_{2r+1} \text{ for } r \leq 15.$$

Then we get

$$H = \sum_{r=0}^{15} b_{2r+1} \sin\left(\frac{m(2r+1)\pi}{n}\right) = 0.0910416766417302 .$$

and

$$B = \sum_{r=0}^{15} \frac{4b_{2r+1}}{(2r+1)\pi} = 0.6412020796.$$

Using (6.7) we have

$$\sigma \leq 0.4550452314.$$

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