STOCHASTIC INTEGRALS WITH RESPECT TO TEMPERED α-STABLE LEVY PROCESS

by

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Stochastic Integrals with Respect to Tempered α-Stable Levy Process

Abstract

by

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As we know, there are many equalities and inequalities for stochastic integrals. Some equalities and inequalities hold when the stochastic integrator has very nice characteristics, like Brownian motion. What will happen to the equalities or inequalities valid for the Brownian motion in the case of other stochastic process, as like proper tempered α-stable Levy process? A proper tempered α-stable Levy process combines both the α-stable and Gaussian trends. In a short time frame it is close to an α-stable process while in a long time frame it approximates a Brownian motion. So, we can find the keys that make analogs of these equalities and inequalities hold.

First I found two equalities and two inequalities that hold for the Brownian motion integral. There are two aspects to prove them. First of all, we consider when the integrand is predictable step process. Base on that it is a finite sum, we can get it from normal random variable inequalities. On the other hand, we need to extend the situation to where integrand is a general predictable process. It involves the problem whether what we deal with is integrable. Secondly, I will research if these equalities and inequalities hold for proper tempered α-stable Levy process. In this step, I will find the space of functions which are integrable for proper tempered α-stable Levy process at first. So that we can find the predictable process which is integrable for proper tempered α-stable Levy process then. And I will research if these equalities and inequalities still hold for integrals of predictable step integrands with respect to proper tempered α-stable Levy processes. At last, I will extend the predictable step condition to general predictable condition. Based on the research about proper tempered α-stable Levy process, there are some tools that make these equalities and inequalities hold. I will prove that a process $X(t)$ which is a Levy process and martingale with $EX(t)^2 = t$ satisfies these equalities and inequalities.

The results are based on the the book [1] by Kwapien and Woyczynski and paper [2] by Rosinski.

1. Introduction

In this thesis, there are two equalities and two inequalities for integrals of predictable integrands with respect to stochastic process. We know these equalities and inequalities hold when the process is Brownian motion. Then we thought what if the process is like Brownian motion but not Brownian motion. For the proper tempered α-stable Levy process, in a long time frame it approximates a Brownian motion. So we tested the equalities and inequalities for a proper tempered α-stable Levy process X. To do this, we need to find whether the predictable processes are X-integrable, where X(t) is a proper tempered α-stable Levy process. Whether a predictable process is X-integrable is determined by the space of regular functions which are X-integrable. To get this space, we define a control measure and a space defined by the control measure. The space and the condition in equalities and inequalities make these predictable processes X-integrable. We didn't calculate the integral with general predictable processes, we used the predictable step processes, and used the predictable step processes to approximate the general predictable processes. For calculating with the predictable step processes, we used some theorems about martingales and obtained the relevant equalities and inequalities in the case of drift $b = 0$ and $EX(t)^2 = t$. As a result, we got these equalities and inequalities for the proper tempered α-stable Levy process. Then we made an effort to extend these result to a Levy process and Martingale with $EX(t)^2 = t$.

In Chapter 2, we introduce some concepts we will use to get the equalities and inequalities. We will introduce the the Levy Characteristics, and then use Levy Characteristics to define control measure so that there is a space about control measure which equals the space of integrable functions. And the concepts of predictable process and proper tempered α-stable process are introduced. Also we will state the basic equalities and inequalities need in further work.

In Chapter 3, we find the spaces of integrable functions with respect to proper tempered α-stable Levy process.

In Chapter 4, we test equalities and inequalities for proper tempered α-stable process, and find that when the proper tempered α-stable Levy process has some supplement at characteristics, these equalities and inequalities hold. Next we consider just a process just Levy process and martingale with $EX(t)^2 = t$. Then we prove the relevant equalities and inequalities hold.

In Chapter 5, we summarize our results, and we point out some future directions of research.

2. Background Information

2.1 Definitions

Here are some definitions we will use to research my topic:

2.1.1 Some Symbols

$p^* = max\{p, p/(p-1)\}$

$S_n^* = max_{1 \le k \le n} |S_k|$

$S^* = sup_{1 \le k < \infty} |S_k|$

$\mathit{\Sigma}^*$ —— maximum or supremum of partial sums

$|X|_0 = E(min\{1, |X|\})$

$[\![X]\!] = X$ if $|X| \le 1$, and $= X/|X|$ if $|X| > 1$

$\mathbf{L}^{det}(dX)$: the class of deterministic functions which are integrable with respect to process X

$\ldots$small$\ldots$small: A is small if and only if B is small: For any $\epsilon > 0$, there exist $\delta > 0$, such that if we have $A < \delta$, then we get $B < \epsilon$. And the converse is true.

2.1.2 Levy Processes

A stochastic process$X = \{X_t \colon\ t \geq 0\}$ is said to be a Levy process if it satisfies the following properties:

1. X_0=0 almost surely,
2. Independence of Increments: For any of $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, X_{t_2}-X_{t_1}, X_{t_3}-X_{t_2},. . . ,X_{t_n}-$X_{t_{n-1}}$ are independent,
3. Stationary increments: For any $s < t$, X_t-X_s is equal in distribution to X_{t-s},
4. Continuity in probability: For any $\epsilon > 0$ and t$\geq$ 0, $\lim\limits_{h\to 0} P(|X_{t+h} - X_t| > \epsilon) = 0$.

2.1.3 Brownian Motion

A Brownian motion is a Levy process X_t such that $X_t - X_s \sim \mathcal{N}(0, t - s)$ (for 0≤s≤t).

2.1.4 Modulars

Let $\mathbf{E}$ be a linear space. A functional $\Phi : \mathbf{E} \to [0, \infty]$ will be called a modular if

- $\Phi(0) = 0$;
- For each $x \in$**E**, the function $g(t) = \Phi(tx)$ is continuous and even on $\mathbf{R}$ and is nondecreasing on $\mathbf{R}^+$

2.1.5 Musielak-Orlicz spaces

Let $(T, \mathcal{A}, \mu)$ be a complete, $\sigma-$finite, separable measure space and let $\varphi : T \times \mathbf{R} \to \mathbf{R}^+$ be such that

1. For every $t \in T$, $\varphi(t, .)$ is a symmetric, continuous function on $\mathbf{R}$ with $\varphi(t, 0) = 0$, and it is nondecreasing on $\mathbf{R}^+$;

2. For every $x \in \mathbf{R}$, $\varphi(., x)$ is $\mathcal{A}$-measurable;

3. φ is of moderate growth, i.e., there exist positive constants C, C_1 such that
$$\varphi(t, Cx) \leq C_1 \varphi(t, x),$$
for all $x \in \mathbf{R}$ and $t \in T$.

Then, for every $\mathcal{A}$-measurable and μ-almost everywhere finite function $f : T \to \mathbf{R}$, the superposition $\varphi(t, f(t))$ is $\mathcal{A}$-measurable, and the formula
$$\Phi(f) = \int_T \varphi(t, f(t))\mu(dt)$$
defines a modular on the space
$$L^{\varphi}(T, \mathbf{A}, \mu) = L^{\varphi} := \{f : \Phi(f) < \infty\} \qquad ,$$
which is called the Musielak-Orlicz space.

2.1.6 Filtration

A filtration $\mathcal{F}$ is an indexed set S_i of subobjects of a given algebraic structure S, with the index i running over some index set I that is a totally ordered set, subject to the condition that if $i \leq j$ in I, $S_i \subseteq S_j$.

2.1.7 $\mathcal{F}(t)$-independent increments

A process $X(t)$, $t \in T$, has $\mathcal{F}(t)$-independent increments, if $X(t)$ is $\mathcal{F}(t)$-measurable for every $t \in T$, and if for any $0 < t < s < t_\infty$, the random variable $X(s) - X(t)$ is independent of $\mathcal{F}(t)$.

2.1.8 Adapted Process

Let

- $(\Omega, \mathcal{F}, P)$ be a probability space;

- $\mathbf{I}$ be an index set with a total order $\leq$;

- $\mathcal{F} = (\mathcal{F}_i)_{i\in I}$ be a filtration of the sigma algebra $\mathcal{F}$;
- $(\mathbf{S}, \Sigma)$ be a measurable space;
- $X : \mathbf{I} \times \Omega \to \mathbf{S}$ be a stochastic process.

The process X is said to be adapted to the filtration $(\mathcal{F}_i)_{i\in I}$, if the random variable $X_i : \Omega \to \mathbf{S}$ is a $(\mathcal{F}_i, \Sigma)$-measurable function for each $i \in I$.

2.1.9 Predictable Process(Continuous-time)

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, then a continuous-time stochastic process $(X_t)_{t\geq 0}$ is predictable, if X considered as mapping from $\Omega \times \mathbf{R}_+$ is measurable with respect to the σ-algebra generated by all left-continuous adapted processes.

2.1.10 Predictable Step Process

F is a predictable step process if it is a finite sum of processes of the form $\xi I_{(s,r]}(t)$, $t \in T$, where the random variable ξ is $\mathcal{F}_s$-measurable.

Integral for such processes

If

$$F(t) = \sum_{k=1}^{n} \xi_k I_{(s_k,r_k]}(t)$$

then

$$\int_T f(t)dX(t) := \sum_{k=1}^{n} \xi_k [X(r_k) - X(s_k)].$$

2.1.11 Martingale

(1)A continuous-time Martingale is a stochastic process X_t with characteristics that for all t

1. $E(|X_t|) < \infty$,
2. $E(X_t|\{X_\tau, \tau \leq s\}) = X_s$ for any s $\leq$ t;

(2)A discrete-time martingale is a discrete-time stochastic process $X_1, X_2, X_3 \dots$ that satisfies for any time n,

1. $E(|X_n|) < \infty$,

2. $E(X_{n+1}|X_1, \dots, X_n) = X_n$.

2.1.12 $(\mathcal{F}_i)$-martingale

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$. A sequence $M_0, M_1, M_2, \dots$ of random variables with values in $\mathbf{R}$ is said to be an $(\mathcal{F}_i)$-martingale if $M_0 = 0$, and $E(M_i|\mathcal{F}_{i-1}) = M_{i-1}$ a.s., for $i = 1, 2, \dots$. And the sequence $\Delta M_n := M_n - M_{n-1}$, $n = 1, 2, \dots$, is its difference sequence.

2.1.13 (Proper) Tempered α-stable processes

A probability measure μ on $\mathbf{R}$ is called tempered α-stable if it is infinitely divisible without Gaussian part and has Levy measure M that can be written as

$$M(du) = u^{-\alpha-1}q(|u|, sgn(u))\sigma(sgn(u))du$$

where $\alpha \in (0, 2)$ and σ is a finite measure on $\{1, -1\}$, and $q : (0, \infty) \times \{-1, 1\} \to (0, \infty)$ is a Borel function such that $q(., sgn(u))$ is completely monotone with $q(\infty, sgn(u)) = 0$ for each $sgn(u) \in \{-1, 1\}$. μ is called a proper tempered α-stable distribution if, in addition to the above, $q(0+, sgn(u)) = 1$ for each $sgn(u) \in \{-1, 1\}$.
The complete monotonicity of $q(., sgn(u))$ means that $(-1)^n \frac{\partial^n}{\partial r^n} q(u, sgn(u)) > 0$ for $u \in \mathbf{R}$, $sgn(u) \in \{-1, 1\}$, and $n = 0, 1, 2, 3 \dots$.
A Levy process $X(t)$ such that $X(1)$ has a (proper) TαS distribution will be called a (proper) TαS Levy process.

2.2 Theorems about and Illustrations for the basic concepts

(Background Information in this section is from [1], [2], [4] and [5].) Considering stochastic integrals, we need an additive stochastic set function which is extendable to $\sigma-$Additive stochastic set function. We have some properties which are equivalent to the $\sigma-$Additive Extendability. About this, we have following theorems:

2.2.1 σ-Additive Extendability for Set Functions

Theorem 1 *Let $\boldsymbol{m} : \mathcal{A}_0 \to \boldsymbol{F}$ be an additive set function, where $\boldsymbol{F}$ is a complete metric linear space. Then the following two conditions are equivalent:*

1. *$\boldsymbol{m}$ can be extended to an $\boldsymbol{F}$-valued measure $\boldsymbol{m} : \mathcal{A} = \sigma(\mathcal{A}_0) \to \boldsymbol{F}$;*

2. *If $A_n \in \mathcal{A}_0$ and $\limsup_{n\to\infty} A_n = \emptyset$, then $\lim_{n\to\infty} \boldsymbol{m}(A_n) = 0$.*

(From [1, Theorem 7.1.1])

2.2.2 σ-Additive Extendability for Stochastic Set Functions

Theorem 2 *Let $\boldsymbol{F} = \boldsymbol{L}^p(\Omega, \mathcal{F}, P)$, $0 \le p < \infty$, and let $\boldsymbol{m} : \mathcal{A}_0 \to \boldsymbol{F}$ be an additive stochastic set function. We difine*

$$\rho_{\boldsymbol{m}}(f) := \sup_{v\in \boldsymbol{S}, |v|\le 1} |\int_T v f d\boldsymbol{m}|,$$

Then the following two conditions are equivalent:

1. *$\boldsymbol{m}$ can be extended to an $\boldsymbol{F}$-valued stochastic measure $\boldsymbol{m} : \mathcal{A} = \sigma(\mathcal{A}_0) \to \boldsymbol{F}$;*

2. *For each sequence of functions $f_n \in \boldsymbol{S}$ with $|f_n| \le 1$ and such that $\lim_{n\to\infty} f_n(t) = 0$ for all $t \in T$, we have that $\lim_{n\to\infty} \boldsymbol{\rho_m}(f_n) = 0$, where $\boldsymbol{S}$ is the class of step functions on $\boldsymbol{T}$ which are $\mathcal{A}_0$-measurable.*

(From [1, Theorem 7.1.2])

2.2.3 Levy Characteristics

(see [1, Section8.2])

Let a process X(t),$t \in T = [0, t_\infty]$, $t_\infty < \infty$, be a real stochastic process with independent increments. And assume its sample paths are right continuous and have left limits. Let $\pi^n = \{(t_k^n : 0 < t_0^n < \cdots < t_{k_n}^n = t_\infty\}$, $n = 1, 2, \ldots$, be a normal sequence of partition of T (i.e., $max_k|t_k^n - t_{k-1}^n| \to 0$ as $n \to \infty$), which is also assumed to be nested, i.e., $\pi^n \subset \pi^{n+1}$, $n = 1, 2, \ldots$. For each $n = 1, 2, \ldots$, the sequence of increments

$$d_k^n := X(t_k^n) - X(t_{k-1}^n),\ k = 1, \ldots, k_n.$$

Let

$$B_n(t) := \sum_{k:t_k^n \leq t} E[\![d_k^n]\!],$$
$$U_n(t) := \sum_{k:t_k^n \leq t} E[\![d_k^n]\!]^2,$$
$$V_n(t) := \sum_{k:t_k^n \leq t} E[\![d_k^n]\!]^2 - (E[\![d_k^n]\!])^2,$$
$$P_n(t) := \sum_{k:t_k^n \leq t} Ef(d_k^n),$$

where $f : \mathbf{R} \to \mathbf{R}$ is a fixed continuous function such that for some $r, c > 0$, $|f(x)| \leq c$, and $|f(x)| = 0$ for $|x| \leq r$.The class of such functions is denoted by $\mathcal{R}_0$.

Assume that (π^n) is a normal nested sequence of partition of T such that all the points of stochastic discontinuity of X are contained in $\cup_{n=1}^{\infty} \pi^n$.(A process Y is said to be stochastically continuous at $t \in T$, if $lim_{s \to t}|Y(t) - Y(s)|_0 = 0$).

Then for each $t \in T$, the limits

$$B(t) := \lim_{n\to\infty} B_n(t),$$
$$U(t) := \lim_{n\to\infty} U_n(t),$$
$$V(t) := \lim_{n\to\infty} V_n(t),$$
$$P(t) := \lim_{n\to\infty} P_n(t),$$

exist and the convergence is uniform on T.

This is proved in [1, Proposition 8.2.2].

Now we consider that the the process is Levy process. Here gives a definition of the Levy characteristics for Levy process.

The first Levy characteristic is the function

$$B(t) := \lim_{n\to\infty} \sum_{k:t_k^n} E[\![d_k^n]\!] = EX(t) = bt, \text{ for some b.}$$

The second Levy characteristic is a measure $\overline{\nu}$ on $(\mathbf{R}\backslash\{0\})$ determined by the following condition: For any $t \in T$ and any $f \in \mathcal{R}_0$,

$$t \int_{\mathbf{R}\backslash\{0\}} f(x)\overline{\nu}(dx) = \lim_{n\to\infty} \sum_{k:t_k^n<t} Ef(d_k^n)$$

Let $f_m(x) = [\![x]\!]^2 I_{x>\frac{1}{m}}(x)$, then

$$\begin{aligned} t\int_{\mathbf{R}\backslash\{0\}} f_m(x)\overline{\nu}(dx) &= \lim_{n\to\infty} \sum_{k:t_k^n<t} Ef_m(d_k^n) \\ &\le \lim_{n\to\infty} \sum_{k:t_k^n<t} E[\![d_k^n]\!]^2 = U(t) \end{aligned}$$

So

$$\begin{aligned} t\int_{\mathbf{R}\backslash\{0\}} [\![x]\!]^2\overline{\nu}(dx) &= \lim_{n\to\infty} t\int_{\mathbf{R}\backslash\{0\}} f_m(x)\overline{\nu}(dx) \\ &\le \lim_{n\to\infty} U(t) = U(t) \end{aligned}$$

Now define the third Levy characteristic C as follows:

$$C(t) := U(t) - t\int_{\mathbf{R}\backslash\{0\}} [\![x]\!]^2\overline{\nu}(dx) = ct \text{ for some } c.$$

Additional theorem: *If X is a Levy process, then the Levy characteristics determine its characteristic function via the Levy-Khinchine formula:*

$$Ee^{iuX(t)} = exp(t(ibu - \tfrac{cu^2}{2} + \int_{\boldsymbol{R}\backslash\{0\}}(e^{iux} - 1 - iu[\![x]\!])\overline{\nu}(dx))$$

This can be obtained from [1, Proposition 8.2.3]

2.2.4 Deterministic Integrands

(see [1, Section8.3])

First, let us introduce the control measure ν for a Levy process by the formula

$$\nu(ds) = |dB(s)| + dC(s) + \textstyle\int_{\mathbf{R}} [\![x]\!] \overline{\nu}(dx)ds \quad ,$$

where B,$\overline{\nu}$ and C are Levy characteristics of X. Using Radon-Nikodym Theorem we define $b(s), c(s)$, so that

$$dB(s) = b(s)\nu(ds), \text{ and } dC(s) = c(s)\nu(ds) \quad ,$$

and the kernel $\hat{\mu}$ so that

$$\overline{\nu}(dx)ds = \hat{\mu}(dx)\nu(ds)$$

We know $b(s)$ and $c(s)$ are constant, then we can replace them by b and c.

For $s \in T$ and $x \in \mathbf{R}$, let

$$k(x) = \textstyle\int_{\mathbf{R}} [\![xu]\!]^2 \hat{\mu}(du) + cx^2$$

$$l'(x) = \textstyle\int_{\mathbf{R}} ([\![xu]\!] - x[\![u]\!])\hat{\mu}(du) + bx$$

$$l(x) = \sup_{|y| \leq |x|} l'(y)$$

$$\varphi(x) = k(x) + l(x)$$

Let $$L^{\varphi}(d\nu) = \{f | \mathit{\Phi}_X(f) := \textstyle\int_T \varphi(f(s))\nu(ds) < \infty\}$$

which is a Musielak-Orlicz space.(We have $\varphi(2x) \leq 5\varphi(x)$)

Let X be a Levy process. The additive stochastic set function $\boldsymbol{m}$ generated by

$$m((s,t]) := X(t) - X(s) \quad ,$$

can be extended to a stochastic measure. Indeed we have for any step function f, the modular $\boldsymbol{\rho_m}(f)$ is small if and only if $\mathit{\Phi}_X(f)$ is small.(from [1, Theorem 8.3.1]) For each sequence of step functions f_n with $|f_n| \leq 1$ and such that $\lim_{n\to\infty} f_n(t) = 0$ for all $t \in T$, we have that

$$\lim_{n\to\infty} \boldsymbol{\mathit{\Phi}}_X(f_n) = \boldsymbol{\mathit{\Phi}}_X(lim_{n\to\infty} f_n) = \boldsymbol{\mathit{\Phi}}_X(0) = 0$$

so we have $\lim_{n\to\infty} \boldsymbol{\rho_m}(f_n) = 0$, which satisfies [Theorem 2, 2], so we have [Theorem 2, 1], meaning $\boldsymbol{m}$ can extend to a stochastic measure.

And we have the following theorem:

Theorem *A function $f : T \to \boldsymbol{R}$ is X-integrable if and only if $f \in \boldsymbol{L}^{\varphi}(d\nu)$*

(see [1, Theorem 8.3.1], [4, Theorem 3.3] and [5])

2.2.5 Predictable Integrands

(See [1, Section 9.1])

Let $X(t)$, $t \in T$, be a Levy process . For a predictable step F, by definition, we know it is X-integrable, and we define $\boldsymbol{\rho}_X(F) := \sup_{V \in \boldsymbol{P_1}} |\int_T VFdX|_0$, where $\mathbf{P}_1$ denotes the class of all predictable processes V such that $|V| \leq 1$. The Levy characteristic B has bounded variation, so $\boldsymbol{\rho}_X(F)$ is small if and only if $|\Phi_X(F)|_0$ is small. Then we have that the associated additive stochastic set function $\boldsymbol{m}$ can extends to a σ-additive stochastic measure. The proof is the similar as the one in last subsection.

If for almost every $\omega \in \Omega$, process $F(.,\omega) \in \mathbf{L}^{det}(dX)$, then predictable process F is X-integrable. And in $\mathbf{L}^{rnd}(dX)$, the set of predictable step processes is dense with respect to the metric given by the $\boldsymbol{\rho}_X(F)$ or $|\Phi_X(F)|_0$.

To research these equalities and inequalities, we calculate them for integrals of predictable step processes. In this way, we need the following theorems:

2.2.6 Martingale Inequalities

Theorem 3 (From [1, Theorem 5.3.1]) *Let $X_1, \ldots, X_n$, and $Y_1, \ldots, Y_n$ be two $(\mathcal{F}_i)-$martingale difference sequences in $\boldsymbol{R}$ such that X_i is subordinated to Y_i, i.e., $|X_i| \leq |Y_i|$ a.s. for $i = 1, 2, \ldots, n$, with $M_n = \sum_{k=1}^{n} X_i$ and $N_n = \sum_{k=1}^{n} Y_i$. Then , for any $t > 0$, and for each $p > 1$,*

1. $tP(M_n^* > t) \leq 2E|N_n|$,

2. $E|M_n|^p \leq (p^* - 1)^p E|N_n|^p$

Theorem 4 (From [1, Theorem 5.6.1]) *Let $M_1, \dots M_n$ be a $(\mathcal{F}_i)$-martingale with values in $\boldsymbol{R}$, with increments (ΔM_i) and the square function $[M_n] = \sum_{i=1}^{n} |\Delta M_i|^2$. Then, for any $1 < p < \infty$,*

$$E[M_n]^{p/2} \leq (p^* - 1)^p E|M_n|^p$$

and

$$E|M_n|^p \leq (p^* - 1)^p E[M_n]^{p/2}$$

Through the properties mentioned above, we can get the two equalities and two inequalities hold for Brownian motion.

2.2.7 Equalities and Inequalities for Brownian Integrals

(see [1, Section 9.2]) Let B be a process with $\mathcal{F}$(t)-independent increments which is also a Brownian motion process and let F be an $\mathcal{F}$(t)-predictable process such that $E \int_T F^2(t)dt < \infty$. Then

$$E \int_T F dB = 0,$$

and

$$E(\int_T F dB)^2 = E \int_T F^2 dt.$$

For each $a > 0$,

$$P(\int_T^* F(t)dB(t) > a) \leq \tfrac{2}{a} E|\int_T F(t)dB(t)|,$$

for each $p > 1$, and $p^* = p \bigvee p'$,

$$\tfrac{1}{p^*-1}(E|\int_T F(t)dB(t)|^p)^{1/p}$$
$$\leq (E(\int_T F^2(t)dt)^{p/2})^{1/p}$$
$$(p^* - 1)(E|\int_T F(t)dB(t)|^p)^{1/p}$$

The specific stochastic process we research is proper tempered α-stable Levy process. The following illustration will introduce the properties of it:

2.2.8 Example of (Proper) Tempered α-stable Distribution

(see [2]) The q in Levy measure of tempered α-stable distribution can be represented as

$$q(|u|, sgn(u)) = \int_0^\infty e^{-|u|s} Q(ds|sgn(u))$$

where $Q(.|sgn(u))$ is a Borel measures on $(0, \infty)$. $Q(., sgn(x))$ are probability measures in case of proper TαS distributions.

Define a measure Q on $\mathbf{R}$ by

$$Q(A) := \int_{\mathbf{R}} I_A(u) Q(d|u||sgn(u))\sigma(sgn(u)),\ A \in \mathscr{B}(\mathbf{R}).$$

We also define a measre R by

$$R(A) := \int_{\mathbf{R}} I_A(\tfrac{x}{|x|^2})|x|^\alpha Q(dx).$$

Then we will have the theorem:(from [2, Theorem2.3])

Theorem *The Levy measure M of a $T\alpha S$ distribution can be written in the form*

$$M(A) := \int_{\boldsymbol{R}} \int_0^\infty I_A(tx) t^{-\alpha-1} e^{-t} dt R(dx),\ A \in \mathscr{B}(\boldsymbol{R})$$

where R is a unique measure on $\boldsymbol{R}$ such that

$$R(\{0\}) = 0 \text{ and } \int_{\boldsymbol{R}} (min\{|x|^2, |x|^\alpha\}) R(dx) < \infty.$$

If R is a measure satisfying $R(\{0\}) = 0$ and $\int_{\boldsymbol{R}} (min\{|x|^2, |x|^\alpha\}) R(dx) < \infty$, then $M(A) := \int_{\boldsymbol{R}} \int_0^\infty I_A(tx) t^{-\alpha-1} e^{-t} dt R(dx)$ defines the Levy measure of a $T\alpha S$ distribution.

M corresponds to a proper $T\alpha S$ distribution if and only in $\int_{\boldsymbol{R}} |x|^\alpha R(dx) < \infty$.

From this theorem, we can research the proper $T\alpha S$ process in terms of $R(dx)$ satisfying these condition.

3. Deterministic Integrands for Proper Tempered α-Stable Levy Process

We research proper tempered α-stable Levy process instead of general Tempered α-stable Levy process, because the proper one has properties which make it easier to study. For example, based on Levy measure, the general one includes stationary $\alpha-$stable processes, but the proper one doesn't include any stationary $\alpha-$stable processes.

3.1 Step 1: Extending to a Stochastic Measure

We know proper tempered α-stable Levy process $X(t)$ has stationary increments, which means there exists fixed constant b such that $E(X(t-s)) = E(X(t)-X(s)) = b(t-s)$. So $B(t) = bt$. We get Levy characteristic B is a function of bounded variation. By statement in [2.2.4] , we know the additive stochastic set function generated by $X(t)$ can extend to a stochastic measure.

3.2 Step 2: Characterization Modular φ

For the Levy characteristics, we have

$$B(t) = bt \text{ and } a = \int_{\mathbf{R}} [\![x]\!]^2 \overline{\nu}(dx) < \infty \text{ and } C(t) = ct$$

so

$$\nu(ds) = \kappa ds, \text{ where } \kappa = |b| + c + a,$$

$$\hat{\mu}(dx) = \kappa^{-1}\nu(dx)$$

$$c(s) = \kappa^{-1}c \text{ and } b(s) = \kappa^{-1}b$$

then

$$k(x) = \kappa^{-1}(\textstyle\int_{\mathbf{R}} [\![xu]\!]^2 \overline{\nu}(du) + cx^2)$$

$$l'(x) = \kappa^{-1}(\textstyle\int_{\mathbf{R}} ([\![xu]\!] - x[\![u]\!])\overline{\nu}(du) + bx)$$

$$l(x) = \sup_{|y| \le |x|} l'(y)$$

$$\varphi(x) = k(x) + l(x)$$

For proper tempered α-stable process, we have $c = 0$.(From [2. Theorem2.9] to get characteristic function and compared to Levy-Khintchine representation)(or just from that Tαs distribution is without Gaussian part). And by statement in [2.2.11], for any Levy measure $\overline{\nu}$ of proper TαS Levy process, there exists a measure R on $\mathbf{R}$ such that $R(\{0\}) = 0$ and $\int_{\mathbf{R}} |x|^\alpha R(dx) < \infty$, then the Levy measure $\overline{\nu}(A) = \int_{\mathbf{R}} \int_0^\infty I_A(tx) t^{-\alpha-1} e^{-t} dt R(dx)$, $A \in \mathscr{B}(\mathbf{R})$. And for any such measure R, $\overline{\nu}(A) = \int_{\mathbf{R}} \int_0^\infty I_A(tx) t^{-\alpha-1} e^{-t} dt R(dx)$ is a Levy measure of a proper TαS Levy process. So we can continue the research using the measure R in place of the Levy measure $\overline{\nu}$.

We have

$$\begin{aligned}
\int_{\mathbf{R}} [\![xu]\!]^2 \overline{\nu}(du) &= \int_{|u| < |\frac{1}{x}|} [\![xu]\!]^2 \overline{\nu}(du) + \int_{|u| \ge |\frac{1}{x}|} [\![xu]\!]^2 \overline{\nu}(du) \\
&= x^2 \int_{\mathbf{R}} u^2 \int_0^{\frac{1}{|ux|}} t^{1-\alpha} e^{-t} dt R(du) + \int_{\mathbf{R}} \int_{\frac{1}{|ux|}}^\infty t^{-1-\alpha} e^{-t} dt R(du) \\
&=: I_1 + I_2
\end{aligned}$$

And for $|x| > 1$,

$$\begin{aligned}\int_{\mathbf{R}} ([\![xu]\!] - x[\![u]\!])\overline{\nu}(du) &= \int_{|\frac{1}{x}|<|u|<1} \overline{\nu}(du) - x\int_{|\frac{1}{x}|<|u|<1} u\overline{\nu}(du) + \int_{|u|\geq 1} \overline{\nu}(du) - x\int_{|u|\geq 1} \overline{\nu}(du) \\ &= \int_{\mathbf{R}} \int_{\frac{1}{|ux|}}^{\frac{1}{|u|}} t^{-1-\alpha}e^{-t}dtR(du) - x\int_{\mathbf{R}} u\int_{\frac{1}{|ux|}}^{\frac{1}{|u|}} t^{-\alpha}e^{-t}dtR(du) \\ &+ \int_{\mathbf{R}} \int_{\frac{1}{|u|}}^{\infty} t^{-1-\alpha}e^{-t}dtR(du) - x\int_{\mathbf{R}} \int_{\frac{1}{|u|}}^{\infty} t^{-1-\alpha}e^{-t}dtR(du) \\ &=: I_3 - I_4 + I_5 - I_6\end{aligned}$$

For I_1, we have the following theorem:

Theorem 3.2 *Let R satisfy that $R(\{0\}) = 0$ and $\int_{\boldsymbol{R}} |x|^{\alpha} R(dx) < \infty$, then*

$$x^2 \int_{\boldsymbol{R}} u^2 \int_0^{\frac{1}{|ux|}} t^{1-\alpha}e^{-t}dtR(du) \sim \beta_1|x|^{\alpha}$$

for large $|x|$ and constant β_1(β_1 can be zero).

Proof:

We have

$$\begin{aligned}\int_{\mathbf{R}} u^2 \int_0^{\frac{1}{|ux|}} t^{1-\alpha}e^{-t}dtR(du) &\leq \int_{\mathbf{R}} u^2 \int_0^{\frac{1}{|ux|}} t^{1-\alpha}dtR(du) \\ &= \int_{\mathbf{R}} u^2 \frac{|u|^{\alpha-2}|x|^{\alpha-2}}{2-\alpha} R(du) \\ &= \frac{|x|^{\alpha-2}}{2-\alpha} \int_{\mathbf{R}} |u|^{\alpha} R(du)\end{aligned}$$

We know $\int_{\mathbf{R}} |x|^\alpha R(dx) < \infty$, so

$$\begin{aligned}
\frac{\partial(\int_{\mathbf{R}} u^2 \int_0^{\frac{1}{|ux|}} t^{1-\alpha} e^{-t} dt R(du))}{\partial |x|} &= \int_{\mathbf{R}} \frac{\partial(u^2 \int_0^{\frac{1}{|ux|}} t^{1-\alpha} e^{-t} dt)}{\partial |x|} R(du) \\
&= \int_{\mathbf{R}} u^2 \frac{1}{|ux|}^{1-\alpha} e^{-\frac{1}{|ux|}} (-1) \frac{1}{|u|} \frac{1}{x^2} R(du) \\
&= -|x|^{\alpha-3} \int_{\mathbf{R}} |u|^\alpha e^{-\frac{1}{|ux|}} R(du) \\
&\sim \beta_0 |x|^{\alpha-3} \text{ for large } |x| \text{ and constant } \beta_0
\end{aligned}$$

Then by L'Hopital's rule, we have

$$\int_{\mathbf{R}} u^2 \int_0^{\frac{1}{|ux|}} t^{1-\alpha} e^{-t} dt R(du) \sim \beta_1 |x|^{\alpha-2} \text{ for large } |x| \text{ and constant } \beta_1$$

So

$$x^2 \int_{\mathbf{R}} u^2 \int_0^{\frac{1}{|ux|}} t^{1-\alpha} e^{-t} dt R(du) \sim \beta_1 |x|^{\alpha} \text{ for large } |x| \text{ and constant } \beta_1$$

If $R(du) = 0$, then $\beta_1 = 0$ obviously.

The same way, we have $I_i \sim \beta_i |x|^\alpha$ for large $|x|$(β_i can be zero), and i=2,3. And when $\alpha \neq 1$, we have $|I_4| \sim \beta_4 |x|^\alpha$ for large $|x|$ (β_4 can be zero). When $\alpha = 1$, $|I_4| \sim \beta_4 |x| log|x|$ for large $|x|$ and some β_4(β_i can be zero).

And we have

$$I_5 \leq \int_{\mathbf{R}} \int_{\frac{1}{|u|}}^{\infty} t^{-1-\alpha} dt R(du) = \frac{1}{\alpha} \int_{\mathbf{R}} |u|^\alpha R(du) < \infty$$

and $I_6 = I_5 x$. And $I_i(x) = I_i(-x)$ for $i = 1, 2, 3$ and $I_i(x) = -I_i(-x)$ for $i = 4, 6$.

3.3 Step 3: Characterization spaces of Integrable Functions

If $I_1 + I_2 = 0$, then we have $R(du) = 0$, so $\varphi(x) = \frac{b}{\kappa}x$. If $b = 0$, then $X(t) = 0$. This is a trivial situation, where $\mathbf{L}^{det} = \mathbf{L}^0(dt)$. For $b \neq 0$, $\mathbf{L}^{det} = \mathbf{L}^1(dt)$.

For $I_1 + I_2 > 0$:

when $\alpha > 1$, we know $lim_{|x|\to\infty}\varphi(x)/|x|^\alpha < \infty$.

And we know

$|I_1 + I_2 + I_3 - I_4| \sim \beta|x|^\alpha$ for some $\beta \neq 0$ when $x \to +\infty$

or

$|I_1 + I_2 + I_3 - I_4| \sim \beta|x|^\alpha$ for some $\beta \neq 0$ when $x \to -\infty$.

So $\mathbf{L}^{det} = \mathbf{L}^\alpha(dt)$;

when $\alpha = 1$, we know

$|I_1 + I_2 + I_3 - I_6 + bx| \sim \beta|x|$ for some $\beta \neq 0$ when $x \to +\infty$

or

$|I_1 + I_2 + I_3 - I_6 + bx| \sim \beta|x|$ for some $\beta \neq 0$ when $x \to -\infty$.

So if $|I_4| = 0$

then $\mathbf{L}^{det} = \mathbf{L}^1(dt)$, else $\mathbf{L}^{det} = \mathbf{L}log\mathbf{L}(dt) := \{f : \int_T |f|log^+|f|dt < \infty\}$;

when $\alpha < 1$, if $I_6 = b$,

then $\mathbf{L}^{det} = \mathbf{L}^\alpha(dt)$, else $\mathbf{L}^{det} = \mathbf{L}^1(dt)$.

Conclusion

1. Let $\alpha > 1$, or $\alpha < 1$ and $I_6 = b$, or $\alpha = 1$ and $|I_4| = 0$. In these cases
$$\varphi(x) \sim \beta |x|^\alpha$$
for large $|x|$ and some constant $\beta \neq 0$, so $\mathbf{L}^{det}(dX) = \mathbf{L}^\alpha(dt)$

2. Let $\alpha = 1$ and $|I_4| = 0$, or $\alpha < 1$ and $I_6 \neq b$, or $R(du) = 0$ and $b \neq 0$,. In these cases
$$\varphi(x) \sim \beta |x|$$
for large $|x|$ and some constant $\beta \neq 0$, so $\mathbf{L}^{det}(dX) = \mathbf{L}^1(dt)$

3. Let $\alpha = 1$ and $|I_4| \neq 0$. In this case,
$$\varphi(x) \sim \beta |x| log |x|$$
for large $|x|$ and some constant β, so $\mathbf{L}^{det}(dX) = \mathbf{L} log \mathbf{L}(dt)$

4. Let $R(du) = 0$ and $b = 0$,In this case
$$\varphi(x) \text{ is constant}$$
so $\mathbf{L}^{det}(dX) = \mathbf{L}^0(dt)$

4. Predictable Integrands for Tempered α-stable Process

In this section we extend the results of Chapter 3 to predictable stochastic integrands.

Theorem 4.1 *If $\mathcal{F}(t)-$predictable process F is that $E\int_T F^2(t)dt < \infty$, then for almost every $\omega \in \Omega$, process $F(.,\omega) \in \boldsymbol{L}^2(dt)$.*

Proof:

Supposed not for almost every $\omega \in \Omega$, process $F(.,\omega) \in \mathbf{L}^2(dt)$. Then let $W = \{\omega \in \Omega : F(.,\omega) \notin \mathbf{L}^2(dt)\}$, we will have $P(W) = \epsilon > 0$. So

$$\begin{aligned} E\int_T F^2(t)dt &= \int_\Omega \int_T F^2(t,\omega)dtP(d\omega) \\ &\geq \int_W \int_T F^2(t,\omega)dtP(d\omega) = \infty \end{aligned}$$

which is contradictary to $E\int_T F^2(t)dt < \infty$. So For almost every $\omega \in \Omega$, process $F(.,\omega) \in \mathbf{L}^2(dt)$.

By the space $\mathbf{L}^{det}(dX)$ discussed before, where X is proper tempered $\alpha-$stable Levy process. we know $\mathbf{L}^2(dt) \subset \mathbf{L}^{det}(dX)$. So we know for almost every $\omega \in \Omega$, process $F(.,\omega) \in \mathbf{L}^2(dt) \subset \mathbf{L}^{det}(dX)$. By statement in [2.2.5], we know this predictable process F is X-integrable.

To discuss what additional conditions we need, the following theorems will be useful:

Theorem 4.2 *Let a $\mathcal{F}(t)$-predictable step process F_s satisfy $E\int_T F_s^2(t)dt < \infty$, and let X be a proper $T\alpha S$ Levy process with $\mathcal{F}(t)$-independent increments, Then*

$$E\int_T F_s dX = 0 \text{ and } E(\int_T F dX)^2 = E\int_T F_s^2 dt.$$

if and only if

$$\text{Levy characteristic } b = 0 \text{ and } E(X(t))^2 = t.$$

Proof:

We have

$$\begin{aligned} E\int_T F_s dX &= E(\sum_{k=1}^{n} \xi_k (X(r_k) - X(s_k))) \\ &= \sum_{k=1}^{n} E(\xi_k (X(r_k) - X(s_k))) \\ &= \sum_{k=1}^{n} E(\xi_k) E(X(r_k) - X(s_k)) \\ &= b\sum_{k=1}^{n} E(\xi_k)(r_k - s_k) \end{aligned}$$

So $E\int_T F_s dX = 0$ if and only if $b = 0$. When $b = 0$, for any predictable step process F_s satisfying $E\int_T F_s^2(t)dt < \infty$, we have $E\int_T F_s dX = 0$.

For a predictable step process F_s satisfying $E\int_T F_s^2(t)dt < \infty$,

$$\begin{aligned}E(\int_T F_s dX)^2 &= E(\sum_{k=1}^{n}\xi_k(X(r_k)-X(s_k)))^2\\&= E(\sum_{k=1}^{n}\xi_k^2(X(r_k)-X(s_k))^2)\\&+ E(\sum_{1\le i<j\le n} 2\xi_i\xi_j[X(r_i)-X(s_i)][X(r_j)-X(s_j)])\\&= E(\sum_{k=1}^{n}\xi_k^2(X(r_k)-X(s_k))^2)(\text{for } b=0)\\&= \sum_{k=1}^{n}\{E\xi_k^2 E[X(r_k)-X(s_k)]^2\}\end{aligned}$$

and

$$\begin{aligned}E\int_T F_s^2 dt &= E(\sum_{k=1}^{n}(\xi_k^2[r_k-s_k]))\\&= \sum_{k=1}^{n}\{E\xi_k^2[r_k-s_k]\}\end{aligned}$$

By comparing, the equalities holds if and only if $E(X(r)-X(s))^2 = r-s$ for any $r > s \in T$,.i.e. $E(X(t))^2 = t$.

So we have

$$E\int_T F_s dX = 0 \text{ and } E(\int_T F_s dX)^2 = E\int_T F_s^2 dt.$$

if and only if

$$\text{Levy characteristic } b = 0 \text{ and } E(X(t))^2 = t.$$

Theorem 4.3 *For a predictable step process F_s and process $X(t)$ as mentioned in [Theorem 4.2] satisfying Levy characteristic $b = 0$ and $E(X(t))^2 = t$.*

We have for each $a > 0$,

$$P(\int_T^* F_s(t)dX(t) > a) \leq \frac{2}{a}E|\int_T F_s(t)dX(t)|, \quad (1)$$

and for each $p > 1$, and $p^* = p \bigvee p'$,

$$\frac{1}{p^*-1}(E|\int_T F_s(t)dX(t)|^p)^{1/p}$$
$$\leq (E(\int_T F_s^2(t)dt)^{p/2})^{1/p}$$
$$\leq (p^*-1)(E|\int_T F_s(t)dX(t)|^p)^{1/p}, \quad (2)$$

Proof:

When $b = 0$, X is a martingale. By Theorem 3, we have $aP(M_n^* > a) \leq 2E|M_n|$, so we have

$$aP(\sum_{k=1}^{n}{}^*(\xi_k[X(r_k) - X(s_k)]) > a) \leq 2E|\sum_{k=1}^{n}\xi_k[X(r_k) - X(s_k)]|$$

which means

$$P(\int_T^* F_s(t)dX(t) > a) \leq \frac{2}{a}E|\int_T F_s(t)dX(t)|.$$

By Theorem 4, we have

$$E|M_n|^p \leq (p^*-1)^p E[M_n]^{p/2}$$

so we have

$$E|\sum_{k=1}^{n}(\xi_k[X(r_k) - X(s_k)])|^p \leq (p^*-1)^p E(\sum_{i=1}^{n}|\xi_k[X(r_k) - X(s_k)]|^2)^{p/2}$$

so $$E|\int_T F_s(t)dX(t)|^p \leq (p^*-1)^p E(\sum_{i=1}^{n}\xi_k^2|X(r_k) - X(s_k)|^2)^{p/2}$$

so $$\frac{1}{p^*-1}(E|\int_T F_s(t)dX(t)|^p)^{1/p} \leq (E(\int_T F_s^2(t)dt)^{p/2})^{1/p}$$

Then we will prove these properties hold when F is a general predictable process:

Theorem 4.4 *For a general $\mathcal{F}(t)$-predictable process F, we have*

$$E\int_T FdX = 0,$$

if for any predictable step process F, we have

$$E\int_T F_s dX = 0.$$

Proof:

For a general predictable process F, we know in $\mathbf{L}^{rnd}$(dX), the set of predictable step processes is dense with respect to the metric given by $\boldsymbol{\rho}_X(F)$.(From [1, Theorem 9.1.1])

So we have there exist step predictable processes F_n such that

$$\lim_{n\to\infty} \boldsymbol{\rho}_X(F_n - F) = 0$$

then we have following consequences:

$$\lim_{n\to\infty} \sup_{V\in\mathbf{P}_1} |\textstyle\int_T V(F_n - F)dX|_0 = 0$$
$$\lim_{n\to\infty} \sup_{V\in\mathbf{P}_1} E|\textstyle\int_T V(F_n - F)dX| = 0$$
$$\lim_{n\to\infty} E|\textstyle\int_T (F_n - F)dX| = 0$$
$$\lim_{n\to\infty} E|\textstyle\int_T (F_n)dX - \int_T (F_n)dX| = 0$$

So for a general $\mathcal{F}$(t)-predictable process F, we have

$$E\textstyle\int_T F dX = 0,$$

and

$$E(\textstyle\int_T F dX)^2 = E\int_T F^2 dt.$$

By the similar way, we can prove the other equalities and the two inequalities hold when F is a general predictable process.

We also have the following theorem providing an easy way to check condition for integrability of predictable processes with respect to general Levy processes:

Theorem 4.5 *Let X(t) be a Levy process and F be a predictable process satisfying* $E\int_T F^2(t)dt < \infty$*, then this predictable process F is X-integrable.*

Proof:

Because

$$|\tfrac{[\![xu]\!] - x[\![u]\!]}{x^2[\![u]\!]^2}| \le 1, \text{ for } |x| > 1$$

and because for each u,

$$\lim_{|x|\to\infty} \tfrac{[\![xu]\!] - x[\![u]\!]}{x^2[\![u]\!]^2} = 0 \quad .$$

we have

$$\lim_{|x|\to\infty} \tfrac{1}{x^2}\textstyle\int_{\mathbf{R}}([\![xu]\!] - x[\![u]\!])\overline{\nu}(du) = \lim_{|x|\to\infty}\int_{\mathbf{R}} \tfrac{[\![xu]\!] - x[\![u]\!]}{x^2[\![u]\!]^2}[\![u]\!]^2\overline{\nu}(du) = 0,$$

So $$lim_{|x|\to\infty} l'(x)/x^2 = 0,$$

then $$lim_{|x|\to\infty} l(x)/x^2 = 0.$$

Because

$$|\frac{[\![xu]\!]^2}{x^2[\![u]\!]^2}| \leq 1, \text{ for } |x| > 1$$

and becasuse for each u,

$$\lim_{|x|\to\infty} \frac{[\![xu]\!]^2}{x^2[\![u]\!]^2} = 0 \qquad .$$

we have

$$\lim_{|x|\to\infty} \frac{1}{x^2} \int_{\mathbf{R}} [\![xu]\!]^2 \overline{\nu}(du) = \lim_{|x|\to\infty} \int_{\mathbf{R}} \frac{[\![xu]\!]^2}{x^2[\![u]\!]^2} [\![u]\!]^2 \overline{\nu}(du) = 0$$

So

$$lim_{|x|\to\infty} k(x)/x^2 < \infty.$$

We can get

$$\lim_{|x|\to\infty} \varphi(x)/x^2 < \infty,$$

which means $\mathbf{L}^2(dt) \subset \mathbf{L}^{det}(dX)$.

Base on Theorem 4.1, when $E\int_T F^2(t)dt < \infty$, we have for almost every $\omega \in \Omega$, process $F(.,\omega) \in \mathbf{L}^2(dt) \subset \mathbf{L}^{det}(dX)$. Based on statement in [2.2.5], we know this predictable process F is X-integrable.

5. Conclusions

Including Brownian motion, we used three steps to approach to the properties of the equalities and the inequalities. The processes that can get the properties show the very good characteristics. In fact, we use only Levy process. Levy process is very special and good for integrals aspect. Stationary increments can make the drift keep the same direction, which gives monotone Levy characteristic B(t). So the B(t) has bounded variation. This is the key of stochastic integrals, because it makes the stochastic measure σ-additive. The characteristics are important, like drift b, when $b = 0$, it is so different from ones when $b \neq 0$. We found that the equalities and inequalities hold when the integrator X(t) is a process which is Levy process with no drift and $EX(t)^2 = t$. If it is a Levy process, then we can find the other two characteristics. But if it has these two charatceristics, does the process need to be Levy process? If not, what characteristics should it have? We calculated the spaces of integrable functions with respect to the proper Tempered α-stable Levy process. We find it is not easy to characterize the spaces. The measure R is restricted a little, and this makes characterizing difficult. We can try another way to characterize instead of control measure way. we know the process defined by R,b,and α. And the complexity of the condition often comes from R. If the conditions about R are more simple, this condition probably by can be easier to understand. We did the calculations for proper TαS Levy processes. When we calculated the functionφ, proper TαS Levy processes can guarantee some integrals are well defined, but general ones don't work.

In the future I would like to extend my results to general TαS Levy process as well as to semimartingale on manifold.(see [3])

Bibliography

[1] Stanislaw Kwapien, and Wojbor A. Woyczynski. "Random Series and Stochastic Integrals: Single and Multiple", Birkhauser Boston,1992

[2] Jan Rosinski. "Stochastic Processes and their Applications" Volume 117, Issue 6, June 2007, Pages 677 – 707

[3] Laurent Schwartz. "Semimartingales and their Stochastic Calculus on Manifolds" Les Press de 1'Universite de Montreal,1984

[4] Balram S. Rajput, and Jan Rosinski, "Spectral representations of infinitely divisible processes",Rel. Fields (1989) 82: 451.

[5] K. Urbanik and W. A. Woyczynski (1967), A random integral and Orlicz spaces, Bull. Acad. Polo. Sci.