CONVEX BODIES WITH SO(2) CONGRUENT PROJECTIONS

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by

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CHAPTER 1

Introduction

The theme addressed by this thesis is easily stated informally and understood intuitively. Throughout the paper, we address issues concerned with the question "if two objects cast the same shadow in any direction, must the objects themselves be the same?" Of course, there is nothing mathematically precise in the preceding sentence, and objects for which the answer appears to be "no" are very easy to find. The unit ball $B = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ and the unit sphere $S = \{x \in \mathbb{R}^3 : |x| = 1\}$ both cast identical shadows, yet there are fundamental differences between the two.

We focus on objects in the intuition-accessible three dimensional space. In order to avoid trivial counterexamples such as the one already discussed, only bounded convex objects are considered. A set is *convex* if it contains the line segment connecting any two points in the set. Without assuming convexity, many of the properties used in this thesis no longer hold, so it would not be possible to reach any substantial conclusions. Convexity has been studied by geometers since Euclid and Archimedes up until the present day, and there is a wide variety of tools from integral geometry available to study convex geometry.

Closely related to the study of convex geometry is the subject called geometric tomography. In his monograph on the subject, Richard Gardner describes tomography as "the area of mathematics dealing with the retrieval of information about a geometric object from data about its sections, or projections, or both" [2] page xvii. The question at the start of this thesis is clearly tomographic in nature, since the concept of a "shadow" is formalized as the orthogonal projection onto a two dimensional hyperplane. In the preface of [3], Golubyatnikov discusses how retrieving information about an object by examining lower dimensional data has applications in fields such as medical imaging and wave optics.

Given that all corresponding projections of our bodies are congruent, we want to conclude the bodies themselves must be congruent. More precisely, we assume that for every orthogonal projection onto a two dimensional hyperplane, there is an isometry which carries one projection into the other, where an isometry is any distance preserving function from the plane to itself. It is well known there are only four types of plane isometries: translation though a vector, rotation about a fixed point, reflection across a line, and a translation followed by a reflection. Two sets $K, L \subset \mathbb{R}^n$ are called parallel if there is some translation $x \in \mathbb{R}^n$ with K + x = L.

The focus of the thesis is on objects whose projections are all rotationally congruent. A rotation about the origin can be represented by orthogonal matrices with determinant one, and the collection of all such matrices form a group called the *special orthogonal group*, and is denoted by SO(2). If a two dimensional set can be rotated about some point into another set, the two sets are called "SO(2) congruent." The approach used in the arguments of the paper are geometric in nature, rather than linear algebraic, so the matrix representation of a rotation is suppressed. An important property about rotations in a plane is that an arbitrary rotation can be decomposed into a rotation about the origin followed by a translation.

It is assumed in the first section that corresponding projections are all parallel, and it is shown the projected bodies themselves are parallel in the three dimensional space. The next two sections thoroughly analyze the book of V. Golubyatnikov which examines the case when all projections are SO(2) congruent or parallel with the goal of clarification and completing the arguments presented. Properties found in the most general case are presented in section 2.2, though no complete conclusion is achieved. In the next section it is assumed that no projection has an SO(2) symmetry, and it is shown this implies the bodies themselves are either parallel or reflected images about some point. After presenting the proof of Golubyatnikov's theorem, the assumption about rotational symmetries is entirely removed in the following section, though the point of rotation for all projections is fixed. The same result under this modified hypothesis holds as was found in Golubyatnikov's theorem.

Many of the definitions and conventions featured in this thesis are standard tools used to study convex geometry, which are provided in pages 2-22 of [2] and pages 2-7 of [3]. While the objects of our paper are convex bodies in three dimensional Euclidean space, we often refer to analogous properties in two dimensional hyperplanes, so we will state the ndimensional version of many of the definitions used. The notation \mathbb{R}^n will be used to denote n dimensional Euclidean space. The unit sphere in \mathbb{R}^n is denoted by $S^{n-1} = \{\xi \in \mathbb{R}^n :$ $\xi \cdot \xi = 1\}$, where $x \cdot y$ represents the standard Euclidean inner product of vectors x and y.

Definition 1. A subset K of \mathbb{R}^n is convex if for any $a, b \in K$, $ta + (1 - t)b \in K$ for any $t \in (0, 1)$. We refer to K as a convex body if it is compact, convex, and has nonempty interior.

For any vector $\xi \in S^{n-1}$, we denote the hyperplane containing all vectors orthogonal containing the origin to ξ by ξ^{\perp} . The great sphere orthogonal to ξ is denoted by $\xi^{\perp} \cap S^{n-1}$. In three dimensions, a great sphere represents a two dimensional unit circle contained in the unit sphere and will be referred to as a great circle. The notion of an orthogonal projection of a convex body can now be formalized as follows:

Definition 2. Let $\xi \in S^2$, and let $K \subset R^3$ be a convex body. The orthogonal projection of K onto ξ^{\perp} , denoted $K|\xi^{\perp}$, is the set

$$K|\xi^{\perp} = \{ y \in \xi^{\perp} : \exists \lambda \in R, y + \lambda \xi \in K \}.$$

The following two functions will be used to provide analytic information about a given convex body.

Definition 3. Let $K \subset \mathbb{R}^n$ be a convex body. The function $h_K : \mathbb{R}^n \mapsto \mathbb{R}$ which is defined by

$$h_K(\xi) = \max\{u \cdot \xi : u \in K\}$$

is called the support function of K. The width function of K is defined by

$$width_K(\xi) = \frac{h_K(\xi) + h_K(-\xi)}{2}$$

The restriction of the support function to the unit sphere is often considered, and an exercise shows that for any two convex bodies K and $L, K \subset L$ if any only if $h_K \leq h_L$ on the whole unit sphere. It follows that a convex body is completely characterized by its support function. Bodies for which the width function is constant on the whole unit sphere are referred to as bodies of constant width, and this property will play an important role in our analysis. In the next section, it will be proven that for a body K, and for any $\xi \in S^2$, there exists a hyperplane orthogonal to ξ which intersects only the boundary of Kand whose distance from the origin is the absolute value of $h_K(\xi)$. These planes are called supporting planes and provide an intuitive geometric interpretation of the support function. Considering these planes in three dimensions, we see that given $\xi \in S^2$ and any $w \in \xi^{\perp} \cap S^2$, a supporting plane orthogonal to ψ in the unit sphere, $h_K(w) = h_{K|\xi^{\perp}}(w)$.

Along with the usual metric defined for vectors in \mathbb{R}^n induced by the inner product, we need some method of determining how "close" two subsets of \mathbb{R}^n are. The Hausdorff metric is a well developed tool for studying the compact subsets of a metric space.

Definition 4. For compact subsets X and Y of \mathbb{R}^n , the **Hausdorff distance** between X and Y is defined by

$$||X - Y|| = \max\{\sup_{x \in X} \inf_{y \in Y} |x - y|, \sup_{y \in Y} \inf_{x \in X} |x - y|\}.$$

Under the right conditions, the Hausdorff metric can be described by the support function defined above. Indeed, [4] provides on page 53 the following alternative characterization of this metric for convex subsets:

Lemma 1. The Hausdorff distance between compact convex $K, L \subset \mathbb{R}^n$ is given by

$$||K - L|| = \sup_{u \in S^{n-1}} |h_K(u) - h_L(u)|$$

A powerful tool used in geometric tomography is known as the spherical Radon transform [2] page 429, and it will be used to prove two lemmas in Section 2.2.

Definition 5. If $f : S^2 \to \mathbb{R}$ is a Borel function, the spherical Radon transform $Rf: S^2 \to R$ of f is defined by

$$Rf(u) = \int_{S^2 \cap u^{\perp}} f(\theta) d\theta.$$

The next set of definitions will be used when the concept of duality is introduced to solve a special case of our problem.

Definition 6. The polar body of a nonempty subset K of \mathbb{R}^n is the set

$$K^* = \{ x \in \mathbb{R}^n : x \cdot y \le 1, \forall y \in K \}.$$

Definition 7. For a convex body K containing the origin and a unit vector $\xi \in S^2$, the section of K orthogonal to ξ is $K \cap \xi^{\perp}$.

A function which will serve a role similar to the support function and will characterize a convex body is called the radial function.

Definition 8. Let $K \subset \mathbb{R}^n$ be a convex body containing the origin. The radial function $\rho_K : S^{n-1} \to R$ is defined by

$$\rho_K(\theta) = \max\{\lambda \in R : \lambda \theta \in K\}.$$

The radial function has the property that for any $\xi \in S^2$ and θ in the great circle orthogonal to ξ ,

$$\rho_K(\theta) = \rho_{K \cap \xi^\perp}(\theta).$$

This property is analogous to the property proven above about the support function and projections of a convex body. The two functions are related by the following identity found in [2] page 20.

Lemma 2. If $K \subset \mathbb{R}^n$ is convex and contains the origin in its interior, then for all $u \in S^{n-1}$

$$\rho_{K^*}(u) = \frac{1}{h_K(u)}$$

Hence, there is a connection between the projections of a body K and the radial function of corresponding dual body which will be exploited later in this thesis under suitable assumptions. It is an exercise that $(K^*)^* = K$ for any convex body K. The purpose of introducing duality is the fact that the dual of a projection is a section, and the transformed sections inherit the assumed SO(2) congruence. Written more precisely, for a convex body K containing the origin, one can show that for any $\xi \in S^{n-1}$, $(K_{|\xi^{\perp}})^* = K^* \cap \xi^{\perp}$ [2] page 22.

CHAPTER 2

SO(2) Congruent Projections

2.1 Parallel Projections

Before investigating the case when projections are SO(2) congruent, it is instructive to examine the case when all projections of a body K are isometric through a parallel translation. Golubyatnikov refers to the result of this section as Süss's lemma [3] page 8, named after the German geometer Wilhelm Süss. The argument presented below originated in a paper by A.D. Alexandrov [1] page 82. Since results in the later sections of the paper will reduce the problem of rotationally congruent projections to the case when projections are parallel, understanding this special case is essential to the arguments used later. The approach used is very geometric in nature, so we first justify the geometric interpretation of the support function.

Lemma 3. Let $K \subset \mathbb{R}^n$ be a convex body, and let $\xi \in S^{n-1}$. Then there is an (n-1)dimensional hyperplane P orthogonal to ξ intersecting K on the boundary, and not the interior, so that the support function is $h_K(\xi) = u \cdot \xi$ for some $u \in K \cap P$.

Proof. Let $\{\xi_1, \ldots, \xi_{n-1}\}$ be a set of mutually orthogonal unit vectors which span an (n-1)dimensional plane orthogonal to ξ and containing the origin. Consider the set of planes $\{P_b : b \in \mathbb{R}\}$ defined by $P_b = \{a_1\xi_1 + \ldots a_{n-1}\xi_{n-1} + b\xi : a_i \in \mathbb{R}\}$. If $z = a_1\xi_1 + \cdots + a_{n-1}\xi_{n-1} + b\xi \in P_b$, the linearity of the inner product operator implies

$$z \cdot \xi = \sum_{i=1}^{n-1} a_i(\xi_i \cdot \xi) + b(\xi \cdot \xi) = b.$$

By compactness, we can pick $b \in \mathbb{R}$ so that $P_b \cap K \neq \emptyset$, but $P_{b'} \cap K = \emptyset$ for all b' > b. We can do this because the *b* in P_b represents the length of the vector which has displaced the plane from the origin. Clearly, P_b does not intersect any interior points, else there is an open ball inside of K centered at this intersection, so there would be another b' > b so that $P_{b'}$ intersects K. Let $u \in P_b \cap K$, then it follows that $z \cdot \xi \leq u \cdot \xi$ for all $z \in K$. If $z \in K$, there is a unique b' so that $z \in P_{b'}$. By our choice of b, it must be that $b' \leq b$, and therefore $z \cdot \xi = b' \leq b = u \cdot \xi$.

Theorem 1. Let $K, L \subset \mathbb{R}^3$ be convex bodies. If for every $\xi \in S^2$ there is a translation $a \in \xi^{\perp}$ so that $a + L_{|\xi^{\perp}} = K_{|\xi^{\perp}}$, then there is a translation $h \in \mathbb{R}^3$ such that h + L = K.

The argument used to prove this theorem originates from Alexandrov's student Lieberman, and is found in [1] page 82.

Proof. Let n_1 and n_2 be an arbitrary pair of orthogonal unit vectors in \mathbb{R}^3 . By assumption, there exists a vector $a' \in n_1^{\perp}$ which takes $L_{|n_1^{\perp}|}$ into $K_{|n_1^{\perp}|}$. That is, $L_{|n_1^{\perp}|} + a' = K_{|n_1^{\perp}|}$. Take some vector $a \in \mathbb{R}^3$ whose projection onto n_1^{\perp} is a', and consider L + a, whose n_1 projection is now equal to the n_1 projection of K.

Now consider the n_2 projections of K and L. We know there is some vector $b' \in n_2^{\perp}$ so that $(L+a)_{|n_2^{\perp}|} = K_{|n_2^{\perp}|}$. If b' is parallel to n_1 , then both the n_1 and n_2 projections coincide. Observe that both bodies lie within the same n_1 projection cylinder, so their n_2 projections onto an orthogonal plane lie within the same parallel lines in n_2^{\perp} determined by the cylinder, which are parallel to n_1 . It follows that an appropriate translation b along the line spanned by n_1 will bring the projection of L + a onto n_2^{\perp} into the projection of K without changing the n_1 projections. That is, $(L + a + b)_{|n_1^{\perp}|} = K_{|n_1^{\perp}|}$ and $(L + a + b)_{|n_2^{\perp}|} = K_{|n_2^{\perp}|}$.

Moreover, the projections in every direction are now equal. For the sake of simplicity, suppose that L is the translated body so that the n_1 and n_2 projections coincide with the projections of K. Let n be an arbitrary unit vector which is not on the plane spanned by n_1 and n_2 . Then there are two supporting planes for both K and L, one of which is parallel to the plane containing n_1 and n and the other parallel to the plane containing n_2 and n. Each one of these supporting planes is parallel to n, and each provides a supporting line in the projection planes for the directions in which the plane is parallel. Since the n_1 and n_2 projections coincide, so do their supporting lines, which implies the supporting planes which made them also coincide. These planes also form parallel supporting lines in the nprojection, so they must coincide. It follows that projections of K and L in the direction of n coincide. For the vectors in the plane spanned by n_1 and n_2 , recall they were arbitrarily chosen orthogonal vectors, so the same argument applied to a different pair of orthogonal unit vectors gives that all projections coincide.

Since a convex body is determined by its support function, if the support planes of Kand L all coincide, then the bodies themselves must be equal. Let $\xi \in S^2$, and let P_{ξ} be the corresponding supporting plane. Then P_{ξ} is the supporting line for any projection orthogonal to ξ of K. Since all projections coincide, that makes P_{ξ} the supporting plane for the projection of L, and thus the supporting plane for L in the direction ξ . Hence, all supporting planes of K and L coincide, which implies that K = L.

This argument can be adapted to any dimension higher than three, and indeed the original paper presents the general version of the theorem. However, the result is false for two dimensional convex bodies. It is easy to see how this argument fails to apply in the two dimensional case. After choosing the two arbitrary orthogonal vectors n_1 and n_2 , an arbitrary vector n was chosen which was not on the plane spanned by n_1 and n_2 in order to show every projection coincided. The directions on the plane spanned by n_1 and n_2 could then be argued for by picking a different set of orthogonal vectors. In the two dimensional plane however, there is no vector in general position not on the plane, and there is no other general initial plane from which to choose.

Beyond indicating the hole in the argument for the plane, we can construct a counterexample. Any projection of a bounded two dimensional connected set onto a one dimensional plane, which is a line, forms a line segment. If we can construct two different two dimensional figures whose projection in every direction is a segment of equal length, then one projection can easily be translated into the other, but the figures are not the same. One candidate is the circle with radius $\frac{r}{2}$ centered at the origin, whose projection in every direction is a segment of length r.

Another object with constant width is known as the Reuleaux triangle [2] page 108. One can be constructed by making an equilateral triangle centered at the origin with one vertex on the positive y-axis and with edge length r. Call this vertex A, the next vertex counter clockwise B, and the next vertex C. Some simple trigonometry shows that the distance from the origin to point A is $\frac{r}{\sqrt{3}}$. Since the vertices are at angles $\frac{\pi}{2}$, $\frac{-5\pi}{6}$, and $\frac{-\pi}{6}$, it follows that

$$A = (0, \frac{r}{\sqrt{3}}), B = (\frac{-r}{2}, \frac{-r}{2\sqrt{3}}), C = (\frac{r}{2}, \frac{r}{2\sqrt{3}})$$

The Reuleaux triangle, L, is the intersection of the circles with centers at A, B, and C all with radius r. That is, $L = BC \cap CA \cap AB$, where

- $BC = \{(x, y) : x^2 + (y \frac{r}{\sqrt{3}})^2 \le r^2\}$, the circle centered at A with an arc through B and C,
- $CA = \{(x, y) : (x + \frac{r}{2})^2 + (y + \frac{r}{2\sqrt{3}}) \le r^2\}$, the circle centered at B with an arc through C and A,
- $AB = \{(x, y) : (x \frac{r}{2})^2 + (y + \frac{r}{2\sqrt{3}}) \le r^2\}$, the circle centered at C with an arc through A and B.

By Lemma 3, The width function $w_L(\xi) = h_L(\xi) + h_L(-\xi)$ can be determined by finding points $u \in L$ where a line perpendicular to $\xi = \xi_{\theta}$ meets the boundary of L and taking $u \cdot \xi$. For an angle $\theta \in [0, \frac{\pi}{3}]$, this point corresponds to the point u on the circle AC at the angle θ from its center B, which is $u = r(\cos \theta, \sin \theta) + B = r(\cos \theta - \frac{1}{2}, \sin \theta - \frac{1}{2\sqrt{3}})$. The same argument shows that for an angle $\theta \in [\frac{2\pi}{3}, \pi]$, $u = r(\cos \theta, \sin \theta) + C = r(\cos \theta + \frac{1}{2}, \sin \theta - \frac{1}{2\sqrt{3}})$ and for $\theta \in [\frac{4\pi}{3}, \frac{5\pi}{3}]$, $u = r(\cos \theta, \sin \theta) + A = r(\cos \theta, \sin \theta + \frac{1}{\sqrt{3}})$. For angles not contained in the arcs of the circles, the vertices will act as the point of support. That is, if $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$. If $\theta \in [\pi, \frac{4\pi}{3}]$, u = B, and if $\theta \in [\frac{5\pi}{3}, 2\pi]$, u = C. Finally, observe that $-\xi_{\theta} = \xi_{\theta+\pi}$. For $\theta \in [0, \frac{\pi}{3}]$, $\theta + \pi \in [\pi, \frac{4\pi}{3}]$, so

$$h_L(\xi_\theta) = r(\cos\theta - \frac{1}{2}, \sin\theta - \frac{1}{2\sqrt{3}}) \cdot (\cos\theta, \sin\theta) = r[1 - \frac{1}{2}\cos\theta - \frac{1}{2\sqrt{3}}\sin\theta]$$

$$h_L(-\xi_\theta) = B \cdot (\cos(\theta + \pi), \sin(\theta + \pi)) = r(\frac{-1}{2}, \frac{-1}{2\sqrt{3}}) \cdot (-\cos\theta, -\sin\theta),$$

Which is $r[\frac{1}{2}\cos\theta + \frac{1}{2\sqrt{3}}\sin\theta]$. Adding gives:

$$width_L(\xi_\theta) = r(1 - \frac{1}{2}\cos\theta - \frac{1}{2\sqrt{3}}\sin\theta) + r(\frac{1}{2}\cos\theta + \frac{1}{2\sqrt{3}}\sin\theta) = r.$$

A similar calculation for any $\theta \in [0, \pi]$ will show that $width_L(\xi_{\theta}) = r$, so the Reuleaux triangle has constant width function taking the value r.

This section closes with a lemma presented in [3] page 9 concerning parallel projections in directions which do not lie in the same plane. It will be used in the argument to prove the central theorem of Section 2.3.

Lemma 4. If $K, L \subset \mathbb{R}^3$ are convex bodies such that there are three noncoplanar unit vectors $\xi_1, \xi_2, \xi_3 \in S^2$ with $K_{|\xi_1^{\perp}} = L_{|\xi_1^{\perp}}, K_{|\xi_2^{\perp}} = L_{|\xi_2^{\perp}}$, and projections onto ξ_3^{\perp} are parallel, then $K_{|\xi_3^{\perp}} = L_{|\xi_3^{\perp}}$.

Proof. For i, j = 1, 2, 3, use the notation $\xi_{i,j}^{\perp}$ to represent the linear subspace $\xi_i^{\perp} \cap \xi_j^{\perp}$, and denote the projections of K and L onto $\xi_{i,j}^{\perp}$ by $K_{|\xi_{i,j}^{\perp}}$ and $L_{|\xi_{i,j}^{\perp}}$ respectively. Projecting K and L onto ξ_1^{\perp} produces corresponding projections by assumption, and so projecting again onto $\xi_{1,3}^{\perp}$ yields the same subset. It follows that $K_{|\xi_{1,3}^{\perp}} = L_{|\xi_{1,3}^{\perp}}$, and an identical argument implies $K_{|\xi_{2,3}^{\perp}} = L_{|\xi_{2,3}^{\perp}}$. By assumption, there is some vector $a \in \xi_3^{\perp}$ such that $K_{|\xi_3^{\perp}} = L_{|\xi_3^{\perp}} + a$. Since it is assumed that the original three vectors are noncoplanar, the linear span of $\xi_{1,3}^{\perp}$ and $\xi_{2,3}^{\perp}$ is ξ_3^{\perp} . Since the projections onto $\xi_{1,3}^{\perp}$ and $\xi_{2,3}^{\perp}$ coincide, it follows that $a_{|\xi_{1,3}^{\perp}}$ and $a_{|\xi_{2,3}^{\perp}}$ are both zero. It follows that a = 0, and therefore $K_{|\xi_3^{\perp}} = L_{|\xi_3^{\perp}}$.

2.2 Lemma of Golubyatnikov

We now turn our attention to the primary purpose of this thesis, which is to examine the case when the orthogonal projections of a convex body in any direction are SO(2) congruent. Suppose that K and L are convex bodies in \mathbb{R}^3 are such that for every $\xi \in S^2$, the orthogonal projection $K_{|\xi^{\perp}}$ can be rotated about some point by an angle in $S^1 = [0, 2\pi)$ into $L_{|\xi^{\perp}}$. It can be assumed without loss of generality that both bodies contain the origin as an interior point, since translating the bodies preserves SO(2) congruence. The primary goal is to show that K and L are either parallel or centrally symmetric about some point. In this section, we will examine how projections behave as unit vectors become arbitrarily close, and then we will present Golubyatnikov's proof of a key lemma in his book concerning the characterization of the unit sphere under this general assumption. We then present a new similar lemma concerning rotationally congruent sections of convex bodies which will be of use later in this thesis.

For any $\xi \in S^2$, we can define the set $\{\phi(\xi)\} \subset S^1$, as in [3] page 17, to be the set of all angles ϕ for which there exists a suitable center about which $K_{|\xi^{\perp}}$ can be rotated into $L_{|\xi^{\perp}}$ through an angle of ϕ . For any $\alpha \in S^1$, we define the set $\phi^{-1}(\alpha) = \{\xi \in S^2 : \alpha \in \{\phi(\xi)\}\}$. The set $\phi^{-1}(0)$ is the set of unit vectors ξ such that the projections are parallel, meaning that $K_{|\xi^{\perp}}$ is a translation of $L_{|\xi^{\perp}}$. In this language, Theorem 1 can be rephrased as "If $S^2 = \phi^{-1}(0)$, then K and L are parallel." The notation for these sets suggests that ϕ represents a function from the sphere in \mathbb{R}^3 into the set of angles on the plane. However, if any projection of K has a nontrivial rotational symmetry, then no such function can exist. We investigate the case when these sets induce a well defined continuous function in the next chapter. However, we can extract useful information about these sets and the convex bodies to which they refer without assuming more about K and L.

The two most important preimages to be considered are $\phi^{-1}(0)$ and $\phi^{-1}(\pi)$, as will be illustrated by the main lemma of this section. Define the body K' to be the body obtained by reflecting K across the origin, and then all projections of K' and L are SO(2) congruent as well as projections of K and L. I claim the role of $\phi^{-1}(0)$ and $\phi^{-1}(\pi)$ are interchanged when considering K or K'. In the plane ξ^{\perp} , define σ_a to be the reflection about the point $a \in \xi^{\perp}$. In particular, $\sigma_0(K_{|\xi^{\perp}}) = K'_{|\xi^{\perp}}$. Suppose $\xi \in \phi^{-1}(\pi)$ with $a \in \xi^{\perp}$ so that $\sigma_a(K_{|\xi^{\perp}}) = L_{|\xi^{\perp}}$. To see that $K'_{|\xi^{\perp}}$ and $L_{|\xi}$ are parallel, it is well known that the product of any two half turns is a translation, so $\sigma_a \sigma_0$ is a translation and

$$\sigma_a \sigma_0(K'_{|\xi^{\perp}}) = \sigma_a \sigma_0 \sigma_0(K_{|\xi^{\perp}}) = \sigma_a(K_{|\xi^{\perp}}) = L_{|\xi^{\perp}}.$$

Therefore, projections of K' and L are parallel in this direction.

Similarly, $\xi \in \phi^{-1}(0)$ means there is some translation taking $K_{|\xi^{\perp}}$ into $L_{|\xi^{\perp}}$. This translation can be decomposed into $\sigma_a \sigma_0$ for some point $a \in \xi^{\perp}$. Since a half turn is an involution, $\sigma_0(K'_{|\xi^{\perp}}) = K_{|\xi^{\perp}}$, which implies

$$L_{|\xi^{\perp}} = \sigma_a \sigma_0(K_{|\xi^{\perp}}) = \sigma_a(K'_{|\xi^{\perp}}),$$

and therefore the projections of K' and L are centrally symmetric about some point. Several results in this thesis are about $\phi^{-1}(0)$ or $\phi^{-1}(\pi)$, whose roles reverse upon considering K'rather than K. The purpose of this discussion is that topological properties of $\phi^{-1}(0)$ are inherited by $\phi^{-1}(\pi)$ because the inverse image of 0 for K' is identical to $\phi^{-1}(\pi)$, and identical reasoning which was used for K applies to K'.

The following lemma will be used to shed light on limits on the unit sphere concerning our bodies. **Lemma 5.** Let K be a convex body in \mathbb{R}^3 . For all $\epsilon > 0$, there is a $\delta > 0$ such that $|w_1 - w| < \delta$ implies the Hausdorff distance between $K_{|w_1^{\perp}}$ and $K_{|w^{\perp}}$ is less than ϵ .

Proof. Suppose w_1 and w are unit vectors in S^2 such that the angle between them is δ , and suppose $u \in S^2$ is arbitrary. If P_u^1 is the supporting plane of $K_{|w_1^{\perp}|}$ in the direction u, and P_u is the supporting plane of $K_{|w^{\perp}|}$ in the direction u, then $|h_{K_{|w^{\perp}|}}(u) - h_{K_{|w^{\perp}}}(u)| =$ $|P_u^1 - P_u|$. Suppose P_u^1 intersects $K_{|w^{\perp}|}$ at the point x, and P_u intersects $K_{|w^{\perp}|}$ at x'. Clearly, $|P_u^1 - P_u| \leq |x - x'|$. Since x and x' correspond to points of support for K, the angle between the planes containing the projections is δ , and the support function of K is continuous, it follows that the angle between x and x' is approximately δ . Denote by M the diameter of the body K. Define vectors y = ax and y' = bx' for positive scalars a and b such that |y| = |y'| = M. The triangle with vertices at the origin, y, and y' is isosceles with two legs of length M separated by an angle of δ . Elementary trigonometry implies that $|y - y'| = 2M \sin(\delta/2)$. Since y and y' are extensions of the vectors x and x' respectively, it follows that $|x - x'| \leq |y - y'| \leq 2M \sin(\delta/2)$, where this bound is independent of the vector u. Since the sine function is continuous and takes the value 0 at 0, for any $\epsilon > 0$, we can pick $\delta > 0$ such that

$$\sup_{u\in S^2} \{ |h_{K_{|w_n^{\perp}}}(u) - h_{K_{|w^{\perp}}}(u)| \} \le 2M \sin(\delta/2) < \epsilon.$$

By the characterization of the Hausdorff distance between convex sets found in Schneider [?], from which it follows that projections are continuous with respect to this metric. \Box Corollary 1. The preimages $\phi^{-1}(0)$ and $\phi^{-1}(\pi)$ are closed sets.

Proof. Note that it suffices to prove $\phi^{-1}(0)$ is closed. Once this is shown, we can consider the body K' discussed above, for which the inverse image of 0 is closed, but this set coincides with $\phi^{-1}(\pi)$. The roles of $\phi^{-1}(\pi)$ and $\phi^{-1}(0)$ are then reversed, so the inverse image of 0 for K' is a closed set, and thus $\phi^{-1}(\pi)$ is closed. Suppose a sequence $\{\xi_n\} \subset \phi^{-1}(0)$ converges to ξ . Then there is a sequence of vectors h_n so that $K_{|\xi_n^{\perp}} = L_{|\xi_n^{\perp}} + h_n$. By compactness, we know that there is some subsequence h_{n_k} which converges to some vector h. We have shown before that if ξ_{n_k} converges to ξ , $K_{|\xi_n^{\perp}}$ converges to $K_{|\xi^{\perp}}$ and $L_{|\xi_n^{\perp}}$ converges to $L_{|\xi^{\perp}}$, from which it follows that $K_{|\xi^{\perp}} = L_{|\xi^{\perp}} + h$. The limit ξ must be in $\phi^{-1}(0)$, and so $\phi^{-1}(0)$ is closed.

Another set defined in [3] page 15 which will play a major role in our analysis is the set $\Sigma \subset S^2$ consisting of all vectors ξ such that $K_{|\xi^{\perp}}$ is a two dimensional body of constant width. Golunyatnikov's lemma states that the unit sphere decomposes into the inverse image of zero, the inverse image of π , and the set Σ . Before the proof of this lemma, first observe that for any two members of Σ , the width of the projections must be the same. Indeed, for $\xi_1 \neq \xi_2$ in S^2 , the great circles $\xi_1^{\perp} \cap S^2$ and $\xi_2^{\perp} \cap S^2$ must intersect. If θ is in this intersection,

$$width_{K_{|\mathcal{E}^{\perp}}}(\theta) = width_{K}(\theta) = width_{K_{|\mathcal{E}^{\perp}}}(\theta),$$

and so the value of the width function on these great circles must be the same. Call this constant width M. Golubyatnikov claims that Σ is a closed set [3] page 15, and we will provide the proof. Consider a sequence $\{\xi_n\}_{n=1}^{\infty} \subset \Sigma$ so that ξ_n converges to ξ . Let $\xi' \in \xi^{\perp} \cap S^2$ be arbitrary. For each n, pick some $\xi'_n \in \xi^{\perp}_n \cap S^2$ so that ξ'_n converges to ξ' . Since the width function is a continuous function on S^2 , it follows that $width_K(\xi'_n) \to width(\xi')$. Each of the widths in this sequence are equal to M, which implies that $width(\xi') = M$. Since $\xi' \in \xi^{\perp} \cap S^2$ was chosen arbitrarily, it follows that $K_{|\xi^{\perp}}$ has constant width M.

The following lemma will be used to prove the main result of this section, which is Lemma 2.1.4 from [3] page 17, whose proof is presented after for the convenience of the reader.

Lemma 6. It is impossible to construct an infinite family of congruent disjoint X figures on the unit sphere. Proof. Let $\{X_c : c \in I\}$ for some index I, and denote the center of X_c by x_c . By congruence, there is an $\epsilon > 0$ so that $|x_{c_1} - x_{c_2}| < \epsilon$ implies that X_{c_1} and X_{c_2} must intersect. Hence, if $\{X_c : c \in I\}$ are all disjoint, there is a collection of neighborhoods on the sphere $\{B(x_c, \epsilon/2) : c \in I\}$ which are disjoint, yet each has the same positive surface area. Since the surface area of the sphere is finite, it must be that I is a finite index.

Lemma 7. If the projections of two convex bodies $K, L \subset \mathbb{R}^3$ are all SO(2) congruent, then

$$S^2 = \phi^{-1}(0) \cup \phi^{-1}(\pi) \cup \Sigma,$$

where Σ is the set of $\xi \in S^2$ such that $K_{|\xi^{\perp}}$ and $L_{|\xi^{\perp}}$ have constant width, where the width is independent of the vector chosen.

Proof. Since all of the projections of K and L are congruent, the relationship $l(\partial K_{|\xi^{\perp}}) = l(\partial L_{|\xi^{\perp}})$ holds, where l is the perimeter of the given projections. Recall that the support function satisfies $h_K(\theta) = h_{K_{|\xi^{\perp}}}(\theta)$ for every $\theta \in \xi^{\perp}$, so the perimeter equality can be rewritten, as in [3] page 14, as

$$\int_{w^{\perp} \cap S^2} \frac{h_{K_{|w^{\perp}}}(\theta) + h_{K_{|w^{\perp}}}(-\theta)}{2} d\theta = \int_{w^{\perp} \cap S^2} \frac{h_{L_{|w^{\perp}}}(\theta) + h_{L_{|w^{\perp}}}(-\theta)}{2} d\theta,$$

which can be further reduced to

$$\int_{w^{\perp} \cap S^2} \frac{h_K(\theta) + h_K(-\theta)}{2} d\theta = \int_{w^{\perp} \cap S^2} \frac{h_L(\theta) + h_L(-\theta)}{2} d\theta.$$

Since this is the spherical Radon transform of an even function, the injectivity of the transform [2] page 430 implies that

$$width_K(\theta) = \frac{h_K(\theta) + h_K(-\theta)}{2} = \frac{h_L(\theta) + h_L(-\theta)}{2} = width_L(\theta)$$

for every θ in S^2 .

Define the set $F = S^2 \setminus (\phi^{-1}(0) \cup \phi^{-1}(\pi) \cup \Sigma)$. Suppose $\xi \in F$ with $a\pi \in \{\phi(\xi)\}$ for some irrational number a. Since $L_{|\xi^{\perp}}$ can be rotated by the angle $a\pi$ into $K_{|\xi^{\perp}}$, it follows that for any θ in $\xi^{\perp} \cap S^2$,

$$width_K(\theta) = width_L(\theta + a\pi) = width_K(\theta + a\pi).$$

A simple induction argument implies that, for a fixed $\theta \in \xi^{\perp} \cap S^2$,

$$width_K(\theta) = width_K(\theta + na\pi), \forall n \in \mathbb{N}.$$

Irrationality of a implies that $\{\theta + na\pi : n \in \mathbb{N}\}$ is a dense subset of $\xi^{\perp} \cap S^2$. Since the width function of K is a continuous function, it follows that the width of K along $\xi^{\perp} \cap S^2$, and this is constant along the projection, which contradicts the fact that $\xi \notin \Sigma$.

Hence, F must be a countable, possibly finite, union of sets F_r , where $F_r = \{\xi \in F : r\pi \in \{\phi(\xi)\}\}$ for some rational number $r \in (0, 1)$. It has been shown that $\phi^{-1}(0), \phi^{-1}(\pi)$, and Σ are closed sets, which implies that F is an open set. Since F is a countable union of closed sets, the Baire category theorem implies that the interior of F_{r_0} must be nonempty for some rational r_0 . If we can show that for all ξ in the interior of $F_{r_0}, K_{|\xi^{\perp}}$ has constant width, this would contradict the fact that F is nonempty.

Suppose there is an ξ in the interior of F_{r_0} such that some w_1 in $\xi^{\perp} \cap S^2$ has a maximal width M and the width function is not constant in any $\xi^{\perp} \cap S^2$ -neighborhood of w_1 . Call w_2 the rotation of w_1 along the great circle by $r_0\pi$. For any point w and positive number x, call S(w, x) the spherical circle of center w and radius $x\pi$.

We claim there is some open arc $l_1 \subset S(w_1, r_0)$ containing w_2 such that $width_K(u) = width_L(u) = M$ for every u in l_1 . Since the interior of F_{r_0} is open, there is an open arc l contained in the interior and in $w_1^{\perp} \cap S^2$ which contains ξ . For an arbitrary $v \in l$, consider the great circle $v^{\perp} \cap S^2$, which must contain both w_1 and $-w_1$. Call w_v, w'_v the points on $v^{\perp} \cap S^2$ which intersect $S(w_1, r_0)$. Then the sets $\{w_v : v \in l\}, \{w'_v : v \in l\}$ are open arcs

of $S(w_1, r_0)$, one of which contains w_2 . Call l_1 the arc containing w_2 . Since each v belongs to F_{r_0} , congruence implies $width_L(w_v) = width_L(w_1) = width_K(w_1) = width_K(w_v) = M$. Hence, this arc l_1 satisfies the claim.

We can reverse the argument to obtain an open arc $l_2 \subset S(w_2, r_0)$ containing w_1 on which the width of K takes the value M. Pick a third point $w_3 \in l_2$ with $|w_1 - w_3| = \epsilon$ for a small ϵ . Similarly, we can construct the circle $S(w_3, r_0)$ and an arc $l_3 \subset S(w_3, r_0)$ containing w_2 on which the width of K is constant on l_3 . Call $X_M = l_1 \cup l_3$, which is an "X" figure on the sphere. Notice the geometry of X_M depends only on the size of the neighborhood containing ξ and the value of ϵ . Since width is continuous, there is a continuum of values cso that there is a vector $w_c \in \xi^{\perp} \cap S^2$ with $width_{K_{|\xi^{\perp}}}(w_c) = c$. An identical construction yields an "X" figure $X_c \subset S^2$ on which the width of K is identically c. For distinct c_1 and $c_2, X_{c_1} \cap X_{c_2} = \emptyset$, so we have constructed an infinite set of disjoint, congruent "X" figures on the sphere. Lemma 6 implies this is impossible, completing the proof of the theorem.

In our attempt to gain more information about the bodies K and L from their projections, our strategy will be to examine the polar bodies K^* and L^* . Since duality transforms projections into sections, and Lemma 2 states the support function of K and the radial function of K^* are related by $h_K(\theta) = 1/\rho_{K^*}(\theta)$ for all θ in S^2 , the following lemma will provide a result analogous to Lemma 7 about sections of convex bodies. Unfortunately, the radial function does not have the same translation invariance property as the width function, so an extra assumption about the rotations is added. The argument follows the same structure used to prove Lemma 7.

Lemma 8. Suppose P and Q are convex bodies in \mathbb{R}^3 such that $P \cap \xi^{\perp}$ is congruent to $Q \cap \xi^{\perp}$ through a rotation about the origin for all $\xi \in S^2$. Then

$$S^2 = \phi^{-1}(0) \cup \phi^{-1}(\pi) \cup \Sigma,$$

where $\{\phi(\xi)\}$ is the set of angles through which $Q \cap \xi^{\perp}$ can be rotated into $P \cap \xi^{\perp}$, and Σ is the set of $\xi \in S^2$ such that

$$\rho_P^2(\theta) + \rho_P^2(-\theta) = c, \forall \theta \in S^2 \cap \xi^{\perp}$$

for some constant c and the same is true for ρ_Q .

Proof. We define the set $F = S^2 \setminus \{\phi^{-1}(0) \cup \phi^{-1}(\pi) \cup \Sigma\}$. Similar arguments as before show that the sets $\phi^{-1}(0), \phi^{-1}(\pi)$, and Σ are all closed sets, so F is an open set. Either F contains some ξ such that some irrational angle $a\pi$ belongs to $\{\phi^{-1}(\xi)\}$, or every rotation is through some rational angle.

Since sections of P and Q are all congruent, $\operatorname{area}(P \cap \xi^{\perp}) = \operatorname{area}(Q \cap \xi^{\perp})$ for every $\xi \in S^2$. By [2] page 410, this equality can be rewritten as

$$\int_{\xi^{\perp} \cap S^2} \frac{\rho_P^2(\theta) + \rho_P^2(-\theta)}{2} d\theta = \int_{\xi^{\perp} \cap S^2} \frac{\rho_Q^2(\theta) + \rho_Q^2(-\theta)}{2} d\theta$$

Since the spherical Radon transform of the two even functions coincide, [2] page 430 implies that

$$\rho_P^2(\theta) + \rho_P^2(-\theta) = \rho_Q^2(\theta) + \rho_Q^2(-\theta)$$

for every member of $\xi^{\perp} \cap S^2$. Since ξ was arbitrary in S^2 , the preceding equality holds for every θ in S^2 .

Suppose $\xi \in S^2$ such that $a\pi \in \{\phi(\xi)\}$ for some irrational number a. It is well known that irrationality of the angle implies the set of rotations generated by $a\pi$ forms a dense subset of ξ^{\perp} . Since $Q \cap \xi^{\perp}$ rotated by $a\pi$ about the origin is equal to $P \cap \xi^{\perp}$, and radial function gives the maximal distance from the origin of a body in a given direction, we get that

$$\rho_P(\theta) = \rho_Q(\theta + a\pi), \forall \theta \in \xi^{\perp}.$$

Using the equality derived relating the radial functions of P and Q, we get

$$\rho_P^2(\theta) + \rho_P^2(-\theta) = \rho_Q^2(\theta) + \rho_Q^2(-\theta) = \rho_P^2(\theta - a\pi) + \rho_P^2(-(\theta - a\pi)).$$

A simple induction argument gives the equation

$$\rho_P^2(\theta) + \rho_P^2(-\theta) = \rho_P^2(\theta - na\pi) + \rho_P^2(-(\theta - na\pi))$$

for every natural number n. It follows that there is a dense subset of ξ^{\perp} on which the continuous function $\rho_P^2(\theta) + \rho_P^2(-\theta)$ is constant, and thus this function is constant on the entire great circle, which contradicts out choice of ξ in F.

We have shown that our set F cannot contain the preimages of any irrational angles, and so it must be a finite or countable union of sets of the form

$$F_r = \{ w \in F : r\pi \in \{\phi(w)\} \}$$

for some rational number $r \in (0, 1)$. Since F is an open set, the Baire category theorem implies at F_{r_0} has nonempty interior for some rational number r_0 . It suffices to show that for any ξ in the interior of F_{r_0} , then $\rho_P^2(w) + \rho_P^2(-w)$ is constant for $w \in \xi^{\perp}$.

We will denote by R_P the function $R_P(\theta) = (\rho_P^2(\theta) + \rho_P^2(-\theta))/2$. Suppose there is a unit vector ξ in the interior of F_{r_0} with a $w_1 \in \xi^{\perp}$ so that $R_P(w_1) = M$ is maximal in $\xi^{\perp} \cap S^2$ and the function R_P is not constant in any neighborhood of w_1 . For any point w in S^2 and rational number r, we call the spherical circle with center w and radius r. In particular, consider the circle $S(w_1, r_0)$, and denote w_2 as the angle obtained by rotating w_1 by r_0 in ξ^{\perp} .

I claim there is an open arc $l_1 \subset S(w_1, r_0)$ containing w_2 such that $R_P(u) = R_Q(u) = M$ for all $u \in l_1$. Since the interior of F_{r_0} is open, there exists an open arc $l \subset w_1^{\perp}$ containing ξ . For any $v \in l$, v is orthogonal to w_1 , so the great circle v^{\perp} intersects $S^2 \cap v^{\perp}$ twice at points w_v, w'_v . Thus, the set of all points $\{w_v, w'_v : v \in l\}$ forms two open arcs on $S(w_1, r_0)$, one of which contains w_2 . Call this arc l_1 . Since v is in the interior of F_{r_0} , we get

$$R_P(w_v) = R_P(w_1) = R_Q(w_1) = R_Q(w_v) = M$$

for every $v \in l$, which gives that l_1 satisfies the claim.

Reversing our argument, we can create an arc $l_2 \,\subset S(w_2, r_0)$ containing w_1 on which the same equation holds. Pick some $w_4 \in l_1$ different from w_2 , and consider a small arc $l_4 \subset S(w_4, r_0)$ which contains w_1 . A similar argument as above implies $R_P(u) = R_Q(u) = M$ for all u in l_4 . Likewise, we can pick some w_3 on l_2 distinct from w_1 and construct an arc $l_3 \subset S(w_3, r_0)$ containing w_2 with $R_P = R_Q = M$ on l_3 . In this way, we obtain two disjoint "X" figures, $X_M = l_1 \cup l_3$ and $X'_M = l_2 \cup l_4$ on which R_P is identically M.

By our choice of w_1 , neither R_P nor R_Q is constant within any $\xi^{\perp} \cap S^2$ -neighborhood of w_1 , continuity of the radial function implies there is a continuum of values c taken by R_P along $\xi^{\perp} \cap S^2$. For some unit vector which takes the value c, a similar construction above yields two "X" figures, call them X_c and X'_c , on which $R_P = R_Q = c$ on the figures. For any distinct values, $c_1 \neq c_2$, it follows that $X_{c_1} \cap X_{c_2} = \emptyset$. We have created a continuum of disjoint "X" figures on the sphere. Since the construction of the figures depends only on the size of the neighborhood containing ξ and the distance of the third point constructed, all of the figures are congruent. By Lemma 6, this is not possible. Thus, we have a contradiction, and so F is empty, which implies that $S^2 = \phi^{-1}(0) \cup \phi^{-1}(\pi) \cup \Sigma$.

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2.3 Asymmetric Projections

This section presents the argument of Goluyatnikov found in [3] page 13-21 concerning convex bodies K and L whose projections have no rotational symmetries. This sacrifices many intuitive examples, such as bodies whose projections are disks or regular polygons, but the result is substantial none the less.

Theorem 2. Suppose K and L are convex bodies in \mathbb{R}^3 such that for all $\xi \in S^2$, $K_{|\xi^{\perp}}$ and $L_{|\xi^{\perp}}$ are SO(2) congruent. If none of the projections of K or L have SO(2) symmetries, then K and L are either parallel or centrally symmetric through some point.

The primary tool used to prove the theorem is the rotation function $\phi: S^2 \to S^1$,

referred to in Section 2.2, which is defined so that $\phi(\xi)$ is the smallest angle ϕ through which $L_{|\xi^{\perp}}$ rotated about some point by ϕ into $K_{|\xi^{\perp}}$. We refer to S^1 as the interval $[-\pi, \pi]$, where a rotation about π is that same as through $-\pi$. An important property of ϕ is that it is an odd function, since the projections in the direction $-\xi$ are mirror images of the projections in the direction of ξ .

Lemma 9. If none of the projections of the bodies K and L have SO(2) symmetries, then the function $\phi: S^2 \to S^1$ is continuous.

Proof. Suppose there is some sequence $\{\xi_n\}$ in S^2 so that $\xi_n \to \xi_0$, but $\phi(\xi_n) \to \phi(\xi_0)$. By compactness, there is a subsequence n_k so that that $\phi(\xi_{n_k}) \to \phi_1 \neq \phi(\xi_0)$. By definition, we know that $\phi(K_{|\xi_0^{\perp}}) = L_{|\xi_0^{\perp}}$. If we can show that $\phi_1(K_{|\xi_0^{\perp}}) = L_{|\xi_0^{\perp}}$, then there is a contradiction since this implies this projection has a rotational symmetry.

To show $\phi_1(K_{|\xi_0^{\perp}}) = L_{|\xi_0^{\perp}}$, it suffices to show $||\phi_1(K_{|\xi_0^{\perp}}) - L_{|\xi_0^{\perp}}||$ is arbitrarily small, where ||.|| is the Hausdorff metric. Let $\epsilon > 0$, and suppose n_k is large. The triangle inequality of the Hausdorff metric implies

$$||\phi_1(K_{|\xi_0^\perp}) - L_{|\xi_0^\perp}|| \le ||\phi_1(K_{|\xi_0^\perp}) - \phi_{n_k}(K_{|\xi_0^\perp})|| + ||\phi_{n_k}(K_{|\xi_0^\perp}) - L_{|\xi_0^\perp}||$$

Since ϕ_1 and ϕ_{n_k} are arbitrarily close, we can bound the first term by ϵ , which implies again by the triangle inequality

$$\leq \epsilon + ||\phi_{n_k}(K_{|\xi_0^{\perp}}) - \phi_{n_k}((K_{|\xi_{n_k}^{\perp}})_{|\xi_0^{\perp}})|| + ||\phi_{n_k}((K_{|\xi_{n_k}^{\perp}})_{|\xi_0^{\perp}}) - L_{|\xi_0^{\perp}}||.$$

To bound the second term, any rotation is an isometry, which implies

$$\begin{split} ||\phi_{n_k}(K_{|\xi_0^{\perp}}) - \phi_{n_k}(K_{|\xi_{n_k}^{\perp}})_{|\xi_0^{\perp}})|| &= ||K_{|\xi_0^{\perp}} - (K_{|\xi_{n_k}^{\perp}})_{|\xi_0^{\perp}}|| \\ \\ &\leq ||K_{|\xi_0^{\perp}} - K_{|\xi_{n_k}^{\perp}}|| + ||K_{|\xi_{n_k}^{\perp}} - (K_{|\xi_{n_k}^{\perp}})_{|\xi_0^{\perp}}|| < \epsilon + ||K_{|\xi_{n_k}^{\perp}} - (K_{|\xi_{n_k}^{\perp}})_{|\xi_0^{\perp}}||. \end{split}$$

This second term can also be bounded by ϵ since the planes ξ_0^{\perp} and $\xi_{n_k}^{\perp}$ are arbitrarily close, so the Hausdorff distance between the projection and its projection onto a close plane must also be arbitrarily small. Therefore,

$$||\phi_1(K_{|\xi_0^{\perp}}) - L_{|\xi_0^{\perp}}|| < 3\epsilon + ||\phi_{n_k}((K_{|\xi_{n_k}^{\perp}})_{|\xi_0^{\perp}}) - L_{|\xi_0^{\perp}}||.$$

Applying the triangle inequality again bounds this by

$$\leq 3\epsilon + ||\phi_{n_k}((K_{|\xi_{n_k}^{\perp}})_{|\xi_0^{\perp}}) - \phi_{n_k}(K_{|\xi_{n_k}^{\perp}})|| + ||\phi_{n_k}(K_{|\xi_{n_k}^{\perp}}) - L_{|\xi_0^{\perp}}||$$

An identical argument as above bounds the middle term, giving an upper bound of

$$<5\epsilon+||\phi_{n_k}(K_{|\xi_{n_k}^{\perp}})-L_{|\xi_0^{\perp}}||=5\epsilon+||L_{|\xi_{n_k}^{\perp}}-L_{|\xi_0^{\perp}}||,$$

where the last equality is from the definition of the angle $\phi(\xi_{n_k})$. Since projections become close in Hausdorff distance as vectors converge by Lemma 5, it follows that $||\phi_1(K_{|\xi_0^{\perp}}) - L_{|\xi_0^{\perp}}|| < 6\epsilon$ for an arbitrary $\epsilon > 0$, and therefore $\phi_1(K_{|\xi_0^{\perp}}) = L_{|\xi_0^{\perp}}$.

The set $\phi^{-1}(0)$ is the set of directions in which the projections of K and L are parallel, while $\phi^{-1}(\pi)$ is the set of directions in which the projections of K and L are congruent through a reflection about some point. Since ϕ is continuous, the fact that these sets are closed is now a triviality. We denote by $[\phi^{-1}(0)], [\phi^{-1}(\pi)]$ the nonisolated points of the preimages of 0 and π respectively. The set $\Sigma \subset S^2$, as before, represents the set of directions ξ such that $K_{|\xi^{\perp}}$ and $L_{|\xi^{\perp}}$ have constant width, which is a closed set. The body K', as defined in Section 2.1, is the reflection of K about the origin.

If $\phi(\xi) = 0$ for every $\xi \in S^2$, the argument in Section 2.1 shows that K and L are parallel. If $\phi(\xi) = \pi$ for every $\xi \in S^2$, then K' and L are parallel to each other, and it follows that K and L are centrally symmetric. Indeed, $\phi^{-1}(0)$ and $\phi^{-1}(\pi)$ share many of the same properties, since we can consider the body K' obtained by reflecting K across the origin as in Section 2.2. Thus, many of the arguments used to prove this theorem will be presented for $\phi^{-1}(0)$ but will apply to $\phi^{-1}(\pi)$ as well.

Suppose $w_0 \in S^2$ such that $0 < \phi(w_0) < \pi$. If no such angle exists, all angles are between π and 2π , but swapping the names of the bodies gives the existence of an angle between 0 and π , so we can assume $0 < w_0 < \pi$ without losing any generality. Consider the set of meridians connecting w_0 and $-w_0$. To prove the following lemma, Golubyatnikov uses the fact that ϕ is odd, along with an argument from algebraic topology. We omit the proof, though the statement is essential to the argument used in this section.

Lemma 10. For the continuous function ϕ , then either $[\phi^{-1}(0)]$ or $[\phi^{-1}(\pi)]$ intersects every meridian which connects w_0 and $-w_0$.

Without loss of generality, assume that $[\phi^{-1}(0)]$ intersects each meridian. Since ϕ is an odd function, $\xi \in [\phi^{-1}(0)]$ implies that $\phi(-\xi) = -\phi(\xi) = 0$, and so the set $[\phi^{-1}(0)]$ is symmetric about the origin. The argument then reduces to the cases when this preimage intersects some meridian more than once, and when it forms a curve parameterized by the meridians. The curve in the latter case can either be a great circle or not, and the argument differs with whether or not appropriate noncoplanar elements of $[\phi^{-1}(0)]$ can be chosen.

Lemma 11. If $[\phi^{-1}(0)]$ is not a great circle on S^2 , then there are two nonparallel vectors w_1, w_2 in $[\phi^{-1}(0)]$ such that on a dense set of the meridians connecting w_0 and $-w_0$, the points of intersection with $[\phi^{-1}(0)]$ are noncoplanar with w_1 and w_2 .

Proof. Suppose the great circle $w_0^{\perp} \cap S^2$ is parameterized by $\{w(t) : t \in [-\pi, \pi)\}$, and consider the meridians m(t) containing w(t) which connects w_0 and $-w_0$. If there is a meridian $m(t_0)$ which intersects $[\phi^{-1}(0)]$ more than once, pick two points on this intersection. Consider the plane containing this meridian and $m(t_0 + \pi)$. Then no other meridian intersects this plane and a dense set exists trivially.

If each meridian intersects $[\phi^{-1}(0)]$ exactly once, the $[\phi^{-1}(0)]$ is homeomorphic to a circle. Fix an arbitrary $w_1 \in [\phi^{-1}(0)]$, and define the great circle which contains the points w_1 and w(t) by $E(w_1, w(t))$ which consists of all unit vectors coplanar with w_1 and $-w_1$. I claim there is some t_1 such that $E(w_1, w(t_1))$ has dense compliment in $[\phi^{-1}(0)]$.

Suppose this claim is false, so that for all t there is an interval I_t contained in $[\phi^{-1}(0)] \cap E(w_1, w(t))$. Since $[\phi^{-1}(0)]$ is not a great circle, there are uncountably many distinct such great circles $E(w_1, w(t))$ If two of the intervals I_{t_1}, I_{t_2} from this distinct family intersect, then they form a cross, and therefore $[\phi^{-1}(0)]$ intersects some meridian more than once. Since $[\phi^{-1}(0)]$ intersects each meridian at exactly once, it follows that there has to be continuum of disjoint intervals contained in $[\phi^{-1}(0)]$, which we know is homeomorphic to a circle, which is not possible.

Therefore, for t_1 chosen by the claim, we can consider $w_2 = w(t_1)$. Then for a dense set of m(t), w(t) is not contained in $E(w_1, w_2)$, and so w(t) is not coplanar with w_1 and w_2 .

If $[\phi^{-1}(0)]$ intersects each meridian, but is not a great circle, let $\xi_1, \xi_2 \in [\phi^{-1}(0)]$ be two such vectors which are obtained in the previous lemma. Translate the body K into K'' so that $K''_{|\xi_1^{\perp}|} = L_{|\xi_1^{\perp}|}$ and $K''_{|\xi_2^{\perp}|} = L_{|\xi_2^{\perp}|}$. Suppose ξ_t is in the dense subset of $[\phi^{-1}(0)]$ which is not coplanar with ξ_1 and ξ_2 . By Lemma 4, $K''_{|\xi_t^{\perp}|} = L_{|\xi_t^{\perp}|}$. It follows that for every vector ξ in any of the great circles $\xi_t^{\perp} \cap S^2$, $h_{K''}(\xi) = h_L(\xi)$. For the dense parameterized set ξ_t , the union of the great circles ξ_t^{\perp} has a dense intersection with $w_0^{\perp} \cap S^2$. By the continuity of the support function, it follows that $h_{K''}(\xi) = h_L(\xi)$ for all $\xi \in w_0^{\perp} \cap S^2$, and so $K_{|w_0^{\perp}|} = L_{|w_0^{\perp}|}$. This contradicts our choice of w_0 with $0 < \phi(w_0) < \pi$, and finishes the proof of Theorem 2 when $[\phi^{-1}(0)]$ is not a great circle.

Suppose that $[\phi^{-1}(0)]$ is a great circle, and recall that it was proven in Lemma 7 that $S^2 = \phi^{-1}(0) \cup \phi^{-1}(\pi) \cup \Sigma$, where Σ is the set of directions in which the projections have

constant width M. The following corollary is stated without proof on page 19 of [3], so the proof is provided below.

Corollary 2. If convex bodies K and L satisfy the conditions of this section with ϕ not constant, and if $[\phi^{-1}(0)]$ or $[\phi^{-1}(\pi)]$ is a great circle, then K and L are bodies of constant width.

Proof. Suppose that $[\phi^{-1}(0)]$ is a great circle. Since ϕ is a continuous function on the sphere, $\phi^{-1}(0)$ and $\phi^{-1}(\pi)$ are disjoint sets so that for every $\xi \in [\phi^{-1}(0)]$ there is an $\epsilon_0 > 0$ so that no point of $\phi^{-1}(\pi)$ lies within $B(\xi, \epsilon_0)$, the ball of center ξ and radius ϵ_0 on the sphere. Lemma 7 implies that for any positive $\epsilon < \epsilon_0$, $B(\xi, \epsilon)$ contains only vectors from $\phi^{-1}(0)$ or Σ . If $B(\xi, \epsilon)$ is contained in $\phi^{-1}(0)$, then this neighborhood contains a nonisolated point not on the great circle $[\phi^{-1}(0)]$, which contradicts the fact that all nonisolated points of $\phi^{-1}(0)$ makes a great circle. Hence, there must exist an element of Σ in the neighborhood $B(\xi, \epsilon)$ for every positive ϵ , which makes ξ a limit point of Σ .

The set Σ is closed, and every $\xi \in [\phi^{-1}(0)]$ is a limit point of Σ , which implies that $[\phi^{-1}(0)]$ is a subset of Σ . Hence for all $\xi \in [\phi^{-1}(0)]$, the projection $K_{|\xi^{\perp}}$ has constant width M. For an arbitrary unit vector $w \in S^2$, the great circle $w^{\perp} \cap S^2$ must intersect the great circle $[\phi^{-1}(0)]$ at some point ξ . Since this implies $\xi \perp w$, we can conclude that $width_K(w) = width_{K_{|\xi^{\perp}}}(w) = M$. Thus, the body K has constant width M.

With this corollary, we can finish the proof of the theorem in the case that $[\phi^{-1}(0)]$ is a great circle $\xi^{\perp} \cap S^2$ for $\xi \in S^2$. Denote by P_1, P_2 the supporting planes of L in the directions $\xi, -\xi$ respectively. Since L has constant width, we can let $x_1 = P_1 \cap L$ and $x_2 = P_2 \cap L$. These points are unique and belong to a segment perpendicular to both P_1 and P_2 , else L contains a segment of length which is longer than the distance between the supporting planes P_1 and P_2 , contradicting the fact that width is constant. Since the corresponding points of support on the body K must be on a segment of equal length parallel to the segment containing x_1 and x_2 , consider the translated the body L'so that these points of support coincide. For all $w \in [\phi^{-1}(0)]$, there is a translation which makes the projections coincide, and it must make these points of supporting points coincide, but there is a unique such translation. Hence, all projections of K and L' along $[\phi^{-1}(0)]$ coincide. Thus, for any $w \in [\phi^{-1}(0)]$, the support functions of K and L' coincide along $w^{\perp} \cap S^2$. Since any unit vector can be obtained in this way, it follows that the support functions of K and L' are identical, and thus K and L' are the same body, which completes the proof of Theorem 2.

2.4 Duality and Fixed Center of Rotation

The assumption that projections lack rotational symmetry in the last theorem was necessary for a technical part of the argument. However, it is shown in this section that the same conclusion can be reached if a suitable center of rotation is fixed in the bodies about which all projections can be rotated. A benefit of this result is that no assumption about the symmetry of projections is required, though the freedom to translate the bodies is lost. Hence, for any direction in which the projections are parallel, they must coincide. The strategy will be to compare projections of the hypothesized bodies K and L while simultaneously comparing the sections of the corresponding dual bodies K^* and L^* .

In order to establish that SO(2) congruence of projections of a body implies SO(2) congruence of sections of the dual body, it can be shown that linear transformations interact very specifically with the dual operation. For any body $B \subset \mathbb{R}^n$ and any linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, we will show for any $u \in S^{n-1}$, $h_{(AB)^*}(u) = h_{(A^{-1})^t B^*}(u)$, and therefore $(AB)^* = (A^{-1})^t B^*$. For any scalar λ and unit vector u, then $\lambda u \in AB$ if and only if $\lambda(A^{-1}u) \in B$, and so $\rho_B(Au) = \rho_{A^{-1}B}(u)$. In particular, for a positive scalar c, $\rho_B(cu) = \rho_{c^{-1}B}(u) = c^{-1}\rho_B(u)$.

The transpose operator A^t is defined by $(Ax) \cdot y = x \cdot (A^t y)$ for all $x, y \in \mathbb{R}^n$, and so

$$h_{AB}(u) = \sup_{x \in AB} \{x \cdot u\} = \sup_{y \in B} \{(Ay) \cdot u\} = \sup_{y \in B} \{y \cdot A^t u\} = h_B(A^t u).$$

Since for any body B and $u \in S^{n-1}$, $h_B(u) = 1/\rho_{B^*}(u)$, it follows that $h_{(AB)^*}(u) = 1/\rho_{AB}(u) = 1/\rho_B(A^{-1}u)$. Define $a = ||A^{-1}u||$, and define the unit vector $\alpha = a^{-1}(A^{-1}u)$, then

$$h_{(AB)^*}(u) = \frac{1}{\rho_B(A^{-1}u)} = \frac{1}{\rho_B(a\alpha)} = \frac{a}{\rho_B(\alpha)} = ah_{B^*}(\alpha) = h_{B^*}(A^{-1}u) = h_{(A^{-1})^tB^*}(u).$$

In conclusion, the support functions of (AB^*) and $(A^{-1})^t B^*$ coincide, and therefore $(AB)^* = (A^{-1})^t B^*$.

If $K_{|\xi^{\perp}}$ is rotated about the origin into $L_{|\xi^{\perp}}$ through an angle ϕ , this is represented by the linear operator

$$\Phi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}.$$

An elementary computation shows that Φ^{-1} is a rotation about $-\phi$, and therefore $(\Phi^{-1})^t = \Phi$. As discussed in the introduction, $(K_{|\xi^{\perp}})^* = K^* \cap \xi^{\perp}$. Since $\phi(K_{|\xi^{\perp}}) = L_{|\xi^{\perp}}$, we can conclude that $(\phi(K_{|\xi^{\perp}}))^* = (L_{|\xi^{\perp}})^* = L^* \cap \xi^{\perp}$, and $(\phi(K_{|\xi^{\perp}}))^* = (\phi^{-1})^t (K_{|\xi^{\perp}})^* = \phi(K^* \cap \xi^{\perp})$. Thus, if projections of K and L are SO(2) congruent by a rotation of ϕ , then the sections of K^* and L^* are congruent by rotating by the same angle.

For the bodies K and L so that projections can be rotated about the origin into each other, Lemma 7 implies that $S^2 = \phi^{-1}(0) \cup \phi^{-1}(\pi) \cup \Sigma$. From the above discussion, Lemma 8 applied to the bodies K^* and L^* gives that $S^2 = \phi_2^{-1}(0) \cup \phi_2^{-1}(\pi) \cup \Sigma_2$, where $\phi_2(\xi)$ is the angle by which sections of the dual bodies orthogonal to ξ can be rotated into each other, and Σ_2 is set of unit vectors ξ with the property that the function $R_{K^*}(\theta) =$ $(\rho_{K^*}^2(\theta) + \rho_{K^*}^2(-\theta))/2$ takes a constant value on $\xi^{\perp} \cap S^2$. It has just been shown that for all $\xi \in S^2$, $\phi(\xi) = \phi_2(\xi)$, and therefore $\xi \notin \phi^{-1}(0) \cup \phi^{-1}(\pi)$ implies $\xi \in \Sigma \cap \Sigma_2$. **Lemma 12.** If the orthogonal projections of convex bodies K and L in \mathbb{R}^3 are congruent through a rotation about the origin, then $S^2 = \phi^{-1}(0) \cup \phi^{-1}(\pi)$.

Proof. If $\xi \in S^2 \setminus (\phi^{-1}(0) \cup \phi^{-1}(\pi))$, then Lemma 7 implies that there is a value γ_1 such that for all $\theta \perp \xi$,

$$\frac{1}{\rho_{K^*}(\theta)} + \frac{1}{\rho_{K^*}(-\theta)} = \gamma_1.$$

Similarly, Lemma 8 implies there is a value γ_2 such that for all such vectors θ ,

$$\rho_{K^*}^2(\theta) + \rho_{K^*}^2(-\theta) = \gamma_2.$$

Consider the equations $\frac{1}{x} + \frac{1}{y} = \gamma_1$ and $x^2 + y^2 = \gamma_2$. We know some positive solution exists to this equation by the above formulas, but we want the solution to be unique. The second equation is a circle of radius $\sqrt{\gamma_2}$, and the other is the graph of the curve $y = \frac{x}{\gamma_1 x - 1}$, which can only intersect the circle once or twice in the first quadrant.

If changing θ yields distinct pairs $(\rho_{K^*}(\theta), \rho_{K^*}(-\theta))$, the continuity of the radial function would imply there is a continuum of solutions, which contradicts the fact that one and two are the only options. Thus, given $\xi \in \Sigma \cap \Sigma_2$, $\rho_{K^*}(\theta) = c$ for some constant c for all $\theta \perp \xi$. Since any two great circles intersect, this same constant holds for all $\xi \in \Sigma$. It follows that these sections are disks, since the distance from the origin is constant on ξ^{\perp} .

Since $(K_{|\xi^{\perp}})^* = K^* \cap \xi^{\perp} = a$ disk, the dual of the dual of a body is the original body, and the dual of a disk is a disk, it follows that $K_{|\xi^{\perp}}$ is a disk. This implies that $\xi \in \phi^{-1}(0) \cup \phi^{-1}(\pi)$ for all $\xi \in \Sigma$.

Given this lemma, it can now be proven that the bodies either coincide or can be reflected about the origin into each other. The argument originated from the proof of a special case of a different theorem in [3] page 22. Golubyatnikov weakens the assumption about asymmetric projections, without completely removing it, and adds that the support functions must be smooth. The portion of the argument used below, however, makes no use of these assumptions and is perfectly applicable to the hypothesis of the following theorem.

Theorem 3. Let $K, L \subset \mathbb{R}^3$ be convex bodies containing the origin so that for all $\xi \in S^2$, $K_{|\xi^{\perp}}$ rotated about the origin through a suitable angle is equal to $L_{|\xi^{\perp}}$. Then K = L or K is the reflection of L about the origin.

Proof. As a result of Lemma 12, $S^2 \setminus (\phi^{-1}(0) \cup \phi^{-1}(\pi))$ is empty, and Theorem 1 finishes the proof if $S^2 \setminus \phi^{-1}(0) = \emptyset$. Suppose $\xi \in S^2 \setminus \phi^{-1}(0)$. If $\phi^{-1}(0)$ is a circle, notice that $S^2 \setminus \phi^{-1}(0) \subset \phi^{-1}(\pi)$ implies that $\phi^{-1}(\pi)$ contains both hemispheres about the great circle $\phi^{-1}(0)$. Since $\phi^{-1}(\pi)$ is a closed set, it must include the great circle $\phi^{-1}(0)$, and therefore $\phi^{-1}(\pi) = S^2$.

If $\phi^{-1}(0)$ intersects every meridian connecting ξ and $-\xi$ but is not a great circle, an argument identical to the proof of Lemma 11 guarantees the existence of nonparallel vectors w_1, w_2 such that a dense set of meridians intersect $\phi^{-1}(0)$ at points noncoplanar with w_1 and w_2 . It follows similarly that $\xi \in \phi^{-1}(0)$, which is a contradiction.

The final case is when there is a meridian m connecting ξ and $-\xi$ such that $\phi^{-1}(0)$ doesn't intersect m or -m. Since $\phi^{-1}(0)$ is closed, m and -m are contained in the open compliment. The compliment of $\phi^{-1}(0)$ is contained in $\phi^{-1}(\pi)$, so it follows that m and -m are contained in an open neighborhood in $\phi^{-1}(\pi)$. We can then pick a great circle in this neighborhood, which does not contain ξ , which is contained in $\phi^{-1}(\pi)$. Hence, $\phi^{-1}(\pi)$ intersects every meridian connecting ξ . If you take the body K' obtained by reflecting Kabout the origin, then the inverse image of 0 for rotations of K' into L intersects every meridian connecting ξ and $-\xi$. The above argument yields the same contradiction, and therefore $S^2 = \phi^{-1}(0)$ or $S^2 = \phi^{-1}(\pi)$.

CHAPTER 3

Conclusion

This thesis addresses an open problem in convex geometry concerning two bodies Kand L which have orthogonal projections that are SO(2) congruent. The goal is to prove that either K is a translation of L or that K is L reflected about some point in \mathbb{R}^3 . Many standard tools of the field which are used in our analysis, such as the support function and polar duality, are presented in the introduction. Since any rotation in the plane can be decomposed into a rotation about the origin followed by a translation, various assumptions about the type of SO(2) congruence are considered separately.

We first present Lieberman's proof of Süss's lemma, which states that if every orthogonal projection of K is a shift of the corresponding orthogonal projection of L, then K and Lare translation congruent in three dimensional space. We then attack the issue of SO(2)symmetry and present the proof of an important lemma of Golubyatnikov's which holds under the most general assumptions. Next, Golubyatnikov's proof that the open question can be solved under the assumption that no projection has nontrivial rotational symmetry is presented in detail. We then solve a new special case of the problem, where it is assumed all of the projections are rotationally congruent without any shifts.

The intent of this paper was to address a very simple geometric question. However, no complete conclusion has been made under the most general conditions. The argument presented breaks down for no obvious geometrical reason. The topological issue of continuity is what requires that no projection has any symmetry in order to prove the result. The question of whether the symmetry assumption can be removed completely remains open. However, we have presented several results which paint a substantial picture of the problem as well as provide insight into what to expect of the general conclusion.

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