VERIFYING HUPPERT'S CONJECTURE FOR THE SIMPLE GROUPS OF LIE TYPE OF RANK TWO

A dissertation submitted to
Kent State University in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy

by

Thomas Philip Wakefield

August 2008

Dissertation written by

Thomas Philip Wakefield

A.B., Youngstown State University, 2002

B.S., Youngstown State University, 2002

M.A., Kent State University, 2004

Ph.D., Kent State University, 2008

Approved by

Donald L. White,	Chair, Doctoral Dissertation Committee
Stephen M. Gagola, Jr.,	Member, Doctoral Dissertation Committee
Mark L. Lewis,	Member, Doctoral Dissertation Committee
Michael Mikusa,	Member, Doctoral Dissertation Committee
Arden Ruttan,	Member, Doctoral Dissertation Committee

Accepted by

Andrew Tonge,	Chair, Department of Mathematical Sciences
Timothy S. Moerland,	Dean, College of Arts and Sciences

TABLE OF CONTENTS

LIST OF TABLES vii

ACKNOWLEDGEMENTS viii

INTRODUCTION 1

1 BACKGROUND RESULTS 10

2 VERIFYING HUPPERT'S CONJECTURE FOR THE TWISTED REE GROUPS OF RANK TWO 13

2.1 Results Concerning the Character Degrees of ${}^2G_2(q^2)$ 13

2.2 Establishing $G' = G''$ when $H \cong {}^2G_2(q^2)$ 14

2.3 Establishing $G'/M \cong H$ when $H \cong {}^2G_2(q^2)$ 16

2.3.1 Key Lemma 17

2.3.2 Eliminating the Alternating Groups for all k 17

2.3.3 Eliminating Sporadic Simple Groups and the Tits Group for all k . . 19

2.3.4 Eliminating the Groups of Lie Type when $k > 1$ 20

2.3.5 Eliminating Simple Groups of Exceptional Lie Type when $k = 1$ . . 22

2.3.6 Eliminating the Groups of Classical Lie Type when $k = 1$ 23

2.4 Establishing $I_{G'}(\theta) = G'$ when $H \cong {}^2G_2(q^2)$ 37

2.5 Establishing $M = \langle 1 \rangle$ when $H \cong {}^2G_2(q^2)$ 38

2.6 Establishing $G = G' \times C_G(G')$ when $H \cong {}^2G_2(q^2)$ 39

3 HUPPERT'S CONJECTURE AND THE FAMILY OF SIMPLE GROUPS $G_2(q)$, $q > 4$ 41

3.1 Results Concerning the Character Degrees of $G_2(q)$ 41

3.2 Establishing $G' = G''$ when $H \cong G_2(q)$ 43

3.3 Establishing $G'/M \cong H$ when $H \cong G_2(q)$ 46

3.3.1 Eliminating the Tits, Sporadic, and Alternating Groups when $k > 1$ 46

3.3.2 Eliminating Sporadic Simple Groups and the Tits Group when $k = 1$ 47

3.3.3 Eliminating A_n when $k = 1$ 49

3.3.4 Eliminating the Groups of Lie Type when $k > 1$ 50

3.3.5 Eliminating Simple Groups of Exceptional Lie Type when $k = 1$ . . 51

3.3.6 Eliminating the Groups of Classical Lie Type when $k = 1$ 53

3.4 Progress Toward Proving $I_{G'}(\theta) = G'$ when $H \cong G_2(q)$ 62

3.4.1 Maximal Subgroups with Structure $\mathrm{SL}_3(q) : 2$ or $\mathrm{SU}_3(\mathrm{q}) : 2$ 66

3.4.2 Maximal Parabolic Subgroups with Structure $[q^5] : \mathrm{GL}_2(q)$ 68

3.5 Establishing $M = \langle 1 \rangle$ when $H \cong G_2(q)$ 73

3.6 Establishing $G = G' \times C_G(G')$ when $H \cong G_2(q)$ 75

4 VERIFYING HUPPERT'S CONJECTURE FOR THE SIMPLE LINEAR GROUPS OF RANK TWO 76

4.1 Results Concerning the Character Degrees of $\mathrm{PSL}_3(q)$ 76

4.2 Establishing $G'/M \cong H$ when $H \cong \mathrm{PSL}_3(q)$ 77

4.2.1 Eliminating the Tits, Sporadic, and Alternating Groups 77

4.2.2 Eliminating the Groups of Lie Type when $k > 1$ 79

4.2.3 Eliminating Groups of Exceptional Lie Type when $k = 1$ 79

4.2.4 Eliminating the Groups of Classical Lie Type when $k = 1$ 80

5 VERIFYING HUPPERT'S CONJECTURE FOR THE UNITARY SIMPLE GROUPS OF RANK TWO 82

5.1 Background Results on the Character Degrees of $\mathrm{PSU}_3(q^2)$ 82

5.2 Establishing $G'/M \cong H$ when $H \cong \mathrm{PSU}_3(q^2)$ 83

5.2.1 Eliminating the Alternating, Sporadic, and Tits Groups for all k . . 83

5.2.2 Eliminating Groups of Lie Type 84

5.2.3 The Case of $\mathrm{PSU}_3(19^2)$ 85

6 VERIFYING HUPPERT'S CONJECTURE FOR THE SIMPLE SYMPLECTIC GROUPS OF RANK TWO IN EVEN CHARACTERISTIC 87

6.1 Results Concerning the Character Degrees of $\mathrm{PSp}_4(q)$, q Even 87

6.2 Establishing $G'/M \cong H$ when $H \cong \mathrm{PSp}_4(q)$, q Even 89

6.2.1 Eliminating the Tits, Sporadic, and Alternating Groups for all k . . 89

6.2.2 Eliminating the Groups of Lie Type when $k > 1$ 90

6.2.3 Eliminating Simple Groups of Exceptional Lie Type when $k = 1$ . . 90

6.2.4 Eliminating the Groups of Classical Lie Type when $k = 1$ 91

6.3 Proving Step 5 for $\mathrm{PSp}_4(q)$, q Even 92

7 VERIFYING HUPPERT'S CONJECTURE FOR THE SIMPLE SYMPLECTIC GROUPS OF RANK TWO IN ODD CHARACTERISTIC 93

7.1 Results Concerning the Character Degrees of $\mathrm{PSp}_4(q)$, q Odd 93

7.2 Establishing $G'/M \cong H$ when $H \cong \mathrm{PSp}_4(q)$, q Odd 94

7.2.1 Eliminating the Tits, Sporadic, and Alternating Groups for all k . . 94

7.2.2 Special Cases in the Elimination of the Groups of Lie Type 95

7.2.3 Case 1 96

7.2.4 Case 2 97

7.2.5 Case 3 98

7.2.6 Case 3: Eliminating the Groups of Lie Type when $k > 1$ 98

7.2.7 Case 3: Eliminating Simple Groups of Exceptional Lie Type when $k = 1$ 99

7.2.8 Case 3: Eliminating the Groups of Classical Lie Type when $k = 1$. 99
7.3 Proving Step 5 for $\mathrm{PSp}_4(q)$, q Odd 100
BIBLIOGRAPHY 103

LIST OF TABLES

1 Simple Groups of Classical Lie Type 2

2 Simple Groups of Exceptional Lie Type 2

3 Huppert's Work 4

2.1 Eliminating Simple Groups of Exceptional Lie Type 23

2.2 Simple Groups of Classical Lie Type 24

2.3 Eliminating $\mathrm{P}\Omega_{2\ell}{}^{+}(q_1)$, $5 \leq \ell \leq 7$ 34

2.4 Maximal Subgroups of ${}^2G_2(q^2)$ 37

3.1 Maximal Subgroups of $G_2(q)$, q Odd 63

3.2 Maximal Subgroups of $G_2(q)$, q Even 65

3.3 Particular Degrees of Maximal Parabolic Subgroups of $G_2(q)$ 72

ACKNOWLEDGEMENTS

This dissertation is due to many whom I owe a huge debt of gratitude. I would especially like to thank the following individuals for their support, encouragement, and inspiration along this long and often difficult journey.

First and foremost, I offer thanks to my advisor Dr. Donald White. This dissertation would not have been possible without your countless hours of advice and support. Thank you for modeling the actions and behaviors of an accomplished mathematician, excellent teacher, and remarkable man. I have learned so much from you during my career at Kent State and aspire to be such a successful mathematician and person. Thank you for your continued support.

The support of the "algebra group" at Kent State has been amazing. A special thank you to Professors Mark Lewis and Steven Gagola for your advice and encouragement. I offer my thanks to the Department of Mathematical Sciences at Kent State University. The supportive environment that it offers its graduate students is remarkable. I would especially like to thank Misty Sommers-Tackett and Virginia Wright for always being available to answer questions and help in any way. I also extend my thanks to Dr. Andrew Tonge, Dr. Joe Diestel, and Dr. Richard Aron for their support and encouragement. I offer my gratitude to the members of the "reform algebra" group, including Dr. Beverly Reed, Dr. Mary Beth Rollick, Joan Shenk, and Mary Lou Britton. I must also thank Antonia Cardwell, Juan Seoane, Tabrina Smith, and Emily Sprague for seeing me through the early years, for giving me the confidence that I could survive the later years, and for continuing to be such wonderful friends.

I have been inspired and motivated by many incredible professors during my college career. In particular, I would like to thank Professor Doug Faires, who introduced me to the beauty of mathematics and challenges me to always do my best; Dr. Nate Ritchey, who taught me to believe in my abilities and always offers sage advice; and Professors Rochelle Ruffer, Tom Smotzer, and Angela Spalsbury, who inspired me to pursue graduate studies in mathematics and continue to offer their support. I thank Dr. Frank Sandomierski for encouraging me to continue for a Ph.D in mathematics and Dr. Neil Flowers for his demonstration of the beauty of algebra and willingness to spend hours at a local coffee shop preparing me for success in graduate-level algebra.

I could not have begun this journey without the limitless love and support of my family. Mom and Dad, thank you for providing us such a supportive environment by celebrating our successes and helping us learn from our mistakes. Grandpa and Grandma, thank you for the lessons on how to live and love and for believing in me through it all. Dan and Sarah, thanks for insisting that I have some fun and relax along the way. I could always count on you for a good laugh. Mr. and Mrs. Snyder, your encouragement and interest in my research and progress through the program have meant so much. This dissertation would never have been completed without the support of such an amazing family.

Finally, I thank my beautiful wife Andrea who lived with me through the most difficult parts of this journey. Our path through graduate school has been long, winding, and often separated by great distance, but it is almost complete! Your limitless patience, support, and willingness to sacrifice have made all the difference.

WAKEFIELD, THOMAS PHILIP, Ph.D., August 2008 PURE MATHEMATICS

VERIFYING HUPPERT'S CONJECTURE FOR THE SIMPLE GROUPS OF LIE TYPE OF RANK TWO (105 pp.)

Director of Dissertation: Donald L. White

In the late 1990s, Bertram Huppert posed a conjecture which, if true, would sharpen the connection between the structure of a nonabelian finite simple group H and the set of its character degrees. Specifically, Huppert made the following conjecture.

Huppert's Conjecture. *Let G be a finite group and H a finite nonabelian simple group such that the sets of character degrees of G and H are the same. Then $G \cong H \times A$, where A is an abelian group.*

To lend credibility to his conjecture, Huppert verified it on a case-by-case basis for many nonabelian simple groups, including the Suzuki groups, many of the sporadic simple groups, and a few of the simple groups of Lie type. Except for the Suzuki groups and the family of simple groups $\mathrm{PSL}_2(q)$, for $q \geq 4$ prime or a power of a prime, Huppert proves the conjecture for specific simple groups of Lie type of small, fixed rank. We extend Huppert's results to all the linear, unitary, symplectic, and twisted Ree simple groups of Lie type of rank two.

In this dissertation, we will verify Huppert's Conjecture for ${}^2G_2(q^2)$ for $q^2 = 3^{2m+1}$, $m \geq 1$, and show progress toward the verification of Huppert's Conjecture for the simple group $G_2(q)$ for $q > 4$. We will also outline our extension of Huppert's results for the remaining families of simple groups of Lie type of rank two, namely $\mathrm{PSL}_3(q)$ for $q > 8$, $\mathrm{PSU}_3(q^2)$ for $q > 9$, and $\mathrm{PSp}_4(q)$ for $q > 7$.

INTRODUCTION

Throughout this dissertation, G will denote a finite group. If F is a finite field of characteristic p, where p is a prime, then we will denote the order of F by q, where q is a power of the prime p. A *simple group* H is a group whose only normal subgroups are $\langle 1 \rangle$ and H itself. Because simple groups form the building blocks of more general groups, the search for all simple groups was an important area of research in group theory throughout the twentieth century. In the early 1980s, group theorists completed the classification of the finite simple groups. The only abelian finite simple groups are $\mathbb{Z}_p$, for prime p. The nonabelian finite simple groups can be classified as the six families of simple groups of classical Lie type, the ten families of exceptional Lie type, the alternating groups A_n for $n \geq 5$, the Tits group ${}^2F_4(2)'$, or one of the 26 sporadic simple groups. Table 1 lists the six families of simple groups of classical Lie type. Note that the rank ℓ of the group varies. Table 2 lists the ten families of simple groups of exceptional Lie type. Note that the rank of these groups is fixed.

A *linear representation* of G is a homomorphism T from G into $\mathrm{GL}_n(F)$, the group of $n \times n$ invertible matrices over a field F. The *character associated with* T is a function $\chi : G \to F$ given by $\chi(g) = \mathrm{trace}(T(g))$. The character χ is said to be *irreducible* if χ cannot be written as the sum of two or more characters. We will denote the set of irreducible characters of G by $\mathrm{Irr}(G)$. The *degree* of the character χ, computed as $\chi(1) = n$, is the rank of the matrix in the representation. We will let $\mathrm{cd}(G)$ denote the set of character degrees of G, i.e., $\mathrm{cd}(G) = \{\chi(1) \ : \ \chi \in \mathrm{Irr}(G)\}$.

The set of character degrees of G can be used to gain some information regarding the structure of G. For example, the set of character degrees can be used to determine if a

Type	Group	Remarks
$A_\ell(q_1)$ $\ell \geq 1$	$\mathrm{PSL}_{\ell+1}(q_1)$	if $\ell = 1$, $q_1 \geq 4$
${}^2A_\ell({q_1}^2)$ $\ell \geq 2$	$\mathrm{PSU}_{\ell+1}({q_1}^2)$	if $\ell = 1$, $\mathrm{PSL}_{\ell+1}(q_1) \cong \mathrm{PSU}_{\ell+1}({q_1}^2)$ if $\ell = 2$, $q_1 > 2$
$B_\ell(q_1)$ $\ell \geq 2$	$\Omega_{2\ell+1}(q_1)$	if $\ell = 2$, $B_2(q_1) \cong C_2(q_1)$ for $q_1 > 2$
$C_\ell(q_1)$ $\ell \geq 2$	$\mathrm{PSp}_{2\ell}(q_1)$	if $\ell = 2$, $q_1 > 2$ if q_1 is even, $B_\ell(q_1) \cong C_\ell(q_1)$
$D_\ell(q_1)$ $\ell \geq 4$	$\mathrm{P}\Omega_{2\ell}{}^+(q_1)$	
${}^2D_\ell({q_1}^2)$ $\ell \geq 4$	$\mathrm{P}\Omega_{2\ell}{}^-(q_1)$	

Table 1: Simple Groups of Classical Lie Type

${}^3D_4(q^3)$	$F_4(q)$
$E_6(q)$	$G_2(q)$
${}^2E_6(q^2)$	${}^2B_2(q^2)$, $q^2 = 2^{2m+1}$, $m \geq 1$
$E_7(q)$	${}^2F_4(q^2)$, $q^2 = 2^{2m+1}$, $m \geq 1$
$E_8(q)$	${}^2G_2(q^2)$, $q^2 = 3^{2m+1}$, $m \geq 1$

Table 2: Simple Groups of Exceptional Lie Type

group G is abelian. A group G is abelian if and only if $\mathrm{cd}(G) = \{1\}$. There is growing interest in the information regarding the structure of G which can be determined from the character degree set of G.

Unfortunately, the character degree set of G cannot be used to completely determine the structure of G. It is possible for non-isomorphic groups to have the same set of character degrees. For example, the non-isomorphic groups D_8 and Q_8 not only have the same set of character degrees, but also share the same character table. The character degree set also cannot be used to distinguish between solvable and nilpotent groups. For example, if G is either Q_8 or S_3, then $\mathrm{cd}(G) = \{1, 2\}$. However, Bertram Huppert conjectured in the late 1990s that the nonabelian simple groups are essentially determined by the set of their character degrees. In [13], he posed the following conjecture.

Huppert's Conjecture. *Let G be a finite group and H a finite nonabelian simple group such that the sets of character degrees of G and H are the same. Then $G \cong H \times A$, where A is an abelian group.*

As abelian groups have only the trivial character degree and the character degrees of $H \times A$ are the products of the character degrees of H and those of A, this result is the best possible. The hypothesis that H is a nonabelian simple group is critical. There cannot be a corresponding result for solvable groups. For example, if we consider the solvable group Q_8, then $\mathrm{cd}(Q_8) = \mathrm{cd}(S_3)$ but $Q_8 \ncong S_3 \times A$ for any abelian group A.

Huppert verified the conjecture on a case-by-case basis for many nonabelian simple groups, including the Suzuki groups, many of the sporadic simple groups, and a few of the simple groups of Lie type [13]. Except for the Suzuki groups and the family of simple groups $\mathrm{PSL}_2(q)$, for $q \geq 4$ prime or a power of a prime, Huppert proves the conjecture for specific simple groups of Lie type of small, fixed rank. Table 3 gives the pre-print and the groups for

Pre-Print	Groups
I	${}^2B_2(q^2)$, $q^2 = 2^{2m+1}$, $m \geq 1$
	$\mathrm{PSL}_2(2^f)$, $f \geq 2$
II	$\mathrm{PSL}_2(p^f)$, p odd
III	$\mathrm{PSU}_3(q)$, $q \leq 9$
IV	$\mathrm{PSL}_3(q)$, $q \leq 8$
V	$\mathrm{PSp}_4(q)$, $q = 3, 4, 5, 7$
	$\mathrm{PSp}_6(2)$
VI	M_{11}; M_{12}; M_{22}; M_{23}; M_{24}
VII	J_1; J_2; J_3; J_4
	$\mathrm{PSU}_4(3)$; $\mathrm{PSU}_5(2)$
	$\mathrm{O}^+(8,2)$; $G_2(3)$; $G_2(4)$; ${}^3D_4(2)$; ${}^2F_4(2)'$
VIII	HS; M$^{\mathrm{c}}$L; He; Ru; O$'$N; Suz; Ly; Th; HN
IX	A_n, $7 \leq n \leq 11$

Table 3: Huppert's Work

which Huppert has verified his conjecture. As a corollary to their results concerning non-divisibility among character degrees, Malle and Moretó [24] also verify Huppert's Conjecture for the groups $\mathrm{PSL}_2(2^f)$ for $f > 1$, ${}^2B_2(2^{2f+1})$ for $f \geq 1$, $\mathrm{PSL}_3(4)$, J_1, and A_7. Huppert's method of proof typically requires verifying the following five steps.

1. Show $G' = G''$. Then if G'/M is a chief factor of G, $G'/M \cong S_1 \times S_2 \times \cdots \times S_k$, where $S_i \cong S$, a nonabelian simple group.

2. Show $G'/M \cong H$.

3. Show that if $\theta \in \mathrm{Irr}(M)$ and $\theta(1) = 1$, then θ is stable under G', which implies $[M, G'] = M'$.

4. Show $M = \langle 1 \rangle$.

5. Show $G = G' \times C_G(G')$. As $G/G' \cong C_G(G')$ is abelian and $G' \cong H$, Huppert's Conjecture is verified.

Our goal is to verify Huppert's Conjecture for all simple groups of Lie type of rank two. We begin by considering the linear and unitary groups. In [14], Huppert verifies his conjecture for the unitary groups $\mathrm{PSU}_3(q^2)$ for $q \leq 9$. In [15], Huppert proves that his conjecture holds for the linear groups $\mathrm{PSL}_3(q)$ for $q \leq 8$. We extend Huppert's results to the families of simple groups $\mathrm{PSL}_3(q)$ for $q > 8$ and $\mathrm{PSU}_3(q^2)$ for $q > 9$ in [33].

In constructing his proofs for the linear and unitary groups, Huppert is able to prove four of the five steps for all q. Huppert's proofs ultimately depend upon special properties of the set of character degrees of the simple group H. These properties are not shared by the families of simple groups $\mathrm{PSL}_3(q)$ for $q > 8$ or $\mathrm{PSU}_3(q^2)$ for $q > 9$.

The restrictions that Huppert places on q are required for his proof of Step 2. Suppose G'/M is a chief factor of G. As $G' = G''$,

$$G'/M \cong S_1 \times S_2 \times \cdots \times S_k,$$

where all $S_i \cong S$, a nonabelian simple group. We need to show that $k = 1$ and $S \cong \mathrm{PSL}_3(q)$ in the linear case and $S \cong \mathrm{PSU}_3(q^2)$ in the unitary case. With his restrictions on q, the character degrees of $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q^2)$ are divisible by at most four distinct primes. As Lempken and Huppert characterized all simple groups whose character degrees are divisible by at most four primes [18], he is able to eliminate most of the nonabelian simple groups as candidates for S. With a shorter list of possibilities for S, he is able to show that $k = 1$ and, ultimately, $S \cong \mathrm{PSL}_3(q)$ in the linear case and $S \cong \mathrm{PSU}_3(q^2)$ in the unitary case.

Generally, the character degrees of $\mathrm{PSL}_3(q)$ for $q > 8$ are divisible by many primes. In [33], we extend Huppert's result to the case $q > 8$. Our method uses properties of the character degree set of $\mathrm{PSL}_3(q)$ which hold for all $q > 8$. We first show that $k = 1$ and then prove that $S \cong \mathrm{PSL}_3(q)$. This establishes Step 2 in Huppert's argument and verifies Huppert's Conjecture for the family of simple groups $\mathrm{PSL}_3(q)$, for $q > 2$ prime or a power of a prime.

Because the character degree sets of $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q^2)$ are so closely related, many of our results also hold for the set of character degrees of $\mathrm{PSU}_3(q^2)$ and allow for verification of Huppert's Conjecture for $\mathrm{PSU}_3(q^2)$ as well. In [33], we establish the following results.

Theorem 1. *Suppose $q > 2$ and the sets of character degrees of G and $\mathrm{PSL}_3(q)$ are the same. Then $G \cong \mathrm{PSL}_3(q) \times A$, where A is an abelian group.*

Theorem 2. *Suppose $q > 2$ and the sets of character degrees of G and $\mathrm{PSU}_3(q^2)$ are the same. Then $G \cong \mathrm{PSU}_3(q^2) \times A$, where A is an abelian group.*

Following the verification of Huppert's Conjecture for the linear and unitary groups of rank two, we next consider the family of symplectic groups of rank two. Again, Huppert verified the conjecture for specific symplectic groups of rank two. In [16], Huppert verifies the conjecture for $\mathrm{PSp}_4(q)$ when $q = 3$, 4, 5, or 7. He is able to prove Steps 1, 3, and 4 hold for all q. We extend his result for all $q > 7$ in [34] by verifying Steps 2 and 5. Because the required properties of the character degree sets of $\mathrm{PSp}_4(q)$ differ for odd q and even q, we consider the two cases separately. We establish the following results in [34].

Theorem 3. *Suppose $q > 4$ is even and G is a finite group such that the sets of character degrees of G and $\mathrm{PSp}_4(q)$ are the same. Then $G \cong \mathrm{PSp}_4(q) \times A$, where A is an abelian group.*

Theorem 4. *Suppose $q > 2$ is prime or a power of an odd prime and G is a finite group such that the sets of character degrees of G and $\mathrm{PSp}_4(q)$ are the same. Then $G \cong \mathrm{PSp}_4(q) \times A$, where A is an abelian group.*

The remaining simple groups of Lie type of rank two are $G_2(q)$ for $q > 2$ and ${}^2G_2(q^2)$ for $q^2 = 3^{2m+1}$, $m \geq 1$. In this dissertation, we begin with the verification of Huppert's Conjecture for the family of simple groups ${}^2G_2(q^2)$ for $q^2 = 3^{2m+1}$, $m \geq 1$. We will prove the following result.

Theorem 5. *Suppose $q^2 = 3^{2m+1}$ for $m \geq 1$ and the sets of character degrees of G and ${}^2G_2(q^2)$ are the same. Then $G \cong {}^2G_2(q^2) \times A$, where A is an abelian group.*

We also will exhibit the work that we have completed toward the verification of Huppert's Conjecture for $G_2(q)$. In [17], Huppert verifies his conjecture for the simple groups $G_2(3)$ and $G_2(4)$. We have made progress in the extension of his result for $q > 4$ and we will outline our results. We have the following conjecture.

Conjecture 1. *Suppose $q > 4$ and the sets of character degrees of G and $G_2(q)$ are the same. Then $G \cong G_2(q) \times A$, where A is an abelian group.*

For the families of simple groups ${}^2G_2(q^2)$ and $G_2(q)$, our method of proof will follow the five steps outlined by Huppert in [13] and used by Huppert in his subsequent attempts to verify the conjecture. We devote Chapter 1 to background results necessary to proceed through the five steps of Huppert's argument. We include many basic results from Clifford Theory.

In Chapter 2, we work through the steps in Huppert's argument to verify Huppert's Conjecture for ${}^2G_2(q^2)$. We begin with results particular to the character degree set which prove useful for later arguments. We then prove Step 1. Assume that the character degree sets of G and ${}^2G_2(q^2)$ are the same. Using the results concerning the character degree sets of these groups, it is possible to show $G' = G''$.

With $G' = G''$, we have that if G'/M is a chief factor of G, then $G'/M \cong S_1 \times S_2 \times \cdots \times S_k$, where $S_i \cong S$, a nonabelian simple group. We then use the properties of the character degree sets and the background results of Chapter 1 to show that $k = 1$ and, ultimately, that $S \cong {}^2G_2(q^2)$. This proves Step 2 in Huppert's argument.

For Step 3 in Huppert's argument, we must show that if $\theta \in \mathrm{Irr}(M)$ and $\theta(1) = 1$, then θ is stable under G'. We argue by contradiction. We suppose that θ is not stable under G'. The inertia subgroup I of θ in G' is thus contained in a maximal subgroup U of G'. Passing to the quotient G'/M, we consider $I/M \leq U/M \lneq G'/M$. As $G'/M \cong {}^2G_2(q^2)$, we have knowledge of the maximal subgroups U/M which could possibly contain I/M. Using this information, it is possible to show that I/M is not contained in a maximal subgroup of G'/M. Hence, $I/M = G'/M$ and $I = G'$. It is this step that is incomplete in the verification of Huppert's Conjecture for $G_2(q)$, $q > 4$.

We then proceed to establish that M is trivial. This is proved using Step 3 and some background results presented in Chapter 1. Again, we proceed by contradiction with the assumption that M is not trivial and show that this is not possible. With Step 4 of Huppert's argument proved, we have that $G' \cong {}^2G_2(q^2)$.

Finally, we establish Huppert's Conjecture holds for ${}^2G_2(q^2)$ by verifying Step 5 in Huppert's argument. We show that $G = G' \times C_G(G')$. As $G/G' \cong C_G(G')$ is abelian and G' is isomorphic to the appropriate group, this will prove Theorem 5. Our argument requires considering G as a subgroup of its automorphism group and again proceeds by contradiction. If $G/(G' \times C_G(G'))$ is nontrivial, then G induces on G' some outer automorphism. Using this and knowledge of the outer automorphism group of $G' \cong {}^2G_2(q^2)$, we establish Step 5 of Huppert's argument.

Hence, taken together, the five steps constitute a proof of Huppert's Conjecture when $H \cong {}^2G_2(q^2)$ for $q^2 = 3^{2m+1}$, $m \geq 1$. We devote Chapter 3 to the nearly complete verification of Huppert's Conjecture for $G_2(q)$ when $q > 4$. We can verify all but the third step for this family of simple groups. Chapters 4, 5, 6, and 7 outline the arguments used to verify Huppert's Conjecture in the case of the linear, unitary, and symplectic simple groups of rank two. More details of these arguments can be found in [33] and [34]. This dissertation, when considered with [33] and [34], shows progress toward the verification of Huppert's Conjecture for the simple groups of Lie type of rank two.

CHAPTER 1

BACKGROUND RESULTS

We will require several lemmas to carry out the proof of Huppert's Conjecture. We begin with the following results from Clifford Theory. These are presented as stated in Lemmas 2 and 3 in [13]. Lemma 1.1(b) is often referred to as Gallagher's Theorem.

Lemma 1.1. *Suppose $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$.*

(a) If $\chi_N = \theta_1 + \cdots + \theta_k$ with $\theta_j \in \mathrm{Irr}(N)$, then k divides $|G : N|$. In particular, if $\chi(1)$ is prime to $|G : N|$, then $\chi_N \in \mathrm{Irr}(N)$.

(b) If $\chi_N \in \mathrm{Irr}(N)$, then $\chi\theta \in \mathrm{Irr}(G)$ for every $\theta \in \mathrm{Irr}(G/N)$.

Lemma 1.2. *Suppose $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$. Let $I = I_G(\theta)$ denote the inertia subgroup of θ in G.*

(a) If $\theta^I = \sum_{i=1}^{k} \phi_i$ with $\phi_i \in \mathrm{Irr}(I)$, then ${\phi_i}^G \in \mathrm{Irr}(G)$. In particular, $\phi_i(1)|G : I| \in \mathrm{cd}(G)$.

(b) If θ allows an extension θ_0 to I, then $(\theta_0\tau)^G \in \mathrm{Irr}(G)$ for all $\tau \in \mathrm{Irr}(I/N)$. In particular, $\theta(1)\tau(1)|G : I| \in \mathrm{cd}(G)$.

We will also need the following result, stated as Lemma 4 in [13] and Lemma 12.3 in [19].

Lemma 1.3. *Let G/N be a solvable factor group of G, minimal with respect to being nonabelian. Then two cases can occur.*

(a) G/N is a p-group for some prime p. Hence there exists $\psi \in \mathrm{Irr}(G/N)$ such that $\psi(1) = p^b > 1$. If $\chi \in \mathrm{Irr}(G)$ and $p \nmid \chi(1)$, then $\chi\tau \in \mathrm{Irr}(G)$ for all $\tau \in \mathrm{Irr}(G/N)$.

(b) G/N *is a Frobenius group with an elementary abelian Frobenius kernel* F/N*. Thus* $|G:F| \in \mathrm{cd}(G)$ *while* $|F:N| = p^a$ *for some prime* p *and* F/N *is an irreducible module for the cyclic group* G/F*, hence* a *is the smallest integer such that* $p^a - 1 \equiv 0 \pmod{|G/F|}$*. If* $\psi \in \mathrm{Irr}(F)$*, then either* $|G:F|\psi(1) \in \mathrm{cd}(G)$ *or* $|F:N|$ *divides* $\psi(1)^2$*. In the latter case,* p *divides* $\psi(1)$*. If no proper multiple of* $|G:F|$ *is a character degree of* G*, then* $\chi(1)$ *divides* $|G:F|$ *for all* $\chi \in \mathrm{Irr}(G)$ *such that* $p \nmid \chi(1)$*.*

The following lemma appears as Corollary 11.29 in [19].

Lemma 1.4. *Let* $N \trianglelefteq G$ *and* $\chi \in \mathrm{Irr}(G)$*. Let* $\theta \in \mathrm{Irr}(N)$ *be a constituent of* χ_N*. Then* $\chi(1)/\theta(1)$ *divides* $|G:N|$*. In particular,*

$$\frac{\chi(1)}{\gcd(\chi(1), |G:N|)} \quad \textit{divides} \quad \theta(1).$$

The following lemma will be used in proving Step 3.

Lemma 1.5. *Let* G *be a finite group and* $N \trianglelefteq G$ *with* $\theta \in \mathrm{Irr}(N)$*. If* θ *extends to an irreducible character of* S*, a subgroup of* $I = I_G(\theta)$ *containing* N*, then there is an irreducible constituent* ϕ *of* θ^I *with* $\phi(1) \geq \theta(1)\psi(1)$ *for all* $\psi \in \mathrm{Irr}(S/N)$*.*

Proof. By hypothesis, there exists $\lambda \in \mathrm{Irr}(S)$ such that $\lambda_N = \theta$. By Gallagher's Theorem, we have

$$\theta^S = \sum_{\psi \in \mathrm{Irr}(S/N)} \psi(1)\lambda\psi.$$

Let $\psi' \in \mathrm{Irr}(S/N)$ be of highest degree among the irreducible characters of S/N and let $\phi \in \mathrm{Irr}(I)$ be a constituent of $(\lambda\psi')^I$. Then ϕ is a constituent of $(\theta^S)^I = \theta^I$ and $\lambda\psi'$ is a constituent of ϕ_S, hence

$$\phi(1) \geq \lambda(1)\psi'(1) = \theta(1)\psi'(1) \geq \theta(1)\psi(1)$$

for all $\psi \in \mathrm{Irr}(S/N)$. $\square$

The following lemma will be useful in proving Step 4. It also appears as Lemma 6 in [13].

Lemma 1.6. *Suppose $M \trianglelefteq G' = G''$ and $\lambda^g = \lambda$ for all $g \in G'$ and all $\lambda \in \mathrm{Irr}(M)$ such that $\lambda(1) = 1$. Then $M' = [M, G']$ and $|M : M'|$ divides the order of the Schur multiplier of G'/M.*

The final lemma will be used to verify Step 5. It is stated and proved as Theorem C in [11].

Lemma 1.7. *Let α be an outer automorphism of the finite simple group G. Then there exists a conjugacy class C of G with $C^\alpha \neq C$.*

We now have the lemmas necessary to begin the verification of Huppert's Conjecture for the twisted Ree groups ${}^2G_2(q^2)$, for $q^2 = 3^{2m+1}$, $m \geq 1$.

CHAPTER 2

VERIFYING HUPPERT'S CONJECTURE FOR THE TWISTED REE GROUPS OF RANK TWO

In this chapter, we verify Huppert's Conjecture for the twisted Ree groups ${}^2G_2(q^2)$, for $q^2 = 3^{2m+1}$, $m \geq 1$. We begin with results concerning the character degrees of this simple group and then proceed through the five steps of Huppert's argument.

2.1 Results Concerning the Character Degrees of ${}^2G_2(q^2)$

As listed in [35], the character degrees of ${}^2G_2(q^2)$ for $q^2 = 3^{2m+1}$, $m \geq 1$, are

$$1,\ (q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3}),\ \frac{1}{6}\sqrt{3}q(q-1)(q+1)(q^2+1-q\sqrt{3}),$$

$$\frac{1}{6}\sqrt{3}q(q-1)(q+1)(q^2+1+q\sqrt{3}),\ \frac{1}{3}\sqrt{3}q(q-1)(q+1)(q^2+1),\ (q-1)(q+1)(q^2+1-q\sqrt{3})(q^2+1),$$

$$(q-1)(q+1)(q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3}),\ q^2(q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3}),\ q^6,$$

$$(q^2+1)(q^2+1+q\sqrt{3})(q^2+1-q\sqrt{3}),\ \text{and}\ (q-1)(q+1)(q^2+1+q\sqrt{3})(q^2+1).$$

We first establish some properties of the set of character degrees of ${}^2G_2(q^2)$ that will enable us to prove Theorem 5.

Lemma 2.1.1. *The only character degree of ${}^2G_2(q^2)$ of the form p^b for prime p and $b \geq 1$ is q^6.*

Proof. As shown in [25], the only nontrivial power of a prime among the degrees of ${}^2G_2(q^2)$ is q^6. $\square$

We will also need to know which pairs of character degrees of ${}^2G_2(q^2)$ are consecutive integers. By examining the degrees of ${}^2G_2(q^2)$, it is possible to prove the following lemma.

Lemma 2.1.2. *The only pair of consecutive integers among the character degrees of* ${}^2G_2(q^2)$ *is* q^6 *and* $(q^2+1)(q^2+1+q\sqrt{3})(q^2+1-q\sqrt{3}) = q^6+1$.

Finally, as q^2 is a power of 3, ${}^2G_2(q^2)$ has three odd degrees, namely

$$q^6,\ (q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3}),\ \text{and}\ q^2(q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3}).$$

Also note that of these three odd degrees of ${}^2G_2(q^2)$, one is a power of the prime 3 while $q^2(q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3})$ is a power of 3 multiple of the other odd degree of G.

2.2 Establishing $G' = G''$ when $H \cong {}^2G_2(q^2)$

Suppose that $G' \neq G''$. Then there exists a solvable factor group G/N of G minimal with respect to being nonabelian. By Lemma 1.3, G/N is a p-group or a Frobenius group.

Case 1: G/N is a p-group for some prime p.

Either $p = 3$ or $p \neq 3$.

Subcase 1(a): Suppose $p = 3$.

Now $(q^2+1)(q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3})$ is a character degree of G and

$$3 \nmid (q^2+1)(q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3}).$$

By Lemma 1.1, if $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = (q^2+1)(q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3})$, then $\chi_N \in \mathrm{Irr}(N)$. As G/N is a nonabelian 3-group, it must have a character degree 3^b with $b > 1$. As the character degrees of G/N are character degrees of G and the only nontrivial power of 3 among the degrees of G is q^6, we have that $q^6 \in \mathrm{cd}(G/N)$. Let $\tau \in \mathrm{Irr}(G/N)$ with $\tau(1) = q^6$. Lemma 1.1 implies G has character degree

$$q^6(q^2+1)(q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3}),$$

which is a contradiction.

Subcase 1(b): G/N is a p-group and $p \neq 3$.

If $\chi \in \mathrm{Irr}(G)$, $\chi(1) = q^6$, then we obtain $\chi\sigma \in \mathrm{Irr}(G)$ for every $\sigma \in \mathrm{Irr}(G/N)$. As G/N is nonabelian, there is some $\sigma \in \mathrm{Irr}(G/N)$ with $\sigma(1) > 1$. This again produces a forbidden degree of G.

Case 2: G/N is a Frobenius group with elementary abelian Frobenius kernel F/N, where $|F : N| = p^a$ for some prime p. In addition, $|G : F| \in \mathrm{cd}(G)$ and $|G : F|$ divides $p^a - 1$.

Subcase 2(a): No proper multiple of $|G : F|$ is in $\mathrm{cd}(G)$.

We will repeatedly use the result of Lemma 1.3 which states if $\chi \in \mathrm{Irr}(G)$, $p \nmid \chi(1)$, then $\chi(1) \mid |G : F|$. Suppose that $p \neq 3$. Thus $p \nmid q^6$, and so $q^6 \mid |G : F|$. As $|G : F| \in \mathrm{cd}(G)$, this implies $|G : F| = q^6$. If $p \nmid (q^2 + 1 - q\sqrt{3})(q^2 + 1 + q\sqrt{3})$, then we have that

$$p \nmid q^2(q^2 + 1 - q\sqrt{3})(q^2 + 1 + q\sqrt{3}),$$

and so this degree divides $|G : F| = q^6$, a contradiction. Hence $p \mid (q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3})$. Using the degree

$$\frac{1}{3}\sqrt{3}q(q-1)(q+1)(q^2+1),$$

by the same reasoning we have

$$p \mid (q-1)(q+1)(q^2+1).$$

As $(q^2 + 1 - q\sqrt{3})(q^2 + 1 + q\sqrt{3}) = q^4 - q^2 + 1$ and $\gcd(q^2 - 1, q^4 - q^2 + 1) = 1$, we must have that $p \mid q^2 + 1$. But $(q^2 + 1)^2 = q^4 + 2q^2 + 1$ and

$$(q^2 + 1)^2 - (q^4 - q^2 + 1) = 3q^2$$

and both $q^2 + 1$ and $q^4 - q^2 + 1$ are relatively prime to 3. Hence $\gcd(q^2 + 1, q^4 - q^2 + 1) = 1$. Thus p cannot divide both $q^2 + 1$ and $q^4 - q^2 + 1$. We have established a contradiction, so $p = 3$.

Since $p = 3$,

$$p \nmid (q-1)(q+1)(q^2+1-q\sqrt{3})(q^2+1)$$

and

$$p \nmid (q-1)(q+1)(q^2+1+q\sqrt{3})(q^2+1)$$

so both of these degrees divide $|G:F|$, which is a character degree of G. This is a contradiction.

Subcase 2(b): A proper multiple of $|G:F|$ is a character degree of G.

Examining the character degree set of G, we must have that

$$|G:F| = (q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3}).$$

If $\tau \in \mathrm{Irr}(G)$ with $\tau(1) = (q-1)(q+1)(q^2+1)(q^2+1-q\sqrt{3})$, then

$$\gcd(\tau(1), |G:F|) \ = q^2+1-q\sqrt{3}.$$

By Lemma 1.4, τ_F has an irreducible constituent ψ such that $\psi(1)$ is divisible by

$$\frac{\tau(1)}{q^2+1-q\sqrt{3}} = (q-1)(q+1)(q^2+1).$$

By Lemma 1.3, if $\psi \in \mathrm{Irr}(F)$, either $|G:F|\psi(1) \in \mathrm{cd}(G)$ or $|F:N|$ divides $\psi(1)^2$. In the first case, we must then have $\psi(1) \leq q^2+1$. In the second case, we get $p \mid \psi(1)$. If $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = q^6$, then $\gcd(\chi(1), |G:F|) = 1$. So $\chi_F \in \mathrm{Irr}(F)$. Since $q^6 > q^2+1$, this forces $p \mid \chi(1)$. Thus $p = 3$. As $3 \nmid |G:F|$ and $3 \nmid \tau(1)$, we conclude that $3 \nmid \psi(1)$. Hence $|G:F|\psi(1) \in \mathrm{cd}(G)$. But this is not the case. Thus $G' = G''$.

2.3 Establishing $G'/M \cong H$ when $H \cong {}^2G_2(q^2)$

We continue by proving Step 2 of Huppert's argument. Recall that Step 2 asserts that if G'/M is a chief factor of G, then $G'/M \cong H$. Suppose G'/M is a chief factor of G. As $G' = G''$ by Step 1,

$$G'/M \cong S_1 \times S_2 \times \cdots \times S_k,$$

where all $S_i \cong S$, a nonabelian simple group. We need to show that $G'/M \cong {}^2G_2(q^2)$. The degrees of S must divide the degrees of G. By the classification of the finite nonabelian simple groups, the possibilities for S include the Tits group, the 26 sporadic simple groups, the alternating groups A_n for $n \geq 5$, the ten families of simple groups of exceptional Lie type, and the six families of simple groups of classical Lie type. We must show that $k = 1$ and eliminate all possibilities for S except ${}^2G_2(q^2)$.

2.3.1 Key Lemma

Using tensor induction, the following lemma is proved in [2].

Lemma 2.3.1. *Let G'/M be a minimal normal subgroup of G/M so that $G'/M \cong S_1 \times \cdots \times S_k$, where $S_i \cong S$, a nonabelian simple group. Let A be the automorphism group of S. If $\sigma \in \mathrm{Irr}(S)$ extends to A, then $\sigma \times \cdots \times \sigma \in \mathrm{Irr}(G'/M)$ extends to G.*

We will exploit this lemma in the following manner. If we can find an irreducible character χ of S which extends to the automorphism group of S, then $\chi(1)^k$ is a degree of G/M, hence also of G.

2.3.2 Eliminating the Alternating Groups for all k

Proposition 2.3.1. *The simple group S is not an alternating group A_n with $n \geq 7$.*

Proof. Recall $G'/M \cong S_1 \times \cdots \times S_k$, where $S_i \cong S$. By Lemma 2.3.1, if S has an irreducible character χ of odd degree which extends to $\mathrm{Aut}(S)$, then $\chi(1)^k$ is an odd degree of G'/M, hence an odd degree of G. We will find two or more irreducible characters of S of distinct odd degrees which extend to $\mathrm{Aut}(S)$, say χ and ϕ, with $\chi(1) > \phi(1)$. The only odd degrees of G are

$$q^6,\ (q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3}),\ \text{and}\ q^2(q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3}).$$

If $\chi(1)^k$ and $\phi(1)^k$ are not powers of 3 or satisfy $\chi(1)^k/\phi(1)^k$ is a power of 3, then we have a contradiction. This follows from the list of odd degrees of G.

Case 1: $S \cong A_7$, $S \cong A_8$, or $S \cong A_{10}$

As shown in the Atlas [6], A_7 and A_8 have irreducible characters of degree 21 and 35 which extend to $\mathrm{Aut}(A_7)$ and $\mathrm{Aut}(A_8)$, respectively. As neither of these degrees is a power of 3 and 35/21 is not an integer, the k^{th} power of these degrees cannot be degrees of G. As shown in the Atlas [6], A_{10} has seven irreducible characters of odd degree which extend to S_{10}, hence their k^{th} powers are degrees of G. As G has only three odd degrees, we have a contradiction.

Case 2: $S \cong A_{2m}$, $m \geq 6$

Adopting the construction and notation of [22], we have the following degrees of irreducible characters of S which extend to $\mathrm{Aut}(S)$:

$$\chi_{1,0}(1) = 2m - 1 \text{ and } \chi_{3,0}(1) = \frac{(2m-1)(m-1)(2m-3)}{3}.$$

As shown in [2], we also have the irreducible character χ of S of degree $\chi(1) = m(2m-3)$ which extends to $\mathrm{Aut}(S)$. Now $\chi_{1,0}(1)$ is odd for any m while $\chi_{3,0}(1)$ is odd for even m and $\chi(1)$ is odd for odd m. Examining the odd degrees of G, we see that $q^6 \neq (2m-1)^k$ since $(2m-1)^k$ is a factor of $(\chi_{3,0}(1))^k$ and q^6 is not a factor of any of the other degrees of G. Further, the quotients of $\chi_{3,0}(1)$ and $\chi(1)$ with $\chi_{1,0}(1)$ yield

$$\frac{\chi_{3,0}(1)}{\chi_{1,0}(1)} = \frac{\frac{1}{3}(2m-1)(m-1)(2m-3)}{2m-1} = \frac{(m-1)(2m-3)}{3}$$

and

$$\frac{\chi(1)}{\chi_{1,0}(1)} = \frac{m(2m-3)}{2m-1},$$

which are not prime or a power of a prime since $\gcd(m-1, 2m-3) = \gcd(m, 2m-1) = 1$ and $\gcd(m, 2m-3) = 1$ or 3. As the other odd degrees of G are powers of a prime multiples of each other, we have a contradiction.

Case 3: $S \cong A_{2m+1}$, $m \geq 4$

Adopting the construction and notation of [22], we have the following degrees of irreducible characters of S which extend to $\mathrm{Aut}(S)$:

$$\chi_{2,0}(1) = m(2m-1),\ \chi_{2,1}(1) = \frac{(2m-1)(2m+1)(2m-3)}{3},$$

and, as constructed in [2],

$$\chi(1) = (m-1)(2m+1).$$

Now $\chi_{2,0}(1)$ is odd for odd m while $\chi_{2,1}(1)$ is odd for all m and $\chi(1)$ is odd for even m. None of these degrees is a power of a prime since $\gcd(m, 2m-1) = \gcd(m-1, 2m+1) = 1$ and $\gcd(2m-1, 2m+1) = 1$ or 2. Further, the quotients of $\chi_{2,1}(1)$ and $\chi_{2,0}(1)$ as well as $\chi_{2,1}(1)$ and $\chi(1)$ yield

$$\frac{\chi_{2,1}(1)}{\chi_{2,0}(1)} = \frac{(2m+1)(2m-3)}{3m}$$

and

$$\frac{\chi_{2,1}(1)}{\chi(1)} = \frac{(2m-1)(2m-3)}{3(m-1)}.$$

Neither of these is an integer as $\gcd(m-1, 2m-1) = 1$ or 3, $\gcd(m-1, 2m-3) = 1$, $\gcd(m, 2m+1) = 1$, and $\gcd(2m-3, m) = 1$ or 3, while $m \geq 4$. So the k^{th} power of these odd degrees cannot be degrees of G. This eliminates from consideration the alternating groups A_n for $n \geq 7$. $\square$

Note that we will consider $A_5 \cong \mathrm{PSL}_2(5)$ and $A_6 \cong \mathrm{PSL}_2(9)$ when we consider the simple groups of classical Lie type.

2.3.3 Eliminating Sporadic Simple Groups and the Tits Group for all k

Proposition 2.3.2. *The simple group S is not one of the sporadic simple groups or the Tits group.*

Proof. Now G has at most 10 distinct, nontrivial character degrees. If an irreducible character of S extends to $\mathrm{Aut}(S)$, then the k^{th} power of its degree must be a degree of G by Lemma 2.3.1. Consulting the Atlas [6], most of the sporadic groups have more than 10 irreducible characters of distinct degrees which extend to $\mathrm{Aut}(S)$. This implies that G has more than 10 distinct character degrees, a contradiction. We only need to consider the following cases of sporadic simple groups with 10 or less extendible characters of distinct degrees.

First consider $S \cong M_{11}$, $S \cong M_{12}$, $S \cong M_{23}$, $S \cong J_1$, $S \cong J_2$, $S \cong J_3$, or $S \cong {}^2F_4(2)'$, the Tits group. Recall that the odd degrees of G consist of a power of 3 and two other degrees, one of which is a multiple of the other. Each of these simple groups has two irreducible characters of distinct odd degrees which are not powers of a prime and are not multiples of each other. Hence, for any $k \geq 1$, the k^{th} power of these degrees cannot be degrees of G. Next, consider $S \cong M_{22}$. The Mathieu group M_{22} has irreducible characters of six distinct, nontrivial, odd degrees which extend. As G has only three odd degrees, it is not possible for the k^{th} power of these degrees to be degrees of G. □

2.3.4 Eliminating the Groups of Lie Type when $k > 1$

To eliminate the groups of Lie type when $k > 1$, we will require the Steinberg character of these groups. The properties of the Steinberg character can be found in [4]. We will rely on the fact that the degree of the Steinberg character is a power of a prime, which is proved in Theorem 6.4.7 of [4].

Lemma 2.3.2. *Let G be a simple group of Lie type with defining characteristic p. If χ is the Steinberg character of G, then $\chi(1) = |G|_p$.*

We will also rely upon the following result, proved in [29] and [30].

Lemma 2.3.3. *(Schmid) Let N be a normal subgroup of a group G, and suppose that N is* *isomorphic to a finite group of Lie type. If θ is the Steinberg character of N, then θ extends to G.*

If S is a simple group of Lie type and χ is the Steinberg character of S, then $\chi(1)$ is a power of the prime p, where p is the defining characteristic of the group. By Lemma 2.3.3, χ extends to the automorphism group of S. Once again appealing to Lemma 2.3.1, we have that $\chi(1)^k$ is a degree of G. As the only composite power of a prime among degrees of G is q^6, we must have that $\chi(1)^k = q^6$. Hence, the defining characteristic of the simple group S must be the same as the prime divisor of q^6, which is 3. Now we will show that $k = 1$ for the simple groups of Lie type. Let $S = S(q_1)$ be defined over a field of q_1 elements.

Proposition 2.3.3. *If $S = S(q_1)$ is a simple group of Lie type, then $k = 1$.*

Proof. Suppose $k \geq 2$. The Steinberg character χ of S extends to $\mathrm{Aut}(S)$ so $\chi(1)^k = q^6$. Let $\chi(1) = {q_1}^j$. As $G'/M \cong S_1 \times \cdots \times S_k$, there is an irreducible character of G'/M found by multiplying $k-1$ copies of χ with another nonlinear irreducible character of S, say $\tau \neq \chi$. Then $(\chi^{k-1}\tau)(1)$ is a "mixed" degree, meaning that the degree is divisible by 3 but is not a power of 3.

As the degrees of G'/M divide the degrees of G, we must have that the degree of this character divides one of the mixed degrees of G. The highest power of q in any mixed degree of G is 2. Now $q^6 = {q_1}^{jk}$ implies $q = {q_1}^{jk/6}$. The power of q_1 in $(\chi^{k-1}\tau)(1)$ is at least $j(k-1)$. We must have that

$$j(k-1) \leq \frac{2jk}{6},$$

which reduces to $2k \leq 3$. Thus $k < 2$, a contradiction. $\square$

Since the sporadic, Tits, and alternating groups have been eliminated as possibilities for S, we have that S is a simple group of Lie type and, thus, $k = 1$. We will now show that $S \cong {}^2G_2(q^2)$ by eliminating all other possibilities for S.

2.3.5 Eliminating Simple Groups of Exceptional Lie Type when $k = 1$

Although we must consider each of the ten families of simple groups of exceptional Lie type separately, we have a general argument that we will use for each of the groups. Unless otherwise stated, all notation is adapted from Section 13.9 of [4].

1. We know that the prime divisor of q and the underlying characteristic of the group S are the same. We also know that the Steinberg character of S extends to $\mathrm{Aut}(S)$, hence its degree is a degree of G, namely q^6.

2. We find a mixed degree of S whose power on q_1 is large. As it is of mixed degree, it must divide one of the mixed degrees of G.

3. By computing q in terms of q_1, the size of the underlying field of S, and examining the relative sizes of the degrees of S and G, we will reach a contradiction.

Proposition 2.3.4. *If $k = 1$ and S is a group of exceptional Lie type, then $S \cong {}^2G_2(q^2)$.*

Proof. We will proceed by case analysis.

Case 1: $S \cong G_2(q_1)$

The Steinberg character of $S \cong G_2(q_1)$ has degree ${q_1}^6$. Hence $q = q_1$. But q_1 is an integer while q is not. Thus $S \ncong G_2(q_1)$.

Case 2: $S \cong {}^2B_2({q_1}^2)$ or $S \cong {}^2F_4({q_1}^2)$, ${q_1}^2 = 2^{2m+1}$, $m \geq 1$

Because q is a power of 3, the underlying prime divisors of q^2 and ${q_1}^2$ are not the same. Thus ${}^2B_2({q_1}^2)$ and ${}^2F_4({q_1}^2)$ cannot possibly be candidates for S.

Case 3: $S \cong {}^2G_2({q_1}^2)$, ${q_1}^2 = 3^{2m+1}$, $m \geq 1$

The Steinberg character of $S \cong {}^2G_2({q_1}^2)$ has degree ${q_1}^6$. As the only power of a prime

$S = S(q_1)$	St(1)	Char of S	Degree
${}^3D_4({q_1}^3)$	${q_1}^{12}$	$\phi''_{1,3}$	${q_1}^7\Phi_{12}(q_1)$
$F_4(q_1)$	${q_1}^{24}$	$\phi_{9,10}$	${q_1}^{10}\Phi_3(q_1)^2\Phi_6(q_1)^2\Phi_{12}(q_1)$
$E_6(q_1)$	${q_1}^{36}$	$\phi_{6,25}$	${q_1}^{25}\Phi_8(q_1)\Phi_9(q_1)$
${}^2E_6({q_1}^2)$	${q_1}^{36}$	$\phi''_{2,16}$	${q_1}^{25}\Phi_8(q_1)\Phi_{18}(q_1)$
$E_7(q_1)$	${q_1}^{63}$	$\phi''_{7,46}$	${q_1}^{46}\Phi_7(q_1)\Phi_{12}(q_1)\Phi_{14}(q_1)$
$E_8(q_1)$	${q_1}^{120}$	$\phi_{8,91}$	${q_1}^{91}\Phi_4(q_1)^2\Phi_8(q_1)\Phi_{12}(q_1)\Phi_{20}(q_1)\Phi_{24}(q_1)$

Table 2.1: Eliminating Simple Groups of Exceptional Lie Type

among degrees of G is q^6, we must have that $q^6 = {q_1}^6$, which implies $q = q_1$. As the character degrees are equal, we have that S could possibly be ${}^2G_2(q^2)$.

Case 4: S is isomorphic to one of the remaining simple groups of exceptional Lie type.

For the remaining simple groups of exceptional Lie type, we will use the same general argument. Recall $S = S(q_1)$ is a simple group of exceptional Lie type defined over a field of q_1 elements. Suppose the Steinberg character of S has degree ${q_1}^j$. By Lemma 2.3.1, $q^6 = {q_1}^j$, so $q^2 = {q_1}^{2j/6}$. For each of the remaining possibilities for S, there is a mixed degree of S whose power on q_1 is greater than $2j/6$. As the mixed degrees of G have power at most $2j/6$ on q_1, we have a contradiction. Let $\Phi_y(q_1)$ denote the y^{th} cyclotomic polynomial in q_1. Table 2.1 exhibits the degree of the Steinberg character of S and a character of appropriate degree which will result in a contradiction. $\square$

2.3.6 Eliminating the Groups of Classical Lie Type when $k = 1$

We are only left with the possibility that $G'/M \cong S$, where S is a simple group of classical Lie type. Once again, we must consider each family of simple groups of classical Lie type separately, but we use the same general argument in each case.

Type	Group	Remarks
$A_\ell(q_1)$ $\ell \geq 1$	$\mathrm{PSL}_{\ell+1}(q_1)$	if $\ell = 1$, $q_1 \geq 4$
${}^2A_\ell({q_1}^2)$ $\ell \geq 2$	$\mathrm{PSU}_{\ell+1}({q_1}^2)$	if $\ell = 1$, $\mathrm{PSL}_{\ell+1}(q_1) \cong \mathrm{PSU}_{\ell+1}({q_1}^2)$ if $\ell = 2$, $q_1 > 2$
$B_\ell(q_1)$ $\ell \geq 2$	$\Omega_{2\ell+1}(q_1)$	if $\ell = 2$, $B_2(q_1) \cong C_2(q_1)$ for $q_1 > 2$
$C_\ell(q_1)$ $\ell \geq 2$	$\mathrm{PSp}_{2\ell}(q_1)$	if $\ell = 2$, $q_1 > 2$ if q_1 is even, $B_\ell(q_1) \cong C_\ell(q_1)$
$D_\ell(q_1)$ $\ell \geq 4$	$\mathrm{P}\Omega_{2\ell}{}^+(q_1)$	
${}^2D_\ell({q_1}^2)$ $\ell \geq 4$	$\mathrm{P}\Omega_{2\ell}{}^-(q_1)$	

Table 2.2: Simple Groups of Classical Lie Type

1. As the Steinberg character of $S = S(q_1)$ extends to the only irreducible character of prime power degree of G, we can determine q in terms of q_1.

2. We find an irreducible character of large mixed degree of S. As the degrees of S divide the degrees of G, this mixed degree of S must divide a mixed degree of G.

3. We will show that the degree of this irreducible character of S cannot divide a mixed degree of G.

Table 2.2 gives the different types of classical groups of Lie type that we must consider.

Proposition 2.3.5. *If $k = 1$, then S is not a simple group of classical Lie type.*

Proof. We will proceed by examining each group separately. We begin with the groups of type A_ℓ.

Groups of Type A_ℓ

In general, we will find a mixed character degree of S that is too large to divide the mixed degrees of G. We will handle the groups of low rank separately as we need to find a large degree of S satisfying some divisibility properties.

Case 1(a): $\ell = 1$, i.e., $S \cong \mathrm{PSL}_2(q_1)$ for $q_1 \geq 4$

The Steinberg character of S, which has degree q_1, extends to $\mathrm{Aut}(S)$, so we must have that $q^6 = q_1$. As shown in [22], an irreducible character of S has degree $q_1 - 1 = q^6 - 1$. This degree must divide a degree of G. Examining the degrees of G, it is clear that $q^6 - 1$ does not divide any degree of G. Note that this eliminates the possibility that $S \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong A_5$ and $S \cong A_6 \cong \mathrm{PSL}_2(9)$.

Case 1(b): $\ell = 2$, i.e., $S \cong \mathrm{PSL}_3(q_1)$

As shown in [32], the degree of the Steinberg character of $\mathrm{PSL}_3(q_1)$ is ${q_1}^3$. Hence $q_1 = q^2$. But the mixed degrees of S are $q^2(q^2+1)$ and $q^2(q^4+q^2+1)$. These must divide $q^2(q^4-q^2+1)$ since this is the only mixed degree of G divisible by q^2. Certainly $q^2 + 1$ does not divide $q^4 - q^2 + 1$, and $q^4 + q^2 + 1$ is relatively prime to $q^4 - q^2 + 1$ so neither of these degrees of S divide degrees of G.

Case 1(c): $\ell = 3$, i.e., $S \cong \mathrm{PSL}_4(q_1)$

When $\ell = 3$, the Steinberg character of S has degree ${q_1}^6$, so $q = q_1$. But q is not an integer while q_1 is an integer. Hence we have a contradiction.

Case 1(d): $\ell \geq 4$, i.e., $S \cong \mathrm{PSL}_{\ell+1}(q_1)$

In this general case, the degree of the Steinberg character of S is ${q_1}^{\ell(\ell+1)/2}$. Thus $q^2 = {q_1}^{\ell(\ell+1)/6}$. From [36], we have that

$$\begin{aligned}\chi_3(1) &= q_1 \frac{({q_1}^2-1)({q_1}^3-1)\cdots({q_1}^{\ell-1}-1)({q_1}^{\ell}-1)({q_1}^{\ell+1}-1)}{({q_1}^2-1)({q_1}^{\ell-1}-1)} \\ &= \begin{cases} q_1({q_1}^4-1)({q_1}^5-1), & \text{if } \ell = 4; \\ q_1({q_1}^3-1)\cdots({q_1}^{\ell-2}-1)({q_1}^{\ell}-1)({q_1}^{\ell+1}-1), & \text{if } \ell > 4 \end{cases}\end{aligned}$$

is a degree of $\mathrm{PGL}_{\ell+1}(q_1)$. Appealing to Lemma 1.4, as χ_3 is an irreducible character of $\mathrm{PGL}_{\ell+1}(q_1)$, if μ is an irreducible constituent of the restriction of χ_3 to $\mathrm{PSL}_{\ell+1}(q_1)$, then $\chi_3(1)/\mu(1)$ divides $[\mathrm{PGL}_{\ell+1}(q_1) : \mathrm{PSL}_{\ell+1}(q_1)]$. Since

$$[\mathrm{PGL}_{\ell+1}(q_1) : \mathrm{PSL}_{\ell+1}(q_1)] = \gcd(\ell+1, q_1-1),$$

$\chi_3(1)/\mu(1)$ divides $q_1 - 1$. Hence $\chi_3(1)/(q_1-1)$ divides $\mu(1)$, a degree of S, and thus must divide a degree of G. As this degree is mixed, it must divide a mixed degree of G.

When $\ell = 4$, we have that $q^2 = {q_1}^{10/3}$. But $q_1({q_1}^2+1)({q_1}^5-1)$ divides no degree of G. When $\ell > 4$, we would like to compare the corresponding part of the mixed degrees of G to the size of the part of $\chi_3(1)/(q_1-1)$ that is relatively prime to 3. To do so, we will factor $q_1 - 1$ from each term of $\chi_3(1)/(q_1-1)$ to get a rough upper bound on the size of this part of $\chi_3(1)/(q_1-1)$. Factoring $q_1 - 1$ from each term gives

$$\begin{aligned}\frac{\chi_3(1)}{q_1-1} &> ({q_1}^2+q_1+1)({q_1}^3+{q_1}^2+q_1+1)\cdots({q_1}^{\ell-3}+{q_1}^{\ell-2}+\cdots+q_1+1) \\ &\quad\cdot({q_1}^{\ell-1}+{q_1}^{\ell-2}+\cdots+q_1+1)({q_1}^{\ell}+{q_1}^{\ell-1}+\cdots+q_1+1) \\ &> {q_1}^{2+3+4+\cdots+(\ell-3)+(\ell-1)+\ell}+1 \\ &= {q_1}^{\ell(\ell+1)/2-1-(\ell-2)}+1 \\ &= {q_1}^{\ell(\ell+1)/2-\ell+1}+1.\end{aligned}$$

Examining the mixed degrees of G, we see that the degree with largest term prime to 3 is $\frac{1}{6}\sqrt{3}q(q^2-1)(q^2+1+q\sqrt{3}) = \frac{1}{6}\sqrt{3}q(q^4+q^3\sqrt{3}-q\sqrt{3}-1)$. So the part of the mixed degree of S that is relatively prime to 3 must be less than or equal to $q^4+q^3\sqrt{3}-q\sqrt{3}-1$ to divide one of these degrees. Replacing q by the appropriate power of q_1 gives

$$q_1{}^{4\ell(\ell+1)/12} + \sqrt{3}q_1{}^{3\ell(\ell+1)/12} - \sqrt{3}q_1{}^{\ell(\ell+1)/12} - 1.$$

We must have that

$$\frac{\ell(\ell+1)}{2} - \ell + 1 \leq \frac{4\ell(\ell+1)}{12},$$

which reduces to $\ell^2 - 5\ell + 6 \leq 0$, and is not satisfied for $\ell > 4$. Thus this degree of S divides no degree of G.

We will use this same general argument to eliminate the other simple groups of classical Lie type. We will find a large mixed degree which divides a mixed degree of S and hence also divides a mixed degree of G. We will compare the size of this degree to the size of the largest mixed degree of G and show that the degree of S is too large to divide a mixed degree of G.

Groups of Type $^2A_\ell$

We will proceed in the same manner as in the linear group case. Again, we will consider the groups of small rank separately.

Case 2(a): $\ell = 2$, i.e., $S \cong \mathrm{PSU}_3(q_1{}^2)$

The degree of the Steinberg character of $\mathrm{PSU}_3(q_1{}^2)$ is $q_1{}^3$. Hence $q^2 = q_1$. The mixed degrees of S are $q^2(q^2-1)$ and $q^2(q^4-q^2+1)$. These must divide $q^2(q^4-q^2+1)$. Now q^2-1 is relatively prime to q^4-q^2+1 so q^2-1 does not divide $q^2(q^4-q^2+1)$.

Case 2(b): $\ell = 3$, i.e., $S \cong \mathrm{PSU}_4({q_1}^2)$

The Steinberg character of S is of degree ${q_1}^6$, implying $q = q_1$. Again, q is not an integer while q_1 is an integer. So this is not possible.

Case 2(c): $\ell = 4$ i.e., $S \cong \mathrm{PSU}_5({q_1}^2)$

The Steinberg character of S is of degree ${q_1}^{10}$, so $q^2 = {q_1}^{10/3}$. As shown in Section 13.8 of [4], the unipotent character of degree

$$\chi^{(1,1,3)}(1) = {q_1}^3({q_1}^2 - q_1 + 1)({q_1}^2 + 1)$$

is a character of $\mathrm{PSU}_5({q_1}^2)$. But this degree is mixed, with a factor of ${q_1}^3$. Thus it must divide $q^2(q^4 - q^2 + 1)$. As q_1 is a power of 3, ${q_1}^2 + 1$ is even, while $q^4 - q^2 + 1$ is odd. So $\chi^{(1,1,3)}(1)$ does not divide $q^2(q^4 - q^2 + 1)$.

Case 2(d): $\ell = 5$ i.e., $S \cong \mathrm{PSU}_6({q_1}^2)$

When $\ell = 5$, the Steinberg character of S has degree ${q_1}^{15}$ so $q^2 = {q_1}^{15/3}$. From Section 13.8 of [4], S has unipotent character $\chi^{(1,2,3)}$ of degree

$$\chi^{(1,2,3)}(1) = {q_1}^4({q_1}^3 - 1)(q_1 - 1)^2({q_1}^2 + 1).$$

This is a mixed degree of S, hence must divide a mixed degree of G. But this degree is mixed, with a factor of ${q_1}^4$. Thus it must divide $q^2(q^4 - q^2 + 1)$. Again, ${q_1}^2 + 1$ is even, while $q^4 - q^2 + 1$ is odd. So $\chi^{(1,2,3)}(1)$ does not divide $q^2(q^4 - q^2 + 1)$.

Case 2(e): $\ell \geq 6$, i.e., $S \cong \mathrm{PSU}_{\ell+1}({q_1}^2)$

In this general case, the degree of the Steinberg character of S is ${q_1}^{\ell(\ell+1)/2}$. Thus

$q^2 = {q_1}^{\ell(\ell+1)/6}$. As shown in [36],

$$\begin{aligned}\chi_3(1) &= q_1 \frac{({q_1}^2-1)({q_1}^3+1)({q_1}^4-1)\cdots({q_1}^{\ell-1}-(-1)^{\ell-1})({q_1}^{\ell}-(-1)^{\ell})({q_1}^{\ell+1}-(-1)^{\ell+1})}{({q_1}^2-1)({q_1}^{\ell-1}-(-1)^{\ell-1})} \\ &= q_1({q_1}^3+1)({q_1}^4-1)\cdots({q_1}^{\ell-2}-(-1)^{\ell-2})({q_1}^{\ell}-(-1)^{\ell})({q_1}^{\ell+1}-(-1)^{\ell+1})\end{aligned}$$

is a degree of $\mathrm{PU}_{\ell+1}({q_1}^2)$. Again, appealing to Lemma 1.4, as χ_3 is an irreducible character of $\mathrm{PU}_{\ell+1}({q_1}^2)$, if μ is an irreducible constituent of the restriction of χ_3 to $\mathrm{PSU}_{\ell+1}({q_1}^2)$, then $\chi_3(1)/\mu(1)$ divides $[\mathrm{PU}_{\ell+1}({q_1}^2) : \mathrm{PSU}_{\ell+1}({q_1}^2)]$. Since

$$[\mathrm{PU}_{\ell+1}({q_1}^2) : \mathrm{PSU}_{\ell+1}({q_1}^2)] = \gcd(\ell+1, q_1+1),$$

$\chi_3(1)/\mu(1)$ divides q_1+1. Hence $\chi_3(1)/(q_1+1)$ divides $\mu(1)$, a degree of S, and thus must divide a degree of G. As this degree is mixed, it must divide a mixed degree of G. If ℓ is even, factoring q_1-1 from all the even degree terms gives

$$\begin{aligned}\frac{\chi_3(1)}{q_1+1} &> ({q_1}^3+{q_1}^2+q_1+1)({q_1}^5+1)({q_1}^5+{q_1}^4+{q_1}^3+{q_1}^2+q_1+1) \\ &\quad \cdots({q_1}^{\ell-3}+1)({q_1}^{\ell-3}+{q_1}^{\ell-4}+\cdots+q_1+1)({q_1}^{\ell-1}+\cdots+1)({q_1}^{\ell+1}+1),\end{aligned}$$

while if ℓ is odd, factoring q_1-1 from all even degree terms gives

$$\begin{aligned}\frac{\chi_3(1)}{q_1+1} &> ({q_1}^3+{q_1}^2+q_1+1)({q_1}^5+1)({q_1}^5+{q_1}^4+{q_1}^3+{q_1}^2+q_1+1) \\ &\quad \cdots({q_1}^{\ell-1}+1)({q_1}^{\ell-1}+{q_1}^{\ell-2}+\cdots+q_1+1)({q_1}^{\ell}+1)({q_1}^{\ell}+\cdots+q_1+1).\end{aligned}$$

The degree of the leading term is

$$[3+5+7+\cdots+(\ell-1)+(\ell+1)]+[5+7+\cdots+(\ell-3)] = \frac{2\ell^2-12}{4}$$

if ℓ is even and

$$[3+5+7+\cdots+\ell]+[5+7+\cdots+\ell]-(\ell-2) = \frac{2\ell^2-10}{4}$$

if ℓ is odd. In this case, the part of the mixed degree of S that is relatively prime to 3 must be less than or equal to

$${q_1}^{4\ell(\ell+1)/12} + \sqrt{3}{q_1}^{3\ell(\ell+1)/12} - \sqrt{3}{q_1}^{\ell(\ell+1)/12} - 1.$$

To divide these small mixed degrees of G, the degree of the leading term of the character degree of S must be less than or equal to $4\ell(\ell+1)/12$. As

$$\frac{2\ell^2-12}{4} < \frac{2\ell^2-10}{4},$$

we must have that

$$\frac{2\ell^2-10}{4} \leq \frac{4\ell(\ell+1)}{12},$$

which reduces to $\ell^2 - 2\ell - 15 \leq 0$ and is not satisfied for $\ell > 5$.

Groups of Type B_ℓ

We will examine the groups of low rank first and then show that S cannot be of type B_ℓ in general in Case 3(b).

Case 3(a): $\ell = 2$, i.e., $S \cong \Omega_5(q_1) \cong \mathrm{PSp}_4(q_1)$

The Steinberg character of S has degree ${q_1}^4$. Thus $q^2 = {q_1}^{4/3}$. As shown in [37], S has degrees $q_1(q_1-1)({q_1}^2+1)$ and $q_1(q_1+1)({q_1}^2+1)$. As $q^2 = {q_1}^{4/3}$, these can only divide $q^2(q^4-q^2+1)$. As q_1 is a power of 3, ${q_1}^2+1$ is even, while q^4-q^2+1 is odd. So these degrees of S do not divide $q^2(q^4-q^2+1)$.

Case 3(b): $\ell \geq 3$, i.e., $S \cong \Omega_{2\ell+1}(q_1)$

In this general case, the degree of the Steinberg character of S is ${q_1}^{\ell^2}$. Thus $q^2 = {q_1}^{\ell^2/3}$. As shown in [37],

$$\chi_1(1) = q_1 \frac{({q_1}^4-1)({q_1}^6-1)\cdots({q_1}^{2(\ell-1)}-1)({q_1}^{2\ell}-1)}{{q_1}^{\ell-1}+1}$$

is a character degree of $\mathrm{SO}_{2\ell+1}(q_1)$. Again, appealing to Lemma 1.4, as

$$[\mathrm{SO}_{2\ell+1}(q_1) : \Omega_{2\ell+1}(q_1)] = \gcd(2, q_1 - 1) = 2,$$

$\chi_1(1)/2$ divides a degree of S, and thus must divide a degree of G. As this degree is mixed, it must divide a mixed degree of G.

When $\ell = 3$, we have that $\chi_1(1) = q_1({q_1}^2 - 1)({q_1}^6 - 1)$. Thus $\chi_1(1)/2$ divides a degree of $S \cong \Omega_7(q_1)$, which must divide a mixed degree of G. In this case, the degree of the Steinberg character of S is ${q_1}^9$, so $q^2 = {q_1}^3$. Thus $\chi_1(1)/2$ cannot divide a degree of G.

When $\ell > 3$, we establish a rough bound on the size of the degree by factoring $q_1 - 1$ from each term and considering only the factors of the degree prime to 3. We obtain

$$\begin{aligned}\frac{1}{2}\chi_1(1) &> ({q_1}^4 - 1)({q_1}^6 - 1)\cdots({q_1}^{2(\ell-2)} - 1)({q_1}^{\ell-1} - 1)({q_1}^{2\ell} - 1)\\ &> ({q_1}^3 + {q_1}^2 + q_1 + 1)({q_1}^5 + {q_1}^4 + \cdots + q_1 + 1)\\ &\qquad \cdots({q_1}^{2\ell-1} + \cdots + q_1 + 1)\\ &> {q_1}^{[3+5+7+\cdots+(2\ell-1)]-(2(\ell-1)-1)+(\ell-2)} + 1\\ &> {q_1}^{\ell^2-\ell+1} + 1.\end{aligned}$$

In this case, the part of the mixed degree of S that is relatively prime to 3 must be less than or equal to

$${q_1}^{4\ell^2/6} + \sqrt{3}{q_1}^{3\ell^2/6} - \sqrt{3}{q_1}^{\ell^2/6} - 1.$$

To divide these small mixed degrees of G, $\ell^2 - \ell + 1$ must be less than or equal to $4\ell^2/6$. Now

$$\ell^2 - \ell + 1 \leq \frac{4\ell^2}{6}$$

reduces to $\ell^2 - 3\ell + 3 \leq 0$, which is not satisfied for $\ell \geq 3$.

Groups of Type C_ℓ

Case 4(a): $\ell = 2$, i.e., $S \cong \mathrm{PSp}_4(q_1)$

As $C_2(q_1) \cong B_2(q_1)$, we are done by Case 3.

Case 4(b): $\ell = 3$ and q_1 is a power of 3, i.e., $S \cong \mathrm{PSp}_6(q_1)$

The Steinberg character of S has degree ${q_1}^9$, so $q^2 = {q_1}^3$. From Section 13.8 of [4], we see that S has unipotent character χ^α of degree

$$\chi^\alpha(1) = {q_1}^2(q_1 - 1)(q_1 + 1)({q_1}^3 + 1)({q_1}^2 + 1)$$

corresponding to the symbol

$$\alpha = \begin{pmatrix} 1 & 3 \\ \\ 0 & 1 \end{pmatrix}.$$

This is a mixed degree of S, hence must divide a mixed degree of G. Thus it must divide $q^2(q^4 - q^2 + 1)$. As q_1 is a power of 3, $\chi^\alpha(1)$ is even, while $q^4 - q^2 + 1$ is odd. So $\chi^\alpha(1)$ does not divide $q^2(q^4 - q^2 + 1)$.

Case 4(c): $\ell \geq 4$ and q_1 is a power of 3, i.e., $S \cong \mathrm{PSp}_{2\ell}(q_1)$

In this general case, the degree of the Steinberg character of S is ${q_1}^{\ell^2}$. Thus $q^2 = {q_1}^{\ell^2/3}$. Once again, from [37],

$$\chi_1(1) = {q_1}^2 \frac{({q_1}^2 + 1)({q_1}^6 - 1)\cdots({q_1}^{2(\ell-1)} - 1)({q_1}^{2\ell} - 1)}{{q_1}^{\ell-2} + 1}$$

is a character degree of $\mathrm{PCSp}_{2\ell}(q_1)$. Again, appealing to Lemma 1.4, as

$$[\mathrm{PCSp}_{2\ell}(q_1) : \mathrm{PSp}_{2\ell}(q_1)] = \gcd(2, q_1 - 1) = 2,$$

$\chi_1(1)/2$ divides a degree of S, and thus must divide a degree of G. As this degree is mixed, it must divide a mixed degree of G. Now

$$\chi_1(1) = \begin{cases} {q_1}^2({q_1}^6 - 1)({q_1}^8 - 1) & \text{if } \ell = 4; \\ \\ {q_1}^2({q_1}^2 + 1)({q_1}^3 - 1)({q_1}^8 - 1)({q_1}^{10} - 1) & \text{if } \ell = 5; \\ \\ {q_1}^2({q_1}^2 + 1)({q_1}^6 - 1)\cdots({q_1}^{2(\ell-3)} - 1) & \\ \\ \quad \cdot({q_1}^{\ell-2} - 1)({q_1}^{2(\ell-1)} - 1)({q_1}^{2\ell} - 1) & \text{if } \ell > 5. \end{cases}$$

When $\ell = 4$ or $\ell = 5$, it is clear that $\chi_1(1)/2$ is too large to divide a mixed degree of G. For $\ell > 5$, we obtain a rough bound on the size of $\chi_1(1)/2$ by factoring out $q_1 - 1$ from each term and only considering the part of the degree prime to 3. We obtain

$$\begin{aligned}\frac{1}{2}\chi_1(1) &> {q_1}^{2+5+7+\cdots+(2(\ell-1)-1)+(2\ell-1)-(2(\ell-2)-1)+((\ell-2)-1)}+1\\ &= {q_1}^{\ell^2-\ell}+1.\end{aligned}$$

In this case, the part of the mixed degree of S that is relatively prime to 3 must be less than or equal to

$${q_1}^{4\ell^2/6}+\sqrt{3}{q_1}^{3\ell^2/6}-\sqrt{3}{q_1}^{\ell^2/6}-1.$$

To divide these small mixed degrees of G, $\ell^2-\ell$ must be less than or equal to $4\ell^2/6$. Now

$$\ell^2-\ell \leq \frac{4\ell^2}{6}$$

reduces to $\ell^2-3\ell \leq 0$, which is not satisfied for $\ell \geq 4$.

Groups of Type D_ℓ

Case 5(a): $\ell = 4$, i.e., $S \cong \mathrm{P}\Omega_8{}^+(q_1)$

The Steinberg character of S has degree ${q_1}^{12}$ so $q^2 = {q_1}^4$. Hence ${q_1}^2 = q$. But q is not an integer, so we have a contradiction.

Case 5(b): $5 \leq \ell \leq 7$, i.e., $S \cong \mathrm{P}\Omega_{2\ell}{}^+(q_1)$

For $5 \leq \ell \leq 7$, we will use the same general argument. Suppose the Steinberg character of S has degree ${q_1}^j$. By Lemma 2.3.1, $q^6 = {q_1}^j$, so $q^2 = {q_1}^{j/3}$. For $5 \leq \ell \leq 7$, there is a mixed degree of $S \cong \mathrm{P}\Omega_{2l}{}^+(q_1)$ whose power on q_1 is greater than $j/3$. As most of the mixed degrees of G have power $j/3$ on q_1, the only mixed degree of G that it could divide is $q^2(q^4-q^2+1)$. But this degree of G is odd while the given degree of S is even. So it is not possible for this degree of S to divide a degree of G. Table 2.3 exhibits the degree of the Steinberg character of S and a character of appropriate degree which will result in a contradiction. All notation is adapted from Section 13.8 of [4].

$S = S(q_1)$	St(1)	$\chi^\alpha(1)$	α
$\mathrm{P\Omega_{10}}^+(q_1)$	${q_1}^{20}$	${q_1}^6\Phi_4(q_1)\Phi_5(q_1)\Phi_8(q_1)$	$\begin{pmatrix}1 & 3\\ 1 & 2\end{pmatrix}$
$\mathrm{P\Omega_{12}}^+(q_1)$	${q_1}^{30}$	${q_1}^8\Phi_3(q_1)\Phi_5(q_1)\Phi_6(q_1)\Phi_8(q_1)\Phi_{10}(q_1)$	$\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}$
$\mathrm{P\Omega_{14}}^+(q_1)$	${q_1}^{42}$	$\frac{1}{2}{q_1}^{12}\Phi_5(q_1)\Phi_6(q_1)\Phi_7(q_1)\Phi_8(q_1)\Phi_{10}(q_1)\Phi_{12}(q_1)$	$\begin{pmatrix}2 & 3 & 4\\ 0 & 1 & 3\end{pmatrix}$

Table 2.3: Eliminating $\mathrm{P\Omega_{2\ell}}^+(q_1)$, $5 \leq \ell \leq 7$

Case 5(c): $\ell \geq 8$, i.e., $S \cong \mathrm{P\Omega_{2\ell}}^+(q_1)$

As shown in [37], a degree of $\mathrm{P}(\mathrm{CO}_{2\ell}^+(q_1)^0)$ is

$$\begin{aligned}\chi_1(1) &= {q_1}^2\frac{({q_1}^2+1)({q_1}^6-1)\cdots({q_1}^{2(\ell-2)}-1)({q_1}^{2(\ell-1)}-1)({q_1}^\ell-1)}{(q_1+1)({q_1}^{\ell-3}+1)}\\ &= {q_1}^2\frac{({q_1}^2+1)({q_1}^6-1)\cdots({q_1}^{(\ell-3)}-1)({q_1}^{2(\ell-2)}-1)({q_1}^{2(\ell-1)}-1)({q_1}^\ell-1)}{q_1+1}.\end{aligned}$$

Again, appealing to Lemma 1.4, as

$$[\mathrm{P}(\mathrm{CO}_{2\ell}^+(q_1)^0) : \mathrm{P\Omega_{2\ell}}^+(q_1)] = \gcd(4, {q_1}^\ell - 1)$$

and q_1 is odd, $\chi_1(1)/2$ or $\chi_1(1)/4$ divides a degree of S.

Now the degree of the Steinberg character of S is ${q_1}^{\ell(\ell-1)}$, so $q^2 = {q_1}^{\ell(\ell-1)/3}$. We obtain a rough bound on the size of $\chi_1(1)/4$ relatively prime to 3 by factoring out $q_1 - 1$ from each term and examining the remaining positive terms. We obtain

$$\begin{aligned}\frac{1}{4}\chi_1(1) &> ({q_1}^2+1)({q_1}^2+q_1+1)({q_1}^7+{q_1}^6+\cdots+q_1+1)\cdots({q_1}^{(\ell-3)-1}+\cdots+1)\\ &\quad\cdot({q_1}^{2(\ell-2)-1}+\cdots+1)({q_1}^{2(\ell-1)-1}+\cdots+1)({q_1}^{\ell-1}+\cdots+q_1+1)\\ &> {q_1}^{2+2+7+\cdots+[2(\ell-1)-1]+(\ell-1)}+1.\end{aligned}$$

The leading term has degree

$$\left(\frac{2(\ell-1)}{2}\right)^2 - 1 - 3 - 5 - [2(\ell-3)-1] + (\ell-1) + [(\ell-3)-1] + 2 + 2 = (\ell-1)^2 - 2 - \ell.$$

In this case, the part of the mixed degree of S that is relatively prime to 3 must be less than or equal to

$$q_1{}^{4\ell(\ell-1)/6} + \sqrt{3}q_1{}^{3\ell(\ell-1)/6} - \sqrt{3}q_1{}^{\ell(\ell-1)/6} - 1.$$

To divide these small mixed degrees of G, $(\ell-1)^2 - 2 - \ell$ must be less than or equal to $4\ell(\ell-1)/6$. Now

$$(\ell-1)^2 - 2 - \ell \leq \frac{4\ell(\ell-1)}{6}$$

reduces to $\ell^2 - 7\ell - 3 \leq 0$, which is not the case for $\ell \geq 8$.

Groups of Type $^2D_\ell$

Case 6(a): $\ell = 4$, i.e., $S \cong \mathrm{P}\Omega_8{}^-(q_1)$

The Steinberg character of S has degree $q_1{}^{12}$ so $q^2 = q_1{}^4$. Hence $q_1{}^2 = q$. But q is not an integer, so we have a contradiction.

Case 6(b): $\ell = 5$, i.e., $S \cong \mathrm{P}\Omega_{10}{}^-(q_1)$

As shown in [37], when $\ell = 5$, we have

$$\begin{aligned}\chi_1(1) &= q_1{}^2\frac{(q_1{}^2+1)(q_1{}^6-1)(q_1{}^8-1)(q_1{}^5+1)}{q_1{}^3+1}\\ &= q_1{}^2(q_1{}^2+1)(q_1{}^3-1)(q_1{}^8-1)(q_1{}^5+1)\end{aligned}$$

is a character degree of $\mathrm{P}(\mathrm{CO}_{10}^-(q_1)^0)$. Again, appealing to Lemma 1.4, as

$$[\mathrm{P}(\mathrm{CO}_{10}^-(q_1)^0) : \mathrm{P}\Omega_{10}{}^-(q_1)] = \gcd(4, q_1{}^5+1)$$

and q_1 is odd, $\chi_1(1)/2$ or $\chi_1(1)/4$ divides a degree of S. Now the degree of the Steinberg character of S is $q_1{}^{20}$. Hence $q^2 = q_1{}^{20/3}$. Examining the degrees of G, we see that it is not possible for this degree of S to divide a degree of G.

Case 6(c): $\ell \geq 6$, i.e., $S \cong \mathrm{P}\Omega_{2\ell}{}^{-}(q_1)$

The degree of the Steinberg character of S is ${q_1}^{\ell(\ell-1)}$, so $q^2 = {q_1}^{\ell(\ell-1)/3}$. For $\ell \geq 6$, as seen in [37],

$$\begin{aligned}\chi_1(1) &= {q_1}^2 \frac{({q_1}^2+1)({q_1}^6-1)\cdots({q_1}^{2(\ell-2)}-1)({q_1}^{2(\ell-1)}-1)({q_1}^{\ell}+1)}{{q_1}^{\ell-2}+1} \\ &= {q_1}^2({q_1}^2+1)({q_1}^6-1)\cdots({q_1}^{2(\ell-3)}-1)({q_1}^{\ell-2}-1)({q_1}^{2(\ell-1)}-1)({q_1}^{\ell}+1)\end{aligned}$$

is a degree of $\mathrm{P}(\mathrm{CO}_{2\ell}^{-}(q_1)^0)$. Again, appealing to Lemma 1.4, as

$$[\mathrm{P}(\mathrm{CO}_{2\ell}^{-}(q_1)^0) : \mathrm{P}\Omega_{2\ell}{}^{-}(q_1)] = \gcd(4, {q_1}^{\ell}+1)$$

and q_1 is odd, $\chi_1(1)/2$ or $\chi_1(1)/4$ divides a degree of S. Factoring out ${q_1}^2(q_1-1)$ from the appropriate factors and finding a rough upper bound on $\chi_1(1)/4$ gives

$$\begin{aligned}\frac{1}{4}\chi_1(1) &> ({q_1}^2+1)({q_1}^5+\cdots+q_1+1)\cdots({q_1}^{2(\ell-3)-1}+\cdots+1)({q_1}^{(\ell-2)-1}+\cdots+1) \\ &\quad \cdot ({q_1}^{2(\ell-1)-1}+\cdots+1)({q_1}^{\ell-1}+\cdots+q_1+1) \\ &> {q_1}^{2+5+7+\cdots+[2(\ell-1)-1]+[\ell-1]}+1.\end{aligned}$$

The leading term has degree

$$\left(\frac{2(\ell-1)}{2}\right)^2 - 1 - 3 - [2(\ell-2)-1] + (\ell-1) + [(\ell-2)-1] + 2 = \ell^2 - 2\ell + 1.$$

In this case, the part of the mixed degree of S that is relatively prime to 3 must be less than or equal to

$${q_1}^{4\ell(\ell-1)/6} + \sqrt{3}{q_1}^{3\ell(\ell-1)/6} - \sqrt{3}{q_1}^{\ell(\ell-1)/6} - 1.$$

To divide these small mixed degrees of G, $\ell^2 - 2\ell + 1$ must be less than or equal to $4\ell(\ell-1)/6$. Now

$$\ell^2 - 2\ell + 1 \leq \frac{4\ell(\ell-1)}{6}$$

reduces to $\ell^2 - 4\ell - 3 \leq 0$, which is not satisfied for $\ell \geq 5$. $\square$

This was the last case to consider to prove that $k = 1$ and $S \cong {}^2G_2(q^2)$.

Maximal Subgroup Structure	Order	Index
$[q^6] : \mathbb{Z}_{q^2-1}$	$q^6(q^2-1)$	q^6+1
$2 \times \mathrm{PSL}_2(q^2)$	$q^2(q^4-1)$	$q^4(q^4-q^2+1)$
$(2^2 : D_{\frac{1}{2}(q^2+1)}) : 3$	$2^2 \cdot 3 \cdot \frac{1}{2}(q^2+1)$	$\frac{1}{6}q^6(q^2-1)(q^4-q^2+1)$
$\mathbb{Z}_{q^2+\sqrt{3}q+1} : \mathbb{Z}_6$	$6(q^2+\sqrt{3}q+1)$	$\frac{1}{6}q^6(q^2+1)(q^2-1)(q^2-\sqrt{3}q+1)$
$\mathbb{Z}_{q^2-\sqrt{3}q+1} : \mathbb{Z}_6$	$6(q^2-\sqrt{3}q+1)$	$\frac{1}{6}q^6(q^2+1)(q^2-1)(q^2+\sqrt{3}q+1)$
${}^2G_2({q_0}^2)$, $q^2 = {q_0}^{2\alpha}$, α prime	${q_0}^6({q_0}^6+1)({q_0}^2-1)$	$\dfrac{{q_0}^{6\alpha}({q_0}^{6\alpha}+1)({q_0}^{6\alpha}-1)}{{q_0}^6({q_0}^6+1)({q_0}^2-1)}$

Table 2.4: Maximal Subgroups of ${}^2G_2(q^2)$

2.4 Establishing $I_{G'}(\theta) = G'$ when $H \cong {}^2G_2(q^2)$

We are now ready to prove Step 3 of Huppert's argument. We must prove if $\theta \in \mathrm{Irr}(M)$, $\theta(1) = 1$, then θ is stable under G'. By Lemma 1.6, this implies $[M, G'] = M'$. We can prove a stronger result for ${}^2G_2(q^2)$. We will remove the restriction that $\theta(1) = 1$ and prove the result for all $\theta \in \mathrm{Irr}(M)$.

Suppose $I_{G'}(\theta) = I \lneq G'$ for some $\theta \in \mathrm{Irr}(M)$. Let U be maximal such that $I \leq U \lneq G'$. If

$$\theta^I = \sum \phi_i,$$

for $\phi_i \in \mathrm{Irr}(I)$, then by Lemma 1.2, $\phi_i(1)|G' : I|$ is a degree of G' and thus divides some degree of G. As $m \geq 1$, $q > 3$ and the list of maximal subgroups of ${}^2G_2(q^2)$ in [21] gives Table 2.4. The notation is as follows: $[q^6]$ denotes an unspecified group of order q^6 and $A : B$ denotes a split extension.

Lemma 2.4.1. *The only maximal subgroups of ${}^2G_2(q^2)$ whose indices divide a degree of G are the maximal parabolic subgroups with structure $[q^6] : \mathbb{Z}_{q^2-1}$.*

Proof. The index of the parabolic subgroups with structure $[q^6] : \mathbb{Z}_{q^2-1}$ divides the degree q^6+1. The indices for all but the last subgroup in Table 2.4 must divide mixed degrees of ${}^2G_2(q^2)$. A close examination of the mixed degrees of ${}^2G_2(q^2)$ shows that these indices have exponents on q too large to divide a mixed degree of G. For the subgroup in the last row of Table 2.4, note that $q^2 = {q_0}^{2\alpha}$ so the power on q_0 in the mixed degrees of ${}^2G_2(q^2)$ is at most 2α. To divide a degree of ${}^2G_2(q^2)$, we must have that the power on q_0 in the index of the subgroup must satisfy $6\alpha - 6 \leq 2\alpha$. This implies $\alpha \leq 3/2$, so $\alpha = 1$, a contradiction. Hence the only maximal subgroups of ${}^2G_2(q^2)$ whose indices divide a degree of G are the parabolic subgroups. $\square$

Recall that we are assuming $I_{G'}(\theta) = I \lneq G'$ for some $\theta \in \mathrm{Irr}(M)$. Let U be maximal such that $I \leq U \lneq G'$. If

$$\theta^I = \sum \phi_i,$$

for $\phi_i \in \mathrm{Irr}(I)$, then by Lemma 1.2, $\phi_i(1)|G' : I|$ is a degree of G'. Hence $\phi_i(1)|G' : U||U : I|$ divides some degree of G. By Lemma 2.4.1, the only possibility is that $|G' : U| = q^6 + 1$. But then $\phi_i(1) = 1$. Hence ϕ_i is an extension of θ to I.

By Lemma 1.2(b), $(\phi_i\tau)^{G'} \in \mathrm{Irr}(G')$ for all $\tau \in \mathrm{Irr}(I/M)$. But then $(q^6+1)\tau(1)$ divides a character degree of G. This forces $\tau(1) = 1$ for all $\tau \in \mathrm{Irr}(I/M)$. Hence I/M is abelian. But I/M contains a Sylow 3-subgroup of ${}^2G_2(q^2)$, which is nonabelian (see [35]). Thus we must have that $I_{G'}(\theta) = G'$.

2.5 Establishing $M = \langle 1 \rangle$ when $H \cong {}^2G_2(q^2)$

We now prove that the subgroup M of G is trivial. By Step 2, we know that $G'/M \cong {}^2G_2(q^2)$. Hence, when paired with this step, we have that $G' \cong {}^2G_2(q^2)$. In general, we will assume that M is not trivial and derive a contradiction. Recall that we were able to prove a stronger result in Step 3. This simplifies the argument in Step 4.

For all q, the Schur multiplier of ${}^2G_2(q^2)$ is trivial. Thus $M' = M$ by Step 3 and

Lemma 1.6. If M is abelian, we are done. Suppose M is nonabelian. Then there is an irreducible character θ of M such that $\theta(1) > 1$. By Step 3, $I_{G'}(\theta) = G'$. As $G'/M \cong {}^2G_2(q^2)$ has trivial Schur multiplier, θ allows an extension θ_0 to G'. Then $\theta_0\tau \in \mathrm{Irr}(G')$ for all $\tau \in \mathrm{Irr}(G'/M)$. Consider $\tau \in \mathrm{Irr}(G'/M)$ with $\tau(1) = q^6$. Then $q^6\theta_0(1) \in \mathrm{cd}(G')$, a contradiction. Hence M is abelian. So $M = M' = \langle 1 \rangle$.

2.6 Establishing $G = G' \times C_G(G')$ when $H \cong {}^2G_2(q^2)$

We can now conclude Huppert's argument and verify the conjecture for the simple groups ${}^2G_2(q^2)$. The previous steps imply $G' \cong {}^2G_2(q^2)$. In this step, we will show that $G = G' \times C_G(G')$. As $G/G' \cong C_G(G')$ and G/G' is abelian, this will prove Huppert's Conjecture for this family of simple groups.

Let $q^2 = 3^f$. Suppose $G' \times C_G(G') \lneq G$. Then G induces on G' some outer automorphism α. By Lemma 1.7, some conjugacy class of G' is not fixed by α. As the irreducible characters of G' separate the conjugacy classes of G', there exists some $\chi \in \mathrm{Irr}(G')$ such that χ is not fixed by α. Let $\psi \in \mathrm{Irr}(G)$ such that $[\psi_{G'}, \chi] > 0$. Then $\psi(1) = e\chi(1)$ where $e > 1$ and e divides $|\mathrm{Out}(G')| = f$. Now $\chi(1) > 1$ and $e\chi(1) \in \mathrm{cd}(G)$. As $\chi \in \mathrm{Irr}(G')$ and $G' \cong {}^2G_2(q^2)$, we have that $\chi(1)$ and $e\chi(1)$ are both character degrees of G. Examining the character degrees of G to find degrees with proper multiples which are also degrees of G shows that

$$\chi(1) = (q^2 + 1 - q\sqrt{3})(q^2 + 1 + q\sqrt{3})$$

and

$$e = q^2 - 1, \ e = q^2, \ \text{or } e = q^2 + 1.$$

As $e \mid f$, we have

$$f \geq e \geq q^2 - 1 = 3^f - 1.$$

Hence $f \geq 3^f - 1$, a contradiction. Thus $G = G' \times C_G(G')$.

This concludes the verification of the five steps of Huppert's argument and proves Theorem 5.

CHAPTER 3

HUPPERT'S CONJECTURE AND THE FAMILY OF SIMPLE GROUPS $G_2(q)$, $q > 4$

In this chapter, we present our results concerning the verification of Huppert's Conjecture for the family of simple groups $G_2(q)$, for $q > 4$. We are able to verify four of the five steps in Huppert's argument. Our argument for Step 3 requires more information regarding the structure of the maximal subgroups of $G_2(q)$ than is currently available and thus remains incomplete. We again begin with some results concerning the character degrees of $G_2(q)$.

3.1 Results Concerning the Character Degrees of $G_2(q)$

We first consider the character degrees of $G_2(q)$ and establish results similar to those in the ${}^2G_2(q^2)$ case. As listed in [5], the character degrees of $G_2(q)$ are

$$1,\ \frac{1}{6}q(q^2-q+1)(q-1)^2,\ \frac{1}{6}q(q^2+q+1)(q+1)^2,$$

$$\frac{1}{3}q(q-1)^2(q+1)^2,\ \frac{1}{3}q(q^2+q+1)(q^2-q+1),\ \frac{1}{2}q(q^2+q+1)(q-1)^2,$$

$$\frac{1}{2}q(q^2-q+1)(q+1)^2,\ (q-1)(q^2+q+1)(q^2-q+1),\ (q+1)(q^2+q+1)(q^2-q+1),$$

$$(q^2+q+1)(q^2-q+1)(q-1)^2,\ (q^2-q+1)(q-1)^2(q+1)^2,\ q(q-1)(q^2+q+1)(q^2-q+1),$$

$$(q-1)(q+1)(q^2+q+1)(q^2-q+1),\ q^6,\ (q^2+q+1)(q-1)^2(q+1)^2,$$

$$q(q+1)(q^2+q+1)(q^2-q+1),\ \text{and}\ (q^2+q+1)(q^2-q+1)(q+1)^2.$$

In addition, there are the following degrees.

If $q \equiv 1 \pmod 6$ or $q \equiv 4 \pmod 6$:

$$(q+1)(q^2-q+1),\ q(q^2-q+1)(q+1)^2,\ \text{and}\ q^3(q+1)(q^2-q+1).$$

If $q \equiv 1 \pmod 6$, $q \equiv 3 \pmod 6$, or $q \equiv 5 \pmod 6$:

$$(q^2+q+1)(q^2-q+1),\ q(q^2+q+1)(q^2-q+1),\ \text{and } q^2(q^2+q+1)(q^2-q+1).$$

If $q \equiv 2 \pmod 6$ or $q \equiv 5 \pmod 6$:

$$(q-1)(q^2+q+1),\ q(q^2+q+1)(q-1)^2,\ \text{and } q^3(q-1)(q^2+q+1).$$

We establish some properties of the set of character degrees of $G_2(q)$ that will enable us to work toward the verification of Huppert's Conjecture for $G_2(q)$. We are first interested in determining which degrees of $G_2(q)$ are nontrivial powers. This reduces to the question of which degrees can be written as y^p, where p is a prime.

Lemma 3.1.1. *The number q^2+q+1 cannot be written in the form y^n for $n > 1$. The number q^2-q+1 is of the form y^n, for $n > 1$, only for $q = 19$.*

Proof. As Nagell showed in [27], $q^2+q+1 = y^p$ has no solutions unless p is 3. In that case, as proved in [23], the only solutions are $(q, y, p) = (18, 7, 3)$ and $(q, y, p) = (-19, 7, 3)$. As q is prime or a power of a prime, we see that neither of these cases is possible. Thus q^2+q+1 cannot be expressed as y^n for $n > 1$. As q^2-q+1 is obtained from q^2+q+1 by replacing q with $-q$, we see that the only solutions to $q^2-q+1 = y^p$ are $(q, y, p) = (-18, 7, 3)$ and $(19, 7, 3)$. As we are concerned with q prime or power of a prime, we see that $19^2-19+1 = 7^3$ is the only solution. $\square$

As shown in Lemma 3.1.1, q^2+q+1 is never a power and q^2-q+1 is a power only when $q = 19$. In addition, as shown in [27], q^2+q+1 and q^2-q+1 can never be written in the form $3y^n$ for any $n > 1$. As $\gcd(q+1, q^2+q+1) = \gcd(q-1, q^2-q+1) = 1$ while $\gcd(q-1, q^2+q+1)$ and $\gcd(q+1, q^2-q+1)$ are either 1 or 3, we see that the products $(q \pm 1)(q^2 \pm q+1)$ and $(q^2+q+1)(q^2-q+1)$ are never nontrivial powers. Now all the degrees of G except q^6 and $\frac{1}{3}q(q-1)^2(q+1)^2$ contain one of these products. As shown in [25], the only power of a prime among the degrees of $G_2(q)$ for $q \neq 3$ is q^6. Again, we will assume $q > 4$ as Huppert has verified the conjecture for $G_2(3)$ and $G_2(4)$ in [17]. We have the following lemma.

Lemma 3.1.2. *For $q > 4$, the only nontrivial powers among the degrees of $G_2(q)$ are q^6 and possibly $\frac{1}{3}q(q-1)^2(q+1)^2$. The only degree of the form p^b for prime p and $b \geq 1$ is q^6.*

We will also need to know which pairs of character degrees of $G_2(q)$ are consecutive integers. By examining the degrees of $G_2(q)$, it is possible to prove the following lemma.

Lemma 3.1.3. *The only pair of consecutive integers among the character degrees of $G_2(q)$, for $q > 2$, is $q^6 - 1$ and q^6.*

Excluding q^6 and 1 from consideration, the only pairs of relatively prime degrees of $G_2(q)$ are possibly

$$\begin{array}{rcll} (q+1)(q^2-q+1) & \text{and} & \dfrac{1}{2}q(q^2+q+1)(q-1)^2 & \text{for } q \equiv 4 \pmod 6; \\ (q^2+q+1)(q^2-q+1) & \text{and} & \dfrac{1}{3}q(q-1)^2(q+1)^2 & \text{for } q \equiv 3 \pmod 6; \\ (q-1)(q^2+q+1) & \text{and} & \dfrac{1}{2}q(q^2-q+1)(q+1)^2 & \text{for } q \equiv 2,5 \pmod 6. \end{array}$$

Hence, excluding 1 and q^6 from consideration, there is at most one pair of relatively prime character degrees of $G_2(q)$. We will frequently use the following well-known result found in [26] and [28].

Lemma 3.1.4. *The only divisors of $q^2 + q + 1$ are 3, but not 3^2, and numbers of the form $1 + 3m$. The only divisors of $q^2 - q + 1$ are 3, but not 3^2, and numbers of the form $1 + 6m$. Thus, the divisors of $q^2 - q + 1$ are 3, but not 3^2, and odd numbers congruent to $1 \pmod 3$.*

3.2 Establishing $G' = G''$ when $H \cong G_2(q)$

Suppose that $G' \neq G''$. Then there exists a solvable factor group G/N of G minimal with respect to being nonabelian. By Lemma 1.3, G/N is a p-group or a Frobenius group.

Case 1: G/N is a p-group for some prime p.

Either $p \mid q$ or $p \nmid q$.

Subcase 1(a): Suppose $p \mid q$.

Now $(q^2+q+1)(q-1)^2(q+1)^2$ is a character degree of G and $p \nmid (q^2+q+1)(q-1)^2(q+1)^2$. By Lemma 1.1, if $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = (q^2+q+1)(q-1)^2(q+1)^2$, then $\chi_N \in \mathrm{Irr}(N)$. As G/N is a nonabelian p-group, it must have a character degree $p^b > 1$. As the character degrees of G/N are character degrees of G and, by Lemma 3.1.2, the only degree of G of the form p^b, for p prime, is q^6, we have that $q^6 \in \mathrm{cd}(G/N)$. Let $\tau \in \mathrm{Irr}(G/N)$ with $\tau(1) = q^6$. Lemma 1.1 implies G has character degree

$$q^6(q^2+q+1)(q-1)^2(q+1)^2,$$

which is a contradiction.

Subcase 1(b): Suppose $p \nmid q$.

By Lemma 1.1, if $\chi \in \mathrm{Irr}(G)$, $\chi(1) = q^6$, then $\chi\sigma \in \mathrm{Irr}(G)$ for every $\sigma \in \mathrm{Irr}(G/N)$. Since G/N is nonabelian, there is some $\sigma \in \mathrm{Irr}(G/N)$ with $\sigma(1) > 1$. Hence we obtain a forbidden degree of G as no degrees of G are proper multiples of q^6.

Case 2: G/N is a Frobenius group with elementary abelian Frobenius kernel F/N, where $|F : N| = p^a$ for some prime p. In addition, $|G : F| \in \mathrm{cd}(G)$ and $|G : F|$ divides $p^a - 1$.

Subcase 2(a): No proper multiple of $|G : F|$ is a character degree of G.

Suppose that $p \nmid q$. Then as $p \nmid q^6$, Lemma 1.3 implies $q^6 \mid |G : F|$. As $|G : F| \in \mathrm{cd}(G)$, this implies $|G : F| = q^6$. If $p \nmid q-1$, then as $\gcd(q+1, q^2+q+1) = 1$, we have that either

$$p \nmid \frac{1}{3}q(q-1)^2(q+1)^2 \quad \text{or} \quad p \nmid \frac{1}{2}q(q^2+q+1)(q-1)^2.$$

Hence one of these degrees divides $|G : F| = q^6$, a contradiction.

We are left with the case where $p \nmid q$ but $p \mid q-1$. If $p = 2$, then as $p \nmid q$, q is odd. Hence $q \equiv 1 \pmod 6$, $q \equiv 3 \pmod 6$, or $q \equiv 5 \pmod 6$. But then $2 \nmid q^2+q+1$ and $2 \nmid q^2-q+1$. So $2 \nmid q^2(q^2+q+1)(q^2-q+1)$. Thus

$$q^2(q^2+q+1)(q^2-q+1)$$

divides $|G : F| = q^6$, a contradiction. If $p = 3$, then $3 \mid q-1$. Thus

$$\frac{1}{6}q(q^2+q+1)(q+1)^2$$

is not divisible by 3 since $3^2 \nmid q^2+q+1$. Hence

$$\frac{1}{6}q(q^2+q+1)(q+1)^2$$

divides $|G : F| = q^6$, a contradiction.

We have $p \nmid q$, $p \mid q-1$ and $p > 3$. Since $p \neq 2$, $p \nmid q+1$. Now $\gcd(q-1, q^2+q+1) = 1$ or 3, so as $p \neq 3$, then

$$p \nmid \frac{1}{6}q(q^2+q+1)(q+1)^2.$$

We obtain that $\frac{1}{6}q(q^2+q+1)(q+1)^2$ divides $|G : F| = q^6$, a contradiction. Hence $p \nmid q$ is not possible.

Suppose next that $p \mid q$. Then

$$p \nmid (q^2-q+1)(q-1)^2(q+1)^2 \text{ and } p \nmid (q+1)(q^2-q+1)(q^2+q+1),$$

so both of these degrees divide $|G : F|$, which is a character degree of G. Examining the character degree set of G, we see that this is not possible.

Subcase 2(b): A proper multiple of $|G : F|$ is a character degree of G.

Consulting the list of possibilities for $|G : F|$, we see that $\gcd(|G : F|, q^6) = 1$ or q. If $\chi \in \mathrm{Irr}(G)$, $\chi(1) = q^6$, then, by Lemma 1.4, χ_F has an irreducible constituent ψ such that q^5 divides $\psi(1)$ and $\psi(1) \mid q^6$. As $|G : F|\psi(1) \notin \mathrm{cd}(G)$, $p^a \mid \psi(1)^2$. Hence $p \mid q$. Since

$|G:F|$ divides p^a-1, $\gcd(|G:F|,p)=1$ and so $|G:F|$ is relatively prime to q. As we are assuming that a proper multiple of $|G:F|$ is a degree of G, $|G:F|$ is one of

$$(q-1)(q^2+q+1)(q^2-q+1),\ (q+1)(q^2+q+1)(q^2-q+1),$$

$$(q+1)(q^2-q+1),\ (q^2+q+1)(q^2-q+1),\ \text{or}\ (q-1)(q^2+q+1).$$

If $\tau\in\mathrm{Irr}(G)$ with $\tau(1)=(q^2-q+1)(q-1)^2(q+1)^2$, then

$$\begin{aligned}
\gcd\big(\tau(1),(q-1)(q^2+q+1)(q^2-q+1)\big)&\text{ divides } 3(q-1)(q^2-q+1);\\
\gcd\big(\tau(1),(q+1)(q^2+q+1)(q^2-q+1)\big)&\text{ divides } 3(q+1)(q^2-q+1);\\
\gcd\big(\tau(1),(q+1)(q^2-q+1)\big)&\text{ equals } (q+1)(q^2-q+1);\\
\gcd\big(\tau(1),(q^2+q+1)(q^2-q+1)\big)&\text{ divides } 3(q^2-q+1);\\
\gcd\big(\tau(1),(q-1)(q^2+q+1)\big)&\text{ divides } 3(q-1).
\end{aligned}$$

By Lemma 1.4, τ_F has an irreducible constituent ψ such that $\psi(1)$ is divisible by

$$\frac{1}{3}(q+1)^2(q-1),\ \frac{1}{3}(q-1)^2(q+1),\ (q-1)^2(q+1),$$

$$\frac{1}{3}(q-1)^2(q+1)^2,\ \text{or}\ \frac{1}{3}(q^2-q+1)(q-1)(q+1)^2,$$

respectively. As $p\nmid\psi(1)$, $|G:F|\psi(1)\in\mathrm{cd}(G)$, but this is not the case. Thus $G'=G''$.

3.3 Establishing $G'/M\cong H$ when $H\cong G_2(q)$

In this section, we will establish Step 2 for the family of simple groups $G_2(q)$ for $q>4$. Many of the arguments are similar to the ${}^2G_2(q^2)$ case, so we will refer to that case when appropriate.

3.3.1 Eliminating the Tits, Sporadic, and Alternating Groups when $k>1$

To eliminate the alternating groups when $k>1$, we need the following result, obtained by the construction outlined in [22].

Lemma 3.3.1. *If $n \geq 7$, then $\mathrm{Irr}(A_n)$ contains at least four nonlinear irreducible characters of different degrees which extend to $\mathrm{Aut}(A_n)$.*

To eliminate the sporadic groups, we need the following result, found by checking the Atlas [6]. Since the Tits group is in the Atlas [6], we will include it with the sporadic groups.

Lemma 3.3.2. *Let S be a sporadic simple group or the Tits group, and let A be the automorphism group of S. Then there exist at least five nonlinear irreducible characters of S of different degrees which extend to A.*

Proposition 3.3.1. *If S is an alternating group A_n with $n \geq 7$, a sporadic simple group, or the Tits group, then $k = 1$.*

Proof. Suppose that $k > 1$ and $S \cong A_n$ for some $n \geq 7$, a sporadic simple group, or the Tits group. Lemmas 3.3.1 and 3.3.2 imply that S has nonlinear characters of distinct degrees, ϕ_1, ϕ_2, and ϕ_3, say, which extend to $\mathrm{Aut}(S)$. By Lemma 2.3.1, ${\phi_1}^k$, ${\phi_2}^k$, and ${\phi_3}^k$ extend to G/M. As shown in Lemma 3.1.2, there are at most two nontrivial powers among the character degrees of G/M. Thus, if $S \cong A_n$ with $n \geq 7$, a sporadic simple group, or the Tits group, then $k = 1$. □

Note that $A_5 \cong \mathrm{PSL}_2(5)$ and $A_6 \cong \mathrm{PSL}_2(9)$ will be considered with the simple groups of classical Lie type. We have proved that if $S \cong A_n$ with $n \geq 7$, a sporadic simple group, or the Tits group, then $k = 1$. In this case, $G'/M \cong S$. We will now show that S cannot be one of these groups.

3.3.2 Eliminating Sporadic Simple Groups and the Tits Group when $k = 1$

Proposition 3.3.2. *The simple group S is not one of the sporadic simple groups or the Tits group.*

Proof. By the same reasoning as in the proof of Proposition 2.3.2, we only need to consider the following cases of sporadic simple groups with 22 or less extendible characters of distinct degrees.

Case 1: $S \cong M_{11}$, $S \cong M_{12}$, $S \cong M_{23}$, $S \cong M_{24}$, or $S \cong J_1$

For each of these sporadic simple groups, irreducible characters of consecutive degrees extend. The higher degree of these consecutive degrees is not a prime power. As stated in Lemma 3.1.3, the only consecutive degrees of G are q^6-1 and q^6. Hence this is not possible.

Case 2: $S \cong M_{22}$

The simple group M_{22} has irreducible characters of relatively prime degrees 45 and 154 which extend to $\mathrm{Aut}(M_{22})$. Examining the list of pairwise relatively prime degrees of G, we see that this is not possible.

Case 3: $S \cong J_2$ or $S \cong J_3$

The simple group J_2 has an irreducible character of degree $225 = 15^2$ which extends to $\mathrm{Aut}(J_2)$ while the simple group J_3 has an irreducible character of degree $324 = 18^2$ which extends to $\mathrm{Aut}(J_3)$. The only nontrivial powers among the degrees of G are q^6 and possibly $\frac{1}{3}q(q-1)^2(q+1)^2$ and we see that neither of these could possibly be 225 or 324.

Case 4: $S \cong HS$, $S \cong O'N$, or $S \cong McL$

The simple group HS has five pairs of irreducible characters of relatively prime degrees which extend to $\mathrm{Aut}(HS)$, while the simple group $O'N$ has three pairs of irreducible characters of relatively prime degrees which extend to $\mathrm{Aut}(O'N)$. The simple group McL has irreducible characters of relatively prime degrees 22 and 5103 as well as 3520 and 5103 which extend to $\mathrm{Aut}(McL)$. As G has at most one pair of relatively prime degrees, we see that it is not possible for $S \cong HS$, $S \cong O'N$, or $S \cong McL$.

Case 5: $S \cong He$

The simple group He has irreducible characters of relatively prime degrees 1275 and 6272 which extend to $\mathrm{Aut}(He)$. Examining the odd degrees of G relatively prime to another degree we see that it is not possible for 1275 to be a degree of G.

Case 6: $S \cong {}^2F_4(2)'$, the Tits group

The Tits group has irreducible characters of eight distinct nontrivial degrees which extend. The only power of a prime which extends is 27. The only character degree of G that is a power of a prime is q^6, so we have a contradiction. □

3.3.3 Eliminating A_n when $k = 1$

Proposition 3.3.3. *The simple group S is not an alternating group A_n with $n \geq 7$.*

Proof. Case 1: $S \cong A_7$, $S \cong A_8$, or $S \cong A_{10}$

As shown in [6], the simple groups A_7, A_8, and A_{10} have irreducible characters of consecutive integer degrees which extend to $\mathrm{Aut}(A_n)$, and hence to G, by Lemma 2.3.1. The higher degree of these consecutive degrees is not a prime power. As stated in Lemma 3.1.3, the only consecutive degrees of G are $q^6 - 1$ and q^6. Hence this is not possible.

Case 2: $S \cong A_{2m}$, $m \geq 6$ or $S \cong A_{2m+1}$, $m \geq 4$

By [2], A_{2m} has irreducible characters of the following degrees which extend to G:

$$\chi_1(1) = \frac{2m(2m-3)}{2} = m(2m-3) = 2m^2 - 3m$$
$$\chi_2(1) = \frac{(2m-1)(2m-2)}{2} = (m-1)(2m-1) = 2m^2 - 3m + 1$$

and A_{2m+1} has irreducible characters of the following degrees which extend to G:

$$\chi_1(1) = (m-1)(2m+1) = 2m^2 - m - 1$$

$$\chi_2(1) = m(2m-1) = 2m^2 - m.$$

In both cases, $\chi_1(1)$ and $\chi_2(1)$ are consecutive integers, with $\chi_2(1)$ larger. But $q^6 \neq (m-1)(2m-1)$, as q^6 is a power of a prime while $\gcd(m-1, 2m-1) = 1$. Similarly, $q^6 \neq m(2m-1)$, as q^6 is a power of a prime while $\gcd(m, 2m-1) = 1$. □

3.3.4 Eliminating the Groups of Lie Type when $k > 1$

Let χ denote the Steinberg character of S. By a similar argument to that presented in the introduction of Section 2.3.4, we must have that $\chi(1)^k = q^6$. Hence, the defining characteristic of the simple group S must be the same as the prime divisor of q^6.

Lemma 3.3.3. *If $S = S(q_1)$ is a simple group of Lie type, and $S \ncong \mathrm{PSL}_2(q_1)$, then S contains an irreducible character of mixed degree.*

Proof. Examining the list of unipotent characters for the simple groups of exceptional Lie type found in Section 13.9 of [4], we see that it is true for these groups. In [36] and [37], such irreducible characters are constructed for the simple groups of classical Lie type. □

Proposition 3.3.4. *If $S = S(q_1)$ is a simple group of Lie type and $S \ncong \mathrm{PSL}_2(q_1)$, then $k = 1$.*

Proof. Suppose $k \geq 2$. The Steinberg character χ of S extends to $\mathrm{Aut}(S)$ so $\chi(1)^k = q^6$. Let $\chi(1) = {q_1}^j$. Since $S \ncong \mathrm{PSL}_2(q_1)$, Lemma 3.3.3 implies that S contains an irreducible character of mixed degree, say τ. As $G'/M \cong S_1 \times \cdots \times S_k$, there is an irreducible character of G'/M found by multiplying $k-1$ copies of χ with τ. Then $(\chi^{k-1}\tau)(1)$ is a mixed degree of G'/M.

As the degrees of G'/M divide the degrees of G, we must have that the degree of this irreducible character divides one of the mixed degrees of G. The highest power of q on any mixed degree of G is 3. Now $q^6 = {q_1}^{jk}$ implies $q = {q_1}^{jk/6}$. The power of q_1 in $(\chi^{k-1}\tau)(1)$ is at least $j(k-1)+1$. We must have that

$$j(k-1)+1 \leq \frac{3jk}{6},$$

which reduces to $3j(k-2)+6 \leq 0$. Thus $k < 2$, a contradiction. Hence $k = 1$ if $S \not\cong \mathrm{PSL}_2(q_1)$. □

We must now eliminate the case when $k > 1$ and $S \cong \mathrm{PSL}_2(q_1)$.

Proposition 3.3.5. *If $S \cong \mathrm{PSL}_2(q_1)$ for $q_1 \geq 4$, then $k = 1$.*

Proof. Recall that S has the character degree $q_1 - 1$ and Steinberg character of degree q_1. Repeating the argument of Proposition 3.3.4, noting that $\chi(1) = q_1$, so $q^6 = {q_1}^k$ and choosing τ to be the irreducible character of S of degree $q_1 - 1$, implies that $k \leq 2$. Suppose that $k = 2$. Then ${q_1}^2 = q^6$ so $q_1 = q^3$. Hence $(q^3-1)^2$ divides a character degree of G. But examining the degrees of G, we see that this is not possible. Hence $k = 1$ in this case as well. □

Since the sporadic, Tits, and alternating groups have been eliminated as possibilities for S, we have that S is a simple group of Lie type and, thus, $k = 1$. We will now show that $S \cong G_2(q)$ by eliminating all other possibilities for S.

3.3.5 Eliminating Simple Groups of Exceptional Lie Type when $k = 1$

Proposition 3.3.6. *If $k = 1$ and S is a group of exceptional Lie type, then $S \cong G_2(q)$.*

Proof. We will examine each of the families of simple groups of exceptional Lie type individually. Again, all notation is adapted from [4].

Case 1: $S \cong G_2(q_1)$

The Steinberg character of $S \cong G_2(q_1)$ has degree ${q_1}^6$. As the only power of a prime among degrees of G is q^6, we must have that $q^6 = {q_1}^6$ which implies $q = q_1$. As the character degrees are equal, we have that S could possibly be $G_2(q)$.

Case 2: $S \cong {}^2B_2({q_1}^2)$, ${q_1}^2 = 2^{2m+1}$, $m \geq 1$

Now

$$\mathrm{cd}({}^2B_2({q_1}^2)) = \{1,\ {q_1}^4,\ {q_1}^4+1,\ ({q_1}^2-1)a,\ ({q_1}^2-1)b,\ ({q_1}^2-1)r\},$$

for

$${q_1}^2 = 2^{2m+1} \geq 8,\ r = \frac{1}{\sqrt{2}}q_1,\ a = {q_1}^2 + \frac{1}{\sqrt{2}}q_1 + 1,\ \text{and } b = {q_1}^2 - \frac{1}{\sqrt{2}}q_1 + 1.$$

As the Steinberg character of S has degree ${q_1}^4$, we have that ${q_1}^4 = q^6$. Then ${q_1}^4+1 = q^6+1$ must divide a degree of G. But this is not the case. Hence $S \ncong {}^2B_2({q_1}^2)$.

Case 3: $S \cong {}^2G_2({q_1}^2)$, ${q_1}^2 = 3^{2m+1}$, $m \geq 1$

Here, ${q_1}^6 = q^6$ so $q = q_1$. But q is an integer while q_1 is not. Thus $S \ncong {}^2G_2({q_1}^2)$.

Case 4: $S \cong {}^2F_4({q_1}^2)$, ${q_1}^2 = 2^{2m+1}$, $m \geq 1$

The Steinberg character of $S \cong {}^2F_4({q_1}^2)$ has degree ${q_1}^{24}$. As the only power of a prime among degrees of G is q^6, we must have that $q^6 = {q_1}^{24}$, which implies $q = {q_1}^4$. Consider the character ϵ'' of ${}^2F_4({q_1}^2)$ of degree

$${q_1}^{10}({q_1}^4 - {q_1}^2 + 1)({q_1}^8 - {q_1}^4 + 1).$$

This degree must divide a mixed degree of G. As $q = {q_1}^4$, this degree must divide $q^3(q^3 \pm 1)$. But $q^3 \pm 1 = {q_1}^{12} \pm 1$ and ${q_1}^8 - {q_1}^4 + 1$ does not divide ${q_1}^{12} \pm 1$. Thus $S \ncong {}^2F_4({q_1}^2)$.

Case 5: S is isomorphic to one of the remaining simple groups of exceptional Lie type.

For the remaining simple groups of exceptional Lie type, we will use the same argument as in Case 4 of Proposition 2.3.4. Each of the remaining possibilities for S has a mixed degree whose power on q_1 is greater than $3j/6$. As the mixed degrees of G have power at most $3j/6$ on q_1, we have a contradiction. Table 2.1 on page 23 exhibits the degree of the Steinberg character of S and a character of S of appropriate degree which will result in a contradiction. □

3.3.6 Eliminating the Groups of Classical Lie Type when $k = 1$

We are only left with the possibility that $G'/M \cong S$, where S is a simple group of classical Lie type. Once again, we must consider each family of simple groups of classical Lie type separately, but we use the same general argument in each case.

Proof. We will proceed by examining each group separately. We begin with the groups of type A_ℓ.

Groups of Type A_ℓ

Case 1(a): $\ell = 1$, i.e., $S \cong \mathrm{PSL}_2(q_1)$ for $q_1 \geq 4$

The Steinberg character of S extends to $\mathrm{Aut}(S)$, so we must have that $q^6 = q_1$. For $q_1 = 4$ or $q_1 = 5$, certainly $q_1 = q^6$ is not possible. For $q_1 > 5$, consider the character of S of degree $q_1 + 1 = q^6 + 1$. This degree must divide a degree of G. Examining the degrees of G, it is clear that $q^6 + 1$ does not divide any of them. Note that this eliminates the possibility that $S \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong A_5$ and $S \cong A_6 \cong \mathrm{PSL}_2(9)$.

Case 1(b): $\ell = 2$, i.e., $S \cong \mathrm{PSL}_3(q_1)$

The degree of the Steinberg character of $\mathrm{PSL}_3(q_1)$ is ${q_1}^3$. Hence $q_1 = q^2$. But the mixed degrees of S are $q^2(q^2+1)$ and $q^2(q^4+q^2+1)$. These must divide $q^3(q^3 \pm 1)$ or $q^2(q^4+q^2+1)$.

Certainly q^2+1 does not divide $q^3 \pm 1$, and q^2+1 is relatively prime to q^4+q^2+1 so it does not divide $q^2(q^4+q^2+1)$ either.

Case 1(c): $\ell = 3$, i.e., $S \cong \mathrm{PSL}_4(q_1)$

When $\ell = 3$, the Steinberg character of S has degree ${q_1}^6$, so $q = q_1$. As shown in [36], S has a character degree which is divisible by $q(q^3-1)(q^2+1)$. This degree of S must divide a mixed degree of G. As q^2+1 is relatively prime to both q^2+q+1 and q^2-q+1, we see that this degree divides no mixed degrees of G.

Case 1(d): $\ell \geq 4$, i.e., $S \cong \mathrm{PSL}_{\ell+1}(q_1)$

In this general case, the degree of the Steinberg character of S is ${q_1}^{\ell(\ell+1)/2}$. Thus $q = {q_1}^{\ell(\ell+1)/12}$. Consider the mixed degrees of $G_2(q)$ whose factor relatively prime to q is of degree four or less as a polynomial in q. The largest of these degrees is $q(q^2+q+1)(q+1)^2 = q(q^4+3q^3+4q^2+3q+1)$. So the part of the mixed degree of S that is relatively prime to q_1 must be less than or equal to $q^4+3q^3+4q^2+3q+1$ to divide one of these degrees. We consider these "smaller" degrees first to eliminate them from consideration. We will eliminate the larger mixed character degrees separately. Replacing q by the appropriate power of q_1 gives

$${q_1}^{4\ell(\ell+1)/12} + 3{q_1}^{3\ell(\ell+1)/12} + 4{q_1}^{2\ell(\ell+1)/12} + 3{q_1}^{\ell(\ell+1)/12} + 1.$$

By the same bound argument as presented in Case 1(d) of Proposition 2.3.5, we must have that

$$\frac{\ell(\ell+1)}{2} - \ell + 1 \leq \frac{4\ell(\ell+1)}{12},$$

which reduces to $\ell^2 - 5\ell + 6 \leq 0$ and is not satisfied for $\ell > 4$. Thus, if $\ell > 4$, this degree must divide

$$q(q+1)(q^2+q+1)(q^2-q+1) \quad \text{or} \quad q(q-1)(q^2+q+1)(q^2-q+1).$$

As $(q+1)(q^2+q+1)(q^2-q+1)$ and $(q-1)(q^2+q+1)(q^2-q+1)$ are both factors of q^6-1, if the degree divides one of these factors, it must divide q^6-1.

Now $q^6-1 = {q_1}^{\ell(\ell+1)/2}-1$. If ℓ is even, $({q_1}^{\ell}-1) \nmid ({q_1}^{\ell(\ell+1)/2}-1)$ since $\ell \nmid \frac{\ell(\ell+1)}{2}$. Similarly, if ℓ is odd, $({q_1}^{\ell+1}-1) \nmid ({q_1}^{\ell(\ell+1)/2}-1)$ since $(\ell+1) \nmid \frac{\ell(\ell+1)}{2}$.

This is a common argument that we will use to eliminate the other simple groups of classical Lie type. We will find a large mixed degree of S which divides a degree of G. We will compare the size of this degree to the largest mixed degree of G whose factor relatively prime to q is of degree four or less as a polynomial in q. The degree of S is too large to divide this mixed degree and any smaller mixed degrees of G. This implies that the mixed degree of S must divide

$$q(q+1)(q^2+q+1)(q^2-q+1) \quad \text{or} \quad q(q-1)(q^2+q+1)(q^2-q+1).$$

But then this mixed degree of S must divide q^6-1. We will show that this is also not possible. Hence no degree of G is divisible by this degree of S, which is a contradiction.

Groups of Type $^2A_\ell$

We will proceed in the same manner as in the linear group case. We will consider the groups of small rank separately.

Case 2(a): $\ell = 2$, i.e., $S \cong \mathrm{PSU}_3({q_1}^2)$

The degree of the Steinberg character of $\mathrm{PSU}_3({q_1}^2)$ is ${q_1}^3$. Hence $q_1 = q^2$. The mixed degrees of S are $q^2(q^2-1)$ and $q^2(q^4-q^2+1)$. These must divide either $q^2(q^4+q^2+1)$ or $q^3(q^3\pm 1)$. Now q^4-q^2+1 is relatively prime to q^4+q^2+1 and does not divide $q^3\pm 1$.

Case 2(b): $\ell = 3$, i.e., $S \cong \mathrm{PSU}_4({q_1}^2)$

The Steinberg character of S is of degree ${q_1}^6$, implying $q = q_1$. As shown in [36], there is a character degree of S that is divisible by

$$\frac{\chi_3(1)}{q+1} = q(q-1)(q^2+1).$$

So $q(q-1)(q^2+1)$ must divide a mixed degree of G. Examining the mixed degrees of G and noting that q^2+1 is relatively prime to both q^2+q+1 and q^2-q+1 and the greatest common divisor of q^2+1 and $q \pm 1$ divides 2, we see that this is not possible.

Case 2(c): $\ell = 4$ i.e., $S \cong \mathrm{PSU}_5({q_1}^2)$

The Steinberg character of S is of degree ${q_1}^{10}$, so $q = {q_1}^{10/6}$. As shown in Section 13.8 of [4], we have that the unipotent character of degree

$$\chi^{(1,1,3)}(1) = {q_1}^3({q_1}^2 - q_1 + 1)({q_1}^2 + 1)$$

is an irreducible character of $\mathrm{PSU}_5({q_1}^2)$. But its degree is mixed, with a factor of ${q_1}^3$. Thus it must divide one of the degrees $q^2(q^4+q^2+1)$ or $q^3(q^3 \pm 1)$. As $q = {q_1}^{5/3}$, we have that $q^3 \pm 1 = {q_1}^5 \pm 1$. Certainly ${q_1}^2+1$ does not divide ${q_1}^5 \pm 1$.

If $q_1 \equiv \pm 1 \pmod 3$, then ${q_1}^2 + 1 \equiv -1 \pmod 3$ and so cannot be a divisor of q^2+q+1 or q^2-q+1 by Lemma 3.1.4. If q_1 is a power of 3, then ${q_1}^2+1$ is even, while q^4+q^2+1 is odd. So it does not divide q^4+q^2+1 in this case either. This argument will be used repeatedly in the following cases.

Case 2(d): $\ell = 5$, i.e., $S \cong \mathrm{PSU}_6({q_1}^2)$,

When $\ell = 5$, the Steinberg character of S has degree ${q_1}^{15}$ so $q = {q_1}^{15/6}$. From Section 13.8 of [4], we see that S has unipotent character $\chi^{(1,2,3)}$ of degree

$$\chi^{(1,2,3)}(1) = {q_1}^4({q_1}^3 - 1)(q_1 - 1)^2({q_1}^2 + 1).$$

This is a mixed degree of S, hence must divide a mixed degree of G. But this degree is mixed, with a factor of ${q_1}^4$. Thus it must divide one of the degrees $q^2(q^4+q^2+1)$ or $q^3(q^3 \pm 1)$.

If $q_1 \equiv \pm 1 \pmod 3$, then ${q_1}^2 + 1 \equiv -1 \pmod 3$ and so cannot be a divisor of q^2+q+1 or q^2-q+1. If q_1 is a power of 3, then ${q_1}^2+1$ is even, while q^4+q^2+1 is odd. So it does not divide q^4+q^2+1.

So this degree of S must divide $q^3(q^3 \pm 1)$. As $q = {q_1}^{5/2}$, we must have that $q_1 = p^{2r}$ for some prime p and positive integer $r \geq 1$. Then $q^3 \pm 1 = p^{15r} \pm 1$, ${q_1}^2+1 = p^{4r}+1$, and ${q_1}^3 - 1 = p^{6r}-1$. Now $p^{6r}-1 \nmid p^{15r}-1$ since $6r \nmid 15r$. Thus we are left with the case where this degree divides $p^{15r}+1$. But $p^{4r}+1 \nmid p^{15r}+1$. Hence this case is eliminated as well.

Case 2(e): $\ell \geq 6$, i.e., $S \cong \mathrm{PSU}_{\ell+1}({q_1}^2)$

In this general case, the degree of the Steinberg character of S is ${q_1}^{\ell(\ell+1)/2}$. Thus $q = {q_1}^{\ell(\ell+1)/12}$. By the same bound argument as presented in Case 2(e) of Proposition 2.3.5, we must have that

$$\frac{2\ell^2-10}{4} \leq \frac{4\ell(\ell+1)}{12},$$

which reduces to $\ell^2 - 2\ell - 15 \leq 0$ and is not satisfied for $\ell > 5$. Since $\ell \geq 6$, this degree must divide q^6-1. Now $q^6 - 1 = {q_1}^{\ell(\ell+1)/2} - 1$. If ℓ is even, $({q_1}^{\ell}-1) \nmid ({q_1}^{\ell(\ell+1)/2}-1)$ since $\ell \nmid \frac{\ell(\ell+1)}{2}$. Similarly, if ℓ is odd, $({q_1}^{\ell+1}-1) \nmid ({q_1}^{\ell(\ell+1)/2}-1)$ since $(\ell+1) \nmid \frac{\ell(\ell+1)}{2}$.

Groups of Type B_ℓ

We will examine the groups of low rank first and then show that S cannot be of type B_ℓ in general in Case 3(b).

Case 3(a): $\ell = 2$, i.e., $S \cong \Omega_5(q_1) \cong \mathrm{PSp}_4(q_1)$

The Steinberg character of S has degree ${q_1}^4$. Thus, $q = {q_1}^{2/3}$. As shown in [37], S has degrees $q_1(q_1-1)({q_1}^2+1)$ and $q_1(q_1+1)({q_1}^2+1)$. As $q = {q_1}^{2/3}$, these can only divide the degrees $q^2(q^4+q^2+1)$ or $q^3(q^3 \pm 1)$ of G. Now $q^3 \pm 1 = {q_1}^2 \pm 1$ so it is clear that these degrees of S must divide

$$q^2(q^4+q^2+1) = q^2(q^2+q+1)(q^2-q+1).$$

If $q_1 \equiv \pm 1 \pmod 3$, then ${q_1}^2+1 \equiv -1 \pmod 3$ and so cannot be a divisor of q^2+q+1 or q^2-q+1. If $q_1 \equiv 0 \pmod 3$, then ${q_1}^2+1$ is even, while $(q^2+q+1)(q^2-q+1)$ is odd. So ${q_1}^2+1$ does not divide $(q^2+q+1)(q^2-q+1)$ in this case either.

Case 3(b): $\ell \geq 3$, i.e., $S \cong \Omega_{2\ell+1}(q_1)$

In this general case, the degree of the Steinberg character of S is ${q_1}^{\ell^2}$. Thus $q = {q_1}^{\ell^2/6}$. By the same bound argument as presented in Case 3(b) of Proposition 2.3.5, we must have that

$$\ell^2 - \ell + 1 \leq \frac{4\ell^2}{6},$$

which reduces to $\ell^2 - 3\ell + 3 \leq 0$. This inequality is not satisfied for $\ell \geq 3$.

Thus, if $\ell \geq 3$, the degree must divide q^6-1. Now $q^6 - 1 = {q_1}^{\ell^2} - 1$. But ${q_1}^{\ell-1}-1$ divides this degree of S and does not divide ${q_1}^{\ell^2}-1$ since $(\ell-1) \nmid \ell^2$.

Groups of Type C_ℓ

Note that if q_1 is even, $C_\ell(q_1) \cong B_\ell(q_1)$, and we are done by the cases in the previous subsection. Thus we may assume that q_1 is odd.

Case 4(a): $\ell = 2$, i.e., $S \cong \mathrm{PSp}_4(q_1)$

As $C_2(q_1) \cong B_2(q_1)$, we are done by Case 3.

Case 4(b): $\ell = 3$ and q_1 odd, i.e., $S \cong \mathrm{PSp}_6(q_1)$

In this case, the Steinberg character has degree ${q_1}^9$, so $q = {q_1}^{3/2}$. From Section 13.8 of [4], we see that S has unipotent character χ^α of degree

$$\chi^\alpha(1) = {q_1}^2(q_1-1)(q_1+1)({q_1}^3+1)({q_1}^2+1)$$

corresponding to the symbol

$$\alpha = \begin{pmatrix} 1 & 3 \\ \\ 0 & 1 \end{pmatrix}.$$

This is a mixed degree of S, hence must divide a mixed degree of G. Thus it must divide

$$q^2(q^2+q+1)(q^2-q+1) \quad \text{or} \quad q^3(q^3 \pm 1).$$

If this degree divides $q^3 \pm 1$, then it divides $(q^3-1)(q^3+1) = q^6 - 1 = {q_1}^9 - 1$. But ${q_1}^2+1$ does not divide ${q_1}^9 - 1$. If $q_1 \equiv \pm 1 \pmod 3$, then ${q_1}^2 + 1 \equiv -1 \pmod 3$ and so cannot be a divisor of q^2+q+1 or q^2-q+1. If $q_1 \equiv 0 \pmod 3$, then ${q_1}^2+1$ is even, while q^4+q^2+1 is odd. So ${q_1}^2+1$ does not divide q^4+q^2+1 either.

Case 4(c): $\ell \geq 4$ and q_1 odd, i.e., $S \cong \mathrm{PSp}_{2\ell}(q_1)$

In this general case, the degree of the Steinberg character of S is ${q_1}^{\ell^2}$. Thus $q = {q_1}^{\ell^2/6}$. By the same bound argument as presented in Case 4(c) of Proposition 2.3.5, we must have that

$$\ell^2 - \ell \leq \frac{4\ell^2}{6},$$

which reduces to $\ell^2 - 3\ell \leq 0$. This inequality is not satisfied for $\ell \geq 4$.

Thus, if $\ell \geq 4$, this degree must divide $q^6 - 1$. Now $q^6 - 1 = {q_1}^{\ell^2} - 1$. But ${q_1}^{\ell-1} - 1$ divides this degree of S and does not divide ${q_1}^{\ell^2} - 1$ since $(\ell - 1) \nmid \ell^2$.

Groups of Type D_ℓ

Case 5(a): $\ell = 4$, i.e., $S \cong \mathrm{P\Omega_8}^+(q_1)$

First suppose $q_1 = 2$. Then we are considering the group $\mathrm{O_8}^+(2)$ in Atlas [6] notation. The Steinberg character of S has degree 2^{12}, so $q = 2^2 = 4$. But we are assuming that $q > 4$.

Next suppose $q_1 = 3$. We are considering the group $\mathrm{O_8}^+(3)$. The Steinberg character of S has degree 3^{12}, so $q = 3^2$. But ${\rm O_8}^+(3)$ has a character degree $\chi_{113}(1) = 716800$, which is larger than any of the degrees of $G_2(9)$.

Suppose $q_1 > 3$. The Steinberg character of S has degree ${q_1}^{12}$ so $q = {q_1}^2$. From [37], we see that S has an irreducible character χ of degree

$$\chi(1) = \frac{1}{2}{q_1}^3(q_1+1)^4({q_1}^2 - q_1 + 1).$$

This is a mixed degree of S, hence must divide a mixed degree of G. But the power of q_1 on this degree of S implies it must divide $q^2(q^4+q^2+1)$ or $q^3(q^3 \pm 1)$. If it divides one of the latter, then it divides

$$(q^3-1)(q^3+1) = q^6 - 1 = {q_1}^{12} - 1.$$

But $(q_1+1)^4$ does not divide ${q_1}^{12} - 1$. Now $q^4+q^2+1 = {q_1}^8 + {q_1}^4 + 1$ and it is clear that $(q_1+1)^4$ does not divide this term either. So it is not possible for this degree of S to divide a degree of G.

Case 5(b): $5 \le \ell \le 7$, i.e., $S \cong {\rm P\Omega_{2\ell}}^+(q_1)$

For $5 \le \ell \le 7$, we will use the same general argument. Suppose the Steinberg character of S has degree ${q_1}^j$. By Lemma 2.3.1, $q^6 = {q_1}^j$, so $q = {q_1}^{j/6}$. For $5 \le \ell \le 7$, there is a mixed degree of $S \cong {\rm P\Omega_{2l}}^+(q_1)$ whose power on q_1 implies it must divide

$$q^2(q^4+q^2+1) \quad \text{or} \quad q^3(q^3 \pm 1).$$

The degree of S can be chosen to be too large to divide $q^3(q^3 \pm 1)$ and to have divisor ${q_1}^2+1$ or ${q_1}^4+1$. If $q_1 \equiv \pm 1 \pmod 3$, then ${q_1}^2+1 \equiv {q_1}^4+1 \equiv -1 \pmod 3$ and so cannot be a divisor of q^2+q+1 or q^2-q+1. If $q_1 \equiv 0 \pmod 3$, then ${q_1}^2+1$ and ${q_1}^4+1$ are even, while q^4+q^2+1 is odd. So it does not divide q^4+q^2+1 in this case either. Table 2.3 on page 34 exhibits the degree of the Steinberg character of S and a character of S of appropriate degree which will result in a contradiction.

Case 5(c): $\ell \geq 8$, i.e., $S \cong \mathrm{P}\Omega_{2\ell}{}^{+}(q_1)$

The degree of the Steinberg character of S is ${q_1}^{\ell(\ell-1)}$, so $q = {q_1}^{\ell(\ell-1)/6}$. By the same bound argument as presented in Case 5(c) of Proposition 2.3.5, we must have that

$$(\ell-1)^2 - 2 - \ell \leq \frac{4\ell(\ell-1)}{6},$$

which reduces to $\ell^2 - 7\ell - 3 \leq 0$ and is not the case for $\ell \geq 8$.

Thus, if $\ell \geq 8$, this degree must divide $q^6 - 1$. Now $q^6 - 1 = {q_1}^{\ell(\ell-1)} - 1$. But ${q_1}^{\ell-2} - 1$ divides this degree of S and does not divide ${q_1}^{\ell(\ell-1)} - 1$ since $(\ell-2) \nmid \ell(\ell-1)$.

Groups of Type ${}^2D_\ell$

Case 6(a): $\ell = 4$, i.e., $S \cong \mathrm{P}\Omega_8{}^{-}(q_1)$

This is the group $\mathrm{O}_8{}^{-}(q_1)$ in Atlas notation. As shown in [37],

$$\chi_1(1) = {q_1}^2(q_1 - 1)(q_1 + 1)({q_1}^2 + q_1 + 1)({q_1}^2 - q_1 + 1)({q_1}^4 + 1)$$

is a degree of $\mathrm{P}(\mathrm{CO}_8^{-}(q_1)^0)$. Again, appealing to Lemma 1.4, as

$$[\mathrm{P}(\mathrm{CO}_8^{-}(q_1)^0) : \mathrm{P}\Omega_8{}^{-}(q_1)] = \gcd(4, {q_1}^4 + 1),$$

if q_1 is even, then $\chi_1(1)$ divides a degree of S. If q_1 is odd, then $\chi_1(1)/2$ or $\chi_1(1)/4$ divides a degree of S. This degree is mixed, so must divide a mixed degree of G. The Steinberg character of S has degree ${q_1}^{12}$, so $q = {q_1}^2$. Examining the possibilities shows that this degree cannot divide any of the mixed degrees of G.

Case 6(b): $\ell = 5$, i.e., $S \cong \mathrm{P}\Omega_{10}{}^{-}(q_1)$

This case can be eliminated by a similar argument to the one presented in Case 6(b) of Proposition 2.3.5.

Case 6(c): $\ell \geq 6$, i.e., $S \cong \mathrm{P}\Omega_{2\ell}{}^{-}(q_1)$

The degree of the Steinberg character of S is ${q_1}^{\ell(\ell-1)}$, so $q = {q_1}^{\ell(\ell-1)/6}$. By the same bound argument as presented in Case 6(c) of Proposition 2.3.5, we must have that

$$\ell^2 - 2\ell + 1 \leq \frac{4\ell(\ell-1)}{6},$$

which reduces to $\ell^2 - 4\ell - 3 \leq 0$, and is not satisfied for $\ell \geq 6$.

Thus, if $\ell \geq 6$, this degree must divide $q^6 - 1$. Now $q^6 - 1 = {q_1}^{\ell(\ell-1)} - 1$. But ${q_1}^{\ell-2} - 1$ divides this degree of S and does not divide ${q_1}^{\ell(\ell-1)} - 1$, since $(\ell - 2) \nmid \ell(\ell-1)$. $\square$

This was the last case to consider to prove that $k = 1$ and $S \cong G_2(q)$.

3.4 Progress Toward Proving $I_{G'}(\theta) = G'$ when $H \cong G_2(q)$

Again, let $\theta \in \mathrm{Irr}(M)$ with $\theta(1) = 1$. If θ is not stable under G', then $I_{G'}(\theta) \lneq G'$. Thus $I_{G'}(\theta)$ is contained in a maximal subgroup of G' and the index of $I_{G'}(\theta)$ in G' divides a degree of G. Hence the index of the maximal subgroup in G' must also divide a degree of G. We wish to show that no such maximal subgroup exists. More formally, suppose $I_{G'}(\theta) = I \lneq G'$ for some $\theta \in \mathrm{Irr}(M)$. Let U be maximal such that $I \leq U \lneq G'$. If

$$\theta^I = \sum \phi_i,$$

for $\phi_i \in \mathrm{Irr}(I)$, then by Lemma 1.2, $\phi_i(1)|G' : I|$ is a degree of G' and thus divides some degree of G.

We will need to find indices of maximal subgroups of $G_2(q)$ which divide character degrees of $G_2(q)$. From a list of maximal subgroups of $G_2(q)$ in [7] and [21], we have Table 3.1 for odd q and Table 3.2 for even q. Let $q = p^f$. Recall that the notation is as follows: $[q^5]$ denotes an unspecified group of order q^5, $A : B$ denotes a split extension, $A \circ B$ denotes a central product, and $A \cdot B$ denotes a non-split extension.

Maximal Subgroup Structure	Order	Index
$[q^5]:\mathrm{GL}_2(q)$	$q^6(q-1)(q^2-1)$	$(q+1)(q^2-q+1)(q^2+q+1)$
$(\mathrm{SL}_2(q)\circ\mathrm{SL}_2(q))\cdot 2$	$\dfrac{2q^2(q^2-1)^2}{2(q-1)^2}$	$q^4(q-1)(q^3-1)(q^2-q+1)$
$2^3\cdot\mathrm{PSL}_3(2)$ (when q is prime)	$2^3\cdot\dfrac{2^3(2^3-1)(2^2-1)}{2}$	$\dfrac{q^6(q^6-1)(q^2-1)}{2^5(2^3-1)(2^2-1)}$
$\mathrm{SL}_3(q):2$	$2q^3(q^3-1)(q^2-1)$	$\frac{1}{2}q^3(q^3+1)$
$\mathrm{SU}_3(q):2$	$2q^3(q^3+1)(q^2-1)$	$\frac{1}{2}q^3(q^3-1)$
$G_2(q_0)$, $q={q_0}^\alpha$, α prime	${q_0}^6({q_0}^6-1)({q_0}^2-1)$	$\dfrac{{q_0}^{6\alpha}({q_0}^{6\alpha}-1)({q_0}^{2\alpha}-1)}{{q_0}^6({q_0}^6-1)({q_0}^2-1)}$
${}^2G_2(q^2)$, $q^2=3^{2m+1}$	$q^6(q^6+1)(q^2-1)$	$q^6(q^2-1)(q^2+1)(q^4+q^2+1)$
$\mathrm{PGL}_2(q)$, $p\geq 7$, $q\geq 11$	$q(q^2-1)$	$q^5(q^6-1)$
$\mathrm{PSL}_2(8)$, $p\geq 5$	$8\cdot(8^2-1)$	$\dfrac{q^6(q^6-1)(q^2-1)}{2^3\cdot 3^2\cdot 7}$
$\mathrm{PSL}_2(13)$, $p\neq 13$	$\frac{1}{2}(13)(13^2-1)$	$\dfrac{q^6(q^6-1)(q^2-1)}{2^2\cdot 3\cdot 7\cdot 13}$
$G_2(2)$, $q\geq 5$ is prime	$2^6(2^6-1)(2^2-1)$	$\dfrac{q^6(q^6-1)(q^2-1)}{2^6\cdot 3^3\cdot 7}$
J_1, $q=11$	$2^3\cdot 3\cdot 5\cdot 7\cdot 11\cdot 19$	$11^5(2^3\cdot 3^2\cdot 5\cdot 37)$

Table 3.1: Maximal Subgroups of $G_2(q)$, q Odd

Lemma 3.4.1. *The only maximal subgroups of $G_2(q)$, for $q > 4$, whose indices divide degrees of G are the parabolic subgroups with structure $[q^5] : \mathrm{GL}_2(q)$ and the subgroups with structure $\mathrm{SL}_3(q) : 2$ and $\mathrm{SU}_3(q) : 2$.*

Proof. We begin by examining the case when q is odd. The index of the parabolic subgroups with structure $[q^5] : \mathrm{GL}_2(q)$ divides the degrees

$$(q+1)(q^2+q+1)(q^2-q+1),\ (q-1)(q+1)(q^2+q+1)(q^2-q+1),$$

$$q(q+1)(q^2+q+1)(q^2-q+1),\ \text{and}\ (q+1)^2(q^2-q+1)(q^2+q+1).$$

The index of the maximal subgroups with structure $(\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)) \cdot 2$ does not divide degrees of G as the power on q in the index of the maximal subgroup is too large. The same reasoning eliminates the maximal subgroups with structure $2^3 \cdot \mathrm{PSL}_3(2)$, as q is odd.

The index of the maximal subgroups with structure $\mathrm{SL}_3(q) : 2$ divides the degree $q^3(q^3+1)$ if $q \equiv 1 \pmod 6$. The index of the maximal subgroups with structure $\mathrm{SU}_3(q) : 2$ divides the degree $q^3(q^3-1)$ if $q \equiv 5 \pmod 6$.

Next consider the indices of maximal subgroups with structure $G_2(q_0)$. Here $\alpha > 1$, so this index must divide a mixed degree of G. Now $6\alpha - 6 > 3\alpha$ when $\alpha > 2$. So this index will not divide a degree of G when $\alpha > 2$. If $\alpha = 2$, we have that the index of the subgroup is ${q_0}^6({q_0}^6+1)({q_0}^2+1)$ and this does not divide ${q_0}^6({q_0}^6 \pm 1)$.

The index of the maximal subgroups with structure ${}^2G_2(q^2)$ is too large to divide the mixed degrees $q^3(q^3+1)$ and $q^3(q^3-1)$ and the power of q is too large to divide the other mixed degrees of G. The index of the maximal subgroups with structure $\mathrm{PGL}_2(q)$ does not divide degrees of G as the power on q in the index of the maximal subgroup is too large. The index of the maximal subgroups with structure $\mathrm{PSL}_2(8)$ and $\mathrm{PSL}_2(13)$ must divide mixed degrees of G. As q is odd, we see that the power on q in the index is too large to allow the index to divide a mixed degree of G.

Maximal Subgroup Structure	Order	Index
$[q^5] : \mathrm{GL}_2(q)$	$q^6(q-1)(q^2-1)$	$(q+1)(q^2-q+1)(q^2+q+1)$
$\mathrm{SL}_3(q) : 2$	$2q^3(q^3-1)(q^2-1)$	$\frac{1}{2}q^3(q^3+1)$
$\mathrm{SU}_3(q) : 2$	$2q^3(q^3+1)(q^2-1)$	$\frac{1}{2}q^3(q^3-1)$
$\mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$	$q^2(q^2-1)^2$	$q^4(q^2-q+1)(q^2+q+1)$
$G_2(2^m)$, f/m prime	$2^{6m}(2^{6m}-1)(2^{2m}-1)$	$\dfrac{2^{6f}(2^{6f}-1)(2^{2f}-1)}{2^{6m}(2^{6m}-1)(2^{2m}-1)}$

Table 3.2: Maximal Subgroups of $G_2(q)$, q Even

If the maximal subgroup of $G_2(q)$ has structure $G_2(2)$, then the index must divide a mixed degree of G. As q is odd, the index will not divide a mixed degree of G if $q \neq 3$ and $q \neq 7$. If $q = 7$, then the power on q in the index is 5 and thus too large to divide a mixed degree of G. If $q = 3$, the mixed degrees of G have power 2 or less on q, so this does not divide a degree of G.

Finally, if the maximal subgroup of $G_2(q)$ has structure of J_1, then the power on 11 is too large to divide a mixed degree of G.

We will now consider the maximal subgroups of $G_2(q)$ for even q. These were determined in [7]. The maximal subgroup structure is summarized in Table 3.2.

The index of the parabolic subgroups with structure $[q^5] : \mathrm{GL}_2(q)$ divides the degrees

$$(q+1)(q^2+q+1)(q^2-q+1),\ (q-1)(q+1)(q^2+q+1)(q^2-q+1),$$

$$q(q+1)(q^2+q+1)(q^2-q+1),\ \text{and}\ (q+1)^2(q^2-q+1)(q^2+q+1).$$

The index of the maximal subgroups with structure $\mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$ does not divide degrees of G as the power on q in the index of the maximal subgroup is too large.

The index of the maximal subgroups with structure $\mathrm{SL}_3(q) : 2$ divides the degree $q^3(q^3 + 1)$ if $q \equiv 4 \pmod 6$. The index of the maximal subgroups with structure $\mathrm{SU}_3(q) : 2$ divides the degree $q^3(q^3-1)$ if $q \equiv 2 \pmod 6$.

Next consider the index of maximal subgroups with structure $G_2(2^m)$. Here $f/m > 1$, so this index must divide a mixed degree of G. Now $6f - 6m > 3f$ when $f/m > 2$. So this index will not divide a degree of G when $f/m > 2$. If $f/m = 2$, then $f = 2m$ and the index of the subgroup is $2^{6m}(2^{6m}+1)(2^{2m}+1)$. Now $q = 2^f = 2^{2m}$ and so the only degrees of G this index could divide are $q^3(q^3 \pm 1) = 2^{6m}(2^{6m} \pm 1)$ and it is clearly too large to do so.

Thus, the only maximal subgroups of $G_2(q)$, for $q > 4$, whose indices divide degrees of G are the parabolic subgroups and the subgroups with structure $\mathrm{SL}_3(q) : 2$ and $\mathrm{SU}_3(q) : 2$. $\square$

3.4.1 Maximal Subgroups with Structure $\mathrm{SL}_3(q) : 2$ or $\mathrm{SU}_3(\mathrm{q}) : 2$

First, consider the maximal subgroups with structure $\mathrm{SL}_3(q) : 2$ or $\mathrm{SU}_3(q) : 2$, whose indices divide the degree $q^3(q^3+1)$ or $q^3(q^3-1)$ respectively. We have

$$\frac{1}{2}q^3(q^3 \pm 1)|U/M : I/M|\phi_i(1) \text{ divides } q^3(q^3 \pm 1).$$

Thus $|U/M : I/M|\phi_i(1)$ divides 2. There are three cases to consider.

Case 1: $|U/M : I/M| = 2$ and $\phi_i(1) = 1$.

Since $|U/M : I/M| = 2$, I/M is a normal subgroup of U/M of index 2. As $\phi_i(1) = 1$, ϕ_i is an extension of θ to I. By Lemma 1.2(b), $(\phi_i\tau)^{G'} \in \mathrm{Irr}(G')$ for all $\tau \in \mathrm{Irr}(I/M)$. Now $(\phi_i\tau)^{G'}(1) = |G' : I|\phi_i(1)\tau(1) \in \mathrm{cd}(G')$, which forces $\tau(1) = 1$ since $|G' : I| = q^3(q^3 \pm 1)$ and this degree must divide a degree of G. Hence I/M is abelian. But $I/M \trianglelefteq U/M$ and $U/M \cong \mathrm{SL}_3(q) : 2$ or $U/M \cong \mathrm{SU}_3(q) : 2$. Since I/M is abelian, it is solvable. But

$$\frac{U/M}{I/M}$$

is of order 2 and thus solvable. But then U/M is solvable, which is a contradiction. So this case is not possible.

Case 2: $|U/M : I/M| = 1$ and $\phi_i(1) = 2$.

In this case, I/M is a maximal subgroup of $G_2(q)$ with structure $\mathrm{SL}_3(q) : 2$ or $\mathrm{SU}_3(q) : 2$. First suppose that $\theta(1) = \phi_i(1)$ for some i. Then ϕ_i is an extension of θ to I and thus $(\phi_i\tau)^{G'} \in \mathrm{Irr}(G')$ for all $\tau \in \mathrm{Irr}(I/M)$. By Lemma 1.2,

$$q^3(q^3 \pm 1)\tau(1) \in \mathrm{cd}(G')$$

and thus must divide a degree of G. Examining the character degrees of G, we see that this forces $\tau(1) = 1$ for all $\tau \in \mathrm{Irr}(I/M)$. This implies I/M is abelian, which is not the case.

Now suppose that $\theta(1) < \phi_i(1)$ for all i. As $\phi_i(1) = 2$, we must have that $\theta(1) = 1$ and $\phi_j(1) = 2$ for all j. Now $I/M \cong \mathrm{SL}_3(q) : 2$ or $I/M \cong \mathrm{SU}_3(q) : 2$. Let $S/M \lneq I/M$ be a Sylow p-subgroup of I/M when q is odd. When q is even, let S/M be the subgroup corresponding to the pre-image of the stabilizer of an imaginary triangle of order $3(q^2+q+1)$ in $\mathrm{PSL}_3(q)$ or of order $3(q^2-q+1)$ in $\mathrm{PSU}_3(q^2)$. This is shown to exist in [20]. In all cases, $\gcd(|S : M|, 2) = 1$.

By Lemma 1.1, if $(\phi_i)_S \in \mathrm{Irr}(S)$, then $(\phi_i)_M \in \mathrm{Irr}(M)$ as $\phi_i(1) = 2$ is relatively prime to $|S : M|$. But $(\phi_i)_M = 2\theta$, so this is not possible.

Thus, $(\phi_i)_S = \lambda_1 + \lambda_2$, where $\lambda_i \in \mathrm{Irr}(S)$, $\lambda_i(1) = 1$. Then λ_1 is an extension of θ to S. By Lemma 1.5, there is an irreducible constituent ϕ of θ^I with $\phi(1) \geq \theta(1)\psi(1)$ for all $\psi \in \mathrm{Irr}(S/M)$. By design, S/M is of odd order. When q is odd, it is the Sylow p-subgroup of I/M, hence is nonabelian. When q is even, it is the pre-image of the stabilizer of an imaginary triangle in $\mathrm{PSL}_3(q)$ of order $3(q^2+q+1)$ or $\mathrm{PSU}_3(q^2)$ of order $3(q^2-q+1)$. This stabilizer is shown to be a Frobenius group in [12]. Hence, S/M is nonabelian in the even case as well. Thus S/M has an irreducible character ψ with $\psi(1) > 2$. But $\phi_i(1) = 2$ for all i, so we have a contradiction to the inequality $\phi_i(1) \geq \theta(1)\psi(1)$.

Case 3: $|U/M : I/M| = 1$ and $\phi_i(1) = 1$.

In this case, I/M is a maximal subgroup of $G_2(q)$ with structure $\mathrm{SL}_3(q):2$ or $\mathrm{SU}_3(q):2$. As $\phi_i(1) = 1$, ϕ_i is an extension of θ to I. By Lemma 1.2, $(\phi_i\tau)^{G'} \in \mathrm{Irr}(G')$ for all $\tau \in \mathrm{Irr}(I/M)$. Now $(\phi_i\tau)^{G'}(1) = |G' : I|\phi_i(1)\tau(1) \in \mathrm{cd}(G')$, which forces $\tau(1) = 1$ or $\tau(1) = 2$. Hence all the irreducible characters of I/M are of degree one or two. But this is not the case, as I/M has structure $\mathrm{SL}_3(q):2$ or $\mathrm{SU}_3(q):2$. In particular, I/M has an irreducible character of degree q^3, which is greater than 2.

Hence it is not possible for U/M to have the structure $\mathrm{SL}_3(q):2$ or $\mathrm{SU}_3(q):2$.

3.4.2 Maximal Parabolic Subgroups with Structure $[q^5]:\mathrm{GL}_2(q)$

Next we will consider the maximal parabolic subgroups. We begin with a couple of remarks. First, if H is a subgroup of $\mathrm{GL}_2(q)$ of index $q-1$, q, or $q+1$, then H is nonabelian. This can be determined from the character table of $\mathrm{GL}_2(q)$. For example, let H be a subgroup of $\mathrm{GL}_2(q)$ of index q. Then H contains all elements of order $q-1$. Let g be an element of order $q-1$. Examining the character table of $\mathrm{GL}_2(q)$, we see that $|C_{\mathrm{GL}_2(q)}(g)| = (q-1)^2$. But

$$|H| = \frac{q(q^2-1)(q-1)}{q} = (q^2-1)(q-1).$$

Thus $|C_{\mathrm{GL}_2(q)}(g)| < |H|$ so H is nonabelian. A similar argument works for the indices $q-1$ and $q+1$.

Second, the character tables of the parabolic subgroups of $G_2(q)$ are available in [1], [8], and [9]. These tables show that $[q^5]:\mathrm{GL}_2(q)$ has character degrees other than 1, $q-1$, q, and $q+1$.

Suppose U/M is a maximal parabolic subgroup of G'/M. We have $I/M \leq U/M \lneq G'/M$. Then

$$(q+1)(q^2-q+1)(q^2+q+1)|U/M : I/M|\phi_i(1)$$

divides a degree of G. So $|U/M : I/M|\phi_i(1)$ divides $q-1,\ q$, or $q+1$.

Suppose first that $\phi_i(1) = 1$ for some i. Then ϕ_i is an extension of θ to I and so $(\phi_i\tau)^{G'} \in \mathrm{Irr}(G')$ for all $\tau \in \mathrm{Irr}(I/M)$. Now $(\phi_i\tau)^{G'}(1) = |G' : I|\phi_i(1)\tau(1)$. If $|U/M : I/M| = 1$, then $I/M \cong U/M$ and this forces I/M to have character degrees $\tau(1)$ which can only be 1, $q-1$, q, or $q+1$. But $[q^5] : \mathrm{GL}_2(q)$ has degrees other than 1, $q-1$, q, and $q+1$. So if $\phi_i(1) = 1$, then $|U/M : I/M| > 1$.

Let $t = |U/M : I/M|$. Then $t\phi_i(1)$ divides $q-1$, q, or $q+1$. Now $U/M \cong [q^5] : \mathrm{GL}_2(q)$ and $I/M \leq U/M$. Examining the maximal subgroups of $\mathrm{GL}_2(q)$ from [20], we have that either $t \mid q-1$, $t \mid q$, or $t = q+1$. We see this from the following argument. Suppose that $t \mid q+1$. Then I/M is a subgroup $U/M \cong [q^5] : \mathrm{GL}_2(q)$ of index dividing $q+1$. So I/M is a subgroup of U/M containing $[q^5]$ and a part of $\mathrm{GL}_2(q)$ of order at least $q(q-1)^2$. Hence I/M contains $Z(\mathrm{GL}_2(q))$ as $|I/M|$ contains the full $(q-1)$-part of the order of $\mathrm{GL}_2(q)$. But examining the orders of maximal subgroups of $\mathrm{GL}_2(q)$ containing $Z(\mathrm{GL}_2(q))$, we see that the only possibility is $t = q+1$ or $t = 1$. Next, we show that if t is a proper divisor of q, then either $t = 1$ or I/M contains $\mathrm{GL}_2(q)$. If t is a proper divisor of q, and I/M does not contain $\mathrm{GL}_2(q)$, then $\mathrm{GL}_2(q)$ has a subgroup whose index is a proper divisor of q. Again, I/M contains $Z(\mathrm{GL}_2(q))$ as $|I/M|$ contains the full $(q-1)$-part of the order of $\mathrm{GL}_2(q)$. But examining the orders of maximal subgroups of $\mathrm{GL}_2(q)$ containing $Z(\mathrm{GL}_2(q))$, we see that the only possibility is $t = 1$. Hence, if t is a nontrivial proper divisor of q, then U/M contains $\mathrm{GL}_2(q)$.

Case 1: $t = q$ or $t = q+1$.

If $t = q$ or $t = q+1$, then $\phi_i(1) = 1$ and ϕ_i is an extension of θ to I. Thus $(\phi_i\tau)^{G'} \in \mathrm{Irr}(G')$

for all $\tau \in \mathrm{Irr}(I/M)$. Now

$$\begin{aligned}
(\phi_i\tau)^{G'}(1) &= |G' : I|\phi_i(1)\tau(1) \\
&= |G'/M : U/M||U/M : I/M|\tau(1) \\
&= (q+1)(q^2-q+1)(q^2+q+1)|U/M : I/M|\tau(1) \\
&= (q+1)(q^2-q+1)(q^2+q+1)t\tau(1),
\end{aligned}$$

which implies $\tau(1) = 1$ as $(q+1)(q^2-q+1)(q^2+q+1)t\tau(1)$ must divide a degree of G. As $\tau(1) = 1$ for all $\tau \in \mathrm{Irr}(I/M)$, I/M is abelian. But any subgroups of index q or $q+1$ in U/M are nonabelian.

Case 2: $t\phi_k(1) \mid q-1$ for all k or $t\phi_k(1) \mid q+1$ for all k.

First note that if $t\phi_k(1) \mid q+1$ for all k, then $t = 1$, as we have shown that either $t = 1$ or $t = q+1$, and $t = q+1$ was eliminated as a possibility in Case 1. Let $R/M \trianglelefteq U/M$ such that $|R : M| = q^5$. As $\gcd(t, q) = 1$, $R \leq I$. Since θ is invariant under I, $\ker(\theta) \trianglelefteq I$ and $\theta(g^{-1}hg) = \theta(h)$ for all $g \in I$ and $h \in M$. Since $\theta(1) = 1$, $\theta(g^{-1}hgh^{-1}) = 1$ and $[I, M] \leq \ker(\theta)$. Hence

$$M/\ker(\theta) \leq Z(I/\ker(\theta)).$$

Since $R \leq I$, $M/\ker(\theta) \leq Z(R/\ker(\theta))$ and

$$\frac{R/\ker(\theta)}{M/\ker(\theta)} \cong R/M,$$

which is a p-group. Hence the irreducible characters of $R/\ker(\theta)$ containing $M/\ker(\theta)$ in their kernels have degrees which are powers of the prime p. But as $M/\ker(\theta)$ is central in $R/\ker(\theta)$, these are all the degrees of $R/\ker(\theta)$. So the degrees of $R/\ker(\theta)$ are powers of p.

As $\phi_k(1) \mid q \pm 1$ and q is a power of p, $\phi_k(1)$ is not a multiple of p. Let

$$(\phi_k)_R = \sum_j \lambda_j,$$

for $\lambda_j \in \mathrm{Irr}(R)$. Now

$$\ker(\theta) = \bigcap_i \ker(\phi_i) \subseteq \ker(\phi_i)_R = \bigcap_j \ker(\lambda_j),$$

so the irreducible constituents of $(\phi_k)_R$ are irreducible characters of $R/\ker(\theta)$ since $\ker(\theta) \leq \ker(\lambda_j)$. As $\phi_k(1)$ is not a multiple of p, while λ_j is a power of p for all j, it must be that some λ_j is linear, say λ_1. Then λ_1 is an extension of θ to R. By Lemma 1.5, there exists an irreducible constituent ϕ of θ^I with $\phi(1) \geq \theta(1)\psi(1)$ for all $\psi \in \mathrm{Irr}(R/M)$. Now $\phi(1) \leq q-1$ if $t\phi_k(1) \mid q-1$ for all k and $\phi(1) \leq q+1$ if $t\phi_k(1) \mid q+1$ for all k.

As $R/M \trianglelefteq U/M$, Lemma 1.4 implies that if $\chi \in \mathrm{Irr}(U/M)$ and $\psi \in \mathrm{Irr}(R/M)$ is a constituent of $\chi_{R/M}$, then

$$\frac{\chi(1)}{\psi(1)} \,\Big|\, |U : R| = |\mathrm{GL}_2(q)|.$$

Now $|\mathrm{GL}_2(q)| = q(q^2-1)(q-1)$ and U/M is a maximal parabolic subgroup of $G_2(q)$.

First let us consider the case where $t\phi_k(1) \mid q-1$ for all k. By examining the character tables of the maximal parabolic subgroups of $G_2(q)$, we see that U/M has an irreducible character χ such that $\chi(1) = q^2 \cdot y$ for some integer y. Indeed, examples of such irreducible characters are listed in Table 3.3. All notation is adapted from [1], [8], and [9]. Hence $\psi(1) \geq q$. But this results in a contradiction. We have $\phi(1) \geq \theta(1)\psi(1)$ for all $\psi \in \mathrm{Irr}(R/M)$. Now $\phi(1) \leq q-1$ and we have found an irreducible character ψ of R/M with $\psi(1) \geq q$.

So we are left with the possibility that $t\phi_k(1) \mid q+1$ for all k. We have that $\phi(1) \leq q+1$ and $\phi(1) \geq \psi(1)$. Suppose $p \neq 3$. Then letting $\chi = \theta_8$ of P, we have that $\psi(1) \geq q^2$. But then $q+1 \geq \phi(1) \geq \psi(1) \geq q^2$, a contradiction. Similarly, if we consider $p > 3$ and the parabolic subgroup Q, letting $\chi = \sum_Q \theta_3$, we see that $q-1$ must divide $\psi(1)$. But $\psi(1)$ is a character degree of R/M, which is a p-group. So $q-1$ cannot divide $\psi(1)$. So we are left with the possibility that $p = 3$ or $p = 2$ and the parabolic subgroup is Q. In those cases, Table 3.3 gives a degree of U/M which forces $\psi(1) \geq q$. As $\phi(1) \mid q+1$ and $\phi(1) \geq \psi(1) \geq q$, this forces $\phi(1) = q+1$ and $t = 1$. We cannot eliminate the cases when $p = 3$ or $p = 2$ and

Parabolic Subgroup	$p \neq 3$	$p = 3$
P	$\theta_8(1) = q^3(q-1)$	$\epsilon_{12}(1) = q^2(q-1)$
Q	$\sum_Q \theta_3(x)(1) = q(q-1)^2(q^2-1)$ if $p > 3$ $\theta_{12}(1) = q^2(q-1)$ if $p > 3$ $\theta_3(x)(1) = q^2(q-1)(q^2-1)$ if $p = 2$	$\theta_{12}(1) = q^2(q-1)$

Table 3.3: Particular Degrees of Maximal Parabolic Subgroups of $G_2(q)$

the parabolic subgroup is Q because, in those cases, there is not an irreducible character of degree divisible by q^3 or $(q-1)^3$. Also, note that this case requires that $\theta(1) = 1$. (In the case when $p = 3$ or $p = 2$ and the parabolic subgroup is Q, we no longer need to assume that all $\phi_k(1) \mid q+1$. Indeed, in this step we prove that $\phi_k(1)$ must divide q or $q+1$ for all k.) So we are left with the possibility that $p = 2$ or $p = 3$, $t = 1$ and $\phi_i(1) = q+1$ for all i.

Case 3: $t\phi_k(1) \mid q$ for all k.

The subgroup structure of $\mathrm{GL}_2(q)$ shows that if the subgroup of index dividing q comes from $\mathrm{GL}_2(q)$, then $t = q$ or $t = 1$. But $t = q$ has been eliminated in Case 1. So if t is a nontrivial proper divisor of q, then I/M contains $\mathrm{GL}_2(q)$.

Suppose that $\phi_k(1) > 1$ for some k so $(\phi_k)_M = e\theta$ where $e \mid q$. We also have that $U = I$ or $|U : I|$ is a proper divisor of q. In either case, I has a subgroup S with $\bar{S} = S/M \cong \mathrm{SL}_2(q)$. If q is even, $\gcd(2, q-1) = 1$, so the Schur multiplier of $\bar{S}$ is trivial. Thus θ extends to an irreducible character θ_0 of S. By Lemma 1.5, there exists i such that $\phi_i(1) \geq \tau(1)$ for all $\tau \in \mathrm{Irr}(S/M)$. But $\mathrm{SL}_2(q)$ has an irreducible character of degree $q+1$ while $\phi_i(1) \leq q$ for all i. The comments preceding Case 1 imply that $t = 1$ and q is even and $\phi_k(1) = 1$ for all k is also not possible.

We are left with the case when q is odd and $t\phi_k(1) \mid q$ for all k. We must also consider the case where $\phi_k(1)$ differs for different values of k.

Summary

- We have proved that $t\phi_k(1) \mid q-1$ for all k is not possible.
- We have proved that $t\phi_k(1) \mid q+1$ for all k is not possible when $p > 3$ and is also not possible when $p = 2$ and the maximal parabolic subgroup is P.
- We have proved that $t\phi_k(1) \mid q$ for all k is not possible if q is even.

The following cases need to be considered:

- ϕ_k has varying degrees for different values of k.
- $t\phi_k(1) \mid q+1$ for all k and $p = 3$ for parabolic subgroups with structure P or Q or $p = 2$ and we are considering the maximal parabolic subgroup Q.
- $t\phi_k(1) \mid q$ for all k and q is odd

3.5 Establishing $M = \langle 1 \rangle$ when $H \cong G_2(q)$

We will assume Step 3 holds for $G_2(q)$. For $q > 4$, the Schur multiplier of $G_2(q)$ is trivial. Thus $M' = M$ by Step 3 and Lemma 1.6. If M is abelian, we are done. Suppose $M = M' \gneq \langle 1 \rangle$. Let $M/N \cong T_1 \times \cdots \times T_k$ be a chief factor of G', where $T_i \cong T$, a nonabelian simple group, and the T_i are transitively permuted by G'. Then $G'/M \cong G_2(q)$ has a subgroup of index k.

Suppose $k > 1$. Examining the list of maximal subgroups of $G_2(q)$, we have, for $q > 4$,

$$k \geq (q^3+1)(q^2+q+1) = (q+1)(q^4+q^2+1).$$

As T_i is a nonabelian simple group, $|T_i|$ is divisible by at least three distinct primes. Hence there is a prime divisor p of $|T_i|$ greater than or equal to 5. By the Itô–Michler Theorem,

p divides a character degree of T_i. Hence T_i has an irreducible character of degree 5 or larger. Take $\psi_i \in \mathrm{Irr}(T_i)$ such that $\psi_i(1) \geq 5$. Then M/N has degree

$$\prod_{i=1}^{k} \psi_i(1) \geq 5^k,$$

and this degree must divide a degree of G. The largest degree of G is

$$(q+1)^2(q^2+q+1)(q^2-q+1) = (q+1)^2(q^4+q^2+1).$$

Thus

$$[(q+1)(q^4+q^2+1)]^2 > (q+1)^2(q^4+q^2+1) \geq 5^{(q+1)(q^4+q^2+1)},$$

a contradiction for all q. Hence $k = 1$. Schreier's Conjecture states that the outer automorphism group of a nonabelian simple group is solvable and has been verified by the classification of finite simple groups. We have $(M/N) \cap C_{G'/N}(M/N) = Z(M/N) = \langle 1 \rangle$ and

$$\frac{N_{G'/N}(M/N)}{C_{G'/N}(M/N)} \leq \mathrm{Aut}(M/N),$$

so

$$\frac{G'/N}{C_{G'/N}(M/N)} \leq \mathrm{Aut}(M/N).$$

As

$$\frac{M/N}{Z(M/N)} \cong M/N \cong \mathrm{Inn}(M/N),$$

we have

$$\frac{G'/N}{M/N \times C_{G'/N}(M/N)} \leq \frac{\mathrm{Aut}(M/N)}{\mathrm{Inn}(M/N)} = \mathrm{Out}(M/N).$$

By Schreier's Conjecture,

$$\frac{G'/N}{M/N \times C_{G'/N}(M/N)}$$

is solvable. But $G' = G''$ by Step 1. So we must have that

$$G'/N = M/N \times C_{G'/N}(M/N) \cong M/N \times G_2(q),$$

which produces forbidden degrees. Hence $M = \langle 1 \rangle$.

3.6 Establishing $G = G' \times C_G(G')$ when $H \cong G_2(q)$

Recall $q = p^f$. Suppose $G' \times C_G(G') \lneq G$. By the same reasoning as presented in Section 2.6, we must have that $\mathrm{cd}(G)$ contains degrees $\chi(1)$ and $e\chi(1)$, where $e > 1$ and e divides $|\mathrm{Out}(G')|$, which is either f or $2f$. Examining the character degrees of G to find degrees with proper multiples which are also degrees of G establishes that $e = 2$, $e = 3$, or $e \geq q - 1$.

If $e \geq q - 1$, then as $e \mid 2f$,

$$2f \geq e \geq q - 1 = p^f - 1 \geq 2^f - 1,$$

so $f \leq 2$. If $f = 2$, then $q = 4$. If $f = 1$, then $q \leq 3$. These cases were considered in [17].

Suppose that $e = 2$. Then the possibilities for $\chi(1)$ are

$$\chi(1) = \frac{1}{2}q(q^2 - q + 1)(q + 1)^2 \quad \text{and} \quad e\chi(1) = q(q^2 - q + 1)(q + 1)^2$$

or

$$\chi(1) = \frac{1}{2}q(q^2 + q + 1)(q - 1)^2 \quad \text{and} \quad e\chi(1) = q(q^2 + q + 1)(q - 1)^2.$$

In each of these cases, $G' \cong G_2(q)$ has only one irreducible character of degree $\chi(1)$, so it is not possible for two characters of $G_2(q)$ of degree $\chi(1)$ to fuse into an irreducible character of G.

Suppose that $e = 3$. Then

$$\chi(1) = \frac{1}{3}q(q^2 + q + 1)(q^2 - q + 1) \quad \text{and} \quad e\chi(1) = q(q^2 + q + 1)(q^2 - q + 1).$$

But $G' \cong G_2(q)$ has only two irreducible characters of degree $\chi(1)$ so it is not possible for three characters of $G_2(q)$ of degree $\chi(1)$ to fuse into an irreducible character of G. Thus $G = G' \times C_G(G')$.

We have proved all but Step 3 for the family of simple groups $G_2(q)$ for all $q > 4$.

CHAPTER 4

VERIFYING HUPPERT'S CONJECTURE FOR THE SIMPLE LINEAR GROUPS OF RANK TWO

In this chapter, we provide a brief outline of the arguments we use to verify Huppert's Conjecture for the simple linear groups of Lie type of rank two. In [15], Huppert verifies his conjecture for $\mathrm{PSL}_3(q)$ for $2 < q \leq 8$. He proves four of the five steps for all q. The restriction that $q \leq 8$ is required to prove Step 2. We prove Step 2 by using properties of the character degree set of $\mathrm{PSL}_3(q)$ which hold for all $q > 8$. We begin by examining the character degrees of $\mathrm{PSL}_3(q)$ and establishing these properties. For a more detailed verification, the reader can refer to [33].

4.1 Results Concerning the Character Degrees of $\mathrm{PSL}_3(q)$

For $q > 4$,

$$\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PSL}_3(q)) = \{1,\, q^3,\ q(q+1),\ (q-1)^2(q+1),\ q(q^2+q+1),\ (q-1)(q^2+q+1),\\ q^2+q+1,\ (q+1)(q^2+q+1),\ \frac{1}{3}(q+1)(q^2+q+1)\}.$$

The last degree appears only if $q \equiv 1 \pmod 3$. We again will rely on properties of this character degree set. In particular, we will need to determine the nontrivial powers among the degrees of G, the nontrivial prime powers among the degrees, and whether the degree set of G contains any consecutive integers.

Lemma 4.1.1. *For $q > 8$, the only nontrivial powers among the degrees of G are q^3 and possibly $(q-1)^2(q+1)$. The only degree of the form p^b for prime p and $b > 1$ is q^3.*

Proof. We begin by examining the product of consecutive integers $q(q+1)$. As proved in [10], this is never a nontrivial power.

As noted in Lemma 3.1.1, q^2+q+1 is not a power. Since $\gcd(q, q^2+q+1) = \gcd(q+1, q^2+q+1) = 1$, we have that q^2+q+1, $q(q^2+q+1)$, and $(q+1)(q^2+q+1)$ are not nontrivial powers.

By a result in [27], $q^2+q+1 = 3y^n$ has only trivial solutions. Thus, $(q-1)(q^2+q+1)$ and $\frac{1}{3}(q+1)(q^2+q+1)$ are not nontrivial powers.

As shown in [25], the only possible powers of a prime among the degrees of G are q^2+q+1 and q^3. But q^2+q+1 is never a power with exponent greater than one, so the only composite prime power among the degrees of G is q^3. □

We will also need to know which pairs of character degrees of G are consecutive integers. By examining the degrees of G, it is possible to prove the following lemma.

Lemma 4.1.2. *The only pairs of consecutive integers among character degrees of G, for $q > 2$, are $q(q+1)$ and q^2+q+1 as well as q^3-1 and q^3.*

4.2 Establishing $G'/M \cong H$ when $H \cong \mathrm{PSL}_3(q)$

4.2.1 Eliminating the Tits, Sporadic, and Alternating Groups

Using Lemma 4.1.1 and a proof similar to that of Proposition 3.3.1 on page 47, we can eliminate the possibility that $k > 1$ and S is an alternating group, sporadic simple group, or the Tits group. For more details, see Proposition 2.8 of [33].

When $k = 1$, we can eliminate the sporadic simple groups and the Tits group by the same reasoning as in the proof of Proposition 2.3.2 on page 19. We only need to consider the sporadic simple groups with eight or less extendible characters of distinct degrees. Hence, we must consider $S \cong M_{11}$, $S \cong M_{12}$, $S \cong J_1$, and $S \cong {}^2F_4(2)'$, the Tits group. Examining the character degree sets of these groups and using the properties of the character degree set of $\mathrm{PSL}_3(q)$, we find that none of these groups is a possible candidate for S. We are left considering $k = 1$ and $S \cong A_n$.

Proposition 4.2.1. *The simple group S is not an alternating group A_n with $n \geq 7$.*

Proof. Recall that $A_5 \cong \mathrm{PSL}_2(5)$ and $A_6 \cong \mathrm{PSL}_2(9)$, and so they will be considered with the groups of classical Lie type. Further, A_7, A_8, and A_{10} can be eliminated using their irreducible characters of consecutive integer degrees which extend to their automorphism groups and comparing them to the consecutive integer degrees of G.

Case 1: $S \cong A_{2m}$, $m \geq 6$

By [2] and [22], A_{2m} has irreducible characters of the following degrees which extend to G:

$$\begin{aligned}
\chi_1(1) &= \frac{8m(m-1)(m-2)(m-3)(2m-3)}{15} \\
\chi_2(1) &= \frac{2m(2m-3)}{2} = m(2m-3) = 2m^2 - 3m \\
\chi_3(1) &= \frac{(2m-1)(2m-2)}{2} = (m-1)(2m-1) = 2m^2 - 3m + 1.
\end{aligned}$$

Now $\chi_2(1)$ and $\chi_3(1)$ are consecutive integers, with $\chi_3(1)$ larger. As $m \geq 6$, $q^3 \neq (m-1)(2m-1)$, as q^3 is a power of a prime while $\gcd(m-1, 2m-1) = 1$.

Thus, $q^2 + q + 1 = (m-1)(2m-1)$ and $q(q+1) = m(2m-3)$. Hence $q(q+1)$ divides $15\chi_1(1)$. Excluding $q(q+1)$, the remaining degrees of G are relatively prime to at least one of q or $q+1$. This implies either $q+1 \mid 15$ or $q \mid 15$. Again, we are assuming $q > 8$, so this is not possible.

Case 2: $S \cong A_{2m+1}$, $m \geq 4$.

By [2] and [22], A_{2m+1} has irreducible characters of the following degrees which extend to G:

$$\chi_1(1) = \frac{2m(m-1)(m-3)(2m-3)(2m+1)}{5}$$

$$\chi_2(1) = (m-1)(2m+1) = 2m^2 - m - 1$$

$$\chi_3(1) = m(2m-1) = 2m^2 - m.$$

Now $\chi_2(1)$ and $\chi_3(1)$ are consecutive integers, with $\chi_3(1)$ larger. Once again $q^3 \neq m(2m-1)$, as q^3 is a power of a prime while $\gcd(m, 2m-1) = 1$.

Thus, $q^2 + q + 1 = m(2m-1)$ and $q(q+1) = (m-1)(2m+1)$. Hence $q(q+1)$ divides $5\chi_1(1)$. Excluding $q(q+1)$, the remaining degrees of G are relatively prime to at least one of q or $q+1$. This implies either $q+1 \mid 5$ or $q \mid 5$. Again, we are assuming $q > 8$, so this is not possible. □

4.2.2 Eliminating the Groups of Lie Type when $k > 1$

Let χ denote the Steinberg character of S. By an argument similar to that presented in the introduction of Section 2.3.4 on page 20, we must have that $\chi(1)^k = q^3$. We can again show that if $S = S(q_1)$ is a simple group of Lie type, then $k = 1$ by an argument similar to that of Proposition 2.3.3.

Proposition 4.2.2. *If $S = S(q_1)$ is a simple group of Lie type, then $k = 1$.*

Since the sporadic, Tits, and alternating groups have been eliminated as possibilities for S, we have that S is a simple group of Lie type and, thus, $k = 1$. We now need to eliminate the groups of exceptional Lie type as possibilities for S.

4.2.3 Eliminating Groups of Exceptional Lie Type when $k = 1$

Proposition 4.2.3. *The group S is not a simple group of exceptional Lie type.*

To establish Proposition 4.2.3, we examine each of the families of simple groups of exceptional Lie type separately, using the general procedure outlined earlier. For $S \cong G_2(q_1)$

and $S \cong {}^2G_2({q_1}^2)$, we find particular mixed degrees of these groups and show that these mixed degrees do not divide either mixed degree of G. For $S \cong {}^2B_2({q_1}^2)$, we show that the largest degree of S is larger than any of the degrees of G. The remaining groups of exceptional Lie type are eliminated by an argument similar to that presented in Case 4 of Proposition 2.3.4.

4.2.4 Eliminating the Groups of Classical Lie Type when $k = 1$

We are only left with the possibility that $G'/M \cong S$, where S is a simple group of classical Lie type.

Proposition 4.2.4. *The simple group $S = S(q_1) \cong \mathrm{PSL}_3(q)$.*

To establish Proposition 4.2.4, we again proceed by examining each family of simple groups of classical Lie type separately. As the Steinberg character of S extends to $\mathrm{Aut}(S)$, the degree of the Steinberg character of S, say ${q_1}^j$, is a character degree of G. As q^3 is the only composite prime power among the degrees of G, we have that ${q_1}^j = q^3$. For groups of low rank, we then typically find a mixed character degree of S that is too large to divide the mixed degree $q(q+1)$ of G and then use the divisibility properties of q^2+q+1 (outlined in Lemma 3.1.4 on page 43) to show that this degree of S cannot divide $q(q^2+q+1)$ either. To eliminate the simple groups of Lie type of higher rank, we use the bounds established on the mixed degrees used in the proof of Proposition 2.3.5 to show that these degrees of S are too large to divide $q(q+1)$. We then show that this degree of S has a factor that cannot divide q^2+q+1, which implies this degree of S cannot divide $q(q^2+q+1)$, the other mixed degree of G. Hence, this degree of S does not divide a degree of G, which is a contradiction. These arguments eliminate all simple groups of classical Lie type as candidates for S except $S \cong \mathrm{PSL}_3(q)$, which establishes the result. For a complete proof of this result, see Proposition 3.12 of [33].

These arguments establish Step 2 of Huppert's argument for all q. As Huppert verified the other four steps, his conjecture is verified for $\mathrm{PSL}_3(q)$.

CHAPTER 5

VERIFYING HUPPERT'S CONJECTURE FOR THE UNITARY SIMPLE GROUPS OF RANK TWO

We will now consider the family of simple groups $\mathrm{PSU}_3(q^2)$. In [14], Huppert verifies his conjecture for this family of simple groups for $q \leq 9$. He is able to establish all but Step 2 for all q. We will establish Step 2 for all q and thus verify Huppert's Conjecture for $\mathrm{PSU}_3(q^2)$ for all q. We will proceed in much the same manner as in the linear case presented in Chapter 4.

5.1 Background Results on the Character Degrees of $\mathrm{PSU}_3(q^2)$

Suppose $q > 9$ and the character degrees of G and $\mathrm{PSU}_3(q^2)$ are the same. We first establish some properties of the set of character degrees of G that will enable us to prove $k = 1$ and ultimately show that $S \cong \mathrm{PSU}_3(q^2)$. We have

$$\begin{aligned}\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PSU}_3(q^2)) = \{&1,\ q^3,\ (q-1)(q+1)^2,\ q(q-1),\ q^2-q+1,\ (q-1)(q^2-q+1),\\ &q(q^2-q+1),\ (q+1)(q^2-q+1),\ \frac{1}{3}(q-1)(q^2-q+1)\}.\end{aligned}$$

The last degree appears only if $q \equiv -1 \pmod 3$. As the character degrees of $\mathrm{PSU}_3(q^2)$ can be obtained from the character degrees of $\mathrm{PSL}_3(q)$ by replacing q by $-q$, many of our previous results hold.

First note that $\gcd(q-1, q+1) = 2$ if q is odd and $\gcd(q^2-q+1, q+1) = 3$ if $q \equiv -1 \pmod 3$. All other factors given in the set of character degrees of $\mathrm{PSU}_3(q^2)$ are pairwise relatively prime. Now let us examine the character degrees of G. Using Lemma 3.1.1 on page 42 and a proof similar to that of Lemma 4.1.1 on page 76, we have the following result.

Lemma 5.1.1. *For $q > 9$, the only nontrivial powers among the degrees of G are q^3, possibly $(q+1)^2(q-1)$, and q^2-q+1 when $q = 19$. The only composite power of a prime among these degrees is q^3 and, when $q = 19$, q^2-q+1.*

We will also need to know which pairs of character degrees of G are consecutive integers. By examining the degrees of G, it is possible to prove the following lemma.

Lemma 5.1.2. *The only pairs of consecutive integers among character degrees of G, for $q > 2$, are $q(q-1)$ and q^2-q+1 as well as q^3 and q^3+1.*

If we handle the case when $q = 19$ separately when dealing with groups of Lie type, we have exactly the same conditions as in the $\mathrm{PSL}_3(q)$ case.

5.2 Establishing $G'/M \cong H$ when $H \cong \mathrm{PSU}_3(q^2)$

5.2.1 Eliminating the Alternating, Sporadic, and Tits Groups for all k

Using Lemma 5.1.1 and a proof similar to that of Proposition 3.3.1, we have the following result. For more details, see Proposition 4.5 of [33].

Proposition 5.2.1. *If S is an alternating group A_n with $n \geq 7$, a sporadic simple group, or the Tits group, then $k = 1$.*

By the same reasoning as in the proof of Proposition 2.3.2, we only need to consider the sporadic simple groups with eight or less extendible characters of distinct degrees. This leaves the cases where $S \cong M_{11}$, $S \cong M_{12}$, $S \cong J_1$, and $S \cong {}^2F_4(2)'$, the Tits group. Examining the character degree sets of these groups and using the properties of the character degree set of $\mathrm{PSU}_3(q^2)$, we find that none of these groups is a possible candidate for S. We have the following result.

Proposition 5.2.2. *The simple group S is not one of the sporadic simple groups or the Tits group.*

By a proof similar to that of Proposition 4.2.1, we have the following result.

Proposition 5.2.3. *The simple group S is not an alternating group A_n with $n \geq 7$.*

5.2.2 Eliminating Groups of Lie Type

Now suppose that $q \neq 19$. Since the Steinberg character of both $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q^2)$ is of degree q^3, and the mixed degrees of $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q^2)$ both have q to the same power, the same argument used in the linear case to show that $k = 1$ will hold in the unitary case as well.

Proposition 5.2.4. *If S is a simple group of Lie type, then $k = 1$.*

The proof of Proposition 5.2.4 is the same as proof of Proposition 4.2.2, replacing $q+1$ by $q-1$ and q^2+q+1 by q^2-q+1.

Proposition 5.2.5. *If $q \neq 19$ then $S \cong \mathrm{PSU}_3(q^2)$.*

Proof. In the $\mathrm{PSL}_3(q)$ case, we eliminated the simple groups of exceptional Lie type when $k = 1$ by finding a mixed degree of S that was too large to divide a mixed degree of G. The same arguments hold for $\mathrm{PSU}_3(q^2)$. The only case that explicitly relied upon the character degrees of $\mathrm{PSL}_3(q)$ occurred when considering ${}^2B_2({q_1}^2)$. In that case we found the largest degree of $\mathrm{PSL}_3(q)$ and showed that it was not large enough to be divisible by the largest character degree of ${}^2B_2({q_1}^2)$. The largest degree of $\mathrm{PSU}_3(q^2)$ is $(q-1)(q+1)^2 = q^3+q^2-q-1$, which is smaller than the largest degree of $\mathrm{PSL}_3(q)$, so the same result holds.

In the $\mathrm{PSL}_3(q)$ case, we eliminated the simple groups of classical Lie type when $k = 1$ by considering the properties of the factors of q^2+q+1 and finding degrees that were too large to divide $q+1$. As the factors of q^2-q+1 share the same properties as those of q^2+q+1 and $q-1$ is smaller than $q+1$, our arguments will also work to establish $S \cong \mathrm{PSU}_3(q^2)$. $\square$

Thus, if $q \neq 19$, we have that $S \cong \mathrm{PSU}_3(q^2)$. We need to consider the case when $q = 19$ separately as it contains two composite prime power character degrees.

5.2.3 The Case of $\mathrm{PSU}_3(19^2)$

Now consider

$$\mathrm{cd}(G) = \ \mathrm{cd}(\mathrm{PSU}_3(19^2)) = \{1,\ 19^3,\ 2\cdot 3^2\cdot 19,\ 2\cdot 3^2\cdot 7^3,\ 7^3\cdot 19,\ 2^2\cdot 5\cdot 7^3,\ 7^3,\ 2^5\cdot 3^2\cdot 5^2\}.$$

If the Steinberg character extends to the character of degree 19^3, then the earlier arguments show $S \cong \mathrm{PSU}_3(19^2)$. So we are left with the possibility that the Steinberg character of S extends to the character of $\mathrm{PSU}_3(19^2)$ of degree 7^3. Then the underlying characteristic of the group S is 7. Most of the simple groups of Lie type have Steinberg characters whose degrees have an exponent greater than 3. So it is not possible for the Steinberg character of S to extend to a character of G of degree 7^3. The groups of Lie type in characteristic 7 with Steinberg characters of degree with exponent 3 or less are $\mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(7^3)$, $\mathrm{PSL}_3(7)$, and $\mathrm{PSU}_3(7^2)$.

Case 1: $S \cong \mathrm{PSL}_2(7)$

In this case, $\mathrm{cd}(S) = \{1,\ 3,\ 6,\ 7,\ 8\}$. First suppose that $k \geq 2$. Then $8^2 = 2^6$ is a character degree of G'/M, hence must divide a character degree of G. But none of the character degrees of $\mathrm{PSU}_3(19^2)$ is divisible by 2^6. Thus $k = 1$.

As shown in [22], irreducible characters of $\mathrm{PSL}_2(7)$ of degrees 6, 7, and 8 extend to $\mathrm{Aut}(\mathrm{PSL}_2(7))$, hence are character degrees of G by Lemma 2.3.1. But 6, 7, and 8 are clearly not character degrees of $\mathrm{PSU}_3(19^2)$. Thus $S \ncong \mathrm{PSL}_2(7)$.

Case 2: $S \cong \mathrm{PSL}_2(7^3)$

In this case, $\mathrm{cd}(S) = \{1,\ 2\cdot 3^2\cdot 19,\ 7^3,\ 2^3\cdot 43,\ 3^2\cdot 19\}$. Thus $k = 1$. Now $2^3\cdot 43$ is a character degree of S, but this divides no character degree of G.

Case 3: $S \cong \mathrm{PSL}_3(7)$

Here $k = 1$ since the degree of the Steinberg character of S is 7^3. Now $7(7^2 + 7 + 1) = 3 \cdot 7 \cdot 19$ is a character degree of S. But this divides no character degree of G.

Case 4: $S \cong \mathrm{PSU}_3(7^2)$

Again, we have that $k = 1$ since the degree of the Steinberg character of S is 7^3. Now $7(7^2 - 7 + 1) = 7 \cdot 43$ is a character degree of S. But this divides no character degree of G.

Thus, if $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PSU}_3(19^2))$, then $S \cong \mathrm{PSU}_3(19^2)$. So in all cases, we have established Step 2 of Huppert's argument for the simple unitary groups of rank two. Hence, Huppert's Conjecture is verified for the family of simple groups $\mathrm{PSU}_3(q^2)$.

CHAPTER 6

VERIFYING HUPPERT'S CONJECTURE FOR THE SIMPLE SYMPLECTIC GROUPS OF RANK TWO IN EVEN CHARACTERISTIC

In this and the next chapter, we outline the arguments used to verify Huppert's Conjecture for the remaining simple groups of Lie type of rank two, namely the symplectic groups. For a more detailed verification, the reader can refer to [34]. In the case of the symplectic groups of rank two, Huppert verifies Steps 1, 3, and 4 for all q in [16]. We will establish Steps 2 and 5 of Huppert's argument for this family of simple groups. Because the character degree set of $\mathrm{PSp}_4(q)$ differs in fundamental ways for even q compared to odd q, we handle odd q and even q separately. We begin in this chapter by examining the symplectic groups $\mathrm{PSp}_4(q)$ for even q. Note that $\mathrm{PSp}_4(2)$ is not simple and Huppert verifies his conjecture for $\mathrm{PSp}_4(4)$ in [16].

Suppose $q > 4$ is a even and the character degrees of G and $\mathrm{PSp}_4(q)$ are the same. We first establish some properties of the set of character degrees of G that will enable us to prove $k = 1$ and ultimately show that $S \cong \mathrm{PSp}_4(q)$.

6.1 Results Concerning the Character Degrees of $\mathrm{PSp}_4(q)$, q Even

Now

$$\begin{aligned}\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PSp}_4(q)) = \{1,\ &\frac{1}{2}q(q^2+1),\ \frac{1}{2}q(q+1)^2,\ q^4,\ \frac{1}{2}q(q-1)^2,\ q^4-1,\\ &(q-1)^2(q^2+1),\ (q^2-1)^2,\ (q+1)(q^2+1),\ (q-1)(q^2+1),\\ &q(q+1)(q^2+1),\ q(q-1)(q^2+1),\ (q+1)^2(q^2+1)\}.\end{aligned}$$

The last degree appears only for $q > 4$.

As q is even, $\gcd(q-1, q+1) = 1$, $\gcd(q-1, q^2+1) = 1$, and $\gcd(q+1, q^2+1) = 1$. Thus, all the factors in the degrees of G listed above are pairwise relatively prime. Zsigmondy's Theorem also assists in the classification of nontrivial powers. This version of the theorem appears in [25].

Lemma 6.1.1. *(Zsigmondy's Theorem) Let $x > 1$ be an integer. For each $n \in \mathbb{N}$, there is a prime ℓ such that ℓ divides $x^n - 1$ and does not divide $x^m - 1$ for $m < n$, except when $x = 2$ and $n = 6$, or $n = 2$ and x is a 2-power. For each $n \in \mathbb{N}$, there is a prime ℓ such that ℓ divides $x^n + 1$ and does not divide $x^m + 1$ for $m < n$, except when $x = 2$, $n = 3$.*

Lemma 6.1.2. *Let q be a 2-power. The number $q + 1$ is a power with prime exponent only when $q = 8$. The number $q - 1$ cannot be written in the form y^p where p is an odd prime. The numbers $q^2 + 1$, $q^2 - 1$, and $q^4 - 1$ are never nontrivial powers.*

Proof. Suppose $q + 1 = y^p$ for some odd integer y and odd prime p. Then $q = y^p - 1 = (y - 1)\Phi_p(y)$, where $\Phi_p(y)$ denotes the p^{th} cyclotomic polynomial. As $y - 1$ is even while $\Phi_p(y)$ is odd, it is impossible for $q = y^p - 1$ to be prime or a power of a prime, since it has at least two distinct prime divisors.

When $p = 2$, we have that $q + 1 = y^2$ so $q = (y - 1)(y + 1)$. Certainly $q \neq 2$. If $q > 2$ is a 2-power, say $q = 2^m$, then $2 \mid y - 1$ and $2 \mid y + 1$. But the only powers of 2 satisfying $y - 1 = 2^m$ and $y + 1 = 2^s$ satisfy $2^{m-1} + 1 = 2^{s-1}$. This only occurs for $m = 1$ and $s = 2$, giving rise to the exceptional case when $q = 8$.

Next, suppose $q - 1 = y^p$ for some odd integer y and odd prime p. Then $q = y^p + 1 = (y+1)R(y)$, where $R(y)$ is an integer. By Lemma 6.1.1, $y^p + 1$ has at least one prime factor that $y + 1$ does not have. So it is impossible for $y^p + 1$ to be prime or a power of a prime, except in the case where $y = 2$, $p = 3$, and thus $q = 9$. But again we are assuming that q is even.

Replacing q by q^2 and appealing to Lemma 6.1.1 and the previous argument, we see that q^2+1 cannot be a prime power.

Next, if $q^2-1=y^p$, then $q^2=y^p+1$ and, appealing to Lemma 6.1.1, we can eliminate the case when p is an odd prime. When $p=2$, we have $q^2-1=y^2$. But $q^2-1=(q-1)(q+1)$ and $\gcd(q-1,q+1)=1$. So for $q^2-1=y^2$, each of the factors $q-1$ and $q+1$ must be perfect squares. But $q+1$ is a perfect square only for $q=8$. Replacing q by q^2 and appealing to Lemma 6.1.1, we see that q^4-1 cannot be a nontrivial power. □

Lemma 6.1.3. *For $q \geq 8$, the only possible nontrivial powers among the degrees of G are q^4, $(q^2-1)^2$, $\frac{1}{2}q(q+1)^2$, and $\frac{1}{2}q(q-1)^2$. The only nontrivial power of a prime among the degrees of G is q^4.*

Proof. As shown in [25], the only power of a prime among the degrees of G is q^4. As the factors of the character degrees of G are pairwise relatively prime, the only way that a degree can be a nontrivial power is if each of its factors is a nontrivial power. Lemma 6.1.2 shows that this is not the case for the remaining degrees of G. □

We will also need to know which pairs of character degrees of G are consecutive integers. By examining the degrees of G, it is possible to prove the following lemma.

Lemma 6.1.4. *The only pair of consecutive integers among the character degrees of G, for $q > 2$, is q^4-1 and q^4.*

Finally, as $q > 2$ is even, we see that G has exactly six nontrivial even degrees and six nontrivial odd degrees.

6.2 Establishing $G'/M \cong H$ when $H \cong \mathrm{PSp}_4(q)$, q Even

6.2.1 Eliminating the Tits, Sporadic, and Alternating Groups for all k

Using Lemma 6.1.3 and a proof similar to that of Proposition 3.3.1, we have the following result. For more details, see Proposition 3.3 of [34].

Proposition 6.2.1. *If S is isomorphic to an alternating group A_n with $n \geq 7$, a sporadic simple group, or the Tits group, then $k = 1$.*

Next we consider S to be a sporadic simple group or the Tits group. We can eliminate most of the sporadic simple groups from consideration. By the same reasoning as presented in the proof of Proposition 2.3.2, we only need to consider sporadic simple groups with 12 or less extendible characters of distinct degrees. Using the consecutive degrees of G, the parity of the degrees of G, and the prime power degree of G, it is possible to eliminate these sporadic simple groups.

Proposition 6.2.2. *The simple group S is not one of the sporadic simple groups or the Tits group.*

By an argument similar to that of Proposition 3.3.3 of Section 3.3.3, we can establish the following result.

Proposition 6.2.3. *The simple group S is not an alternating group A_n with $n \geq 7$.*

6.2.2 Eliminating the Groups of Lie Type when $k > 1$

Let χ denote the Steinberg character of S. By an argument similar to that presented in the introduction of Section 2.3.4, we must have that $\chi(1)^k = q^4$. Using this and a proof similar to that of Proposition 2.3.3, we have the following result.

Proposition 6.2.4. *If $S = S(q_1)$ is a simple group of Lie type, then $k = 1$.*

6.2.3 Eliminating Simple Groups of Exceptional Lie Type when $k = 1$

We now have that $G'/M \cong S$, where $S = S(q_1)$ is a simple group of Lie type defined over the field of q_1 elements. We now want to show that $S \cong \mathrm{PSp}_4(q)$. We begin by eliminating the possibility that S is a simple group of exceptional Lie type.

Proposition 6.2.5. *The group S is not a simple group of exceptional Lie type.*

To establish this result, consider $S \cong {}^2G_2({q_1}^2)$, $S \cong G_2(q_1)$, and $S \cong {}^2B_2({q_1}^2)$ separately. As the characteristic of the underlying field of ${}^2G_2({q_1}^2)$ is 3, while the characteristic of the underlying field of $\mathrm{PSp}_4(q)$ for even q is 2, ${}^2G_2({q_1}^2)$ is eliminated as a candidate for S. For $S \cong G_2(q_1)$, it is possible to find a mixed degree of $G_2(q_1)$ that divides no mixed degree of G. Finally, for ${}^2B_2({q_1}^2)$, the prime power degree of G forces $q = q_1$, which is a contradiction as q is an integer while q_1 is not. The remaining groups of exceptional Lie type are eliminated by an argument similar to that presented in Case 4 of Proposition 2.3.4.

6.2.4 Eliminating the Groups of Classical Lie Type when $k = 1$

We are left only with the possibility that $G'/M \cong S$, where S is a simple group of classical Lie type.

Proposition 6.2.6. *The simple group* $S \cong \mathrm{PSp}_4(q)$.

To establish Proposition 6.2.6, we again proceed by examining each family of simple groups of classical Lie type separately. As the Steinberg character of S extends to $\mathrm{Aut}(S)$, the degree of the Steinberg character of S, say ${q_1}^j$, is a character degree of G. As q^4 is the only prime power among the degrees of G, we have that ${q_1}^j = q^4$. For groups of low rank, we then typically find a mixed character degree of S that is too large to divide the mixed degrees of G. The mixed degrees of G are

$$\frac{1}{2}q(q^2+1),\ \frac{1}{2}q(q+1)^2,\ \frac{1}{2}q(q-1)^2,\ q(q+1)(q^2+1),\ \text{and } q(q-1)(q^2+1).$$

To eliminate the simple groups of Lie type of higher rank, we use the bounds established on the mixed degrees used in the proof of Proposition 2.3.5 to show that these degrees of S are too large to divide the mixed degrees of G, which is a contradiction. These arguments eliminate all simple groups of classical Lie type as candidates for S except $S \cong \mathrm{PSp}_4(q)$, which establishes the result. For a complete proof of this result, see Proposition 8.1 of [34].

This was the last case to consider to prove that $k = 1$ and $S \cong \mathrm{PSp}_4(q)$. This verifies

Step 2 in Huppert's argument.

6.3 Proving Step 5 for $\mathrm{PSp}_4(q)$, q Even

To complete the verification of Huppert's Conjecture for the family of groups $\mathrm{PSp}_4(q)$ for even q, we must establish Step 5 of Huppert's argument. This requires proving that $G = G' \times C_G(G')$. Suppose that $q = 2^f$. As $G' \cong \mathrm{PSp}_4(q)$ and q is even, $|\mathrm{Out}(G')| = f$. By the same reasoning as presented in Section 2.6, we must have that $\mathrm{cd}(G)$ contains degrees $\chi(1)$ and $e\chi(1)$ for some $e > 1$. Examining all possibilities for $\chi(1)$ and e shows that

$$\chi(1) = \frac{1}{2}q(q^2+1),\ \chi(1) = (q-1)(q^2+1), \quad \text{or} \quad \chi(1) = (q+1)(q^2+1),$$

and

$$e = 2(q-1),\ e = 2(q+1),\ e = q-1,\ e = q,\ \text{or}\ e = q+1.$$

As $e \mid f$, we have

$$f \geq e \geq q-1 = 2^f - 1.$$

Thus $f \geq 2^f - 1$, which implies $f = 1$. Hence there is no nontrivial outer automorphism, a contradiction. Thus $G = G' \times C_G(G')$ and Step 5 is verified.

With Steps 2 and 5 verified, Huppert's Conjecture is established for $\mathrm{PSp}_4(q)$ for even q.

CHAPTER 7

VERIFYING HUPPERT'S CONJECTURE FOR THE SIMPLE SYMPLECTIC GROUPS OF RANK TWO IN ODD CHARACTERISTIC

In this chapter, we conclude the dissertation by outlining the proof of Huppert's Conjecture for the family of simple symplectic groups in odd characteristic. Again, we must establish Steps 2 and 5 of Huppert's argument. In [16], Huppert verifies his conjecture for $\mathrm{PSp}_4(q)$ when $q = 3$, 5, or 7, so we will consider the case when $q > 7$.

7.1 Results Concerning the Character Degrees of $\mathrm{PSp}_4(q)$, q Odd

For odd $q > 7$,

$$\begin{aligned}\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PSp}_4(q)) = \{&1,\ (q^2-1)^2,\ (q-1)(q^2+1),\ q(q-1)(q^2+1),\ \frac{1}{2}(q^4-1), q(q^2+1),\\ &\frac{1}{2}(q^2+1), \frac{1}{2}q(q^2+1),\ \frac{1}{2}q^2(q^2+1),\ \frac{1}{2}q(q+1)^2,\\ &\frac{1}{2}q(q-1)^2,\ q^4,\ q^4-1,\ (q+1)(q^2+1),\ q(q+1)(q^2+1),\\ &\frac{1}{2}(q-1)^2(q^2+1),\ (q-1)^2(q^2+1),\ \frac{1}{2}(q+1)^2(q^2+1),\\ &(q+1)^2(q^2+1), \frac{1}{2}(q+\epsilon)(q^2+1),\ \frac{1}{2}q(q+\epsilon)(q^2+1)\},\end{aligned}$$

with $\epsilon = (-1)^{(q-1)/2}$. First we determine which degrees of G are nontrivial powers of a prime.

Lemma 7.1.1. *For $q > 7$, the only nontrivial powers of a prime among the degrees of G are q^4 and possibly $\frac{1}{2}(q^2+1)$. In particular, $\frac{1}{2}(q^2+1)$ is possibly prime, a square of a prime, or $q = 239$ and $\frac{1}{2}(239^2+1) = 13^4$.*

Proof. As shown in [25], the only possible powers of a prime among the degrees of G are q^4 and $\frac{1}{2}(q^2+1)$. Let us examine the second case more closely. If $\frac{1}{2}(q^2+1) = p^d$, where p is prime and $d > 1$, then $q^2 + 1 = 2p^d$. As shown in [3], this Diophantine equation has no solution for integers $q > 1$, $p \geq 1$, and $d \geq 3$ odd. Further, $q^2 + 1 = 2p^4$ only has solution $(x, p) = (239, 13)$. Suppose $d > 4$ is even. If $4 \mid d$ and (x_1, p_1) satisfies ${x_1}^2 + 1 = 2{p_1}^d$, then ${x_1}^2 + 1 = 2\left({p_1}^{d/4}\right)^4$. Hence ${p_1}^{d/4} = 13$, a contradiction. If $4 \nmid d$, then d is divisible by an odd number, say $d = 2m$, where m is odd. If (x_2, p_2) satisfies ${x_2}^2 + 1 = 2{p_2}^d$, then ${x_2}^2 + 1 = 2\left({p_2}^{d/m}\right)^m$. Hence $(x_2, {p_2}^{d/m})$ is a solution to $q^2 + 1 = 2p^m$, a contradiction. Thus, $q^2 + 1 = 2p^d$ has no solution for even $d > 4$. Hence, if $\frac{1}{2}(q^2 + 1)$ is a power of a prime, it is either prime, a square of a prime, or $q = 239$ and $\frac{1}{2}(239^2 + 1) = 13^4$. $\square$

We will also need to know which pairs of character degrees of G are consecutive integers. By examining the degrees of G, it is possible to prove the following lemma.

Lemma 7.1.2. *The only pair of consecutive integers among the character degrees of G, for $q > 2$, is $q^4 - 1$ and q^4.*

Finally, we will need to know which degrees of G are odd. As q is odd, the only odd degrees of G are

$$q^4,\ \frac{1}{2}(q^2+1),\ \frac{1}{2}q(q^2+1),\ \text{and}\ \frac{1}{2}q^2(q^2+1).$$

Notice that the only odd degrees of G are the prime power q^4 or are prime power multiples of $\frac{1}{2}(q^2 + 1)$. Note that the odd degree $\frac{1}{2}q^2(q^2 + 1)$ is the only mixed degree of G divisible by q^2.

7.2 Establishing $G'/M \cong H$ when $H \cong \mathrm{PSp}_4(q)$, q Odd

7.2.1 Eliminating the Tits, Sporadic, and Alternating Groups for all k

By the same reasoning as in the proof of Proposition 2.3.1 of Section 2.3.2, it is possible to eliminate the alternating groups from consideration for S for any $k \geq 1$. Consider the possibility that S is a sporadic simple group or the Tits group. By the same reasoning as in the proof of Proposition 2.3.2, we only need to consider sporadic simple groups with 20 or less extendible characters of distinct degrees. Using the number of odd degrees of G and the properties of these odd degrees, it is possible to eliminate these sporadic simple groups and the Tits group from consideration. This leads to the following result.

Proposition 7.2.1. *The simple group S is not an alternating group A_n with $n \geq 7$, a sporadic simple group, or the Tits group.*

7.2.2 Special Cases in the Elimination of the Groups of Lie Type

Thus S is a simple group of Lie type. If χ is the Steinberg character of S, then $\chi(1)$ is a power of the prime p, where p is the defining characteristic of the group. By Lemma 2.3.3, χ extends to the automorphism group of S. Once again appealing to Lemma 2.3.1, we have that $\chi(1)^k$ is a degree of G. As the only composite powers of a prime among degrees of G are q^4 and possibly $\frac{1}{2}(q^2+1)$, we must have that $\chi(1)^k = q^4$ or $\chi(1)^k = \frac{1}{2}(q^2+1)$. In the latter case, if $q \neq 239$, we must have that $\chi(1)^k = p^2$ or $\chi(1)^k = p$. Thus, if $\chi(1)^k = \frac{1}{2}(q^2+1)$ and $q \neq 239$, then $k = 1$ or $k = 2$. As the only simple group with Steinberg character of degree p^2 is $\mathrm{PSL}_2(p^2)$, we have the possibilities that $k = 1$ and $S \cong \mathrm{PSL}_2(p^2)$ or $k \leq 2$ and $S \cong \mathrm{PSL}_2(p)$. In all other cases, the defining characteristic of the simple group S must be the same as the prime divisor of q^4. Hence we have the following cases.

Case 1: $q = 239$ and the k^{th} power of the degree of the Steinberg character of S is $\frac{1}{2}(239^2 + 1) = 13^4$.

Case 2: $S \cong \mathrm{PSL}_2(q_1)$ and the k^{th} power of the degree of the Steinberg character of S is q^4 or $\frac{1}{2}(q^2+1)$.

Case 3: $S \ncong \mathrm{PSL}_2(q_1)$ and the k^{th} power of the degree of the Steinberg character of S is q^4.

We begin with the assumption that $q = 239$ and the k^{th} power of the degree of the Steinberg character of S is $\frac{1}{2}(239^2+1) = 13^4$.

7.2.3 Case 1

When $q = 239$, we have that $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PSp}_4(239))$ has two prime power degrees, namely 239^4 and $\frac{1}{2}(239^2+1) = 13^4$. Suppose that the k^{th} power of the degree of the Steinberg character of S is 13^4.

Most of the simple groups of Lie type have Steinberg characters whose degrees are powers of a prime with exponent greater than 4. So it is not possible for the k^{th} power of these degrees to be 13^4.

The groups of Lie type in characteristic 13 with Steinberg characters of degree with exponent dividing 4 are $\mathrm{PSL}_2(13)$, $\mathrm{PSL}_2(13^2)$, $\mathrm{PSL}_2(13^4)$, and $\mathrm{PSp}_4(13)$.

Subcase 1(a): $S \cong \mathrm{PSL}_2(13^r)$ for $r = 1$, 2, or 4

For each possible value of r and resulting value of k, we can find $\chi \in \mathrm{Irr}(\mathrm{PSL}_2(13^r))$ such that $\chi(1)^k$ does not divide a degree of G, a contradiction.

Subcase 1(b): $S \cong \mathrm{PSp}_4(13)$

In this case, $k = 1$ since the degree of the Steinberg character of S is 13^4. All the character degrees of $\mathrm{PSp}_4(13)$ divide degrees of G. The outer automorphism group of $\mathrm{PSp}_4(13)$ has order 2 and consists of the trivial automorphism and a diagonal automorphism. Extending $\mathrm{PSp}_4(13)$ by the diagonal automorphism results in the group $\mathrm{PCSp}_4(13)$. Examining the irreducible characters of $\mathrm{PCSp}_4(13)$, we see that the unipotent characters of $\mathrm{PSp}_4(13)$ extend to $\mathrm{PCSp}_4(13)$, hence must be character degrees of G by Lemma 2.3.1. This results in many contradictions. For example, the unipotent character of $\mathrm{PSp}_4(13)$ of degree

$$\frac{1}{2}13(13^2+1) = 1105 = 5 \cdot 13 \cdot 17$$

extends to $\mathrm{PCSp}_4(13)$, and thus must be a degree of G. But examining the degrees of

$\mathrm{PSp}_4(239)$, we see that 1105 is not a character degree of G. So Case 1 is not possible.

7.2.4 Case 2

Subcase 2(a): $S \cong \mathrm{PSL}_2(q_1)$ for $q_1 \geq 5$ prime.

Suppose that $k = 1$. The degree of the Steinberg character of $\mathrm{PSL}_2(q_1)$ is q_1. As q_1 is prime, it is impossible for $q_1 = q^4$. Next, suppose that $q_1 = \frac{1}{2}(q^2+1)$. Now $q_1 \neq 5$ since this implies $q = 3$ and we are assuming that $q > 7$. Suppose $q_1 \geq 7$. Irreducible characters of degrees $q_1 - 1$, q_1, and $q_1 + 1$ extend to the automorphism group of S, hence must be character degrees of G. But $q_1 - 1$, q_1, and $q_1 + 1$ are three consecutive integers, and $\mathrm{cd}(G)$ does not contain three consecutive integers for $q > 2$. Hence $k \neq 1$.

Now consider the possibility that $k = 2$. We could have that ${q_1}^2 = q^4$ or ${q_1}^2 = \frac{1}{2}(q^2+1)$. Again, as q_1 is prime, it is not possible for ${q_1}^2 = q^4$. Suppose ${q_1}^2 = \frac{1}{2}(q^2+1)$. Note that $q_1 = 5$ is not possible (as we assume $q > 7$). Consider $q_1 > 5$. By Lemma 2.3.1, $(q_1-1)^2$ and $(q_1+1)^2$ are both character degrees of G. Hence, the sum

$$(q_1-1)^2 + (q_1+1)^2 = 2{q_1}^2 + 2 = q^2 + 3$$

is the sum of two character degrees of G. Examining the character degrees of G, we see that no two degrees of G sum to q^2+3. So this is not possible.

Subcase 2(b): $S \cong \mathrm{PSL}_2(q_1)$ for $q_1 \geq 4$ composite.

Suppose first that $k = 1$. The Steinberg character of S extends to $\mathrm{Aut}(S)$, hence to G, so first suppose that $q_1 = q^4$. Replacing q_1 by q^4 gives $q^4 + 1 \in \mathrm{cd}(S)$. But $q^4 + 1$ does not divide any of the character degrees of G, so we have a contradiction. If $q_1 = \frac{1}{2}(q^2+1)$, then

$$q_1 + 1 = \frac{1}{2}(q^2+1) + 1 = \frac{1}{2}(q^2+3)$$

must divide a character degree of G. But this is not possible. In particular, $q_1 = 4$ eliminates $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong A_5$ and $q_1 = 9$ eliminates $\mathrm{PSL}_2(9) \cong A_6$.

Now consider the possibility that $k = 2$. We could have that ${q_1}^2 = q^4$ or ${q_1}^2 = \frac{1}{2}(q^2+1)$. Suppose first that ${q_1}^2 = \frac{1}{2}(q^2+1)$. By Lemma 3.1.2, $\frac{1}{2}(q^2+1)$ is a square of a prime while q_1 is assumed to be composite. So this is not possible. Now suppose that ${q_1}^2 = q^4$. Then $q_1 = q^2$ and $q^2+1 \in \mathrm{cd}(S)$. As $k = 2$, $(q^2+1)^2$ is a character degree of G'/M, hence must divide a degree of G. But it is clear that $(q^2+1)^2$ divides no degree of G.

Finally, consider the possibility that $k > 2$. As we are assuming $q \neq 239$, ${q_1}^k = \frac{1}{2}(q^2+1)$ is not possible. Hence ${q_1}^k = q^4$, which implies $q = {q_1}^{k/4}$. Consider the irreducible character of G'/M found by multiplying $k-1$ copies of the Steinberg character with a character of S of degree $q_1 - 1$. As the degree of this character must divide a degree of G, we find that $k - 1 \leq \frac{2k}{4}$, which implies $k \leq 2$, a contradiction.

7.2.5 Case 3

We have shown that neither Case 1 nor Case 2 is possible. As Case 3 requires substantially more detailed arguments than the previous two cases, we will devote the following sections to Case 3. We are assuming $S \not\cong \mathrm{PSL}_2(q_1)$ and the k^{th} power of the degree of the Steinberg character of S is q^4. First we will show that $k = 1$ for the simple groups of Lie type.

7.2.6 Case 3: Eliminating the Groups of Lie Type when $k > 1$

Let χ denote the Steinberg character of S. Since we assume $\chi(1)^k = q^4$ in this case, a proof similar to that of Proposition 2.3.3 establishes the following result.

Proposition 7.2.2. *If $S = S(q_1)$ is a simple group of Lie type and $S \not\cong \mathrm{PSL}_2(q_1)$, then $k = 1$.*

Since the sporadic, Tits, and alternating groups have been eliminated as possibilities for S, we have that S is a simple group of Lie type and, thus, $k = 1$. We will now show that $S \cong \mathrm{PSp}_4(q)$ by eliminating all other possibilities for S.

7.2.7 Case 3: Eliminating Simple Groups of Exceptional Lie Type when $k = 1$

We have proved that $G'/M \cong S$, where $S = S(q_1)$ is a simple group of Lie type defined over the field of q_1 elements. We now want to show that $S \cong \mathrm{PSp}_4(q)$. We will begin by eliminating the possibility that S is a simple group of exceptional Lie type.

Proposition 7.2.3. *The group S is not a simple group of exceptional Lie type.*

To establish this result, we find characters of mixed degrees for most of the simple groups of exceptional Lie type which are too large to divide the mixed degrees of G. For ${}^2B_2({q_1}^2)$, ${q_1}^2 = 2^{2n+1}$, $n \geq 1$, and ${}^2F_4({q_1}^2)$, ${q_1}^2 = 2^{2m+1}$, $m \geq 1$, it is enough to note that the defining characteristic of S must be odd. Hence these groups cannot be candidates for S.

7.2.8 Case 3: Eliminating the Groups of Classical Lie Type when $k = 1$

We are only left with the possibility that $G'/M \cong S$, where S is a simple group of classical Lie type.

Proposition 7.2.4. *The simple group $S \cong \mathrm{PSp}_4(q)$.*

To establish Proposition 7.2.4, we again proceed by examining each family of simple groups of classical Lie type separately. As the Steinberg character of S extends to G, the degree of the Steinberg character of S, say ${q_1}^j$, is a character degree of G. Since we are in Case 3, we are assuming ${q_1}^j = q^4$. The mixed degrees of G are

$$q(q-1)(q^2+1),\ q(q^2+1),\ \frac{1}{2}q(q^2+1),\ \frac{1}{2}q^2(q^2+1),\ \frac{1}{2}q(q+1)^2,$$
$$\frac{1}{2}q(q-1)^2,\ q(q+1)(q^2+1),\ \text{and}\ \frac{1}{2}q(q+\epsilon)(q^2+1),$$

for $\epsilon = (-1)^{(q-1)/2}$. For groups of low rank, we then typically find a mixed character degree of S that is too large to divide all the mixed degrees of G except $q^2(q^2+1)/2$, and then use the divisibility properties of $q^2(q^2+1)/2$ (particularly that this degree of G is odd while the constructed degree of S is even) to show that this degree of S cannot divide $q^2(q^2+1)/2$ either. To eliminate the simple groups of Lie type of higher rank, we use the bounds established on the mixed degrees used in the proof of Proposition 2.3.5 to show that these degrees of S are too large to divide all the mixed degrees of G except $q^2(q^2+1)/2$. Again employing the parity argument, it is possible to show that the degree of S does not divide $q^2(q^2+1)/2$. Hence, this degree of S does not divide a degree of G, which is a contradiction. These arguments eliminate all simple groups of classical Lie type as candidates for S except $S \cong \mathrm{PSp}_4(q)$, which establishes the result. For a complete proof of this result, see Proposition 8.1 of [34].

This was the last case to consider to prove that $k = 1$ and $S \cong \mathrm{PSp}_4(q)$. This verifies Step 2 in Huppert's argument.

7.3 Proving Step 5 for $\mathrm{PSp}_4(q)$, q Odd

All that remains to establish Huppert's Conjecture for $\mathrm{PSp}_4(q)$, for $q > 7$ prime or a power of an odd prime, is to verify Step 5 in Huppert's argument. We have shown that $G'/M \cong \mathrm{PSp}_4(q)$. In [16], Huppert proved that $M = \langle 1 \rangle$ so we have that $G' \cong \mathrm{PSp}_4(q)$. Suppose $q = p^n$. Then $|\mathrm{Out}(G')| = 2n$ and every element of $\mathrm{Out}(G')$ can be written as the composition of diagonal and field automorphisms. Suppose $G' \times C_G(G') \lneq G$. By Lemma 1.7, G induces on G' some outer automorphism α. All notation is adapted from [31]. Let $\mathbf{F}_{q^4}^{\times} = \langle \kappa \rangle$, $\theta = \kappa^{q^2+1}$, $\gamma = \theta^{q+1}$, and $\eta = \theta^{q-1}$. Then $|\gamma| = q-1$ and $|\eta| = q+1$. Let

$$\delta = \begin{bmatrix} \gamma^{-1} & & & \\ & 1 & & \\ & & \gamma^{-1} & \\ & & & 1 \end{bmatrix}.$$

It can be shown that all diagonal automorphisms of G are induced by conjugation by powers of δ and conjugation by δ^2 is an inner automorphism of $\mathrm{PSp}_4(q)$. So modulo inner automorphisms, conjugation by δ is of order 2. Now α can be written as the composition of diagonal and field automorphisms. Let $\alpha = \phi^k\delta^t$, for some $0 \leq k \leq n-1$ and $t = 0$ or $t = 1$. We will examine the field automorphisms ϕ^k and diagonal automorphism δ in some detail. Note that δ refers to both the matrix element defined above and the diagonal automorphism of $\mathrm{PSp}_4(q)$. The meaning of δ will be clear from context.

It can be shown that δ interchanges the conjugacy classes A_{21} and A_{22}, A_{41} and A_{42}, C_{21} and C_{22}, and D_{21} and D_{22}, while these classes remain fixed under the field automorphisms. Hence, if $\alpha = \phi^k\delta$ for some $0 \leq k \leq n-1$, then α does not fix the conjugacy classes A_{21}, A_{22}, A_{41}, A_{42}, C_{21}, C_{22}, D_{21}, and D_{22}. Examining the character table of $G' \cong \mathrm{PSp}_4(q)$, we see that G' has characters θ_1 and θ_2 which are not fixed by α and must fuse in G. In particular, we have that the stabilizer of θ_1 is $\mathrm{PSp}_4(q)\langle\phi\rangle$. Hence, $I_G(\theta_1)/\mathrm{PSp}_4(q)$ is cyclic of order n. By Corollary 6.11(a) of [19], θ_1 extends to $I_G(\theta_1)$ and then induces irreducibly to G. As G contains α, $G \neq I_G(\theta_1)$. Let $\psi \in \mathrm{Irr}(G)$ such that $[\psi_{G'}, \theta_1] > 0$. Then $\psi(1) = e\theta_1(1)$, where $e > 1$ and $e \mid 2n$. But θ_1 has degree $\frac{1}{2}q^2(1+q^2)$ and no proper multiple of $\theta_1(1)$ is a degree of G. So we have a contradiction.

Thus $\alpha = \phi^k$ for some k satisfying $1 \leq k \leq n-1$. Consider the conjugacy class $B_6(1)$ of $\mathrm{PSp}_4(q)$. We can establish that $B_6(1)$ is moved by α. So $B_6(1)$ is moved by α to another class of the form $B_6(s)$, where $s \neq 1$.

We now claim that the irreducible character $\chi_2[2]$ (denoted by $\chi_2(2)$ in [31]) is not fixed by α^{-1}. On the class $B_6(1)$, this character has value $(1+q)\beta_2$, where $\beta_i = \eta^i + \eta^{-i}$. If $\chi_2[2] = \chi_2[2]^{\alpha^{-1}}$, then as

$$\begin{aligned}
\chi_2[2]^{\alpha^{-1}}(B_6(1)) &= \chi_2[2](B_6(1)^{\phi^k}) \\
&= \chi_2[2](B_6(s)) \quad \text{for some } s \neq 1 \\
&= (1+q)\beta_{2s} \\
&= (1+q)(\eta^{2s} + \eta^{-2s}),
\end{aligned}$$

this implies

$$(1+q)(\eta^2+\eta^{-2}) = (1+q)(\eta^{2s}+\eta^{-2s}).$$

It can be shown that

$$\eta^2+\eta^{-2} = \eta^{2s}+\eta^{-2s}$$

implies $p^n+1 \mid 2(p^k+1)$ or $p^n+1 \mid 2(p^k-1)$. Thus $p^n+1 \leq 2(p^k+1) \leq 2p^{n-1}+2$, hence $p^n - 2p^{n-1} \leq 1$ so $p^{n-1}(p-2) \leq 1$. But $p \geq 3$ and $n \geq 2$ so $p^{n-1}(p-2) \geq p \geq 3$, a contradiction. So $\chi_2[2]$ is not fixed by α^{-1}.

Now δ, the diagonal automorphism of $\mathrm{PSp}_4(q)$, fixes the conjugacy classes on which $\chi_2[2]$ is nonzero. In particular, we have that the stabilizer of $\chi_2[2]$ is $\mathrm{PSp}_4(q)\langle\delta\rangle$, where δ is the diagonal automorphism. Hence, $I_G(\chi_2[2])/\mathrm{PSp}_4(q)$ is cyclic of order 2. By Corollary 6.11(a) in [19], we have that $\chi_2[2]$ extends to $I_G(\chi_2[2])$ and then induces irreducibly to G. As G contains α, $G \neq I_G(\chi_2[2])$. Let $\psi \in \mathrm{Irr}(G)$ such that $[\psi_{G'}, \chi_2[2]] > 0$. Then $\psi(1) = e\chi_2[2](1)$, where $e > 1$ and $e \mid 2n$. But $\chi_2[2]$ has degree q^4-1 and no proper multiple of $\chi[2](1)$ is a degree of G. So we have a contradiction.

Thus, $\alpha = 1$ and $G = G' \times C_G(G')$, which verifies Step 5 in Huppert's argument. With Steps 2 and 5 verified, Huppert's Conjecture is proved for the family of simple groups $\mathrm{PSp}_4(q)$ for odd q.

BIBLIOGRAPHY

[1] J. An and S. Huang, Character tables of parabolic subgroups of the Chevalley groups of type G_2, *Comm. Alg.* **34** (2006), 1763–1792.

[2] M. Bianchi, D. Chillag, M.L. Lewis, and E. Pacifici, Character degree graphs that are complete graphs, *Proc. AMS.* **135** (2007), 671–676.

[3] Z. Cao and X. Dong, On Terai's Conjecture, *Proc. Japan Acad. Ser. A Math. Sci.* **74** (1998), 127–129.

[4] R.W. Carter, "Finite Groups of Lie Type: Conjugacy Classes and Complex Characters," Wiley, New York, 1985.

[5] B. Chang and R. Ree, The characters of $G_2(q)$, *Symposia Mathematica (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome)* **13** (1972), 395–413.

[6] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, "Atlas of Finite Groups," Oxford University Press, London, 1984.

[7] B. Cooperstein, Maximal subgroups of $G_2(2^n)$, *J. Algebra* **70** (1981), 23–36.

[8] H. Enomoto, The characters of the finite Chevalley group $G_2(q)$, $q = 3^f$, *Japan J. Math.* **2** (1976), 191–248.

[9] H. Enomoto and H. Yamada, The characters of $G_2(2^n)$, *Japan J. Math.* **12** (1986), 325–377.

[10] P. Erdös and J.L. Selfridge, The product of consecutive integers is never a power, *Illinois J. of Math.* **19** (1975), 292–301.

[11] W. Feit and G.M. Seitz, On finite rational groups, *Illinois J. of Math.* **33** (1989), 103–131.

[12] M. Hestenes, Singer groups, *Canad. J. Math.* **22** (1970), 492–513.

[13] B. Huppert, Some simple groups which are determined by the set of their character degrees I, *Illinois J. of Math.* **44** (2000), 828–842.

[14] B. Huppert, Some simple groups, which are determined by the set of their character degrees III, pre-print.

[15] B. Huppert, Some simple groups, which are determined by the set of their character degrees IV, pre-print.

[16] B. Huppert, Some simple groups, which are determined by the set of their character degrees V, pre-print.

[17] B. Huppert, Some simple groups, which are determined by the set of their character degrees VII, pre-print.

[18] B. Huppert and W. Lempken, Simple groups, whose order is divisible by only four primes, *Proc. of F. Scorina Gomel State Univ.* **16** (2000), 64–75.

[19] I.M. Isaacs, "Character Theory of Finite Groups," Dover, New York, 1994.

[20] O. King, The subgroup structure of finite classical groups in terms of geometric configurations, *London Mathematical Society Lecture Note Series 327* (2005), 29–56.

[21] P. Kleidman, The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups, *J. Algebra* **117** (1988), 30–71.

[22] M.L. Lewis and D.L. White, Nonsolvable groups satisfying the one-prime hypothesis, *Algebra Represent. Theory* **10** (2007), 379–412.

[23] W. Ljunggren, Einige Bermerkungen über die Darstellung ganzer Zahlen durch binäre kubische Formen mit positiver Diskriminante, *Acta Math.* **75** (1943), 1–21.

[24] G. Malle and A. Moretó, Non-divisibility among character degrees II: nonsolvable groups, *J. London Math. Soc.* **76** (2007), 667–682.

[25] G. Malle and A.E. Zalesskiĭ, Prime power degree representations of quasi-simple groups, *Arch. Math.* **77** (2001), 461–468.

[26] W. McDaniel, On multiple prime divisors of cyclotomic polynomials, *Math. Comp.* **28** (1974), 847–850.

[27] T. Nagell, Des équations indéterminées $x^2 + x + 1 = y^n$ et $x^2 + x + 1 = 3y^n$, *Norsk. mat. Foren. Skr. Serie I 2* (1921), 12–14.

[28] T. Nagell, "Introduction to Number Theory," Wiley, New York, 1951.

[29] P. Schmid, Extending the Steinberg representation, *J. Algebra* **150** (1992), 254–256.

[30] P. Schmid, Rational matrix groups of a special type, *Linear Algebra Appl.* **71** (1985), 289–293.

[31] M.A. Shahabi and H. Mohtadifar, The characters of the finite projective symplectic group $\mathrm{PSp}(4, q)$, *London Mathematical Society Lecture Note Series 305* (2001), 496–527.

[32] W. Simpson and J. Sutherland Frame, The character tables for $\mathrm{SL}(3, q)$, $\mathrm{SU}(3, q^2)$, $\mathrm{PSL}(3, q)$ and $\mathrm{PSU}(3, q^2)$, *Canad. J. Math.* **25** (1973), 486–494.

[33] T.P. Wakefield, Verifying Huppert's Conjecture for $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q^2)$, submitted.

[34] T.P. Wakefield, Verifying Huppert's Conjecture for $\mathrm{PSp}_4(q)$ when $q > 7$, submitted.

[35] H. Ward, On Ree's series of simple groups, *Trans. Amer. Math. Soc.* **121** (1966), 62–89.

[36] D.L. White, Degree graphs of simple linear and unitary groups, *Comm. Alg.* **34** (2006), 2907–2921.

[37] D.L. White, Degree graphs of simple orthogonal and symplectic groups, *J. Algebra* **319** (2008), 833–845.