ON BODIES WHOSE SHADOWS ARE RELATED VIA RIGID MOTIONS

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by

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CHAPTER 1

Introduction to the Subjects of Convexity and Geometric Tomography

Convexity and Geometric Tomography are extremely simple and natural notions in mathematics which have intrigued humans for thousands of years. Convexity can be traced back to Archimedes, and Aristotle used geometric tomography. Both subjects are still important areas of mathematics in today's world, in particular, convexity has applications to linear programming and geometric tomography can be used in CAT scans and X-rays.

Convexity is the study of "nice" shapes, convex shapes to be precise. A convex body is considered "nice" because it has no holes nor any loops, it is defined to have the property that if you consider any two points in the body then the line connecting those two points is also contained in the body, see Figure 1.1. These shapes occur in the real world and in nature frequently.

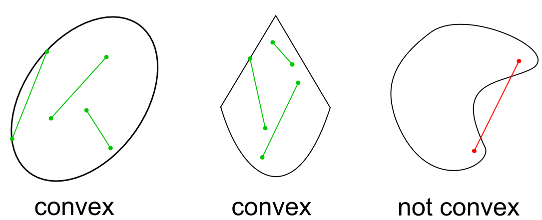

Figure 1.1: Examples of convex and not convex shapes.

Archimedes used convex bodies to show that the inner curve has smaller length than the outer one if the inner figure is convex, see Figure 1.2. On the other hand, if it is not convex then no conclusion can be made.

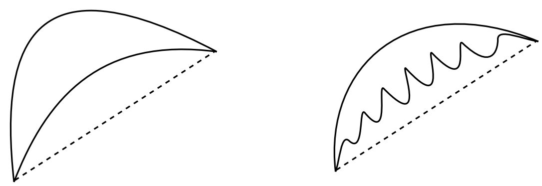

Figure 1.2: On the left, the inner curve is shorter than the outer curve, the opposite is true for the right.

Geometric tomography can be thought of as the detectives of geometry, the investigators are given only some information about the original figure in various subspaces and are asked to make conclusions about the figure in the ambient space. For instance, if you know the shape of every shadow that an object has, can you say for certain what the object is? Mostly geometric tomography deals with knowing information about the shadows or sections of a body, and then concluding things about the original body.

Aristotle used geometric tomography when he reasoned why the Earth had to be spherical. His solution was that the shadow of the Earth onto the moon was circular during a lunar eclipse, see Figure 1.3.

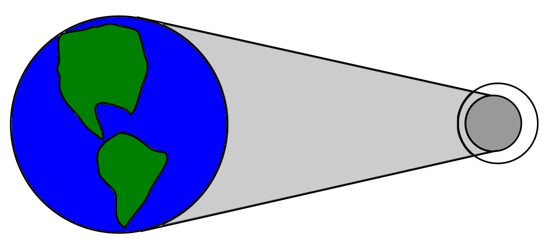

Figure 1.3: Earth's shadow projected onto the moon.

To give a glimpse into the problems I discuss in this dissertation, consider the following problem. Say for instance, that every shadow I make can be rotated and translated to be contained in a corresponding tree's shadow, see Figure 1.4. Does that mean that I can be rotated and translated to be contained in the tree?

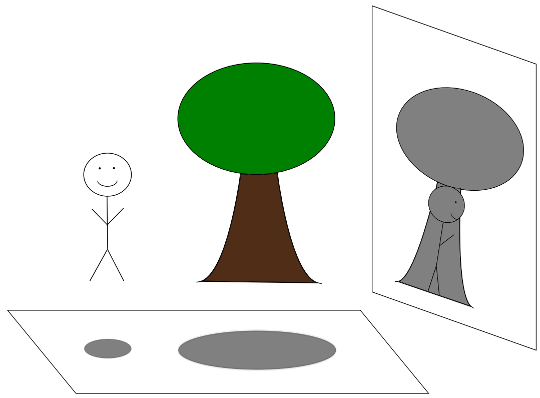

Figure 1.4: The tree's shadow, and my shadow.

Another way this problem could be stated is, if I can rotate and translate my shadow to hide behind the tree in every direction, does that mean I can hide inside the tree?

While convexity and geometric tomography have problems that are simple to formulate and understand, intuition is somewhat deceptive in "obvious problems", and hence the beauty of the subjects shines through.

Now I will state the problems I consider more precisely.

CHAPTER 2

Introduction

My dissertation is split into two major parts.

The first part deals with problems about bodies with directly congruent projections or sections. In particular, in this part I address the following problems (cf. [6, Problem 3.2, page 125 and Problem 7.3, page 289]).

Problem 1. *Suppose that* $2 \leq k \leq n-1$ *and that* K *and* L *are convex bodies in* $\mathbb{R}^n$ *such that the projection* $K|H$ *is directly congruent to* $L|H$ *for all* $H \in \mathcal{G}(n,k)$*. Is* K *a translate of* $\pm L$*?*

Problem 2. *Suppose that* $2 \leq k \leq n-1$ *and that* K *and* L *are star bodies in* $\mathbb{R}^n$ *such that the section* $K \cap H$ *is directly congruent to* $L \cap H$ *for all* $H \in \mathcal{G}(n,k)$*. Is* K *a translate of* $\pm L$*?*

Here I say that $K|H$, the projection of K onto H, is directly congruent to $L|H$ if there exists a special orthogonal transformation (rotation) $\varphi \in SO(k,H)$ in H such that $\varphi(K|H)$ is equal to a translate of $L|H$; $\mathcal{G}(n,k)$ stands for the Grassmann manifold of all k-dimensional subspaces in $\mathbb{R}^n$.

The second part addresses similar problems with the equality given by the directly congruent condition changed to containment.

Problem 3. *Suppose that* $2 \leq k \leq n-1$ *and that* K *and* L *are convex bodies in* $\mathbb{R}^n$ *such that the projection* $K|H$ *can be rotated to be contained in a translate of* $L|H$ *for all* $H \in \mathcal{G}(n,k)$*.*

(a) Can K *be rotated to be contained in a translate of* L*?*

(b) Is $vol_n(K) \leq vol_n(L)$*?*

Problem 4. *Suppose that $2 \le k \le n-1$ and that K and L are star bodies in $\mathbb{R}^n$ such that the section $K \cap H$ can be rotated to be contained in a translate of $L \cap H$ for all $H \in \mathcal{G}(n,k)$.*

(a) Can K be rotated to be contained in a translate of L?

(b) Is $vol_n(K) \le vol_n(L)$?

2.1 Summary of Chapters

My dissertation is organized as follows. For the remaining part of the Introduction, I state known results, results proved in this dissertation and open questions related to Problems 1−4. In Chapter 3 I will provide preliminary information that is used throughout. Then in Chapter 4 I prove results related to Problems 1 and 2. Chapter 5 is devoted to results regarding Problems 3 and 4.

2.2 On Bodies with Directly Congruent Projections or Sections

2.2.1 Known Results Related to Problem 1 and Problem 2

If the corresponding projections are translates of each other, or if the bodies are convex and the corresponding sections are translates of each other, the answers to Problems 1 and 2 are known to be affirmative [6, Theorems 3.1.3 and 7.1.1], (see also [1], [26]). Besides, for Problem 1, with $k = n-1$, Hadwiger established a more general result and showed that it is not necessary to consider projections onto all $(n-1)$-dimensional subspaces; the hypotheses need only be true for one fixed subspace H, together with all subspaces containing a line orthogonal to H. In other words, one requires only a "ground" projection on H and all corresponding "side" projections, see Figure 2.1. Moreover, Hadwiger noted that in $\mathbb{R}^n$, $n \ge 4$, the ground projection might be dispensed with (see [9], and [6, pages 126–127]).

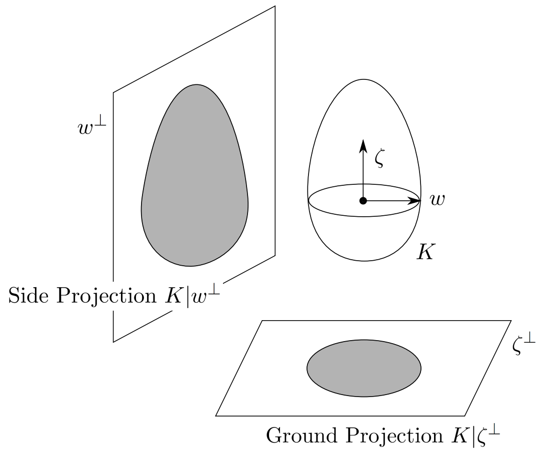

Figure 2.1: Side projection $K|w^\perp$ and ground projection $K|\zeta^\perp$.

If the corresponding projections (sections) of convex (star-shaped) bodies are rotations of each other, the results in the case $k = 2$ were obtained by Ryabogin in [27]; see also [20].

Golubyatnikov [8] obtained several interesting results related to Problem 1 in the cases $k = 2, 3$ [8, Theorem 2.1.1, page 13; Theorem 3.2.1, page 48]. In particular, he gave an affirmative answer to Problem 1 in the case $k = 2$ if the projections of K and L have no direct rigid motion symmetries.

If the bodies are centrally symmetric, then the answers to Problems 1 and 2 are known to be affirmative. In the case of projections they are the consequence of the Aleksandrov Uniqueness Theorem about convex bodies, having equal volumes of projections (see Theorem 12); in the case of sections they follow from the Generalized Funk Theorem (see Theorem 10).

In Chapter 4, I follow the ideas from Golubyatnikov [8] and Ryabogin [27] to obtain several Hadwiger-type results related to both Problems 1 and 2 in the case $k = 3$ and $n \geq 4$.

2.2.2 Heuristics in $\mathbb{R}^3$, the Idea of Golubyatnikov

Let me give a simple heuristic result in $\mathbb{R}^3$. Its understanding will help to understand the main results in $\mathbb{R}^4$. For a complete statement of the result in $\mathbb{R}^3$, see Chapter 4, Theorem 20.

Suppose $K, L \subset \mathbb{R}^3$ are convex bodies. The *diameter* of a body is the maximum distance between any two points in the body. Suppose that K and L have only one diameter. In Figure 2.2 there is a body that has 1 diameter and a body that has infinitely many diameters.

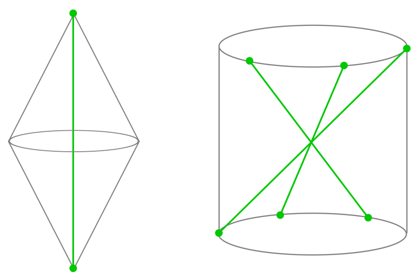

Figure 2.2: The left body has one diameter, the right has infinitely many diameters.

Suppose that the diameter $d_K(\zeta)$ is parallel to $\zeta \in S^2$.

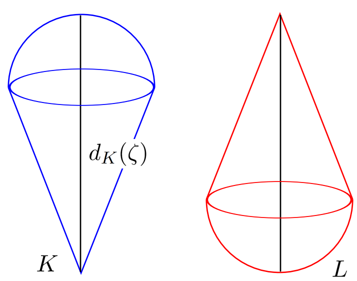

Figure 2.3: Diameter $d_K(\zeta)$ of K.

Consider the "side" projections of K and L, *i.e.*, the projections onto the subspaces $w^\perp$ that contain ζ, and suppose that $K|w^\perp$ and $L|w^\perp$, for every $w^\perp \ni \zeta$, are directly congruent.

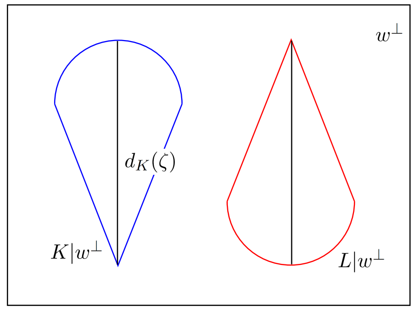

Figure 2.4: Directly congruent projections $K|w^\perp$ and $L|w^\perp$.

In other words, every projection $K|w^\perp$, $w^\perp \ni \zeta$, can be rotated and translated to be equal to $L|w^\perp$. Notice that if these projections are directly congruent then the diameters must be parallel (if the diameters were not parallel, there would exist a side projection such that L's diameter becomes smaller when projected onto it). Hence, the only valid rotations are the identity and the rotation φ_w^π by π. This restriction on the angle of rotation makes the problem almost trivial for bodies having one diameter and no projections with π-rotational symmetries ($\varphi_w^\pi(K|w^\perp) = K|w^\perp + a$ for some $a \in w^\perp$).

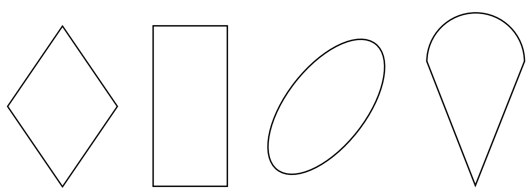

Figure 2.5: The first 3 sets have π-rotational symmetry, the last does not.

Thus, for such bodies, it can easily be seen that

$$K = \pm L + b \text{ for some } b \in \mathbb{R}^3. \tag{2.1}$$

I would like to briefly mention that the same idea could be applied to bodies $K, L \subset \mathbb{R}^3$, that have a countable number of diameters (see Chapter 4 Theorem 20). In fact, even this assumption can be weakened (see Remark 5 in Section 4.6). In addition, the class of bodies that have countably many diameters is large (dense in the class of all convex bodies, even the class of convex bodies with 1 diameter are dense in the class of all convex bodies).

2.2.3 **Results About Bodies with Directly Congruent Projections**

My goal in this section is to state and briefly discuss my results related to Problem 1 and Problem 2 in $\mathbb{R}^n$, for $n \geq 4$ and $k = 3$.

In order to formulate these results I introduce some notation and definitions. Let $n \geq 4$ and let S^{n-1} be the unit sphere in $\mathbb{R}^n$. I will use the notation $w^\perp$ for the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to $w \in S^{n-1}$. I will also denote by $\mathcal{O} = \mathcal{O}_\zeta \in O(n)$ the orthogonal transformation satisfying $\mathcal{O}|_{\zeta^\perp} = -I|_{\zeta^\perp}$, and $\mathcal{O}(\zeta) = \zeta$, see Figure 2.6.

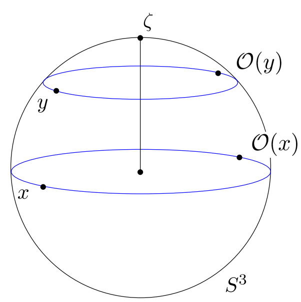

Figure 2.6: Orthogonal transformation $\mathcal{O}$.

Let D be a subset of $H \in \mathcal{G}(n, 3)$, and let $\xi \in (H \cap S^{n-1})$. I say that D has a rigid motion symmetry if $\varphi(D) = D + a$ for some vector $a \in H$ and some non-identical orthogonal transformation $\varphi \in O(3, H)$ in H. In addition, I say that D has a $(\xi, \alpha\pi)$-rotational symmetry if $\varphi(D) = D + a$ for some vector $a \in H$ and some rotation $\varphi \in SO(3, H)$ by the angle $\alpha\pi$, $\alpha \in \mathbb{R} \setminus \{2\mathbb{Z}\}$, satisfying $\varphi(\xi) = \xi$. In the particular case when the angle of rotation is π, I say that D has a (ξ, π)-rotational symmetry.

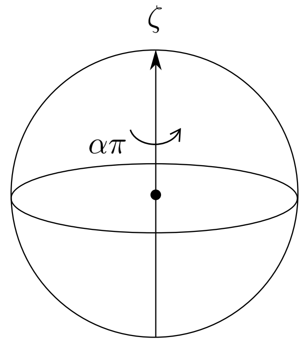

Figure 2.7: Rotation about axis ζ by angle α.

I start with the following 4-dimensional result.

Theorem 1. *Let K and L be two convex bodies in $\mathbb{R}^4$ having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$, such that the "side" projections $K|w^\perp$, $L|w^\perp$ onto all subspaces $w^\perp$ containing ζ are directly congruent. Assume also that these projections have no (ζ, π)-rotational symmetries and no (u, π)-rotational symmetries for any $u \in (\zeta^\perp \cap w^\perp \cap S^3)$. Then $K = L + b$ or $K = \mathcal{O}L + b$ for some $b \in \mathbb{R}^4$.*

If, in addition, the "ground" projections $K|\zeta^\perp$, $L|\zeta^\perp$, are directly congruent and do not have rigid motion symmetries, then $K = L + b$ for some $b \in \mathbb{R}^4$.

I would like to mention the fact that $K = L + b$ or $K = \mathcal{O}L + b$ is the direct analog to the 3-dimensional statement (2.1).

I state a straight n-dimensional generalization of Theorem 1 as a corollary.

Corollary 1. *Let K and L be two convex bodies in $\mathbb{R}^n$, $n \geq 4$, having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$ such that the "side" projections $K|H$, $L|H$ onto all 3-dimensional subspaces H containing ζ are directly congruent. Assume also that these projections have no (ζ, π)-rotational symmetries and no (u, π)-rotational symmetries for any $u \in (\zeta^\perp \cap H \cap S^{n-1})$. Then $K = L + b$ or $K = \mathcal{O}L + b$ for some $b \in \mathbb{R}^n$.*

If, in addition, the "ground" projections $K|G$, $L|G$ onto all 3-dimensional subspaces G

of $\zeta^{\perp}$, are directly congruent and have no rigid motion symmetries, then $K = L + b$ for some $b \in \mathbb{R}^n$.

In particular, I have the following result.

Theorem 2. *If K and L are convex bodies in $\mathbb{R}^n$, $n \geq 4$, having countably many diameters, and directly congruent projections onto* **all** *3-dimensional subspaces, and if the "side" and "ground" projections related to one of the diameters satisfy the conditions of the above corollary, then K and L are translates of each other.*

Theorem 2 was proved by Golubyatnikov [8, Theorem 3.2.1, page 48] under the stronger assumptions that the "side" projections have no direct rigid motion symmetries. Theorem 1 and Corollary 1 under the same stronger assumptions are implicitly contained in his proof. To weaken the symmetry conditions on the "side" projections I replace the topological argument from [8] with an analytic one based on ideas from [27] (compare [8, pages 48–52] with Proposition 2 in Section 4.2).

I note that the assumption about countability of the sets of the diameters of K and L can be weakened. Instead, one can assume, for example, that these sets are subsets of a countable union of the great circles containing ζ (see the remark after Lemma 15 in Section 4.4). I also note that the set of bodies considered in the above statements contains the set of all polytopes whose three dimensional projections do not have rigid motion symmetries. This set of polytopes without symmetries is an everywhere dense set with respect to the Hausdorff metric in the class of all convex bodies in $\mathbb{R}^n$, $n \geq 4$. For the convenience of the reader I prove this at the end of Chapter 4.

2.2.4 Results About Bodies with Directly Congruent Sections

The analytic approach also allows me to obtain results related to Problem 2 (see [6, pages 288-290, open problems 7.1, 7.3, and Note 7.1]).

Theorem 3. *Let K and L be two star-shaped bodies with respect to the origin in $\mathbb{R}^4$, having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$ containing the origin, such that for all subspaces $w^\perp$ containing ζ, the "side" sections $K \cap w^\perp$, $L \cap w^\perp$, are directly congruent. Assume also that these sections have no (ζ, π)-rotational symmetries and no (u, π)-rotational symmetries for any $u \in (\zeta^\perp \cap w^\perp \cap S^3)$. Then $K = L + b$ or $K = \mathcal{O}L + b$ for some $b \in \mathbb{R}^4$ parallel to ζ.*

As in the case of projections, I state a straight n-dimensional generalization of Theorem 3 as a corollary.

Corollary 2. *Let K and L be star-shaped bodies with respect to the origin in $\mathbb{R}^n$, $n \geq 4$, having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$ containing the origin, such that for all 3-dimensional subspaces H containing ζ, the "side" sections $K \cap H$, $L \cap H$ are directly congruent. Assume also that these sections have no (ζ, π)-rotational symmetries and no (u, π)-rotational symmetries for any $u \in (\zeta^\perp \cap H \cap S^3)$. Then $K = L + b$ or $K = \mathcal{O}L + b$ for some $b \in \mathbb{R}^n$ parallel to ζ.*

2.2.5 Questions for Future Research Related to Problems 1 and 2

In general, Problems 1 and 2 are open. I am unaware of any results related to them in the case $k \geq 4$. It is my belief that if $n = 4$, the answer to Problems 1 and 2 is that $K = L + a$ for some $a \in \mathbb{R}^4$.

2.3 On Bodies Related via Containment of Rotated Projections or Sections

2.3.1 Known Results Related to Problem 3 and Problem 4

In [16], D. Klain studied the questions related to Problem 3(a) with translations only. He proved, in particular, that in this case if $k = n - 1$, the answer is negative in general, for any dimension. A counterexample is obtained by considering a ball B, together with the dilated simplex $(1 + \epsilon)T$, where T is the simplex inscribed in B. His idea is that for any $\varepsilon > 0$, the dilated simplex $(1+\varepsilon)T$ is not contained in the ball nor can be translated to fit inside, but if ε is small enough, all the projections of $(1+\varepsilon)T$ on hyperplanes can be translated to fit inside the corresponding projections of the ball, see Figure 2.8. Klain also proved that if both bodies are centrally symmetric, the answer to Problem 3(a) for translations only is affirmative, [17].

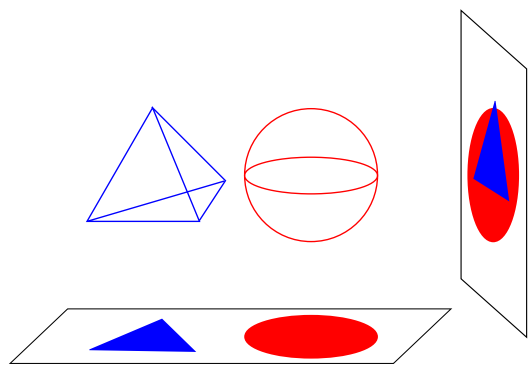

Figure 2.8: $(1+\varepsilon)T$ and B.

In addition, Klain showed that Problem 3(b) for translations has a negative answer for $k = n-1$, in [16]. However, in this case, he found a class of bodies such that if L belongs to that class, Problem 3(b) has an affirmative answer.

Problem 3 is related to the well-known Shephard's Problem (see [33]).

Shephard's Problem: *Let K, L be origin symmetric convex bodies in $\mathbb{R}^n$. If for every $\xi \in S^{n-1}$, $vol_{n-1}(K|\xi^{\perp}) \leq vol_{n-1}(L|\xi^{\perp})$, does it follow that $vol_n(K) \leq vol_n(L)$?*

It was proven independently by Petty [23] and Schneider [31] that the answer to Shephard's Problem is negative in general in dimension $n \geq 3$. In other words, a body K may have greater volume than another body L, even if all projections of K have smaller $(n-1)$-dimensional volume than the corresponding projections of L. In fact, K may be taken to be a ball, while L is a centrally symmetric double cone (see Figure 2.9 and [6, Theorem 4.2.4]). Petty and Schneider also proved that the answer is affirmative under the additional assumption that the body L is a *projection body* (see Definition 2 in Chapter 3).

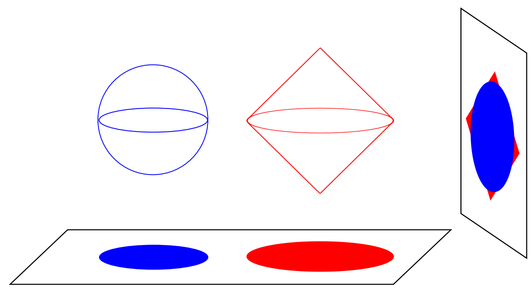

Figure 2.9: K is the ball and L is a double cone.

Observe that if K and L are two origin symmetric convex bodies for which the answer to Problem 3(a) (and trivially for Problem 3(b)) is affirmative, then Shephard's problem for K and L also has an affirmative answer.

Regarding sections, Problem 4 is related to the well-known Busemann-Petty Problem, see [4] and [18].

Busemann-Petty's Problem: *Let K, L be origin symmetric convex bodies. If for every $\xi \in S^{n-1}$, $vol_{n-1}(K \cap \xi^{\perp}) \leq vol_{n-1}(L \cap \xi^{\perp})$, does it follow that $vol_n(K) \leq vol_n(L)$?*

It was proven that the answer to Busemann-Petty's Problem is negative in general in dimension $n \geq 5$ [18]. In other words, a body K may have greater volume than another body L, even if all central sections of K have smaller $(n-1)$-dimensional volume than the corresponding sections of L.

Observe that if K and L are two origin symmetric convex bodies for which the answer to Problem 4(a) (and trivially for Problem 4(b)) is affirmative, then Busemann-Petty's problem for K and L also has an affirmative answer.

2.3.2 Results Related to Problem 3 and Problem 4 for Rotations

In this section I will state and briefly discuss several major results of Chapter 5. I first note that in both Problem 3 and Problem 4 for rotations only, the counterexamples for

Klain's and Shephard's problems, mentioned above, will not work. Indeed, since one of the bodies is a ball, it is invariant under rotations, as well as all of its projections and sections.

I start with counterexamples giving a negative answer to both Problem 3(a) and Problem 4(a) for rotations. The first counterexample is in $\mathbb{R}^3$.

Counterexample 1. *Let $C \subset \mathbb{R}^3$ be the cylinder around the z-axis, centered at the origin, with radius r and height $2r$, where $\frac{1}{2} < r \leq \sqrt{2-\sqrt{3}} = 0.5176\ldots$ Let K be the double cone obtained by rotating the triangle with vertices $(0,0,\pm 1)$ and $(1,0,0)$ around the z-axis. Then the sections (projections) of C can be rotated to be contained in the corresponding section (projection) of K, however the cylinder C itself can never be rotated to be contained in the double cone K.*

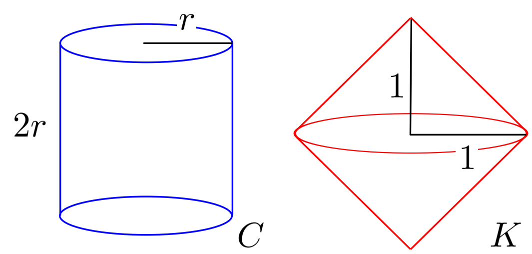

Figure 2.10: Cylinder C and double cone K.

This counterexample is interesting because both C and K are centrally symmetric, and hence, unlike the case of translations proved by Klain, Problem 3(a) for rotations does not have an affirmative answer for centrally symmetric bodies.

The next counterexample works in all dimensions but is less intuitive. In this counterexample I follow the ideas of Kuzminykh [19] and Nazarov.

Counterexample 2. *Given the unit sphere in $\mathbb{R}^n$ where $n \geq 3$, I will perturb it by adding bump functions to create two convex bodies K, L. I place the bumps on K so that they form a $(n-1)$-dimensional simplex on the surface of K, but no such simplex configuration of bumps will appear on the surface on L, see Figure 2.11. Here, every $(n-1)$-dimensional* *section of K (projection of L^*) can be rotated to be contained in the corresponding section of L (projection of K^*) , however K itself can never be rotated to be contained in L (L^* can never be rotated to be contained in K^*).*

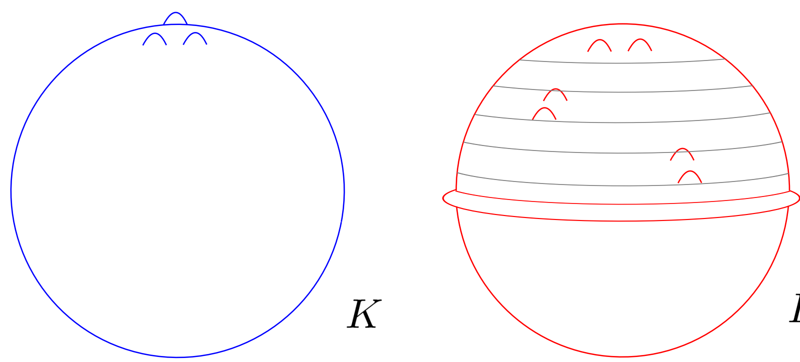

Figure 2.11: Bumps on the sphere.

For a complete description of the body L, see Subsection 5.1.2.

In both Counterexamples 1 and 2, $vol_n(C) \leq vol_n(K)$ and $vol_n(K) \leq vol_n(L)$ (respectively) and hence they do not provide a counterexample to Problem 3(b) or Problem 4(b). The following result shows that Problem 4(b) has an affirmative answer in the case of rotations.

Theorem 4. *Let K and L be two star bodies in $\mathbb{R}^n$, $n \geq 2$, such that for every $\xi \in S^{n-1}$, there exists a rotation $\varphi_\xi \in SO(n-1, \xi^\perp)$ such that*

$$\varphi_\xi(K \cap \xi^\perp) \subseteq L \cap \xi^\perp.$$

Then,

$$vol_n(K) \leq vol_n(L). \tag{2.2}$$

The proof is quite easy, due to the fact that it is possible to simply express the volume of the body in terms of the radial function (see (3.8) and Section 5.2, Theorem 4 for the proof). Unfortunately, no such simple formula exists for the volume of the body in terms of the support function (the formula that does exist requires taking the derivatives of the support function, which makes it harder to use).

I have the following partial result (compare with heuristics in Section 2.2.2).

Theorem 5. *Let K, L be convex bodies in $\mathbb{R}^3$ with countably many diameters, and the diameters of K and L are of equal length. Assume that there exists a diameter $d_K(\xi_0)$, such that for every $w \in \xi_0^\perp$, there exists $\varphi_w \in SO(2, w^\perp)$ and $a_w \in w^\perp$ such that $\varphi_w(K|w^\perp) \subseteq L|w^\perp + a_w$. If either K or L is centrally symmetric then $K \subseteq L + a$ for some $a \in \mathbb{R}^3$.*

In Section 5.2, I will also state results about other classes of convex bodies for which the answer to Problem 3(b) is affirmative in $\mathbb{R}^3$. For example, bodies whose averages of their support function are equal (for rotations only, Lemma 23), and bodies of equal constant width (Lemma 24).

2.3.3 Questions for Future Research Related to Problems 3 and 4

Problem 4 is open in the case of translations. Another open problem is Problem 3(b) for rotations. Besides questions about translations and rotations, the next idea would be to consider similar questions for reflections, *i.e.*, replacing the special orthogonal group $SO(k, H)$ with the orthogonal group $O(k, H)$ (cf. [6, Problem 3.2, page 125 and Problem 7.3, page 289]).

CHAPTER 3

Preliminaries

At the beginning of this chapter I discuss standard notation that will be used throughout the remaining chapters. After that, I state preliminary information that is used implicitly and explicitly throughout my dissertation. In particular, material from the areas of harmonic analysis, differential geometry, convex geometry, and topology. The final section is additional material that will also be needed.

3.1 Notation

I will use the following standard notation. The unit sphere in $\mathbb{R}^n$ $(n \geq 2)$, sometimes referred as the $(n-1)$-dimensional sphere, is S^{n-1}. Given $w \in S^{n-1}$, the hyperplane orthogonal to w and passing through the origin will be denoted by $w^{\perp} = \{x \in \mathbb{R}^n : x \cdot w = 0\}$, where $x \cdot w = x_1 w_1 + \cdots + x_n w_n$ is the usual inner product in $\mathbb{R}^n$. The notation of the orthogonal group $O(n)$ and the special orthogonal group, also known as the group of rotations, $SO(n)$ in $\mathbb{R}^n$ is standard. If $\mathcal{U} \in O(n)$ is an orthogonal matrix, I will write $\mathcal{U}^t$ for its transpose.

Refer to Figure 3.1 for the next two definitions. Given $\xi \in S^{n-1}$, the great $(n-2)$-dimensional sub-sphere of S^{n-1} that is perpendicular to ξ will be denoted by $S^{n-2}(\xi) = \{\theta \in S^{n-1} : \theta \cdot \xi = 0\}$. Similarly, for $\zeta \in S^3$ and $t \in [-1, 1]$, the parallel to $S^2(\zeta)$ at height t will be denoted by $S^2_t(\zeta) = S^3 \cap \{x \in \mathbb{R}^4 : x \cdot \zeta = t\}$. Observe that when $t = 0$, $S^2_0(\zeta) = S^2(\zeta)$.

The Grassmann manifold of all k-dimensional subspaces in $\mathbb{R}^n$ will be denoted by $\mathcal{G}(n, k)$. Let E be a two or three-dimensional subspace of $\mathbb{R}^n$. I will write $\varphi_E \in SO(2, E)$, or

$\varphi_E \in SO(3, E)$, meaning that there exists a choice of an orthonormal basis in $\mathbb{R}^n$ and a rotation $\Phi \in SO(n)$, with a matrix written in this basis, such that the action of Φ on E is the rotation φ_E in E, and the action of Φ on $E^\perp$ is trivial, *i.e.*, $\Phi(y) = y$ for every $y \in E^\perp$ (here $E^\perp$ stands for the orthogonal complement of E). Similar notation will be used for $\varphi_E \in O(3, E)$. I will also denote by $O(3, S^2(w))$, $SO(3, S^2(w))$, the orthogonal transformations in the 3-dimensional subspace spanned by the great subsphere $S^2(w)$ of S^3. The restriction of a transformation $\varphi \in O(n)$ onto the subspace of smallest dimension containing $F \subset S^{n-1}$ will be denoted by $\varphi|_F$. I stands for the identity transformation.

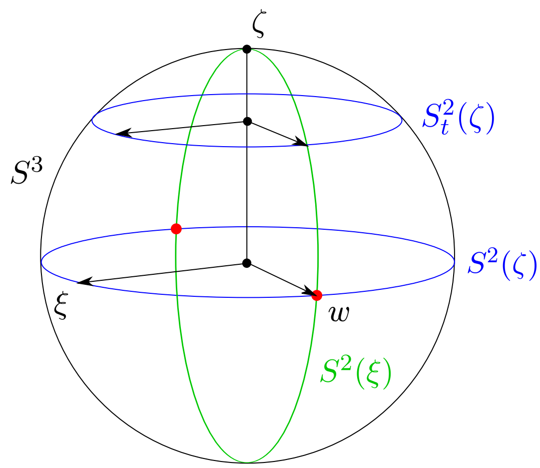

Figure 3.1: The great spheres $S^2(\zeta)$ and $S^2(\xi)$

I will also denote by $\mathcal{O} = \mathcal{O}_\zeta \in O(n)$ the orthogonal transformation satisfying $\mathcal{O}|_{\zeta^\perp} = -I|_{\zeta^\perp}$, and $\mathcal{O}(\zeta) = \zeta$, see Figure 3.2.

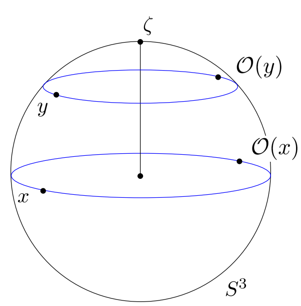

Figure 3.2: Orthogonal transformation $\mathcal{O}$.

The boundary of K will be denoted by ∂K, the interior of K will be denoted by $int(K)$, and the closure of K is denoted by $\bar{K}$.

The even part of a function f is denoted by $f_e(x) = \frac{f(x)+f(-x)}{2}$. The odd part of a function f is denoted by $f_o(x) = \frac{f(x)-f(-x)}{2}$.

3.2 Harmonic Analysis

There is a famous theorem proved by P. Funk in 1915 that I will need to use (cf. Theorem C.2.4 from [6, page 430]),

Theorem 6. *If f is a continuous even function on S^{n-1} such that for all $u \in S^{n-1}$,*

$$\int_{u^\perp \cap S^{n-1}} f(v)\, dv = 0$$

then $f = 0$.

I will also use the *Funk transform*, [10, Chapter III, §1],

$$Rf(w) = R_\zeta f(w) = \int\limits_{S^2(w)\cap S^2(\zeta)} f(\theta) d\theta, \qquad w \in S^2(\zeta). \tag{3.1}$$

Here $d\theta$ stands for the Lebesgue measure on the one-dimensional great circle $E = S^2(w) \cap S^2(\zeta)$ of $S^2(\zeta)$.

3.3 Differential Geometry

For the next definitions I refer to [6, page 25]. Let K be smooth and $x \in \partial K$. Suppose that u is the outer unit normal vector to K at x. The *Gauss map* g from ∂K to S^{n-1} is defined by $g(x) = u$. The *tangent space* of K at x is the translate $H_u - x = u^\perp$ of the supporting hyperplane to K with outer normal vector u. The differential $W_x = dg_x$ of the Gauss map is a linear map from this tangent space to itself. The eigenvalues of W_x are called the *principal curvatures* of K at x. In 2-dimensions, curvature can be described as the reciprocal of the radii of the osculating circle. In 3-dimensions the principal curvatures are the maximum and minimum values of the curvature at the point x. In other words, they measure how the surface bends by different amounts in different directions at that point, see [14]. In higher dimensions, say $K \subset \mathbb{R}^{n+1}$, if the principal curvatures at p are ordered as follows, $k_1(p) \leq k_2(p) \leq \cdots k_n(p)$, in the directions $e_1, \ldots, e_n$ (called *principal directions*) respectively, then $k_n(p)$ is the maximum value of curvature. Next, k_{n-1} is the maximum value of curvature at point p for all vectors that are perpendicular to e_n, and so on, see [35, page 86]. The product of the principal curvatures is called the *Gaussian curvature* of K at x. If this value is positive, then K is said to have *positive Gaussian curvature.*

Define $C^2_+(\mathbb{R}^n)$ to be the set of convex bodies in $\mathbb{R}^n$ having a positive Gaussian curvature.

The following theorem is a result of Schneider, [32]. For the definition of a polytope and Hausdorff distance $\delta(K, P)$ see Section 3.4.

Theorem 7. *Let $K \in C^2_+(\mathbb{R}^n)$, $n \geq 3$. Then, for $v \to \infty$,*

$$\delta(K, P_v^*) \approx c_n \, v^{-\frac{2}{n-1}} \Big(\int\limits_{\partial K} \sqrt{G_K(\sigma)} d\sigma \Big)^{\frac{2}{n-1}},$$

where P_v^* is a polytope with vertices on the boundary ∂K, not unique in general, for which $\delta(K, P_v^*)$ equals the infimum of the Hausdorff distance $\delta(K, P)$ over all convex polytopes P contained in K that have at most v vertices, c_n is a constant depending on the dimension, and $G_K(\sigma)$ is the Gaussian curvature of K at $\sigma \in \partial K$.

I see, in particular, that the amount of vertices of P_v^* gets larger, provided the Hausdorff distance between K and P_v^* gets smaller.

3.4 Convex Geometry

I refer to [6, Chapter 1] for the next definitions involving convex and star-shaped bodies. A *body* in $\mathbb{R}^n$ is a compact set which is equal to the closure of its non-empty interior. A *convex body* is a body K such that for every pair of points in K, the segment joining them is contained in K. A *convex polytope* is the convex hull of finitely many points, where the *convex hull* of a set X is the smallest convex set that contains all of X.

A *Euclidean ball* (sometimes referred to as a ball) in $\mathbb{R}^n$ is defined to be $B = B_r = \{x = (x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 = r^2\}$. An *ellipsoid* in $\mathbb{R}^n$ is defined to be $\{(x_1, \ldots, x_n) : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_n^2}{a_n^2} = 1\}$, where $a_i \in \mathbb{R}$ and are called the *semi-principal axes.* A *standard simplex* T in $\mathbb{R}^n$ is the convex hull of the points $e_1, \ldots, e_{n+1}$ in $\mathbb{R}^{n+1}$ where e_i is the vector that has zeros in every coordinate except for the ith which is equal to 1.

Let K be a subset of $\mathbb{R}^n$, and $w \in S^{n-1}$, then the *projection of* K *onto* $w^\perp$ is

$$K|w^\perp = \{x \in w^\perp : x + \lambda w \in K \text{ for some } \lambda \in \mathbb{R}\},$$

see Figure 3.3 and [36, page 307].

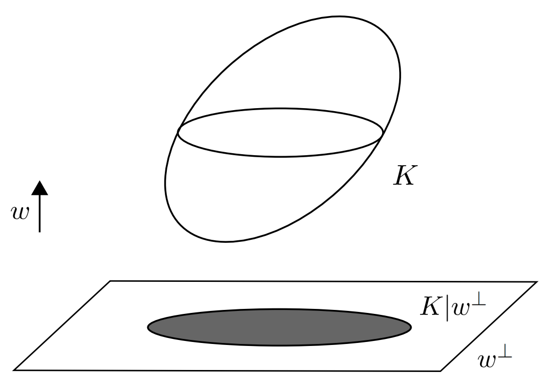

Figure 3.3: Projection of K onto $w^\perp$.

Projections have the following property, $(K - a)|w^\perp = K|w^\perp - a|w^\perp$ where $a|w^\perp$ is the projection of a onto $w^\perp$. A projection of a polytope is a polytope. It can be proven that all the vertices of a projected polytope are the projections of vertices from the original polytope.

The *central section* of K with respect to the direction $w \in S^{n-1}$ is $K \cap w^\perp$, *i.e.*, the slice of K when it is cut through the origin perpendicular to w. I will refer to central sections simply as sections. For sections, the following property holds, $(K - a) \cap w^\perp = K \cap w^\perp - a$ where $a \in w^\perp$.

I refer to [6, Chapter 1] for the next definitions. For $x \in \mathbb{R}^n$, the *support function* of a convex body K is defined as $h_K(x) = \max\{x \cdot y : \ y \in K\}$ (see page 16 in [6]). The support function is continuous, uniquely determines the body, and $h_{K_1} \leq h_{K_2}$ if and only if $K_1 \subseteq K_2$. I will repeatedly use the following well-known properties of the support function. For every convex body $\tilde{K}$ and χ_w is a rotation in $w^{\perp}$,

$$h_{\tilde{K}|w^{\perp}}(x) = h_{\tilde{K}}(x) \ \text{ and } \ h_{\chi_w(\tilde{K}|w^{\perp})}(x) = h_{\tilde{K}|w^{\perp}}(\chi_w^t(x)), \quad \forall x \in w^{\perp}, \tag{3.2}$$

(see, for example, [6, (0.21), (0.26), pages 17–18]). The average of the support function of body K in $\mathbb{R}^n$ is simply $\int_{S^{n-1}} h_K(\theta)\, d\theta$.

Definition 1. *Let $\delta(K, P)$ be the Hausdorff distance, or Hausdorff metric, between the convex bodies K and P in $\mathbb{R}^n$, $n \geq 2$,*

$$\delta(K, P) = \max_{\theta \in S^{n-1}} |h_K(\theta) - h_P(\theta)|.$$

A well-known property of the Hausdorff metric is any convex body can be approximated in the Hausdorff metric by convex bodies with positive Gaussian curvature.

The *width function* $w_K(x)$ of K in the direction $x \in S^{n-1}$ is defined as $\omega_K(x) = h_K(x) + h_K(-x)$. The segment $[z, y] \subset K$, parallel to $\zeta \in S^{n-1}$, is called the *diameter* of the body K if $|z - y| = \max_{\{\theta \in S^{n-1}\}} \omega_K(\theta)$. I will denote it by $d_K(\zeta)$. The length of the diameter will be denoted as follows, $\text{diam}(K) = \max_{\{\theta \in S^{n-1}\}} \omega_K(\theta)$. I say that a convex body $K \subset \mathbb{R}^n$ has *countably many diameters* if the width function ω_K reaches its maximum on a countable subset of S^{n-1}. In addition, a body has *constant width* if its width function is constant.

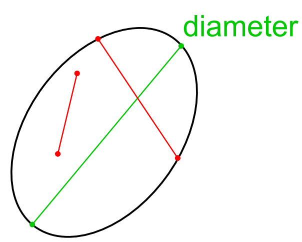

Figure 3.4: The diameter.

A set $S \subset \mathbb{R}^n$ is said to be *star-shaped at a point* p if the line segment from p to any point in S is contained in S. Let $x \in \mathbb{R}^n \setminus \{0\}$, and let $K \subset \mathbb{R}^n$ be a star-shaped set with respect to the origin. The *radial function* ρ_K in the direction $x \in S^{n-1}$ is defined as $\rho_K(x) = \max\{c : cx \in K\}$. Here the line through x and the origin is assumed to meet K, ([6, page 18]). The radial function uniquely determines a star body, and $\rho_{K_1} \leq \rho_{K_2}$ if and only if $K_1 \subseteq K_2$. I will use the well-known properties of the radial function (see, for example, [6, (0.33), page 20])

$$\rho_{\tilde{K}\cap w^\perp}(\theta) = \rho_{\tilde{K}}(\theta), \quad \rho_{\chi_w(\tilde{K}\cap w^\perp)}(\theta) = \rho_{\tilde{K}\cap w^\perp}(\chi_w^{-1}(\theta)), \qquad \forall \theta \in w^\perp \cap S^3. \tag{3.3}$$

The radial function of the unit sphere is the constant function 1. Define $\varphi_{\xi,\delta}$ to be a *smooth bump function* defined on S^{n-1}, supported in a small disk on the surface of S^{n-1} with center at $\xi \in S^{n-1}$ and with radius δ. The function $\varphi_{\xi,\delta}$ is invariant under rotations that fix the direction ξ, and its maximum height at the point ξ is 1. For small enough ε, the body K whose radial function is $1 + \varepsilon\, \varphi_{\xi,\delta}(u)$ is convex, since its curvature will be positive.

The segment $[z, y] \subset K$, parallel to $\zeta \in S^{n-1}$, is called the *diameter* of the star-shaped body K if $|z - y| = \max\limits_{\{[a,b]\subset K\}} |a - b|$. As in the case of a convex body, I will denote this diameter by $d_K(\zeta)$.

For a subset E of $\mathbb{R}^n$, the polar set of E is defined as $E^* = \{x : x \cdot y \leq 1 \text{ for every } y \in E\}$ (see [6, pages 20-22]). When K is a convex body containing the origin, the same is true of K^* (which is called the polar body of K), and the following relation between the support function of K and the radial function of K^* exists: For every $u \in S^{n-1}$,

$$\rho_{K^*}(u) = 1/h_K(u). \tag{3.4}$$

For any linear transformation $\phi \in GL(n)$, I have

$$h_{\phi K}(u) = h_K(\phi^t u). \tag{3.5}$$

A similar relation

$$\rho_{\phi K}(u) = \rho_K(\phi^{-1}u) \tag{3.6}$$

holds for the radial function. Combining (3.5), (3.6) and (3.4), it follows that $h_{(\phi K)^*}(u) = h_{\phi^{-t}K^*}(u)$ (see [6, page 21]); this gives the identity $(\phi K)^* = \phi^{-t}K^*$ for the polar of a linear transformation of the body K.

Using the properties of the support function and radial function, and (3.4), I note the polarity relation between sections and projections,

$$\rho_{K^*\cap w^\perp}(u) = \rho_{(K|w^\perp)^*}(u) \quad \text{for all} u \in w^\perp. \tag{3.7}$$

Now consider Theorem 3.1.1 from [6, page 99].

Theorem 8. *Let $1 \le k \le n-1$, and let K be a compact convex set in $\mathbb{R}^n$. Then K is determined by all its projections $K|H$, where $H \in \mathcal{G}(n,k)$. In fact, K is determined by its projections on all 2-dimensional subspaces containing a given line through the origin.*

Two bodies K and L are *homothetic* if K is a dilation and translation of L, see [6, page 5]. A similar theorem to Theorem 8 is also true for sections, see Theorem 7.1.1 from [6, page 270].

Theorem 9. *Suppose that K, L are compact convex sets in $\mathbb{R}^n$, containing the origin in their relative interiors. Let $2 \le k \le n-1$. If $K \cap S$ is homothetic to (or a translate of) $L \cap S$ for each $S \in \mathcal{G}(n,k)$, then K is homothetic to L (or a translate of L, respectively).*

The Lebesgue measure on $\mathbb{R}^n$ is also called the n-dimensional volume. From this fact, it is easy to see that the Lebesgue measure is invariant under translations, rotations and reflections. The n-dimensional volume of a body K is denoted as $vol_n(K)$. Some formulas that will help to compute it are the following.

The first formula relates volume to the radial function (see [18, page 16])

$$vol_n(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(\theta)d\theta, \tag{3.8}$$

Let K be a star body, then the volume of sections can be computed by using polar coordinates and the Funk transform with $f = \frac{1}{n-1}\rho_K^{n-1}$, namely,

$$vol_{n-1}(K \cap \zeta^{\perp}) = \frac{1}{n-1}\int_{S^{n-2}(\zeta)} \rho_K^{n-1}(\theta)\, d\theta.$$

Definition 2. *A projection body P is a convex body such that there exists a convex body $K \subset \mathbb{R}^n$ for $n \geq 2$ such that $h_P(u) = vol_{n-1}(K|u^{\perp})$ for all $u \in S^{n-1}$, see [6, Section 4.1].*

The Generalized Funk Theorem is as follows, see [6, Theorem 7.2.6, page 281].

Theorem 10 (Generalized Funk Theorem)**.** *Let K, L be origin symmetric star bodies in $\mathbb{R}^n$. If $vol_{n-1}(K \cap w^{\perp}) = vol_{n-1}(L \cap w^{\perp})$ for all $w \in S^{n-1}$, then $K = L$.*

Recall the definition of the *Gauss map* g from Section 3.3. For the next definitions I refer to [6, page 395]. Define $g^{-1}(K, S^{n-1})$ to be the set of points in ∂K at which there is an outer unit normal vector. Now define the *surface area measure* of a convex body K to be

$$S(K) = vol_{n-1}(g^{-1}(K, S^{n-1})).$$

Thus, the surface area measure and the Gauss map are related. It can be seen that $S(K)$ is the $(n-1)$-dimensional volume of the *surface area* of K, and can also be defined by

$$S(K) = vol_{n-1}(\partial K).$$

Additionally, the surface area of K can be expressed in terms of K's projections, see [6, page 408].

Theorem 11 (Cauchy's surface area formula)**.** *Let K be a convex body in $\mathbb{R}^n$, then*

$$S(K) = \frac{1}{vol_{n-1}(B)}\int_{S^{n-1}} vol_{n-1}(K|u^{\perp})\, du,$$

where B is the unit Euclidean ball in $\mathbb{R}^{n-1}$.

The following is a formula that relates volume, surface area, and width for bodies K of constant width w in $\mathbb{R}^3$, namely,

$$2vol_3(K) = wS(K) - \frac{2\pi}{3}w^3, \tag{3.9}$$

see [5, page 66].

Next I state Aleksandrov's uniqueness theorem, see [6, Theorem 3.3.1, page 111].

Theorem 12 (Aleksandrov's uniqueness theorem)**.** *Let K, L be compact convex bodies in $\mathbb{R}^n$. If the surface area measure of K is equal to the surface area measure of L, i.e., $S(K) = S(L)$ then K is a translate of L.*

3.5 Topology

I use many properties of open and closed sets. For instance, the complement of a closed set is open and the union and intersection of a finite number of closed sets is closed. Another property I use is that if the topological space A is connected then the only sets that are both open and closed are the empty set and A.

A set A is called *nowhere dense* in a topological space Y, if the closure of A has an empty interior, see [25, page 42].

Theorem 13 (Baire Category Theorem)**.** *No complete metric space can be written as a countable union of nowhere dense sets, cf. [25, page 43].*

To say this another way, if A is a complete metric space and can be written as the countable union of sets A_i, then there exists an A_j such that $int(\bar{A}_j) \neq \emptyset$.

A set $A \subset B$ is *everywhere dense* in B if every ball in B contains an element from A. In other words, the closure of A is equal to B. In particular, if $B \backslash A$ has measure zero, then it follows that A is everywhere dense in B. Additionally, if $B \backslash A$ has dimension smaller than B then $B \backslash A$ has measure zero with respect to B. If A is everywhere dense in B and if there exists C, closed, where $A \subset C \subset B$, then $C = B$.

A *manifold* is a topological space that resembles the Euclidean space near each point. A *differentiable manifold* is a manifold that locally is close to being a linear space in the sense that one can apply calculus on it. A *compact manifold* is a manifold that is compact as a topological space, see [15]. It is known that the n-dimensional sphere is a compact differentiable manifold, [15].

A *line field* associated to a space A is a function that to each point in A assigns a line. A *tangent line field* is a line field associated to a space A, where each line is tangent to A at the assigned point. The *Euler characteristic* is a number that describes a topological space's shape or structure regardless of the way it is bent, see [11]. The Euler characteristic is 2 for any 3-dimensional convex polyhedron, *i.e.*, $V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces. It is also known that the Euler characteristic is 2 for a 2-dimensional sphere.

In Chapter 4, I will need the following result of Hopf, [21], [28].

Theorem 14. *If a compact differentiable manifold M admits a continuous tangent line field, then the Euler characteristic of M is zero.*

I will now discuss a definition taken from [27] in a way that is more convenient for me (refer to Figure 3.5 for the next definition).

Definition 3. *Let $\alpha \in (0,1)$ and let $\mathbf{S}_1$, $\mathbf{S}_2$ be any two spherical circles in the standard metric of $S^2(\zeta)$, both of radius $\alpha\pi$. The union $\mathfrak{l} \cup \mathfrak{m}$ of two open arcs $\mathfrak{l} \subset \mathbf{S}_1$ and $\mathfrak{m} \subset \mathbf{S}_2$ will be called a spherical X-figure if the angle between arcs is in $(0, \frac{\pi}{4})$, the length of the arcs is less than $\alpha\pi$, and the arcs intersect at their centers only, $\mathfrak{l} \cap \mathfrak{m} = \{x\}$. The point $x \in S^2(\zeta)$ will be called the center of the X-figure.*

For this definition, the requirements for the angle between the arcs is so that the X-figure is a "skinny" X-figure, and the requirements for the length of the arcs is so that the X-figure is not too large.

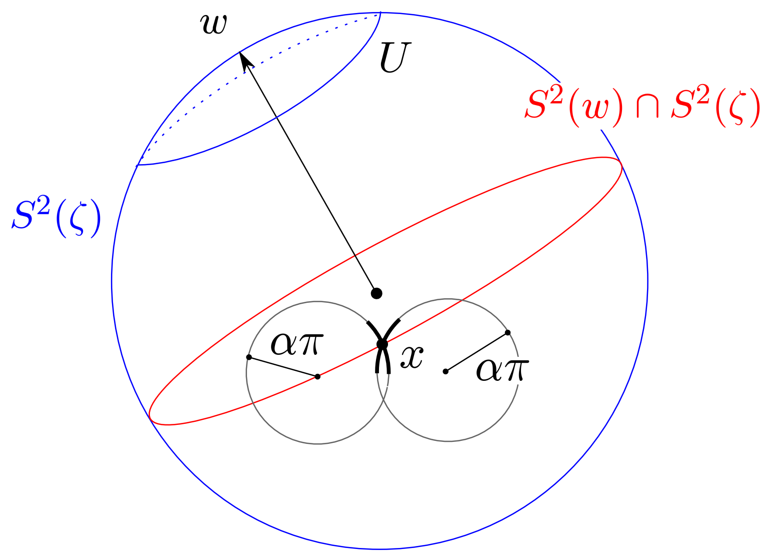

Figure 3.5: The spherical X-figures from Definition 3.

Later I will show that a set that consists of points from these spherical X-figures that is contained in the red one-dimensional circle $S^2(w) \cap S^2(\zeta)$ is open and closed. Hence from the property mentioned above I can show that the set is either the empty set or the whole space.

3.6 Additional Definitions and Results

Define the *even (and odd) part of a function with respect to orthogonal transformation* $\mathcal{O}$ to be

$$f_{\mathcal{O},e}(\theta) = \frac{f(\theta) + f(\mathcal{O}\theta)}{2}, \qquad f_{\mathcal{O},o}(\theta) = \frac{f(\theta) - f(\mathcal{O}\theta)}{2},$$

respectively, where $\mathcal{O}$ fixes direction ζ and $\mathcal{O}|_{\zeta^\perp} = -I|_{\zeta^\perp}$.

Proposition 1. *Let φ be a rotation of angle θ around the origin in $\mathbb{R}^2$ where θ is irrational. Then the closure of the set $\{x \in S^1 : \varphi^n(x_0) = x, n \in \mathbb{Z}, \exists x_0 \in S^1\}$ is equal to S^1.*

Suppose $\varphi \in SO(n)$ then φ can be represented as a n by n matrix where the $\det(\varphi) = 1$. From this it can be seen that every entry a_{ij} is bounded, namely $|a_{ij}| \leq 1$.

The following theorem is well known (cf. [24] and [34, pages 17-18]).

Theorem 15. *The composition of two rotations by π about axes that are separated by an angle θ, is a rotation by 2θ about an axis perpendicular to the axes of the given rotations.*

I will need the results of Radin and Sadun [24].

Theorem 16. *Let $\mathcal{A}$ and $\mathcal{B}$ be rotations of finite order of Euclidean 3-space, about axes that are themselves separated by an angle which is a rational multiple of π. Then, the 2-generator subgroup of $SO(3)$, generated by $\mathcal{A}$ and $\mathcal{B}$, is infinite and dense, except in the following cases: if one generator has order 1, the group is cyclic; if one generator has order 2 and the axes are orthogonal, the group is dihedral; and if both generators have order 4 and the axes are orthogonal, the group is the symmetries of the cube.*

Later I will use the fact that

$$\dim(SO(n)) = \dim(O(n)) = \frac{n(n-1)}{2}. \tag{3.10}$$

Definition 4. *A body K in $\mathbb{R}^n$ is directly congruent to L if there exists a special orthogonal transformation (rotation) $\varphi \in SO(n)$ and $a \in \mathbb{R}^n$ such that $\varphi(K) = L + a$.*

I say that $K|H$, the projection of K onto H, is directly congruent to $L|H$ if there exists a special orthogonal transformation (rotation) $\varphi \in SO(k, H)$ in H such that $\varphi(K|H)$ is equal to a translate of $L|H$. Similarly for directly congruent sections.

A body $K \subset \mathbb{R}^n$ is said to be *origin symmetric* if $x \in K$ then $-x \in K$. Similarly, K is said to be *centrally symmetric* if there exists $p \in \mathbb{R}^n$ such that K is invariant under point reflection about p. In other words, if $K + a$ is such that p is translated to the origin, where $a \in \mathbb{R}^n$, then $K + a$ is origin symmetric.

A *body of revolution* is a body in $\mathbb{R}^n$ that is obtained by rotating a curve in $H \in \mathcal{G}(n, n-1)$ around an axis that lies in H.

Next, I define the notion of *rigid motion symmetry.*

Definition 5. *Let D be a subset of $H \in \mathcal{G}(n,3)$. I say that D has a rigid motion symmetry if $\varphi(D) = D + a$ for some vector $a \in H$ and some non-identical orthogonal transformation $\varphi \in O(3,H)$ in H. Similarly, D has a direct rigid motion symmetry if $\varphi(D) = D + a$ for some vector $a \in H$ and some non-trivial rotation $\varphi \in SO(3,H)$.*

In some cases I specifically say that D has a ξ*-rotational symmetry* if $\varphi(D) = D + a$ for some vector $a \in H$ and some rotation $\varphi \in SO(3,H)$ by the angle $\alpha\pi$, $\alpha \in \mathbb{R} \setminus \{2\mathbb{Z}\}$, satisfying $\varphi(\xi) = \xi$. In the particular case when the angle of rotation is π, I say that D has a (ξ,π)*-rotational symmetry*.

In $\mathbb{R}^2$ I say that a body D has π*-rotational* symmetry if $\varphi(D) = D + a$ for some vector $a \in \mathbb{R}^2$ and some rotation $\varphi \in SO(2)$ by angle π. In this case, π-rotational symmetry is equivalent to being centrally symmetric. Additionally, if D has a π-rotational symmetry and $a = 0$ then D is origin symmetric.

Similarly I define what it means for a function to have rotational symmetry. Let f be a continuous function on S^3 and let $\xi \in S^3$. I say that the restriction of f onto $S^2(\xi)$ (or just f) has a $(\zeta,\alpha\pi)$-rotational symmetry if for some rotation $\varphi_\zeta^{\alpha\pi} \in SO(3, S^2(\xi))$ by the angle $\alpha\pi$ around vector $\zeta \in S^2(\xi)$, one has

$$f \circ \varphi_\zeta^{\alpha\pi} = f \qquad \text{on} \quad S^2(\xi).$$

In particular, if $\alpha = 1$, I say that f has a (ζ,π)-rotational symmetry on $S^2(\xi)$.

A *permutation* is the rearrangement of all the members of a set. A permutation can be written as the disjoint union of cycles. A j*-cycle* is a cycle that rearranges j elements.

Fubini's Theorem states when it is possible to compute a double integral using iterated integrals and when it is possible to switch the limits of integration.

Theorem 17 (Fubini's Theorem)**.** *One may switch the order of integration if the double* *integral yields a finite answer when the integrand is replaced by its absolute value, i.e.*

$$\int_X \left(\int_Y f(x,y)\,dy \right) dx = \int_Y \left(\int_X f(x,y)\,dx \right) dy = \int_{X\times Y} f(x,y)\,d(x,y)$$

Next I state the definition of the Hausdorff dimension, [12].

Definition 6. *Given any subset E of $\mathbb{R}^n$ and $\alpha \geq 0$, the exterior α-dimensional Hausdorff measure of E is defined by $m_\alpha^*(E) = \lim_{\delta\to 0^+} \inf \mathcal{H}_\alpha^\delta(E)$, where*

$$\mathcal{H}_\alpha^\delta(E) := \inf\{ \sum_{k=1}^{\infty} (\, diam\, F_k)^\alpha : \; E \subset \bigcup_{k=1}^{\infty} F_k, \quad diam\, F_k \leq \delta\},$$

and $diam\,(S) = \sup_{x,y\in S} |x-y|$ stands for the length of the diameter of S. The Hausdorff dimension of E is $dim_H(E) = \inf\{\alpha > 0 : \; m_\alpha^*(E) = 0\}$.

Now for the implicit function theorem, see [13]. For this I will need the Jacobian matrix, *i.e.*,

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}.$$

A well known property of determinants is if the matrix is an upper triangular matrix, then the determinant is the product of the diagonal entries.

Theorem 18 (Implicit Function Theorem)**.** *Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function, and let $\mathbb{R}^{n+m}$ have coordinates (x, y). Fix a point (a, b) with $f(a, b) = 0$. If the Jacobian matrix that involves $f_1, \ldots, f_m$ and $y_1, \ldots, y_m$ has a nonzero determinant, then there exists an open set U containing a, an open set V containing b, and a unique continuously differentiable function $g : U \to V$ such that*

$$\{(x, g(x)) : x \in U\} = \{(x, y) \in U \times V : f(x, y) = 0\}.$$

In other words, if there is $n+m$ equations with n variables then $\mathcal{M} = \{(x, y) \in U \times V : f(x, y) = 0\} \subset \mathbb{R}^{n+m}$ has dimension m.

CHAPTER 4

On Bodies with Directly Congruent Projections and Sections

Chapter 4 is organized as follows. In the first section I give a heuristic idea of the proof of the main results. In Section 4.2 and Section 4.3 I formulate and prove the main auxiliary results Proposition 2 and Proposition 3. Section 4.4 is devoted to the proof of Theorem 1 and Corollary 1. Theorem 3 and Corollary 2 are proved in Section 4.5. In Section 4.6, I state and prove the analogous statement to Theorem 1 in $\mathbb{R}^3$. Then in Section 4.7, I show that the set of bodies that are considered in the main results is dense in the set of all convex bodies. The way I do this is by proving that the set of polytopes in $\mathbb{R}^n$, $n \geq 4$, with 3-dimensional projections having no rigid motion symmetries is dense in the Hausdorff metric in the class of all convex bodies in $\mathbb{R}^n$.

4.1 Additional Heuristics in $\mathbb{R}^3$

To motivate and explain the proof ideas of this chapter, I consider the analogous theorem to Theorem 1 in $\mathbb{R}^3$, for a full detailed proof see Theorem 20 in Section 4.6. Suppose that K and L are two convex bodies in $\mathbb{R}^3$ and have one diameter each. Assume that K has its diameter in the ζ direction. Assume that the "side" projections $K|w^\perp$ and $L|w^\perp$ onto all subspaces $w^\perp$ containing ζ are directly congruent. In addition, assume also that these projections are not centrally symmetric. Then it easily follows that $K = \pm L + b$ for some $b \in \mathbb{R}^3$.

For if the diameters of K and L are not parallel,

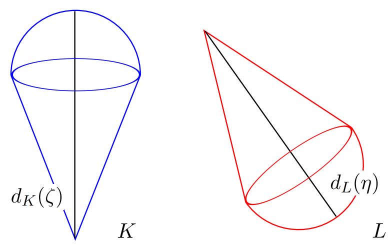

Figure 4.1: K and L, diameters not parallel.

then looking at the projections $K|w^\perp$ and $L|w^\perp$ where $w^\perp$ contains ζ,

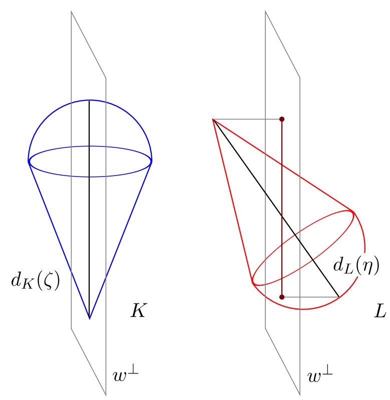

Figure 4.2: K and L and $w^\perp$.

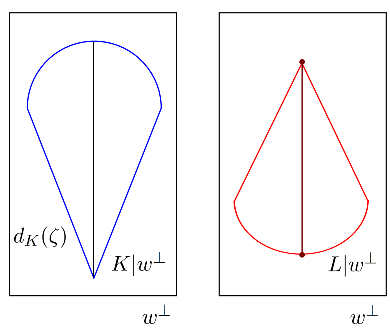

Figure 4.3: $K|w^\perp$ and $L|w^\perp$.

and knowing that these projections are directly congruent, I get a contradiction. Thus, the diameters of K and L are parallel.

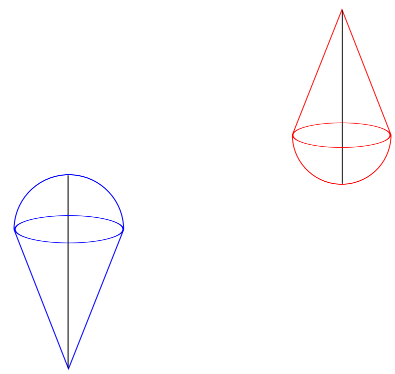

Figure 4.4: K and L.

Next I translate K and L to make the diameters coincide and be centered at the origin.

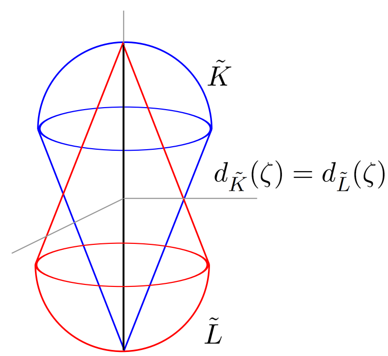

Figure 4.5: Translated K and L.

In a sense this step is "separating" translations and rotations. Next, I consider any 2-dimensional projection of the translated bodies $\tilde{K}$ and $\tilde{L}$ that contains the diameter.

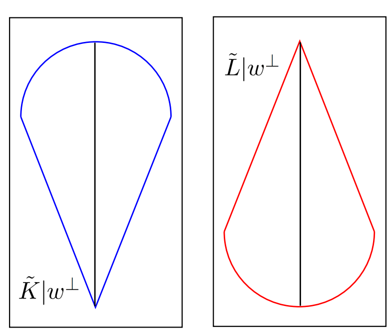

Figure 4.6: Trick of Golubyatnikov.

The direct rigid motion given by the statement of the theorem must fix this diameter. There are only two possibilities, namely, that the rigid motion is the identity, or a rotation by π. Due to the lack of the corresponding symmetries, these cases are mutually exclusive. If all rigid motions are the identity, then $\tilde{K} = \tilde{L}$. Alternatively, if all rigid motions reflect the diameter, then $\tilde{K} = -\tilde{L}$.

The proof for Theorem 1 in $\mathbb{R}^4$ is much more involved, yet the idea remains the same.

4.2 First Result About a Functional Equation on S^3

In [27, page 3429, Theorem 1], Ryabogin proves the following theorem.

Theorem 19. *Let F and G be two continuous functions on the unit sphere S^{n-1} in $\mathbb{R}^n$, $n \geq 3$, and let their restrictions to any one-dimensional great circle E coincide after some rotation $\phi_E \in SO(2)$ of this circle: $f(\phi_E(\theta)) = g(\theta)$ for every $\theta \in E$. Then, $f(\theta) = g(\theta)$ or $f(\theta) = g(-\theta)$ for all $\theta \in S^{n-1}$.*

The main result of this section is a related statement for S^3, which, in my opinion, has independent interest.

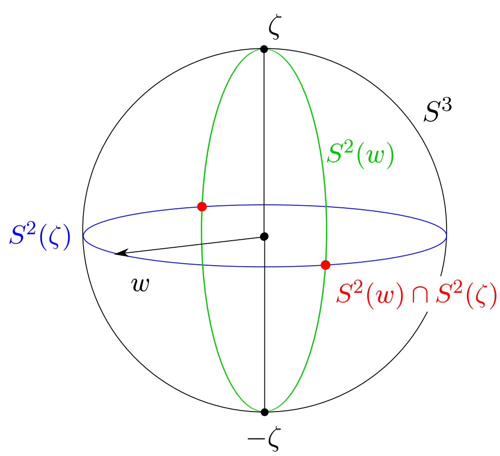

Figure 4.7: S^3.

Proposition 2. *Let f and g be two continuous functions on S^3. Assume that for some $\zeta \in S^3$ and for every $w \in S^2(\zeta)$ there exists a rotation $\varphi_w \in SO(3, S^2(w))$, verifying*

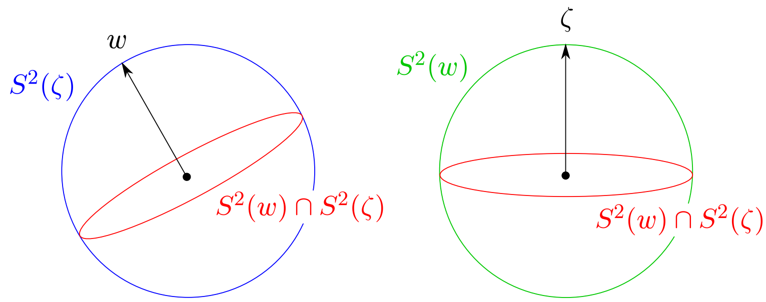

Figure 4.8: $S^2(w) \cap S^2(\zeta)$.

$\varphi_w(\zeta) = \zeta$, and

$$f \circ \varphi_w = g \qquad \textit{on} \quad S^2(w), \tag{4.1}$$

(see Figures 4.7 and 4.8). Then either $f = g$ on S^3 or $f(\theta) = g(\mathcal{O}\theta)$ $\forall \theta \in S^3$, where $\mathcal{O} \in O(4)$ is the orthogonal transformation satisfying $\mathcal{O}|_{S^2(\zeta)} = -I$, and $\mathcal{O}(\zeta) = \zeta$.

4.2.1 Auxiliary Observations

The direction $\zeta \in S^3$ will be fixed throughout the proof. I start with an easy observation about the geometry of the three dimensional sphere.

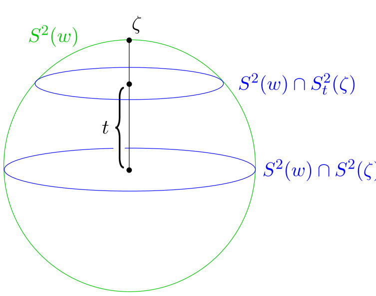

Figure 4.9: Parallels.

Lemma 1. *Let $\zeta \in S^3$ and let $\xi \in S^2(\zeta)$. Then*

$$S^3 = \bigcup_{\{w \in S^2(\xi) \cap S^2(\zeta)\}} S^2(w). \tag{4.2}$$

Proof. As it is shown in Figure 4.9, for any $w \in S^2(\zeta)$, the two-dimensional sphere $S^2(w)$

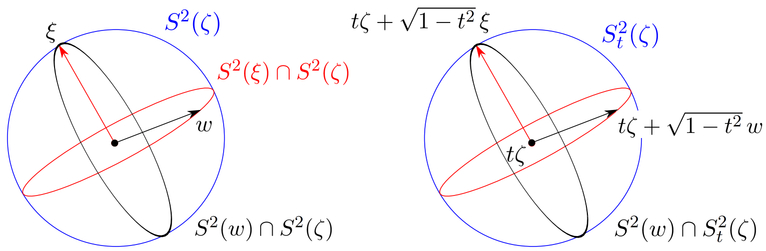

Figure 4.10: $S^2(\zeta)$ and $S_t^2(\zeta)$.

can be written as the union of all one-dimensional parallels $S^2(w) \cap S_t^2(\zeta)$, $t \in [-1, 1]$, *i.e.*

$$S^2(w) = \bigcup_{\{t\in[-1,1]\}} (S^2(w) \cap S_t^2(\zeta)). \tag{4.3}$$

On the other hand, I can write the two-dimensional sphere $S^2(\zeta)$ as the union of all meridians $S^2(w) \cap S^2(\zeta)$ containing a fixed direction $\xi \in S^2(\zeta)$, for every $w \in S^2(\xi) \cap S^2(\zeta)$ (see Figure 4.10, left), by

$$S^2(\zeta) = \bigcup_{\{w\in S^2(\xi)\cap S^2(\zeta)\}} (S^2(w) \cap S^2(\zeta)),$$

and, rescaling, the same is true for every $S_t^2(\zeta)$, $t \in [-1, 1]$ (see Figure 4.10, right, where the points on the sphere are found using the Pythagorean Theorem). I would like to note that $t\zeta + \sqrt{1-t^2}\xi \in S^2(w) \cap S_t^2(\zeta)$ because $\zeta, \xi \in S^2(w)$, and similarly $t\zeta + \sqrt{1-t^2}w \in S^2(\xi) \cap S_t^2(\zeta)$. Thus, I have

$$S_t^2(\zeta) = \bigcup_{\{w\in S^2(\xi)\cap S^2(\zeta)\}} (S^2(w) \cap S_t^2(\zeta)) \qquad \forall t \in [-1, 1]. \tag{4.4}$$

Indeed, let $w \in S^2(\xi) \cap S^2(\zeta)$ then $S^2(w)$ and $S_t^2(\zeta)$ intersect in a 1-dimensional circle (refer to Figure 3.1), hence the right hand side is contained in the left hand side. To show the opposite containment, let $x \in S_t^2(\zeta)$ then there exists $w \in S^2(\xi) \cap S^2(\zeta)$ such that x is in the meridian $S^2(w) \cap S_t^2(\zeta)$ (see Figure 4.10, right), thus proving (4.4).

Combining (4.3) and (4.4), I obtain

$$S^3 = \bigcup_{\{t\in[-1,1]\}} S_t^2(\zeta) = \bigcup_{\{t\in[-1,1]\}} \bigcup_{\{w\in S^2(\xi)\cap S^2(\zeta)\}} (S^2(w) \cap S_t^2(\zeta)) =$$

$$= \bigcup_{\{w\in S^2(\xi)\cap S^2(\zeta)\}} \bigcup_{\{t\in[-1,1]\}} (S^2(w)\cap S_t^2(\zeta)) = \bigcup_{\{w\in S^2(\xi)\cap S^2(\zeta)\}} S^2(w).$$

□

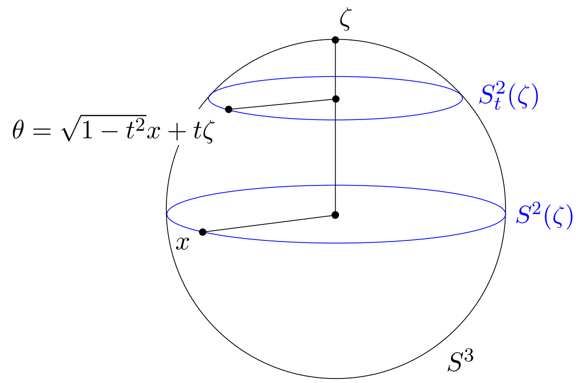

Figure 4.11: θ on $S_t^2(\zeta)$.

Let $\mathcal{O} \in O(4)$ be an orthogonal transformation, satisfying $\mathcal{O}|_{S^2(\zeta)} = -I$, and $\mathcal{O}(\zeta) = \zeta$. Observe that $\mathcal{O}|_{S^2(w)}$ commutes with every rotation $\varphi_w \in SO(3, S^2(w))$, such that $\varphi_w(\zeta) = \zeta$, where $w \in S^2(\zeta)$. Indeed, from the fact that rotations are distance preserving and the properties of the functions I have,

$$\begin{aligned}
(\mathcal{O}|_{S^2(w)} \circ \varphi_w)(\theta) &= (\mathcal{O}|_{S^2(w)} \circ \varphi_w)(\sqrt{1-t^2}x + t\zeta) \\
&= \sqrt{1-t^2}(\mathcal{O}|_{S^2(w)} \circ \varphi_w)(x) + t(\mathcal{O}|_{S^2(w)} \circ \varphi_w)(\zeta) \\
&= \sqrt{1-t^2}\varphi_w(-x) + t\zeta \\
&= \sqrt{1-t^2}(\varphi_w \circ \mathcal{O}|_{S^2(w)})(x) + t(\varphi_w \circ \mathcal{O}|_{S^2(w)})(\zeta) \\
&= (\varphi_w \circ \mathcal{O}|_{S^2(w)})(\theta),
\end{aligned}$$

since $\varphi_w(x) \in S^2(w) \cap S^2(\zeta)$.

Remark 1. *Notice that if $\varphi_w(\zeta) = -\zeta$, then $\mathcal{O}|_{S^2(w)}$ still commutes with φ_w. Just apply the same argument as above with $\varphi_w(\zeta) = -\zeta$.*

It is clear that any function f on S^3 can be decomposed in the form

$$f(\theta) = \frac{f(\theta) + f(\mathcal{O}\theta)}{2} + \frac{f(\theta) - f(\mathcal{O}\theta)}{2} = f_{\mathcal{O},e}(\theta) + f_{\mathcal{O},o}(\theta), \quad \theta \in S^3, \tag{4.5}$$

where I will call $f_{\mathcal{O},e}$, $f_{\mathcal{O},o}$, the even and odd parts of f with respect to $\mathcal{O}$. Since $\mathcal{O}^2 = I$, I have

$$f_{\mathcal{O},e}(\theta) = f_{\mathcal{O},e}(\mathcal{O}\theta), \qquad f_{\mathcal{O},o}(\theta) = -f_{\mathcal{O},o}(\mathcal{O}\theta). \tag{4.6}$$

It is also clear that every $\theta \in S^3$ belongs to $S_t^2(\zeta)$ for some $t \in [-1, 1]$, *i.e.*, can be written in the form

$$\theta = \sqrt{1-t^2}x + t\zeta, \tag{4.7}$$

for some $t \in [-1, 1]$ and $x \in S^2(\zeta)$ (see Figure 4.11).

Let $t \in [-1, 1]$. For any function f on S^3, define the function F_t on $S^2(\zeta)$,

$$F_t(x) = F_{t,\zeta}(x) = f(\sqrt{1-t^2}x + t\zeta), \qquad x \in S^2(\zeta), \tag{4.8}$$

which is the restriction of f onto $S_t^2(\zeta)$. Observe that

$$(F_t)_e(x) = \frac{f(\sqrt{1-t^2}x + t\zeta) + f(-\sqrt{1-t^2}x + t\zeta)}{2} = \frac{f(\theta) + f(\mathcal{O}\theta)}{2},$$

where θ is as in (4.7), and similarly for $(F_t)_o(x)$, *i.e.*,

$$(F_t)_e(x) = f_{\mathcal{O},e}(\theta), \qquad (F_t)_o(x) = f_{\mathcal{O},o}(\theta). \tag{4.9}$$

Note that $(F_t)_e(x) = (F_t)_e(-x)$ for every $x \in S^2(\zeta)$.

As seen in Lemma 1, every one-dimensional great circle of $S^2(\zeta)$ is of the form $S^2(w) \cap S^2(\zeta)$ for some $w \in S^2(\zeta)$. To simplify the notation, I will denote such great circles by

$$E = E_{\zeta,w} = S^2(w) \cap S^2(\zeta) = \{\theta \in S^3 : \theta \cdot \zeta = \theta \cdot w = 0\}.$$

Since $\varphi_w(\zeta) = \zeta$ and $\varphi_w(S^2(w)) = S^2(w)$, I have

$$\varphi_w(E_{\zeta,w}) = \varphi_w(S^2(w) \cap S^2(\zeta)) = S^2(w) \cap S^2(\zeta) = E_{\zeta,w}.$$

Thus, for every $t \in [-1, 1]$, and for the corresponding one-dimensional equator $E = E_{\zeta,w}$ of $S^2(\zeta)$, there is a rotation $\phi_E \in SO(2, E)$, which is the restriction to E of the rotation

$\varphi_w \in SO(3, S^2(w))$ given by the conditions of Proposition 2, and which satisfies

$$F_t \circ \phi_E(x) = G_t(x) \qquad \forall x \in E, \tag{4.10}$$

(see Figure 4.10). Here G_t is defined from g similarly to F_t in (4.8). Indeed, let $x \in E$ and $\theta = \sqrt{1-t^2}x + t\zeta \in S^2(w) \cap S^2_t(\zeta)$, then $(f \circ \varphi_w)(\theta) = g(\theta)$ and hence

$$\begin{aligned} G_t(x) &= g(\sqrt{1-t^2}x + t\zeta) = (f \circ \varphi_w)(\sqrt{1-t^2}x + t\zeta) \\ &= f(\sqrt{1-t^2}\varphi_w(x) + t\zeta) = f(\sqrt{1-t^2}\phi_E(x) + t\zeta) = (F_t \circ \phi_E)(x). \end{aligned}$$

4.2.2 Auxiliary Lemmata

I will use the *Funk transform*, see (3.1),

$$Rf(w) = R_\zeta f(w) = \int\limits_{S^2(w)\cap S^2(\zeta)} f(\theta)d\theta, \qquad w \in S^2(\zeta).$$

Here $d\theta$ stands for the Lebesgue measure on the one-dimensional great circle $E = S^2(w) \cap S^2(\zeta)$ of $S^2(\zeta)$.

Lemma 2. *Let f and g be as in Proposition 2. Then $f_{\mathcal{O},e} = g_{\mathcal{O},e}$.*

Proof. Let $w \in S^2(\zeta)$, and let $\varphi_w \in SO(3, S^2(w))$ be such that (4.1) holds. Then, $\phi_E = \varphi_w|_{S^2(w)\cap S^2(\zeta)} \in SO(2, E)$ is the corresponding rotation in $E = S^2(w) \cap S^2(\zeta)$. By the rotation invariance of the Lebesgue measure on E and (4.10),

$$\int\limits_E F_t(x)dx = \int\limits_E F_t \circ \phi_E(x)dx = \int\limits_E G_t(x)dx, \qquad \forall t \in [-1, 1]. \tag{4.11}$$

Hence, $R_\zeta F_t(w) = R_\zeta G_t(w)$ for every $w \in S^2(\zeta)$. Thus, $(F_t)_e(x) = (G_t)_e(x)$ for every $x \in S^2(\zeta)$ (apply Theorem 6 to $(F_t)_e - (G_t)_e$ and $S^2(\zeta)$ instead of S^{n-1}). Using the first relation in (4.9), its analogue for g, and (4.3), I have

$$f_{\mathcal{O},e}(\theta) = (F_t)_e(x) = (G_t)_e(x) = g_{\mathcal{O},e}(\theta),$$

which is true for all $t \in [-1, 1]$ and all $x \in S^2(\zeta)$. Hence the desired result is obtained. $\square$

Remark 2. *By Lemma 2, I can assume that my functions f and g are odd with respect to $\mathcal{O}$. Indeed, I need to show either $f = g$, which suffices to show the odd parts of the functions are equal, or show $f(\theta) = g(\mathcal{O}\theta)$, which using the first relation in (4.6) gives the same equation with the original functions replaced with their odd parts. In order to simplify the notation, from now on I will write f and g instead of $f_{\mathcal{O},o}$ and $g_{\mathcal{O},o}$. I will also write F_t for $(F_t)_o$ and G_t for $(G_t)_o$.*

Let $\varphi_w^{\alpha\pi}$ be the rotation of the sphere $S^2(w)$ by the angle $\alpha\pi$ around ζ, *i.e.*, $\varphi_w^{\alpha\pi}(\zeta) = \zeta$. By this I mean that $\varphi_w^{\alpha\pi}$ is the restriction to the 3-dimensional subspace spanned by $S^2(w)$ of a rotation $\Phi \in SO(4)$ with the following properties: $\Phi(\zeta) = \zeta$, $\Phi(w) = w$, if $\{x, y, w, \zeta\}$ is a positively oriented orthonormal basis of $\mathbb{R}^4$, then for every $v \in (span\{x, y\} \cap S^3) = S^2(w) \cap S^2(\zeta)$, the angle between the vectors v and $\varphi_w^{\alpha\pi}(v) \in S^2(w) \cap S^2(\zeta)$ is $\alpha\pi$, and if α is not an integer, $\{v, \varphi_w^{\alpha\pi}(v), w, \zeta\}$ form a positively oriented basis of $\mathbb{R}^4$.

For any $\alpha \in \mathbb{R}$, consider the sets Ξ_α, defined as

$$\{w \in S^2(\zeta) : \exists \varphi_w^{\alpha\pi} \in SO(3, S^2(w)) \text{ such that } f \circ \varphi_w^{\alpha\pi} = g \text{ on } S^2(w)\}. \tag{4.12}$$

Observe that $\Xi_0 = \{w \in S^2(\zeta) \text{ such that } f = g \text{ on } S^2(w)\}$, and

$$\Xi_1 = \{w \in S^2(\zeta) : f(\theta) = g(\mathcal{O}\theta) \quad \forall \theta \in S^2(w)\}. \tag{4.13}$$

My aim is to show that $S^2(\zeta) = \Xi_0 \cup \Xi_1$. This will be achieved in Lemma 8 if I prove the following Lemmata.

Lemma 3. *The set Ξ_α is closed.*

Proof. If $\Xi_\alpha = \emptyset$ then it is closed, hence I can assume that Ξ_α is not empty.

Let $(w_l)_{l=1}^\infty$ be a sequence of elements of Ξ_α converging to $w \in S^2(\zeta)$ as $l \to \infty$, and let θ be any point on $S^2(w)$. Consider a sequence $(\theta_l)_{l=1}^\infty$ of points $\theta_l \in S^2(w_l)$ converging to θ as $l \to \infty$.

(It is readily seen that such a sequence exists. Indeed, let $B_{\frac{1}{l}}(\theta)$ be a Euclidean ball centered at θ of radius $\frac{1}{l}$, where $l \in \mathbb{N}$. Since $S^2(w_m) \to S^2(w)$ as $m \to \infty$, for each $l \in \mathbb{N}$ there exists $m = m(l)$ such that

$$S^2(w_m) \cap B_{\frac{1}{l}}(\theta) \neq \emptyset.$$

Choose any $\theta_l \in S^2(w_{m(l)}) \cap B_{\frac{1}{l}}(\theta)$ and rename $w_{m(l)}$ to be w_l since $w_{m(l)}$ converges to w. Then $\theta_l \to \theta$ as $l \to \infty$).

By the definition of Ξ_α, I see that

$$f \circ \varphi_{w_l}(\theta_l) = g(\theta_l) \qquad \theta_l \in S^2(w_l), \quad l \in \mathbb{N}. \tag{4.14}$$

Passing to a subsequence if necessary, I can assume that the sequence of rotations $(\varphi_{w_l})_{l=1}^{\infty}$, $\varphi_{w_l} = \varphi_{w_l}^{\alpha\pi} \in SO(3, S^2(w_l))$, is convergent, say, to $\varphi_w \in SO(3, S^2(w))$ because all the entries in the corresponding rotation matrices are bounded and w_l converges to w. Writing out the matrices of rotations $\varphi_{w_l}^{\alpha\pi}$ in the corresponding orthonormal bases $\{x_l, y_l, w_l, \zeta\}$, $x_l, y_l \in S^2(w_l) \cap S^2(\zeta)$, and passing to the limit as $l \to \infty$ I see that φ_w is the rotation by the angle $\alpha\pi$ and the limit of (4.14) is $f \circ \varphi_w(\theta) = g(\theta)$. Since the choice of $\theta \in S^2(w)$ was arbitrary, I obtain $w \in \Xi_\alpha$, and the result follows. □

Lemma 4. *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then $\Xi_\alpha \subset \Xi_0$.*

Proof. Let $w \in \Xi_\alpha$. Following the ideas of Schneider [30], I claim at first that $f^2 = g^2$ on $S^2(w)$. Indeed, since f and g are odd with respect to $\mathcal{O}$, f^2 and g^2 are even with respect to $\mathcal{O}$ because

$$\begin{aligned} f^2(\theta) &= \frac{f^2(\theta) + f^2(\mathcal{O}\theta) - f(\theta)f(\mathcal{O}\theta) - f(\theta)f(\mathcal{O}\theta)}{4} \\ &= \frac{f^2(\theta) + f^2(\mathcal{O}\theta)}{2}, \end{aligned}$$

from the right equation in (4.6), and similarly for g^2. Here, both f^2 and g^2 satisfy the conditions of Proposition 2, since

$$(f^2 \circ \varphi_w)(\theta) = f(\varphi_w(\theta))^2 = (g(\theta))^2 = g^2(\theta).$$

Thus, by Lemma 2, I obtain that $f^2 = g^2$ on $S^2(w)$.

This gives the following equation,

$$f^2 \circ \varphi_w(\theta) = g^2(\theta) = f^2(\theta) \qquad \forall \theta \in S^2(w).$$

Iterating for any $k \in \mathbb{Z}$,

$$f^2 \circ \varphi_w^k(\theta) = f^2 \circ \varphi_w^{k-1}(\theta) = \cdots = f^2(\theta) \qquad \forall \theta \in S^2(w),$$

and using the fact that for every $\theta \in S^2(w)$, the orbit of $(\varphi_w^k(\theta))_{k\in\mathbb{Z}}$ is dense on every parallel of $S^2(w)$ orthogonal to ζ, I obtain that the restrictions of f^2 and g^2 onto $S^2(w)$ are invariant under rotations leaving ζ fixed. In other words, f^2 and g^2 are constant, say k, on every parallel of $S^2(w)$ orthogonal to ζ. This implies that $f(\theta)$ and $g(\theta)$ must be $\pm\sqrt{k}$ on these parallels, but since f and g are continuous, $f = \pm\sqrt{k}$ and $g = \pm\sqrt{k}$. Notice that this says that f and g are invariant under rotations that leave ζ fixed, *i.e.*, $f \circ \varphi_w = f$. Hence, using (4.1) I have $f = g$ on $S^2(w)$, and therefore $w \in \Xi_0$. Since w from Ξ_α was chosen arbitrarily, I obtain the desired result. □

In Lemma 4, I have shown that rotations whose angle is an irrational multiple of π are not relevant under the assumptions of Proposition 2. My next goal is to prove that rational multiples are not relevant either, except for the rotations by the angles 0 and π. This will be achieved in Lemma 8, by means of a topological argument, which is based on one definition and two Lemmata from [27] (see Lemmata 5 and 6 below). The argument will show that for each $t \in (-1, 1)$ and an appropriate $w \in S^2(\zeta)$, the subset of the great circle $S^2(w) \cap S^2(\zeta)$ where the functions $F_t = G_t$ are equal to each other, is open. Since such a set is closed by definition, and it is non-empty, I will conclude that F_t equals G_t on this large circle. Using (4.3) I will obtain that $f = g$ on the corresponding $S^2(w)$, which will give the desired result.

I will reformulate the corresponding statements from [27] in a way that is more convenient for me here. Recall the following definition stated in the Preliminaries and refer to

Figure 4.12.

Definition 7. *Let $\alpha \in (0,1)$ and let $\mathbf{S}_1$, $\mathbf{S}_2$ be any two spherical circles in the standard metric of $S^2(\zeta)$, both of radius $\alpha\pi$. The union $\mathfrak{l} \cup \mathfrak{m}$ of two open arcs $\mathfrak{l} \subset \mathbf{S}_1$ and $\mathfrak{m} \subset \mathbf{S}_2$ will be called a spherical X-figure if the angle between arcs is in $(0, \frac{\pi}{4})$, the length of the arcs is less than $\alpha\pi$, and the arcs intersect at their centers only, $\mathfrak{l} \cap \mathfrak{m} = \{x\}$. The point $x \in S^2(\zeta)$ will be called the center of the X-figure.*

For this definition, the requirements for the angle between the arcs is so that the X-figure is a "skinny" X-figure, and the requirements for the length of the arcs is so that the X-figure is not too large.

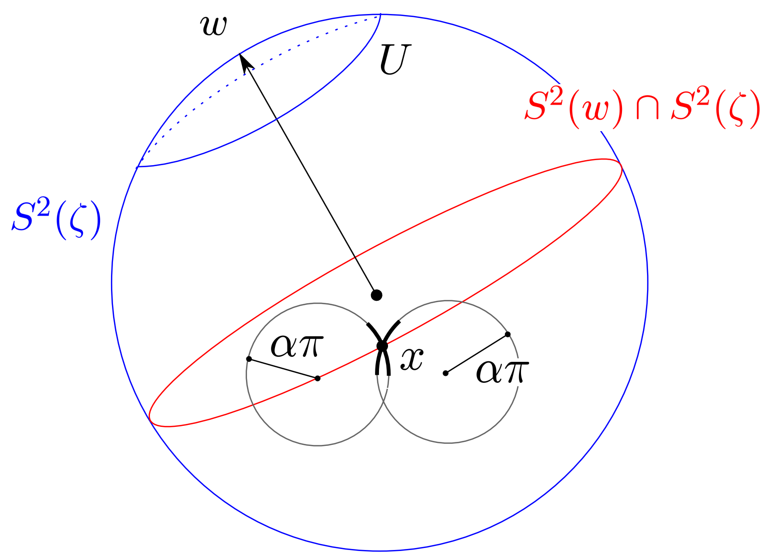

Figure 4.12: The spherical X-figures from Definition 7.

Let $t \in (-1,1)$, F_t be a function on $S^2(\zeta)$, and x be the center of a spherical X-figure. If for every $u \in X$ I have $F_t(u) = F_t(x)$, I will say that there exists an X-figure $X_{F_t(x)} \subset S^2(\zeta)$. The following result is Lemma 10 from [27] with $f = F_t$, $g = G_t$, $f_e = F_t^2$, and $S^2 = S^2(\zeta)$.

Lemma 5. *Let $t \in (-1,1)$, and let F_t and G_t be two continuous functions on $S^2(\zeta)$. Assume that there is an open spherical cap $U \subset \Xi_{\frac{p}{q}}$, with $\frac{p}{q} \in (0,1) \cap \mathbb{Q}$, such that for every*

$w \in U$, there exists a rotation $\phi_w = \phi_{w,\zeta}$ of the great circle $S^2(w) \cap S^2(\zeta)$ by the angle $\frac{p}{q}\pi$, verifying

$$F_t \circ \phi_w(x) = G_t(x) \qquad \forall x \in S^2(w) \cap S^2(\zeta). \tag{4.15}$$

Then, for every $x \in S^2(w) \cap S^2(\zeta)$ there exists an X-figure $X_{F_t^2(x)} \subset S^2(\zeta)$, with one of the arcs of $X_{F_t^2(x)}$ being orthogonal to $S^2(w) \cap S^2(\zeta)$. Moreover, for every $x, y \in S^2(w) \cap S^2(\zeta)$ there exist X-figures $X_{F_t^2(x)}$, $X_{F_t^2(y)} \in S^2(\zeta)$, such that

$$\Theta(X_{F_t^2(x)}) = X_{F_t^2(y)},$$

where $\Theta \in SO(3, S^2(\zeta))$ is such that $\Theta(w) = w$ and $\Theta(x) = y$.

In other words, if there is an open spherical cap $U \subset \Xi_{\frac{p}{q}}$ then for every $w \in U$ and every $x \in S^2(w) \cap S^2(\zeta)$ there exists an X-figure whose center is x and all the X-figures of this type are congruent.

Lemma 6. *Let $t \in (-1, 1)$, and let F_t, G_t, and U be as above. Then, for every $w \in U$ there exists a constant c such that $F_t^2(x) = G_t^2(x) = c$ for every $x \in S^2(w) \cap S^2(\zeta)$.*

Observe that since any two great circles of $S^2(\zeta)$ intersect, the above constant is actually independent of $w \in U$.

The idea of the proof is, assuming that Lemma 6 is not true, to use Lemma 5 to show the existence of an uncountable family of *disjoint* spherical $X_{F_t^2(x)}$-figures, $x \in S^2(w) \cap S^2(\zeta)$ (with F_t^2 being constant on the corresponding figure). But this family cannot exist: if the X-figures are disjoint, one can find a collection of disjoint open balls, each centered at the center of the corresponding X-figure; I obtained the uncountable collection of disjoint open balls, which gives the desired contradiction. Next, the points $x \in S^2(w) \cap S^2(\zeta)$ that are in $X_{F_t^2(x)}$ form an open set, because the spherical X-figures intersect. The exact details can be found in the proof of Lemma 12, [27], starting 17 lines from below on page 3438 until the end of the proof on page 3439 (use F_t^2 instead of f_e and $S^2(w) \cap S^2(\zeta)$ instead of $\xi^\perp$).

Lemma 7. *Let $t \in (-1,1)$, and let F_t, G_t, and U be as above. Then $f = g = 0$ on $S^2(w)$ for every $w \in U$.*

Proof. Let w be any point in U, and let $t \in (-1,1)$. By Remark 2, f and g are odd with respect to $\mathcal{O}$ on S^3. Using the second relation in (4.9), I see that F_t, G_t are odd on $S^2(\zeta)$, *i.e.*, $F_t(x) = -F_t(-x)$ and similarly for G_t. From this fact, if $F_t(x_0) > 0$ then $F_t(-x_0) < 0$, and hence by continuity, there exist x_1, $x_2 \in S^2(w) \cap S^2(\zeta)$ such that $F_t(x_1) = G_t(x_2) = 0$. By Lemma 6, $F_t^2(x) = G_t^2(x) = 0$ for every $x \in S^2(w) \cap S^2(\zeta)$, which implies $F_t(x) = G_t(x) = 0$ for every $x \in S^2(w) \cap S^2(\zeta)$. Since t was arbitrary, the previous statement is true for all $t \in (-1,1)$. Next, using (4.8) and the continuity of f and g, I see that the last statement is indeed true for all $t \in [-1,1]$. Finally, using (4.3) and (4.8) again, I conclude that $f = g = 0$ on $S^2(w)$. □

Now I am ready to prove

Lemma 8. *The sets $S^2(\zeta) = \Xi_0 \cup \Xi_1$ are equal.*

Proof. Assume the contrary, the set $A := S^2(\zeta) \setminus (\Xi_0 \cup \Xi_1)$ is not empty. By Lemma 4, $\Xi_\alpha \cap A = \emptyset$, provided $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Hence, A may be written as

$$A = \bigcup_{\{\frac{p}{q} \in \mathbb{Q}, \frac{p}{q} \neq 0,1\}} (\Xi_{\frac{p}{q}} \cap A).$$

By Lemma 3, all $\Xi_{\frac{p}{q}}$ are closed and A is open. Hence, by the Baire Category Theorem, (Theorem 13), there exists $\frac{p}{q} \in \mathbb{Q}$ where $\frac{p}{q} \neq 0,1$ such that $int(\Xi_{\frac{p}{q}} \cap A) \neq \emptyset$. I can assume that there exists an open spherical cap $U \subseteq (\Xi_{\frac{p}{q}} \cap A)$ such that for every $w \in U$, there is a rotation $\varphi_w^{\frac{p}{q}\pi} \in SO(3, S^2(w))$ such that

$$f \circ \varphi_w^{\frac{p}{q}\pi} = g \text{ on } S^2(w).$$

In particular, for any $t \in (-1,1)$, and for every large circle $E = S^2(w) \cap S^2(\zeta)$ of $S^2(\zeta)$ there exists a rotation $\phi_w \in SO(2,E)$ by the angle $\frac{p}{q}\pi$ such that (4.15) holds. Changing the orientation if necessary, I can assume that p/q is strictly between 0 and 1.

By Lemma 6, $F_t^2(x) = G_t^2(x) = c$ for every $x \in S^2(w) \cap S^2(\zeta)$, and by Lemma 7 I have $f = g = 0$ on $S^2(w)$. Hence, $w \in \Xi_0$, which is impossible, since $w \in A$. The result follows. □

I need one more simple lemma.

Lemma 9. *Let $\zeta \in S^3$, $\xi \in S^2(\zeta)$. Assume that*

$$(S^2(\xi) \cap S^2(\zeta)) \cap \Xi_0 \cap \Xi_1 = \emptyset.$$

Then, either

$$(S^2(\xi) \cap S^2(\zeta)) \subset (\Xi_0 \setminus \Xi_1) \quad \textit{or} \quad (S^2(\xi) \cap S^2(\zeta)) \subset (\Xi_1 \setminus \Xi_0). \tag{4.16}$$

Proof. By Lemma 8,

$$S^2(\zeta) = (\Xi_0 \setminus \Xi_1) \cup (\Xi_0 \cap \Xi_1) \cup (\Xi_1 \setminus \Xi_0). \tag{4.17}$$

By assumption, $(S^2(\xi) \cap S^2(\zeta)) \cap (\Xi_0 \cap \Xi_1) = \emptyset$. Therefore,

$$(S^2(\xi) \cap S^2(\zeta)) \subset ((\Xi_0 \setminus \Xi_1) \cup (\Xi_1 \setminus \Xi_0)).$$

If (4.16) is not true, then

$$S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1) \neq \emptyset, \qquad \text{and} \qquad S^2(\xi) \cap S^2(\zeta) \cap (\Xi_1 \setminus \Xi_0) \neq \emptyset.$$

Take any $w_1 \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1)$ and $w_2 \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_1 \setminus \Xi_0)$. Rotating if necessary I can assume that

$$S^2(\xi) \cap S^2(\zeta) = \{w = w(t) \in S^3 : w(t) = (\cos t, \sin t, 0, 0), \quad t \in [0, 2\pi]\},$$

and

$$w_1 = (\cos t_1, \sin t_1, 0, 0), \qquad w_2 = (\cos t_2, \sin t_2, 0, 0),$$

for some $t_1, t_2 \in [0, 2\pi]$, $t_1 < t_2$. Now define

$$t^* := \sup\{t \in [t_1, t_2) : w(t) \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1)\}, \qquad w^* := w(t^*).$$

I have two possibilities,

$$(a) \quad w^* \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1), \qquad (b) \quad w^* \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_1 \setminus \Xi_0).$$

If (a) is true, then $w^* \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_1 \setminus \Xi_0)$ due to the fact that $w(t) \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_1 \setminus \Xi_0)$ for all $t_2 > t > t^*$, and $S^2(\xi) \cap S^2(\zeta) \cap \Xi_1$ is closed. But then,

$$w^* \in (\Xi_0 \setminus \Xi_1) \cap (\Xi_1 \setminus \Xi_0), \tag{4.18}$$

which is impossible since the set is empty.

If (b) is true, then for every $l \in \mathbb{N}$ there exists a $t_l \in [t^* - \frac{1}{l}, t^*)$ such that $w_l = w(t_l) \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1)$, (otherwise, there would be an l such that for every $t \in [t^* - \frac{1}{l}, t^*]$ I have $w(t) \notin S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1)$, and t^* is not a supremum). Since $w_l \to w^*$ as $l \to \infty$ and $S^2(\xi) \cap S^2(\zeta) \cap \Xi_0$ is closed, I again have (4.18). Hence, (4.16) is proved. $\square$

4.2.3 **Proof of Proposition 2**

By Lemma 8, I have that $S^2(\zeta) = \Xi_0 \cup \Xi_1$. If I assume that $\Xi_1 = \emptyset$, then $S^2(\zeta) = \Xi_0$, and therefore $f(\theta) = g(\theta)$ for every $\theta \in S^3$. On the other hand, if $\Xi_0 = \emptyset$, I have that $S^2(\zeta) = \Xi_1$, which means that $f(\theta) = g(\mathcal{O}\theta)$ for every $\theta \in S^3$. Hence, in these two situations I obtain the desired conclusion.

Now I can assume that both Ξ_0, Ξ_1 are not empty. It is impossible for them to not overlap, hence $\Xi_0 \cap \Xi_1 \neq \emptyset$. Indeed, let w be a point on the boundary of Ξ_0, ($w \in \Xi_0$, since Ξ_0 is closed). Then for every $l \in \mathbb{N}$, the set $B_{\frac{1}{l}}(w) \cap S^3$ contains a point w_l from Ξ_1. But then $w_l \to w$ as $l \to \infty$, hence $w \in \Xi_1$, and $w \in \Xi_0 \cap \Xi_1$.

I shall consider two cases:

1) There exists $\xi \in S^2(\zeta)$ such that $\Xi_0 \cap \Xi_1 \cap S^2(\xi) = \emptyset$.

2) For every $x \in S^2(\zeta)$ I have $\Xi_0 \cap \Xi_1 \cap S^2(x) \neq \emptyset$.

Consider the first case. Using (4.17) and Lemma 9, I obtain (4.16). If the first relation in (4.16) holds, then, by Lemma 1, I have $S^3 = \bigcup_{\{w \in \Xi_0\}} S^2(w)$, and $f(\theta) = g(\theta)$ for every $\theta \in S^3$. If the second relation in (4.16) holds, then, using Lemma 1 again, I obtain $S^3 = \bigcup_{\{w \in \Xi_1\}} S^2(w)$, and $f(\theta) = g(\mathcal{O}\theta)$ for every $\theta \in S^3$. This concludes the first case scenario.

Next consider the second case. I claim that

$$S^2(\zeta) = \bigcup_{\{u \in (\Xi_0 \cap \Xi_1)\}} (S^2(u) \cap S^2(\zeta)). \tag{4.19}$$

Indeed, let $x \in S^2(\zeta)$. By the hypothesis of the second case, the set $\Xi_0 \cap \Xi_1 \cap S^2(x)$ is non-empty. Let $u \in \Xi_0 \cap \Xi_1 \cap S^2(x)$. This says that u and x are perpendicular, *i.e.*, $x \in S^2(u)$, and hence $x \in S^2(u) \cap S^2(\zeta)$, from which it follows that

$$x \in \bigcup_{\{u \in (\Xi_0 \cap \Xi_1)\}} (S^2(u) \cap S^2(\zeta)),$$

thus proving (4.19).

Using (4.19) and an argument similar to the one in the proof of Lemma 1, I conclude that

$$S^3 = \bigcup_{\{u \in (\Xi_0 \cap \Xi_1)\}} S^2(u). \tag{4.20}$$

Indeed, I can create a similar equation as (4.19), namely if $t \in [-1, 1]$ then

$$S_t^2(\zeta) = \bigcup_{\{u \in (\Xi_0 \cap \Xi_1)\}} (S^2(u) \cap S_t^2(\zeta)). \tag{4.21}$$

To prove this equality, let $\theta \in S_t^2(\zeta)$. Then $\theta = \sqrt{1-t^2}x + t\zeta$ for some $x \in S^2(\zeta)$ and hence $\Xi_0 \cap \Xi_1 \cap S^2(x) \neq \emptyset$. Let $u \in \Xi_0 \cap \Xi_1 \cap S^2(x)$ then $u \in S^2(x) \cap S^2(\zeta)$ and $x \in S^2(u) \cap S^2(\zeta)$ which implies that $\theta = \sqrt{1-t^2}x + t\zeta \in S^2(u) \cap S_t^2(\zeta)$. Thus, (4.21) holds.

Now I use (4.21) and (4.3) to finish (4.20),

$$\begin{aligned} S^3 &= \bigcup_{\{t\in[-1,1]\}} S_t^2(\zeta) = \bigcup_{\{t\in[-1,1]\}} \bigcup_{\{u\in(\Xi_0\cap\Xi_1)\}} (S^2(u)\cap S_t^2(\zeta)) \\ &= \bigcup_{\{u\in(\Xi_0\cap\Xi_1)\}} \bigcup_{\{t\in[-1,1]\}} (S^2(u)\cap S_t^2(\zeta)) \\ &= \bigcup_{\{u\in(\Xi_0\cap\Xi_1)\}} S^2(u) \end{aligned}$$

It is easy to see that if (4.20) holds, then f and g are *zero* on S^3, and I am done. Indeed, let $\theta \in S^3$. Then $\theta \in S^2(w)$ for some $w \in (\Xi_0 \cap \Xi_1)$. Using (4.13) I see that $f(\theta) = g(\theta) = g(\mathcal{O}\theta)$. Since g is odd with respect to $\mathcal{O}$ I have the property $g(\theta) = -g(\mathcal{O}\theta)$ and hence $g(\theta) = f(\theta) = 0$. Since θ was arbitrary, I have proved that if (4.20) holds, then $f = g = 0$ on S^3.

Thus, in all possible cases, I have shown that if f and g are odd with respect to $\mathcal{O}$, then either $f(\theta) = g(\theta)$ for every $\theta \in S^3$, or $f(\theta) = g(\mathcal{O}\theta)$ for every $\theta \in S^3$ (see Remark 2). Proposition 2 is proved. □

4.3 Another result about a functional equation on S^3

In this section I prove Proposition 3, which is a consequence of Proposition 2. In order to formulate it I recall the following definition.

Let f be a continuous function on S^3 and let $\xi \in S^3$. I say that the restriction of f onto $S^2(\xi)$ (or just f) has a $(\zeta, \alpha\pi)$-rotational symmetry if for some rotation $\varphi_\zeta^{\alpha\pi} \in SO(3, S^2(\xi))$ by the angle $\alpha\pi$ around vector $\zeta \in S^2(\xi)$, one has

$$f \circ \varphi_\zeta^{\alpha\pi} = f \qquad \text{on} \quad S^2(\xi).$$

In particular, if $\alpha = 1$, I say that f has a (ζ, π)-rotational symmetry on $S^2(\xi)$.

Proposition 3. *Let f and g be two continuous functions on S^3. Assume that for some $\zeta \in S^3$ and for every $w \in S^2(\zeta)$ there exists a rotation $\varphi_w \in SO(3, S^2(w))$, verifying*

$\varphi_w(\zeta) = \zeta$, or $\varphi_w(\zeta) = -\zeta$, and

$$f \circ \varphi_w = g \qquad \text{on} \quad S^2(w). \tag{4.22}$$

Assume also that f and g have no (ζ, π)-rotational symmetries and no (u, π)-rotational symmetries for any $u \in S^2(\zeta) \cap S^2(w)$. Then either $f = g$ on S^3 or $f(\theta) = g(\mathcal{O}\theta)$ $\forall \theta \in S^3$, where $\mathcal{O} \in O(4)$ is the orthogonal transformation satisfying $\mathcal{O}|_{S^2(\zeta)} = -I$, and $\mathcal{O}(\zeta) = \zeta$.

4.3.1 Auxiliary Lemmata

Consider the sets

$$\Xi = \{w \in S^2(\zeta) : \quad (4.22) \quad \text{holds with} \quad \varphi_w(\zeta) = \zeta\} \tag{4.23}$$

and

$$\Psi = \{w \in S^2(\zeta) : \quad (4.22) \quad \text{holds with} \quad \varphi_w(\zeta) = -\zeta\}. \tag{4.24}$$

My final goal is to show that $S^2(\zeta) = \Xi$, which will be achieved in Lemmata 10−14. Then I will be able to invoke Proposition 2.

Lemma 10. *The sets Ξ and Ψ are closed, and $\Xi \cup \Psi = S^2(\zeta)$.*

Proof. I prove that Ξ is closed. I can assume that Ξ is non-empty. Let $(w_l)_{l=1}^{\infty}$ be a sequence of elements of Ξ converging to $w \in S^2(\zeta)$, and let θ be any point on $S^2(w)$. As in the proof of Lemma 3, I see that there exists a sequence $(\theta_l)_{l=1}^{\infty}$, $\theta_l \in S^2(w_l)$, converging to θ as $l \to \infty$. Then,

$$f \circ \varphi_{w_l}(\theta_l) = g(\theta_l), \qquad \varphi_{w_l}(\zeta) = \zeta, \quad \forall l = 1, 2, \dots. \tag{4.25}$$

Using the compactness of $SO(4)$ and passing to a subsequence if necessary, I can assume that the sequence $(\varphi_{w_l})_{l=1}^{\infty}$ of rotations is convergent, say, to φ_w. Passing to the limit in (4.25) as $l \to \infty$, and using the fact that θ is an arbitrary point in $S^2(w)$, I obtain (4.22) with $\varphi_w(\zeta) = \zeta$. Hence, $w \in \Xi$.

The proof of the fact that Ψ is closed is very similar. One has only to repeat the above arguments with $\varphi_{w_l}(\zeta) = -\zeta$ instead of the second equality in (4.25).

By conditions of the proposition it is clear that $\Xi \cup \Psi = S^2(\zeta)$. □

Lemma 11. *The sets Ξ and Ψ remain the same if, instead of the pair f, g in equation (4.22), I take $f_{\mathcal{O},o}$, $g_{\mathcal{O},o}$.*

Proof. I claim at first that the conclusion of Lemma 2 (and Remark 2) remain valid, *i.e.* f and g can be assumed to be odd with respect to $\mathcal{O}$. Indeed, let $w \in S^2(\zeta)$, and let $\varphi_w \in SO(3, S^2(w))$ be such that (4.22) holds. Denoting $E = S^2(w) \cap S^2(\zeta)$, if $w \in \Xi$, then, $\phi_E := \varphi_w|_E \in SO(2, E)$ is the corresponding rotation in $S^2(w) \cap S^2(\zeta)$. If $w \in \Psi$, then $\psi_E := \varphi_w|_E \in O(2, E)$ is the corresponding reflection with respect to $u \in S^2(w) \cap S^2(\zeta)$, see Figure 4.13. By the rotation and reflection invariance of the Lebesgue measure on E, I see that (4.11) holds (with ψ_E instead of ϕ_E if $w \in \Psi$). Thus, one can repeat the rest of the argument in the proof of Lemma 2, to see that $f_{\mathcal{O},e} = g_{\mathcal{O},e}$. The claim follows.

Since the even parts $f_{\mathcal{O},e}$, $g_{\mathcal{O},e}$ are equal, and φ_w commutes with $\mathcal{O}$ (see Remark 1), and (4.32), I conclude that (4.22) holds for $f_{\mathcal{O},o}$, $g_{\mathcal{O},o}$. Indeed,

$$f_{\mathcal{O},e} \circ \varphi_w(\theta) = \frac{f(\varphi_w(\theta)) + f(\mathcal{O}\varphi_w(\theta))}{2} = \frac{g(\theta) + g(\mathcal{O}\theta)}{2} = g_{\mathcal{O},e}(\theta),$$

and from here I can prove the statement of the lemma. □

Remark 3. *In order to simplify the notation, from now on I will write f and g instead of $f_{\mathcal{O},o}$ and $g_{\mathcal{O},o}$.*

As I did in the proof of Proposition 2, for $\alpha \in \mathbb{R}$, I consider the sets Ξ_α, defined by (4.12) with $\varphi_w^{\alpha\pi}(\zeta) = \zeta$. I see that the conditions of Lemma 3 are satisfied, hence the sets Ξ_α are closed.

Lemma 12. *I have $S^2(\zeta) = \Xi_0 \cup \Xi_1 \cup \Psi$ and $(\Xi_0 \cup \Xi_1) \cap \Psi = \emptyset$.*

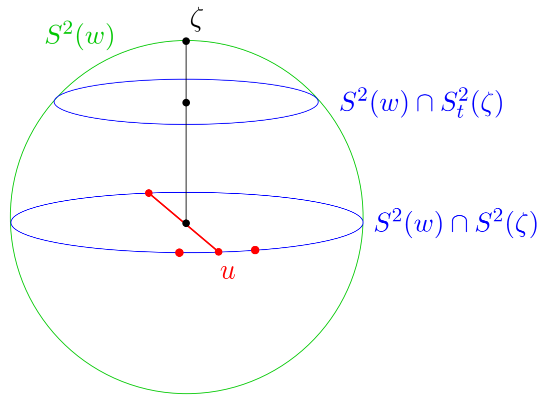

Figure 4.13: If $\psi_E \in O(2, E)$ the red point gets reflected to the opposite red point.

Proof. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then using Lemma 4 I obtain that $\Xi_\alpha \subset \Xi_0$. Also, arguing as in the proof of Lemma 8 (with $A = S^2(\zeta) \setminus (\Xi_0 \cup \Xi_1 \cup \Psi)$), I obtain that the only possible rational values for α are 0 and 1.

Now I show that $(\Xi_0 \cup \Xi_1) \cap \Psi = \emptyset$. If this is not true, let w be any element of $(\Xi_0 \cup \Xi_1) \cap \Psi$. Using the definition of Ξ and Ψ, I have

$$f \circ \varphi_w = g, \qquad f \circ \psi_w = g \qquad \text{on} \quad S^2(w),$$

where φ_w, $\psi_w \in SO(3, S^2(w))$ are rotations satisfying $\varphi_w(\zeta) = \zeta$, $\psi_w(\zeta) = -\zeta$.

If $w \in \Xi_0$, then φ_w is trivial, and I have $f = f \circ \psi_w$ on $S^2(w)$. Since any 3-dimensional rotation has a one-dimensional invariant subspace, there exists $u \in S^2(w) \cap S^2(\zeta)$ such that $\psi_w(u) = u$. This means that f has a (u, π)-symmetry, which is impossible by the assumptions of Proposition 3.

If $w \in \Xi_1$, then φ_w is the rotation of angle π around ζ, while ψ_w is the rotation of angle π around $u \in S^2(w) \cap S^2(\zeta)$. Since $\varphi_w^{-1} = \varphi_w$, it follows that $f = f \circ \varphi_w \circ \psi_w$. Recall Theorem 15, namely the composition of two rotations by π about axes that are separated by an angle β, is a rotation by 2β about an axis perpendicular to the axes of the given rotations. Since ζ and u are perpendicular, I conclude that $\varphi_w \circ \psi_w$ is a rotation by π around $v \in S^2(w) \cap S^2(u) \cap S^2(\zeta)$. Hence, f has a (v, π)-symmetry, which is impossible by the assumptions of Proposition 3. Thus, $(\Xi_0 \cup \Xi_1) \cap \Psi = \emptyset$, and the lemma is proved. $\square$

Hence from this lemma, I see $S^2(\zeta) = \Xi_0 \cup \Xi_1$ or $S^2(\zeta) = \Psi$.

To prove the next lemma recall Theorem 16 from the Preliminaries. Let $\mathcal{A}$ and $\mathcal{B}$ be rotations of finite order of Euclidean 3-space, about axes that are themselves separated by an angle which is a rational multiple of π. Then, the 2-generator subgroup of $SO(3)$, generated by $\mathcal{A}$ and $\mathcal{B}$, is infinite and dense, except in the following cases: if one generator has order 1, the group is cyclic; if one generator has order 2 and the axes are orthogonal, the group is dihedral; and if both generators have order 4 and the axes are orthogonal, the group is the symmetries of the cube.

Lemma 13. *Let $S^2(\zeta) = \Psi$. Then $\forall w \in S^2(\zeta)$ there exists a unique rotation $\varphi_w \in SO(3, S^2(w))$ by the angle π around some $u \in S^2(w) \cap S^2(\zeta)$.*

Proof. If for some $w \in S^2(\zeta)$ there were two different rotations, $\tilde{\varphi}_1 \neq \tilde{\varphi}_2$, around $u_1 \neq \pm u_2$, $u_1, u_2 \in S^2(w) \cap S^2(\zeta)$, then I would have had

$$f \circ \tilde{\varphi}_1(\theta) = g(\theta), \quad f \circ \tilde{\varphi}_2(\theta) = g(\theta) \qquad \forall \theta \in S^2(w). \tag{4.26}$$

In other words, $f \circ \tilde{\varphi}_1(\theta) = f \circ \tilde{\varphi}_2(\theta)$ for every $\theta \in S^2(w)$. As in the proof of the previous lemma, I see that $f = f \circ \tilde{\varphi}_1 \circ \tilde{\varphi}_2$ on $S^2(w)$, where $\tilde{\varphi}_1 \circ \tilde{\varphi}_2$ must be the rotation by 2β around ζ, and β is the angle between u_1 and u_2. Hence, f would have a $(\zeta, 2\beta)$-rotational symmetry.

I claim that β cannot be rational nor irrational multiple of π (except the case $\beta = \frac{\pi}{2}$, which is excluded by the conditions of the Theorem).

Indeed, if β is a rational multiple of π, $\beta \neq \frac{\pi}{2}$, by the remarks before the Lemma 13, I see that the 2-generator subgroup, generated by $\tilde{\varphi}_1$, $\tilde{\varphi}_2$, is dense in $SO(3, S^2(w))$. Using (4.26), I obtain

$$f^2 \circ \tilde{\varphi}_1(\theta) = g^2(\theta) = f^2 \circ \tilde{\varphi}_2(\theta) \qquad \forall \theta \in S^2(w), \tag{4.27}$$

which implies

$$f^2 \circ \tilde{\varphi_1} \circ \tilde{\varphi_2}(\theta) = f^2(\theta) \qquad \forall \theta \in S^2(w). \tag{4.28}$$

Since for every $\theta \in S^2(w)$ the sequence of points $\tilde{\varphi_1}(\theta)$, $\tilde{\varphi_2}(\theta)$, $\tilde{\varphi_1} \circ \tilde{\varphi_2}(\theta)$,..., generated by the words with letters $\tilde{\varphi_1}$, $\tilde{\varphi_2}$, is dense in $S^2(w)$, the functions f^2 and g^2 must be identically constant (zero, since they are the squares of the odd functions) on $S^2(w)$, and hence f and g must equal zero. Then, $w \in \Xi_0$, which, by the previous lemma, is impossible.

If β is an irrational multiple of π, then, using (4.28) and the argument which is similar to the one in Lemma 4, I see that f is constant on every parallel of $S^2(w)$ orthogonal to ζ. This implies that f has a (ζ, π)-rotational symmetry, which is impossible due to the hypothesis of the proposition.

Thus, the rotation $\varphi_w \in SO(3, S^2(w))$ must be unique, and the lemma is proved. □

To prove this Lemma 14 I will need Theorem 14. The idea of the proof of the following statement is taken from [8, Lemma 3.2.1, page 48, and the third paragraph on page 51].

Lemma 14. *I have* $S^2(\zeta) = \Xi_0 \cup \Xi_1$.

Proof. Since $S^2(\zeta)$ is connected, it cannot be written as a disjoint union of two closed sets. Using Lemma 12, I see that either $S^2(\zeta) = \Xi_0 \cup \Xi_1$, or $S^2(\zeta) = \Psi$. I will prove that second case does not occur, by showing that the assumption $S^2(\zeta) = \Psi$ leads to the existence of a continuous tangent line field on $S^2(\zeta)$, which is impossible due to Theorem 14.

Assume that $S^2(\zeta) = \Psi$ and let $\mathbb{A}$ be the function assigning to each $w \in S^2(\zeta)$ the rotation $\mathbb{A}(w) = \varphi_w \in SO(3, S^2(w))$ by the angle π around some $u \in S^2(w) \cap S^2(\zeta)$, $\varphi_w(\zeta) = -\zeta$. By the previous lemma the map $\mathbb{A}$ is well-defined. I claim that $\mathbb{A}$ is continuous. Let $(w_l)_{l=1}^{\infty}$ be a convergent sequence of directions from $S^2(\zeta)$, with $\lim\limits_{l \to \infty} w_l = w$, and let $(\varphi_l)_{l=1}^{\infty}$ be the corresponding sequence of rotations in $S^2(w_l)$,with $\varphi_l(\zeta) = -\zeta$, for every $l \in \mathbb{N}$. First, I prove that $(\varphi_l)_{l=1}^{\infty}$ is convergent. Let $(\theta_l)_{l=1}^{\infty}$, $\theta_l \in S^2(w_l)$, be a sequence converging to any point $\theta \in S^2(w)$ as $l \to \infty$ (the existence of such a sequence can be shown as in Lemma 3). Since each φ_l is embedded in $SO(4)$ and $SO(4)$ is compact, every sequence has a convergent subsequence. If $(\varphi_l)_{l=1}^\infty$ were not convergent, then there would exist two subsequences $(\varphi_{m_l})_{l=1}^\infty$ and $(\varphi_{j_l})_{l=1}^\infty$, with $\tilde{\varphi_1} := \lim_{l\to\infty} \varphi_{m_l} \neq \lim_{l\to\infty} \varphi_{j_l} =: \tilde{\varphi_2}$. Using the assumptions of Proposition 3 on the corresponding equators $S^2(w_{m_l})$, $S^2(w_{j_l})$, I have

$$f \circ \varphi_{w_{m_l}}(\theta_{m_l}) = g(\theta_{m_l}), \qquad f \circ \varphi_{w_{j_l}}(\theta_{j_l}) = g(\theta_{j_l}).$$

Passing to the limit in the above equalities and using the fact that θ was an arbitrary point in $S^2(w)$, I obtain (4.26). As I saw in the proof of the previous lemma, this is impossible. This contradiction shows that the sequence $(\varphi_l)_{l=1}^\infty$ is convergent.

To show that $\mathbb{A}$ is continuous, it remains to prove that $\lim_{l\to\infty} \varphi_l = \varphi_w$. Assume that the last equality is not true, and let $\lim_{l\to\infty} \varphi_l = \tilde{\varphi_1} \neq \varphi_w$. Then I have (4.26) with $\tilde{\varphi_2} = \varphi_w$, which is impossible (as seen before). Thus, $\mathbb{A}$ is continuous.

Consider now the map $\mathbb{B}$ assigning to each $w \in S^2(\zeta)$ the one-dimensional invariant subspace $\mathcal{Y}(w)$ of the corresponding rotation $\varphi_w \in SO(3, S^2(w))$, $\varphi_w(\zeta) = -\zeta$. By a similar argument as the one used for $\mathbb{A}$, the map $\mathbb{B}$ is well-defined and continuous. Observe also that $\mathcal{Y}(w) \subset (w^\perp \cap \zeta^\perp)$. Thus, assuming that $S^2(\zeta) = \Psi$, I have constructed a *continuous tangent line field* $\mathcal{Y}(w)$ on $S^2(\zeta)$. Since the Euler characteristic of the two-dimensional sphere is 2, this contradicts Theorem 14. The proof of the lemma is finished. □

4.3.2 Proof of Proposition 3.

I have $S^2(\zeta) = \Xi_0 \cup \Xi_1$ and hence I can apply Proposition 2 to finish the proof.

4.4 Proofs of Theorem 1 and Corollary 1

The main idea of the proof is more easily understood if I consider the case in which each of the bodies K and L has exactly one diameter $d(\zeta)$ (cf. [8, pages 51–52]. First, I show that the diameters of K and L must be parallel (Lemma 15), and that I can translate the bodies to make the diameters coincide and be centered at the origin (Lemma 16). Next, for any 3-dimensional projection of the translated bodies $\tilde{K}$ and $\tilde{L}$ that contains the diameter, the direct rigid motion given by the statement of Theorem 1 must fix this diameter. There are only two possibilities, namely, that the rigid motion is a rotation around the diameter, or a rotation around a line perpendicular to the diameter. In other words, I reduced matters to Proposition 3 with $f = h_{\tilde{K}}$ and $g = h_{\tilde{L}}$.

Throughout this section, the direction $\zeta \in S^3$ will be fixed.

4.4.1 **Auxiliary Lemmata**

By the conditions of Theorem 1, the projections $K|w^{\perp}$ and $L|w^{\perp}$ are directly congruent for every $w \in S^2(\zeta)$. Hence, for every $w \in S^2(\zeta)$ there exists $\chi_w \in SO(3, S^2(w))$ and $a_w \in w^{\perp}$ such that

$$\chi_w(K|w^{\perp}) = L|w^{\perp} + a_w. \tag{4.29}$$

During this part of my dissertation, I will use the well-known properties of the support function (3.2).

Let $\mathcal{A}_K \subset S^3$ be a set of directions parallel to the diameters of K. Observe that K has at most one diameter parallel to a given direction $\theta \in S^3$ (for, if a *convex* K has two parallel diameters $d_1(\theta)$, $d_2(\theta)$, then K contains a parallelogram Y with sides $d_1(\theta)$, $d_2(\theta)$, and one of the diagonals of Y is longer than $d_1(\theta)$).

My first goal is to reduce matters to rotations fixing the one-dimensional subspace containing ζ. I will do this by showing that for most of the directions $w \in S^2(\zeta)$ the projections $K|w^{\perp}$ and $L|w^{\perp}$ have exactly one diameter, parallel to ζ. I define

$$\Omega = \{w \in S^2(\zeta) : \quad (\mathcal{A}_K \cup \mathcal{A}_L) \cap S^2(w) = \{\pm\zeta\}\}. \tag{4.30}$$

In other words, $w \in \Omega$ means that $K|w^{\perp}$ and $L|w^{\perp}$ only have one diameter that is the length of $d_K(\zeta)$.

Lemma 15. *Let K and L be as in Theorem 1, and let $\zeta \in \mathcal{A}_K$. Then $\zeta \in \mathcal{A}_L$ and Ω is everywhere dense in $S^2(\zeta)$. Moreover, for every $w \in \Omega$ I have $\chi_w(\zeta) = \pm\zeta$ and $\omega_K(\zeta) = \omega_L(\zeta)$.*

Proof. Using (3.2), I see that the length of diameters $d_{K|w^\perp}(\zeta)$ and $d_K(\zeta)$ is the same for every $w \in S^2(\zeta)$. For completeness I include the proof, if $w \in S^2(\zeta)$ then

$$\max_{\{\theta \in S^2(w)\}} w_{K|w^\perp}(\theta) \leq \max_{\{\theta \in S^3\}} w_K(\theta) = w_K(\zeta) = h_K(\zeta) + h_K(-\zeta) =$$

$$= h_{K|w^\perp}(\zeta) + h_{K|w^\perp}(-\zeta) = w_{K|w^\perp}(\zeta).$$

Thus, $\max_{\{\theta \in S^2(w)\}} w_{K|w^\perp}(\theta) = w_{K|w^\perp}(\zeta)$ and $d_{K|w^\perp}(\zeta)$ is the same length as $d_K(\zeta)$.

Let ξ be any element of $\mathcal{A}_L$, and let $w \in S^2(\zeta)$ be such that $\zeta, \xi \in S^2(w)$. Since $K|w^\perp$ and $L|w^\perp$ are directly congruent, and the length of the diameters does not change under rigid motions, I have $\omega_K(\zeta) = \omega_L(\xi)$.

I will prove that Ω is everywhere dense in $S^2(\zeta)$. Suppose $\xi \in (S^2(\zeta) \setminus \Omega)$. Then there exists $\eta \in \mathcal{A}_K \cup \mathcal{A}_L$, $\eta \neq \pm\zeta$ such that $\eta, \zeta \in S^2(\xi)$. Hence $\xi \in S^2(\eta) \cap S^2(\zeta)$, and

$$(S^2(\zeta) \setminus \Omega) \subseteq \bigcup_{\{\eta \in \mathcal{A}_K \cup \mathcal{A}_L, \eta \neq \pm\zeta\}} \left(S^2(\eta) \cap S^2(\zeta)\right).$$

Since the right-hand side of the above inclusion is a countable union of one-dimensional circles, the measure of $S^2(\zeta) \setminus \Omega$ is zero. Hence, Ω is everywhere dense in $S^2(\zeta)$.

Next I show that $\zeta \in \mathcal{A}_K$ implies $\zeta \in \mathcal{A}_L$. By definition of Ω, I have $\mathcal{A}_K \cap S^2(w) \subseteq (\mathcal{A}_K \cup \mathcal{A}_L) \cap S^2(w) = \{\pm\zeta\}$ for every $w \in \Omega$ and since $\zeta \in \mathcal{A}_K$, I have $\mathcal{A}_K \cap S^2(w) = \{\pm\zeta\}$. If $\mathcal{A}_L \cap S^2(w) = \emptyset$, then $L|w^\perp$ has no diameter but $\chi_w(K|w^\perp)$ does, where χ_w is as in (4.29). This contradicts the fact that $K|w^\perp$ and $L|w^\perp$ are directly congruent. Thus, $\mathcal{A}_L \cap S^2(w) = \{\pm\zeta\}$ and hence $\zeta \in \mathcal{A}_L$.

Finally, assume that for some $w \in \Omega$ I have $\chi_w(\zeta) \neq \pm\zeta$. Then $\chi_w(K|w^\perp)$ has a diameter in a direction $\eta \neq \pm\zeta$. Since $\chi_w(K|w^\perp)$ and $L|w^\perp$ are translations of each other, $L|w^\perp$ must have a diameter parallel to η, which is impossible. Hence for every, $w \in \Omega$ I have $\chi_w(\zeta) = \pm\zeta$, and $\omega_K(\zeta) = \omega_{K|w^\perp}(\zeta) = \omega_{\chi_w(K|w^\perp)}(\zeta) = \omega_{L|w^\perp}(\zeta) = \omega_L(\zeta)$. The result follows. $\square$

Remark 4. *The previous lemma remains valid if, instead of the condition about countability of the diameters of the bodies, one assumes that, say, the sets of diameters of K and L are countable unions of large circles containing ζ. The only fact that was used in the proof is that the set of the directions $w \in S^2(\zeta)$, such that $d_K(\zeta)$ and $d_L(\zeta)$ are the only diameters of the projections $K|w^\perp$ and $L|w^\perp$, is dense in $S^2(\zeta)$.*

My next goal is to "separate" translations from rotations. I translate the bodies K and L by vectors a_K, $a_L \in \mathbb{R}^4$, to obtain $\tilde{K} = K + a_K$ and $\tilde{L} = L + a_L$ such that their diameters $d_{\tilde{K}}(\zeta)$ and $d_{\tilde{L}}(\zeta)$ coincide and are centered at the origin.

Lemma 16. *Let χ_w be the rotation given by (4.29), and let $w \in \Omega$. Then the function $\varphi_w := (\chi_w)^t$ verifies $\varphi_w(\zeta) = \pm\zeta$ and*

$$h_{\tilde{K}} \circ \varphi_w(\theta) = h_{\tilde{L}}(\theta) \qquad \forall \theta \in S^2(w). \tag{4.31}$$

Proof. Define $b_w = \chi_w(a_K|w^\perp) - a_L|w^\perp + a_w$, where $a_K|w^\perp$, $a_L|w^\perp$ are projections of vectors a_K, a_L, onto $w^\perp$. Then (4.29) holds with $\tilde{K}$ and $\tilde{L}$ instead of K and L, and b_w instead of a_w. This can be seen by using properties of projections and the fact that rotations are isometries. I claim at first that $b_w = 0$ for all $w \in \Omega$. In other words,

$$\chi_w(\tilde{K}|w^\perp) = \tilde{L}|w^\perp. \tag{4.32}$$

Indeed, using the definition of $\tilde{K}$ and $\tilde{L}$, and Lemma 15, for every $w \in \Omega \subset S^2(\zeta)$ I have

$$d_{\tilde{K}|w^\perp}(\zeta) = d_{\tilde{K}}(\zeta) = d_{\tilde{L}}(\zeta) = d_{\tilde{L}|w^\perp}(\zeta)$$

and

$$\chi_w(d_{\tilde{K}}(\zeta)) = d_{\tilde{K}}(\zeta).$$

from which it follows that

$$d_{\tilde{K}|w^\perp}(\zeta) = \chi_w(d_{\tilde{K}|w^\perp}(\zeta)) = d_{\tilde{L}|w^\perp}(\zeta) + b_w = d_{\tilde{K}|w^\perp}(\zeta) + b_w.$$

Thus, $b_w = 0$ and (4.32) holds for every $w \in \Omega$. Then,

$$h_{\chi_w(\tilde{K}|w^\perp)}(x) = h_{\tilde{L}|w^\perp}(x) \qquad \forall x \in w^\perp,$$

together with (3.2) gives us the desired conclusion. □

I define Ξ and Ψ similarly as before,

$$\Xi = \{w \in S^2(\zeta) : \quad (4.31) \quad \text{holds with} \quad \varphi_w(\zeta) = \zeta\} \tag{4.33}$$

and

$$\Psi = \{w \in S^2(\zeta) : \quad (4.31) \quad \text{holds with} \quad \varphi_w(\zeta) = -\zeta\}. \tag{4.34}$$

By Lemma 16 we have $\Omega \subset (\Xi \cup \Psi)$, hence $\Xi \cup \Psi \neq \emptyset$.

I claim that $S^2(\zeta) = \Xi$. Indeed, by Lemma 10 with $f = h_{\tilde{K}}$ and $g = h_{\tilde{L}}$ I see that the sets Ξ and Ψ are closed. Since $\Xi \cup \Psi$ is closed, and the set $\Omega \subset (\Xi \cup \Psi)$ is everywhere dense in $S^2(\zeta)$ by Lemma 15, I conclude that $\Xi \cup \Psi = S^2(\zeta)$. Thus, using the fact that (4.31) and (4.32) are equivalent, I have reduced the matters to Proposition 3.

4.4.2 Proof of Theorem 1

By Lemma 14, for every $w \in S^2(\zeta)$ there exists either a trivial rotation, or a rotation by the angle π, $\varphi_w^\pi \in SO(3, S^2(w))$, $\varphi_w^\pi(\zeta) = \zeta$, such that (4.31) holds. Applying Proposition 3 with $f = h_{\tilde{K}}$ and $g = h_{\tilde{L}}$ I obtain that either $h_{\tilde{K}} = h_{\tilde{L}}$ on S^3 or $h_{\tilde{K}}(\theta) = h_{\tilde{L}}(\mathcal{U}\theta)$ for every $\theta \in S^3$, where $\mathcal{U} \in O(4)$ is the orthogonal transformation satisfying $\mathcal{U}|_{S^2(\zeta)} = -I$, and $\mathcal{U}(\zeta) = \zeta$. Letting $\mathcal{O} = \mathcal{U}^t$, it follows from (3.2) that $h_{\tilde{K}}(\mathcal{U}\theta) = h_{\mathcal{O}\tilde{K}}(\theta)$ for every $\theta \in S^3$, and therefore either $K + a_K = L + a_L$ or $K + a_K = \mathcal{O}L + \mathcal{O}(a_L)$. This proves the first part of the theorem.

Assume, in addition, that the ground projections $K|\zeta^\perp$, $L|\zeta^\perp$, are direct rigid motions of each other. Then, there exists $\chi_\zeta \in SO(3, S^2(\zeta))$ and $a_\zeta \in \zeta^\perp$ such that

$$\chi_\zeta(K|\zeta^\perp) = L|\zeta^\perp + a_\zeta.$$

If $K = \mathcal{O}L + b$ holds, then I have

$$K|\zeta^\perp = (\mathcal{O}L)|\zeta^\perp + b|\zeta^\perp = -L|\zeta^\perp + b|\zeta^\perp.$$

The last two equations imply that $\chi_\zeta(K|\zeta^\perp) - a_\zeta = -K|\zeta^\perp + b|\zeta^\perp$, and $K|\zeta^\perp$ has a rigid motion symmetry, which is impossible by my assumptions. Thus, I conclude that $K = L + b$ and the proof of Theorem 2 is finished. □

4.4.3 **Proof of Corollary 1**

First I note that if $d_K(\zeta)$ exists then $d_L(\zeta)$ does as well. The argument is similar to the one in Lemma 15. Indeed, in this case I define

$$\Omega = \{H \in \mathcal{G}(n, 3) : \zeta \in H \text{ and } (\mathcal{A}_K \cup \mathcal{A}_L) \cap H = \{\pm\zeta\}\}$$

where $\mathcal{A}_K$ is the same as before, *i.e.*, the directions of the diameters of K. By definition $\mathcal{A}_K \cap H \subseteq (\mathcal{A}_K \cup \mathcal{A}_L) \cap H = \{\pm\zeta\}$ for every $H \in \Omega$. Since $\zeta \in \mathcal{A}_K$ then $\mathcal{A}_K \cap H = \{\pm\zeta\}$ and hence if $\mathcal{A}_L \cap H = \emptyset$ then there are no diameters in $L|H$ which is impossible because $K|H$ and $L|H$ are directly congruent. Thus $\zeta \in \mathcal{A}_L$ and $d_L(\zeta)$ exists.

Next I translate the bodies K and L by vectors a_K, $a_L \in \mathbb{R}^n$, to obtain $\tilde{K} = K + a_K$ and $\tilde{L} = L + a_L$ such that the origin is the center of $d_{\tilde{K}}(\zeta) = d_{\tilde{L}}(\zeta)$. Let J be an arbitrary 4-dimensional subspace of $\mathbb{R}^n$, containing ζ. Observe that $\tilde{K}|J$ and $\tilde{L}|J$ satisfy the conditions of Theorem 1 with $\tilde{K}|J$ and $\tilde{L}|J$ instead of K and L. By Theorem 1 I have $\tilde{K}|J = \tilde{L}|J$ or $\tilde{K}|J = \mathcal{O}_J(\tilde{L}|J)$ where $\mathcal{O}_J \in O(4, J)$, $\mathcal{O}_J|_{\zeta^\perp} = -I$ and $\mathcal{O}_J(\zeta) = \zeta$. If there existed two different 4-dimensional subspaces J_1 and J_2, such that $\tilde{K}|J_1 = \tilde{L}|J_1$ and $\tilde{K}|J_2 = \mathcal{O}_{J_2}(\tilde{L}|J_2)$, then $\tilde{L}$ would have a 3-dimensional projection with a (ζ, π)-rotational symmetry. Indeed, assume that $J_1 \cap J_2$ is a 3-dimensional subspace. Then,

$$\tilde{L}|(J_1 \cap J_2) = (\tilde{L}|J_1)|(J_1 \cap J_2) = (\tilde{K}|J_1)|(J_1 \cap J_2) = (\tilde{K}|J_2)|(J_1 \cap J_2)$$

$$= (\mathcal{O}_{J_2}(\tilde{L})|J_2)|(J_1 \cap J_2) = \mathcal{O}_{J_2}|_{J_1}(\tilde{L}|(J_1 \cap J_2)),$$

and $\tilde{L}|(J_1 \cap J_2)$ has a (ζ, π)-rotational symmetry, contradicting the assumptions of the corollary. Hence, either $\tilde{K}|J = \tilde{L}|J$ for every J, or $\tilde{K}|J = \mathcal{O}_J(\tilde{L}|J)$ for every J. If I am in the second case, let $\mathcal{O} \in O(n)$ such that $\mathcal{O}|_{\zeta^\perp} = -I$ and $\mathcal{O}(\zeta) = \zeta$. Then I have that $\mathcal{O}|_J = \mathcal{O}_J$. Since J was arbitrary, the projections of $\tilde{K}$ and $\tilde{L}$ onto all two-dimensional subspaces containing ζ coincide or are reflections of each other (with respect to the line containing ζ). Using Theorem 8 I have $\tilde{K} = \tilde{L}$ or $\tilde{K} = \mathcal{O}\tilde{L}$. Thus, $K = L + a_L - a_K$ or $K = \mathcal{O}L + \mathcal{O}(a_L) - a_K$.

Now assume that the dimension of $J_1 \cap J_2$ is 2. In this case, let $\{\zeta, v_1, v_2, v_3\}$ be an orthonormal basis of J_1, and $\{\zeta, v_1, v_2', v_3'\}$ be an orthonormal basis of J_2. Define J_0 to be the 4 dimensional subspace with basis $\{\zeta, v_1, v_2, v_2'\}$. Then, both $J_1 \cap J_0$ and $J_2 \cap J_0$ have dimension 3, and the above argument can be used. A similar argument can be used if the dimension $J_1 \cap J_2$ is 1.

Finally, assume that, in addition, the "ground" projections $K|G$, $L|G$ onto all 3-dimensional subspaces G of $\zeta^\perp$, are directly congruent and have no rigid motion symmetries, then, using Theorem 1, I see that $\tilde{K}|J = \tilde{L}|J$ for an arbitrary 4-dimensional subspace J. Hence, the projections of $\tilde{K}$ and $\tilde{L}$ onto all two-dimensional subspaces containing ζ coincide. Using Theorem 8 I have $\tilde{K} = \tilde{L}$. Thus, $K + a_K = L + a_L$ and the corollary is proved. □

4.5 Proofs of Theorem 3 and Corollary 2

The proofs are slightly different from the ones about projections. I recall that I consider star-shaped bodies with respect to the origin. The direction $\zeta \in S^3$ will be fixed through the proof. By the conditions of Theorem 3, the sections $K \cap w^\perp$ and $L \cap w^\perp$ are directly congruent for every $w \in S^2(\zeta)$. Hence, for every $w \in S^2(\zeta)$ there exists $\chi_w \in SO(3, S^2(w))$ and $a_w \in w^\perp$ such that

$$\chi_w(K \cap w^\perp) = (L \cap w^\perp) + a_w. \tag{4.35}$$

Let $l(\zeta)$ denote the one-dimensional subspace containing ζ. As in Section 4.4, I use the notation $\mathcal{A}_K \subset S^3$ for the set of directions that are parallel to the diameters of K. I consider the set Ω^r, which is defined in the same way as Ω, by (4.30), with K and L being star-shaped. Here I choose the notation Ω^r to stress that I am concerned about sections and that instead of considering the support function I will be considering the radial function, hence the superscript r. I will also use the notation $v_K(\zeta) = \rho_K(\zeta) + \rho_K(-\zeta)$.

4.5.1 Auxiliary Lemmata

My first goal is to reduce matters to rotations leaving $l(\zeta)$ fixed. I will do this by showing that for most of the directions $w \in S^2(\zeta)$, the sections $K \cap w^\perp$ and $L \cap w^\perp$ have exactly one diameter contained in $l(\zeta)$.

For this part of my dissertation I will use the well-known properties of the radial function, see (3.3).

Lemma 17. *Let K and L be as in Theorem 3. Then L has a diameter $d_L(\zeta) \subset l(\zeta)$, and Ω^r is everywhere dense in $S^2(\zeta)$. Moreover, for every $w \in \Omega^r$ I have $\chi_w(\zeta) = \pm\zeta$ and $v_K(\zeta) = v_L(\zeta)$.*

Proof. The proof that Ω^r is everywhere dense in $S^2(\zeta)$ is exactly as the one of Lemma 15 with Ω^r instead of Ω.

I will show that $\zeta \in \mathcal{A}_K$ implies $\zeta \in \mathcal{A}_L$. By definition of Ω^r, I have $\mathcal{A}_K \cap S^2(w) = \{\pm\zeta\}$ for every $w \in \Omega^r$. If $\mathcal{A}_L \cap S^2(w) = \emptyset$, then $L \cap w^\perp$ has no diameters that are the length of $d_K(\zeta)$ and $d_K(\zeta) \subset K \cap w^\perp$. This contradicts the fact that $K \cap w^\perp$ and $L \cap w^\perp$ are directly congruent. Thus, $\mathcal{A}_L \cap S^2(w) = \{\pm\zeta\}$.

Now I show that there exists $d_L(\zeta) \subset l(\zeta)$. Assume that this is not true. Then, for each diameter $d_L(\zeta)$ parallel to ζ, the linear subspace $\mathrm{span}(d_L(\zeta))$ is two dimensional. Let $\mathcal{R}(\zeta)$ be the union of all such two-dimensional subspaces, which is a countable union by the conditions of Theorem 3. Since $\mathcal{A}_L$ is also countable, there exists $w \in S^2(\zeta)$ such that $w^\perp \cap \mathcal{R}(\zeta) = l(\zeta)$ and $w^\perp$ does not contain any direction $\eta \neq \zeta$ that is parallel to a diameter of L. But then L does not have a diameter in $w^\perp$, while K does. This contradiction shows that there exists $d_L(\zeta) \subset l(\zeta)$.

Finally, assume that for some $w \in \Omega^r$ I have $\chi_w(\zeta) \neq \pm\zeta$. Then $\chi_w(K \cap w^\perp)$ has a diameter in a direction $\eta \neq \pm\zeta$. Since $\chi_w(K \cap w^\perp)$ and $L \cap w^\perp$ are translations of each other, $L \cap w^\perp$ must have a diameter parallel to η, which is impossible by the definition of Ω^r. Hence, for all $w \in \Omega^r$ I have $\chi_w(\zeta) = \pm\zeta$, and $v_K(\zeta) = v_L(\zeta)$, since both $d_K(\zeta)$, $d_L(\zeta)$ are subsets of $l(\zeta)$. The result follows. □

My next goal is to separate translations from rotations. I translate the bodies K and L by the vectors a_K, $a_L \in \mathbb{R}^4$, which are parallel to ζ, to obtain $\tilde{K} = K + a_K$ and $\tilde{L} = L + a_L$, with $d_{\tilde{K}}(\zeta) = d_{\tilde{L}}(\zeta)$ and the origin at the center of these diameters.

Lemma 18. *For every $w \in \Omega^r$ there exists $\varphi_w = \chi_w^{-1} \in SO(3, S^2(w))$, $\varphi_w(\zeta) = \pm\zeta$, such that*

$$\rho_{\tilde{K}} \circ \varphi_w(\theta) = \rho_{\tilde{L}}(\theta) \qquad \forall \theta \in S^2(w). \tag{4.36}$$

Proof. Define $b_w = \chi_w(a_K) - a_L + a_w$. Then (4.35) holds with $\tilde{K}$ and $\tilde{L}$ instead of K and L, and b_w instead of a_w using properties of sections. I first claim that $b_w = 0$ for all $w \in \Omega^r$. In other words, for all $w \in \Omega^r$ I have

$$\chi_w(\tilde{K} \cap w^\perp) = \tilde{L} \cap w^\perp \tag{4.37}$$

for some $\chi_w \in SO(3, S^2(w))$, $\chi_w(\zeta) = \pm\zeta$. Indeed, using the definition of $\tilde{K}$ and $\tilde{L}$, and

Lemma 17, for every $w \in \Omega^r \subset S^2(\zeta)$ I have

$$d_{\tilde{K}\cap w^\perp}(\zeta) = d_{\tilde{K}}(\zeta) = d_{\tilde{L}}(\zeta) = d_{\tilde{L}\cap w^\perp}(\zeta)$$

and

$$\chi_w(d_{\tilde{K}}(\zeta)) = d_{\tilde{K}}(\zeta),$$

where $d_{\tilde{K}}(\zeta)$ and $d_{\tilde{L}}(\zeta)$ are contained in $l(\zeta)$. Therefore,

$$d_{\tilde{K}\cap w^\perp}(\zeta) = \chi_w(d_{\tilde{K}\cap w^\perp}(\zeta)) = d_{\tilde{L}\cap w^\perp}(\zeta) + b_w = d_{\tilde{K}\cap w^\perp}(\zeta) + b_w.$$

Thus, $b_w = 0$ and (4.37) holds. Then, $\rho_{\chi_w(\tilde{K}\cap w^\perp)}(x) = \rho_{\tilde{L}\cap w^\perp}(x)$ for all $x \in w^\perp$. In particular, I have that $\rho_{\chi_w(K\cap w^\perp)}(\theta) = \rho_{\tilde{L}\cap w^\perp}(\theta)$ for all $\theta \in S^2(w)$. I now use properties of the radial function, see (3.3), to conclude the proof. □

Consider the sets

$$\Xi^r = \{w \in S^2(\zeta) : \quad (4.36) \quad \text{holds with} \quad \varphi_w(\zeta) = \zeta\}$$

and

$$\Psi^r = \{w \in S^2(\zeta) : \quad (4.36) \quad \text{holds with} \quad \varphi_w(\zeta) = -\zeta\}.$$

By Lemma 17, I have $\Omega^r \subset (\Xi^r \cup \Psi^r)$, hence $(\Xi^r \cup \Psi^r) \neq \emptyset$. Similarly to the arguments in the proof of Lemma 11, it can be shown that the sets Ξ^r and Ψ^r remain the same if, instead of the pair $\rho_{\tilde{K}}$, $\rho_{\tilde{L}}$, I take $(\rho_{\tilde{K}})_{\mathcal{O},o}$, $(\rho_{\tilde{L}})_{\mathcal{O},o}$.

Let $\varphi_w^{\alpha\pi}$ be the rotation of the sphere $S^2(w)$ by the angle $\alpha\pi$ around ζ. For any $w \in S^2(\zeta)$, and any $\alpha \in \mathbb{R}$, I consider the sets Ξ^r_α, which are defined by (4.12) with $f = (\rho_{\tilde{K}})_{\mathcal{O},o}$ and $g = (\rho_{\tilde{L}})_{\mathcal{O},o}$.

4.5.2 **Proof of Theorem 3**

By Lemma 17, I have $\Omega^r \subset (\Xi^r \cup \Psi^r)$, hence $\Xi^r \cup \Psi^r = S^2(\zeta)$. Now I can apply Proposition 3 and Proposition 2 (with $f = \rho_{\tilde{K}}$, $g = \rho_{\tilde{L}}$, and $\Xi = \Xi^r$, $\Psi = \Psi^r$) obtaining that either $\rho_{\tilde{K}} = \rho_{\tilde{L}}$ on S^3, or $\rho_{\tilde{K}}(\theta) = \rho_{\tilde{L}}(\mathcal{U}\theta)$ for all $\theta \in S^3$. Here $\mathcal{U} \in O(4)$ is an orthogonal transformation, satisfying $\mathcal{U}|_{S^2(\zeta)} = -I$ and $\mathcal{U}(\zeta) = \zeta$. In the first case, $\tilde{K} = \tilde{L}$, and in the second, $\tilde{K} = \mathcal{O}\tilde{L}$, where $\mathcal{O} = \mathcal{U}^{-1}$. Thus, $K = L + a_L - a_K$, or $K = \mathcal{O}L + \mathcal{O}(a_L) - a_K$. This finishes the proof of Theorem 3. □

4.5.3 Proof of Corollary 2

The proof is similar to the one of Corollary 1. One has only to consider the sections $\tilde{K} \cap J$, $\tilde{L} \cap J$, instead of the projections $\tilde{K}|J$, $\tilde{L}|J$, and Theorem 9, instead of Theorem 8. □

4.6 Congruent Projections in $\mathbb{R}^3$

Now I would like to discuss the specific statement for $\mathbb{R}^3$, and give a rigorous proof of the statement. For the heuristics of the statement of this theorem, see Section 2.2.2.

Theorem 20. *Let K and L be two convex bodies in $\mathbb{R}^3$ having countably many diameters. Assume that there exists a diameter $d_K(\xi_0)$, such that the "side" projections $K|w^\perp$, $L|w^\perp$ onto all subspaces $w^\perp$ containing ξ_0 are directly congruent. Assume also that these projections are not centrally symmetric. Then $K = \pm L + b$ for some $b \in \mathbb{R}^3$.*

I refer to the discussion in Section 4.1 for the idea of this proof. Notice that in $\mathbb{R}^2$, centrally symmetric is equivalent to having a π-rotational symmetry.

Proof. I first observe that K and L have at most one diameter parallel to a given direction (for the same reasoning as in Section 4.4.1).

Next I show that L must also have a diameter in the direction ξ_0. If this is not the case, then there exists a plane H that contains ξ_0 and none of the directions of the diameters of L. Then $K|H$ contains a diameter of K (namely $d_K(\xi_0)$) and $L|H$ does not contain a diameter of L, hence $K|H$ can never be rotated into $L|H$. This contradiction shows that L has a diameter $d_L(\xi_0)$.

Let $\{\xi_0, \xi_1, \ldots\}$ be the countable set of directions of the diameters of K and the diameters of L, and H_i be the plane that contains the directions ξ_0 and ξ_i. Next consider the set

$$\Lambda = \{w \in S^1(\xi_0) : w \notin H_i \text{ for all } i\}.$$

I claim Λ is everywhere dense in $S^1(\xi_0)$. The proof of this is hidden in the proof of Theorem 5 in Section 5.2.1. I prove this by showing $S^1(\xi_0) \setminus \Lambda$ is nowhere dense. Let $w \in (S^1(\xi_0) \setminus \Lambda)$. Then $w \in H_k$ for some k. Then there are two cases that can happen. In the first case there exists a neighborhood of planes that contains ξ_0 around H_k such that this neighborhood does not contain any other H_i for $i \neq k$. In the second case every neighborhood of planes that contain ξ_0 around H_k there is some other H_ℓ. If the first case occurs then I am done. If the second occurs then for every neighborhood there is some plane P_ℓ that is not H_i for every i. Thus, there exists $w_\ell \in P_\ell \cap \Lambda$, and in this case I am done as well. Therefore Λ is everywhere dense in $S^1(\xi_0)$.

Remark 5. *For the proof of Theorem 20, all I need is Λ is everywhere dense. Hence as long as this happens, I could have more than countably many diameters.*

Next I "separate" translations and rotations. I translate K and L so that their diameters $d_K(\zeta_0), d_L(\zeta_0)$ are equal and are centered at the origin. I can do this for the same reasons as in Lemma 16 in Section 4.4. Name the translated bodies $\tilde{K}$ and $\tilde{L}$. In addition, for the same reasoning as in Lemma 16 in Section 4.4, $\varphi_w(\tilde{K}|w^\perp) = \tilde{L}|w^\perp$, *i.e.* there is no translation.

Denote H'_w to be the plane that contains w and ξ_0. Now for all $w \in \Lambda$ the only diameter of $\tilde{K}|H'_w$ is $d_{\tilde{K}}(\xi_0)$, and hence the direct rigid motion must fix this diameter. The only two rotations that do this are the identity and a rotation about the origin by π. Notice in the first case, $\tilde{K}|H'_w = \tilde{L}|H'_w$ and in the second, $\tilde{K}|H'_w = -\tilde{L}|H'_w$ since a rotation by π about the origin in 2 dimensions is the same as a reflection about the origin. Define

$$\Xi = \{w \in S^1(\xi_0) : \tilde{K}|H'_w = \tilde{L}|H'_w\}$$

and

$$\Psi = \{w \in S^1(\xi_0) : \tilde{K}|H'_w = -\tilde{L}|H'_w\}.$$

I note that Ξ and Ψ are closed. The proof of this is the standard argument when proving a set is closed and can be seen in more detail in Lemma 10 in Section 4.4. From the definitions of Ξ and Ψ I have $\Lambda \subseteq \Xi \cup \Psi \subseteq S^1(\xi_0)$, and since $\Xi \cup \Psi$ is closed and Λ is everywhere dense, I have $\Xi \cup \Psi = S^1(\xi_0)$.

Next I claim that $\Xi \cap \Psi = \emptyset$. Indeed, if this was not the case, let $w \in \Xi \cap \Psi$. Then I have

$$\tilde{L}|H'_w = \tilde{K}|H'_w = -\tilde{L}|H'_w,$$

which implies that $\tilde{L}|H'_w$ is origin symmetric, and hence $L|H'_w$ is centrally symmetric. This contradicts my assumption, thus $\Xi \cap \Psi = \emptyset$.

Therefore, $\Xi = S^1(\xi_0)$ or $\Psi = S^1(\xi_0)$. If $\Xi = S^1(\xi_0)$ then let $\theta \in S^2$ then $\theta \in H'_w$ for some $w \in S^1(\xi_0)$. Hence

$$h_{\tilde{K}}(\theta) = h_{\tilde{K}|H'_w}(\theta) = h_{\tilde{L}|H'_w}(\theta) = h_{\tilde{L}}(\theta).$$

Thus, $\tilde{K} = \tilde{L}$.

If $\Psi = S^1(\xi_0)$, let $\theta \in S^2$ then $\theta \in H'_w$ for some $w \in S^1(\xi_0)$, note also that $-\theta \in H'_w$. Hence

$$h_{\tilde{K}}(\theta) = h_{\tilde{K}|H'_w}(\theta) = h_{-\tilde{L}|H'_w}(\theta) = h_{(-\tilde{L})|H'_w}(\theta) = h_{-\tilde{L}}(\theta).$$

Thus, $\tilde{K} = -\tilde{L}$. Therefore, from both cases, $K = \pm L + b$. □

This concludes the major results of Chapter 4.

4.7 Polytopes Without 3-dimensional Projections Symmetries

For this section, I prove that the class of polytopes whose 3-dimensional projections do not have rigid motion symmetries is dense in the set of all convex bodies. This implies that the class of bodies considered in Corollary 1 is dense in the set of all convex bodies. I will do this by showing that any convex body can be approximated by polytopes without 3-dimensional projections that have rigid motion symmetries. The idea of the proof of Proposition 4 is due to Mark Rudelson.

Recall Definition 1 where $\delta(K, P)$ is the Hausdorff distance between the convex bodies K and P in $\mathbb{R}^n$, $n \geq 2$,

$$\delta(K, P) = \max_{\theta \in S^{n-1}} |h_K(\theta) - h_P(\theta)|.$$

The goal is to prove

Proposition 4. *Any convex body K in $\mathbb{R}^n$, $n \geq 4$, can be approximated in the Hausdorff metric, by polytopes without 3-dimensional projections that have rigid motion symmetries.*

Since polytopes have finitely many diameters, Proposition 4 shows that the set of bodies satisfying the conditions of Corollary 1 contains the set of polytopes which is dense in the set of all convex bodies.

Proposition 4 is not a new result (see [8, page 48]). An abstract geometric proof of this fact can be given [22]. However, for the completeness of this dissertation and the convenience of the reader, I include an elementary proof. The idea is, assuming that K has positive Gaussian curvature, to observe first that K can be approximated by polytopes whose 3-dimensional projections have many vertices. If a polytope has a 3-dimensional projection with a rigid motion symmetry, then I use (4.38) to form a system of linear equations, and use the implicit function theorem to prove that these polytopes form a "manifold" of small dimension.

4.7.1 Auxiliary Results

I will need the following two lemmata. Let $C^2_+(\mathbb{R}^n)$ be the set of convex bodies in $\mathbb{R}^n$ having a positive Gaussian curvature. It is well-known, that any convex body can be approximated in the Hausdorff metric by convex bodies $K \in C^2_+(\mathbb{R}^n)$ [29, pages 158-160]. Hence, I can assume that $K \in C^2_+(\mathbb{R}^n)$.

Recall, Theorem 7 from Chapter 3. In particular, the closer K and P are in terms of Hausdorff distance, the more vertices P will have.

The next known statement will be used to show that the same is true for all 3-dimensional projections of K.

Lemma 19. *Let $K \in C^2_+(\mathbb{R}^n)$, $n \geq 4$. Then $K|H \in C^2_+(H)$, where $K|H$ is the projection of K onto $H \in \mathcal{G}(n,3)$.*

Proof. Let x be any point on the boundary of K. Changing the coordinates if necessary I can assume that x is the origin and the tangent hyperplane to K at x is the $(x_1, \dots, x_{n-1})$-hyperplane. From the fact that K is smooth, I can assume that for a small enough neighborhood around the origin K can be described as part of an ellipsoid. Using the Taylor decomposition of the boundary of K near the origin I have

$$x_n = f(x_1, \dots, x_{n-1}) = k_1 x_1^2 + \dots + k_{n-1} x_{n-1}^2 + o(x),$$

where $k_j > 0$, $j = 1, \dots, n-1$, are the main curvatures of the boundary at x (see Section 3.3), and $\frac{o(x)}{|x|} \to 0$ as $|x| \to 0$. Consider the ball B,

$$B = \{x \in \mathbb{R}^n : x_1^2 + \dots + x_{n-1}^2 + (x_n - \frac{1}{k})^2 = \frac{1}{k^2}\}, \qquad k = \min_{j=1,\dots,n-1} k_j.$$

Since the main curvatures are the reciprocals of the main radii of curvature I see that in a small enough neighborhood W of the origin, $K \cap W$ is contained in B. Let $u \in S^{n-1}$ be such that $u_n = 0$, *i.e.*, u is the unit vector contained in the $(x_1, \dots, x_{n-1})$-hyperplane, and let $H_u \in \mathcal{G}(n,3)$ be contained in the $(x_1, \dots, x_{n-1})$-hyperplane, and orthogonal to u. Observe that the boundary of the projection $(K \cap W)|H_u$ is contained in the 3-dimensional ball of radius $\frac{1}{k}$, which is the projection of B. Since the main curvatures of the boundary of $(K \cap W)|H_u$ are the reciprocals of the radii of curvature, I see that the main curvatures of $(K \cap W)|H_u$ at the origin are positive. Since x was an arbitrary point on the boundary of K, the result follows. □

To formulate my last auxiliary lemma, I recall the definition of the Hausdorff dimension, Definition 6 in Chapter 3. Given any subset E of $\mathbb{R}^n$ and $\alpha \geq 0$, the *exterior α-dimensional Hausdorff measure* of E is defined by $m^*_\alpha(E) = \lim_{\delta \to 0^+} \inf \mathcal{H}^\delta_\alpha(E)$, where

$$\mathcal{H}^\delta_\alpha(E) := \inf\{ \sum_{k=1}^{\infty} (\operatorname{diam} F_k)^\alpha : \ E \subset \bigcup_{k=1}^{\infty} F_k, \quad \operatorname{diam} F_k \leq \delta\},$$

and $\operatorname{diam}(S) = \sup_{x,y \in S} |x-y|$ stands for the length of the diameter of S. The *Hausdorff dimension* of E is $dim_H(E) = \inf\{\alpha > 0 : \ m^*_\alpha(E) = 0\}$.

Lemma 20. *Let $\mathcal{M}$ be a smooth manifold of dimension k in $\mathbb{R}^m$, $m \geq 3$, $k \leq m-2$, and let $\mathcal{M}|H$ be the orthogonal projection of $\mathcal{M}$ onto a l-dimensional subspace H, $k < l \leq m-1$. Then the Hausdorff dimension of $\mathcal{M}|H$ does not exceed the dimension of $\mathcal{M}$.*

Proof. Let $\delta > 0$ and let $\bigcup_{j=1}^{\infty} F_j$, $\operatorname{diam}(F_j) \leq \delta$, be a covering of $\mathcal{M}$. Since $\bigcup_{j=1}^{\infty} (F_j|H)$ is a covering of $\mathcal{M}|H$, and $\operatorname{diam}(F_j|H) \leq \operatorname{diam}(F_j) \leq \delta$, I see that

$$\sum_{j=1}^{\infty} (\operatorname{diam}(F_j|H))^\alpha \leq \sum_{j=1}^{\infty} (\operatorname{diam}(F_j))^\alpha,$$

and $m^*_\alpha(\mathcal{M}|H) \leq m^*_\alpha(\mathcal{M})$. The result follows. □

4.7.2 Proof of Proposition 4

To prove the proposition it is enough to show that each P^*_v, having sufficiently many vertices, can be approximated by polytopes without any 3-dimensional projection rigid motion symmetries. I will do this by proving that the set of polytopes having v vertices *with* 3-dimensional projection rigid motion symmetries is a nowhere dense set contained in the set of all polytopes having v vertices.

Define $\mathcal{P}_v$ to be the set of polytopes in $\mathbb{R}^n$, $n \geq 4$, with v vertices $p_1, p_2, \dots, p_v$. I see that $\mathcal{P}_v$ can be parametrized by points from $\mathbb{R}^{nv}$, with $p_j = (p_{1j}, \dots, p_{nj}) \in \mathbb{R}^n$, $j = 1, \dots, v$, and I can identify $\mathcal{P}_v$ with an open domain in $\mathbb{R}^{nv}$.

I denote by Π_v the set of polytopes in $\mathcal{P}_v$ that have a 3-dimensional projection with rigid motion symmetries. My goal is to show that Π_v is nowhere dense in $\mathcal{P}_v$, provided that v is large enough. I can partition Π_v into equivalence classes such that two polytopes are in the same class if there is a rigid motion in $\mathbb{R}^n$ taking one to the other. Letting H_0 be the (x_1, x_2, x_3)-plane in $\mathbb{R}^n$, each equivalence class can be represented by a polytope whose projection on H_0 has rigid motion symmetries. Let me define $\mathcal{Q}_v$ to be the set of these representatives, *i.e.*,

$$\mathcal{Q}_v = \{Q \in \mathcal{P}_v : \exists \varphi_{H_0} \in O(3, H_0), \varphi_{H_0} \neq I, \exists a_{H_0} \in \mathbb{R}^3 \text{ such that}$$

$$\varphi_{H_0}(Q|H_0) + a_{H_0} = Q|H_0\}. \tag{4.38}$$

Observe that every $P \in \Pi_v$ can be written as $P = \phi(Q) + b$ for some $\phi \in O(n)$, $Q \in \mathcal{Q}_v$, $b \in \mathbb{R}^n$, and hence can be represented as the triple $(Q, \phi, b) \in \mathcal{Q}_v \times O(n) \times \mathbb{R}^n$. Thus using (3.10),

$$\dim(\Pi_v) \leq \dim(\mathcal{Q}_v) + \dim(O(n)) + n = \dim(\mathcal{Q}_v) + \frac{n(n+1)}{2}. \tag{4.39}$$

All that remains is to find the dimension of $\mathcal{Q}_v$. Consider the set $\mathcal{M} = \mathcal{M}(\mathcal{Q}_v)$ of all triples

$$(Q, \varphi_{H_0}, a_{H_0}) \in \mathbb{R}^{nv} \times O(3, H_0) \times \mathbb{R}^3,$$

satisfying (4.38).

Let $H \in \mathcal{G}(n, 3)$. Since for every $\theta \in H \cap S^{n-1}$, I have $h_{K|H}(\theta) = h_K(\theta)$ (3.2), $K|H$ can be approximated in the Hausdorff metric by polytopes $P_v^*|H$. Let V_H be the number of vertices of $P_v^*|H$. Using Lemma 19 and Theorem 7, I can assume that

$$t_0 := \min_H V_H > 5 + \frac{n(n+1)}{2}. \tag{4.40}$$

Lemma 21. *The set $\mathcal{M}$ is manifold in $\mathbb{R}^{nv+6}$ with dimension at most $(nv+5-t_0)$, provided that v is so large that $t_0 > 5 + \frac{n(n+1)}{2}$.*

Proof. Let Q be a polytope in $\mathcal{Q}_v$ and consider its projection $Q|H_0$, which is also a polytope with t vertices, where $t \geq t_0$. I will write the assumption that $Q|H_0$ has rigid motion symmetries as a system of linear equations that equal zero precisely at the vertices of $Q|H_0$, and explicitly compute the determinant of its Jacobian matrix to show that it is nonzero. The Implicit Function Theorem (Theorem 18) will allow me to obtain the result.

Since any rigid motion maps a vertex into a vertex, an equation, similar to (4.38), can be written for the corresponding vertices $q_i|H_0$ of $Q|H_0$,

$$q_i|H_0 = \varphi_{H_0}(q_{j(i)}|H_0) + a_{H_0}, \tag{4.41}$$

where φ_{H_0} is a nonidentical orthogonal transformation whose 3×3 matrix has coordinates $(o_{l,m})_{l,m=1,2,3}$, and j is a permutation on the set $\{1, \dots t\}$, which indicates that the $j(i)$-th vertex gets mapped to the i-th vertex. As it is well known, a permutation can be written as a product of cycles. I will consider two cases: cycles of length one, and cycles of length greater than one.

Assume that the vertex $q_i|H_0$ is mapped to itself, *i.e.*, $q_i|H_0 = \varphi_{H_0}(q_i|H_0) + a_{H_0}$. Since φ_{H_0} is not the identity, given a basis e_1, e_2, e_3 of H_0, there exists $r \in \{1, 2, 3\}$ such that $\varphi_{H_0}(e_r) \neq e_r$. For this r, consider the function $F_{ri} : \mathbb{R}^{nv} \times O(3, H_0) \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$F_{ri}(x_{11}, \dots, x_{nv}, \varphi_{H_0}, a_{H_0}) = ((x_{1i}, x_{2i}, x_{3i}) - \varphi_{H_0}(x_{1i}, x_{2i}, x_{3i}) - a_{H_0})_r.$$

$$= x_{ri} - o_{r1}x_{1i} - o_{r2}x_{2i} - o_{r3}x_{3i} - (a_{H_0})_r.$$

Since the right hand side depends only on the variables x_{1i}, x_{2i}, x_{3i}, I see that $\frac{\partial F_{ri}}{\partial x_{ks}} = 0$ for all $s \neq i$ and all k, while $\frac{\partial F_{ri}}{\partial x_{ri}} \neq 0$ because $\varphi_{H_0}(e_r) \neq e_r$. Thus, this cycle forms a (1×1)-Jacobian block whose entry is not 0.

Next, suppose that the cycle is of length k and permutates the vertices $q_{i_1}, q_{i_2}, \ldots, q_{i_k}$, (for $\ell < k$, $q_{i_{\ell+1}}$ gets mapped to q_{i_ℓ} and q_{i_1} is mapped back to q_{i_k}). Consider the system of $3(k-1)$ functions $F_{rs} : \mathbb{R}^{nv} \times O(3, H_0) \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$F_{rs}(x_{11}, \ldots, x_{nv}, \varphi_{H_0}, a_{H_0}) = ((x_{1s}, x_{2s}, x_{3s}) - \varphi_{H_0}(x_{1j(s)}, x_{2j(s)}, x_{3j(s)}) - a_{H_0})_r$$

for $r = 1, 2, 3$ and for $s = i_1, i_2, \ldots, i_{k-1}$.

I will order the variables in such a way that the Jacobian block corresponding to this cycle will be upper triangular. I note that for $r = 1, 2, 3$, and $s = i_1, \ldots, i_{k-1}$, F_{rs} depends on the variables x_{rs} and $x_{kj(s)}$ for $k = 1, 2, 3$. Thus, $\frac{\partial F_{rs}}{\partial x_{ks}} = 0$ for $k \neq r$, and $\frac{\partial F_{rs}}{\partial x_{k\ell}} = 0$ for all $\ell \neq s$, $\ell \neq j(s)$ and all k. Order the Jacobian block as follows, $x_{1i_1}, x_{2i_1}, x_{3i_1}, x_{1i_2}, \ldots, x_{3i_{k-1}}$. Since $\frac{\partial F_{rs}}{\partial x_{rs}} = 1$, the diagonal entries are all 1. In addition, the variables $x_{kj(s)}$ occur after x_{rs}, so the Jacobian block is upper triangular. Therefore, the determinant of this block is equal to 1. Thus, the Jacobian of the system of equations is a block diagonal matrix with nonzero determinant.

I observe that the number of equations in this system depends on the decomposition of the permutation j into cycles. Each 1-cycle gives one equation, while each cycle of length $k > 1$ contributes $3(k-1)$ equations to the system. Hence, the smallest possible number of equations in the system is $3 + (t-2)$, which occurs if the decomposition of the permutation j into cycles contains only one two-cycle and all the rest are one-cycles. By the Implicit Function Theorem (Chapter 3, Theorem 18), I can express at least $t+1$ variables x_{rs} as functions of the coordinates of φ_{H_0}, a_{H_0} and at most $nv - (t+1)$ other variables. Since $t \geq t_0$, this shows that the dimension of the manifold $\mathcal{M}$ in $\mathbb{R}^{nv+6}$ is at most

$$(nv + \dim(O(3)) + \dim(H_0) - (t_0 + 1)) = nv + 3 + 3 - t_0 - 1 = nv + 5 - t_0.$$

□

I am now ready to prove my goal.

Lemma 22. *The set* Π_v *is nowhere dense in* $\mathcal{P}_v$.

Proof. By definition, $\mathcal{Q}_v$ is equal to the projection of $\mathcal{M}$ onto $\mathbb{R}^{nv}$ and by Lemmata 20 and 21,

$$\dim(\mathcal{Q}_v) = \dim(\mathcal{M}|\mathbb{R}^{nv}) \leq \dim(\mathcal{M}) \leq nv + 5 - t_0.$$

Hence, using (4.39), I have $\dim(\Pi_v) \leq nv + 5 - t_0 + \frac{n(n+1)}{2}$. Finally, (4.40) yields $\dim(\Pi_v) < \dim(\mathcal{P}_v) = nv$. $\square$

To complete the proof of Proposition 4, I use Theorem 7 to approximate $K \in C^2_+(\mathbb{R}^n)$ in the Hausdorff metric, by polytopes P_v^* with v so large that $t_0 > 5 + \frac{n(n+1)}{2}$. By Lemma 22, I can approximate P_v^* by polytopes without 3-dimensional projections that have rigid motion symmetries. $\square$

CHAPTER 5

On Bodies Related Via Containment of Rotated Projections or Sections

Chapter 5 is organized as follows. In Section 5.1, I present bodies that give a negative answer to Problem 3(a) and Problem 4(a) for rotations. The first counterexample is in $\mathbb{R}^3$, and consists of a cylinder C and a double cone K. I note that both the cylinder and double cone are centrally symmetric bodies, and hence, unlike the case of translations proved by Klain, Problem 3(a) for rotations does not have an affirmative answer for centrally symmetric bodies. The second example, which works in general dimension n, is given by appropriately chosen perturbations of two balls, following ideas of Kuzminykh [19] and Nazarov. However, none of these counterexamples are counterexamples to Problem 3(b) or Problem 4(b) for rotations.

In Section 5.2, I prove that the answer to Problem 4(b) for rotations has an affirmative answer (Theorem 4). However, for the case of projections, the argument only allows me to conclude the relation $vol(K^*) \geq vol(L^*)$ for the polar bodies. I obtain partial positive answers for Problem 3(a) for rotations, in $\mathbb{R}^3$, assuming a Hadwiger type additional condition on the bodies K and L (see [9], and similar to results in Chapter 4), while Problem 3(b) for rotations remains open in the general case.

5.1 Counterexamples for Problem 3(a) and Problem 4(a) for Rotations

5.1.1 Counterexample in $\mathbb{R}^3$.

My first counterexample is three-dimensional, and it is provided by two centrally symmetric convex bodies, a cylinder C and a double cone K. I show that all *sections* of the C can be rotated to fit into the corresponding sections of the body K, and that C cannot be rotated to fit inside K. Due to the relations (3.5), (3.6), and polarity (3.7), this will imply that all projections of K^* fit into the corresponding projections of C^* after a rotation, while no rotation of K^* is included in C^*. Thus, my two bodies provide at the same time counterexamples for Problems 3(a) and 4(a) for rotations. For the convenience of the reader I repeat the statement that appears in the Introduction.

Counterexample 1. *Let $C \subset \mathbb{R}^3$ be the cylinder around the z-axis, centered at the origin, with radius r and height $2r$, where $\frac{1}{2} < r \leq \sqrt{2-\sqrt{3}} = 0.5176\ldots$ Let K be the double cone obtained by rotating the triangle with vertices $(0,0,\pm 1)$ and $(1,0,0)$ around the z-axis, see Figure 5.1. Then the sections (projections) of C can be rotated to be contained in the corresponding section (projection) of K, however the cylinder C itself can never be rotated to be contained in the double cone K.*

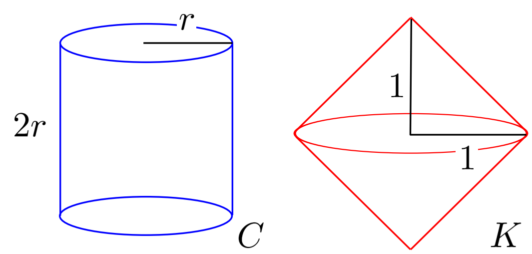

Figure 5.1: Cylinder C and double cone K.

This counterexample is interesting because both C and K are centrally symmetric, and hence, unlike the case of translations proved by Klain, Problem 3(a) for rotations does not have an affirmative answer for centrally symmetric bodies.

As mentioned previously, it will be enough to show that every section of the cylinder C by a plane passing through the origin can be rotated to be included in the corresponding plane section of K. Observe that the polar body of C is the double cone obtained by rotating the triangle with vertices $(0,0,\pm 1/r)$ and $(1/r,0,0)$ around the z-axis. The polar body of K is a cylinder with radius 1 and height 2. Hence, the dilation of C^* by a factor r is equal to K, and similarly the dilation of K^* by r is equal to C. By proving that all sections of C can be rotated to fit into the sections of K, I am in fact proving that all projections of C are included in the corresponding projections of K after a rotation. Here I present a sketch of the argument, with the detailed calculations shown in Section 5.3.

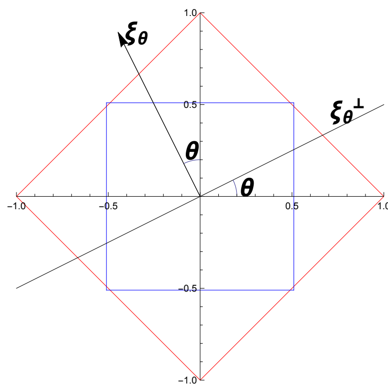

Figure 5.2: Section of the cylinder C and the double cone K through a vertical plane containing the axis of revolution.

Since C and K are centrally symmetric bodies of revolution, it is enough to study their sections by planes perpendicular to $\xi_\theta = (-\sin(\theta), 0, \cos(\theta))$, where $\theta \in [0, \pi/2]$ is the vertical angle from the axis of revolution (see Figure 5.2). The radial function of the section of the double cone K by $\xi_\theta^\perp$ is

$$\rho_{K_\theta}(u) = \frac{\sec(u)}{\sin(\theta) + \sqrt{\tan^2(u) + \cos^2(\theta)}}. \tag{5.1}$$

For the cylinder, when $\theta \in [0, \pi/4]$, the section by $\xi^\perp$ is an ellipse with semiaxes of length $r \sec\theta$ (for $u = 0$) and r (for $u = \pi/2$). Its radial function is

$$\rho_{C_\theta}(u) = \frac{r\sec(u)}{\sqrt{\tan^2(u) + \cos^2(\theta)}}. \tag{5.2}$$

On the other hand, when $\theta \in (\pi/4, \pi/2]$, the section of the cylinder looks like an ellipse with semiaxes of length $r \sec\theta$ (along the x-axis) and r (along the y-axis), that has been truncated by two vertical lines at $x = \pm r\csc(\theta)$. Its radial function is

$$\rho_{C_\theta}(u) = \begin{cases} r\sec(u)\csc(\theta) & 0 \le u \le u_0, \\ \frac{r\sec(u)}{\sqrt{\tan^2(u)+\cos^2(\theta)}} & u_0 \le u \le \pi/2, \end{cases} \tag{5.3}$$

where $u_0 = \arctan(\sqrt{\sin^2(\theta) - \cos^2(\theta)})$.

Let $\theta_0 = \arctan\left(\frac{1-r}{r}\right)$. For $\theta \in [0, \theta_0]$, the section $C \cap \xi_\theta^\perp$ is contained in $K \cap \xi_\theta^\perp$ and there is nothing to prove (see Figure 5.2). For $\theta \in (\theta_0, \pi/4]$, $C \cap \xi_\theta^\perp$ is not a subset of $K \cap \xi_\theta^\perp$. However, a rotation by $\pi/2$ of the section of the cylinder is contained in the section of the cone. Indeed, from equation (5.2) it can easily be seen that the rotation by $\pi/2$ of the section of the cylinder has radial function

$$\widetilde{\rho}_{C_\theta}(u) = \frac{r\csc(u)}{\sqrt{\cot^2(u) + \cos^2(\theta)}}. \tag{5.4}$$

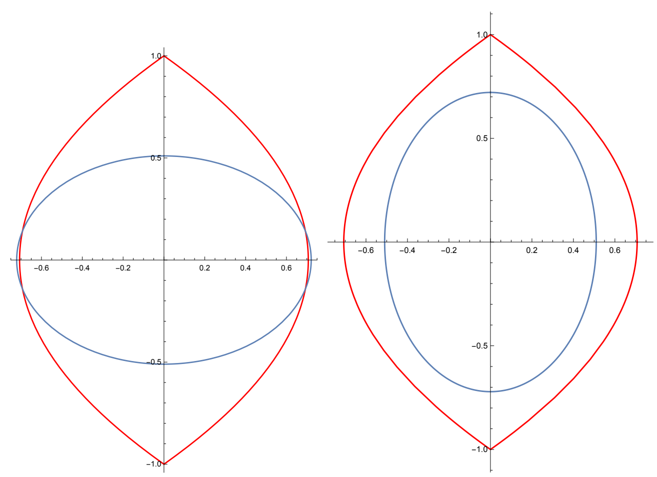

Figure 5.3: Left: For $\theta \in (\theta_0, \pi/4]$, the section of the cylinder is not a subset of the section of the cone. Right: The section of the cylinder has been rotated 90 degrees. Here $r = 0.51$, $\theta = \pi/4$.

I will prove that $\widetilde{\rho}_{C_\theta}(u) < \rho_{K_\theta}(u)$, for every $u \in [0, \pi/2]$, $\theta \in (\theta_0, \pi/4]$. The crucial observation is that for fixed u, $\widetilde{\rho}_{C_\theta}(u)$ is an increasing function of $\theta \in [0, \pi/4]$, while $\rho_{K_\theta}(u)$

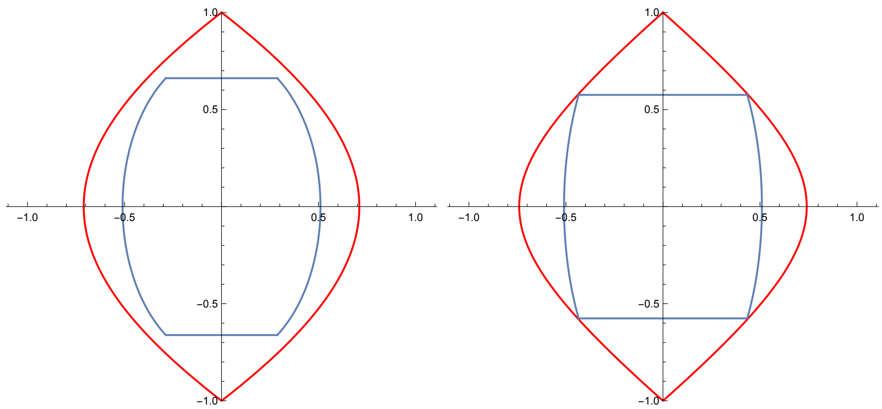

Figure 5.4: Section of the cone and the 90-degree rotation of the section of the cylinder for $r = 0.51$. For the left figure, $\theta \in (\pi/4, \theta_1)$; for the right figure $\theta = \theta_1$.

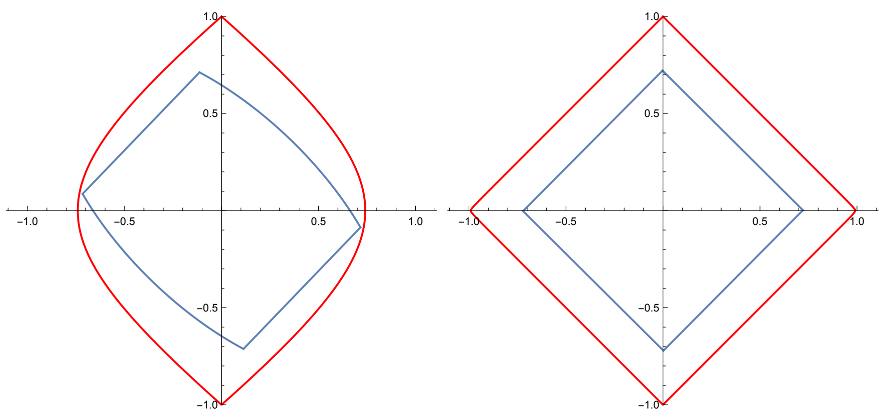

Figure 5.5: In both figures, $r = 0.51$. The left figure shows the same sections as Figure 5.4 (right), but the section of the cylinder has been rotated 45 degrees. The right figure shows the case where $\theta = \pi/2$.

is decreasing. It is enough, therefore, to show that $\widetilde{\rho}_{C_{\pi/4}}(u) < \rho_{K_{\pi/4}}(u)$ for $u \in [0, \pi/2]$. Figure 5.3 shows this situation for $r = 0.51$. The calculations are in the Section 5.3.

When $\theta \in (\pi/4, \pi/2]$, the section $C \cap \xi_\theta^\perp$ is never contained in $K \cap \xi_\theta^\perp$, but if $r \leq \sqrt{2 - \sqrt{3}}$, there exist angles $\theta_1, \theta_2 \in (\pi/4, \pi/2)$, with $\theta_2 \leq \theta_1$, such that for $\theta \in (\pi/4, \theta_1]$ a rotation by $\pi/2$ of the section of the cylinder is contained in the section of the double cone, and for $\theta \in [\theta_2, \pi/2]$ a rotation of angle u_0 of the section of the cylinder is contained in the section of the double cone. The reason why this is true is that when $\theta = \pi/4$, the $\pi/2$ rotation of the section $C \cap \xi_{\pi/4}^\perp$ is strictly contained within $K \cap \xi_{\pi/4}^\perp$, which implies the same, by continuity, for θ in some interval $(\pi/4, \theta_1]$; on the other hand, when $\theta = \pi/2$ and both sections are squares, a rotation by $\pi/4 = u_0(\pi/2)$ of the section of the cylinder is strictly included in the section of the cone, and by continuity the same is true on some interval $[\theta_2, \pi/2]$. The calculations in Section 5.3 show that for $r \in (1/2, \sqrt{2-\sqrt{3}}]$, $\theta_2 \leq \theta_1$ and hence all sections of the cylinder can be rotated to fit within the corresponding section of the cone. Figures 5.4 and 5.5 illustrate both cases. This concludes Counterexample 1.

5.1.2 Counterexample in $\mathbb{R}^n$

The idea of the next counterexample belongs Kuzminykh [19] and Nazarov. The counterexample works in all dimensions but is less intuitive. For the convenience of the reader I repeat the statement that appears in the Introduction.

Counterexample 2. *Given the unit sphere in $\mathbb{R}^n$ where $n \geq 3$, I will perturb it by adding bump functions to create two convex bodies K, L. I place the bumps on K so that they form a simplex on the surface of K, but no such simplex configuration of bumps will appear on the surface on L, see Figure 5.6. Here, every $(n-1)$-dimensional section of K (projection of L^*) can be rotated to be contained in the corresponding section of L (projection of K^*), however K itself can never be rotated to be contained in L (and similarly L^* can never be rotated to be contained in K^*).*

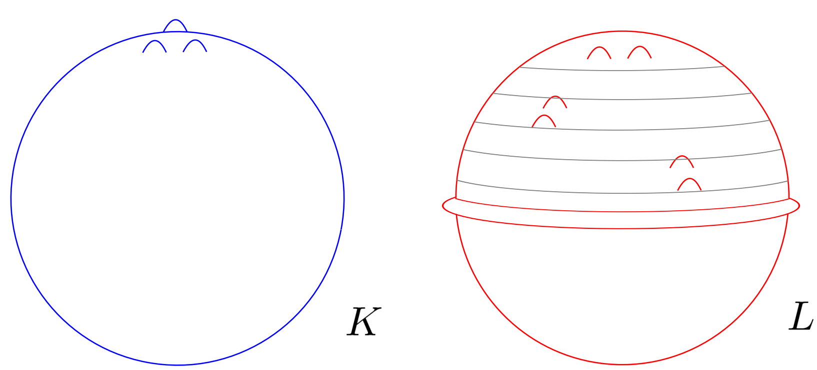

Figure 5.6: Bumps on the sphere.

I will prove the counterexample for sections, then one can prove the counterexample for projections by considering the polar bodies.

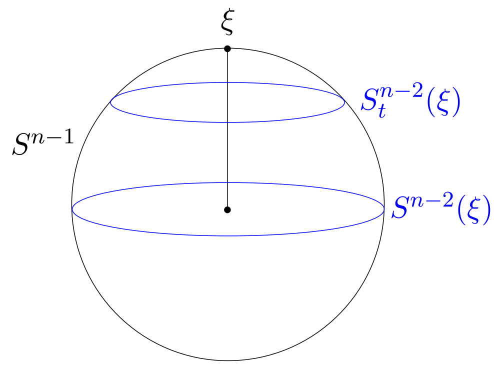

Figure 5.7: Parallels.

Recall the following notation, given $\xi \in S^{n-1}$, the great $(n-2)$-dimensional subsphere of S^{n-1} that is orthogonal to ξ will be denoted by $S^{n-2}(\xi) = \{\theta \in S^{n-1} : \theta \cdot \xi = 0\}$. For $t \in [-1, 1]$, the subsphere that is parallel to $S^{n-2}(\xi)$ and is at height t will be denoted by $S_t^{n-2}(\xi)$ (see Figure 5.7).

Recall that the radial function of the unit sphere is the constant function 1. I consider a smooth bump function $\varphi_{\xi,\delta}$ defined on S^{n-1}, supported in a small $(n-1)$-dimensional spherical ball D_ξ on the surface of S^{n-1} with center at $\xi \in S^{n-1}$ and with radius δ. The function $\varphi_{\xi,\delta}$ is invariant under rotations that fix the direction ξ, and its maximum height at the point ξ is 1. The body whose radial function is $1+\varepsilon\,\varphi_{\xi,\delta}(u)$ is convex, since its curvature will be positive provided that ε is small enough (here the computations are similar to [7, page 267]).

The first body K is defined to be the unit sphere with n bumps placed on the surface, so that their centers form a regular $(n-1)$-dimensional spherical simplex. Its radial function is

$$\rho_K(x) = 1 + \sum_{j=1}^{n} \frac{\varepsilon}{10^3} \varphi_{\xi_j,\delta}(x), \qquad x \in S^{n-1},$$

i.e. each bump is supported on a $(n-1)$-dimensional spherical ball of radius δ (to be chosen later), and has height $\frac{\varepsilon}{10^3}$, where ε is small enough so that the body whose radial function is $1 + \varepsilon\varphi_{x,\delta}$ is convex. Here I see that ε depends on n. I assume that the vertex ξ_1 is the north pole, and that $v < \frac{4-n}{10^3}$ is the spherical distance between the vertices of the simplex. Given any two vertices of the simplex ξ_i and ξ_j, with $i \neq j$, consider the lune formed by the union of all $(n-2)$-dimensional great spheres passing through any two points $x \in D_{\xi_i}$ and $y \in D_{\xi_j}$. Let a be the maximum width of the lune. I can find the width of a by considering similar Euclidean triangles on the parallels of S^{n-1} that include a and 2δ. First I note some distances by considering two spherical triangles on the surface of the $(n-1)$-dimensional sphere (see figure 5.8). The spherical height of the smaller triangle is $\frac{v}{2}$ and spherical base is 2δ. The spherical height of the larger triangle is $\frac{\pi}{2}$ and the spherical base is a. (To find the lengths and prove that the Euclidean triangles are similar, I see that they are both isosceles and the smaller one has two legs of length $\sin\left(\frac{v}{2}\right)$ and the other leg is slightly smaller than 2δ and bigger than δ. The larger triangle has two legs of length 1, and the remaining leg is slightly smaller than a and bigger than $\frac{a}{2}$. The angle opposite the non-congruent sides is the same in both triangles, hence both triangles are similar.) By using the properties of similar Euclidean triangles, I see that $\frac{2\delta}{v} < a < \frac{4\pi\delta}{v}$ (noting that $\frac{v}{\pi} \leq \sin\left(\frac{v}{2}\right) \leq \frac{v}{2}$). If I choose $\delta = v^4$, it follows that $a < 4\pi v^3$. I consider a because it will correspond with the sections of K that I will rotate.

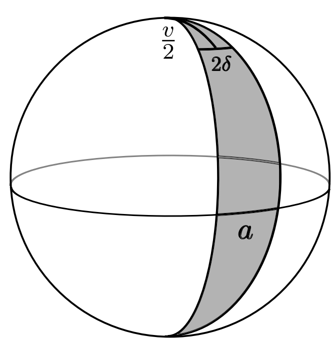

Figure 5.8: A lune.

To guarantee a rotation of a section K is contained in the corresponding section of L I place "larger" bumps on L in the specific places.

Define L to be the unit sphere with bumps placed on the surface in the following way. For every center $\xi_i \neq \xi_1$ of a bump function on K, I place the same bump function on L. Thus, by construction, any section of K that passes through any combination of bumps *except* for the bump whose center is the north pole ξ_1, is automatically contained in the corresponding section of L.

Now to take care of any section of K that passes through any combination of bumps *including* the bump whose center is the north pole ξ_1. First, I split the top half of the sphere into 2^n layers. For $k = 1, \dots, 2^n - 1$, the k-th layer L_k is the spherical ring placed between the parallels $S^{n-2}_{t_{k-1}}(\xi_1)$ and $S^{n-2}_{t_k}(\xi_1)$, where $t_k = \frac{k}{2^n}$. The top layer is the spherical cap centered at the north pole, and above the parallel $S^{n-2}_{t_{2^n-1}}(\xi_1)$. Observe that the $(n-1)$ bumps I have already placed are all on the top layer, since $\delta = v^4 < v$ and $v < \frac{4^{-n}}{10^3}$, while the spherical radius of the top layer is $\arccos(1 - \frac{1}{2^n})$. (To check this, I first note that it is enough to show is that the distance between two centers of the bumps, v, plus the radius of a bump, δ, is smaller then the spherical radius of the top layer, *i.e.* $v + \delta < \arccos\left(1 - \frac{1}{2^n}\right)$. Second, I note $v^4 < v < \frac{4^{-n}}{10^3}$, and consider the Taylor expansion of $\arccos(1-x)$ centered at $x = 0$, concluding that $\sqrt{2x} < \arccos(1-x)$. Next I observe that $\sqrt{2^{n+1}} < 10^3 4^n$, which implies that $\frac{2 \cdot 4^{-n}}{10^3} < \sqrt{2\left(\frac{1}{2^n}\right)}$, and thus $v + \delta < \arccos\left(1 - \frac{1}{2^n}\right)$.)

For every odd k, the layer L_k will contain no bumps. Next, I place bump functions on the even layers in a special way so that if a section of K goes through j bumps *including* the north pole ξ_1, the section of K can be rotated to be contained in the corresponding section of L. For each $2 \leq j \leq n-1$, and for each configuration of j vertices of the simplex in K, one of which is the north pole, I place on a layer L_k with k even an identical configuration of vertices (*i.e.* a rotation of the original configuration into L_k, see figure 5.6). On each vertex x I place the bump function $\varepsilon\varphi_{x,\widetilde{\delta}}$, where $\widetilde{\delta} = v^2$. I note that these "larger" bumps still do not overlap, since the radii of two bumps, $2\widetilde{\delta}$, is smaller than the distance between the two centers of the bumps, v, *i.e.* $2\widetilde{\delta} < v$. (To check this, $2 < 10^3 \cdot 4^n$ which implies that $v < \frac{4^{-n}}{10^3} < \frac{1}{2}$, and hence $2\widetilde{\delta} = 2v^2 < v$.) The definition of v guarantees that the layers are wide enough to contain each configuration of bumps. Indeed, the smallest spherical height of any layer L_k is L_1, which is equal to $\arcsin\left(\frac{1}{2^n}\right)$. Thus, it is enough to show that the distance between the two centers of the bumps, v, plus the radii of both of the bumps, $2\widetilde{\delta}$, is smaller than $\arcsin\left(\frac{1}{2^n}\right)$, *i.e.* $v + 2\widetilde{\delta} < \arcsin\left(\frac{1}{2^n}\right)$. (First, note that $v + 2\widetilde{\delta} = v + 2v^2 < \frac{3 \cdot 4^{-n}}{10^3}$, and considering the Taylor expansion of $\arcsin(x)$ centered at $x = 0$, I have $x < \arcsin(x)$. Then observe that $3 \cdot 2^n < 10^3 4^n$, which implies that $\frac{3 \cdot 4^{-n}}{10^3} < \frac{1}{2^n}$, and hence $v + 2\widetilde{\delta} < \arcsin\left(\frac{1}{2^n}\right)$.)

Since $\widetilde{\delta} >> a$, every section of K that intersects j of the bumps *including* the bump whose center is the north pole ξ_1 will be contained after a rotation in the corresponding section of L. Here, $\widetilde{\delta} > a$ because $4\pi < 10^3 4^n$ which implies, $a < 4\pi v^3 < v^2 = \widetilde{\delta}$. On the other hand, since $\widetilde{\delta} << v$, no layer can contain n bumps of smaller height $\varepsilon/10^3$ placed in the shape of the original simplex on K. One way to see this is to construct the bumps that are on K in a layer L_k. For every "larger" bump I could only place one "smaller" bump to ensure that all the centers of the "smaller" bumps are a distance of v away from each other. Hence it is impossible for a spherical simplex to be formed in L_k.

Finally, I define a function ψ_{ξ_1} as the function obtained by placing a bump functions on L in the following way. Suppose $x \in S^{n-2}(\xi_1)$, then place the bump function $\frac{\varepsilon}{10^3}\varphi_{x,\delta}$. Repeat this for all $x \in S^{n-2}(\xi_1)$. One can think of this function as "sliding" the bump function $\frac{\varepsilon}{10^3}\varphi$ around the equator $S^{n-2}(\xi_1)$ of L. This guarantees that every section of K that passes through only the bump function whose center is the north pole ξ_1, can be rotated by angle $\pi/2$ into the corresponding section of L. This concludes the n-dimensional counterexample.

5.2 Sections, Projections, and Volumes

5.2.1 Theorems and Lemmata

Both Counterexamples 1 and 2 to Problems 3(a) and 4(a) for rotations presented in Section 5.1 have $vol_n(C) \leq vol_n(K)$ and $vol_n(K) \leq vol_n(L)$, respectively and hence do not provide a negative answer to Problems 3(b) and 4(b) for rotations. However, one can obtain the desired relation $vol_n(K) \leq vol_n(L)$ if the *sections* of K are assumed to fit into the corresponding sections of L after rotation. This is proved in the next theorem. On the other hand, Theorem 21 below shows that my assumptions on the projections of K and L only imply that the volume of the polar body K^* is larger than the volume of L^*, but gives me no relation between the volumes of K and L. Besides this fact, I find classes of convex bodies for which I have an affirmative answer for Problem 3(b).

For the convenience of the reader I repeat the statement that appears in the Introduction.

Theorem 4. *Let K and L be two star bodies in $\mathbb{R}^n$, $n \geq 2$, such that for every $\xi \in S^{n-1}$, there exists a rotation $\varphi_\xi \in SO(n-1, \xi^\perp)$ such that*

$$\varphi_\xi(K \cap \xi^\perp) \subseteq L \cap \xi^\perp.$$

Then,

$$vol_n(K) \leq vol_n(L).$$

Proof. By hypothesis, for every $\xi \in S^{n-1}$ there exists a rotation $\varphi_\xi \in SO(n-1, \xi^\perp)$ such that

$$\rho_{\varphi_\xi(K\cap\xi^\perp)}(\theta) \leq \rho_{L\cap\xi^\perp}(\theta) \quad \forall \theta \in \xi^\perp.$$

By (3.6), this is equivalent to

$$\rho_K(\varphi_\xi^t(\theta)) \leq \rho_L(\theta) \quad \forall \theta \in \xi^\perp.$$

Raising to the power n, integrating, and using the rotation invariance of the Lebesgue measure, I obtain

$$\int\limits_{\xi^{\perp}\cap S^{n-1}} \rho_K^n(\varphi_\xi^t(\theta))d\theta = \int\limits_{\xi^{\perp}\cap S^{n-1}} \rho_K^n(\theta)d\theta \leq \int\limits_{\xi^{\perp}\cap S^{n-1}} \rho_L^n(\theta)d\theta.$$

Averaging over the unit sphere, I have

$$\int\limits_{S^{n-1}} d\xi \int\limits_{\xi^{\perp}\cap S^{n-1}} \rho_K^n(\theta)d\theta \leq \int\limits_{S^{n-1}} d\xi \int\limits_{\xi^{\perp}\cap S^{n-1}} \rho_L^n(\theta)d\theta.$$

Finally, using Fubini's Theorem and the formula for the volume in terms of the radial function, see (3.8),

$$vol_n(K) = \frac{1}{n}\int\limits_{S^{n-1}} \rho_K^n(\theta)d\theta \leq \frac{1}{n}\int\limits_{S^{n-1}} \rho_L^n(\theta)d\theta = vol_n(L),$$

I obtain the result. □

For the next theorem I use the standard notation $int(K)$ to stand for the interior of K. The proof is very similar to that of Theorem 4.

Theorem 21. *Let K and L be two convex bodies in $\mathbb{R}^n$, $n \geq 2$, such that $0 \in int(K)\cap int(L)$, and for every $\xi \in S^{n-1}$, there exists a rotation $\varphi_\xi \in SO(n-1,\xi^{\perp})$ such that*

$$\varphi_\xi(K|\xi^{\perp}) \subseteq L|\xi^{\perp}.$$

Then,

$$vol_n(K^*) \geq vol_n(L^*).$$

Proof. By hypothesis, for every $\xi \in S^{n-1}$ there exists a rotation $\varphi_\xi \in SO(n-1,\xi^{\perp})$ such that

$$h_{\varphi_\xi(K|\xi^{\perp})}(\theta) \leq h_{L|\xi^{\perp}}(\theta) \quad \forall\theta \in \xi^{\perp}.$$

By (3.4) and (3.5), this is equivalent to

$$\rho_{K^*}(\varphi_\xi^t(\theta)) \geq \rho_{L^*}(\theta) \quad \forall\theta \in \xi^{\perp}.$$

Raising to the power n, integrating, and using the rotation invariance of the Lebesgue measure, I obtain

$$\int_{\xi^\perp\cap S^{n-1}} \rho^n_{K^*}(\varphi^t_\xi(\theta))d\theta = \int_{\xi^\perp\cap S^{n-1}} \rho^n_{K^*}(\theta)d\theta \geq \int_{\xi^\perp\cap S^{n-1}} \rho^n_{L^*}(\theta)d\theta.$$

Averaging over the unit sphere, I have

$$\int_{S^{n-1}} d\xi \int_{\xi^\perp\cap S^{n-1}} \rho^n_{K^*}(\theta)d\theta \geq \int_{S^{n-1}} d\xi \int_{\xi^\perp\cap S^{n-1}} \rho^n_{L^*}(\theta)d\theta.$$

Finally, using Fubini's Theorem and (3.8), I obtain the desired result. □

In order to obtain a positive answer to Problem 3(b), I need to impose additional conditions on the bodies K, L. I do this in Theorem 5, following ideas of Hadwiger [9] as in Chapter 4, by assuming the existence of a diameter $d_K(\xi_0)$ of K in a fixed direction ξ_0, such that the hypotheses of Problem 3 hold on every plane that contains that diameter. Recall that $K|w^\perp$ (resp. $L|w^\perp$) is called a *side projection* of K (resp. of L) if $w \in \xi_0^\perp$ (see Figure 5.9).

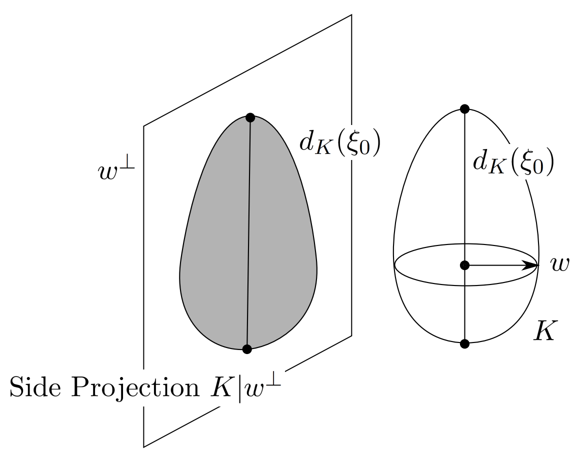

Figure 5.9: Side projections.

For the convenience of the reader I repeat the statement that appears in the Introduction (cf. with Theorem 20 in Chapter 4).

Theorem 5. *Let K, L be convex bodies in $\mathbb{R}^3$ with countably many diameters, and the diameters of K and L are of equal length. Assume that there exists a diameter $d_K(\xi_0)$, such that for every $w \in \xi_0^\perp$, there exists $\varphi_w \in SO(2, w^\perp)$ and $a_w \in w^\perp$ such that $\varphi_w(K|w^\perp) \subseteq L|w^\perp + a_w$. If either K or L is centrally symmetric then $K \subseteq L + a$ for some $a \in \mathbb{R}^3$.*

The idea of this proof is similar to those given in Section 2.2.2.

Proof. First I note that there exists a diameter of L in the ξ_0 direction and that K and L have at most one diameter parallel to a given direction for the same reasons as in Theorem 20 in Chapter 4.

Next I "separate" translations and rotations. I translate K and L so that their diameters $d_K(\xi_0), d_L(\xi_0)$ are equal and are centered at the origin. I can do this for the same reasons as in Lemma 16 in Section 4.4. Name the translated bodies $\tilde{K}$ and $\tilde{L}$. In addition, for the same reasoning as in Lemma 16 in Section 4.4, $\varphi_w(\tilde{K}|w^\perp) \subseteq \tilde{L}|w^\perp$.

Let D be the countable set of all directions of the diameters of $\tilde{K}$ and $\tilde{L}$, excluding ξ_0. For $w \in \xi_0^\perp$, let $w^\perp$ be a plane containing no direction in D (clearly, $w^\perp$ contains ξ_0). Since $\varphi_w(\tilde{K}|w^\perp) \subseteq \tilde{L}|w^\perp$, it follows that φ_w is either the identity or a rotation by π about the origin. If φ_w is the identity, then $\tilde{K}|w^\perp \subseteq \tilde{L}|w^\perp$. If φ_w is a rotation by π, then $-\tilde{K}|w^\perp \subseteq \tilde{L}|w^\perp$. But either $\tilde{K}$ or $\tilde{L}$ is centrally symmetric and their diameters are centered at the origin this implies either $\tilde{K}$ or $\tilde{L}$ is origin-symmetric. This means I can obtain $\tilde{K}|w^\perp \subseteq \tilde{L}|w^\perp$ also in this case. Thus, for every $\theta \in S^2$ such that $\theta \in w^\perp$ and $w^\perp$ does not contain any direction in D, I have that $h_{\tilde{K}}(\theta) \leq h_{\tilde{L}}(\theta)$.

Let H_i be the plane that contains ξ_0 and $\xi_i \in D$, and assume that $\theta \in S^2 \cap H_i$. Since there are only countably many such H_i's, I can choose a sequence $\{\theta_j\}$ of points in S^2, converging to θ, such that none of the θ_j are contained in $\cup_{i \geq 1} H_i$. Hence, $h_{\tilde{K}}(\theta_j) \leq h_{\tilde{L}}(\theta_j)$, and by the continuity of the support function, $h_{\tilde{K}}(\theta) \leq h_{\tilde{L}}(\theta)$.

Thus, I have $h_{\tilde{K}}(\theta) \leq h_{\tilde{L}}(\theta)$ for all $\theta \in S^2$, so $\tilde{K} \subseteq \tilde{L}$ and hence $K \subseteq L + a$ where

$a \in \mathbb{R}^3$. $\square$

I now present two related results that use different hypotheses.

For the first one I will need the following theorem from [27],

Theorem 22. *Let K and L be two convex bodies in $\mathbb{R}^3$ containing the origin in their interior. Then $K = \pm L$, provided the projections $K|H, L|H$ onto any two-dimensional subspace H of $\mathbb{R}^3$ are rotations of each other around the origin.*

Lemma 23. *Let K, L be two convex bodies in $\mathbb{R}^3$, such that*

$$\forall \xi \in S^2 \quad \exists \varphi_\xi \in SO(2, \xi^\perp): \quad \varphi_\xi(K|\xi^\perp) \subseteq L|\xi^\perp,$$

and

$$\int_{S^2} h_K = \int_{S^2} h_L.$$

Then $K = \pm L$.

Proof. Assume that $\varphi_\xi(K|\xi^\perp)$ is strictly contained in $L|\xi^\perp$. By continuity, there is a open set of directions in S^2 where the containment is strict. Integrating, $\int_{S^2} h_K < \int_{S^2} h_L$, contradicting my hypothesis. Therefore, for every $\xi \in S^2$, there exists $\varphi_\xi \in SO(2, \xi^\perp)$ such that $\varphi_\xi(K|\xi^\perp) = L|\xi^\perp$. Thus by Theorem 22, I conclude that $K = \pm L$. $\square$

Recall the following formula that relates volume, surface area $S(\cdot)$ and width for bodies K of constant width w in $\mathbb{R}^3$, namely,

$$2vol_3(K) = wS(K) - \frac{2\pi}{3}w^3, \tag{5.5}$$

see Chapter 3 Equation (3.9).

Lemma 24. *Let K, L be two convex bodies of equal constant width in $\mathbb{R}^3$, such that*

$$\forall \xi \in S^2 \quad \exists \varphi_\xi \in SO(2, \xi^\perp) \quad \exists a_\xi \in \xi^\perp: \quad \varphi_\xi(K|\xi^\perp) \subseteq L|\xi^\perp + a_\xi.$$

Then $vol_3(K) \leq vol_3(L)$.

Proof. The assumption on the projections implies that the surface area of K is less than or equal to the surface area of L, see Cauchy's surface area formula (Theorem 11 in the Preliminaries). By (5.5), I conclude that $vol_3(K) \leq vol_3(L)$. $\square$

5.3 The Sections of the Cylinder and Cone in $\mathbb{R}^3$

Here I provide the calculations for the example in Section 5.1.

5.3.1 Determining the Radial Function of the Boundary Curves of the Sections of K and C.

The upper half of the cone has equation $z = 1 - \sqrt{x^2 + y^2}$, and the plane $\xi_\theta^\perp$ has equation $z = \tan(\theta)x$. The curve of intersection in parametric equations is given by

$$r_{K,\theta}(t) = \langle (1-z)\cos(t), (1-z)\sin(t), z \rangle$$

where $z = \tan(\theta)(1-z)\cos(t)$ (from the equation of the plane). Solving for z in this last equation, I obtain $z = \frac{\tan(\theta)\cos(t)}{1+\tan(\theta)\cos(t)}$, and therefore

$$r_{K,\theta}(t) = \left\langle \frac{\cos(t)}{1+\tan(\theta)\cos(t)}, \frac{\sin(t)}{1+\tan(\theta)\cos(t)}, \frac{\tan(\theta)\cos(t)}{1+\tan(\theta)\cos(t)} \right\rangle.$$

This curve is still expressed as a subset of $\mathbb{R}^3$, so now I will write it as a two dimensional curve on the plane $\xi_\theta^\perp$. The vectors $\langle 1,0,0 \rangle$ and $\langle 0,1,0 \rangle$ project onto $\vec{e}_{1,\theta} = \left\langle \frac{1}{\sqrt{1+\tan^2(\theta)}}, 0, \frac{\tan(\theta)}{\sqrt{1+\tan^2(\theta)}} \right\rangle = \langle \cos(\theta), 0, \sin(\theta) \rangle$ and $\vec{e}_{2,\theta} = \langle 0,1,0 \rangle$ on the plane $z = \tan(\theta)x$. Therefore, for $t \in [0, \pi/2]$, the parametric curve written on this basis becomes

$$\widetilde{r}_{K,\theta}(t) = \left(\frac{\cos(t)\sec(\theta)}{1+\tan(\theta)\cos(t)} \right) \vec{e}_{1,\theta} + \left(\frac{\sin(t)}{1+\tan(\theta)\cos(t)} \right) \vec{e}_{2,\theta}.$$

Finally, it will be more convenient to express it in polar coordinates. Setting $\widetilde{r}_{K,\theta}(t) = \rho_{K_\theta}(u)\cos(u)\vec{e}_{1,\theta} + \rho_{K_\theta}(u)\sin(u)\vec{e}_{2,\theta}$ and solving, I obtain that the radial function of the section $K \cap \xi_\theta^\perp$ is

$$\rho_{K_\theta}(u) = \frac{\sec(u)}{\sin(\theta) + \sqrt{\tan^2(u) + \cos^2(\theta)}},$$

for $u \in [0, \pi/2]$. The function is extended evenly to $[-\pi/2, 0]$. It can easily be checked that $\rho'_{K_\theta}(u) \geq 0$ when $\theta \in [0, \pi/4]$, and thus $\rho_{K_\theta}(u)$ is an increasing function of u on $[0, \pi/2]$, with minimum value $\rho_{K_\theta}(0) = \frac{1}{\sin\theta + \cos\theta}$, and maximum value $\rho_{K_\theta}(\pi/2) = 1$. Also, for fixed $u \in [0, \pi/2]$, ρ_{K_θ} is a decreasing function of $\theta \in [0, \pi/4]$. In contrast, when $\theta \in (\pi/4, \pi/2]$, $\rho_{K_\theta}(u)$ has a local maximum at $u = 0$ and a local (and absolute) minimum at $u_0 = \arctan(\sqrt{\sin^2(t) - \cos^2(t)})$, with value $\rho_{K_\theta}(u_0) = 1/\sqrt{2}$. Its absolute maximum is $\rho_{K_\theta}(\pi/2) = 1$.

Similarly, I calculate the radial function of $C \cap \xi_\theta^\perp$. The intersection of the cylinder with the plane $z = \tan(\theta)x$, for $\theta \in [0, \pi/4]$, is an ellipse with parametrization

$$r_{C,\theta}(t) = \langle r\cos(t), r\sin(t), r\cos(t)\tan(\theta)\rangle .$$

In terms of the basis $\{\vec{e}_{1,\theta}, \vec{e}_{2,\theta}\}$, the parametrization is given by

$$\widetilde{r}_{C,\theta}(t) = r\cos(t)\sec(\theta)\vec{e}_{1,\theta} + r\sin(t)\vec{e}_{2,\theta},$$

and the radial function is

$$\rho_{C_\theta}(u) = \frac{r\sec(u)}{\sqrt{\tan^2(u) + \cos^2(\theta)}},$$

and evenly extended on $[-\pi/2, 0]$. The section is an ellipse with semiaxes of length $r\sec\theta$ (for $u = 0$) and r (for $u = \pi/2$), and the radial function is strictly decreasing on $u \in [0, \pi/2]$. It is also useful to note that for fixed u, ρ_{C_θ} is an increasing function of $\theta \in [0, \pi/4]$.

When $\theta \in [\pi/4, \pi/2]$, the plane cuts the top and bottom of the cylinder, and I obtain the following radial function:

$$\rho_{C_\theta}(u) = \begin{cases} r\sec(u)\csc(\theta) & 0 \leq u \leq u_0, \\ \frac{r\sec(u)}{\sqrt{\tan^2(u)+\cos^2(\theta)}} & u_0 \leq u \leq \pi/2, \end{cases}$$

where $u_0 = \arctan(\sqrt{\sin^2(\theta) - \cos^2(\theta)})$. The section looks like an ellipse with semiaxes of length $r\sec\theta$ (along the x-axis) and r (along the y-axis), that has been truncated by two vertical lines at $x = \pm r\csc(\theta)$. Here $\rho_{C_\theta}(u)$ has a local minimum at $u = 0$, is strictly increasing on $(0, u_0)$, reaches a local (and absolute) maximum at $u = u_0$ with $\rho_{C_\theta}(u_0) = \sqrt{2}r$, and is decreasing on $(u_0, \pi/2)$. The absolute minimum is $\rho_{C_\theta}(\pi/2) = r$. Observe that the absolute maximum of ρ_{C_θ} occurs at the same point as the absolute minimum of ρ_{K_θ}, and that $\sqrt{2}r = \rho_{C_\theta}(u_0) > \rho_{K_\theta}(u_0) = 1/\sqrt{2}$, since $r > 1/2$, thus reflecting the fact that for $\theta > \pi/4$, the section of the cylinder is not contained in the section of the cone. Figure 5.10 shows the graphs of $\rho_{K_\theta}(u)$ and $\rho_{C_\theta}(u)$ with $u \in [0, \pi/2]$, for $r = 0.51$. On the left, $\theta = \pi/4$; on the right, $\pi/4 < \theta < \pi/2$.

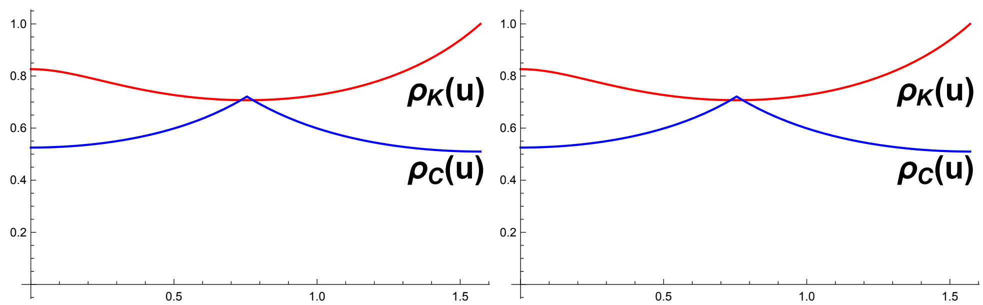

Figure 5.10: Left: $\theta = \pi/4$; Right: $\pi/4 < \theta < \pi/2$.

Now I am ready to compare the sections of the cylinder and the cone on each plane $\xi_\theta^\perp$. As noted in Section 5.1, if $\theta_0 = \arctan\left(\frac{1-r}{r}\right)$, for $\theta \in [0, \theta_0]$, the section of the cylinder is contained in the section of the cone and there is nothing to prove. For $\theta \in (\theta_0, \pi/4]$, the section of the cylinder is not contained in the section of the cone, but a 90-degree rotation of the section of the cylinder is contained in the section of the cone. Since for fixed u, $\widetilde{\rho}_{C_\theta}(u)$ is increasing as a function of $\theta \in (\theta_0, \pi/4]$, while $\rho_{K_\theta}(u)$ is decreasing, it is enough to show that for $u \in [0, \pi/2]$, $\widetilde{\rho}_{C_{\pi/4}}(u) < \rho_{K_{\pi/4}}(u)$. Here, $\widetilde{\rho}_{C_{\pi/4}}(u)$ is the radial function of the 90-degree rotation of the section of the cone, as defined in equation (5.4). I want to show that

$$\frac{r^2\csc^2(u)}{\frac{1}{2} + \cot^2(u)} < \frac{\sec^2(u)}{\left(\frac{1}{\sqrt{2}} + \sqrt{\tan^2(u) + \frac{1}{2}}\right)^2}. \tag{5.6}$$

This can be rearranged as

$$r^2\left(1+\tan^2(u)+\sqrt{2}\sqrt{\tan^2(u)+\frac{1}{2}}\right)<\tan^2(u)\left(\frac{1}{2}+\cot^2(u)\right),$$

or

$$\sqrt{2}\,r^2\sqrt{\tan^2(u)+\frac{1}{2}}<\tan^2(u)\left(\frac{1}{2}-r^2\right)+(1-r^2).$$

Squaring both sides, I obtain

$$0<\frac{1}{4}(1-2r^2)^2\tan^4 u+(1-3r^2)\tan^2 u+(1-2r^2),$$

a quadratic equation on $\tan^2 u$ whose discriminant is $(1-3r^2)^2-(1-2r^2)^3=r^4(8r^2-3)$. But this expression is negative for $r\in(\frac{1}{2},\sqrt{2-\sqrt{3}}]$, and thus (5.6) holds.

5.3.2 Calculation of the angles θ_1,θ_2

As noted in Section 5.1, when $\theta=\pi/4$, the 90 degree rotation of $C\cap\xi_\theta^\perp$ is strictly contained in the section of the double cone, and by continuity the same is true for $\theta\in(\pi/4,\theta_1)$ for some angle θ_1. Similarly, for $\theta=\pi/2$ the rotation of the section of the cylinder by $u_0=\pi/4$ is strictly contained in the section of the double cone, and thus the same must hold for $\theta\in(\theta_2,\pi/2)$. Here I compute θ_1 and θ_2, and prove that $\theta_2<\theta_1$, allowing me to always rotate the section of the cylinder to fit into the section of the cone.

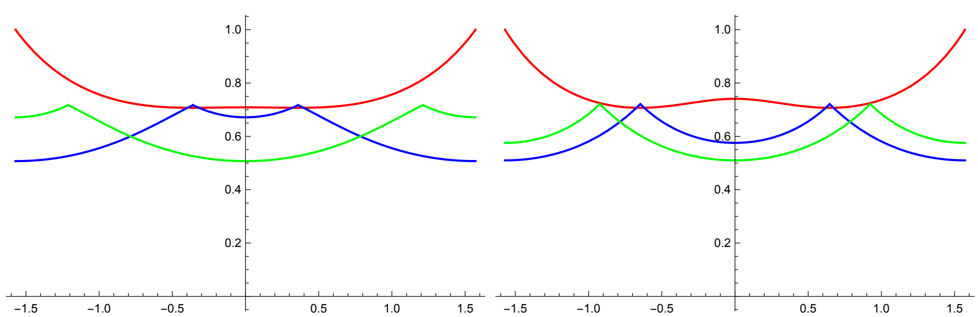

Figure 5.11: The radial functions of the sections of the cone (red), cylinder (blue) and the rotation of the cylinder by $\pi/2$ (green). In both figures, $\theta\in(\pi/4,\theta_1)$. On the left, θ is close to $\pi/4$; on the right, θ is close to θ_1.

Let

$$\widetilde{\rho}_{C_\theta}(u) = \begin{cases} \frac{r\csc(u)}{\sqrt{\cot^2(u)+\cos^2(\theta)}} & 0 \le u \le \pi/2 - u_0, \\ r\csc(u)\csc(\theta) & \pi/2 - u_0 \le u \le \pi/2, \end{cases}$$

be the radial function of the 90-degree rotation of the section of the cylinder for $\theta \in (\pi/4, \pi/2]$. Observing Figures 5.4 and 5.11, I notice that the sections of the cone and the cylinder will touch first at the "corner" point $u = \pi/2 - u_0$, where $\widetilde{\rho}_{C_\theta}(u)$ has its maximum. Thus, I will define θ_1 as the angle such that $\widetilde{\rho}_{C_{\theta_1}}(\pi/2 - u_0) = \rho_{K_{\theta_1}}(\pi/2 - u_0)$. As seen above, $\widetilde{\rho}_{C_{\theta_1}}(\pi/2 - u_0) = \sqrt{2}r$, while for the cone I have

$$\rho_{K_\theta}(\pi/2 - u_0) = \frac{\sqrt{2}}{(1 + \sqrt{-1 - 2\sec(2\theta)})\sqrt{\sin^2(\theta) - \cos^2(\theta)}}.$$

These two expressions will be equal if

$$\begin{aligned} r^{-2} &= (1 + \sqrt{-1 - 2\sec(2\theta_1)})^2 \left(\sin^2(\theta_1) - \cos^2(\theta_1)\right) \\ &= 2 - 2\cos(2\theta_1)\sqrt{-1 - 2\sec(2\theta_1)}, \end{aligned}$$

or equivalently, $-4\cos(2\theta)\,(2 + \cos(2\theta)) = (2 - r^{-2})^2$, which is a quadratic equation on $\cos(2\theta)$, with solutions $-1 \pm \sqrt{1 - \frac{(2-r^{-2})^2}{4}}$. Only the positive sign makes sense, and I obtain that the two radial functions are equal at $u = \pi/2 - u_0$ only for $\theta = \theta_1$, where

$$\theta_1 = \frac{1}{2}\arccos\left(-1 + \frac{\sqrt{4r^2 - 1}}{2r^2}\right).$$

Now I compute θ_2. Let $\widehat{\rho}_{C_\theta}(u) = \rho_{C_\theta}(u - u_0)$. By the above considerations on ρ_{C_θ}, the two absolute maxima of $\widehat{\rho}_{C_\theta}$ happen at $u = 0$ and $u = 2u_0$; the local minima happen at $u = -\pi/2 + u_0$ and at $u = u_0$, (see Figure 5.12). At the point $u = 0$ where $\widehat{\rho}_{C_\theta}$ has a maximum with value $\sqrt{2}r$, ρ_{K_θ} has a local maximum with value $1/(\sin\theta + \cos\theta)$. The two values coincide for $\theta_2 = \frac{1}{2}\arcsin\left(1/(2r^2) - 1\right)$, and $\widehat{\rho}_{C_\theta}(0) < \rho_{K_\theta}(0)$ for $\theta > \frac{1}{2}\arcsin\left(1/(2r^2) - 1\right)$. I claim that $\widehat{\rho}_{C_\theta}(u) < \rho_{K_\theta}(u)$ for every $u \in [-\pi/2, \pi/2]$ and $\theta \in (\theta_2, \pi/2]$. In fact, the slope at $u = 0$ for ρ_K is zero, while for $\widehat{\rho}'_{C_\theta}(0+)$ is negative, so it decreases faster; both functions attain their local minimum at $u = u_0$, with $\widehat{\rho}_{C_\theta}(u_0) = r \csc\theta$ and $\rho_{K_\theta}(u_0) = 1/\sqrt{2}$. But $r \csc\theta_2 < 1/\sqrt{2}$ for $r \in (1/2, \sqrt{2-\sqrt{3}})$, and $r \csc\theta$ is decreasing in θ. Hence the cylinder function stays below the cone up to $u = u_0$. Additionally, at the other maximum for the cylinder, $\widehat{\rho}_{C_\theta}(2u_0) < \rho_{K_\theta}(2u_0)$.

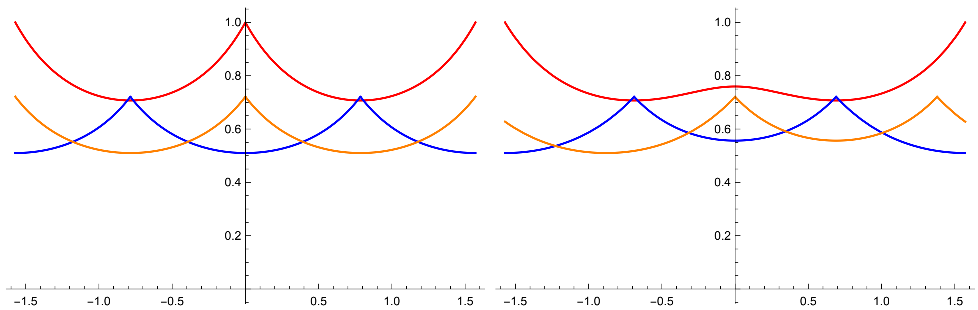

Figure 5.12: The radial functions of the sections of the cone (red), cylinder (blue) and the rotation of the cylinder by u_0 (orange). The left figure shows the case $\theta = \pi/2$, and the right one $\theta = \theta_2$.

Finally, let me check that $\theta_2 < \theta_1$ for $r \in (1/2, \sqrt{2-\sqrt{3}})$. Indeed, $\cos(2\theta_1) = -1 + \frac{\sqrt{4r^2-1}}{2r^2}$, while $\cos(2\theta_2) = -\frac{\sqrt{4r^2-1}}{2r^2}$, and the angles will be equal if $\frac{\sqrt{4r^2-1}}{r^2} = 1$, or $r^4 - 4r^2 + 1 = 0$, which has solutions $r = \pm\sqrt{2 \pm \sqrt{3}}$. Since for $r = 1/2$, $\pi/4 = \theta_2 < \theta_1 = \pi/2$, the same relation holds for $r \in (1/2, \sqrt{2-\sqrt{3}})$. I have proved that all sections of the cylinder can be rotated into the corresponding section of the double cone.

5.3.3 The Cylinder Can Never Be Rotated To Be Contained in the Double Cone

Lemma 25. *No three-dimensional rotation of the cylinder C fits inside the cone K.*

Proof. By construction, $C \nsubseteq K$. Since both C and K are origin symmetric and rotational symmetric, it is enough to consider rotations of C around the x-axis by an angle $\varphi \in (0, \frac{\pi}{2}]$. I will show that for each angle $\varphi \in (0, \frac{\pi}{2}]$, there is a point $P(\varphi)$ on the top rim of C, that remains outside of K after a rotation by the angle φ around the x-axis. Consider the point $P(\varphi) = (r\cos\alpha_0, r\sin\alpha_0, r)$, where $\alpha_0 = \arcsin\left(\frac{1-\cos\varphi}{\sin\varphi}\right)$. The rotation of angle φ maps $P(\varphi)$ to the point $R(\varphi) = (r\cos\alpha_0, r\sin\alpha_0\cos\varphi - r\sin\varphi, r\sin\alpha_0\sin\varphi + r\cos\varphi) = \left(\frac{r\sqrt{\sin^2\varphi-(1-\cos\varphi)^2}}{\sin\varphi}, \frac{r(\cos\varphi-1)}{\sin\varphi}, r\right)$. Note that the z-coordinate is positive, hence it will be enough to show that $R(\varphi)$ is outside the top part of the cone K, whose equation is $z = 1-\sqrt{x^2+y^2}$. But it is clear that

$$1-\sqrt{\left(\frac{r\sqrt{\sin^2\varphi-(1-\cos\varphi)^2}}{\sin\varphi}\right)^2+\left(\frac{r(\cos\varphi-1)}{\sin\varphi}\right)^2} = 1-r < \frac{1}{2} < r.$$

Therefore, $R(\varphi)$ is outside the cone and no three-dimensional rotation of the cylinder fits inside the cone.

□

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