GEOMETRIC PROPERTIES OF ORBITS OF INTEGRAL OPERATORS

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by

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INTRODUCTION

Integral operators are one of the better studied operators. They are useful in many disciplines, occurring as Fourier or Laplace transforms, convolution operators, or as plainly as an indefinite integral operator. We will direct out attention to the latter form of an integral operator. One of the more general forms is the Fredholm operator of the first kind. Suppose that E is one of the spaces $C[a,b]$ or $L^p[a,b]$. Let $k(x,t)$ be a function defined on the rectangle $[a,b] \times [a,b]$. For $f \in E$, we define the above Fredholm operator by $F_1 f = g$, where

$$g(x) = \int_a^b k(x,t) f(t) dt.$$

A somewhat more general Fredholm operator, that of the second kind, is defined by $F_2 f = g$, where

$$g(x) = f(x) - \int_a^b k(x,t) f(t) dt.$$

Such integral operators are useful in the study of differential equations (see Taylor [21]).

If we modify the kernel in the Fredholm operator in a special way, we obtain another special class of integral operators, called the Volterra operators. If $k(x,t) = 0$ whenever $t > x$, we have a Volterra operator (call it V_1) of type one. We define $V_1 f = g$, by

$$g(x) = \int_a^x k(x,t) f(t) dt.$$

There is also a Volterra analog of F_2, defined similarly. If we further require that

$$k(x,t) = \begin{cases} 0 & \text{if } t > x \\ 1 & \text{if } t \le x \end{cases}$$

and $a = 0$, we find the Volterra operator T, which in some sense is the prototypical indefinite integral operator. It is given by

$$(Tf)(x) = \int_0^x f(t)dt.$$

Much is know about the Volterra operator: It has an adjoint T^* given by

$$(T^*f)(x) = \int_x^1 f(t)dt,$$

which satisfies $(Tx, y) = (x, T^*y)$ on the Hilbert space $L^2[0,1]$. It is a compact operator, meaning that the image of the closed unit ball is relatively compact (i.e. its closure is compact). It has spectrum $\sigma(T) = \{0\}$ and spectral radius 0, is quasinilpotent but not nilpotent, and has operator norm $\|T\| = 2/\pi$. The latter result may be found in Little [13], while the others may be found in Rudin [18].

Early in the history of functional analysis, the invariant subspace problem was posed. Given a Banach space B and an operator S on that space, does there exist a non-trivial subspace M such that $SM \subset M$? This question was answered in the negative by Enflo [4], who constructed a Banach space and an operator on the subspace which had no non-trivial invariant subspaces. Beyond the existence of invariant subspaces, one wonders about the nature of the invariant subspaces of a given operator. Invariant subspaces are easily obtained with the notion of the orbit of an operator. Given a Banach space B, an operator S on B, and an element $x \in B$, the n-th iterate of x under S is $S^n x$, where $S^1 x = Sx$. The collection of all iterates of x under S is called the orbit of x under S and is denoted by $\mathrm{Orb}(S, x)$. A so-called elementary invariant subspace for the operator S is the closure of the linear span of the orbit. That is, $\overline{\mathrm{span}}\{\mathrm{Orb}(S, x)\}$ in an invariant subspace for S with the fixed element x. (For a discussion of these notions, see Beauzamy [2].)

The orbit of a fixed element under a given operator may exhibit some special cyclic properties. As before, let B be a Banach space, let S be a bounded linear operator on B, and fix $x \in B$. The operator S is called cyclic, and x is said to be a cyclic vector for S, if $\text{span}\{\text{Orb}(S, x)\}$ is dense in B. An example of such an operator is the multiplication operator on $C[a, b]$ with the vector 1. This is just a restatement of the classical Weierstrass Approximation Theorem. Our Volterra operator is also cyclic for $C[a, b]$ with the vector 1. Two related notions are that of hyper-cyclicity and super-cyclicity, both of which the Volterra operator does not exhibit. An operator S is hyper-cyclic if its orbit is dense in B and is super-cyclic if the projective span of the orbit is dense in B. That is, if $\{\alpha_n T^n x : n \in \mathbb{Z}, \alpha \in \mathbb{R}\}$ is dense in B. This last result was proved independently by Saavedra and Lerena in [19] and by Gutierrez and Rodriguez in [11].

A Schauder basis for a Banach space B is a subset of non-zero vectors $\{x_n\}_{n=1}^{\infty}$ of B such that for each $x \in B$, there exists a unique collection $\{\alpha_n\}_{n=1}^{\infty}$ of scalars such that

$$x = \sum_{n=1}^{\infty} \alpha_n x_n.$$

Any such space will have the approximation property (any operator can be approximated by finite-rank operators) and so will also exhibit operators with non-trivial invariant subspaces. It will be shown that, under very general conditions on the starting element, an orbit of the Volterra operator cannot be a Schauder basis for its closed linear span. Lacunary subsequences of the orbit, however, will be seen to be Schauder bases for their closed linear span.

CHAPTER 1

Iterates of the Volterra Operator

We consider here orbits of the integral operator T on $C[0,1]$ with the supremum norm. Given $f \in C[0,1]$, the integral operator is defined by

$$Tf(x) = \int_0^x f(t)dt.$$

The n-th iterate of T on f is $T^n f = T(T^{n-1}f)$, where $T^0 f = f$. The orbit of f under T is the collection of all iterates of T on f. This will be denoted by $\text{Orb}(T,f)$ so that $\text{Orb}(T,f) = \{T^n f\}_{n=0}^{\infty}$. The supremum norm will be denoted by $\|\cdot\|$ and the L_p norm will be denoted by $\|\cdot\|_p$. Two function sequences (f_n) and (g_n) are *asymptotically equal on* $[a,b]$ if $\lim_{n\to\infty} f_n(x)/g_n(x) = 1$ for all $x \in [a,b]$. We will denote this by $f_n \sim g_n$. We say (f_n) is *asymptotically less that or equal* (g_n) if $\lim_{n\to\infty} f_n(x)/g_n(x) \leq 1$ on $[a,b]$. This is denoted by $f_n \lesssim g_n$.

The computation of $T^n f$ is a tedious and cumbersome affair, so we start with a tool to make it more manageable. This formula may be found in Taylor ([21] p. 291) without proof, so we offer our own proof here.

Proposition 1. *If $f \in C[0,1]$, then*

$$T^n f(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt.$$

Proof. This result follows from induction and Fubini's theorem. The base case with $n = 1$ is just the definition of the integral operator, so assume the result holds for an arbitrary n. Then

$$
\begin{aligned}
T^{n+1}f(x) &= T(T^n f)(x) \\
&= \int_0^x T^n f(t)dt \\
&= \int_0^x \left[\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s)ds\right] dt \\
&= \frac{1}{(n-1)!}\int_0^x f(s)\left[\int_s^x (t-s)^{n-1}dt\right] ds \\
&= \frac{1}{n!}\int_0^x f(s)\left[(t-s)^n\right]_{t=s}^x ds \\
&= \frac{1}{n!}\int_0^x (x-s)^n f(s)ds.
\end{aligned}
$$

$\square$

Corollary 1. *Let $p \geq 0$. Then*

$$
T^n x^p = \frac{\Gamma(p+1)}{\Gamma(n+p+1)} x^{n+p}.
$$

Proof.

$$
\begin{aligned}
T^n x^p &= \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} t^p dt \\
&= \frac{x^{n+p-1}}{(n-1)!}\int_0^x \left(1-\frac{t}{x}\right)^{n-1}\left(\frac{t}{x}\right)^p dt.
\end{aligned}
$$

Make the substitution $u = t/x$. Then

$$
\begin{aligned}
T^n x^p &= \frac{x^{n+p}}{(n-1)!}\int_0^1 (1-u)^{n-1} u^p du \\
&= \frac{x^{n+p}}{(n-1)!}\frac{\Gamma(n)\Gamma(p+1)}{\Gamma(n+p+1)} \\
&= \frac{\Gamma(p+1)}{\Gamma(n+p+1)} x^{n+p}.
\end{aligned}
$$

Here we have noticed that the last integral is the beta function

$$
B(x,y) = \int_0^1 (1-t)^{x-1} t^{y-1} dt,
$$

which is related to the gamma function by $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. $\square$

Example 1. For $p = 0$ we have that $T^n 1(x) = x^n/n!$.

Example 2. The exponential function has the Taylor series expansion $e^x = \sum_{j=0}^{\infty} x^j/j!$. This expansion converges uniformly in x so that

$$T^n e^x = T^n \left(\sum_{j=0}^{\infty} \frac{x^j}{j!} \right) = \sum_{j=0}^{\infty} T^n \frac{x^j}{j!} = \sum_{j=0}^{\infty} \frac{x^{n+j}}{(n+j)!}.$$

Although they are not continuous, we will have occasion to make use of the orbits of characteristic and step functions. Note that Proposition 1 still applies.

Corollary 2. *Let* $0 \leq a < b \leq 1$ *and let* $\chi_{[a,b]}$ *be the characteristic function on* $[a, b]$*. Then*

$$T^n \chi_{[a,b]}(x) = \begin{cases} 0 & x < a \\ \frac{1}{n!}(x-a)^n & a \leq x \leq b \\ \frac{1}{n!}[(x-a)^n - (x-b)^n] & x > b. \end{cases}$$

Proof. Suppose that $x < a$. Then

$$\begin{aligned} T^n \chi_{[a,b]}(x) &= \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} \chi_{[a,b]}(x) dx \\ &= \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} \cdot 0 \cdot dx \\ &= 0. \end{aligned}$$

This holds for $x = a$ as well, since the support of the integrand is then a set of measure zero.

Now suppose that $a < x \leq b$. Then

$$\begin{aligned}
T^n \chi_{[a,b]}(x) &= \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} \chi_{[a,b]}(x) dx \\
&= \frac{1}{(n-1)!} \int_0^a (x-t)^{n-1} \chi_{[a,b]}(x) dx \\
&\qquad + \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \chi_{[a,b]}(x) dx \\
&= 0 + \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} dx \\
&= \frac{1}{(n-1)!} \left[-\frac{1}{n} (x-t)^n \right]_{t=a}^{x} \\
&= \frac{1}{n!} (x-a)^n.
\end{aligned}$$

Finally suppose that $x > b$. Then

$$\begin{aligned}
T^n \chi_{[a,b]}(x) &= \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} \chi_{[a,b]}(x) dx \\
&= \frac{1}{(n-1)!} \int_0^a (x-t)^{n-1} \chi_{[a,b]}(x) dx \\
&\qquad + \frac{1}{(n-1)!} \int_a^b (x-t)^{n-1} \chi_{[a,b]}(x) dx \\
&\qquad\qquad + \frac{1}{(n-1)!} \int_b^x (x-t)^{n-1} \chi_{[a,b]}(x) dx \\
&= 0 + \frac{1}{(n-1)!} \int_a^b (x-t)^{n-1} dx \\
&\qquad\qquad + \frac{1}{(n-1)!} \int_b^x (x-t)^{n-1} \cdot 0 dx \\
&= \frac{1}{(n-1)!} \left[-\frac{1}{n} (x-t)^n \right]_{t=a}^{b} \\
&= \frac{1}{n!} \left[(x-a)^n - (x-b)^n \right].
\end{aligned}$$

$\square$

Corollary 3. *Let $E_j = [a_j, b_j] \subset [0,1]$ for $j = 1, \ldots, m$, where $b_{j-1} \leq a_j$ for all j. Let f be the simple function $f(x) = \sum_{j=1}^m \alpha_j \chi_{E_j}(x)$, where $\alpha_j \in \mathbb{R}$. Then*

$$T^n f(x) = \sum_{j=1}^m \alpha_j T^n \chi_{E_j}(x).$$

Proof. This is immediate from the linearity of T^n and Corollary 2. $\square$

We will now look for where the norm of $T^n f$ is attained. We begin by noticing that intervals near 0 contribute more to the norm than those intervals closer to 1.

Lemma 1. *Let $a_1 < a_2$, b_1, $b_2 \leq 1$ and m, $M > 0$. Then*

$$T^n m\chi_{[a_1,b_1]}(x) \geq T^n M\chi_{[a_2,b_2]}(x)$$

for sufficiently large n. Furthermore,

$$\lim_{n\to\infty} \frac{T^n M\chi_{[a_2,b_2]}(x)}{T^n m\chi_{[a_1,b_1]}(x)} = 0,$$

for all $x > a_1$.

Proof. There are three cases to consider and all are proved in a similar manner, so we will look at just one. Suppose that $0 \leq a_1 < a_2 < b_1 < b_2 \leq 1$. Then $\chi_{[a_1,b_1]}(x) \geq \chi_{[a_1,a_2]}(x)$ and $\chi_{[a_2,b_2]}(x) \leq \chi_{[a_2,1]}(x)$ for all $x \in [0,1]$. We thus need

$$T^n m\chi_{[a_1,b_1]}(x) - T^n M\chi_{[a_2,b_2]}(x) \geq T^n m\chi_{[a_1,a_2]}(x) - T^n M\chi_{[a_2,1]}(x) > 0.$$

This happens when

$$m[(x-a_1)^n - (x-a_2)^n] > M(x-a_2)^n$$

which requires

$$n > \ln \frac{m+M}{m} / \ln \frac{x-a_1}{x-a_2}.$$

The denominator is minimized when $x = 1$, so we need

$$n > \ln \frac{m+M}{m} / \ln \frac{1-a_1}{1-a_2}.$$

We will now establish that $\lim_{n\to\infty} T^n M\chi_{[a_2,b_2]}(x)/T^n m\chi_{[a_1,b_1]}(x) = 0$. There are five sub-cases to look at.

Sub-Case 1. If $x \le a_1$ then the numerator and denominator are both zero and so the quotient is undefined.

Sub-Case 2. Suppose that $a_1 < x \le a_2$. Then $T^n M\chi_{[a_2,b_2]}(x) = 0$ and $T^n m\chi_{[a_1,b_1]}(x) = \frac{m}{n!}(x-a_1)^n$ so that

$$\lim_{n\to\infty} \frac{T^n M\chi_{[a_2,b_2]}(x)}{T^n m\chi_{[a_1,b_1]}(x)} = 0.$$

Sub-Case 3. Suppose that $a_2 < x \le b_1$. Then $T^n M\chi_{[a_2,b_2]}(x) = \frac{M}{n!}(x-a_2)^n$ and $T^n m\chi_{[a_1,b_1]}(x) = \frac{m}{n!}(x-a_1)^n$. Now

$$\begin{aligned}\lim_{n\to\infty} \frac{T^n M\chi_{[a_2,b_2]}(x)}{T^n m\chi_{[a_1,b_1]}(x)} &= \lim_{n\to\infty} \frac{M(x-a_2)^n}{m(x-a_1)^n} \\ &= \lim_{n\to\infty} \frac{M}{m}\left(\frac{x-a_2}{x-a_1}\right)^n \\ &= 0\end{aligned}$$

since $(x-a_2)/(x-a_1) < 1$.

Sub-Case 4. Suppose that $b_1 < x \le b_2$. Then $T^n M\chi_{[a_2,b_2]}(x) = \frac{M}{n!}(x-a_2)^n$ and $T^n m\chi_{[a_1,b_1]}(x) = \frac{m}{n!}[(x-a_1)^n - (x-b_1)^n]$. Now

$$\begin{aligned}\lim_{n\to\infty} \frac{T^n M\chi_{[a_2,b_2]}(x)}{T^n m\chi_{[a_1,b_1]}(x)} &= \lim_{n\to\infty} \frac{M(x-a_2)^n}{m[(x-a_1)^n - (x-b_1)^n]} \\ &= \lim_{n\to\infty} \frac{M}{m}\left(\frac{x-a_2}{x-a_1}\right)^n \frac{1}{1-\left(\frac{x-b_1}{x-a_1}\right)^n} \\ &= 0\end{aligned}$$

since $(x-a_2)/(x-a_1) < 1$ and $(x-b_1)/(x-a_1) < 1$.

Sub-Case 5. Suppose that $x > b_2$ in case that $b_2 < 1$. Then $T^n M\chi_{[a_2,b_2]}(x) = \frac{M}{n!}[(x -$

$a_2)^n - (x-b_2)^n]$ and $T^n m\chi_{[a_1,b_1]}(x) = \frac{m}{n!}[(x-a_1)^n - (x-b_1)^n]$. Now

$$\begin{aligned}\lim_{n\to\infty} \frac{T^n M\chi_{[a_2,b_2]}(x)}{T^n m\chi_{[a_1,b_1]}(x)} &= \lim_{n\to\infty} \frac{M[(x-a_2)^n - (x-b_2)^n]}{m[(x-a_1)^n - (x-b_1)^n]} \\ &= \lim_{n\to\infty} \frac{M}{n}\left(\frac{x-a_2}{x-a_1}\right)^n \frac{1-\left(\frac{x-b_2}{x-a_2}\right)^n}{1-\left(\frac{x-b_1}{x-a_1}\right)^n} \\ &= 0\end{aligned}$$

since $(x-a_2)/(x-a_1) < 1$, $(x-b_2)/(x-a_2) < 1$ and $(x-b_1)/(x-a_1) < 1$.

The other two cases are $a_1 < b_1 \le a_2 < b_2$ and $a_1 < a_2 < b_2 \le b_1$. □

This lemma has an application in the following monotonicity result for a certain class of functions. Let $a = \inf\{x \in [0,1] : f(x) \neq 0\}$ and let c be the least zero of f in $(a,1]$. If f has no zeros in $(a,1]$, take c to be 1. Notice that for any continuous function f that is not identically 0 on $[0,1]$, $a < c$. It may or may not happen that $f(a) = 0$. We will refer to any interval $(a,b) \subset (a,c)$ as an *initial part of the support of* f. We will also say that f *starts out positive* if $f > 0$ on the initial part of its support. That f *starts out negative* means that $-f$ starts out positive.

Proposition 2. *If f starts out positive (negative), then the orbits of f end up positive (negative) after a finite number of iterations of T. In fact, the orbits of f are eventually monotonic and $\|T^n f\|$ is attained at 1.*

Proof. If f is strictly positive or negative on its support then there is nothing to do, so suppose it's not. Suppose also that f starts off positive and that (α,β) is the initial part of the support of f. Then there exists $[a,b] \subset (\alpha,\beta)$ and $\epsilon > 0$ such that $f(x) \ge \epsilon$ for $x \in [a,b]$. Let $M = \sup_{x\in[b,1]} |f(x)|$ and define

$$g(x) = \epsilon \cdot \chi_{[a,b]} \quad \text{and} \quad h(x) = -M \cdot \chi_{[b,1]}.$$

Then $f(x) \ge g(x) + h(x)$ and so $T^n f(x) \ge T^n g(x) + T^n h(x)$. Now, by Lemma 1,

$$T^n f(x) \ge T^n g(x) + T^n h(x) > 0$$

for sufficiently large n. Once $T^n f$ is positive, integrating once more yields a monotonic function. If f starts off negative, apply the above to $-f$. $\square$

It should be noted that the above proposition does not necessarily hold if 0 is an accumulation point of zeros of f and f alternates signs around those zeros. In particular, such a function need not have a preference towards positivity or negativity under T.

Proposition 3. *There exists a function $F \in C[0,1]$ such that $T^n f(x)$ alternates sign for all n.*

Proof. We will construct a function with the desired properties. Let $E_j = \left(\frac{1}{j+1}, \frac{1}{j}\right]$ for $j \in \mathbb{N}$ and define

$$f_m(x) = \sum_{j=1}^{m} (-1)^{j+1} \chi_{E_j}(x),$$

also for $m \in \mathbb{N}$. Then, if $x \in E_{j_0}$,

$$Tf_m(x) = (-1)^{j_0+1}\left(x - \tfrac{1}{j_0+1}\right) + \sum_{j=j_0+1}^{m} (-1)^{j+1}\left(\tfrac{1}{j} - \tfrac{1}{j+1}\right)$$

by Corollary 3. It is clear that each Tf_m is continuous and linear on each interval E_j, $j = 1, \ldots, m$. Also, Tf_m has a zero in each interval E_j since Tf_m has opposite signs for $x = 1/k$ and $x = 1/(k+1)$. This is so since $\sum_{j=k}^{m}(-1)^j(1/j - 1/(j+1))$ alternates signs for successive values of k. The sequence $\{Tf_m(x)\}_{m=1}^{\infty}$ converges uniformly in x to the function

$$F(x) = (-1)^{j_0+1}\left(x - \frac{1}{j_0+1}\right) + \sum_{j=j_0+1}^{\infty} (-1)^{j+1}\left(\frac{1}{j} - \frac{1}{j+1}\right)$$

since

$$|F(x) - Tf_m(x)| = \left|\sum_{j=m+1}^{\infty} (-1)^{j+1}\left(\frac{1}{j} - \frac{1}{j+1}\right)\right|$$

tends to 0 as the tail of a convergent series. That F alternates sign on each interval E_j follows since each Tf_m does. Finally,

$$T^n F(x) = T^n\left(\lim_{m\to\infty} Tf_m(x)\right) = \lim_{m\to\infty} T^{n+1} f_m(x)$$

This convergence is uniform in x since T preserves uniform convergence. Each function T^nF alternates sign on $[0,1]$ so that F has no preference towards positivity or negativity under T. This is so since

$$T^nF(1/k) = \frac{1}{(n+1)!}\sum_{j=k}^{\infty}(-1)^{j+1}\left[\left(\frac{1}{k}-\frac{1}{j+1}\right)^{n+1}-\left(\frac{1}{k}-\frac{1}{j}\right)^{n+1}\right]$$

alternates sign for successive values of k.

Note that some values of F can be calculated. For example,

$$F(1) = 2\ln 2 - 1.$$

We begin by finding the series expansion for $\ln(1+x)$, which is obtained by first differentiating the function.

$$\begin{aligned}\frac{d}{dx}\ln(1+x) &= \frac{1}{1+x}\\ &= \frac{1}{1-(-x)}\\ &= \sum_{j=0}^{\infty}(-x)^j.\end{aligned}$$

Integrating now returns $\ln(1+x)$ and its series expansion.

$$\begin{aligned}\ln(1+x) &= \int\sum_{j=0}^{\infty}(-x)^j\,dx\\ &= \sum_{j=0}^{\infty}\int(-x)^j\,dx\\ &= \sum_{j=0}^{\infty}(-1)^j\frac{x^{j+1}}{j+1}.\end{aligned}$$

The interchange of the sum and integral is valid since the series is uniformly convergent on its radius of convergence. We can now see that

$$\ln 2 = \ln(1+1) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}.$$

We can now calculate $F(1)$:

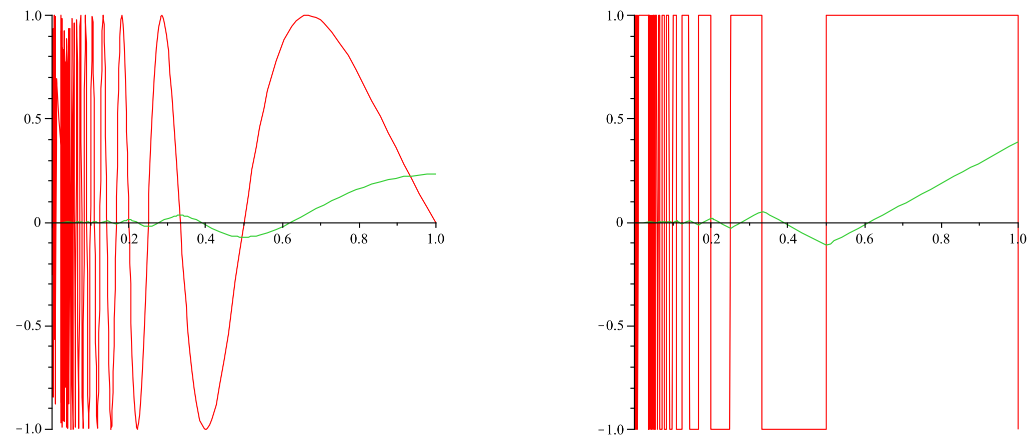
$$
\begin{aligned}
F(1) &= \sum_{j=1}^{\infty}(-1)^{j+1}\left(\frac{1}{j}-\frac{1}{j+1}\right) \\
&= \sum_{j=1}^{\infty}\left(\frac{(-1)^{j+1}}{j}+\frac{(-1)^{j+2}}{j+1}\right) \\
&= \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}+\sum_{j=1}^{\infty}\frac{(-1)^{j}}{j+1}
\end{aligned}
$$

This last step is valid since the series is absolutely convergent. Now make the substitution $j = k+1$ in the first series.

$$
\begin{aligned}
F(1) &= \sum_{k=0}^{\infty}\frac{(-1)^{k+2}}{k+1}+\sum_{j=1}^{\infty}\frac{(-1)^{j}}{j+1}+1-1 \\
&= \sum_{k=0}^{\infty}\frac{(-1)^{k}}{k+1}+\sum_{j=0}^{\infty}\frac{(-1)^{j}}{j+1}-1 \\
&= \ln 2+\ln 2-1 \\
&= 2\ln 2-1.
\end{aligned}
$$

$\square$

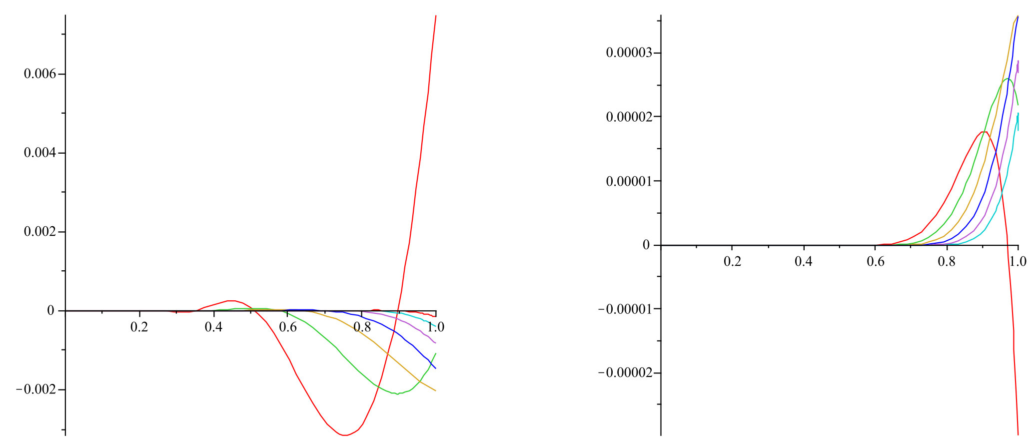
The preceding example was designed to mimic the behavior of $\sin(1/x)$. More accurately, it is mimicking $-\sin(\pi/x)$ and the function F above is $T\,\mathrm{sgn}(-\sin(\pi/x))$ almost everywhere. The graphs of $-\sin(\pi/x)$ and $T(-\sin(\pi/x))$ and $f=\lim_{m\to\infty} f_m$ and F are shown below for comparison.



(a) $-\sin(\pi/x)$ and $T(-\sin(\pi/x))$ (b) f and $F = Tf$

FIGURE 1

The next figures show some of the early oscillatory behavior of the orbits of $-\sin(\pi/x)$.



(a) Iterates 3–9 (b) Iterates 10–15

FIGURE 2

It must be noted that each graph has been scaled by a factor of $(n-1)!$, according to the iterate, so that many graphs may be seen at once.

We will now acquire bounds on the norm of the iterates.

Proposition 4.

(a) If $f \in C[0,1]$, then there are constants m and M_1 such that

$$m/(2^{n^2+2n}n!) \leq \|T^{n+3}f\| \leq M_1/(n+3)!.$$

(b) If $f \in L_p[0,1]$, then there are constants m and M_2 such that

$$m/(2^{n^2+2n}n!\sqrt[p]{2^{n-1}}) \leq \|T^{n+3}f\|_p \leq M_2/((n+2)!\sqrt[p]{np+2p+1});$$

if f happens to be bounded, then

$$\|T^{n+3}f\|_p \leq M_1/((n+3)!\sqrt[p]{np+3p+1}).$$

Proof. Suppose that f is integrable and bounded on $[0,1]$. Since f is bounded there so that $|f(x)| \leq M_1$ for some positive number M_1. Thus

$$\begin{aligned}
\|T^n f\| &= \sup_{x\in[0,1]} |T^n f(x)| \\
&\leq \sup_{x\in[0,1]} |T^n M_1| \\
&= \frac{M_1}{n!},
\end{aligned}$$

and

$$\begin{aligned}
\|T^n f\|_p^p &\leq \|T^n M_1\|_p^p \\
&= \int_0^1 \left|\frac{M_1}{n!}x^n\right|^p dx \\
&= \left(\frac{M_1}{n!}\right)^p \int_0^1 x^{np} dx \\
&= \left(\frac{M_1}{n!}\right)^p \left.\frac{x^{np+1}}{np+1}\right|_0^1 \\
&= \left(\frac{M_1}{n!}\right)^p \frac{1}{np+1}.
\end{aligned}$$

If $f \in L_p[0,1]$ and f is not bounded, then

$$\begin{aligned}\|T^{n+3}f\|_p^p &\le \|T^{n+2}(Tf)\|_p^P \\ &\le \|T^{n+2}M_2\|_p^p \\ &\le \left(\frac{M_2}{(n+2)!}\right)^p \frac{1}{np+2p+1}\end{aligned}$$

so that

$$\|T^{n+3}f\|_p \le \frac{M_2}{(n+2)!\sqrt[p]{np+2p+1}},$$

where $M_2 = \sup_{x\in[0,1]} |Tf(x)|$.

There is more work to do for the lower bound. Suppose that $f \neq 0$ is integrable on $[0,1]$. Let $F = T^{n+3}f$ so that $F \in C^{(n+2)}[0,1]$. Since F is continuous and not identically 0, there is an $a \in (0,1)$ and an $\eta > 0$ such that $F \neq 0$ on $(a-\eta, a+\eta)$. Choose $\epsilon < \eta$ so that $F \neq 0$ on $A = [a-\epsilon, a+\epsilon]$. Since $F \in C^{(n+2)}[0,1]$, F has a series development

$$\begin{aligned}F(x) = F(a) + (x-a)F'(a) + \frac{(x-a)^2}{2!}F''(a) + \cdots \\ + \frac{(x-a)^n}{n!}F^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}F^{(n+1)}(a) + R_{n+2}(x),\end{aligned}$$

where

$$R_{n+2}(x) = \frac{F^{(n+2)}(\mu_{n+2})}{(n+2)!}(x-a)^{n+2},$$

and μ_{n+2} is between a and x. Let P be the polynomial part of F so that

$$P(x) = F(a) + (x-a)F'(a) + \cdots + \frac{(x-a)^{n+1}}{(n+1)!}F^{(n+1)}(a)$$

on A. Transform P on A to $\widetilde{P}$ on $[-1,1]$ by

$$\widetilde{P}(x) = P(\epsilon x + a)$$

so that

$$\|\widetilde{P}\|_{[-1,1]} = \|P\|_A.$$

Let $a_{n+1} = \epsilon^{n+1} F^{(n+1)}(a)/(n+1)!$ and define

$$\widehat{P}(x) = \widetilde{P}/|a_{n+1}|$$

so that $\widehat{P}$ has a leading coefficient of 1. The normalized Chebyshev polynomials $\widetilde{T}_n$ have minimal supremum norm on $[-1,1]$ amongst all n-th degree polynomials with leading coefficient 1 (see Rivlin [16]). In fact, given any such polynomial p,

$$\|p\|_{[-1,1]} \geq \|\widetilde{T}_n\|_{[-1,1]} = \begin{cases} 2^{1-n}, & n > 0; \\ 1, & n = 0. \end{cases}$$

In particular, for $n > 0$,

$$\|\widehat{P}\|_{[-1,1]} \geq 2^{1-n}.$$

Consequently,

$$\|P\|_A = \|\widetilde{P}\|_{[-1,1]} = |a_{n+1}|\|\widehat{P}\| \geq \frac{\epsilon^{n+1}|F^{(n+1)}(a)|}{2^n(n+1)!}.$$

Now,

$$\begin{aligned} \|F\|_A &= \|P + R_{n+2}\| \\ &\geq |\|P\| - \|R_{n+2}\|| \\ &\geq \left| \frac{\epsilon^{n+1}|F^{(n+1)}(a)|}{2^n(n+1)!} - \|R_{n+2}\| \right| \end{aligned}$$

and

$$\begin{aligned} \|R_{n+2}\| &\leq \frac{|F^{(n+2)}(\mu_{n+2})|}{(n+2)!}\epsilon^{n+2} \\ &= |Tf(\mu_{n+2})|\frac{\epsilon^{n+2}}{(n+2)!}. \end{aligned}$$

Therefore

$$\begin{aligned} \|F\| \geq \|F\|_A &\geq \left| \frac{\epsilon^{n+1}|F^{(n+1)}(a)|}{2^n(n+1)!} - \frac{\epsilon^{n+2}|Tf(\mu_{n+1})|}{(n+2)!} \right| \\ &= \left| \frac{\epsilon^{n+1}|T^2f(a)|}{2^n(n+1)!} - \frac{\epsilon^{n+2}|Tf(\mu_{n+1})|}{(n+2)!} \right| \\ &= \frac{\epsilon^{n+1}}{(n+1)!} \left| \frac{|T^2f(a)|}{2^n} - \frac{\epsilon|Tf(\mu_{n+2})|}{n+2} \right| \end{aligned}$$

Take $\epsilon = 1/2^n$, then

$$\begin{aligned}\frac{\epsilon^{n+1}}{(n+1)!}\left|\frac{|T^2f(a)|}{2^n} - \frac{\epsilon|Tf(\mu_{n+2})|}{n+2}\right| &= \frac{\epsilon^{n+1}}{(n+1)!}\left|\frac{|T^2f(a)|}{2^n} - \frac{|Tf(\mu_{n+2})|}{2^n(n+2)}\right| \\ &\sim \left(\frac{1}{2^n}\right)^{n+1}\frac{1}{(n+1)!}\frac{|T^2f(a)|}{2^n} \\ &= \frac{|T^2f(a)|}{2^{n^2+2n}(n+1)!}.\end{aligned}$$

Thus

$$\|T^{n+3}f\| \geq \frac{|T^2f(a)|}{2^{n^2+2n}(n+1)!}.$$

This bound holds on the interval $(a-1/2^n, a+1/2^n)$, so that

$$\begin{aligned}\|T^{n+3}f\|_p &\geq \left\|\frac{|T^2f(a)|}{2^{n^2+2n}(n+1)!}\chi_{(a-1/2^n,a+1/2^n)}(x)\right\|_p \\ &= \frac{m}{2^{n^2+2n}n!\sqrt[p]{2^{n-1}}}.\end{aligned}$$

Let $m = |T^2f(a)|$. □

So far we have examples of functions whose norms attain the lower and upper bounds. But is there a function whose norm is strictly between these bounds?

Example 3. Consider the function $e^{-1/x}$. We can't evaluate $\|T^n e^{-1/x}\|$ directly, so we will bound it above and below by approximating functions and evaluate the norms of their iterates. To this end, let $\epsilon > 0$. Define $g(x) = e^{-1/\epsilon}\chi_{[\epsilon,1]}(x)$ and $h(x) = e^{-1/\epsilon}\chi_{[0,\epsilon]} + e^{-1}\chi_{(\epsilon,1]}$. Then $g(x) \leq e^{-1/x} \leq h(x)$ on $[0,1]$ so that $\|T^n g\| \leq \|T^n e^{-1/x}\| \leq \|T^n h\|$. By Corollary 2

$$T^n g(x) = \begin{cases} 0 & x < \epsilon \\ e^{-1/\epsilon}\frac{(x-\epsilon)^n}{n!} & x \geq \epsilon \end{cases},$$

and

$$||T^n g|| = e^{-1/\epsilon}\frac{(1-\epsilon)^n}{n!}.$$

This provides us with a lower bound for $\|T^n f\|$ but we want an ϵ that maximizes it. Differentiating with respect to ϵ gives

$$\begin{aligned}\frac{d}{d\epsilon}||T^n g|| &= \frac{1}{n!}\cdot\frac{1}{\epsilon^2}e^{-1/\epsilon}(1-\epsilon)^n - \frac{1}{(n-1)!}e^{-1/\epsilon}(1-\epsilon)^{n-1}\\ &= \frac{1}{n!}e^{-1/\epsilon}(1-\epsilon)^{n-1}\left(\frac{1}{\epsilon^2}(1-\epsilon)-n\right)\\ &= \frac{1}{n!}e^{-1/\epsilon}(1-\epsilon)^{n-1}(\epsilon^{-2}-\epsilon^{-1}-n),\end{aligned}$$

which is zero when $\epsilon^{-1} = \frac{1+\sqrt{1+4n}}{2}$ so that $\epsilon = \frac{2}{1+\sqrt{1+4n}} \sim \frac{1}{\sqrt{n}}$. Now $||T^n g||$ is maximized with $\epsilon \sim \frac{1}{\sqrt{n}}$ and

$$\begin{aligned}||T^n g|| &\sim \frac{e^{-\sqrt{n}}}{n!}\left(1-\frac{1}{\sqrt{n}}\right)^n\\ &= \frac{e^{-\sqrt{n}}}{n!}\left[\left(1-\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right]^{\sqrt{n}}\\ &\sim \frac{e^{-2\sqrt{n}}}{n!}\end{aligned}$$

since $\left(1-\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} \sim e^{-1}$.

For the upper bound,

$$T^n h(x) = \begin{cases} \frac{e^{-1/\epsilon}}{n!}x^n & \text{if } 0 \le x \le \epsilon \\ \frac{e^{-1/\epsilon}}{n!}\left[x^n-(x-\epsilon)^n\right] + \frac{e^{-1}}{n!}(x-\epsilon)^n & \text{if } \epsilon < x < 1 \end{cases},$$

so $\|T^n h\| = \frac{e^{-1/\epsilon}}{n!}\left[1-(1-\epsilon)^n\right] + \frac{e^{-1}}{n!}(1-\epsilon)^n$. This gives an upper bound for $\|T^n e^{-1/x}\|$, but we are seeking an ϵ that minimizes the expression. Differentiating with respect to ϵ gives

$$\begin{aligned}\frac{d}{d\epsilon}\|T^n h\| &= \frac{1}{n!}\frac{1}{\epsilon^2}e^{-1/\epsilon}[1-(1-\epsilon)^n] + \frac{1}{n!}e^{-1/\epsilon}n(1-\epsilon)^{n-1} - \frac{e^{-1}}{n!}n(1-\epsilon)^{n-1}\\ &= \frac{e^{-1/\epsilon}}{n!}\left\{\epsilon^{-2}[1-(1-\epsilon)^n] + n(1-\epsilon)^{n-1} + e^{1/\epsilon-1}n(1-\epsilon)^{n-1}\right\}\\ &= \frac{e^{-1/\epsilon}}{n!}\left\{\epsilon^{-2}[1-(1-\epsilon)^n] + n(1-\epsilon)^{n-1}(1-e^{1/\epsilon-1})\right\}\end{aligned}$$

This expression has a zero that is asymptotically $1/\sqrt{n}$. The derivative is negative for $\epsilon < 1/\sqrt{n}$ and is positive for $\epsilon > 1/\sqrt{n}$. We thus get a minimum for $\|T^n h\|$ when $\epsilon \sim 1/\sqrt{n}$.

CHAPTER 2

Asymptotic Equivalence of Iterates

The fact that the bulk of the weight of $\|T^n f\|$ is carried by the initial support of f is highlighted by the asymptotic equivalence of $T^n f$ and $T^n f\chi_{[a,b]}$, where (a, b) is an initial part of the support of f.

Proposition 5. *Suppose that the closure of the initial part of the support of f is $[a, b]$ and that f starts off positive or negative. If $[a, c] \subset [a, b]$, then $T^n f(x) \sim T^n f\chi_{[a,c]}(x)$. In particular, $\|T^n f\| \sim \|T^n f\chi_{[a,c]}\|$.*

Proof. Let $c \in (a, b)$. Then

$$\begin{aligned} T^n f(x) &= T^n f\chi_{[0,c]}(x) + T^n f\chi_{[c,1]}(x) \\ &= T^n f\chi_{[a,c]}(x) + T^n f\chi_{[c,1]}(x) \end{aligned}$$

for all $x \in [0, 1]$. If $x > a$ then

$$\frac{T^n f(x)}{T^n f\chi_{[a,c]}} = 1 + \frac{T^n f\chi_{[c,1]}(x)}{T^n f\chi_{[a,c]}(x)}.$$

By Lemma 1 $\frac{T^n f\chi_{[c,1]}(x)}{T^n f\chi_{[a,c]}(x)} \to 0$ as $n \to \infty$ and so

$$\lim_{n\to\infty} \frac{T^n f(x)}{T^n f\chi_{[a,c]}(x)} = 1.$$

In particular, since f starts off positive or negative, $\|T^n f\| = T^n f(1)$ and $\|T^n f\chi_{[a,c]}\| = T^n f\chi_{[a,c]}(1)$, so that

$$\lim_{n\to\infty} \frac{\|T^n f\|}{\|T^n f\chi_{[a,c]}\|} = 1.$$

□

Proposition 6. *Let* $p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_{\ell+1}x^{\ell+1} + a_\ell x^\ell$. *Then* $T^n p(x) \sim T^n a_\ell x^\ell$ *and, in particular,* $\|T^n p\| \sim \|T^n a_\ell x^\ell\|$.

Proof. $T^n p(x) = a_m \frac{m!}{(n+m)!} x^{n+m} + \cdots + a_{\ell+1} \frac{(\ell+1)!}{(n+\ell+1)!} x^{n+\ell+1} + a_\ell \frac{\ell!}{(n+\ell)!} x^{n+\ell}$ and $T^n a_\ell x^\ell = a_\ell \frac{\ell!}{(n+\ell)!} x^{n+\ell}$ by Corollary 1. Now

$$\frac{T^n p(x)}{T^n c x^\ell} = \frac{a_m m! x^{m-1}}{a_\ell \ell! (n+m)\cdots(n+\ell+1)} + \cdots + \frac{a_{\ell+1}(\ell+1)! x}{a_\ell \ell! (n+\ell+1)} + 1 \to 1$$

as $n \to \infty$. Note that this convergence is uniform on $[0, 1]$. Since the norms are attained at 1, we have in particular that $\|T^n p\| \sim \|T^n a_\ell x^\ell\|$. □

Corollary 4. *Let* p *and* q *be two polynomials with the same least significant term. Then* $T^n p(x) \sim T^n q(x)$.

Proof. Suppose that $p(x) = a_{m_1} x^{m_1} + \cdots + a_{\ell+1} x^{\ell+1} + c x^\ell$ and $q(x) = b_{m_2} x^{m_2} + \cdots + b_{\ell+1} x^{\ell+1} + c x^\ell$, where $\ell \le \min\{m_1, m_2\}$. Then $T^n p(x) \sim T^n c x^\ell$ and $T^n q(x) \sim T^n c x^\ell$ by Proposition 6. Hence $T^n p(x) \sim T^n q(x)$. □

This is interesting. Since the orbits of two polynomials whose least significant terms coincide are asymptotically equal, one wonders if the same holds true for the Taylor series expansion of an arbitrary function. That is, is the orbit of any function with a Taylor series expansion equal to the orbit of the first, least significant term of that expansion?

Example 4. Consider the exponential function e^x. The series expansion at 0 is $e^x = \sum_{j=0}^{\infty} x^j / j!$ so that

$$T^n e^x = \sum_{j=0}^{\infty} \frac{x^{n+j}}{(n+j)!} = \frac{x^n}{n!} + \sum_{j=1}^{\infty} \frac{x^{n+j}}{(n+j)!}.$$

We want to show that $T^n e^x \sim T^n 1(x)$ for all x.

$$\begin{aligned}
T^n e^x / T^n 1(x) &= \left(\frac{x^n}{n!} + \sum_{j=1}^{\infty} \frac{x^{n+j}}{(n+j)!} \right) / \frac{x^n}{n!} \\
&= 1 + \sum_{j=1}^{\infty} \frac{n!}{(n+j)!} x^j \\
&= 1 + \frac{1}{n+1} \sum_{j=1}^{\infty} \frac{(n+1)!}{(n+j)!} x^j .
\end{aligned}$$

The last series converges for all $x \in [0,1]$ and is bounded above by 2:

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{(n+1)!}{(n+j)!} x^j &\leq \sum_{j=1}^{\infty} \frac{(n+1)!}{(n+j)!} \\
&= \frac{(n+1)!}{(n+1)!} + \frac{(n+1)!}{(n+2)!} + \sum_{j=3}^{\infty} \frac{(n+1)!}{(n+j)!} \\
&= 1 + \frac{1}{n+2} + \sum_{j=3}^{\infty} \frac{1}{\underbrace{(n+j)\cdots(n+2)}_{j-1}} \\
&\leq 1 + \frac{1}{n} + \sum_{j=3}^{\infty} \frac{1}{\underbrace{n \cdots n}_{j-1}} \\
&= \sum_{j=1}^{\infty} \frac{1}{n^{j-1}} \\
&= \frac{1}{1 - \frac{1}{n}} \\
&= \frac{n}{n-1} \\
&\leq 2.
\end{aligned}$$

Hence $T^n e^x \sim T^n 1(x)$. In particular, $\|T^n e^x\| \sim 1/n!$.

Example 5. In a manner analogous to the preceding example,

$$T^n \sin x \sim T^n x = x^{n+1}/(n+1)!.$$

Theorem 1. *Suppose that f has a series expansion $f(x) = \sum_{j=0}^{\infty} f^{(j)}(0)\frac{x^j}{j!}$ and that f starts off positive. Let n_0 be the least integer for which $f^{(n_0)}(0) \neq 0$. Then*

$$T^n f(x) \sim f^{(n_0)}(0)x^{n+n_0}/(n+n_0)!$$

for all $x \neq 0$. In particular,

$$\|T^n f\| \sim \left|f^{(n_0)}(0)\right| /(n+n_0)!.$$

Note: It may be that $n_0 = 0$, where $f^{(0)} = f$. Also, the derivatives of f may eventually vanish so that the sum is finite.

Proof. The series expansion of f is $f(x) = \sum_{j=n_0}^{\infty} f^{(j)}(0)\frac{x^j}{j!}$ so that

$$T^n f(x) = \sum_{j=n_0}^{\infty} f^{(j)}(0)\frac{1}{(n+j)!}x^{n+j}.$$

Now

$$\begin{aligned}
&T^n f(x)/(f^{(n_0)}(0)x^{n+n_0}/(n+n_0)!) \\
&= \frac{(n+n_0)!}{f^{(n_0)}(0)x^{n+n_0}}\left(\sum_{j=n_0}^{\infty}\frac{f^{(j)}(0)x^{n+j}}{(n+j)!}\right) \\
&= \sum_{j=n_0}^{\infty}\frac{f^{(j)}(0)}{f^{(n_0)}(0)}\frac{(n+n_0)!}{(n+j)!}x^{j-n_0} \\
&= 1 + \frac{1}{n+n_0+1}\sum_{j=n_0+1}^{\infty}\frac{f^{(j)}(0)}{f^{(n_0)}(0)}\frac{(n+n_0+1)!}{(n+j)!}x^{j-n_0} \\
&\to 1
\end{aligned}$$

as $n \to \infty$ since the series converges. $\square$

Example 6. Let $r(x) = x^2 + x^{3/2}$. Then $T^n r(x) \sim T^n x^{3/2}$ by Corollary 1:

$$\begin{aligned}
T^n r(x)/T^n x^{3/2} &= \left(\frac{\Gamma(3)}{\Gamma(n+3)}x^{n+2} + \frac{\Gamma(5/2)}{\Gamma(n+5/2)}x^{n+3/2}\right)/\frac{\Gamma(5/2)}{\Gamma(n+5/2)}x^{n+3/2} \\
&= \frac{\Gamma(3)\Gamma(n+5/2)}{\Gamma(n+3)\Gamma(5/2)}x^{1/2} + 1 \\
&\to 1
\end{aligned}$$

as $n \to \infty$. The point here is that even though only the first derivative at 0 exists, the orbits of the function are asymptotically equal to the least significant term, as with the polynomials. This is seen in the next example as well.

Example 7. Let $s(x) = x^{3/2} + 1$. Then $T^n s(x) \sim T^n 1(x)$ by Corollary 1:

$$\begin{aligned} T^n s(x)/T^n 1(x) &= \left(\frac{\Gamma(5/2)}{\Gamma(n+5/2)} x^{n+5/2} + \frac{1}{\Gamma(n+1)} x^n \right) / \frac{1}{\Gamma(n+1)} x^n \\ &= \frac{\Gamma(5/2)\Gamma(n+1)}{\Gamma(n+5/2)} x^{5/2} + 1 \\ &\to 1 \end{aligned}$$

as $n \to \infty$.

The asymptotic behavior of functions without series expansions at 0 remains a mystery. Such functions obviously defy the proof presented. Two such functions are $e^{-1/x}$ and $T\sin(1/x)$. The first function has a zero derivative of all orders at 0 and hence has no series expansion about 0. The growth of this function is slower than that of any power function, however large the power. The sine function has no expansion about 0 as the derivatives aren't even defined there. Beyond that, the chronic oscillation of the orbits of the sine function defies even the calculation of the norm.

CHAPTER 3

Orbits Yielding Basic Sequences

We want to determine if there exists $f \in C[0,1]$ so that $\mathrm{Orb}(T, f)$ is a Schauder basis for $\overline{\mathrm{span}}\,\mathrm{Orb}(T, f)$. Now, while the Schauder system is a basic sequence for $C[0,1]$ (it is, after all, a Schauder basis for it) it is clearly not the orbit of any $C[0,1]$ function under the integral operator. The actual question we wish to address is whether or not there exists a function $f \in C[0,1]$ whose orbit $(T^n f)$ yields a basic sequence. The following proposition will be our main tool in addressing this question.

Proposition 7. *If (x_n) is a normalized basic sequence in a Banach space X, then*

$$\|x_n - x_{n+1}\| \nrightarrow 0.$$

Proof. Since (x_n) is a basic sequence, there is an $M > 0$ such that for every sequence (α_j) of scalars and m_1 and $m_2 \in \mathbb{N}$ with $m_1 \leq m_2$,

$$\left\| \sum_{j=1}^{m_1} \alpha_j x_j \right\| \leq M \left\| \sum_{j=1}^{m_2} \alpha_j x_j \right\|.$$

In this particular situation, take $m_1 = n$, $m_2 = n+1$, $\alpha_n = 1$, $\alpha_{n+1} = -1$, and $\alpha_j = 0$ for every $j < n$. Then

$$1 = \|x_n\| \leq M \|x_n - x_{n+1}\|$$

so that

$$\|x_n - x_{n+1}\| \geq 1/M.$$

$\square$

We begin with an example that motivates the exclusion of a large class of possible functions.

Example 8. The distance between successive normalized orbits of 1 is

$$\left\| \frac{T^n 1(x)}{\|T^n 1\|} - \frac{T^{n+1} 1(x)}{\|T^{n+1} 1\|} \right\| = \|x^n - x^{n+1}\|.$$

To find the norm, notice that $x^n - x^{n+1}$ is maximized when $x = n/(n+1)$. Hence

$$\begin{aligned} \|x^n - x^{n+1}\| &= \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} \\ &= \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} \\ &\sim e^{-1}/n. \end{aligned}$$

This example motivates the following two lemmas.

Lemma 2. *Let* $[a, b] \subset [0, 1]$ *and* $f = \chi_{[a,b]}$. *Then*

$$\left\| \frac{T^n f}{\|T^n f\|} - \frac{T^{n+1} f}{\|T^{n+1} f\|} \right\| \sim e^{-1}/n.$$

Proof. Let

$$f_n(x) = \frac{T^n f(x)}{\|T^n f\|} - \frac{T^{n+1} f(x)}{\|T^{n+1} f\|}.$$

Then

$$\begin{aligned} f_n(x) &= \frac{(x-a)^n - (x-b)^n}{(1-a)^n - (1-b)^n} - \frac{(x-a)^{n+1} - (x-b)^{n+1}}{(1-a)^{n+1} - (1-b)^{n+1}} \\ &= \left(\frac{x-a}{1-a}\right)^n \frac{1 - \left(\frac{x-b}{x-a}\right)^n}{1 - \left(\frac{1-b}{1-a}\right)^n} - \left(\frac{x-a}{1-a}\right)^{n+1} \frac{1 - \left(\frac{x-b}{x-a}\right)^{n+1}}{1 - \left(\frac{1-b}{1-a}\right)^{n+1}} \\ &\sim \left(\frac{x-a}{1-a}\right)^n - \left(\frac{x-a}{1-a}\right)^{n+1}. \end{aligned}$$

This last expression is maximized when $x = x_n = \frac{n+a}{n+1} = \frac{n}{n+1}(1-a) + a$. Now

$$\begin{aligned} \|f_n\| &\sim \left(\frac{x_n - a}{1-a}\right)^n - \left(\frac{x_n - a}{1-a}\right)^{n+1} \\ &\sim \left(\frac{x_n - a}{1-a}\right)^n \left(1 - \frac{x_n - a}{1-a}\right) \\ &= \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) \\ &\sim e^{-1}/n. \end{aligned}$$

□

Lemma 3. *Let $E_j = (a_j, b_j] \subset [0,1]$ for $j = 1, \ldots, m$, where $b_{j-1} \leq a_j$ for $j > 1$. Let f be the step function $f(x) = \sum_{j=1}^{m} \alpha_j \chi_{E_j}(x)$, where $\alpha_j \in \mathbb{R}$. Then*

$$\left\| \frac{T^n f}{\|T^n f\|} - \frac{T^{n+1} f}{\|T^{n+1} f\|} \right\| \sim e^{-1}/n.$$

Proof. According to Proposition 1

$$\frac{T^n f(x)}{\|T^n f\|} - \frac{T^{n+1} f(x)}{\|T^{n+1} f\|} \sim \frac{T^n \chi_{E_1}(x)}{\|T^n \chi_{E_1}\|} - \frac{T^{n+1} \chi_{E_1}(x)}{\|T^n \chi_{E_1}\|}.$$

so that

$$\left\| \frac{T^n f}{\|T^n f\|} - \frac{T^{n+1} f}{\|T^{n+1} f\|} \right\| \sim \left\| \frac{T^n \chi_{E_1}}{\|T^n \chi_{E_1}\|} - \frac{T^{n+1} \chi_{E_1}}{\|T^n \chi_{E_1}\|} \right\| \sim e^{-1}/n$$

by Lemma 2. □

These two lemmas, together with the preceding proposition, show that the orbit of no characteristic or step function is a basic sequence. As a consequence, no function with a step function as a derivative has an orbit that is a basic sequence. Since the characteristic functions are dense in $C[0,1]$, we suspect that this result extends to all $C[0,1]$ functions. Unfortunately, the preceding lemma cannot be used to extend the result: The more closely an arbitrary $C[0,1]$ function is approximated, the larger n must be to obtain the $O(1/n)$ bound. Functions that have a series expansion at 0, however, are easily handled.

Lemma 4. *Let $p \geq 0$. Then*

$$\left\| \frac{T^n x^p}{\|T^n x^p\|} - \frac{T^{n+1} x^p}{\|T^{n+1} x^p\|} \right\| \sim e^{-1}/n.$$

Proof. By Corollary 1, $T^n x^p = \Gamma(p+1)x^{n+p}/\Gamma(n+p+1)$ so that $\|T^n x^p\| = \Gamma(p+1)/\Gamma(n+p+1)$. Now

$$\left\| \frac{T^n x^p}{\|T^n x^p\|} - \frac{T^{n+1} x^p}{\|T^{n+1} x^p\|} \right\| = \|x^{n+p} - x^{n+p+1}\|.$$

This expression is maximized when $x = np/(np+1)$. Thus

$$\left\| \frac{T^n x^p}{\|T^n x^p\|} - \frac{T^{n+1} x^p}{\|T^{n+1} x^p\|} \right\| = \left(\frac{np}{np+1} \right)^{np-1} \left(1 - \frac{np}{np+1} \right)$$
$$\sim e^{-1}/n.$$

□

Theorem 2. *Suppose that f has a series expansion at 0. Then*

$$\left\| \frac{T^n f}{\|T^n f\|} - \frac{T^{n+1} f}{\|T^{n+1} f\|} \right\| \sim e^{-1}/n.$$

Proof. Let $f^{(n_0)}$ be the first non-zero derivative of f at 0. (It is possible that $n_0 = 0$.) By Theorem 1,

$$\left\| \frac{T^n f(x)}{\|T^n f\|} - \frac{T^{n+1} f(x)}{\|T^{n+1} f\|} \right\| \sim \left\| \frac{T^n f^{(n_0)}(0) x^{n_0}/n_0!}{\|T^n f^{(n_0)}(0) x^{n_0}/n_0!\|} - \frac{T^{n+1} f^{(n_0)}(0) x^{n_0}/n_0!}{\|T^{n+1} f^{(n_0)}(0) x^{n_0}/n_0!\|} \right\|$$
$$= \left\| \frac{T^n x^{n_0}}{\|T^n x^{n_0}\|} - \frac{T^{n+1} x^{n_0}}{\|T^{n+1} x^{n_0}\|} \right\|.$$

By Lemma 4,

$$\left\| \frac{T^n x^{n_0}}{\|T^n x^{n_0}\|} - \frac{T^{n+1} x^{n_0}}{\|T^{n+1} x^{n_0}\|} \right\| \sim e^{-1}/n.$$

□

Lemma 5 (Asymptotic Squeeze Theorem)**.** *Suppose that $g_n(x) \le f_n(x) \le h_n(x)$ and that $g_n(x) \sim h_n(x)$ for all $x \in [a, b]$. Then $f_n(x) \sim g_n(x)$ for all $x \in [a, b]$.*

Proof.

$$1 \le \lim_{n\to\infty} \frac{f_n(x)}{g_n(x)} \le \lim_{n\to\infty} \frac{h_n(x)}{g_n(x)} = 1.$$

Thus $\lim_{n\to\infty} f_n(x)/g_n(x) = 1$ so that $f_n(x) \sim g_n(x)$. □

Before we proceed, we need a definition. We will say that f has *Hölder growth* at a if there exist p_1 and p_2 with $0 < p_1 < p_2$ such that $f(a) + x^{p_1} \le f(x) \le f(a) + x^{p_2}$.

Theorem 3. *Let $f \in C[0,1]$. Suppose that $f(0) \neq 0$ and that f is of Hölder growth at 0. Then*

$$\left\| \frac{T^n f}{\|T^n f\|} - \frac{T^{n+1} f}{\|T^{n+1} f\|} \right\| \sim e^{-1}/n.$$

Proof. Suppose that $f(0) = a > 0$ and that the growth is positive. Let $[0,b]$ be an initial part of the support of f such that $f(0) \leq f(x) \leq f(0) + x^p$ on $[0,b]$ for some $p > 0$. That is, $f(0)\chi_{[0,b]}(x) \leq f(x)\chi_{[0,b]}(x) \leq [f(0) + x^p]\chi_{[0,b]}(x)$ for all $x \in [0,1]$. Then $T^n f(0)\chi_{[0,b]}(x) \leq T^n f\chi_{[0,b]}(x) \leq T^n[f(0)+x^p]\chi_{[0,b]}(x)$. Since $T^n f(0)\chi_{[0,b]}(x) \sim T^n[f(0)+x^p]\chi_{[0,b]}(x)$, Lemma 5 gives $T^n f\chi_{[0,b]}(x) \sim T^n f(0)\chi_{[0,b]}(x)$. In particular, $\|T^n f\chi_{[0,b]}\| \sim \|T^n f(0)\chi_{[0,b]}\|$. Now

$$\begin{aligned} \frac{T^n f(x)}{\|T^n f\|} - \frac{T^{n+1} f(x)}{\|T^{n+1} f\|} &\sim \frac{T^n f\chi_{[0,b]}(x)}{\|T^n f\chi_{[0,b]}\|} - \frac{T^{n+1} f\chi_{[0,b]}(x)}{\|T^{n+1} f\chi_{[0,b]}\|} \\ &\sim \frac{T^n f(0)\chi_{[0,b]}(x)}{\|T^n f(0)\chi_{[0,b]}\|} - \frac{T^{n+1} f(0)\chi_{[0,b]}(x)}{\|T^{n+1} f(0)\chi_{[0,b]}\|} \\ &= \frac{T^n \chi_{[0,b]}(x)}{\|T^n \chi_{[0,b]}\|} - \frac{T^{n+1} \chi_{[0,b]}(x)}{\|T^{n+1} \chi_{[0,b]}\|} \end{aligned}$$

Finally,

$$\begin{aligned} \left\| \frac{T^n f(x)}{\|T^n f\|} - \frac{T^{n+1} f(x)}{\|T^{n+1} f\|} \right\| &= \left\| \frac{T^n \chi_{[0,b]}(x)}{\|T^n \chi_{[0,b]}\|} - \frac{T^{n+1} \chi_{[0,b]}(x)}{\|T^{n+1} \chi_{[0,b]}\|} \right\| \\ &\sim e^{-1}/n \end{aligned}$$

by Lemma 2. $\square$

Theorem 4. *Let $f \in C[0,1]$ and suppose that $f > 0$ on $[0,1]$. Fix $\epsilon > 0$. Then*

$$\left\| \frac{T^n f}{\|T^n f\|} - \frac{T^{(1+\epsilon)n} f}{\|T^{(1+\epsilon)n} f\|} \right\| \nrightarrow 0.$$

Proof. Consider $x = 1 - 1/(n+1) = x_n$ and let $\eta > 0$.

$$\begin{aligned}
\frac{T^{n+1}f(x_n)}{\|T^{n+1}f\|} &= \frac{\frac{1}{n!}\int_0^{x_n}\left(1-\frac{1}{n+1}-t\right)^n f(t)dt}{\frac{1}{n!}\int_0^1 (1-t)^n f(t)dt} \\
&\sim \frac{\int_0^{\eta}\left(1-\frac{1}{(1-t)(n+1)}\right)^n (1-t)^n f(t)dt}{\int_0^{\eta}(1-t)^n f(t)dt} \\
&\geq \frac{\int_0^{\eta}\left(1-\frac{1}{(1-\eta)(n+1)}\right)^n (1-t)^n f(t)dt}{\int_0^{\eta}(1-t)^n f(t)dt} \\
&= \left(1-\frac{1}{(1-\eta)(n+1)}\right)^n \\
&\to e^{-1/(1-\eta)}.
\end{aligned}$$

For the second term we have

$$\begin{aligned}
\frac{T^{(1+\epsilon)n+1}f(x_n)}{\|T^{(1+\epsilon)n+1}f\|} &= \frac{\frac{1}{\Gamma((1+\epsilon)n+1)}\int_0^{x_n}\left(1-\frac{1}{n+1}-t\right)^{(1+\epsilon)n} f(t)dt}{\frac{1}{\Gamma((1+\epsilon)n+1)}\int_0^1 (1-t)^{(1+\epsilon)n} f(t)dt} \\
&\sim \frac{\int_0^{1/2}\left(1-\frac{1}{(1-t)(n+1)}\right)^{(1+\epsilon)n} (1-t)^{(1+\epsilon)n} f(t)dt}{\int_0^{1/2}(1-t)^{(1+\epsilon)n} f(t)dt} \\
&\leq \frac{\int_0^{1/2}\left(1-\frac{1}{n+1}\right)^{(1+\epsilon)n} (1-t)^{(1+\epsilon)n} f(t)dt}{\int_0^{1/2}(1-t)^{(1+\epsilon)n} f(t)dt} \\
&\to e^{-(1+\epsilon)}.
\end{aligned}$$

Hence

$$\left\|\frac{T^{n+1}f}{\|T^{n+1}f\|} - \frac{T^{(1+\epsilon)n+1}f}{\|T^{(1+\epsilon)n+1}f\|}\right\| \geq e^{-(1-\eta)^{-1}} - e^{-(1+\epsilon)}$$

in the limit. For this to be positive we need $\eta < \epsilon/(1+\epsilon)$. □

While the preceding theorem tells us that subsequences of the orbits are bounded away from one another, we do not necessarily get basic subsequences. We will have to look to lacunary subsequences of the orbit to find basic sequences.

CHAPTER 4

Lacunary Subsequences

The following 1969 result of Gurariy and Macaev [10] provided us with a list of necessary and sufficient conditions for a sequence $(x_n)_{n=-\infty}^{\infty} \subset C[0,1]$ to be basic. With this we find that a lacunary subsequence of an orbit is basic. A sequence $(n_k)_{k=-\infty}^{\infty}$ of positive numbers is called *lacunary* if $\inf_k n_{k+1}/n_k = r > 1$. The number r is called the *index of lacunarity*. Let (x_{n_k}) be a sequence in a Banach space with norm $\|\cdot\|$. The sequence (x_{n_k}) is *separated* if $\inf_{j \neq k} \|x_{n_j} - x_{n_k}\| > 0$. It is called *uniformly minimal* if the distance between x_j and the closed linear span of the remaining elements is positive. The sequence is *basic* if it is a Schauder basis for its closed linear span.

Theorem 5. *Let $(n_k)_{k=-\infty}^{\infty}$ be a positive, increasing sequence. The following are equivalent.*

i. The sequence (n_k) is lacunary.

ii. The sequence (x^{n_k}) is separated in $C[0,1]$.

iii. The sequence (x^{n_k}) is uniformly minimal in $C[0,1]$.

iv. The sequence (x^{n_k}) is basic in $C[0,1]$.

v. The sequence (x^{n_k}) is equivalent to the usual basis in c.

The following proposition provides us with more basic sequences than just the lacunary power sequences above. It is stated for Schauder bases, but clearly holds for basic sequences.

Proposition 8. *Suppose that (x_n) is a Schauder basis for the Banach space X and that (λ_n) is a sequence of non-zero scalars. Then $(\lambda_n x_n)$ is also a Schauder basis for X.*

From this proposition, we can see that a normalized basic sequence is also basic.

Corollary 5. *If (x_n) is a basis for a Banach space, then $(x_n/\|x_n\|)$ is a normalized basic sequence for the space.*

Gurariy and Macaev's Theorem 5, together with the preceding proposition, shows us that a lacunary subsequence of the orbit of a power function and a normalization of such is also a basic sequence.

Corollary 6. *Let ℓ be a non-negative integer. If (n_k) is a lacunary sequence, then $(T^{n_k}x^\ell)$ and $(T^{n_k}x^\ell/\|T^{n_k}x^\ell\|)$ are both basic sequences.*

Given a basic sequence, we see that a permutation of that sequence is also basic if is small enough in the sense of the following theorem.

Proposition 9. *Suppose that X is a Banach space, that (x_n) is a normal basic sequence in X, that K is the basis constant for (x_n), and that (y_n) is a sequence in X such that $\sum_n \|x_n - y_n\| < 1/(2K)$. Then (y_n) is a basic sequence equivalent to (x_n). If (x_n) is a basis for X, then so is (y_n).*

Example 9. The orbit of 1 has a lacunary basic subsequence, as does the orbit of $1 + x$. This can be calculated directly, but here we will use the preceding proposition. Let

$$x_n(x) = \frac{T^n 1(x)}{\|T^n 1\|} = x^n$$

and let

$$y_n(x) = \frac{[T^n(1+x)](x)}{\|T^n(1+x)\|} = \frac{\frac{1}{n!}x^n + \frac{1}{(n+1)!}x^{n+1}}{\frac{1}{n!} + \frac{1}{(n+1)!}}.$$

Then

$$x_n(x) - y_n(x) = \frac{x^n - x^{n+1}}{n+2}$$

and $\|x_n - y_n\|$ is obtained when $x = 1 - \frac{1}{n+1}$. Hence

$$\begin{aligned}\|x_n - y_n\| &= \frac{1}{n+2}\left(1 - \frac{1}{n+1}\right)^n \left(\frac{1}{n+1}\right) \\ &< \frac{e^{-1}}{(n+1)(n+2)} \\ &< \frac{1}{(n+1)(n+2)}\end{aligned}$$

and so

$$\sum_{n=1}^{\infty} \|x_n - y_n\| < \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{2}.$$

Since the series converges, there is a lacunary subsequence (n_k) such that

$$\sum_{n=1}^{\infty} \|x_{n_k} - y_{n_k}\| < \frac{1}{2K},$$

where K is the basis constant of (x_n). Hence both (x_{n_k}) and (y_{n_k}) are both basic sequences.

It is of interest to note the angle between elements of some specific subsequences of the orbit of 1. Let $f : \mathbb{N} \to \mathbb{N}$ be an unbounded strictly increasing function.

$$\left\| \frac{T^{f(n)}1}{\|T^{f(n)}1\|} - \frac{T^{f(n+1)}1}{\|T^{f(n+1)}1\|} \right\| = \left\| x^{f(n)} - x^{f(n+1)} \right\|.$$

This difference is maximized when

$$x = \left(\frac{f(n)}{f(n+1)}\right)^{1/(f(n+1)-f(n))}.$$

Now,

$$\begin{aligned}\left\| x^{f(n)} - x^{f(n+1)} \right\| &= \left(\tfrac{f(n)}{f(n+1)}\right)^{\frac{f(n)}{f(n+1)-f(n)}} - \left(\tfrac{f(n)}{f(n+1)}\right)^{\frac{f(n+1)}{f(n+1)-f(n)}} \\ &= \left(\tfrac{f(n)}{f(n+1)}\right)^{\frac{f(n)}{f(n+1)-f(n)}} \left(1 - \tfrac{f(n)}{f(n+1)}\right)\end{aligned}$$

We will look at different functions f with various growth rates.

If $f(n) = kn$ for some $k \neq 0$, then the norm is

$$\begin{aligned}\left(\frac{kn}{k(n+1)}\right)^n \left(1 - \frac{kn}{k(n+1)}\right) &= \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) \\ &\sim e^{-1}/n\end{aligned}$$

as before.

Let $a > 1$ and let $f(n) = a^n$ so that f grows at an exponential rate. The norm is then

$$\begin{aligned}\left(\frac{a^n}{a^{n+1}}\right)^{\frac{a^n}{a^{n+1}-a^n}}\left(1-\frac{a^n}{a^{n+1}}\right) &= \left(\frac{1}{a}\right)^{\frac{1}{a-1}}\left(\frac{a-1}{a}\right) \\ &= (a-1)\left(\frac{1}{a}\right)^{\frac{a}{a-1}}.\end{aligned}$$

As a concrete example, for $a = 2$ the norm is $1/4$ for all n.

If $f(n) = n!$, then the norm is

$$\begin{aligned}\left(\frac{n!}{(n+1)!}\right)^{\frac{n!}{(n+1)!-n!}}\left(1-\frac{n!}{(n+1)!}\right) &= \left(\frac{1}{n+1}\right)^{\frac{1}{n}}\left(1-\frac{1}{n+1}\right) \\ &\sim \frac{n}{n+1},\end{aligned}$$

which tends to 1 at the rate of $1/n$.

Finally, if f is doubly exponential, say $f(n) = 2^{2^n}$, then the norm is

$$\begin{aligned}\left(\frac{2^{2^n}}{2^{2^{n+1}}}\right)^{\frac{2^{2^n}}{2^{2^{n+1}}-2^{2^n}}}\left(1-\frac{2^{2^n}}{2^{2^{n+1}}}\right) &= \left(\frac{1}{2^{2^n}}\right)^{\frac{1}{2^{2^n}-1}}\left(1-\frac{1}{2^{2^n}}\right) \\ &\sim 1-\frac{1}{2^{2^n}}.\end{aligned}$$

This tends to 1 at an exponential rate.

CHAPTER 5

Uniform Minimality

Let $\{e_j\}_{j=0}^{\infty}$ be a sequence in a Banach space X. We say that this sequence is *uniformly minimal* if there exists $\rho = \rho(\{e_j\}) > 0$ such that

$$\|e_j - \text{span}_{k>j}(e_k)\| \geq \rho.$$

The goal here is to determine if the orbit of any $f \in C[0,1]$ is uniformly minimal.

We wish now to find a number α ($\neq 0$) that minimizes

$$\left\| \frac{T^n 1}{\|T^n 1\|} - \alpha \frac{T^{n+1} 1}{\|T^{n+1} 1\|} \right\| = \|x^n - \alpha x^{n+1}\|.$$

Let $f(x) = x^n - \alpha x^{n+1}$. Then

$$f'(x) = nx^{n-1} - \alpha(n+1)x^n = x^{n-1}[n - \alpha(n+1)x].$$

Since $f(0) = 0$, $|f|$ is maximized when

$$x = \frac{n}{\alpha(n+1)}.$$

Since we are seeking zeros in $[0,1]$, we must have $\alpha \geq 1$. Now,

$$\begin{aligned}
\|f\| &= \left(\frac{n}{\alpha(n+1)}\right)^n \left(1 - \alpha\frac{n}{\alpha(n+1)}\right) \\
&= \left(\frac{1}{\alpha}\right)^n \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} \\
&\sim \alpha^{-n} e^{-1}/n
\end{aligned}$$

or

$$\|f\| = |1 - \alpha|.$$

Case 1:

If $\alpha \gtrsim 1$, then $(1/\alpha)^n \to 0$ exponentially. Since $\|f\| \geq |1-\alpha|$, $\|f\| = |1-\alpha|$.

Case 2:

If $\alpha \sim 1$, then we have three subcases to consider.

Subcase 1:

If $\alpha = 1 + k/n^p$, where $p > 1$, and k is any non-zero real number, then $\alpha^{-n} \to 1$ as $n \to \infty$ so that $\|f\| \sim e^{-1}/n$.

Subcase 2:

If $\alpha = 1 + k/n^p$, where $p < 1$, and k is any non-zero real number, then $\alpha^{-n} \to 0$ as $n \to \infty$ so that $|1-\alpha| \gtrsim \alpha^{-n}e^{-1}/n$. Hence $\|f\| = |1-\alpha|$.

Subcase 3:

Suppose that $\alpha = 1 + k/n$, where k is any non-zero real number. Now $\|f\| \sim e^{-k}e^{-1}/n = e^{-k-1}/n$. Since $\|f\| \geq |1-\alpha|$, we need to choose k so that

$$e^{-k-1}/n \geq |1-\alpha| = |1-(1+k/n)| = |k|/n.$$

We thus need the k that satisfies

$$e^{-k-1} = k.$$

This equation has a solution given by $k \approx 0.27846$.

Example 10.

$$\left\| \frac{T^n 1}{\|T^n 1\|} - 2\frac{T^{n+1}1}{\|T^{n+1}1\|} + \frac{T^{n+2}1}{\|T^{n+2}1\|} \right\| = \|x^n - 2x^{n+1} + x^{n+2}\|.$$

Differentiating with respect to x yields

$$\begin{aligned}\frac{d}{dx}(x^n - 2x^{n+1} + x^{n+2}) &= nx^{n-1} - 2(n+1)x^n + (n+2)x^{n+1} \\ &= x^{n-1}[n - 2(n+1)x + (n+2)x^2]\end{aligned}$$

The zeros of the quadratic factor are

$$x = \frac{n}{n+2}, 1.$$

The norm is obtained when $x = n/(n+2)$ since $x = 1$ gives a value of zero. Now

$$\|x^n - 2x^{n+1} + x^{n+2}\| = \left(\frac{n}{n+2}\right)^n \frac{4}{(n+2)^2} \sim 4e^{-2}/n^2.$$

The point here is that the norm of the difference of an iterate and a linear combination of subsequent iterates can tend to zero at a rate faster than $O(1/n)$. Perhaps it is possible to find a linear combination that tends to zero faster than a polynomial rate.

Example 11.

$$\left\| \frac{T^n 1}{\|T^n 1\|} - \frac{1}{2}\frac{T^{n+1}1}{\|T^{n+1}1\|} - \frac{1}{2}\frac{T^{n+2}1}{\|T^{n+2}1\|} \right\| = \left\|x^n - \tfrac{1}{2}x^{n+1} - \tfrac{1}{2}x^{n+2}\right\|.$$

Differentiating with respect to x yields

$$\begin{aligned}\frac{d}{dx}\left(x^n - \tfrac{1}{2}x^{n+1} - \tfrac{1}{2}x^{n+2}\right) &= nx^{n-1} - \tfrac{1}{2}(n+1)x^n + \tfrac{1}{2}(n+2)x^{n+1} \\ &= \tfrac{1}{2}x^{n-1}[2n - (n+1)x - (n+2)x^2]\end{aligned}$$

The zeros of the quadratic factor are

$$x = \frac{n+1}{n+2} \cdot \frac{-1 \pm \sqrt{9 - 8/(n+1)^2}}{2},$$

of which only

$$x = \frac{n+1}{n+2} \cdot \frac{\sqrt{9 - 8/(n+1)^2} - 1}{2} \qquad (\sim (n+1)/(n+2))$$

is in $[0,1]$. The norm is obtained here since $x = 1$ gives a value of zero. Now

$$\left\| x^n - \tfrac{1}{2}x^{n+1} - \tfrac{1}{2}x^{n+2} \right\| \sim \tfrac{3}{2}e^{-1}/n.$$

Example 12.

$$\left\| \frac{T^n 1}{\|T^n 1\|} - 3\frac{T^{n+1}1}{\|T^{n+1}1\|} + \frac{9}{4}\frac{T^{n+2}1}{\|T^{n+2}1\|} \right\| = \|x^n - 3x^{n+1} + \tfrac{9}{4}x^{n+2}\|.$$

Differentiating with respect to x yields

$$\begin{aligned} \frac{d}{dx}\left(x^n - 3x^{n+1} + \tfrac{9}{4}x^{n+2}\right) &= nx^{n-1} - 3(n+1)x^n + \tfrac{9}{4}(n+2)x^{n+1} \\ &= x^{n-1}\left[n - 3(n+1)x + \tfrac{9}{4}(n+2)x^2\right] \\ &= \tfrac{1}{4}x^{n-1}\left[9(n+2)x^2 - 12(n+1)x + 4n\right] \end{aligned}$$

The zeros of the quadratic factor are

$$x = \frac{2n}{3(n+2)}, \ \frac{2}{3}.$$

The norm is not obtained at either of these values, however: substituting $x = 2n/(3(n+2))$ gives an expression that is $\sim 4e^{-2}\left(\frac{2}{3}\right)^n/n^2$ while $x = 2/3$ evaluates to 0. The norm is thus obtained when $x = 1$ so that

$$\left\| \frac{T^n 1}{\|T^n 1\|} - 3\frac{T^{n+1}1}{\|T^{n+1}1\|} + \frac{9}{4}\frac{T^{n+2}1}{\|T^{n+2}1\|} \right\| = \|x^n - 3x^{n+1} + \tfrac{9}{4}x^{n+2}\| = 1/4.$$

Example 13.

$$\left\| \frac{T^n 1}{\|T^n 1\|} + \frac{1}{2}\frac{T^{n+1}1}{\|T^{n+1}1\|} - \frac{1}{2}\frac{T^{n+2}1}{\|T^{n+2}1\|} \right\| = \left\| x^n + \tfrac{1}{2}x^{n+1} - \tfrac{1}{2}x^{n+2} \right\|.$$

Differentiating with respect to x yields

$$\frac{d}{dx}\left(x^n + \tfrac{1}{2}x^{n+1} - \tfrac{1}{2}x^{n+2}\right) = -\tfrac{1}{2}x^{n-1}\left[(n+2)x^2 - (n+1)x - 2n\right]$$

The zeros of the quadratic factor are

$$x = \frac{(n+1) \pm \sqrt{9n^2 + 18n + 1}}{2(n+2)}.$$

Since these zeros are asymptotically -1 and 2, respectively, the norm is attained at $x = 1$. Thus

$$\left\| \frac{T^n 1}{\|T^n 1\|} + \frac{1}{2}\frac{T^{n+1} 1}{\|T^{n+1} 1\|} - \frac{1}{2}\frac{T^{n+2} 1}{\|T^{n+2} 1\|} \right\| = \|x^n + \tfrac{1}{2}x^{n+1} - \tfrac{1}{2}x^{n+2}\| = 1.$$

Example 14.

$$\left\| \frac{T^n 1}{\|T^n 1\|} + \frac{T^{n+1} 1}{\|T^{n+1} 1\|} + \frac{T^{n+2} 1}{\|T^{n+2} 1\|} \right\| = \|x^n + x^{n+1} + x^{n+2}\|.$$

Differentiating with respect to x yields

$$\frac{d}{dx}\left(x^n + x^{n+1} + x^{n+2}\right) = x^{n-1}\left[(n+2)x^2 + (n+1)x + n\right]$$

The zeros of the quadratic factor are

$$x = \frac{-(n+1) \pm \sqrt{-3n^2 - 6n + 1}}{2(n+2)},$$

neither of which are real. The norm is thus obtained at $x = 1$ so that

$$\left\| \frac{T^n 1}{\|T^n 1\|} + \frac{T^{n+1} 1}{\|T^{n+1} 1\|} + \frac{T^{n+2} 1}{\|T^{n+2} 1\|} \right\| = \|x^n + x^{n+1} + x^{n+2}\| = 3.$$

These five examples illustrate two important points. Let

$$f(x) = x^n - \alpha x^{n+1} - \beta x^{n+2}$$

so that

$$\left\| \frac{T^n 1}{\|T^n 1\|} - \alpha\frac{T^{n+1} 1}{\|T^{n+1} 1\|} - \beta\frac{T^{n+2} 1}{\|T^{n+2} 1\|} \right\| = \|f\|.$$

Finding the norm thus amounts to maximizing $|f|$ on $[0,1]$. The last three examples show that if the critical numbers of f are not real, lie outside of $[0,1]$, or are asymptotically interior to $[0,1]$, then the norm is attained when $x = 1$ and

$$\left\| \frac{T^n 1}{\|T^n 1\|} - \alpha\frac{T^{n+1} 1}{\|T^{n+1} 1\|} - \beta\frac{T^{n+2} 1}{\|T^{n+2} 1\|} \right\| = |1 - \alpha - \beta|.$$

All such differences are thus $O(1)$.

The first two examples show that at least on critical number of f must be in $[0,1]$, asymptotically 1, and $f(1)=0$ to get a norm tending to 0. In the first example we found that the norm is $O(1/n^2)$ and in the second the norm was found to be $O(1/n)$. The difference here is the requirement that $f'(1)=0$. This results in a slower growth of the function around 1 and hence a smaller norm. The minimum norm for $f(x)$ is thus given by the equations $f(1)=0$ and $f'(1)=0$. We will find that this is the general situation for the difference of an arbitrary span.

We wish to calculate

$$\left\| \frac{T^n 1}{\|T^n 1\|} - \sum_{j=1}^{m} \alpha_j \frac{T^{n+j} 1}{\|T^{n+j} 1\|} \right\| = \left\| x^n - \sum_{j=1}^{m} \alpha_j x^{n+j} \right\|,$$

where $\alpha_j \neq 0$ for each $j = 1 \dots m$. We also wish to find the α_j that minimize the norm. Let $f(x) = x^n - \sum_{j=1}^{m} \alpha_j x^{n+j}$. The α_j we are looking for are given by $f(1) = f'(1) = \cdots = f^{(m-1)}(1) = 0$. This yields a system of m equations in the m variables α_j, $j = 1, \dots, m$. The surprising solution to this system is given by the m-th row of Pascal's triangle, starting with the second entry and alternating sign from positive to negative. Thus we find that $f(x) = x^n(1-x)^m$. It is easily seen that this f satisfies the conditions $f(1) = f'(1) = \cdots = f^{(m-1)}(1) = 0$. The critical numbers are given by

$$\begin{aligned} f'(x) &= nx^{n-1}(1-x)^m - x^n m(1-x)^{m-1} \\ &= x^{n-1}(1-x)^{m-1}[n(1-x) - mx] \\ &= x^{n-1}(1-x)^{m-1}(n - nx - mx). \end{aligned}$$

The one we are interested in is $x = \frac{n}{n+m}$ since f evaluates to zero at the others. Now

$$\begin{aligned} \|f\| &= \left(\frac{n}{n+m}\right)^n \left(1 - \frac{n}{n+m}\right)^m \\ &= \left(\frac{n}{n+m}\right)^n \left(\frac{m}{n+m}\right)^m \\ &\sim m^m e^{-m}/n^m \end{aligned}$$

This shows that $\|f\|$ is $O(1/m!)$ by Stirling's formula so that the orbit of f is not uniformly minimal.

CHAPTER 6

L_p Spaces

We wish to extend the above results to the $L_p[0,1]$ spaces, $p \geq 1$. We will first look at our ubiquitous motivating example $f = 1$.

Example 15. Recall that for the constant function 1, $T^n 1(x) = \frac{1}{n!} x^n$ and

$$\begin{aligned} \|T^n 1\|_p &= \left(\int_0^1 \left| \tfrac{x^n}{n!} \right|^p dx \right)^{\frac{1}{p}} \\ &= \tfrac{1}{n!} \left(\tfrac{1}{np+1} \right)^{\frac{1}{p}}. \end{aligned}$$

The difference between successive normalized orbits is

$$\begin{aligned} &\left\| \frac{T^n 1}{\|T^n 1\|_p} - \frac{T^{n+1} 1}{\|T^{n+1} 1\|_p} \right\|_p^p \\ &\quad = \left\| (np+1)^{\frac{1}{p}} x^n - (np+p+1)^{\frac{1}{p}} x^{n+1} \right\|_p^p \\ &\quad = \int_0^1 \left| (np+1)^{\frac{1}{p}} x^n - (np+p+1)^{\frac{1}{p}} x^{n+1} \right|^p dx \\ &\quad = \int_0^{b(p)} \left[(np+1)^{\frac{1}{p}} x^n - (np+p+1)^{\frac{1}{p}} x^{n+1} \right]^p dx \\ &\qquad + \int_{b(p)}^1 \left[(np+p+1)^{\frac{1}{p}} x^{n+1} - (np+1)^{\frac{1}{p}} x^n \right]^p dx, \end{aligned}$$

where

$$b(p) = \left(\tfrac{np+1}{np+p+1} \right)^{\frac{1}{p}}.$$

To evaluate these integrals, we will make use of Euler's Beta Function

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

and the identity

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

$$\begin{aligned}
&\int_0^{b(p)} \left[(np+1)^{\frac{1}{p}} x^n - (np+p+1)^{\frac{1}{p}} x^{n+1}\right]^p dx \\
&= (np+1)\int_0^{b(p)} x^{np}\left[1-\left(\tfrac{np+p+1}{np+1}\right)^{\frac{1}{p}} x\right]^p dx \\
&= (np+1)\left(\tfrac{np+1}{np+p+1}\right)^{\frac{np}{p}} \int_0^{b(p)} \left[\left(\tfrac{np+p+1}{np+1}\right)^{\frac{1}{p}} x\right]^{np} \left[1-\left(\tfrac{np+p+1}{np+1}\right)^{\frac{1}{p}} x\right]^p dx.
\end{aligned}$$

Make the substitution $t = \left(\tfrac{np+p+1}{np+1}\right)^{\frac{1}{p}} x$ so that

$$\begin{aligned}
&\int_0^{b(p)} \left[(np+1)^{\frac{1}{p}} x^n - (np+p+1)^{\frac{1}{p}} x^{n+1}\right]^p dx \\
&= (np+1)\left(\tfrac{np+1}{np+p+1}\right)^{\frac{np+1}{p}} \int_0^1 t^{np}(1-t)^p dt \\
&= (np+1)\left(\tfrac{np+1}{np+p+1}\right)^{\frac{np+1}{p}} \frac{\Gamma(np+1)\Gamma(p+1)}{\Gamma(np+p+2)} \\
&= \left(\tfrac{np+1}{np+p+1}\right)^{\frac{np+1}{p}} \frac{\Gamma(np+2)\Gamma(p+1)}{\Gamma(np+p+2)} \\
&\sim e^{-1}\Gamma(p+1)\frac{1}{(np)^p}.
\end{aligned}$$

Now for the second integral.

$$\begin{aligned}
&\int_{b(p)}^1 \left[(np+p+1)^{\frac{1}{p}} x^{n+1} - (np+1)^{\frac{1}{p}} x^n\right]^p dx \\
&= (np+1)\int_{b(p)}^1 x^{np}\left[\left(\tfrac{np+p+1}{np+1}\right)^{\frac{1}{p}} x-1\right]^p dx \\
&= (np+1)\left(\tfrac{np+1}{np+p+1}\right)^{\frac{np}{p}} \int_{b(p)}^1 \left[\left(\tfrac{np+p+1}{np+1}\right)^{\frac{1}{p}} x\right]^{np} \left[\left(\tfrac{np+p+1}{np+1}\right)^{\frac{1}{p}} x-1\right]^p dx
\end{aligned}$$

Again, make the substitution $t = \left(\frac{np+p+1}{np+1}\right)^{\frac{1}{p}} x$. Then

$$\begin{aligned}
&\int_{b(p)}^{1} \left[(np+p+1)^{\frac{1}{p}} x^{n+1} - (np+1)^{\frac{1}{p}} x^{n}\right]^p dx \\
&= (np+1)\left(\tfrac{np+1}{np+p+1}\right)^{\frac{np+1}{p}} \int_1^{b(p)^{-1}} t^{np}(t-1)^p dt \\
&\leq (np+1)\left(\tfrac{np+1}{np+p+1}\right)^{\frac{np+1}{p}} \int_1^{b(p)^{-1}} t^{np}\left[b(p)^{-1}-1\right]^p dt \\
&= (np+1)\left(\tfrac{np+1}{np+p+1}\right)^{\frac{np+1}{p}} \left[b(p)^{-1}-1\right]^p \tfrac{1}{np+1}\left\{\left[b(p)^{-1}\right]^{np+1}-1\right\} \\
&\sim e^{-1}\tfrac{1}{(np)^p}(e-1) \\
&= (1-e^{-1})\tfrac{1}{(np)^p}
\end{aligned}$$

Hence

$$\begin{aligned}
\left\|\tfrac{T^n 1}{\|T^n 1\|_p} - \tfrac{T^{n+1}1}{\|T^{n+1}1\|_p}\right\|_p^p &\lesssim e^{-1}\Gamma(p+1)\tfrac{1}{(np)^p} + (1-e^{-1})\tfrac{1}{(np)^p} \\
&= \tfrac{1}{(np)^p}\left\{e^{-1}\Gamma(p+1)+1-e^{-1}\right\} \\
&= \tfrac{1}{(np)^p}\left\{e^{-1}\left[\Gamma(p+1)-1\right]+1\right\}
\end{aligned}$$

and

$$\left\|\tfrac{T^n 1}{\|T^n 1\|_p} - \tfrac{T^{n+1}1}{\|T^{n+1}1\|_p}\right\|_p \lesssim \tfrac{1}{np}\sqrt[p]{e^{-1}\left[\Gamma(p+1)-1\right]+1}.$$

As with the sup norm, we find that the norm of the orbits is asymptotically equal to the norm of the orbits of the initial part of the function.

Example 16. We find that $\|T^n 1\|_p \sim \|T^n \chi_{[0,a]}\|_p$, where $0 < a < 1$. Recall that

$$T^n \chi_{[0,a]}(x) = \begin{cases} \frac{1}{n!}x^n & \text{if } x \leq a; \\ \frac{1}{n!}[x^n - (x-a)^n] & \text{if } x > a. \end{cases}$$

The p-th power of the norm is

$$\begin{aligned}
\|T^n\chi_{[0,a]}\|_p^p &= \int_0^1 |T^n\chi_{[0,a]}(x)|^p dx \\
&= \int_0^a |T^n\chi_{[0,a]}(x)|^p dx + \int_a^1 |T^n\chi_{[0,a]}(x)|^p dx \\
&= \int_0^a \left|\frac{1}{n!}x^n\right|^p dx + \int_a^1 \left|\frac{1}{n!}[x^n - (x-a)^n]\right|^p dx \\
&= \frac{1}{(n!)^p}\int_0^a x^{np} dx + \frac{1}{(n!)^p}\int_a^1 [x^n - (x-a)^n]^p dx \\
&= \frac{1}{(n!)^p}\left\{\frac{a^{np+1}}{np+1} + \int_a^1 x^{np}\left[1 - \left(1 - \frac{a}{x}\right)^n\right]^p dx\right\}.
\end{aligned}$$

We need to get bounds on the second integral as it cannot be calculated directly. First is the upper bound:

$$\int_a^1 x^{np}\left[1 - \left(1 - \frac{a}{x}\right)^n\right]^p dx \le \int_a^1 x^{np} dx = \frac{1}{np+1}(1 - a^{np+1}).$$

Next is the lower bound:

$$\begin{aligned}
\int_a^1 x^{np}\left[1 - \left(1 - \frac{a}{x}\right)^n\right]^p dx &\ge \int_a^1 x^{np}[1 - (1-a)^n]^p dx \\
&= [1 - (1-a)^n]^p \frac{1}{np+1}\, x^{np+1}\Big|_{x=a}^{x=1} \\
&= [1 - (1-a)^n]^p \frac{1}{np+1}(1 - a^{np+1})
\end{aligned}$$

Now,

$$\begin{aligned}
\|T^n\chi_{[0,a]}\|_p^p &\le \frac{1}{(n!)^p}\left\{\frac{a^{np+1}}{np+1} + \frac{1}{np+1} - \frac{a^{np+1}}{np+1}\right\} \\
&= \frac{1}{(n!)^p}\frac{1}{np+1}
\end{aligned}$$

so that

$$\|T^n\chi_{[0,a]}\|_p \le \frac{1}{n!}\sqrt[p]{\frac{1}{np+1}},$$

and

$$\|T^n\chi_{[0,a]}\|_p^p \ge \frac{1}{(n!)^p}\left\{\frac{1}{np+1}(1 - a^{np+1})[1 - (1-a)^n]^p\right\}$$

so that

$$\|T^n\chi_{[0,a]}\|_p \geq \frac{1}{n!}\sqrt[p]{\frac{1}{np+1}(1-a^{np+1})[1-(1-a)^n]^p}.$$

From the upper bound

$$\lim_{n\to\infty}\frac{\|T^n\chi_{[0,a]}\|_p}{\|T^n 1\|_p} \leq \lim_{n\to\infty}\frac{\frac{1}{n!}\sqrt[p]{\frac{1}{np+1}}}{\frac{1}{n!}\sqrt[p]{\frac{1}{np+1}}} = 1$$

and from the lower bound

$$\begin{aligned}\lim_{n\to\infty}\frac{\|T^n\chi_{[0,a]}\|_p}{\|T^n 1\|_p} &\geq \lim_{n\to\infty}\frac{\frac{1}{n!}\sqrt[p]{\frac{1}{np+1}(1-a^{np+1})[1-(1-a)^n]^p}}{\frac{1}{n!}\sqrt[p]{\frac{1}{np+1}}}\\ &= \lim_{n\to\infty}\sqrt[p]{(1-a^{np+1})[1-(1-a)^n]^p}\\ &= 1.\end{aligned}$$

Thus $\|T^n 1\|_p \sim \|T^n\chi_{[0,a]}\|_p$.

We will next check the sharpness of this result. We'll look first at $p=1$ then $p=2$. For $p=1$ we get:

$$\begin{aligned}&\left\|\frac{T^n 1}{\|T^n 1\|_1} - \frac{T^{n+1}1}{\|T^{n+1}1\|_1}\right\|_1\\ &\quad= \int_0^{b(1)}\left[(n+1)x^n-(n+2)x^{n+1}\right]dx\\ &\qquad+\int_{b(1)}^1\left[(n+2)x^{n+1}-(n+1)x^n\right]dx\\ &\quad= \left[x^{n+1}-x^{n+2}\right]_0^{b(1)} + \left[x^{n+2}-x^{n+1}\right]_{b(1)}^1\\ &\quad= \left(\frac{n+1}{n+2}\right)^{n+1}\left(1-\frac{n+1}{n+2}\right)+\left(\frac{n+1}{n+2}\right)^{n+1}\left(1-\frac{n+1}{n+2}\right)\\ &\quad\sim 2e^{-1}/n.\end{aligned}$$

The formula above gives $1/n$ for $p=1$ for a difference of $(1-2e^{-1})/n \approx 0.26/n$.

Now let $p = 2$.

$$\begin{aligned}
&\left\| \frac{T^n 1}{\|T^n 1\|_2} - \frac{T^{n+1} 1}{\|T^{n+1} 1\|_2} \right\|_2^2 \\
&\qquad = \int_0^{b(2)} \left[(2n+1)^{\frac{1}{2}} x^n - (2n+3)^{\frac{1}{2}} x^{n+1} \right]^2 dx \\
&\qquad\quad + \int_{b(2)}^1 \left[(2n+3)^{\frac{1}{2}} x^{n+1} - (2n+1)^{\frac{1}{2}} x^n \right]^2 dx.
\end{aligned}$$

The first integral is $\sim 2e^{-1}/(4n^2)$. From the second integral we get

$$\begin{aligned}
&\int_{b(2)}^1 \left[(2n+3)^{\frac{1}{2}} x^{n+1} - (2n+1)^{\frac{1}{2}} x^n \right]^2 dx \\
&= 2 \left[1 - \frac{(2n+3)^{1/2}(2n+1)^{1/2}}{2n+2} \right] - \left(\frac{2n+1}{2n+3} \right)^{\frac{2n+1}{3}} \frac{1}{(2n+3)(n+1)} \\
&\sim \frac{1}{4n^2} - \frac{2e^{-1}}{4n^2}.
\end{aligned}$$

Putting these together we find that

$$\left\| \frac{T^n 1}{\|T^n 1\|_2} - \frac{T^{n+1} 1}{\|T^{n+1} 1\|_2} \right\|_2^2 \sim \frac{1}{4n^2}$$

so that

$$\left\| \frac{T^n 1}{\|T^n 1\|_2} - \frac{T^{n+1} 1}{\|T^{n+1} 1\|_2} \right\|_2 \sim \frac{1}{2n}.$$

From the formula for the $p = 2$ norm we get $\sqrt{e^{-1}+1}/(2n)$, for a difference of $(\sqrt{e^{-1}+1} - 1)/(2n) \approx 0.085/n$.

Questions: Is this approximation sharp for any p? Do these differences increase without bound?

For $p = 3$, the norm is $\sim \sqrt[3]{12e^{-1} - 2}/(3n)$. The formula gives $\sqrt[3]{5e^{-1}+1}/(3n)$ for a difference of $(\sqrt[3]{5e^{-1}+1} - \sqrt[3]{12e^{-1} - 2}\,)/(3n) \approx 0.025/n$.

For $p = 4$, the norm is $\sim \sqrt[4]{9}/(4n)$. The formula gives $\sqrt[4]{23e^{-1} - 1}/(4n)$ for a difference of $\sqrt[4]{23e^{-1} - 1}/(4n) - \sqrt[4]{9}/(4n) \approx -0.020/n$.

CHAPTER 7

Conclusions and Further Research

We have shown that the orbit of a sufficiently regular function under the integral operator cannot be a Schauder basis for its closed linear span. Here we mean sufficiently regular in the sense that the function starts off positive or negative. In fact, it is enough that some function in an orbit has a non-zero derivative of some order at 0.

There are orbits for which some of the proofs do not hold. Some such orbits are those where all the functions have 0 as an accumulation point of zeros and alternate sign around those zeros (e.g. $\sin(1/x)$). Also, those orbits whose functions lack non-zero derivatives of all orders at 0 defy the asymptotic analysis presented in some of the proofs, as is the case with the orbit of $e^{-1/x}$. However, the estimate on the norms of the member functions of an orbit in Proposition 4 works for all orbits. It is unknown whether this is the best possible estimate.

It seems that the further study of pathological orbits like $\{T^n \sin(1/x)\}$ and $\{T^n e^{-1/x}\}$ may be easier in the L_2 setting. In this space we have additional tools at our disposal. Not only does an inner product become available, but we also gain the use of the adjoint of the integral operator. It is the availability of the adjoint that may allow us to sidestep the problematic behavior of pathological orbits.

In light of the theorem of Gurariy and Macaev, we see that lacunary subsequences of a sufficiently regular orbit are basic. Since a sequence $\{x^n\}$ is basic if and only if $\{\lambda_n x^n\}$ is basic, where $\lambda_n \neq 0$ for any n, a subsequence $\{\lambda_{n_k} x^{n_k}\}$ is basic if and only if $\{x^{n_k}\}$ is lacunary. Coupled with the results presented in this dissertation, we find that the orbit of a regular $C[0,1]$ function with a series development about 0 has a subsequence that is basic.

While some results analogous to those in this dissertation remain incomplete in the

L_p spaces, the initial analysis provided agrees with the established results and leads us to expect similar behavior from orbits in the L_p spaces. This work may be extended to other integral operators as well. Generic Volterra and Fredholm operators with arbitrary kernels deserve consideration, and one can generalize the measure as well. There are many open problems concerning these operators similar to those stated here. Hopefully, those problems will be readily solved following the analysis given here.

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