LACUNARY POWER SEQUENCES AND EXTREMAL VECTORS

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by

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For their immeasurable support, understanding and patience,

to my wife:	Nitsuh Beyene
and our kids:	Helina, Biniam and Kidist

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INTRODUCTION

In this thesis, we study two topics. The first part deals with basic sequences in relation to lacunary power sequences of multiplication operators in the spaces $C[a, b]$ and $L_p[a, b]$ and the second part deals with the rectifiability property of a curve arising from best approximation of non-invertible operators in Hilbert spaces.

The space $C[a, b]$ is the Banach space of functions continuous on the interval $[a, b]$ equipped with the supremum norm: $\|f\|_{C[a,b]} := \sup\limits_{t\in[a,b]} |f(t)|$.

The space $L_p[a, b]$, $1 \le p < \infty$ is the Banach space of functions f for which $\int_a^b |f(t)|^p dt < \infty$ and the $L_p[0, 1]$ norm is given by: $\|f\|_{L_p[a,b]} : \left(\int_a^b |f(t)|^p dt\right)^{1/p}$

When there is no ambiguity, we will simply write $\|f\|$ for $\|f\|_{C[a,b]}$ and $\|f\|_{L_p[a,b]}$

The well known approximation theorem of Weierstrass (1885) tells us that the polynomials $P(t) = \sum_0^\infty \alpha_k t^k$ are dense in the spaces of continuous functions, $C[0, 1]$. In the more general setting, Müntz's theorem as stated by T. Erdélyi and W. B. Johnson [12] asserts that for a sequence $\{\lambda_j\}_{j=0}^\infty$ with $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$, the linear span of $\{t^{\lambda_0}, t^{\lambda_1}, \ldots\}$ is dense in $C[0, 1]$ and $L_p[0, 1]$, $1 \le p < \infty$ if and only if $\sum_{j=1}^\infty \frac{1}{\lambda_j} = \infty$.

Definition 0.0.1 *Let T be a linear operator on a Banach space X, and let $x \in X$. x is said to be cyclic in X if the linear span of the orbit of x for T is dense in X.*

Put in the language of the above definition, Weierstrass' theorem says that for the multiplication operator $T : C[0, 1] \to C[0, 1]$ given by $(Tf)(t) = tf(t)$, the span of *the orbit $\{\mathbf{1}, T\mathbf{1}, T^2\mathbf{1}, \ldots\} = \{t^n\}_{n=0}^{\infty}$ of the constant function $f(t) = \mathbf{1}$ with respect to T is dense in $C[0,1]$. That is, $\mathbf{1}$ is cyclic for T.*

A natural question is to ask if parts of the orbits of cyclic vectors in Banach spaces are Schauder bases of the spaces or even basic sequences. As far as we know, this problem has not been studied extensively.

In this thesis, we study more general multiplication operators. So, multiplication by t in Weierstrass' theorem is replaced by multiplication by a more general function $h(t)$ and instead of the constant function **1**, we consider the orbit of a more general function.

Definition 0.0.2 *An infinite sequence $\{x_i\}$ of non-zero vectors in a Banach space X is said to be a Schauder basis of X if for each $x \in X$, there is a unique sequence of scalars $\{\lambda_i\}$ such that $x = \sum_{i=1}^{\infty} \lambda_i x_i$. It is said to be a basic sequence in X if it is a Schauder basis of its closed linear span, denoted by $[x_i] := \overline{span\{x_i\}}$. If $\|x_i\| = 1$ for all i, we say that the basic sequence is normalized.*

Definition 0.0.3 *An increasing sequence of nonnegative real numbers $\{\lambda_k\}_1^{\infty}$ is said to be lacunary if $\inf_k \dfrac{\lambda_{k+1}}{\lambda_k} = \rho > 1$. The number ρ is said to be the index of lacunarity of $\{\lambda_k\}_1^{\infty}$.*

In 1966, V.I. Gurariy and V.I. Matsaev [15] showed that for the operator of multiplication by t on $C[0,b]$ and $L_p[0,b]$, with $1 \le p < \infty$ and $b > 0$, the power sequence $\{t^{\lambda_k}\}_{k=-\infty}^{\infty}$, is a basic sequence if and only if the sequence $\{\lambda_k\}$ is lacunary. This means that lacunary parts of orbits of the constant function **1** under the multiplication by t operator in $C[0,b]$ and $L_p[0,1]$ are basic sequences and vice versa. In the sequel, λ_k need not be integers.

In chapter 1, we show that if an increasing sequence $\{\lambda_k\}_1^\infty$ of positive real numbers is such that $\sup\limits_k(\lambda_{k+1}-\lambda_k)<\infty$ then the sequence $\left\{t^{\lambda_k}\right\}_1^\infty$ is not a basic sequence. In fact, we will show that $\left\|\frac{t^k f(t)}{\|t^k f(t)\|}-\frac{t^{k+1} f(t)}{\|t^{k+1} f(t)\|}\right\|=\mathcal{O}(k^{-1})$, where the norm is the sup-norm in $C[0,1]$ or the L_p-norm.

In chapter 2, we extend the results of Gurariy and Matsaev to the operator of multiplication by a general function h in the spaces $C[a,b]$ and $L_p[a,b]$, by using and extending the techniques employed by Gurariy and Matsaev. In chapter 3, we apply the results of Gurariy and Lusky [14], and P. Borwein and T. Erdélyi [3] to get the generalizations in different ways. Borwein and Erdélyi used Remez inequality and Müntz theorem to show that if E represents either of $C[0,1]$ or $L_p[0,1]$, and if $\sum 1/\lambda_k<\infty$, then there exists a positive constant c such that $\|p\|_{E(0,a)}\leq c\|p\|_{E(A)}$ for any polynomial p in the linear span of $\{t^{\lambda_k}\}$ and any subset A of $[0,a]$ with small Lebesgue measure. This result gives rise to an efficient alternate way for the extension. In chapter 4, we introduce a new technique to show that lacunary block sequences - which are more general than lacunary power sequences - are uniformly minimal and even are basic sequences. Chapter 5 deals with rectifiability property of the curve $\gamma:\epsilon\mapsto Ty_\epsilon$, where y is the backward minimal vector associated to ϵ.

CHAPTER 1

SEQUENCES OF FINITE GAP AND RATE OF CONVERGENCE

1.1 Introduction

Let $\{\lambda_k\}$ be an infinite, increasing sequence of nonnegative real numbers with $\lambda_0 = 0$. Consider the multiplication operator T given by $(Tf)(t) := tf(t)$ for all $f \in C[0,1]$. If $f(t) \equiv \mathbf{1}$ for all $t \in [0,1]$, then the classical Müntz theorem [5, p.171] tells us that the linear span of $\{T^{\lambda_k}\mathbf{1}\}_0^\infty$ is dense in $C[0,1]$ if and only if $\sum_0^\infty \frac{1}{\lambda_k} = \infty$. A particular case of this is the well-known Weierstrass approximation theorem, which says that the polynomials $P(t) := \sum_0^\infty \alpha_k t^k$ are dense in $C[0,1]$. That is, the linear closed span, $\overline{span}\{t^k\}_0^\infty = C[0,1]$.

On the other hand, it is known that $\{t^k\}_0^\infty$ is not a Schauder basis of $C[0,1]$, and hence it is not a basic sequence. Indeed, the sequence $\{t^{\lambda_k}\}$ does not satisfy the following well-known characterization of basic sequences, which can be found in [13, p. 169] and [19, p. 359].

Theorem 1.1.1 *A sequence of non-zero elements $\{x_k\}_1^\infty$ in a Banach space X is a basic sequence in X if and only if there exists a positive constant K satisfying the inequality*

$$\left\| \sum_1^{m_1} \alpha_k x_k \right\|_X \leq K \left\| \sum_1^{m_2} \alpha_k x_k \right\|_X \tag{1.1}$$

for every pair of positive integers $m_1,\ m_2$ with $m_1 < m_2$ and any sequence of scalars $\alpha_1, \alpha_2, \cdots, \alpha_{m_2}$. K is called the basic constant.

This is equivalent to saying that there exists a $K > 0$, such that for every $x = \sum_1^\infty \alpha_k x_k \in \overline{span}\{x_k\}$, and every $m \in \mathbb{N}$,

$$\left\| \sum_1^m \alpha_k x_k \right\|_X \leq K \|x\|_X$$

Corollary 1.1.2 *Suppose $\{x_k\}$ is a basic sequence in a Banach space X. If $\{\lambda_k\}$ is an infinite sequence of nonzero scalars, then $\{\lambda_k x_k\}$ is also a basic sequence.*

Definition 1.1.3 *An infinite sequence $\{x_k\}$ of non-zero elements in a Banach space X is said to be separated if $\inf_{i\neq j} \|x_i - x_j\|_X > 0$.*

The following is an immediate consequence.

Corollary 1.1.4 *Every normalized basic sequence $\{x_i\}$ in a Banach space X is separated.*

PROOF: Suppose $\inf_{i\neq j} \|x_i - x_j\|_X = 0$. Let $K \in \mathbb{N}$. Then, there exist m_1, m_2 such that $\|x_{m_1} - x_{m_2}\|_X < 1/(K+1)$. But then, choosing $\alpha_{m_1} = 1$, $\alpha_{m_2} = -1$ and $\alpha_k = 0$ for $m_1 < k < m_2$, we obtain

$$K \left\| \sum_1^{m_2} \alpha_k x_k \right\|_X < \frac{K}{K+1} < 1 = \left\| \sum_1^{m_1} \alpha_k x_k \right\|_X$$

Consequently, $\{x_k\}$ is not basic sequence.

■

Let $E[0,1]$ represent either $C[0,1]$ or $L_p[0,1]$ and $\|x\|_E$ be the corresponding norm.

Corollary 1.1.5 *$\{t^k\}_0^\infty$ is not a basic sequence in $E[0,1]$.*

PROOF: The $C[0,1]$-case:

$$\left\|t^k - t^{k+1}\right\| = \frac{1}{k}\left(\frac{k}{k+1}\right)^{k+1} = \mathcal{O}(k^{-1})$$

The $L_p[0,1]$-case: Let

$$\Delta_k(t) := \frac{t^k}{\|t^k\|} - \frac{t^{k+1}}{\|t^{k+1}\|} \;=\; \frac{t^k}{\|t^{k+1}\|}\,(A_k \;-\; t)$$

where $A_k \;=\; \dfrac{\|t^{k+1}\|}{\|t^k\|} \;=\; \sqrt[p]{\dfrac{kp+1}{kp+p+1}}$.

Then,

$$|\Delta_k(t)| \;=\; \begin{cases} \Delta_k(t) & \text{if } 0 \le t \le A_k \\ -\Delta_k(t) & \text{if } A_k < t \le 1 \end{cases}$$

$$\|\Delta_k\|^p \;=\; \int_0^1 |\Delta_k(t)|^p\,dt \;=\; \int_0^{A_k} (\Delta_k(t))^p\,dt + \int_{A_k}^1 (-\Delta_k(t))^p\,dt$$

For $p = 1,\ A_k \;=\; \dfrac{k+1}{k+2}$, and hence

$$\|\Delta_k\| \;=\; \frac{2}{k+2}A_k^{k+1} \;=\; \mathcal{O}(k^{-1}).$$

For $p > 1$,

$$\begin{aligned}
\|\Delta_k\|^p &= \int_0^{A_k} (\Delta_k(t))^p \, dt \; + \; \int_{A_k}^1 (-\Delta_k(t))^p \, dt \\
&\leq \left| \Delta_k \left(\frac{k}{k+1} A_k \right) \right|^p A_k \; + \; |\Delta_k(1)|^p \, (1 - A_K) \\
&= \mathcal{O}(k^{1-p})
\end{aligned}$$

Thus,

$$\|\Delta_k\| \leq \mathcal{O}(k^{-1+1/p})$$

$\blacksquare$

In the following section, we will show that corollary 1.1.4 extends to more general sequences of finite gap.

1.2 Rate of Convergence for Power sequences of Finite Gap

We will say that an increasing sequence of nonnegative reals $\{\lambda_k\}_0^\infty$ is a sequence of finite gap or of supremum gap M if $sup_k(\lambda_{k+1} - \lambda_k) = M < \infty$.

For example, $\{k\}_0^\infty$ is of finite gap while the sequence $\{\lambda^k\}_0^\infty$, ($\lambda > 1$) is not.

A strictly increasing sequence $\{\lambda_k\}_0^\infty$ of positive scalars is of supremum gap M, if and only if $\{c\lambda_k\}_0^\infty$ is of supremum gap cM for every positive scalar c. Also, if $\{\lambda_k\}_0^\infty$ is of supremum gap M then, $\lim\limits_{k\to\infty} \dfrac{\lambda_{k+1}}{\lambda_k} = 1$. Thus, $\{\lambda_k\}$ is not lacunary.

Let $\{\lambda_k\}$ be of a finite gap and let E be one of $C[0,1]$ or $L_p[0,1]$. For a nonnegative $h \in E$, consider the operator of multiplication by by h. Let

$$\Delta_k(t) := \frac{h^{\lambda_k} f(t)}{\|h^{\lambda_k} f\|_E} \; - \; \frac{h^{\lambda_{k+1}} f(t)}{\left\|h^{\lambda_{k+1}} f\right\|_E} \tag{1.2}$$

We will show that

(i) $\|\Delta_k\|_C \longrightarrow 0$ as $k \longrightarrow \infty$ for any $f \in C[0,1]$.

(ii) if $h(t) = t$ and $f(t) = (1-t)^m$, for a fixed positive integer m, then $\|\Delta_k\|_C$ is of the order n^{-1} but not of order $n^{-(1+\varepsilon)}$ for any $\varepsilon > 0$.

(iii) if $h(t) = t$ and either f is positive, or has positive m^{th} derivative near 1, for some m, then $\|\Delta_k\|_C$ is of order n^{-1}.

(iv) If the sequence $\{t^{\lambda_k} f\}$ is a basic sequence then $\{\lambda_k\}$ is lacunary.

If the product hf is the zero function, there is nothing to prove. Thus, assume that $(hf)(t)$ is not identically zero.

First we need two lemmas. Let $S := \{t \in [0,1] : f(t) \neq 0\}$ Then,

$$\left\|h^{\lambda_k} f\right\|_{C(S)} = \sup_{t\in S} \left|h^{\lambda_k}(t) f(t)\right|, \text{ and } \|\Delta_k\|_{C[0,1]} = \|\Delta_k\|_{C(S)}$$

Hence, we may as well assume that $f(t) \neq 0$ for all $t \in [0,1]$. We can also assume that $\|h\|_C = 1$ and $\|f\|_C = 1$.

Lemma 1.2.1 *Suppose $\{\lambda_k\}$ is a sequence of positive scalars. If $\{t_k\}$ is a sequence in $[0,1]$ such that there is a constant $\varepsilon > 0$ with $\left|h^{\lambda_k}(t_k) f(t_k)\right| \geq \varepsilon \|h^{\lambda_k} f\|_{C(S)}$, for all $k \in \mathbb{N}$, then $h(t_k) \to 1$ as $k \to \infty$.*

PROOF: Note that the continuity of h and f, guarantees the existence of such a sequence $\{t_k\}$. Assume there exists some $\delta > 0$ such that $|h(t_k)| \leq 1 - \delta$, for all

$k \in \mathbb{N}$. Since $\|h\| = 1$ we can choose some $t_0 \in (0,1)$ such that $|h(t_0)| > 1-\delta^2$. Then,

$$\begin{aligned} 0 < \varepsilon(1-\delta^2)^{\lambda_k}|f(t_0)| &< \varepsilon|h^{\lambda_k}(t_0)f(t_0)| \\ &\leq \varepsilon\|h^{\lambda_k}f\| \\ &\leq \left|h^{\lambda_k}(t_k)f(t_k)\right| \\ &\leq (1-\delta)^{\lambda_k}\,|f(t_k)| \\ &\leq (1-\delta)^{\lambda_k} \;\text{ since }\; \|f\| = 1 \end{aligned}$$

So, $\varepsilon(1+\delta)^{\lambda_k}|f(t_0)| < 1,$ for all $k \in \mathbb{N}$, which is a contradiction since $\varepsilon > 0$, $|f(t_0| > 0$ and $(1+\delta)^{\lambda_k} \to \infty$ as $k \to \infty$.

∎

Lemma 1.2.2 *Suppose* $\{\lambda_k\}$ *is a sequence of supremum gap* m*. Then,*

$$\frac{\left\|h^{\lambda_{k+1}}f\right\|}{\|h^{\lambda_k}f\|} \to 1, \text{ as } k \to \infty$$

PROOF: Assume $\|h\| = 1$. By continuity, suppose $\left\|h^{\lambda_k}f\right\| = |h^{\lambda_k}(t_k)f(t_k)|$ for all $k = 0, 1, 2, \cdots$. Then, by lemma 1.2.1, for each small $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that $|h^m(t_k)| > 1-\varepsilon$, whenever $k > K_\varepsilon$.

Therefore, for all $k > K_\varepsilon$,

$$\begin{aligned} (1-\varepsilon)\left\|h^{\lambda_k}f\right\| &= (1-\varepsilon)|h^{\lambda_k}(t_k)f(t_k)| \\ &< |h^{m+\lambda_k}(t_k)f(t_k)| \\ &\leq \left|h^{\lambda_{k+1}}(t_k)f(t_k)\right|, \text{ since } \lambda_{k+1} \leq m+\lambda_k \text{ and } |h(t_k)| \leq 1 \\ &\leq \left\|h^{\lambda_{k+1}}f\right\| \end{aligned}$$

Thus, $1 - \varepsilon < \frac{\|h^{\lambda_{k+1}} f\|}{\|h^{\lambda_k} f\|} \leq 1$, for all $k > K_\varepsilon$. Letting $\varepsilon \to 0$, gives the required result.

■

Theorem 1.2.3 *Suppose $\{\lambda_k\}$ is a sequence of finite gap. Let $f, h \in C[0,1]$ be such that h is nonnegative and the product hf is not the zero function. Then*

$$\inf_k \|\Delta_k\|_c := \inf_k \left\| \frac{h^{\lambda_k} f}{\|h^{\lambda_k} f\|} - \frac{h^{\lambda_{k+1}} f}{\|h^{\lambda_{k+1}} f\|} \right\| = 0 \tag{1.3}$$

In particular, $\{h^{\lambda_k} f\}$ is not basic sequence in $C[0,1]$

PROOF: Suppose $\{z_k\} \subset [0,1]$ is such that

$$\|\Delta_k\| = |\Delta_k(z_k)|$$

Let $A_k = \left\| h^{\lambda_k} f \right\| = \sup_{0 \leq x \leq 1} |(hf)(x)|$ for $n = 1, 2, \cdots$. Then,

$$\begin{aligned}
|\Delta_k(z_k)| &= \left| A_k^{-1} \cdot \left(h^{\lambda_k} f\right)(z_k) - A_{\lambda_{k+1}}^{-1} \cdot \left(h^{\lambda_{k+1}} f\right)(z_k) \right| \\
&\leq \left| A_k^{-1} \cdot \left(h^{\lambda_k} f\right)(z_k) - A_k^{-1} \cdot \left(h^{\lambda_{k+1}} f\right)(z_k) \right| + \\
&\qquad + \left| A_k^{-1} \cdot \left(h^{\lambda_{k+1}} f\right)(z_k) - A_{k+1}^{-1} \cdot \left(h^{\lambda_{k+1}} f\right)(z_k) \right| \\
&= A_k^{-1} \cdot \left| \left(h^{\lambda_k} f\right)(z_k) \right| \cdot (1 - h^m(z_k)) + \\
&\qquad + A_{k+1}^{-1} \cdot \left| \left(h^{\lambda_{k+1}} f\right)(z_k) \right| \cdot \left| A_{k+1} A_k^{-1} - 1 \right| \\
&\leq A_k^{-1} \left| \left(h^{\lambda_k} f\right)(z_k) \right| \cdot |1 - h^m(z_k)| + \left| A_{k+1} A_k^{-1} - 1 \right|
\end{aligned}$$

Now, by lemma 1.2.2, the second term goes to 0 as $n \to \infty$. For the first term, since $\{A_k^{-1} \left| \left(h^{\lambda_k} f\right)(z_k) \right|\}$ is bounded, there is a subsequence $\left\{ A'_{\lambda_k} \left| \left(h^{n'_k} f\right)(z'_k) \right| \right\}$ that converges, say to a number c.

If $c = 0$, then $A'_k \left|(h^{\lambda_k} f)(z_k)\right| |1 - h^m(z_k)|$ converges to 0, we are done.

If $c > 0$, then $\left|\left(h^{\lambda_k} f\right)(z_k)\right| \geq \frac{c}{2}\left\|h^{\lambda_k} f\right\|$ for all k, sufficiently large. But then, by lemma 1.2.1, $h^m(z_k) \to 1$, and we are done.

∎

In particular, we have

Theorem 1.2.4 *If $\{\lambda_k\}$ is a sequence of finite gap, then for any $m \in \mathbb{N}$,*

$$\|\Delta_k\| := \left\| \frac{t^{\lambda_k}(1-t)^m}{\|t^{\lambda_k}(1-t)^m\|} - \frac{t^{\lambda_{k+1}}(1-t)^m}{\left\|t^{\lambda_{k+1}}(1-t)^m\right\|} \right\| = \mathcal{O}\left(\frac{1}{\lambda_k}\right)$$

If $\inf_k(\lambda_{k+1} - \lambda_k) \geq \delta > 0$ for some δ, then $\|\Delta_k\|$ is not of order $\lambda_k^{-1-\epsilon}$, for any $\epsilon > 0$

PROOF: Suppose $sup_k(\lambda_{k+1} - \lambda_k) = M$. Then,

$$A_k := \max_{0 \leq t \leq 1} \left|t^{\lambda_k}(1-t)^m\right| = \left(\frac{\lambda_k}{m + \lambda_k}\right)^{m+\lambda_k} \left(\frac{m}{\lambda_k}\right)^m = \mathcal{O}\left(\frac{1}{\lambda_k^m}\right) \tag{1.4}$$

By lemma 1.2.2, for each $\varepsilon > 0$, there is a $K_\varepsilon > 0$ such that $\left|1 - \frac{A_k}{A_{k+1}}\right| < \varepsilon$ for all $k > K_\varepsilon$. So, for each $t \in [0, 1]$,

$$\left|\left(t^{\lambda_k} - \frac{A_k}{A_{k+1}} t^{\lambda_{k+1}}\right) - \left(t^{\lambda_k} - t^{\lambda_{k+1}}\right)\right| < \varepsilon t^{\lambda_k}, \text{ for } k > K_\varepsilon$$

Consequently for $k > K_\varepsilon$,

$$\begin{aligned} |\Delta_k| &= \frac{(1-t)^m}{A_k}\left|\left(t^{\lambda_k} - \frac{A_k}{A_{k+1}}t^{\lambda_{k+1}}\right)\right| \\ &< \frac{(1-t)^m}{A_k}\left(\left|t^{\lambda_k} - t^{\lambda_{k+1}}\right| + \varepsilon t^{\lambda_k}\right) \\ &= \frac{(1-t)^m}{A_k}\left(\left(t^{\frac{1}{2}\lambda_k} + t^{\frac{1}{2}\lambda_{k+1}}\right)\left(t^{\frac{1}{2}\lambda_k} - t^{\frac{1}{2}\lambda_{k+1}}\right) + \varepsilon t^{\lambda_k}\right) \\ &\leq \frac{2t^{\frac{1}{2}\lambda_k}(1-t)^m}{A_k}\left(t^{\frac{1}{2}\lambda_k} - t^{\frac{1}{2}\lambda_{k+1}} + \varepsilon\right) \end{aligned} \tag{1.5}$$

Now, $\frac{2}{A_k}t^{\frac{1}{2}\lambda_{k+1}}(1-t)^m = \mathcal{O}(1)$ and since $\lambda_{k+1} - \lambda_k \leq M$, $t^{\frac{1}{2}\lambda_k} - t^{\frac{1}{2}\lambda_{k+1}} = \mathcal{O}\left(\frac{1}{\lambda_k}\right)$. This proves the first assertion of the theorem. On the other hand, if $\lambda_{k+1} - \lambda_k$ becomes arbitrarily small for large k, $\|\Delta_k\|$ can be of smaller order.

For example, consider the increasing sequence $\{\lambda_k\}$, where $\lambda_{2^n+m} = n + \frac{m}{2^n}$ for $m = 0, 1, 2, \cdots, 2^n - 1$ and $n = 0, 1, 2, \cdots$. Since $\lambda_{2^n+m+1} - \lambda_{2^n+m} = \frac{1}{2^n}$,

$$\|\Delta_{2^n+m}\| = \mathcal{O}\left(\frac{1}{n \cdot 2^n}\right)$$

∎

Corollary 1.2.5 *Suppose $\{\lambda_k\}$ is a sequence of finite gap. If $1 \leq p < \infty$, then*

$$\begin{aligned} \|\Delta_k\|_{L_p} &:= \left\|\frac{t^{\lambda_k}(1-t)^m}{\|t^{\lambda_k}(1-t)^m\|_{L_p}} - \frac{t^{\lambda_{k+1}}(1-t)^m}{\left\|t^{\lambda_{k+1}}(1-t)^m\right\|_{L_p}}\right\|_{L_p} \\ &= \mathcal{O}\left(\lambda_k^{-1}\right) \end{aligned} \tag{1.6}$$

PROOF: Without loss of generality, we may assume that $\lambda_1 = 1$. For each $k \in \mathbb{N}$, let N_k and N be non-negative integers such that $N_k \leq p\lambda_k < N_k{+}1$ and $N \leq pm < N{+}1$.

Then using integration by parts, we have

$$\begin{aligned}\left\|t^{\lambda_k}(1-t)^m\right\|_{L_p}^p &= \int_0^1 t^{p\lambda_k}(1-t)^{pm}dt \leq \int_0^1 t^{N_k}(1-t)^N dt \\ &= \frac{N!}{\prod_{j=0}^{N}(j+1+N_k)} = \mathcal{O}\left(\frac{1}{(1+N_k)^{N+1}}\right) = \mathcal{O}\left(\lambda_k^{-pm}\right)\end{aligned}$$

So,

$$\left\|t^{\lambda_k}(1-t)^m\right\|_{L_p} = \mathcal{O}\left(\lambda_k^{-m}\right)$$

Consequently, for large k, putting $A_k := \left\|t^{\lambda_k}(1-t)^m\right\|_C$

$$\left\|t^{\lambda_k}(1-t)^m\right\|_{L_p} \approx C\left\|t^{\lambda_k}(1-t)^m\right\|_C = CA_k \tag{1.7}$$

Therefore, for large k, there exists a positive constant C such that

$$\|\Delta_k\|_{L_p}^p \approx \frac{1}{C}\int_0^1 \left|\frac{t^{\lambda_k}(1-t)^m}{A_k} - \frac{t^{\lambda_k}(1-t)^m}{A_{k+1}}\right|^p dt \tag{1.8}$$

Which means,

$$\|\Delta_k\|_{L_p}^p = \mathcal{O}\left(\lambda_k^{-1}\right)$$

■

Again, if $\lambda_{k+1}-\lambda_k$ goes to 0 rapidly, then $\|\Delta_k\|_{L_p}^p$ is of smaller order.

Corollary 1.2.6 *Suppose **0** $\neq h \in C[0,1]$ is differentiable in $(0,1)$. Then, for any sequence $\{\lambda_k\}$ of finite gap,*

$$\left\|\frac{h^{\lambda_k}}{\|h^{\lambda_k}\|} - \frac{h^{\lambda_{k+1}}}{\|h^{\lambda_{k+1}}\|}\right\|_C = \mathcal{O}\left(\frac{1}{\lambda_k}\right)$$

PROOF: With out loss of generality, we may assume that $\|h\| = 1$. Let $\Delta_k(t) :=$

$h^{\lambda_k}(t) - h^{\lambda_{k+1}}(t)$. Then, Δ_k attains its maximum value at a number $t_k \in [0,1]$ such that either $h(t_k) = \left(\dfrac{\lambda_k}{\lambda_{k+1}}\right)^{\frac{1}{\lambda_{k+1}-\lambda_k}}$ or $h'(t_k) = 0$

<u>Case i</u>: $h(t_k) = \left(\dfrac{\lambda_{k+1}}{\lambda_k}\right)^{\frac{1}{\lambda_{k+1}-\lambda_k}}$ implies that

$$|\Delta_k(t_k)| \;=\; \frac{\lambda_{k+1}-\lambda_k}{\lambda_k}\left(\frac{\lambda_k}{\lambda_{k+1}}\right)^{\dfrac{\lambda_{k+1}}{\lambda_{k+1}-\lambda_k}} \;\leq\; \frac{M}{\lambda_k}$$

<u>Case ii</u>: Suppose $h'(t_k) = 0$. If $h(t_k) = 0$, then $\left(h^{\lambda_k}\right)'(t_k) = 0$ and hence, $|\Delta_k(t_k)| = 0$. If $0 < |h(t_k)| = \gamma \leq 1$, then $|\Delta_k(t_k)| = \gamma^{\lambda_k}\left(1 - \gamma^{\lambda_{k+1}-\lambda_k}\right) \;\leq\; \gamma^{\lambda_k} \;\leq \mathcal{O}\left(\lambda_k^{-1}\right)$.
In any case, $\|\Delta_k\| \leq \mathcal{O}\left(\lambda_k^{-1}\right)$

■

Corollary 1.2.7 *Suppose $h(t) = t$ and $f \in C[0,1]$ with $|f(1)| > 0$. Then, for any sequence $\{\lambda_k\}$ of finite gap,*

$$\|\Delta_k\| := \left\| \frac{t^{\lambda_k} f}{\|t^{\lambda_k} f\|} - \frac{t^{\lambda_{k+1}} f}{\left\|t^{\lambda_{k+1}} f\right\|} \right\| \leq \mathcal{O}\left(\lambda_k^{-1}\right)$$

PROOF: Since f is bounded, for each fixed t with $0 \leq t < 1$, $t^{\lambda_k} f(t) \to 0$ as $k \to \infty$, which means that eventually, $\left\|t^{\lambda_k} f\right\| \to |f(1)|$. Thus, for sufficiently large k,

$$\left|\Delta_k(1-\lambda_k^{-1})\right| \;\approx\; \frac{1}{|f(1)|}\,|f(t)|\left(t^{\lambda_k} - t^{\lambda_{k+1}}\right) \leq \frac{\|f\|}{|f(1)|}\left(t^{\lambda_k} - t^{\lambda_{k+1}}\right) \leq \mathcal{O}\left(\lambda_k^{-1}\right)$$

The last inequality holds since $0 < \lambda_{k+1} - \lambda_k \leq M < \infty$ and $h(t) = t^{\lambda_k} - t^{\lambda_{k+1}}$ implies that $\|h\| \leq \left(1 - \dfrac{\lambda_{k+1}-\lambda_k}{\lambda_{k+1}}\right)\dfrac{M}{\lambda_k} = \mathcal{O}(\frac{1}{\lambda_k})$.

■

Lemma 1.2.8 *Suppose $f \in C[0,1]$ is n times continuously differentiable in some interval $(1-\delta, 1]$ where $f^{(n)}(1)$ is the left side n^{th} derivative of f at 1 and $f^{(0)}(1) = f(1)$. If $f^{(k)}(1) = 0$ for $k = 0, 1, 2, \cdots, n-1$ and $f^{(n)}(1) \neq 0$, then there are positive constants c and d such that*

$$c\,(1-t)^n \;\leq\; |f(t)| \;\leq\; d\,(1-t)^n \quad \textit{for all } t \textit{ sufficiently near } 1$$

PROOF: The proof is by inductive construction.
By continuity, the case $n = 0$ is trivial. For $n = 1$, again by continuity of f', $|f'(1)| = L_1 > 0$, implies that there exists $\delta_1 > 0$ such that for all $t \in (1-\delta_1, 1)$,

$$\frac{1}{2}L_1 \leq f'(t) \leq \frac{3}{2}L_1 \tag{1.9}$$

Then for all $t < s$ in $(1-\delta_1, 1)$, by the mean value theorem for derivatives, there exists a γ with $t \leq \gamma \leq s$ such that

$$|f(s) - f(t)| = \left|f'(\gamma)\right| |s-t| \tag{1.10}$$

Using inequality (1.10) and letting $s \to 1$, we get

$$\frac{1}{2}L_1(1-t) \leq f(t) \leq \frac{3}{2}L_1(1-t), \text{ for all } t \in (1-\delta_1, 1) \tag{1.11}$$

If $f(1) = f'(1) = 0$ and $f''(1) \neq 1$, similar arguments as above give us

$$\frac{1}{2}L_2(1-t) \leq f'(t) \leq \frac{3}{2}L_2(1-t), \text{ for all } t \in (1-\delta_2, 1) \tag{1.12}$$

for some $L_2 > 0$ and $\delta_2 \in (0,1]$. Then, another application of the mean value theorem gives

$$(1.13) \qquad \frac{1}{2}L_2(1-t)^2 \le f(t) \le \frac{3}{2}L_2(1-t)^2, \text{ for all } t \in (1-\delta_2, 1)$$

Continuing this way gives

$$(1.14) \qquad \frac{1}{2}L_n(1-t)^m \le f(t) \le \frac{3}{2}L_n(1-t)^m, \text{ for all } t \in (1-\delta_n, 1)$$

■

Lemma 1.2.8 says that under the given conditions on f, $|f(t)| \approx c\,(1-t)^n$ for t sufficiently near 1 and some positive constant c.

Corollary 1.2.9 *Suppose* $h(t) = t$ *and* $f \in C[0,1]$ *has a bounded positive* m^{th} *derivative in some interval* $(1-\delta, 1)$ *. Then, for any sequence* $\{\lambda_k\}$ *of finite gap ,* $\|\Delta_k\| = \mathcal{O}\left(\lambda_k^{-1}\right)$

PROOF: If $\left\|t^{\lambda_k} f\right\| = \left|t_k^{\lambda_k} f(t_k)\right|$, then by similar arguments as in lemma 1.2.1, $t_k \to 1$ as $k \to \infty$. On the other hand, by lemma 1.2.8, for all t sufficiently near 1, $|f(t)| \approx c(1-t)^m$, for some constant $c > 0$. Thus, for large k,

$$\|\Delta_k\| \approx \left\| \frac{t^{\lambda_k}(1-t)^m}{\|t^{\lambda_k}(1-t)^m\|} - \frac{t^{\lambda_{k+1}}(1-t)^m}{\|t^{\lambda_{k+1}}(1-t)^m\|} \right\|$$

But then, theorem 1.2.3 gives the required result.

■

Corollary 1.2.10 *Let* $\{\lambda_k\}_1^\infty$ *be a positive increasing sequence and let* $\mathbf{0} \ne f \in C[0,1]$*. If the sequence* $\{t^{\lambda_k} f\}_1^\infty$ *is a basic sequence, then* $\{\lambda_k\}_1^\infty$ *is lacunary.*

PROOF: Suppose $\{t^{\lambda_k} f\}$ is a basic sequence. Then, by corollary 1.1.4, there exists a $\delta > 0$ such that

$$\|\Delta_k\| := \left\| \frac{t^{\lambda_k} f}{\|t^{\lambda_k} f\|} - \frac{t^{\lambda_{k+1}} f}{\|t^{\lambda_{k+1}} f\|} \right\| > \delta, \text{ for all } k = 1, 2, 3, \cdots$$

On the other hand, for each k, considering the sequence $\{j + \lambda_k\}_{j=0}^{\infty}$, and applying corollary 1.2.4, we get

$$\left\| \frac{t^{\lambda_k} f}{\|t^{\lambda_k} f\|} - \frac{t^{1+\lambda_k} f}{\|t^{1+\lambda_k} f\|} \right\| \approx c\lambda_k^{-1},$$

for k large and some $c > 0$.

Now if $\lfloor \lambda_{k+1} - \lambda_k \rfloor = p_k$ is the largest integer less than or equal to $\lambda_{k+1} - \lambda_k$, the triangle inequality gives

$$\|\Delta_k\| \leq (1 + p_k) c\lambda_k^{-1}$$

Thus, $(\lambda_{k+1} - \lambda_k)\, c\lambda_k^{-1} \geq (1 + p_k)c\lambda_k^{-1} \geq \delta$, which implies that $\lambda_{k+1} \geq \lambda_k(1 + \delta)$. That is, $\dfrac{\lambda_{k+1}}{\lambda_k} \geq 1 + \delta > 1$.

∎

CHAPTER 2

LACUNARY POWER SEQUENCES IN $\mathrm{C}[\mathrm{a}, \mathrm{b}]$ and $\mathrm{L}_{\mathrm{p}}[\mathrm{a}, \mathrm{b}]$

The first section of this chapter contains known results on the power sequences of the constant function 1 under the multiplication operator $T : f \mapsto tf$ in $C[0, 1]$. The last fours sections discuss generalizations and theorems 2.2.3, 2.3.1, 2.4.1 and 2.5.1 give min results of such generalizations to lacunary power sequences of a general function under a general multiplication operator in $C[a, b]$ and $L_p[a, b]$.

2.1 Lacunary Power Sequences of t in $C[0, 1]$

In 1966, V.I. Gurariy and V.I. Matsaev [15] established necessary and sufficient conditions for a sequence $\{t^{\lambda_k}\}_{k=-\infty}^{\infty}$, $t \in [0, b]$ to be a basic sequence in $C[0, b]$ and $L_p[0, b]$.

Recall that separated sequences were defined in chapter 1 to be sequences $\{x_j\}$ such that $\inf_{j \neq k} \|x_j - x_k\| > 0$.

Definition 2.1.1 *We say that a sequence $\{x_j\}$ in a Banach space X is uniformly minimal if there exists a positive number $\beta = \beta(\{x_j\})$ depending only on the sequence $\Lambda = \{x_j\}_j$ such that* $\inf\{\|x_j - f\| : f \in S = \overline{span}\{x_k\}, k \neq j;\ j, k = 1, 2, 3, \cdots\} > \beta$

Definition 2.1.2 *(Gurariy and Matsaev [15]) Two Banach spaces E and G are isomorphic if there exists a bounded operator T, having a bounded inverse, that maps E onto G. Let $\{e_i\}_{-\infty}^{\infty}$ and $\{g_i\}_{-\infty}^{\infty}$ be sequences in E and G, and let E_1 and*

G_1 be their closed linear spans. The sequences $\{e_i\}_{-\infty}^{\infty}$ and $\{g_i\}_{-\infty}^{\infty}$ are said to be equivalent if there exists an isomorphism T mapping E_1 onto G_1 such that

$$Te_i = g_i,\ i = 0, \pm 1, \pm 2, \ldots.$$

Let $\ell_p,\ 1 \leq p < \infty$ and ℓ_∞ be spaces of sequences $\{x_i\}_{-\infty}^{\infty}$ for which respectively,

$$\sum_{-\infty}^{\infty} |x_i|^p < \infty,\ \sup_i |x_i| < \infty,$$

with the natural definition of algebraic operations and norms

$$\left\|\{x_i\}_{-\infty}^{\infty}\right\| = \left(\sum_{-\infty}^{\infty} |x_i|^P\right)^{\frac{1}{p}}, \quad \left\|\{x_i\}_{-\infty}^{\infty}\right\| = \sup_i |x_i|$$

Let c be the subspace of ℓ_∞ containing sequences $\{x_i\}_{-\infty}^{\infty}$ such that $\lim_{k\to-\infty} x_k = 0$ and $\lim_{k\to+\infty} x_k$ exists. The basis in c is the sequence $\{g_i\}_{-\infty}^{\infty}$:

$$g_i = \left\{x_k^{(i)}\right\}_{k=-\infty}^{\infty} \quad \text{where} \quad x_k^{(i)} = \begin{cases} 1 & \text{for } k \geq i \\ 0 & \text{for } k < i \end{cases}$$

Theorem 2.1.3 *(Gurariy and Matsaev, [15]) Let $\Lambda = \{\lambda_k\}_{-\infty}^{\infty}$ be a positive, increasing sequence. Let $\mathcal{B} = \left\{t^{\lambda_k}\right\}_{-\infty}^{\infty}$. The following conditions are equivalent.*

(i) *Λ is lacunary*

(ii) *$\mathcal{B}$ is separated in $C[0,1]$*

(iii) *$\mathcal{B}$ is uniformly minimal in $C[0,1]$*

(iv) *$\mathcal{B}$ is a basic sequence in $C[0,1]$*

(v) *$\mathcal{B}$ is equivalent to the usual basis in c*

Theorem 2.1.4 *(Gurariy and Matsaev [15]) Let* $\{\lambda_k\}_{-\infty}^{\infty}$ *be an increasing sequence with* $\lambda_k > -1/p,\ (1 \le p < \infty),\ k = 0, \pm 1, \pm 2, \ldots,\ .$ *Let* $\mathcal{B} = \left\{(\lambda_k + 1/p)^{1/p} t^{\lambda_k}\right\}_{-\infty}^{\infty}$*. The following conditions are equivalent.*

(i) $\{\lambda_k + 1/p\}_{-\infty}^{\infty}$ *is lacunary*

(ii) $\mathcal{B}$ *is separated in* $L_p[0,1]$

(iii) $\mathcal{B}$ *is uniformly minimal in* $L_p[0,1]$

(iv) $\mathcal{B}$ *is a basic sequence in* $L_p[0,1]$

(v) $\mathcal{B}$ *is equivalent to the natural basis in* l_p

As Gurariy and Matsaev remarked in the same paper, the results are true for $C[0,b]$ and $L_p[0,b]$ for any $b > 0$, which immediately follows using the transformation $t \mapsto t/b$. The results are also true for one-sided sequences $\left\{t^{\lambda_k}\right\}_1^{\infty}$ or $\left\{t^{\lambda_k}\right\}_{-1}^{-\infty}$

In this thesis, we consider generalizations of these results for one-sided sequences.

Lemma 2.1.5 *Suppose* $M = \{a_{ij}\}_{i,j=1}^{\infty}$ *is an infinite matrix. Define an operator* $T : \ell_1 \to \ell_1$ *given by* $Ax = Mx$*. If there exists a positive number* λ *such that* $\sup_i \sum_{j=1}^{\infty} |a_{ij}| = \lambda$*, then* $\|T\| \le \lambda$*.*

The proof follows from theorem 25, in the book *Inequalities* by G. Hardy, J. E. Littlewood and G. Pólya [16, P31] that says: If $p \ge 1$ and $a_{ij} \ge 0$ for all $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, M$ then

$$\left(\sum_j^M \left(\sum_i^N a_{ij} \right)^p \right)^{1/p} \le \sum_i^N \left(\sum_j^M a_{ij}^p \right)^{1/p}.$$

2.2 Lacunary Power Sequences of t in $C[a, b]$

In this section, we extend theorem 2.1.3 to the space $C[a, b]$, $0 \leq a < b$.

Observe that due to the transformation $t \to bt$ from $[\frac{a}{b}, 1]$ to $[a, b]$, it is sufficient to prove that the results are true on $C[a, 1]$ for any $a \in (0, 1)$. Again due to the following lemma, it suffices to prove the result for any one particular $a \in (0, 1)$.

Lemma 2.2.1 *If there exists some $a \in (0, 1)$ such that $\{t^{\lambda_k}\}$ is a basic sequence in $C[a, 1]$ for every lacunary sequence $\{\lambda_k\}$, then the same is true in $C[b, 1]$ for any $b \in (0, 1)$.*

PROOF: This result follows from the facts that: (i) if $a, b \in (0, 1)$ then $c := log_b a > 0$; (ii) the transformation $t \to t^c$ maps $[b, 1]$ onto $[a, 1]$; and (iii) $\{\frac{1}{c}\lambda_k\}$ is lacunary.

■

Lemma 2.2.2 *If $\{x_j\}$ is a basic sequence in a Banach space X, then every infinite subsequence $\{y_i\}$ of $\{x_i\}$ is itself a basic sequence in X.*

Theorem 2.2.3 *Let $\Lambda = \{\lambda_k\}_1^\infty$ be a positive increasing sequence. Let $0 < a < 1$ and let $\mathcal{B} = \left\{t^{\lambda_k}\right\}_{-\infty}^{\infty}$. The following statements are equivalent.*

(i) Λ *is lacunary.*

(ii) $\mathcal{B}$ *is separated in* $C[a, 1]$

(iii) $\mathcal{B}$ *is uniformly minimal in* $C[a, 1]$

(iv) $\mathcal{B}$ *is a basic sequence in* $C[a, 1]$

(v) *$\mathcal{B}$ is equivalent to the summing basis $\{x_i\}$ where $x_i = (0, 0, \ldots, \overset{i}{1}, 1, 1, \ldots)$ of c.*

PROOF: A part of this proof is an adoption of the methods used by Gurariy and Matsaev [15]. We will assume that $\lambda_1 = 1$, for otherwise, we can use the transformation $t \longmapsto t^{1/\lambda_1}$.

Clearly, condition (iii) implies condition (ii). Also condition (ii) implies condition (i), for if $m < n$, then

$$\left\|t^{\lambda_m} - t^{\lambda_n}\right\|_{[a,1]} \leq \left\|t^{\lambda_m} - t^{\lambda_n}\right\|_{[0,1]} = \left(\frac{\lambda_n}{\lambda_m} - 1\right)\left(\frac{\lambda_m}{\lambda_n}\right)^{\frac{\lambda_n}{\lambda_n - \lambda_m}} \leq e^{-1}\left(\frac{\lambda_n}{\lambda_m} - 1\right).$$

If $\{\lambda_k\}$ is not lacunary, then there will be a subsequence $\{\lambda_{k_\ell}\}$ of $\{\lambda_k\}$ such that

$$\left\|t^{\lambda_{k_{\ell+1}}} - t^{\lambda_{k_\ell}}\right\| \longrightarrow 0, \text{ as } \ell \longrightarrow \infty$$

Hence, $\{t^{\lambda_k}\}$ will not be separated.

To show condition (i) implies condition (iii), it suffices to show that there is a positive number D depending only on the index of lacunarity γ such that

$$\left\|\sum_{k=1}^{\infty} \alpha_k t^{\lambda_k}\right\|_{C[a,1]} \geq D \left\|\{\alpha_k\}_{k=1}^{\infty}\right\|_{\ell_\infty} \tag{2.1}$$

for any sequence $\{\alpha_k\}_{k=1}^{\infty}$ that has finitely many nonzero terms.

To this end, let P be a polynomial of the form $P(u) = \sum_{\ell=1}^{m} c_\ell u^\ell$ and let $M(P) = m \cdot \sup\{|c_\ell| : \ell = 1, 2, ..., m\}$

Then,

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty}\alpha_j P\left(u^{\lambda_j}\right)\right\|_{C[a^{1/m},1]} &= \left\|\sum_{j=1}^{\infty}\alpha_j\sum_{\ell=1}^{m}c_\ell u^{\ell\lambda_j}\right\|_{C[a^{1/m},1]} = \left\|\sum_{\ell=1}^{m}c_\ell\sum_{j=1}^{\infty}\alpha_j u^{\ell\lambda_k}\right\|_{C[a^{1/m},1]} \\
&= \sup_{a^{\frac{\ell}{m}}\le t\le 1}\left|\sum_{\ell=1}^{m}c_\ell\sum_{j=1}^{\infty}\alpha_j t^{\lambda_j}\right| \le \left|\sum_{\ell=1}^{m}c_\ell\right|\cdot\sup_{a\le t\le 1}\left|\sum_{j=1}^{\infty}\alpha_j t^{\lambda_j}\right| \\
&\le M(P)\sup_{a\le t\le 1}\left|\sum_{j=1}^{\infty}\alpha_j t^{\lambda_j}\right| = M(P)\left\|\sum_{j=1}^{\infty}\alpha_j t^{\lambda_j}\right\|_{C[a,1]}
\end{aligned}$$

Therefore, choosing $a = e^{-1}$,

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty}\alpha_j t^{\lambda_j}\right\|_{C[e^{-1},1]} &\ge \frac{1}{M(p)}\left\|\sum_{j=1}^{\infty}\alpha_j P\left(u^{\lambda_j}\right)\right\|_{C[e^{-1/m},1]} \\
&\ge \frac{1}{M(P)}\cdot\sup_k\left|\sum_{j<k}\alpha_j P\left(e^{-\lambda_j/\lambda_k}\right)\right|
\end{aligned} \tag{2.2}$$

Let the operator T be defined on ℓ_∞ by the matrix $\{m_{jk}\}_{j,k=1}^{\infty}$:

$$m_{jk} = \begin{cases} P\left(e^{-\lambda_j/\lambda_k}\right) & \text{for } j<k \\ 0 & \text{for } j\ge k \end{cases}$$

relative to the naturale basis $e = \{e_i\}_{i=1}^{\infty}$, where $e_1 = (1,0,0,\ldots),\ e_2 = (0,1,0,\ldots),\ \ldots$.

If I is the identity operator on ℓ_∞, then (2.2) can be written as

$$\left\|\sum_{1}^{\infty}\alpha_k t^{\lambda_k}\right\|_{C[e^{-1},1]} \ge \frac{1}{M(P)}\left\|(I+T)\{\alpha_k\}_1^{\infty}\right\|_{\ell_\infty}. \tag{2.3}$$

Let $\gamma := \inf_k \frac{\lambda_{k+1}}{\lambda_k}$ be the index of lacunarity. Choose a polynomial P with the properties:

(i) $P(e^{-1}) = 1$

(ii) P is decreasing in the interval $(e^{-1/m\gamma}, 1)$, and

(iii) $P(u) \leq \frac{\gamma-1}{2}(1-u)$ for $e^{-1/m\gamma} \leq u \leq 1$

For an appropriate m , $P(u) := \left(\dfrac{1-u}{1-e^{-1}}\right)^m$ is one such polynomial.

Since $\frac{\lambda_j}{\lambda_k} \leq \gamma^{j-k}$ for $j < k$ and $1 - e^{ct} \leq t$ for all $c, t \in (0,1)$, using (ii) and (iii) above, for each fixed k we get

$$\begin{aligned}
\sum_{j<k} m_{jk} &= \sum_{j<k} P\left(e^{-\frac{\lambda_j}{m\lambda_k}}\right) \leq \sum_{j<k} P\left(e^{-\frac{1}{m}\gamma^{j-k}}\right) \\
&\leq \frac{\gamma-1}{2}\sum_{j<k}\left(1 - e^{-\frac{1}{m}\gamma^{k-j}}\right) \leq \frac{\gamma-1}{2}\sum_{l=1}^{\infty}\gamma^{-l} = \frac{1}{2}.
\end{aligned}$$

Since $\displaystyle\sum_{j<k} m_{jk} = 0$, for each k, $\displaystyle\sum_{j=1}^{\infty} m_{jk} \leq \frac{1}{2}$. By similar arguments, for any j , $\displaystyle\sum_{k=1}^{\infty} m_{jk} \leq \frac{1}{2}$.

So by lemma 2.1.5, $\|T\| \leq \frac{1}{2}$ which means $\|(I+T)^{-1}\| \leq \frac{1}{1-\|T\|} \leq 2$. Therefore, from (2.3) we get

$$\left\|\sum_{1}^{\infty} \alpha_k t^{\lambda_k}\right\|_{C[e^{-1},1]} \geq \frac{1}{2M(P)} \|\{\alpha_k\}_1^{\infty}\|_{\ell_\infty},$$

which is what we wanted in (2.1).

To show that (i) and (iii) implies (iv), we will show that the projection maps P_n

given by

$$P_n\left(\sum_1^{\infty}\alpha_k t^{\lambda_k}\right) \;=\; \sum_1^{n}\alpha_k t^{\lambda_k},\ n=1,2,3,\cdots$$

are uniformly bounded.

By (iii) there exists $\beta > 0$, depending only on the sequence $\Lambda = \{\lambda_k\}_k$ such that the distance between t^{λ_i} and the linear space generated by the sequence $\{t^{\lambda_k}\}_{k\neq i}$ is not less than β. Clearly $\beta \le 1$ and

(2.4) $$\sup\{|\alpha_k| \;:\; f=\sum_k \alpha_k t^{\lambda_k},\ \|f\|\le 1\} \le \frac{1}{\beta}$$

Let $\beta_n := \left(1-\frac{1}{\lambda_n}\right)$. Define a linear transformation T_{β_n} on $M(\Lambda)$ by

$$T_{\beta_n}(f)(t) := f(\beta_n t)$$

By the arguments in the proof of corollary 6.2.4 in Gurariy and Lusky [15, p. 81], $\lim\limits_{n\to\infty}\|T_{\beta_n}f-f\|=0$. Accordingly,

(2.5) $$\sup_n \|T_{\beta_n}\| < \infty$$

Moreover, using the fact that $e^{a^x} \ge e\cdot a^x > a^x$ for all $a>0,\ x\ge 0$,

$$\sum_{k=n+1}^{\infty}\beta_n^{\lambda_k} \;=\; \sum_{k=n+1}^{\infty}\left(1-\frac{1}{\lambda_n}\right)^{\lambda_k} \le \sum_1^{n} e^{-\gamma^{(n-k)}} \le \sum_{k=n+1}^{\infty}\gamma^{-(n-k)} \le \frac{\gamma}{\gamma-1}$$

and

$$\sum_{1}^{n}\left(1-\beta_n^{\lambda_k}\right) \leq \sum_{1}^{n}\lambda_k\left(1-\beta_n\right) = \sum_{1}^{n}\frac{\lambda_k}{\lambda_n} \leq \sum_{1}^{n}\left(\frac{1}{\gamma}\right)^{n-k} \leq \frac{\gamma}{\gamma-1}$$

Consequently,

$$\|P_n f - T_{\beta_n} f\| \leq \sum_{1}^{n}\left(1-\beta_n^{\lambda_k}\right)|\alpha_k| + \sum_{k=n+1}^{\infty}\beta_n^{\lambda_k}|\alpha_k| \leq \frac{2\gamma}{\beta(\gamma-1)}$$

Therefore, by (2.6),

$$\|P_n\| \leq \|T_{\beta_n}\| + \frac{2\gamma}{\beta(\gamma-1)} < \infty$$

This means that the sequence of projections, $\{P_n\}$ is uniformly bounded in $C[a,1]$. As a result, $\{t^{\lambda_k}\}$ is a basic sequence in $C[a,1]$.

■

Corollary 2.2.4 *If $\{\lambda_k\}_1^\infty$ is a lacunary sequence, then for $0 < a < b$ the sequence $\{t^{\lambda_k}\}_1^\infty$ is basic in the space $C[a,b]$.*

PROOF: Let $0 < a < b$. Let m_1, m_2 be positive integers such that $m_1 < m_2$ and let $\{\alpha_k\}$ be any sequence. By theorem 2.2.3, there exists a positive number K such that

$$\begin{aligned}
\left\|\sum_1^{m_1} \alpha_k t^{\lambda_k}\right\|_{C[a,b]} &= \left\|\sum_1^{m_1} \alpha_k b^{\lambda_k}\left\{b^{-1}t\right\}^{\lambda_k}\right\|_{C[a,b]} \\
&= \left\|\sum_1^{m_1} \left(\alpha_k b^{\lambda_k}\right) u^{\lambda_k}\right\|_{C[\frac{a}{b},1]} \\
&\le K\left\|\sum_1^{m_2} \alpha_k b^{\lambda_k} u^{\lambda_k}\right\|_{C[\frac{a}{b},1]} \\
&= K\left\|\sum_1^{m_2} \alpha_k t^{\lambda_k}\right\|_{C[a,b]}
\end{aligned}$$

■

2.3 Lacunary Power Sequences of h in $C[a,b]$

Here, we consider generalizations to a multiplication operator by a general function $h \in C[a,b]$, namely, $T : C[a,b] \longrightarrow C[a,b]$ such that $(Tf)(t) := h(t)f(t)$.

Theorem 2.3.1 *Let $0 \le a < b$. Suppose $h \in C[a,b]$ is non-negative and not constant on any subinterval of $[a,b]$. Suppose $\{\lambda_k\}_1^\infty$ is a lacunary sequence. Then $\left\{h^{\lambda_k}\right\}_1^\infty$ is a basic sequence in $C[a,b]$.*

PROOF: Suppose $h([a,b]) = [c,d]$ for some $0 \le a < b$ and $0 \le c < d$. Let n be a positive integer and $\alpha_1, \alpha_2, \ldots \alpha_n$ be scalars. Then, the function $\sum\limits_1^n \alpha_k h^{\lambda_k} \in C[a,b]$ attains its maximum value, $\|\sum\limits_1^n \alpha_k h^{\lambda_k}\|$ at some point, say $t_n \in [a,b]$. Suppose $h(t_L) = u_n$. That is, $\left\|\sum\limits_1^n \alpha_k h^{\lambda_k}\right\|_{C[a,b]} = \left|\sum\limits_1^n \alpha_k u_n^{\lambda_k}\right|$.

Since h is onto $[c,d]$, for every $u \in [c,d]$, there exists a $t_u \in [a,b]$ such that $h(t_u) = u$. This means that for all $u \in [c,d]$,

$$\left|\sum_1^n \alpha_k u^{\lambda_k}\right| = \left|\sum_1^n \alpha_k h^{\lambda_k}(t_u)\right| \leq \left|\sum_1^n \alpha_k h^{\lambda_k}(t_n)\right| = \left|\sum_1^n \alpha_k u_n^{\lambda_k}\right|.$$

That is,

$$\left\|\sum_1^n \alpha_k h^{\lambda_k}\right\|_{C[a,b]} = \left\|\sum_1^n \alpha_k u^{\lambda_k}\right\|_{C[c,d]}.$$

Therefore, by theorem 2.2.4 we conclude that $\{h^{\lambda_k}\}_1^\infty$ is a basic sequence in $C[a,b]$

■

2.4 Lacunary Power Sequences of t in $L_p[a,b]$

Here we show that the result of Gurariy and Matsaev extends to $L_p[a,b]$ for all $0 \leq a < b$ and to sequences $\{h^{\lambda_k}\}_1^\infty$, where h satisfies certain conditions.

Lemma 2.4.1 *Let $\{\lambda_k\}$ be a positive increasing sequence. The following are equivalent.*

(i) $\{\lambda_k\}$ *is lacunary*

(ii) $\{c\lambda_k\}$ *is lacunary for all* $c > 0$.

(iii) $\{\lambda_k + b\}$ *is lacunary for any* $b > 0$.

PROOF: Equivalence of (i) and (ii) is clear. On the other hand, since $\dfrac{\lambda_{k+1}}{\lambda_k} > \dfrac{\lambda_{k+1}+b}{\lambda_k+b}$ for all $b > 0$, (iii) $\Rightarrow$ (i) follows. Conversely, if $\dfrac{\lambda_{k+1}}{\lambda_k} \geq 1+\varepsilon$, then $\dfrac{\lambda_{k+1}+b}{\lambda_k+b} \geq 1 + \dfrac{\lambda_k}{\lambda_k+b} \geq 1 + \dfrac{\lambda_1}{\lambda_1+b}\varepsilon$ and so (i) $\Rightarrow$(iii).

■

Theorem 2.4.2 *Suppose* $1 \leq p < \infty$ *and* $\{\lambda_k\}_1^\infty$ *is a lacunary sequence. If* $0 \leq a < b$*, then* $\left\{t^{\lambda_k}\right\}_1^\infty$ *is a basic sequence in the space* $L_p[a, b]$.

For the proof, observe that letting $t = bu$,

$$\int_a^b \left|\sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1}\; t^{\lambda_k}\right|^p dt \;=\; \int_{\frac{a}{b}}^1 \left|\sum_1^\infty \alpha_k b^{\frac{1}{p}+\lambda_k}\; \sqrt[p]{p\lambda_k + 1}\; u^{\lambda_k}\right|^p du$$

Since $\frac{a}{b} < 1$, it suffices to show that the sequence $\left\{t^{\lambda_k}\right\}$ is basic in the space $L_p\,[b, 1]$ for every b with $0 < b < 1$

We will do this in two steps. First we show the existence of one such a and then we prove the general case. Then we prove it for any $b \in (0, 1)$.

Lemma 2.4.3 *Suppose* $\{\lambda_k\}_1^\infty$ *is a lacunary sequence. If* $1 \leq p < \infty$*, then there exists a number* $a > 0$ *such that* $\left\{t^{\lambda_k}\right\}_1^\infty$ *is a basic sequence in the space* $L_p[a, 1]$.

PROOF: By Gurariy and Matsaev [15], for any $a > 0$, the sequence $\left\{t^{\lambda_k}\right\}_1^\infty$ is a basic sequence in the space $L_p[0, a]$ equivalent to the natural basis of ℓ_p. Thus, there exist positive numbers A, B, such that

$$A^p \sum_1^\infty |\alpha_k|^p \leq \int_0^1 \left|\sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1} t^{\lambda_k}\right|^p dt \leq B^p \sum_1^\infty |\alpha_k|^p \tag{2.6}$$

for all sequences $\{\alpha_k\}_1^\infty$ with finitely many nonzero terms.

For $0 < a < 1$,

$$\begin{aligned}
\int_0^a \left| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1} t^{\lambda_k} \right|^p dt &\leq \int_0^a \sum_1^\infty |\alpha_k|^p \sum_1^\infty (p\lambda_k + 1)\, t^{p\lambda_k} dt \\
&\leq \sum_1^\infty |\alpha_k|^p \sum_1^\infty a^{p\lambda_k + 1} \\
&\leq \sum_1^\infty |\alpha_k|^p \sum_1^\infty a^k \\
&= \frac{a}{1-a} \sum_1^\infty |\alpha_k|^p
\end{aligned}$$

Since $\frac{x}{1-x}$ is increasing in $(0,1)$ and converges to 0 as $x \to 0^+$, for all $a \in (0,1)$ such that $\frac{a}{1-a} \leq \frac{1}{2} A^p$,

$$\begin{aligned}
\int_a^1 \left| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1} t^{\lambda_k} \right|^p dt &= \int_0^1 \left| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1} t^{\lambda_k} \right|^p dt - \\
&\quad - \int_0^a \left| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1} t^{\lambda_k} \right|^p dt \\
&\geq \frac{1}{2} A^p \sum_1^\infty |\alpha_k|^p .
\end{aligned}$$

On the other hand,

$$\int_a^1 \left| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1} t^{\lambda_k} \right|^p dt \leq \int_0^1 \left| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1} t^{\lambda_k} \right|^p dt \leq B^p \sum_1^\infty |\alpha_k|^p$$

Thus there exists a number $a \in (0,1)$ and two positive numbers C_1 and C_2 such that for any sequence $\{\alpha_k\}_1^\infty$ with finitely many nonzero terms,

$$C_1 \left\| \{\alpha_k\}_1^\infty \right\|_{\ell_p} \leq \left\| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1} t^{\lambda_k} \right\|_{L_p[a,1]} \leq C_2 \left\| \{\alpha_k\}_1^\infty \right\|_{\ell_p} \tag{2.7}$$

That is, the sequence is equivalent with the natural basis of ℓ_p. Consequently, there exists an $a > 0$ such that $\{t^{\lambda_k}\}$ is a basic sequence in the space $L_p[a,1]$

■

Now we show that the same is true for any $0 < b < 1$.

Lemma 2.4.4 *Under the hypothesis of the theorem, the sequence* $\{t^{\lambda_k}\}$ *is basic in the space* $L_p[b,1]$ *for every* b *with* $0 < b < 1$.

PROOF: Let $0 < b < 1$. By lemma 2.4.3, there exist $a \in (0,1)$ and positive numbers C_1, C_2 such that

$$C_1^p \sum_1^\infty |\alpha_k|^p \leq \int_a^1 \left| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1} t^{\lambda_k} \right|^p dt \leq C_2^p \sum_1^\infty |\alpha_k|^p$$

for all sequences $\{\alpha_k\}_1^\infty$ with finitely many nonzero terms.

Clearly,

$$\left\| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1} t^{\lambda_k} \right\|_{L_p[b,1]} \leq \left\| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1} t^{\lambda_k} \right\|_{L_p[0,1]} \leq C_2 \left\| \{\alpha_k\}_1^\infty \right\|_{\ell_p} \tag{2.8}$$

for some $C_2 > 0$.

On the other hand, putting $c = \log_a b$, and replacing t by $t^{c\lambda_k}$ we have

$$\begin{aligned} \int_b^1 \left| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1}\, t^{\lambda_k} \right|^p dt &= c \int_a^1 \left| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1}\, t^{c\lambda_k} \right|^p t^{c-1} dt \\ &\geq c \int_a^1 \left| \sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1}\, t^{c\lambda_k} \right|^p dt. \end{aligned}$$

Thus there exists a positive number C_1 such that

$$C_1 \left\|\{\alpha_k\}_1^\infty\right\|_{\ell_p} \leq \left\|\sum_1^\infty \alpha_k \sqrt[p]{p\lambda_k + 1}\, t^{\lambda_k}\right\|_{L_p[b,1]} . \tag{2.9}$$

Therefore, by (2.8) and (2.9), the sequence $\left\{t^{\lambda_k}\right\}_1^\infty$ is a basic sequence in the space $L_p[b,1]$ for any $0 < b < 1$.

■

2.5 Lacunary Power Sequences of h in $L_p[a,b]$

Theorem 2.5.1 *Suppose $h : [a,b] \to [c,d], (c > 0)$ is increasing, differentiable and $\frac{1}{K} \leq h'(t) \leq K$ for some $K > 0$. Let $\{\lambda_k\}_{k=1}^\infty$ be a positive increasing sequence. Then for any sequence $\{\alpha_k\}_1^\infty$ and any $n \in \mathbb{N}$,*

$$\frac{1}{\sqrt[p]{K}} \left\|\sum_1^n \alpha_k u^{\lambda_k}\right\|_{L_p[c,d]} \leq \left\|\sum_1^n \alpha_k h^{\lambda_k}\right\|_{L_p[a,b]} \leq \sqrt[p]{K} \left\|\sum_1^n \alpha_k u^{\lambda_k}\right\|_{L_p[c,d]}$$

Consequently, $\{h^{\lambda_k}\}_1^\infty$ is a basic sequence in $L_p[a,b]$ for $0 \leq a < b$ and $1 \leq p < \infty$, provided that $\{\lambda_k\}$ is lacunary.

PROOF: Using the transformation $t \mapsto (b-a)t + a$, we may assume that $[a,b] = [0,1]$. Let $L = d - c$. For each positive integer N and $m = 1, 2, \cdots, N$, let $S_{m,N} = \left\{t : c + \frac{m-1}{N}L \leq h(t) \leq c + \frac{m}{N}L\right\}$, and $I_{m,N} = \left[c + \frac{m-1}{N}L, c + \frac{m}{N}L\right]$

Since h is increasing and $\frac{1}{K} \leq h'(t) \leq K$, for all t, we see that

$$\frac{L}{KN} \leq \mu(S_{m,N}) \leq \frac{KL}{N} \tag{2.10}$$

That is,

$$\frac{1}{K}\mu(I_{m,N}) \leq \mu(S_{m,N}) \leq K\mu(I_{m,N}).$$

Now, let $\{\alpha_k\}_1^n$ be a finite sequence of scalars. Since $p(t) := \sum_1^n \alpha_k h^{\lambda_k}$ is uniformly continuous on $[a, b]$, for every $\varepsilon > 0$ we can choose an $N > 0$, large enough such that

$$\left| \left| \sum_{k=1}^n \alpha_k h^{\lambda_k}(t) \right|^p - \left| \sum_{k=1}^n \alpha_k u^{\lambda_k} \right|^p \right| < \varepsilon$$

for all $t \in S_{m,N}$ and all $u \in I_{m,N}$, for all $m = 1, 2, \cdots, N$.

That is, for each $u \in I_{m,N}$,

$$\left| \sum_{k=1}^n \alpha_k u^{\lambda_k} \right|^p - \varepsilon < \left| \sum_{k=1}^n \alpha_k h^{\lambda_k}(t) \right|^p < \left| \sum_{k=1}^n \alpha_k u^{\lambda_k} \right|^p + \varepsilon,$$

Integrating with respect to t over $S_{m,N}$, and using (2.10),

$$\frac{L}{KN}\left(\left| \sum_1^n \alpha_k u^{\lambda_k} \right|^p - \varepsilon \right) < \int_{S_{m,N}} \left| \sum_1^n \alpha_k h^{\lambda_k}(t) \right|^p dt < \frac{KL}{N}\left(\left| \sum_1^n \alpha_k u^{\lambda_k} \right|^p + \varepsilon \right)$$

and integrating with respect to u over $I_{m,N}$, we obtain,

$$\begin{aligned} \frac{L}{KN}\left(\int_{I_{m,N}} \left| \sum_1^n \alpha_k u^{\lambda_k} \right|^p du - \frac{L}{N}\varepsilon \right) &< \frac{L}{N}\int_{S_{m,N}} \left| \sum_1^n \alpha_k h^{\lambda_k}(t) \right|^p dt \\ &< \frac{KL}{N}\left(\int_{I_{m,N}} \left| \sum_1^n \alpha_k u^{\lambda_k} \right|^p du + \frac{L}{N}\varepsilon \right) \end{aligned}$$

Adding up,

$$\begin{aligned}\frac{L}{K}\sum_{m=1}^{N}\int_{I_{m,N}}\left|\sum_1^n \alpha_k u^{\lambda_k}\right|^p du - \frac{L}{K}\varepsilon &< L\sum_{m=1}^{N}\int_{S_{m,N}}\left|\sum_1^n \alpha_k h^{\lambda_k}(t)\right|^p dt \\ &< KL\sum_{m=1}^{N}\int_{I_{m,N}}\left|\sum_1^n \alpha_k u^{\lambda_k}\right|^p du + KL\varepsilon\end{aligned}$$

Dividing by L and letting $n \to \infty$,

$$\frac{1}{K}\int_c^d\left|\sum_1^n \alpha_k u^{\lambda_k}\right|^p du - \frac{1}{K}\varepsilon < \int_b^a\left|\sum_1^n \alpha_k h^{\lambda_k}(t)\right|^p dt < K\int_c^d\left|\sum_1^n \alpha_k u^{\lambda_k}\right|^p du + K\varepsilon$$

Letting $\varepsilon \to 0$,

$$\frac{1}{K}\int_c^d\left|\sum_1^n \alpha_k u^{\lambda_k}\right|^p du \leq \int_a^b\left|\sum_1^n \alpha_k h^{\lambda_k}(t)\right|^p dt \leq K\int_c^d\left|\sum_1^n \alpha_k u^{\lambda_k}\right|^p du$$

Therefore,

$$\frac{1}{\sqrt[p]{K}}\left\|\sum_1^n \alpha_k u^{\lambda_k}\right\|_{L_p[c,d]} \leq \left\|\sum_1^n \alpha_k h^{\lambda_k}(t)\right\|_{L_p[0,1]} \leq \sqrt[p]{K}\left\|\sum_1^n \alpha_k u^{\lambda_k}\right\|_{L_p[c,d]}$$

Thus, $\left\{h^{\lambda_k}\right\}_1^\infty$ is a basic sequence in $L_p[a,b]$, whenever $\{\lambda_k\}$ is lacunary.

■

CHAPTER 3

LACUNARY POWER SEQUENCES : AN ALTERNATE APPROACH

In this chapter, we use the results of Gurariy and Lusky [14] and Borwein and T. Erdélyi [3], [5] to obtain generalizations of Gurariy and Matsaev, which in general are similar to those we had in chapter 2. We will use the notations: $E[a,b]$ to represents one of $C[0,1]$ or $L_p[0,1]$, $E(A)$ to represent one of $C(A)$ or $L_p(A)$, $1 \leq p < \infty$, and $M(\Lambda) =$ span$\{t^{\lambda_1}, t^{\lambda_2}, \ldots\}$, called the Müntz space associated with the sequence $\Lambda = \{t^{\lambda_1}, t^{\lambda_2}, \ldots\}$. The polynomials $p(t) = \sum_1^n \alpha_k t^{\lambda_k}$ in $M(\Lambda)$ are said to be Müntz polynomials on the sequence Λ. We will show that,

(i) If f is nonzero almost everywhere in some interval $(1-\delta, 1) \subset (0,1)$, then every lacunary orbit $\{t^{\lambda_k} f(t)\}_1^\infty$ of f under the multiplication operator $T(f)(t) := tf(t)$ is a basic sequence in $E[0,1]$

(ii) For the case $f = \mathbf{1}$, the interval $[0,1]$ will be extended to $[a,b]$, where $0 < a < b$

(iii) Under certain conditions on h, lacunary power sequences $\{h^{\lambda_k}\}$ are basic sequences in $E[a,b]$

3.1 Known Results

Lemma 3.1.1 *(V.I. Gurariy, W. Lusky [14, p.82]) Define $\eta : [0,1] \to [0,1]$ by $\eta(y) = 4y(1-y)$. There exists a positive integer M, independent of n such that*

$$\sum_{k=1,\, k\neq n}^{\infty} \eta(2^{-\lambda_k/\lambda_n})^M \leq \frac{1}{2} \quad \textit{for every } n \in \mathbb{N} \tag{3.1}$$

Putting $c_l = 4^M \binom{M}{l}(-1)^l$, we get

$$\eta(2^{-\lambda/\lambda_n})^M = \sum_{1}^{M} c_l \cdot 2^{-\lambda(l+M)/\lambda_n} \quad \text{and} \quad \sum_{l=0}^{M} |c_l| = 8^M \tag{3.2}$$

We will also use the following two theorems by Borwein and T. Erdélyi [3]).

Theorem 3.1.2 *(Borwein and T. Erdélyi, [3]) Suppose* $\sum_1^\infty 1/\lambda_k < \infty$. *Let* $s > 0$. *Then there exists a constant* c *depending only on* $\Lambda = \{\lambda_k\}_1^\infty$ *and* s *(and not on* a, A, *or the "length" of* p*) so that*

$$\|p\|_{C[0,a]} \le c\|p\|_{C(A)}$$

for every $p(t) = \sum_1^n \alpha_k t^{\lambda_k} \in M(\Lambda)$ *and for every set* $A \subset [a,1]$ *with* $\mu(A) > s$.

Consequently, for each $a \in (0,1)$, there exists a positive number C such that

$$C\,\|p\|_{C[0,1]} \le \|p\|_{C[a,1]} \le \|p\|_{C[0,1]}$$

for every Müntz polynomial p.

Theorem 3.1.3 *(Borwein and T. Erdélyi, [3]) Suppose* $\sum_1^\infty 1/\lambda_k < \infty$. *Let* $s > 0$ *and* $q \in (0,\infty)$. *Then there exists a constant* c *depending only on* $\Lambda := \{\lambda_k\}_0^\infty, s,$ *and* q *(and not on* $a,\ A$, *or "length" of* p*) so that*

$$\|p\|_{C[0,a]} \le c\|p\|_{L_q(A)}$$

for every $p \in M(\Lambda)$ *and for every set* $A \subset [a,1]$ *with* $\mu(A) > s$.

Consequently, for each $a \in (0,1)$, there exists a positive number C such that

$$C\,\|p\|_{L_p[0,1]} \leq \|p\|_{L_p[a,1]} \leq \|p\|_{L_p[0,1]}$$

for every Müntz polynomial p.

Now, for each $\rho \in (0,1)$, define an operator $T_\rho : M(\Lambda) \to M(\Lambda)$ by $T_\rho(p)(t) = p(\rho t)$.

3.2 Lacunary Power Sequences in $C[0,1]$ and $L_p[0,1]$

Corollary 3.2.1 *If* $0 < a, \delta < 1$, *then there exists a constant* $c > 0$, *depending only on* Λ, a, ρ *and* δ *such that*

$$\mu(\{x \in [a,1] : |p(x)| \geq c\|T_\rho p\|_{E[0,1]}\}) \geq (1-a)(1-\frac{\delta}{4}) \tag{3.3}$$

for all $p \in M(\Lambda)$ *and for all* $\rho \in (0,1]$, *where* μ *is the Lebesgue measure.*

PROOF: Observe that $\|T_\rho p\|_{C[0,1]} \leq \|p\|_{[0,1]}$ for all $p \in M(\Lambda)$. Now, let $A_N := \{x \in [a,1] : |p(x)| < N^{-1}\|T_\rho p\|_{E[0,1]}, \|p\|_{C[0,1]} \leq 1\}$.

If (3.3) were not true, then for each $N > 0$ there would exist some $p \in M(\Lambda)$ with $\|p\|_{C[0,1]} \leq 1$ and $\rho \in (0,1)$ such that $\mu(A_N) \geq \frac{1}{4}(1-a)\delta =: s > 0$. Then, by theorems 3.1.2 and 3.1.3 there exists a constant $c > 0$, independent of N and p such that $\|p\|_{E[0,1]} \leq c\|p\|_{E(A_N)}$. On the other hand, by definition of A_N, $\|p\|_{E(A_N)} \leq N^{-1}\|T_\rho p\|_{C[0,1]}$. Combining these conditions, we get $1 \leq cN^{-1}$ for all $N > 0$, which is impossible.

■

Corollary 3.2.1 says that for each $\rho \in (0,1)$, there exists a constant c, independent of p such that $\|p\|_{E[a,1]} \geq c\|T_\rho p\|_{E[0,1]}$. Therefore, there exists a $C > 0$ such that

$$\sup_{0<\rho<1} \|T_\rho\|_{E[0,1]} < C < \infty \tag{3.4}$$

For a fixed $h \in E[0,1]$ and $0 < \rho < 1$, define $T_{\rho;h} : M(\Lambda; h) \to M(\Lambda; h)$ by $T_{\rho;h}(F) := (T_\rho p) \cdot h$, where $F(t) := p(t)h(t) \in M(\Lambda; h) = \{p(t)h(t) : p \in span\{t^{\lambda_k}\}\}$.

Lemma 3.2.2 *Suppose $h \in E[0,1]$ is such that $h(t) \neq 0$ $a.e.$ in some interval $(1-\delta, 1)$. Then, for each $0 < a < 1$, there exist a constant $c > 0$ such that*

$$\mu\left(\{x \in [a,1] : |h(x)| > c\|h\|_{E[0,1]}\}\right) \geq \frac{3}{4}(1-a)\delta, \tag{3.5}$$

where μ is the Lebesgue measure.

PROOF: Let $A := [a,1] \cap [1-\delta, 1]$. If (3.5) were not true, then, for all $c > 0$, we would have

$$\mu\left(\{x \in A : |h(x)| > c\|h\|_{E[0,1]}\}\right) \leq \frac{3}{4}(1-a)\delta \tag{3.6}$$

Consequently,

$$\begin{aligned}
(1-a)\delta &\leq \mu(A) \\
&= \mu\left(\{x \in A : |h(x)| > c\|h\|_{E[0,1]}\}\right) + \mu\left(\{x \in A : |h(x)| \leq c\|h\|_{E[0,1]}\}\right) \\
&\leq \frac{3}{4}(1-a)\delta + \mu(\{x \in A : |h(x)| \leq c\|h\|_{E[0,1]}\})
\end{aligned}$$

Letting $c \to 0$, gives $0 < \frac{1}{4}(1-a)\delta \leq \mu(\{x \in A : |h(x)| = 0\})$, contradicting the hypothesis.

■

Theorem 3.2.3 *Suppose* $h \in E[0,1]$ *is such that* $h(t) \neq 0$ $a.e.$ *in some interval* $(1-\delta, 1)$*. Then, for every* $a \in (0,1)$*, there is a constant* c *that depends only on* δ, a*, and* Λ*, but not on* p *such that*

$$\|T_{\rho;h}F\|_{E[0,1]} \ \leq \ c\|F\|_{E[a,1]}, \quad \textit{for all } F \ \in \ M(\Lambda; h), \textit{ for all } \rho \in (0,1) \tag{3.7}$$

Consequently,

$$\sup_{0<\rho<1} \|T_{\rho;h}\|_{E[0,1]} < C < \infty \quad \textit{for some } \ C > 0 \tag{3.8}$$

PROOF: Fix $0 < a < 1$. For each $p \in M(\Lambda)$ and $k > 0$, let

$$A_k := \{x \in [a,1] : |h(x)| > k\|h\|_{E[0,1]}\}$$

and

$$B_{k,p} := \{x \in [a,1] : |p(x)| > k\|T_{\rho;h}p\|_{E[0,1]}, \|p\|_{C[0,1]} \leq 1\}$$

By corollary 3.2.1 and lemma 3.2.2, there is a positive constant c such that $\mu(A_c) \geq \frac{3}{4}(1-a)\delta$ and $\mu(B_{c,p}) \geq 1 - a - \frac{1}{4}(1-a)\delta$. Then, for all $p \in M(\Lambda)$,

$$\begin{aligned} \mu\left(\{x \in [a,1] : |ph(x)| > c\|T_{\rho;h}ph\|_{E[0,1]}\}\right) \quad &\geq \quad \mu(A_c \cap B_{c,p}) \\ &= \quad \mu(A_c) + \mu(B_{c,p}) - \mu(A_c \cup B_{c,p}) \\ &\geq \quad \frac{1}{2}(1-a)\delta \ > \ 0 \end{aligned}$$

Since $p \in M(\Lambda)$ is arbitrary, we get, $\|T_{\rho;h}F\|_{E[0,1]} \leq c^{-1}\|F\|_{E[a,1]}$, for all $p \in M(\Lambda)$, which implies (3.8).

■

Theorem 3.2.4 *Suppose $f \in C[0,1]$, and $f(t) \neq 0$ a.e. Then, the sequence $\{t^{\lambda_k} f(t)\}_1^\infty$ is a basic sequence in $E[0,1]$.*

PROOF: Let $A_k = \|T^{\lambda_k} h\|_{E[0,1]}$ and $e_k = A_k t^{\lambda_k} h(t)$, $k = 1, 2, \cdots$. Then $\|e_k\| = \|e_k\|_{E[0,1]} = 1$, for all $k \in \mathbb{N}$. First, we will show that $\{A_k t^{\lambda_k} h(t)\}_1^\infty$ is uniformly minimal. To this end, let $F_N(t) = \sum_1^N \alpha_k A_k e_k \in [M(\Lambda)]$.

Suppose

$$|\alpha_n|\|e_n\| = \sup\{|\alpha_k|\|e_k\| : k = 1, 2, \cdots, N\}$$

From lemmas 3.1.1 and theorem 3.2.3,

$$\begin{aligned}
\frac{1}{2}|\alpha_n|\|e_n\| &\leq |\alpha_n|\|e_n\| - \sum_{k \neq n} \eta(2^{\frac{-\lambda_k}{\lambda_n}})^M |\alpha_k|\|e_k\| \\
&\leq \|\sum_1^N \eta\left(2^{-\lambda_k/\lambda_n}\right)^M \alpha_k e_k\| \\
&\leq \sum_0^M |c_k| \|\sum_1^N \alpha_k A_k (2^{\frac{-(k+M)}{\lambda_n}} t)^{\lambda_k} h(t)\| \\
&= 8^M \|T_{\rho;h} F\|, \text{where } 2^{-\frac{k+M}{\lambda_n}} \\
&\leq 8^M \|T_{\rho;h}\| \|F\|_{E[0,1]} \\
&\leq 8^M c \|F\|_{E[0,1]}
\end{aligned}$$

Since $\|e_k\| = 1$, we obtain $|\alpha_k| \leq 2 \cdot 8^M c \|F\|_{E[0,1]}$, for all $k = 1, 2, \cdots$.

Equivalently,

$$\left\| e_j - \sum_{k=1, k\neq j}^{N} \alpha_k e_k \right\| \geq (2c)^{-1} \cdot 8^{-M} > 0, \tag{3.9}$$

for all $k = 1, 2, \cdots, N$ and for all $N = 1, 2, \cdots$. Therefore, $\{A_k t^{\lambda_k} h(t)\}_1^{\infty}$ is uniformly minimal.

To finish the proof of the theorem, we will show that the projection maps $P_n \left(\sum_1^{\infty} \alpha_k e_k\right) = \sum_1^n \alpha_k e_k$ are uniformly bounded.

For each projection P_n, consider the corresponding operator T_{ρ_n}, where $\rho_n < 1$ is to be chosen below. Then,

$$\begin{aligned} \|P_n F - T_{\rho_n} F\| &\leq \left(\sum_1^n (1 - \rho_n^{\lambda_k}) + \sum_{k=n+1}^{\infty} \rho_n^{\lambda_k} \right) |\alpha_k| \|e_k\| \\ &\leq \left(\sum_1^n \lambda_k (1 - \rho_n) + \sum_{k=n+1}^{\infty} \rho_n^{\lambda_k} \right) 2c \cdot 8^M \|F\|_{E[0,1]} \end{aligned} \tag{3.10}$$

This is using (3.7) and the fact that $1 - x^r \leq r(1-x)$ for all $x \in (0,1)$ and $r > 0$.

Now, if $\gamma := \inf_k \frac{\lambda_{k+1}}{\lambda_k}$, choosing ρ_n such that $1 - \frac{1}{\lambda_n} \leq \rho_n \leq 1 - \frac{1}{\lambda_{n+1}}$, gives

(i) $\sum_1^n \lambda_k (1 - \rho_n) \leq \sum_1^n \frac{\lambda_k}{\lambda_n} \leq \sum_1^n \gamma^{k-n} \leq \frac{\gamma}{\gamma - 1}$, and

(ii) $\rho_n^{\lambda_k} \leq \left(1 - \frac{1}{\lambda_{n+1}}\right)^{\lambda_k} = \left[\left(1 - \frac{1}{\lambda_{n+1}}\right)^{\lambda_{n+1}}\right]^{\frac{\lambda_k}{\lambda_{n+1}}} \leq \left(\frac{1}{e}\right)^{\frac{\lambda_k}{\lambda_{n+1}}} \leq \left(\frac{1}{e}\right)^{\gamma^{k-n-1}}$.

Thus, $\sum_{k=n+1}^{\infty} \rho_n^{\lambda_k} \leq \frac{1}{e}$. Therefore, using (3.7) and (3.9), we see that $\|P_n\| \leq C$ for some constant C, for all $n = 1, 2, 3, \cdots$, which is what we wanted.

■

3.3 Lacunary Power Sequences in $C[a,b]$ and $L_p[a,b]$

An application of theorem 3.2.3, gives us that if $\Lambda = \{\lambda_k\}$ is lacunary then $\{t^{\lambda_k}\}$ is basic sequence in $E[0,1]$. The converse is also true.

Theorem 3.3.1 *Let $\{\lambda_k\}_1^\infty$ be a positive increasing sequence, Then, $\{t^{\lambda_k}\}$ is basic sequence in $E[a,b]$ if and only if $\{\lambda_k\}$ is a lacunary sequence.*

PROOF: Without loss of generality, suppose $b = 1$, for otherwise, we can consider the transformation $t \to t/b$.

Suppose $\{\lambda_k\}$ is lacunary. Let β_1, β_2 $(\beta_1 < \beta_2)$ be any two positive integers and let $\{\alpha_k\}_1^\infty$ be a sequence of scalars. Then,

$$\left\| \sum_1^{\beta_1} \alpha_k t^{\lambda_k} \right\|_{C[a,1]} \leq \left\| \sum_1^{\beta_1} \alpha_k t^{\lambda_k} \right\|_{E[0,1]}$$

By theorems 2.1.3, 3.1.2 and 3.1.3, there exist constants c and $M > 0$, such that

$$\left\| \sum_1^{\beta_1} \alpha_k t^{\lambda_k} \right\|_{E[0,1]} \leq M \left\| \sum_1^{\beta_2} \alpha_k t^{\lambda_k} \right\|_{E[0,1]} \leq Mc \left\| \sum_1^{\beta_2} \alpha_k t^{\lambda_k} \right\|_{C[a,1]}$$

Therefore, $\{t^{\lambda_k}\}$is a basic sequence.

Conversely, assume $\{\lambda_k\}$ is a positive increasing sequence that is not lacunary. Let $\varepsilon > 0$. Consider a decreasing sequence of scalars $\{\varepsilon_k\}$ such that $\varepsilon \to 0$ and $\prod_1^\infty \varepsilon_k \leq 1 + \varepsilon$. Since $\{\lambda_k\}$ is not lacunary, there exists a subsequence $\{\beta_k\}$ such that $\frac{\beta_{k+1}}{\beta_k} \le 1 + \varepsilon_k$. We claim that

$$\|t^{\beta_k} - t^{\beta_{k+1}}\|_{C[a,1]} \longrightarrow 0, \quad as\ k \to \infty,$$

which implies that $\{t^{\beta_k}\}$ is not a basic sequence.

To prove this claim, observe that $f(t) := t^{\beta_k} - t^{\beta_k}$. Then, using calculus,

$$\|f\|_{C[0,1]} = \left(\frac{\beta_k}{\beta_{k+1}}\right)^{\frac{\beta_k}{\beta_{k+1}-\beta_k}} \left(1 - \frac{\beta_k}{\beta_{k+1}}\right),$$

attained at $t_0 = \left(\frac{\beta_k}{\beta_{k+1}}\right)^{\frac{1}{\beta_{k+1}-\beta_k}}$.

Using the conditions on $\{\beta_k\}$, we see that

$$t_0 > \left(\frac{1}{1+\varepsilon_k}\right)^{\frac{1}{\varepsilon_k \beta_k}} \to 1 \text{ and } \|f\|_{[0,1]} \le 1 - \frac{1}{1+\varepsilon_k} \to 0.$$

Thus, for k sufficiently large, we have $t_0 > a$ and hence, $\|f\|_{[a,1]} = |f(t_0)| \le \|f\|_{[0,1]} \to 0$. This means that, $\|f\|_{C[a,1]} = \|t^{\beta_k} - t^{\beta_{k+1}}\|_{C[a,1]} \to 0$.

■

Theorem 3.3.2 *Suppose $h \in C[a,b]$ is nonnegative and not constant on any subinterval of $[a,b]$, where $0 \le a < b$. Then $\{h^{\lambda_k}\}_1^\infty$ is a basic sequence in $C[a,b]$ if and only if $\{\lambda_k\}$ is lacunary.*

PROOF: Let E represent one of $C[0,1]$ or $L_p[0,1]$. If $\{\lambda_k\}$ is lacunary, then by theorem 3.2.4, $\{h^{\lambda_k}\}$ is a basic sequence. So, $\{h^{\lambda_k}\}_1^\infty$ is a basic sequence in $C[a,b]$. Without loss of generality, assume $b = 1$ and $\|h\|_{C(a,1)} = 1$. Suppose $h([a,1]) = [c,d]$ for some $0 \le a < b$ and $0 \le c < d$. Let L be a positive integer and $\alpha_1, \alpha_2, ...\alpha_L$

be scalars. Then, the function $\sum_1^L \alpha_k h^{\lambda_k} \in C[a,b]$ attains its maximum value, $\|\sum_1^L \alpha_k h^{\lambda_k}\|$ at some point, say $t_L \in [a,b]$. Suppose $h(t_L) = u_L$

That is, $\left\|\sum_1^L \alpha_k h^{\lambda_k}\right\|_{C[a,b]} = \left|\sum_1^L \alpha_k u_L^{\lambda_k}\right|$

Since h is onto $[c,d]$, for every $u \in [c,d]$, there exists a $t_u \in [a,b]$ such that $h(t_u) = u$. This means that for all $u \in [c,d]$,

$$\left|\sum_1^L \alpha_k u^{\lambda_k}\right| = \left|\sum_1^L \alpha_k h^{\lambda_k}(t_u)\right| \leq \left|\sum_1^L \alpha_k h^{\lambda_k}(t_L)\right| = \left|\sum_1^L \alpha_k u_L^{\lambda_k}\right|$$

That is,

$$\left\|\sum_1^L \alpha_k h^{\lambda_k}\right\|_{C[a,b]} = \left\|\sum_1^L \alpha_k u^{\lambda_k}\right\|_{C[c,d]}$$

Therefore, by theorem 3.3.1 we conclude that $\{h^{\lambda_k}\}_1^\infty$ is a basic sequence in $C[a,b]$

■

CHAPTER 4

LACUNARY BLOCK SEQUENCES

4.1 Introduction

I. Singer, 1970 [20] and J. Lindenstrauss and L. Tzafriri, 1977 [17] used of block sequences to obtain new basic sequences from given basic sequences. Every block sequence $\{y_k\}_1^\infty$ in a Banach space X defined as follows from a basic sequence $\{x_n\}_1^\infty$ is itself a basic sequence whose basic constant is not larger than that of $\{x_n\}_1^\infty$.

Definition 4.1.1 *Let $\{x_n\}_1^\infty$ be a basic sequence in a Banach space X. A sequences of non-zero vectors $\{y_k\}_1^\infty$ in X of the form*

$$y_k = \sum_{j=m_k}^{m_{k+1-1}} a_j x_j$$

with $\{a_j\}_1^\infty$ scalars and $m_1 < m_2 < \ldots$an increasing sequence of integers is called a block basic sequence with respect to $\{x_n\}_1^\infty$

In this chapter, we define a special class of block sequences of Müntz polynomials $\{p_k\}$ in $C[0,1]$ with respect to a sequence of scalars Λ and show that $\{p_k\}$ is a basic sequence sequences if and only if Λ is lacunary.

4.2 Main Result

Definition 4.2.1 *Let* $0 < \lambda_1 < \lambda_2 < \cdots$. *We say that* $\{p_j\}$ *is a positive block sequence of powers of* t *of logarithmic length* $\leq \varepsilon$ *if*

$$p_k = \sum_{j=m_k}^{m_{k+1}-1} a_j t^{\lambda_j} \tag{4.1}$$

where $a_j \geq 0$ *and* $\dfrac{m_{k+1}-1}{m_k} < 1+\varepsilon$.

We say that the sequence is lacunary with lacunarity index r if $\frac{\lambda_{m_k}}{\lambda_{m_k-1}} > r$ for all k.

Theorem 4.2.2 *If for every* $r > 1$ *there is an* $\varepsilon > 0$ *such that if* $\{p_j\}$ *is a lacunary positive block sequence of powers of* t *in* $[0,1]$ *of lacunarity index* r *and logarithmic block length* $< \varepsilon$, *then* $\{p_j\}$ *is uniformly minimal.*

PROOF: Without loss of generality, we assume that $\{p_j\}$ is normalized. Consider a sum $\sum_{j=1}^{N} b_j p_j$. We assume that $\max_{1\leq j\leq N} |b_j| = b_\ell = 1$. We will show that, given lacunarity index r, there is an $\varepsilon > 0$ and a C such that

$$\left\| \sum_{j=1}^{N} b_j p_j \right\| \geq C$$

By making the transformation $t \longmapsto t^\alpha$ for suitable α, we can, without loss of generality, assume that

$$p_l = \sum_{K_{l-1}+1}^{K_l} a_j t^{\lambda_j}$$

where $\lambda_{K_{l-1}+1} = n$ and $\lambda_{K_l} < n(1+\varepsilon)$ and $\lambda_1 > \sqrt{n}$ and we can have n arbitrarily large.

We will work with the m^{th} derivative $(\sum b_j p_j)^{(m)}$ of $\sum b_j p_j$. Let C_1, C_2 be positive numbers and $m, n \in \mathbb{N}$.

Lemma 4.2.3 *Assume that $f'(t) > C_1 \cdot n$ on an interval I of length $C_2 \cdot \frac{1}{n}$. Then, $|f(t)| > \dfrac{C_1 C_2}{4}$ on an interval of length $\dfrac{C_2}{4} \cdot \dfrac{1}{n}$*

Corollary 4.2.4 *Assume that $f^{(m)}(t) > C_1 \cdot n^m$ on an interval I of length $C_2 \cdot \frac{1}{n}$. Then*

$$|f(t)| > \frac{C_1 \cdot C_2^m}{4^{\frac{m(m+1)}{2}}}$$

on an interval of length $\dfrac{C_2}{4^m} \cdot \frac{1}{n}$.

We will show that, given $r > 0$, there is an $\varepsilon = \varepsilon(r)$, an $m = m(r)$, a $C_1 = C_1(r)$ and a $C_2 = C_2(r)$, all depending only on r such that

$$\left| \left(\sum b_j p_j \right)^{(m)} \right| > C_1 \cdot n^m$$

on an interval of length $> C_2 \cdot \frac{1}{n}$. By corollary 4.2.4, this will give the proof of the theorem.

We now have

Lemma 4.2.5 *If $p_l^{(m)}(t) \geq A \sum_{j>l} |b_j| p_j^{(m)}$ for $t = t_1$, then $p_l^{(m)}(t) \geq A \sum_{j>l} |b_j| p_j^{(m)}$ for $t < t_0$*

and

Lemma 4.2.6 *If $p_l^{(m)}(t) \geq A \sum_{j<l} |b_j| p_j^{(m)}$ for $t = t_0$, then $p_l^{(m)} \geq A \sum_{j<l} |b_j| p_j^{(m)}$ for $t > t_0$.*

PROOF: Both these lemmas are immediate consequences of the fact that

$$\text{if } t_1 > t_2 > 0 \text{ and } n_1 > n_2, \text{ then } \frac{t_1^{n_1}}{t_1^{n_2}} > \frac{t_2^{n_1}}{t_2^{n_2}}$$

and equivalently,

$$\text{if } t_1 < t_2 \text{ and } n_1 > n_2, \text{ then } \frac{t_1^{n_1}}{t_1^{n_2}} < \frac{t_2^{n_1}}{t_2^{n_2}}$$

■

We choose m such that $r^m > 2 \cdot 10^4$. We will now consider the point $t_0 = 1 - \frac{k}{n}$ where

$$k = \frac{m \ln r - \ln 1000}{1 + \varepsilon - \frac{1}{r(1+\varepsilon)}}$$

which gives $k(1+\varepsilon) = m \ln r + \frac{k}{r(1+\varepsilon)} - \ln 1000$. At this point,

$$\begin{aligned} f_l^{(m)}(t_0) &\geq n(n-1) \cdot \ldots \cdot (n-m+1) \cdot \left(1 - \frac{k}{n}\right)^{n(1+\varepsilon)-m} \\ &> \frac{1}{2} n^m \cdot e^{-k(1+\varepsilon)}, \text{ if } n \text{ is large enough} \end{aligned}$$

Moreover, for $j \geq 1$, we have

$$\begin{aligned} p_{l-j}^{(m)}(t_0) &\leq \frac{n}{r^j} \left(\frac{n}{r^{jm}} - 1\right) \cdot \ \ldots \ \cdot \left(1 - \frac{k}{n}\right)^{nr^{-j}(1+\varepsilon)^{-j}-m} \\ &< 2\frac{n^m}{r^{jm}} \cdot e^{-kr^{-j}(1+\varepsilon)^{-j}} \text{ if } n \text{ is large enough} \end{aligned}$$

This gives

$$\ln \left(\frac{p_{l-j}^{(m)}(t_0)}{p_l^{(m)}(t_0)} \right) < \ln 4 - jm \ln r - \frac{k}{r^j(1+\varepsilon)^j} + m \ln r + \frac{k}{r(1+\varepsilon)} - \ln 1000$$

For $j = 1$, this is $-\ln 250$. The term $jm \ln r$ decreases by $m \ln r > 10$ every time j increases by 1. The term $-\dfrac{k}{r^j(1+\varepsilon)^j}$ is increasing with j and has its largest increase when j goes from 1 to 2. Putting in the value of k, the increase is then $\dfrac{m \ln r - \ln 100}{r(1+\varepsilon)} < m \ln r - \ln 100$. This gives that at t_0 we have $p_l^{(m)}(t_0) > 10 \cdot \sum_{j=1}^{l-1} |b_j| p_j^{(m)}(t_0)$ and so by lemma **??** this holds in the interval $[t_0, 1]$.

Now, consider $k = \dfrac{m \ln r + \ln 1000}{r - 1 - \varepsilon}$ and $t_1 = 1 - \frac{k}{n}$. If ε is small enough, we obviously have $t_0 < t_1$ and the length of the interval $[t_0, t_1]$ is $C_2 \cdot \frac{1}{n}$ where C_2 and ε just depends on r.

For $p_{l+i}^{(m)}, i \geq 1$, we have

$$p_{l+i}^{(m)}(t) < nr^i(1+\varepsilon)^i \left(nr^i(1+\varepsilon)^i - 1\right) \cdot \ \ldots \ \cdot \left(nr^i(1+\varepsilon)^i - m + 1\right) t^{nr^i - m}$$

and so at $t = 1 - \frac{k}{n}$ we have $p_{l+i}^{(m)}(t) < 2n^m r^{im}(1+\varepsilon)^{im} \cdot e^{-kr^i}$.

This gives

$$\ln \left(\frac{p_{l+i}^{(m)}(t_1)}{p_l^{(m)}(t_1)} \right) < \ln 2 + mi \ln(r(1+\varepsilon)) - kr^i + \ln 2 + k(1+\varepsilon)$$

The argument is the same as for $\sum_{j<l} |b_j| p_j^{(m)}$ and so the theorem is proved.

■

By the techniques we used in the proof of theorem 3.3.2, the above theorem implies that if $\{\lambda_k\}_1^\infty$ is lacunary, then $\{p_k\}_1^\infty$ is basic sequence in $C[0,1]$

CHAPTER 5

RECTIFIABILITY AND EXTREMAL VECTORS

5.1 Introduction

Let H be a Hilbert spaces over $\mathbb{C}$, and let $T : H \to H$ be a bounded linear operator with dense range $\mathcal{R}(T)$, but not onto.

Definition 5.1.1 *(Ansari and Enflo, 1998 [1]) Let $x_0 \in H,\ x_0 \neq 0,\ 0 < \varepsilon < \|x_0\|$ and $n \in \mathbb{N}$. There exists a unique vector $y^{\varepsilon}_{n,x_0}\ \epsilon H$ such that*

$$\|T^n y^{\varepsilon}_{n,x_0} - x_0\| \ \leq \ \varepsilon$$

and

$$\|y^{\varepsilon}_{n,x_0}\| \ = \ \inf\{\|y\| : \|T^n y - x_0\| \ \leq \ \varepsilon\}.$$

Such vectors y^{ε}_{n,x_0} are called backward minimal vectors of T^n. In case of no ambiguity, we simple write y_ε or y_n.

Clearly,

$$\|Ty^{\varepsilon}_{n,x_0} \ - \ x_0\| \ = \ \varepsilon.$$

Ansari and Enflo [1] proved the following orthogonality equations.

Theorem 5.1.2 *There exists a constant $\delta_\varepsilon < 0$ such that*

$$y_\varepsilon = \delta_\varepsilon T^*(Ty_\varepsilon - x_0) \tag{5.1}$$

Applying T on both sides of (5.1),

$$Ty_\varepsilon = \delta_\varepsilon TT^*TY_\varepsilon - \delta_\varepsilon TT^*x_0.$$

Thus, isolating Ty_ε and adding $x_0 - x_0$,

$$\begin{aligned} Ty_\varepsilon &= -\delta_\varepsilon(I-\delta_\varepsilon TT^*)^{-1}TT^*x_0 \\ &= x_0 - \left[(I-\delta_\varepsilon TT^*)^{-1}(I-\delta_\varepsilon TT^*)x_0 + \delta_\varepsilon(I-\delta_\varepsilon TT^*)^{-1}TT^*x_0\right] \\ &= x_0 - (I-\delta_\varepsilon TT^*)^{-1}x_0 \end{aligned}$$

Replacing δ_ε by $-\delta_\varepsilon$, (note: $-\delta_\varepsilon > 0$) it follows that

(5.2) $$Ty_\varepsilon = x_0 - (I+\delta_\varepsilon TT^*)^{-1}x_0$$

In the same paper, S. Ansari and P. Enflo showed that for each $x_0 \in H \setminus \{0\}$, the functions $\varepsilon \longrightarrow y_\varepsilon$ and $\varepsilon \longrightarrow \delta_\varepsilon$ are analytic on $(0, \|x_0\|)$. P. Enflo and T. Hõim [10] proved that the function $f(\varepsilon) := \|y_\varepsilon\|$ is convex and has derivative

$$\frac{d}{d\varepsilon}f(\varepsilon) = -\frac{\|y_\varepsilon\|}{\|Ty_\varepsilon\|cos\theta_\varepsilon}$$

where θ_ε is the angle between $Ty_\varepsilon - x_0$ and Ty_ε.

Observe that if $x_0 \in H,\ x_0 \notin \mathcal{R}(T)$, then $\|y_\varepsilon\| \to \infty$, and $\delta_\varepsilon \to \infty$ as $\varepsilon \to 0$ and $\|y_\varepsilon\| \to 0,\ \delta_\varepsilon \to 0$ as $\varepsilon \to \|x_0\|$

PROBLEM: Is the curve $\varepsilon \to Ty_\varepsilon$ rectifiable on $(0, \|x_0\|)$ for every Hilbert space H and every non-invertible linear map T on H with dense range?

5.2 Rectifiability in $\mathbf{L}_2[0,1]$

Let $a \in L_2[0,1]$ be nonzero function. Suppose T is the multiplication operator on $L_2[0,1]$ given by $Tf(t) := a(t)f(t)$. Then, $T^* = T$ and hence, $(I+\delta TT^*)^{-1}(f)(t) = \frac{1}{1+\delta a^2(t)} f(t)$. For each $x_0 \notin \mathcal{R}(T)$ and each $\varepsilon \in (0, \|x_0\|)$, there exists a unique minimal vector y_ε. If $x_0 \equiv 1$, equation (5.2) gives us

$$Ty_\varepsilon(t) = \left(1 - \frac{1}{1+\delta_\varepsilon a^2(t)}\right) x_0(t) \tag{5.3}$$

Thus, the curve $\varepsilon \to Ty_\varepsilon$ is rectifiable for $\varepsilon \in (0, \|x_0\|)$ if and only if the map $\delta \to g_\delta(t) := \frac{1}{1+\delta a^2(t)}$ is rectifiable on $\delta \in (0,\infty)$, where $t \in [0,1]$. We have the following theorem.

Theorem 5.2.1 *If $a(t) = t^\alpha$ for $\alpha > 0$, then the map $\delta \to g_\delta$ is rectifiable for $\delta \in (0,\infty)$.*

PROOF: Consider a partition $\mathcal{P} = \{\delta_k : 0 = \delta_1 < \delta_2 < \cdots\}$ of $(0,\infty)$. Let $\Delta\delta_k := \delta_{k+1} - \delta_k$. Then,

$$\|g_{\delta_k} - g_{\delta_{k+1}}\|^2 = \Delta\delta_k^2 \int_0^1 \frac{t^{4\alpha}}{(1+\delta_k t^{2\alpha})^2(1+\delta_{k+1}t^{2\alpha})^2} dt \tag{5.4}$$

If $\delta_k \in [0,1]$ we get $(1+\delta_k t^{2\alpha})^2 \geq 1,$ for all k and so

$$\|g_{\delta_k} - g_{\delta_{k+1}}\| \leq \left(\Delta\delta_k^2 \int_0^1 t^{4\alpha} dt\right)^{1/2} = \frac{\Delta\delta_k}{\sqrt{4\alpha+1}} \leq \Delta\delta_k$$

Therefore, for any partition $\mathcal{P} = \{\delta_k : 0 = \delta_1 < \delta_2 < \cdots < \delta_n = 1\}$ of $(0,1)$,

$$\sup_{\mathcal{P}} \sum_1^\infty \|g_{\delta_k} - g_{\delta_{k+1}}\| \leq \sup_{\mathcal{P}} \sum_1^n \Delta\delta_k = \sup_{\mathcal{P}} \sum_1^n (\delta_{k+1} - \delta_k) = 1$$

Thus, it suffices to show rectifiability for $\delta_k \in [1,\infty)$. To this end, putting $u =$

$1 + \delta_k t^{2\alpha}$ in (5.4), we get

$$
\begin{aligned}
\|g_{\delta_k} - g_{\delta_{k+1}}\|^2 &\leq \frac{\Delta\delta_k^2}{2\alpha\ \delta_k^{2+1/2\alpha}} \int_1^{1+\delta_k} \frac{(u-1)^{1+1/2\alpha}}{u^4} du \\
&< \frac{\Delta\delta_k^2}{2\alpha\ \delta_k^{2+1/2\alpha}} \int_1^{1+\delta_k} u^{-3+1/2\alpha} du \\
&= \begin{cases} \frac{2\Delta\delta_k^2}{\delta_k^4}\ ln(1+\delta_k), & \text{if } \alpha = \frac{1}{4} \\ \frac{\Delta\delta_k^2}{2\alpha}\ \frac{\delta^{2+1/2\alpha}}{-2+1/2\alpha} \left[(1+\delta_k)^{-2+1/2\alpha} - 1\right], & \text{if } \alpha \neq \frac{1}{4} \end{cases} \\
&\leq \begin{cases} \frac{2\Delta\delta_k^2}{\delta_k^4}\ ln(1+\delta_k), & \text{if } \alpha = \frac{1}{4} \\ \frac{1}{4\alpha-1}\ \frac{\Delta\delta_k^2}{\delta_k^{2+1/2\alpha}} \left(1 - \frac{1}{(1+\delta_k)^{2-1/2\alpha}}\right), & \text{if } \alpha > \frac{1}{4} \\ \frac{2^{-2+1/2\alpha}}{1-4\alpha}\ \frac{\Delta\delta_k^2}{\delta_k^4} \left(1 - \frac{1}{\delta_k^{-2+1/2\alpha}}\right), & \text{if } \alpha < \frac{1}{4} \end{cases} \\
&< \begin{cases} \frac{2\Delta\delta_k^2}{\delta_k^{3+1/2}}, & \text{if } \alpha = \frac{1}{4} \\ \frac{\Delta\delta_k^2}{4\alpha-1}\ \frac{1}{\delta_k^{2+1/2\alpha}}, & \text{if } \alpha > \frac{1}{4} \\ \frac{\Delta\delta_k^2}{1-4\alpha}\ \frac{1}{\delta_k^4}, & \text{if } \alpha < \frac{1}{4} \end{cases}
\end{aligned}
$$

In any case,

$$
\|g_{\delta_k} - g_{\delta_{k+1}}\| < M \frac{\Delta\delta_k^2}{\delta_k^{1+p}}
$$

for some constants $M, p > 0$.

Now, for any partition $\mathcal{P} = \{\delta_k\}$ of $[0, \infty)$, taking $\mathcal{P}' = \mathcal{P} \cup \mathbb{N}$, if necessary, we get

$$
\begin{aligned}
\sum_k^\infty \|g_{\delta k} - g_{\delta_{k+1}}\| &< M \sum_1^\infty \frac{\Delta\delta_k}{\delta_k^{1+p}} < M \sum_{n=1}^\infty \sum_{\delta_k,\ \delta_{k+1} \in [n,n+1]} \frac{\Delta\delta_k}{\delta_k^{1+p}} \\
&\leq M \sum_{n=1}^\infty \frac{1}{n^{1+p}} \sum_{\delta_k,\ \delta_{k+1} \in [n,n+1]} \Delta\delta_k \leq M \sum_{n=1} \frac{1}{n^{1+p}} = S < \infty
\end{aligned}
$$

Therefore,

$$
\sup_{\mathcal{P}} \sum_1^\infty \|g_{\delta_k} - g_{\delta_{k+1}}\| \leq S < \infty
$$

Using a similar argument, it can be shown that the curve is rectifiable for every polynomial $a \in L_2[0, 1]$.

■

Theorem 5.2.2 *Let $I_n := (\frac{1}{n+1}, \frac{1}{n}]$, where $n \in \mathbb{N}$, $h \in L_\infty[0,1]$ and let S_n be the essential supremum of h on I_n. Suppose*

(i) $M_n := \frac{S_n}{S_{n+1}} \to \infty$ *as* $n \to \infty$

(ii) $\frac{|h(t)|}{S_n} \to 1$ *as* $n \to \infty$ $a.e.$ *on* I_n

Then, the curve $\delta \to g_\delta = \frac{1}{1+\delta h^2}$ is not rectifiable on $[0,\infty)$

Such a function h exists in $L_\infty[0,1]$. For example, $h(t) := \exp(-1/\exp(1/t^2))$ is one such function.

PROOF: Consider the partition δ_k of $[S_1^{-2},\infty)$ with $\delta_k = S_k^{-2}$. Then, there exists some K_0 such that

$$\begin{aligned} \|g_{\delta_k} - g_{\delta_{k+1}}\| &= \left(\int_0^1 \left(\frac{1}{1+S_k^{-2}h^2(t)} - \frac{1}{1+M_k^2 S_k^{-2}h^2(t)}\right)^2 dt\right)^{1/2} \\ &\geq \left(\int_{\frac{1}{k+1}}^{\frac{1}{k}} \left(\frac{1}{2}-\frac{1}{4}\right)^2 dt\right)^{1/2} = \frac{1}{4}\frac{1}{\sqrt{k(k+1)}}, \quad \text{for all } k \geq K_0 \end{aligned}$$

Therefore,

$$\sum_1^\infty \|g_{\delta_k} - g_{\delta_{k+1}}\| \geq \frac{1}{4}\sum_{k\geq K_0} \frac{1}{\sqrt{k(k+1)}} = \infty$$

■

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