COX-ROSS-RUBINSTEIN OPTION PRICING MODEL WIYH DEPENDENT JUMP SIZES

by

SOUHA FARES

Submitted in partial fulfillment of the requirements
For the degree of Doctorate of Philosophy

Thesis Advisor: Dr. Manfred Denker   Dr. Wojbor Woyczynski

Department of Statistics

CASE WESTERN RESERVE UNIVERSITY

May 2011

CASE WESTERN RESERVE UNIVERSITY

SCHOOL OF GRADUATE STUDIES

We hereby approve the thesis/dissertation of

Souha Fares

candidate for the Doctorate of Philosophy degree * degree *.

(signed) Dr. Wojbor Woyczynski
(chair of the committee)

Dr. Manfred Denker

Dr. Patricia Williamson

Dr. David Gurarie

(date) 03/18/2011

*We also certify that written approval has been obtained for any proprietary material contained therein.

DEDICATION

I would like to dedicate my thesis to my family. Words cannot express how grateful I am to my parents, Abdelmassih and Afaf, my sisters and best friends, Rima, Jihane and Manal and my brothers in law Elie, Maroun and Georges. I could not have asked for a more loving family, and I would not be here without their love, unfailing support and encouragement. I would also like to dedicate it to my lovely nieces and nephew, Talia, Christopher and Selena.

Table of Contents

ACKNOWLEDGEMENTS

Foremost, I would like to express my sincere gratitude to my advisor Professor Manfred Denker, without whose guidance, time and encouragement this thesis would not have been possible. Professor Denker was always accessible and willing to help; in every moment of struggle he would come up with a brilliant idea and make the research progress. Many thanks are due to Professor Wojbor Woyczynski for his useful collaboration and insight throughout this project. He was always ready to listen and give the right advice. I was delighted to work with Professor Patricia Williamson during her classes and as a member of my committee. I am very appreciative for her careful reading of my manuscript and constructive criticism. I would also like to thank Professor David Gurarie for being on my committee and for his insightful opinions on my final presentation. I am deeply grateful to the Statistics department's assistant Mrs. Sharon Dingess for listening, encouraging and helping all graduate students. Finally, many thanks go to Starbucks Retail Cedar Fairmount as a big part of my work was done there.

CRR Option Pricing Model with Dependent Jump Sizes

Abstract

by

Souha Fares

Options are very important derivative securities in the financial market and the option pricing theory is used in most areas in finance. Numerous researchers have contributed to the theory of option pricing. Cox, Ross and Rubinstein presented a discrete time option pricing formula that has, in the limit, the notorious Black-Scholes formula. Kan extended the CRR model by representing the changes in the stock price by the sequence of random variables X_t. She assumed the$X_t's$ to be independent and introduced the multinomial model.

In this thesis, we extend the CRR model assuming a dependency between the jump sizes of the stock price. We have chosen this approach because of its relevance to the stock market. We show the option price to have a similar expression as in the independent case. In addition, we introduce new limiting theorems using Fourier inversion method and perturbation theory of linear operators. Finally we describe a limit of the new option price.

Introduction

Financial markets are basic structures of financial theory. They can be separated into *derivative securities* and *underlying*. A derivative can be seen as a contract that derives most of its value from some underlying asset, reference rate or index. There are several types of derivatives, including financial derivatives, which is the focus of this thesis.

In practice, financial derivatives cover a diverse spectrum of underlyings, including stocks, bonds, exchange rates, interest rates..... The major types of financial derivatives are forward contracts, futures, options and swaps.

Over the past few years derivative securities have become essential tools for corporations and investors alike. Derivatives facilitate the transfer of financial risks.

Options are very important derivative securities in financial markets, "almost everything in finance can be written in terms of options"[P. Ritchken]. Any investment which provides some kind of protection actually includes an option feature. Fore more details about financial markets one can refer to [16, 26, 42, 36, 19, 37, 34, 10, 26].

Many people think that options and futures are recent inventions, in fact, options have a long history, going back to ancient Greece. However they remained a vague financial instrument until 1973 when the option exchange has been introduced. And at the same time the option pricing theory underwent a revolutionary changing, especially after Black-Scholes offered in their paper [6] a first satisfactory model of option pricing.

Robert Merton and Myron Scholes were awarded the Nobel Prize for Economics in 1997 to honour their contributions to option pricing. Unfortunately, Fischer Black, who has also given his name and contributions, had passed away two years before. The Black-Scholes formula is still used nowadays.

However, the Black and Scholes model has some restrictions, after all, the derivation of the Black-Scholes equation, and hence, the closed-form solutions for some options, assume a continuously trading strategy which is not feasible in the market in order to hedge the portfolio that has been constructed.

In 1978 Sharpe [45] has partially developed a simplified approach to option pricing and suggested the advantages of using the discrete-time approach.

In 1979 Cox, Ross and Rubinstein [11] presented a discrete-time option pricing model known as the binomial model which has as a limiting case the Black-Scholes formula. The binomial model assumes that the stock price at each time moment can go either up or down by the multiplication of two factors called u and d.

Kan [22] extended the binomial model and considered the case where at each time moment the stock price is changed not only by multiplication by the two factors u and d, but due to the embedding of the process $\{X_t\}_{t \leq T^*}$, the changes of the stock price are modeled by a variety of possible values uX_t and dX_t, $t \leq T^*$. Therefore, the assumptions on the sequence $\{X_t\}_{t \leq T^*}$ play a crucial role.

The generalized CRR stock price model was given by $S_t = \xi_{t-1} S_{t-1}$ for all $t \leq T^*$ where $S_1 = S_0 \xi_0$ and S_0 is a positive constant and $\xi_t = X_t \nu_t, \forall t \leq T^*$.

Kan assumed that the random variables $\{X_t\}_{t \leq T^*}$ and $\{\nu_t\}_{t \leq T^*}$ are mutually independent and conditioned on the values that the random sequence $\{X_t\}_{t \leq T^*}$ can take.

She assumed that the sequence $\{X_t\}_{t \le T^*}$ takes k possible values in the set she referred to as the multinomiality set $\mathbb{C}_k = (c_1, \ldots, c_k)$. Therefore it is obvious that the probability distribution p_n of the vector of occurrences $(n_1, \ldots, n_k)$ of these k values is the multinomial distribution.

A local limit theorem by Richter [38] was then used in order to find the limit of the option price. The theorem says that the multinomial distribution converges to the normal distribution. The following asymptotic yield to a “general” form of the Black-Scholes formula as a limit of the Conditional Generalized CRR option price.

Estimating procedures as the Hull-White algorithm in [19] were used to estimate the multinomiality parameters. The results showed that embedding the multinomial parameters gave a better approximation of the stock price (based on real data) than the binomial model. It was also evident how the multinomiality parameters influenced the option price.

However the assumption of the independence between the jump sizes is not relevant to the actual stock price behavior: it is obvious that the changes in the stock price over time are somehow dependent; the price of a stock in a certain financial period can affect its price on the next one.

In this thesis we develop a new option pricing model, which we refer to as the “Generalized CRR Option Price Model with Dependent Jump Sizes”. Based on Kan’s model, we want to find a new model that describes the reality better.

In **Chapter 1** we introduce and define the financial problem of option pricing. We give a brief history about the option pricing theory and evolution of stock price models, define options and present some basic aspects of the financial market. Then we present the discrete time stock pricing models and corresponding option prices.

We also introduce the idea behind our model. Based on the assumption of dependence between the sizes of the jumps of the stock price, we extend the Conditional Generalized Option Pricing Model" derived by Kan [22] and show that the expression of the option and stock price in the dependent case are the same as in the independent case.

Starting from this point we want to show that the limiting behavior is also similar. In **Chapter 2** we prove a Richter type theorem based on Fourier analysis with a somewhat small region, however this will permit the generalization in **Chapter 4**. The proof of this theorem involves the characteristic function of p_n. The statement of the theorem is the following

Theorem 2.2.3: Let X be a random variable which can assume k different values $\beta_1, \ldots \beta_k$ with probabilities $P(X = \beta_j) = p_j,\ \ j = 1, \ldots k$. Let $Z_n = (n_1, \ldots, n_k)$ be the vector of occurrences of these k possible outcomes in n trials of X.

If the n trials are independent, there exists a region of points

$$G'_n = \left\{ x = (x_1, \ldots, x_k) \in \mathbb{R}^k; |x_j| \leq \sqrt{A \frac{\log(n)}{n}} \right\}$$

for some $A > 0$ such that the following holds

$$P\left(Z_n = (n_1, ..., n_k)\right) = p_n(n_1, \ldots, n_k) \sim \frac{1}{\sqrt{2\pi n}^{k-1} \sqrt{p_1 \ldots p_k}} e^{-\frac{1}{2}\sum_{j=1}^{k} \frac{n x_j^2}{p_j}}, \qquad x \in G'_n.$$

The use of **Theorem 2.2.3** require the use of the characteristic function of Z_n which is known to be the multinomial distribution in the independent case.

In **Chapter 3** we use perturbation theory of linear operators and find an expression of the characteristic function of Z_n without the assumption of independence.

For more details about perturbation theory of linear operators one can refer to [24, 23, 13, 40, 1, 39, ?].

Let

$$T : [0,1] \to [0,1]$$

be a piecewise monotone and expanding transformation on the unit interval. This means that there are finitely many intervals so that T is monotone and differentiable on each interval and the derivative on each (open) interval can be extended to its closure. Moreover, the derivative has a modulus which is bounded below by $\Lambda > 1$. The sequence of jumps will be considered as a stochastic stationary sequence of the form $\tilde{f} \circ T, \tilde{f} \circ T^2, \ldots, \tilde{f} \circ T^n$, T being an operator and $\tilde{f}$ a function defined on the unit interval with values in $\mathbb{R}^k$. Stationarity is understood with respect to an invariant, absolutely continuous measure. We assume that the transformation is weakly mixing with respect to this transformation. We let $S_n \tilde{f} = \sum_{k=0}^{n-1} \tilde{f} \circ T^k, n \geq 1$ and $S_0 \tilde{f} = \tilde{0}$ and introduce what is referred to as a "Characteristic Function Operator", as in [39, 1] and we denote it by $P_{\tilde{f}}(it)$, $t \in \mathbb{C}$. The perturbation theory of $P_{\tilde{f}}$ gives us the expression of the characteristic function we need which is that of $S_n \tilde{f}$, which will play the role of the Z_n in Theorem 2.2.3:

$$\int_0^1 P_{\tilde{f}}^n(i\tilde{t})\mathbb{1}\mathrm{d}\mu = \int_0^1 e^{i\langle \tilde{t}, S_n \tilde{f}\rangle}\mathrm{d}\mu,$$

where μ denotes the unique absolutely continuous measure with respect to Lebesgue measure. Of course, using perturbation theory, the maximal eigenvalue of the operator $P_{\tilde{f}}(0)$ has to be unique, which can be expressed by the condition that μ is weakly mixing.

The expression of the characteristic function of $S_n \tilde{f}$ turns out to be similar to that of a multinomial distribution. We substitute our expression in the proof of the Richter-type local limit theorem and get the following asymptotic that holds when the n trials are dependent.

Theorem 3.5.2 Let T be a piecewise monotone and expanding transformation of the unit interval and μ be the weakly mixing invariant probability, absolutely continuous with respect to Lebesgue measure on $[0,1]$. Let X be a μ-random variable which can assume k different values $\beta_1, \dots \beta_k$ with probabilities $\mu(X = \beta_j) = p_j,\ \ j = 1, \dots k$. Let $S_n X$ be the vector of occurrences of these k possible outcomes in n iterations of X under T. Let G_n denote the region of points $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ for which $|x_j| \leq \sqrt{\frac{A \log n}{n}}$ for $j+1, \dots, k$, where $A > 0$:

As $(x_1, \dots, x_k) = \frac{1}{n}(n_1 - np_1, \dots, n_k - np_k) \in G_n$ and $n \to \infty$ there exists a $(k-1)-$multivariate normal distribution with mean $\tilde{0}$ and covariance matrix Σ such that

$$\mu\left(S_n X = (n_1, ..., n_k)\right) = p_n(n_1, \dots, n_k) \sim \frac{1}{(2\pi n)^{(k-1)/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}\tilde{x}'\Sigma^{-1}\tilde{x}\right\}$$

In **Chapter 4** we find the expression of the the option price when the stock price follows the generalixed CRR model with dependent jump sizes and its limiting behaviour. We follow Kan's procedure, use **Theorem 3.5.2** and make other necessary change. We obtain the following results.

Proposition 4.1.1 The generalized CRR option price with dependent jumps is given by the following formula

$$\begin{aligned} \hat{C}_{T-m} &= S_{T-m} \sum_{j=0}^{m} \sum_{\{M_1,\dots,M_k; \sum M_j = m\}} \sum_{I(m,j,N_{c_1},\dots,N_{c_k})} \overline{P}(J^{(j)}, T, z)\mathbf{P}(N_{c_1} = m_1, \dots, N_{c_k} = m_k) \\ &- \frac{K}{\hat{r}^m} \sum_{j=0}^{m} \sum_{\{M_1,\dots,M_k; \sum M_j = m\}} \sum_{I(m,j,N_{c_1},\dots,N_{c_k})} P(J^{(j)}, T, z)\mathbf{P}(N_{c_1} = m_1, \dots, N_{c_k} = m_k). \end{aligned} \tag{0.0.1}$$

Theorem 4.2.1 For $x \in \mathbb{X}, i = 1, \ldots, k$ as $n \to \infty$ the following asymptotic holds true

$$\begin{aligned}\hat{C}_{T-m} \sim\ & S_{T-m} \sum_{j=0}^{m} \sum_{\{M_1,\ldots,M_k;\sum M_j = m\}} \sum_{I(m,j,N_{c_1},\ldots,N_{c_k})} \overline{P}(J^{(j)}, T) \frac{1}{(2\pi m)^{\frac{k-1}{2}} |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} x' \Sigma^{-1} x\right\} \\ & - \frac{K}{\hat{r}^m} \sum_{j=0}^{m} \sum_{\{M_1,\ldots,M_k;\sum M_j = m\}} \sum_{I(m,j,N_{c_1},\ldots,N_{c_k})} P(J^{(j)}, T) \frac{1}{(2\pi m)^{\frac{k-1}{2}} |\Sigma|^{(k-1)/2}} \exp\left\{-\frac{1}{2} x' \Sigma^{-1} x\right\};\end{aligned} \tag{0.0.2}$$

where Σ is the covariance matrix of a zero mean $(k-1)$-normal distribution as in chapter 3.

Chapter 1

Option Pricing Theory and the CRR Option Price Model

1.1 Options

An option is a contract between a buyer and a seller that gives the buyer the right, but not the obligation, to buy or to sell a particular asset (the underlying asset), on or before the option's expiration time, at an agreed price, the strike price. [48]

Options are standard examples of *derivative securities*, that is, securities whose value depends on the prices of other more basic securities, referred to as *primary securities* or *underlying assets*. The *underlying security* of an option is fixed and cannot be changed [48]. The underlying financial instruments include the bank accounts, bonds, stocks . . . [36] and the derivatives include options, futures, swaps, warrants ... [46].

Options are traded only on a single security, which may be a stock or an index, currency, commodity, another option . . . . In this thesis we will consider options on stock as the best understood and most popular form of listed options trading [48].

Options are divided into call and put contracts. A call option gives its holder (or purchaser) the right to buy a security from the call writer (or seller) at a specified price. Conversely, a put option gives its holder the right to sell a security to the put writer at the exercise price. In return for granting the option, the seller collects a payment (the premium) from the buyer. When an option could only be exercised on the contract's expiration date, it is called a European option. An American option may be exercised at any time during the contract's life. There exists also Exotic options, Bermudian, Vanilla and Barrier options.

1.1.1 History of options

Options have existed -at least in concept- since antiquity. The aged history of options is going way back to the Romans and the Phoenicians, who used contracts similar to options in shipping. There is also remarkable evidence that Thales of Miletus (624BC-547BC), a Greek mathematician and philosopher, used options to secure a low price for olive presses in advance of the harvest. The concept was formalized in Japan with the first physical futures exchange, the Yodoya rice market in Osaka in 1650. Option contracts became also popular as hedging and speculative devices in the Dutch tulip market. These contracts continually resurfaced on most of the major security markets throughout the world [3].

It wasn't until publication of the Black-Scholes (1973)[6] option pricing formula that a theoretically consistent framework for pricing options became available. That framework was a direct result of work by Robert Merton as well as Black and Scholes [32, 30].

Prior to the publication of the Black-Scholes model, the quest for a valuation formula that would describe option prices reflected one of the most elusive goals in financial economics [22].

Option pricing theory-also called Black-Scholes theory or derivatives pricing theory traces its roots to Bachelier (1900)[2] who invented Brownian motion to model options on French government bonds. In order to describe how option contracts are priced, Bachelier needed to describe the underlying distribution of stock prices. By modeling successive price changes in a very specific way, he used the central limit theorem to derive a normal distribution for stock price movements. The particular stochastic process that Bachelier used to describe stock price changes is now known as Brownian motion (or a Wiener process). It has the characteristic that the increments in the process (the movement of stock prices) are independent random variables. Further, the increments are normally distributed with a zero mean and a variance that is proportional to the length of the time span involved. Combined, these two features imply what is known as a "stationary" process. Brownian motion is used to characterize stock price movements because stock prices are said to have "no memory" [47].

Research picked up in the 1960's. Typical of efforts during this period is Samuelson (1965)[41]. He considered long-term equity options, and used geometric Brownian motion to model the random behavior of the underlying stock. Based upon this, he modeled the random value of the option at exercise. The model required two assumptions. The first was the expected rate of return α for the stock price. The second was the rate β at which the option's value at exercise should be discounted back to the pricing date. These two factors depended upon the unique risk characteristics of, respectively, the underlying stock and the option. Neither factor was observable in the market place; depending upon their degree of risk aversion, different observers might propose different values for the factors. Accordingly, Samuelson's formula was largely arbitrary. It offered no means for a buyer and seller with different risk aversions to agree on a price for an option. Black and Scholes got around the problem with a completely new approach.

The Black-Scholes (1972)[6] option pricing formula prices European put or call options on a stock that does not pay a dividend or make other distributions. The formula assumes the underlying stock price follows a geometric Brownian motion with constant volatility. It is historically significant as the original option pricing formula published by Black and Scholes in their paper [6].

The Black-Scholes model has some restrictions. A constant risk-free interest rate r

and a constant volatility *sigma* do not seem to be realistic. After all, the derivation of the Black-Scholes equation, and hence, the closed-form solutions for some options, assume a continuously trading strategy which is not feasible in the market in order to hedge the portfolio that has been constructed.

Until then only continuous models were considered. In 1978 Sharpe [45] introduced the advantages of using the discrete-time approach to option pricing.

In 1979 Cox, Ross and Rubinstein [11] presented a discrete-time option pricing formula, the Binomial Model, for pricing a call option on a stock which doesn't pay any dividends and has, as a special limiting case, the celebrated Black-Scholes model, which has previously been derived only by much more difficult methods. This model is categorized as a Lattice Model or Tree Model because of the graphical representation of the stock price and option price over the large number of intervals or steps, during the time period from valuation to expiration, which are used in computing the option price. At each step, the stock price will either move up, or down, with a probability defined by the volatility of the stock. The Cox Ross and Rubinstein model is generalized to the multinomial case. Limits are investigated and shown to yield the Black-Scholes formula in the case of continuous sample paths for a wide variety of complete market structures.

In 1977, Merton derived a formula for the discontinuous case. In this case, the limiting formula requires the replacement of jump probabilities by the Arrow-Debreu prices. For more details about the Arrow-Debreu model one can refer to [35, 31].

The mathematics behind some of the models cited above will be discussed in section 2.

1.1.2 Language of options

A European option written on a stock that pays no dividends during the option's lifetime is a financial security that gives its holder the right (but not the obligation) to buy the underlying stock on a predetermined date and for a prespecified price. The predetermined price, K, is called the strike or exercise price, the terminal date, T, is called the expiry date or maturity, the option's price is called a premium.

The proximity between the strike price and the current market value determine the option's value along with the amount of time remaining until expiration. When the underlying's stock current value is higher than the strike price, the call option is in the money. When the price is lower than the strike, the call is out of the money. When it's exactly equal to the strike, the option is at the money. Other characteristics that also affect the theoretical price of an option are the amount of time remaining to expiration, the current price of the underlying stock, the risk-free interest rate over the life of the option and the volatility of the underlying. The most elusive and hard to understand part of premium value is due to the level of volatility in the underlying stock. Intuitively, price volatility is a measure of the amount and intensity of price fluctuation. The more volatile a stock's price, the more often and intensely it fluctuates.

In this thesis we will consider stock price models in order to calculate the theoretical price of the option using a specific option pricing model.

In the theory of option pricing, a stock price model is, roughly speaking, a mathematical description of the relationship between the current price of a stock and its possible future prices.

The perfect model would be predictive, it would tell you the future price of a stock based on its present value and possibly some auxiliary data. The stock price models employed in options pricing are not predictive but probabilistic. That is, they do not make precise statements about what the future stock price will be, but instead, they assume a distribution of future prices derived from historical data, current market conditions, and possibly other relevant data.

1.2 The Evolution of the Cox-Ross-Rubinstein Stock Price Model

1.2.1 Discrete-time security markets

This thesis deals with *finite markets*, that is discrete-time models of financial markets. We consider the number of dates to be finite, so there is no loss of generality if we take the set of date $\mathcal{T} = \{0, 1, \ldots, T^*\}$. Let $\Omega = (\omega_1, \ldots, \omega_d)$ be an arbitrary finite set and $\mathcal{F} = \mathcal{F}_{T^*}$ be the $\sigma-$field of all subsets of Ω.

We consider a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F})_{t=0}^{T^*}$, where $\mathbb{P}$ is an arbitrary probability measure over $(\Omega, \mathcal{F}_{T^*})$ such that $\mathbb{P}(\omega_j) > 0$ for every $j = 1, \ldots d$ [36].

We assume that the securities market operates under conditions of " uncertainty " that can be described in the probabilistic framework in terms of a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$.

We consider a $(B, S)-$market formed by a risky asset S, referred to as a stock and a risk-free asset B, called a savings account (or bond). Bonds are fairly popular in many countries mainly because the interest on bonds is fixed and payable on a regular basis and the repayment of the entire loan at a specified time is guaranteed [46].

We will consider a European call option written on one share of stock S, which doesn't pay any dividends during the option's lifetime. We will denote by S_T the stock price at the terminal date T and S_t the stock price at any time moment $t \leq T$. This option is equivalent to the claim Y with payoff at time T, contingent on the stock price S_T

$$Y = (S_T - K) = max\{S_T - K, 0\}$$

where K is the exercise price [22].

The call option value at expiry time T equals $C_T = (S_T - K)^+$. We want to evaluate the option's price C_t at any time moment $t \leq T$.

We will now give several definitions relating to the financial position of an investor in such a (B, S)-market. The following definitions are from [36, 46].

1.2.2 Definitions

Definition 1.2.1 *A predictable stochastic sequence $\phi = (\alpha, \beta)$ where $\alpha = (\alpha_t(w))_{t \geq 0}$ and $\beta = (\beta_t(w))_{t \geq 0}$ with $\mathcal{F}_{t-1}$ measurable $\alpha_t(w)$ and $\beta_t(w)$ for all $t \geq 0$ ($\mathcal{F}_{-1} = \mathcal{F}_0$) is called an investment portfolio on the (B,S)-market.*

For any $t \leq T$, α_t stands for the number of shares of stock held during the period $[t, t+1)$ and β_t for the dollar investment in the savings account during this period. Sometimes the *investment portfolio* is called an *investment* or *trading strategy* instead.

The idea is to construct a portfolio at time 0 that replicates exactly the option's terminal payoff at time T.

Here the investment strategy $\phi(t)$ has to be determined on the basis of information available before time t, which means that investor selects his portfolio $\phi(t)$ after observing the prices S_{t-1} [22].

Definition 1.2.2 *The value of an investment portfolio V at time t is the stochastic sequence*

$$V(\phi) = (V_t(\phi))_{t \geq 0}$$

where

$$V_t(\phi) = \alpha_t S_t + \beta_t$$

and $V_0(\phi) = \alpha_1 S_0 + \beta_1$. *The process* $V_t(\phi)$ *is called the wealth of the trading strategy* ϕ

Definition 1.2.3 *A trading strategy* ϕ *is called self-financing if*

$$\alpha_{t-1} S_t + \beta_{t-1}\hat{r} = \alpha_t S_t + \beta_t \forall t \leq T^*$$

In other words, a self-financing strategy is a strategy that draws no money. The portfolio is recombined in such a way that its value remains the same. When new prices are quoted at time t, the investor adjusts his portfolio from ϕ_{t-1} to ϕ_t without any withdrawals or inputs of funds concerning the wealth of portfolio [22].

Denote as Φ a linear space off all stock-bonds portfolios ϕ, then consider a security market model $\mathcal{M} = (B, S, \Phi)$.

Definition 1.2.4 *We say that a security pricing model* $\mathcal{M}$ *is arbitrage-free if there is no portfolio* $\phi \in \Phi$ *for which*

$$V_0(\phi) = 0 \quad \textit{and} \quad \mathcal{P}\{V_T(\phi) > 0\} > 0$$

A portfolio ϕ *for which definition 1.2.4 is satisfied is called an arbitrage opportunity.*

Definition 1.2.5 *A strong arbitrage opportunity is a portfolio* ϕ *for which*

$$V_0(\phi) < 0 \textit{ and } V_T(\phi) \geq 0.$$

In other words an arbitrage possibility consists of the existence of a trading strategy such that - starting from an initial investment zero, the resulting contingent claim is non negative and not identically equal to zero. An arbitrage opportunity exists if it is possible to make a gain that is guaranteed to be at least equal to the risk free rate of return, with a chance of making a greater gain.

Although arbitrage opportunities do exist in real markets, they are usually very small and quickly eliminated, therefore the no arbitrage assumption is reasonable in financial theory.

Definition 1.2.6 *A replicating strategy for the contingent claim Y which is paid off at time T is a self-financing trading strategy ϕ such that $V_T(\phi) = Y$.*

The replicating strategy can not be unique. There is usually a class of trading strategies which replicate Y .

Definition 1.2.7 *We say that a contingent claim Y is attainable in $\mathcal{M}$ if it admits at least one replicating strategy.*

Definition 1.2.8 *A market $\mathcal{M}$ is called complete if any contingent claim Y is attainable in $\mathcal{M}$, or, equivalently, if for every $\mathcal{F}_T$ -measurable random variable Y , where T is the expiration time for a claim Y , there exists at least one trading strategy $\phi \in \Phi$ such that $V_T(\phi) = Y$.*

In other terms, a complete market is a market in which all payoffs can be obtained by trading the securities available in the market. If the financial instruments available in a market were not sufficiently rich and diverse to permit such a speculation, the market would be deemed incomplete. From this, we see that completeness of the financial market is an idealization that is most likely unobtainable in practice, however it is a very desirable property. Only under market completeness, any European claim can be priced by arbitrage and its price process can be replicated by means of a replicating self-financing strategy.

Definition 1.2.9 *Suppose that the security market $\mathcal{M}$ is arbitrage-free. Then the rational price of Y is called the arbitrage price of Y .*

It is worth noting that in order to determine the cost of a call option, we do not need to know the probability of the rise or fall of the stock price. All investors agree on the range of future price fluctuations but they may have different assessments of the corresponding subjective probabilities. We only assume that they prefer more wealth to less [22].

In their paper (1979) [11], Cox, Ross and Rubinstein presented a simple discrete time model for valuing options, beginning by assuming that the stock price follows a multiplicative binomial process over discrete periods. Their model was known as the classical CRR option pricing model and was later extended in many ways, we will discuss some of them in the sections that follow.

1.2.3 Classical discrete Cox-Ross-Rubinstein model

The Cox-Ross-Rubinstein model (CRR model) is a discrete time model of financial (B, S)-market during the time interval $[0, T^*] = 0, \ldots, T^*$, where T^* is some positive natural number, with two primary traded securities, namely a risk-free bond with interest rate r over each time period and a stock with initial price S_0 and whose price process is modeled as a strictly positive discrete-time process $S = (St)_{t \leq T}$. It is assumed that S_t is F_t-adapted, i.e. random variables S_t are F_t-measurable for $t \in [0, T]$. The price process of a bond (risk-free investment) is defined as

$$B_t = (1 + r)^t, \forall t \leq T^*. \tag{1.2.1}$$

It is categorized as a Lattice Model or Tree Model because of the graphical representation of the stock price and option price over the large number of intervals or steps, during the time period from valuation to expiration, which are used in computing the option price. Binomial models of n-period financial markets are of considerable practical and theoretical interest since they allow, due to their completeness, pricing formulas. At each step, the stock price will either move up, or down, with a probability defined by the volatility of the stock [11].

The *savings account* is assumed to have a constant rate of return r over each time period $[t, t + 1]$; its price process B is given by

$$B_t = (1 + r)^t = \hat{r}^t, \forall t = 0, \ldots, T^*. \tag{1.2.2}$$

Definition 1.2.10 *The stock price in the classical discrete Cox-Ross-Rubinstein model is given by the following formula [36]*

$$S_t = \xi_{t-1} S_{t-1} for\ all t \leq T^* \tag{1.2.3}$$

where:

- $\xi_t, t \leq T^*$ *are i.i.d random variables taking two possible values* u *and* d *with probabilities* p *and* $1 - p$*, respectively.*

- $d < 1 + r < u$ *are given real numbers*

- S_0 *is a strictly positive constant.*

It's apparent under the present assumptions that the random variables $\xi_t, t \leq T^*$ are mutually independent random variables on common probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with probability law [36]

$$\mathbb{P}\{S_{t+1} = uS_t / S_0, S_1, \ldots, S_t\} = \mathbb{P}\{\xi_{t+1} = u\} = p \quad \forall\, t \leq T^*$$

and

$$\mathbb{P}\{S_{t+1} = dS_t / S_0, S_1, \ldots, S_t\}, = \mathbb{P}\{\xi_{t+1} = d\} = 1 - p \quad \forall\, t \leq T^*$$

Assuming that the financial security market $\mathcal{M}$ is complete and arbitrage free and using a recursive pricing procedure (backward induction), Marek Musiela and Marek Rutkowski [?] derived an explicit formula for the arbitrage price of a European call option in the classical discrete CRR model, that turns out to be independent of the choice of the probabilities with which the stock moves up or down and is uniquely determined by the assumed values of the stock price.

Before we give the expression of the corresponding option price, we will introduce the following notations as in [36].

For a fixed natural number m,

$$a_m(x) = \inf\{j \in \mathbb{N} \backslash x u^j d^{m-j} > K\}, \quad a^d = a_m(dx), \;\; a^u = a_m(ux)$$

$$\Delta_m(x, j) = \binom{m}{j} p^j (1-p)^{m-j} (u^j d^{m-j} x - K)$$

In [36], it was shown that for any $m = 1, \ldots, T$, the arbitrage price for a European call option at time $t = T - m$ is given by the Cox Ross Rubinstein valuation formula

$$C_{T-m} = S_{T-m} \sum_{j=a}^{m} \binom{m}{j} \overline{p}^j (1-\overline{p})^{m-j} - \frac{K}{\hat{r}^m} \sum_{j=a}^{m} \binom{m}{j} p^j (1-p)^{m-j}, \tag{1.2.4}$$

where $a = a(S_{T-m})$, $\quad p = \dfrac{\hat{r} - d}{u - d}$, $\quad$ and $\overline{p} = \dfrac{pu}{\hat{r}}$

In their book [36], Musiela and Rutkowski showed that the Black-Scholes option valuation formula (1.2.5) can be obtained from the CRR option valuation formula (1.2.4). For a fixed, real number $T > 0$ and for n of the form $n = 2^k$, their asymptotic procedure consisted in dividing the interval $[0, T]$ into n intervals I_j, $\quad j = 0, \ldots, n-1$. Then, examining the asymptotic properties of the CRR model, when the number of steps (i.e n) goes to infinity and the size of time steps goes to zero in an appropriate way, they showed that the limit of the CRR option price C_{T-m} (1.2.4) is C_t given by the Black-Scholes formula (1.2.5)

$$C_t = S_t N(d_1(S_t, T-t)) - Ke^{-r(T-t)} N(d_2(S_t, T-t)), \tag{1.2.5}$$

where

$$d_1(s,t) = \frac{\ln(s/K) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}},$$

$$d_2(s,t) = d_1(s,t) - \sigma\sqrt{t} = \frac{\ln(s/K) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}},$$

and N stands for the standard Gaussian cumulative distribution function

$$N(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}du \quad \forall\, x \in \mathbb{R}$$

1.2.4 Generalized discrete Cox-Ross-Rubinstein model

Since the Black-Scholes model disagrees with reality in a number of ways, some of them significant, it is useful to think of ways to extend the classical CRR model and get a model closer to practice.

Kan [22] generalized the binomial model in a way that at each time moment the stock price is changed not only by multiplication by two possible values u and d, but by embedding a process$\{X_t\}_{t\leq T}$ that takes different values at different time moments and the changes of stock price are modeled by a variety of possible values uX_t and $dX_t, t \leq T$. The assumptions on the random sequence $\{X_t\}_{t\leq T}$ play a crucial role.

A new formula for stock price was derived, along with corresponding option price formula called "Generalized CRR option pricing model".

Definition 1.2.11 *[22] The model of stock price process is called generalized Cox-Ross-Rubinstein stock price model if S is defined as follows*

$$S_t = \xi_{t-1} S_{t-1} \textit{for all} t \leq T^* \tag{1.2.6}$$

where:

- *$S_1 = S_0 \xi_0$ and S_0 is a positive constant*
- *$\xi_t = X_t \nu_t, \forall t \leq T^*$ where $\{\nu_t\}_{t\leq T^*}$ are Bernoulli random variables taking values u and d with corresponding probabilities p and $1-p$. Also assume that random variables $\{X_t\}_{t\leq T^*}$ and $\{\nu_t\}_{t\leq T^*}$ are mutually independent.*

It is also apparent that the random variables $\xi_t, t \leq T^*$, are mutually independent random variables on common probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with probability law

$$\mathbb{P}\{X_t\nu_t = xu/X_t = x\} = p = 1 - \mathbb{P}\{X_t\nu_t = xd/X_t = x\} \quad \forall\, t \leq T^*.$$

Before giving the expression for the option price, some notations are introduced

Let $\Gamma_m = \{1, \ldots, m\}$. For any fixed $m \in \mathbb{N}$ and $j \in \Gamma_m$ $I_{j,m}$ denotes the following random set

$$I_{j,m}(x) = \left\{ J \subset \Gamma_m, |J| = j, x \prod_{k\in J} \xi^u_{T-k} \prod_{k\in \overline{J}} \xi^d_{T-k} > K \right\}, \tag{1.2.7}$$

where $|J|$ stands for the cardinality of the set J and $\overline{J}$ stands for the complement of J. Also $\Delta_m(x)$ is defined as

$$\Delta_m(x) = \frac{1}{\hat{r}^m} \sum_{j=0}^{m} \sum_{J \in I^p_{j,m}(x)} \prod_{k \in J} p_{T-k} \prod_{k \notin J} q_{T-k} \left(x \prod_{k \in J} \xi^u_{T-k} \prod_{k \notin J} \xi^d_{T-k} - K \right)$$

For any fixed natural number m, $a_m(x) = inf\{j : I_{j,m}(x) \neq \emptyset\}$

and

$$p_{T-k} = \frac{\hat{r} - \xi^d_{T-k}}{\xi^u_{T-k} - \xi^d_{T-k}}, \;\; \overline{p}_{T-k} = \frac{\xi^u_{T-k}}{\hat{r}} p_{T-k}$$

$$q_{T-k} = 1 - p_{T-k},, \;\; \overline{q}_{T-k} = 1 - \overline{p}_{T-k}$$

Also, to shorten the lengthy expressions, the following notations are introduced;

$$\overline{P}(J^{(j)}, T) = \prod_{k \in J} \overline{p}_{T-k} \prod_{k \notin J} \overline{q}_{T-k}$$

$$P(J^{(j)}, T) = \prod_{k \in J} p_{T-k} \prod_{k \notin J} q_{T-k}$$

$$\Xi(J^{(j)}, T) = \prod_{k \in J} \xi^u_{T-k} \prod_{k \notin J} \xi^d_{T-k}$$

It was shown that [22] for any $m = 1, \ldots, T$, the arbitrage price of a European call option at time $t = T - m$ provided that the stock price process follows generalized CRR stock price model defined by (?) is given by the CRR valuation formula

$$C_{T-m} = S_{T-m} \sum_{j=0}^{m} \left(\sum_{J \in I_{j,m}(S_{T-m})} \overline{P}(J^{(j)}, T) - \frac{K}{\hat{r}^m} \sum_{J \in I_{j,m}(S_{T-m})} P(J^{(j)}, T) \right) \quad (1.2.8)$$

for $m = 0, \ldots, T$.

1.2.5 Conditional generalized CRR option pricing model

In the generalized CRR model (1.2.6), the sequence $\{X_t\}_{t \leq T}$ plays a crucial role. Different assumptions about the distribution of the family of random variables $X_k, k = 0, \ldots, m$, yield different models for the option price, some of them can be complicated.

Since the generalized CRR model is an extension of the binomial model, where the jumps of the stock price between two time moments are described not only by the multiplication by the two values u and d, but by embedding the sequence $\{X_t\}_{t \leq T}$ which takes different values at each time moment, it is straighforward to consider the multinomial distribution as a generalization of the binomial.

Kan considered the generalized CRR option price provided that the random variables $X_t, t = 1, \ldots, T$ are independent and take values in the set

$$\mathbb{C}_k = (c_1, \ldots, c_k), \;\; k \in \mathbb{N}, \;\; c_i > 0, \;\; \forall i = 1, \ldots, k, \tag{1.2.9}$$

where the events $\{X_t = c_j\}, t = 1, \ldots, T, j = 1, \ldots, k$, are equally likely with probabilities $p_1 = \ldots = p_k = \frac{1}{k}$, c_{j_i} are in $\mathbb{C}_k, i = 1, \ldots, k$, and are called multinomiality parameters.

The mean of the conditional expectation (1.2.8), C_{T-m} given the random sequence $X_{T-m}, \ldots, X_T$ was considered to find the corresponding option price

$$\tilde{C}_{T-m} = E\{E\{C_{T-m} | X_{T-m}, \ldots, X_T\}\} = E(C_{T-m}). \tag{1.2.10}$$

The number of $c_i^\prime s$ occurred in the sequence $X_{T-m}(\omega), \ldots, X_T(\omega)$ for a fixed ω is denoted by $N_{c_1}, \ldots, N_{c_k}$. It is a sequence of random variables taking values in the set $\Gamma_m \bigcup \{0\}$ and satisfying the condition $N_{c_1} + \ldots + N_{c_k} = m$.

Since the random variables $(X_t)_{t \leq T}$, it is known that the sequence $N_{c_1}, \ldots, N_{c_k}$ follows a multinomial distribution with parameters m and p. Based on this, the corresponding option pricing model was called the multinomial model.

Because it is unknown which combination of of $N_{c_1}, \ldots, N_{c_k}$ occurs in practice, the averaging over all possible combinations of $N_{c_1}, \ldots, N_{c_k}$ was considered.

Using an asymptotic procedure similar to [36] and limit theorems for the multinomial distribution [38], the limit of the option price (**??**) was obtained. It was referred to as the generalized Black-Scholes option valuation formula.

Remark. We will discuss the Black-Scholes type limit of the multinomial option price (4.1.7) in details in chapter 4.

The only point we will mention for now is that in finding the expression of this limit, the quantities T, m, S_{T-m}, r and σ need to be known. All quantities are observable directly except for the volatility parameter σ. The constant volatility assumed in the Black-Scholes model was not satisfactory anymore [17].

It was shown that volatility models of asset returns have an important effect on pricing options. One can model the volatility in different ways. Historical Volatility is one way to measure price fluctuation over time; it uses historical (daily, weekly, monthly, quarterly, and yearly) price data to empirically measure the volatility of a market or instrument in the past . The implied volatility of an option contract is the volatility implied by the market price of the option based on an option pricing model. In general Continuous-time models model the volatility as a stochastic process. Such models are called the stochastic volatility models. The discrete-time approach to model stochastic volatility is based on autoregressive random variance models called ARCH (or GARCH) models. For more about volaitility models one can refer to

[8, 42, 43, 50, 33].

In the asymptotic procedure introduced in finding the limit of the multinomial option price, the volatility parameter of stock return was modelled as the stochastic process

$$\sigma_t = \sigma(\log X_t + 1), \forall t = 0, \ldots, T$$

where $X_t, t = 0, \ldots, T$ take values in the set $\mathbb{C}_k$ defined in (1.2.9).

1.2.6 Binomial versus Multinomial option pricing models

The binomial and multinomial stock price models are both discrete time models. At each time moment, the stock price moves up or down by the multiplication of two (binomial) or k factors (multinomial).

After estimating the multinomiality parameters $c_1, \ldots, c_k$, both stock price models were applied to several raw financial data, under different volatilities ranging from high to low. The results showed that including the multinomial parameters clearly gave a better approximation for real data especially with data with higher volatility.

The results also showed that the values of the multinomiality parameters have a significant effect on the option price; including a multinomial parameter, the multinomial option price was closer to the actual payoff of the option than the binomial.

As we already mentioned, the option prices corresponding to both models have in the limit a Black-Scholes type formula. Both the classical and generalized Black-Scholes formulae overestimate the payoff of the option; however the prices given by the generalized formula are closer to the actual payoff of the option.

Since the multinomial model gave a better approximation of the actual prices, the idea of improving it in a way that is closer to the reality of financial markets is worth considering.

As we already mentioned, the assumptions on the sequence of random variables $\{X_t\}_{t \leq T}$ representing the changes in the stock price, have the main effect in defining the model.

Stock prices change every day as a result of market forces as the supply and demand. And what makes investors want to buy or sell a stock is influenced by many factors that can be social, political, economical...

Many theories tried to explain the way stock prices move the way they do and unfortunately no one theory that can explain everything. However, depending on the factors affecting the changes in the stock price, it is logical and closer to practice to consider that the jumps from a day to another are dependent.

In this thesis, we will consider the generalized CRR stock price model (1.2.6) where the random variables $\{X_t\}_{t \leq T}$ are no longer assumed to be independent over a certain period $[T - m, T]$; that is the idea behind the title "CRR Option Pricing Model with Dependent Jumps".

Chapter 2

A Limit Theorem for Option Price Convergence

The limit of the conditional generalized option price formula was, as we mentioned inCchapter 1, is what was referred to as the generalized Black-Scholes formula. An important theorem used in finding this limit goes to Richter [38]; this theorem states that the limit of the probability distribution of a multinomial random variable is a mutivariate normal distribution (we will give the statement of the theorem below). An important feature of the multinomial distribution is the independence between the trials, which was the case assumed in order to find the limit of the option price: the random variables $\{X_t\}_{t \leq T}$ representing the jump sizes of the stock price were assumed to be independent.

In this chapter, we will give an alternative proof of a Richter-type local limit theorem using the Fourier Inversion method [40, 7, 14, 29]. In this proof the independence of the multinomial trials is used only to find the expression of the characteristic function of a multinomial random variable. In Chapter 3, we will find an expression of the characteristic function of dependent random variables using perturbation theory of linear operators. This expression turns out to be similar to that of the independent case, a fact that will make the Richter-type local limit theorem valid even when the jump sizes are dependent.

We will start by giving the statement of the theorem and explaining how it was applied in finding the generalized Black-Scholes limit for the conditional generalized CRR option price.

2.1 Richter's Local Limit Theorem

Theorem 2.1.1 *[38] Let X be a random variable which can assume k different values $\beta_1, \ldots \beta_k$ with probabilities $P(X = \beta_j) = p_j,\ j = 1, \ldots k$. Let Z_n be the vector of occurrences of these k possible outcomes in n independent trials of X. Then as is known Z_n follows a multinomial distribution with parameters n and $p = (p_1, \ldots, p_k)$. Let G_n denote the region of points $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ for which $|x_j| \leq An^{-\beta}$ for $j + 1, \ldots, k$, where $A > 0$ and $1/3 < \beta < 1/2$ are arbitrarily chosen (but fixed afterwards):*

As $(x_1, \ldots, x_k) = \frac{1}{n}(n_1 - np_1, \ldots, n_k - np_k) \in G_n$ *and* $n \to \infty$ *we have that*

$$P\left(Z_n = (n_1, ..., n_k)\right) = p_n(n_1, \ldots, n_k) \sim \frac{1}{\sqrt{2\pi n}^{k-1}\sqrt{p_1 \ldots p_k}} e^{-\frac{1}{2}\sum_{j=1}^{k} \frac{nx_j^2}{p_j}}$$

where the sign "$\sim$ " denotes the following relation as $n \to \infty$

$A(n) \sim B(n)$ *if and only if* $\frac{A(n)}{B(n)} \to 1$ *as* $n \to \infty$.

Remark. Note that the statement in the previous theorem is uniform in the region

G_n. It states formally that

$$\lim_{n\to\infty} \sup_{(x_1,...,x_k)\in G_n} \left| p_n(n_1,...,n_k)(2\pi n)^{(k-1)/2}(p_1\cdot ...\cdot p_k)^{1/2} e^{\frac{1}{2}\sum_{j=1}^k \frac{nx_j^2}{p_j}} - 1 \right| = 0.$$

Remark. Richter's proof of the multinomial local limit theorem in [38] uses Stirling's formula

$$n! \sim \sqrt{2\pi n} n^n e^{-n}$$

to approximate the exact distribution $p(n_1,...,n_k)$, which is multinomial. Below we want to prove this theorem without using this approximation and without knowing the exact form of the distribution. This leads to the traditional method of using Fourier analysis, a standard tool to prove local limit theorems. The price we have to pay is that the uniformity region becomes smaller. The advantage of the new method of proof lies in the fact that it holds in more general situations when the independence assumption is dropped.

Using this theorem, the following limit was found. For the full proof one can refer to [22].

$$\hat{C}_{T-n} = S_{T-n} \sum_{\mathcal{M}_{2k}(n,p,x)} \frac{1}{(2\pi)^{k/2}} \frac{1}{\sqrt{(\sum_{i=1}^k)\overline{p}_{k+1}\cdots\overline{p}_{2k}}} \exp\left\{ \frac{(\sum_{i=1}^k \overline{x}_i)^2}{\sum_{i=1}^k \overline{p}_i} + \sum_{i=k+1}^{2k} \frac{\overline{x}_i^2}{\overline{p}_i} \right\}$$

$$-\frac{K}{\hat{r}^n} \sum_{\mathcal{M}_{2k}(n,\overline{p},x)} \frac{1}{(2\pi)^{k/2}} \frac{1}{\sqrt{(\sum_{i=1}^k)p_{k+1}\cdots p_{2k}}} \exp\left\{ \frac{(\sum_{i=1}^k x_i)^2}{\sum_{i=1}^k p_i} + \sum_{i=k+1}^{2k} \frac{x_i^2}{p_i} \right\}$$

where

$$\mathcal{M}_{2k}(n,p,x) := \begin{cases} (n_1+n_2+\ldots+n_k, n_{k+1},\ldots,n_{2k}) : m_1+\ldots+n_{2k} = n \\ x \le n_1+\ldots+n_k \le n(p_1+\ldots+p_k) + An^\gamma, \\ np_i + An^\gamma \le n_i \le mp_i + An^\gamma, \forall i = k+1,\ldots,2k \end{cases}$$

$$= \begin{cases} (\mathbf{n}_1^k, \mathbf{m}_{k+1}, \ldots, \mathbf{n}_{2k}) : \mathbf{n}_1^k + \ldots + \mathbf{n}_{2k} = n \\ x \leq \mathbf{n}_1^k \leq n(p_1 + \ldots + p_k) + An^\gamma, \\ np_i + An^\gamma \leq \mathbf{n}_i \leq np_i + An^\gamma, \forall i = k+1, \ldots, 2k \end{cases}$$

for $A > 0$t, $\frac{1}{2} < \gamma < \frac{2}{3}, k \leq \frac{n}{2}$

and

$$x_i := \frac{n_i - np_i}{\sqrt{n}}, \quad \overline{x}_i := \frac{n_i - n\overline{p}_i}{\sqrt{n}}$$

The set $\mathbb{X} := \{(x_1, \ldots, x_{2k}) : |x_i| \leq An^{\frac{1}{2}-\beta} \;\; \forall i = 1, \ldots, 2k\}$ where $\frac{1}{3} < \beta < \frac{1}{2}$.

Notice that:

$\sum_{i=1}^{2k} n_i = n$, $\sum_{i=1}^{2k} p_i = \sum_{i=1}^{2k} \overline{p}_i = 1$ so $\sum_{i=1}^{2k} x_i = \sum_{i=1}^{2k} \overline{x}_i = 0$

2.2 A New Proof of a Richter-Type Local Limit Theorem

In this section we prove a Richter type theorem based on Fourier analysis with a somewhat small region, however this will permit the generalization in Chapter 4. The statement of the theorem follows.

Definition 2.2.1 *Let X be a random variable which can assume k different values $\beta_1, \ldots \beta_k$ with probabilities $P(X = \beta_j) = p_j$, $j = 1, \ldots k$. Let Z_n be the vector of occurrences of these k possible outcomes in n trials of X.*

We say that the sequence Z_n satisfies a Richter's type theorem if there exist a region G_n such that as $(x_1, \ldots, x_k) = \dfrac{1}{n}(n_1 - np_1, \ldots, n_k - np_k) \in G_n$and a normal distribution $(0, \Sigma)$such that as $n \to \infty$ we have

$$n^{(k-1)/2} p_n(n_1, \ldots, n_k) \sim \frac{1}{\sqrt{2\pi}^{k-1} \sqrt{|\Sigma|}} e^{-\frac{1}{2}\tilde{x}'\Sigma^{-1}\tilde{x}},$$

Where the sign "$\sim$ " denotes the following relation as $n \to \infty$:

$A(n) \sim B(n)$ if and only if $\dfrac{A(n)}{B(n)} \to 1$ as $n \to \infty$

Remark. If the n trials are independent and G_n is the region of points $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ for which $|x_j| \leq An^{-\beta}$ for $j + 1, \ldots, k$, where $A > 0$ and $1/3 < \beta < 1/2$ are arbitrarily chosen (but fixed afterwards), then as is known Z_n follows a multinomial distribution with parameters n and $p' = (p_1, \ldots, p_k)$ and we have Richter's local limit theorem 2.1.1,

$$p_n(n_1, \ldots, n_k) \sim \frac{1}{\sqrt{2\pi n}^{k-1} \sqrt{p_1 \ldots p_k}} e^{-\frac{1}{2}\sum_{j=1}^{k} \frac{nx_j^2}{p_j}}.$$

Observe the following easy fact.

Lemma 2.2.2 *For each $n \in \mathbb{N}$*

$$\sup_{(x_1,\ldots,x_k)\in G_n} \left| (2\pi n)^{(k-1)/2} (p_1 \cdot \ldots \cdot p_k)^{1/2} e^{\frac{1}{2}\sum_{j=1}^{k} \frac{nx_j^2}{p_j}} \right| \leq C_0^{-1} n^{(k-1)/2} e^{A_0 n^{1-2\beta}},$$

where

$$C_0^{-1} = (2\pi)^{(k-1)/2} (p_1 \cdot \ldots \cdot p_k)^{1/2}$$

and

$$A_0 = \frac{1}{2}\sum_{j=1}^{k} \frac{1}{p_j}.$$

In order to prove Richter's theorem by estimating the difference of the left and right hand side we need to show that uniformly in $(x_1, ..., x_k) \in G_n$,

$$\begin{aligned} & \left| p_n(n_1, ..., n_k) - \frac{1}{(2\pi n)^{(k-1)/2}(p_1 \cdot ... \cdot p_k)^{1/2}} e^{-\frac{1}{2}\sum_{j=1}^k \frac{nx_j^2}{p_j}} \right| \\ = \; & o\left(C_0 n^{-(k-1)/2} e^{-A_0 n^{1-2\beta}}\right). \end{aligned}$$

We shall see below that the differences cannot be estimated well enough in that region, but we show that this method suffices for a smaller region.

In this section we prove

Theorem 2.2.3 *If the trials are independent, there exist regions of points*

$$G'_n = \left\{ x = (x_1, \ldots, x_k) \in \mathbb{R}^k; |x_j| \leq \sqrt{A \frac{\log(n)}{n}} \right\}$$

for $n \geq 1$ where $A > 0$ denotes some fixed constant, independent of n, such that theorem 2.1.1 holds

$$p_n(n_1, \ldots, n_k) \sim \frac{1}{\sqrt{2\pi n}^{k-1} \sqrt{p_1 \ldots p_k}} e^{-\frac{1}{2}\sum_{j=1}^k \frac{nx_j^2}{p_j}}, \qquad x \in G'_n.$$

Remark. As we will see in the proof, the left hand side differs from the right hand side but at most $o(n^{-\eta})$ for some $\eta > 1/3$. So we may choose $A \sum_{j=1}^k \frac{1}{p_j} < 2\eta$. In fact we have as in Lemma 2.2.2

$$\sup_{(x_1, \ldots, x_k) \in G_n} \left| (2\pi n)^{(k-1)/2} (p_1 \cdot \ldots \cdot p_k)^{1/2} e^{\frac{1}{2}\sum_{j=1}^k \frac{nx_j^2}{p_j}} \right| \leq C_0^{-1} n^{(k-1)/2} n^{\frac{A}{2}\sum_{j=1}^k p_j^{-1}},$$

where $C_0^{-1} = (2\pi)^{(k-1)/2}(p_1 \cdot \ldots \cdot p_k)^{\frac{1}{2}}$.

In the rest of the chapter we prove theorem 2.2.3.

Note: First we notice that $G'_n \subset G_n$, so what we can prove in the region G_n holds automatically in the region G'_n.

Let $p_1, ..., p_k$ and n be fixed.

We need to show that uniformly in $(x_1, ..., x_k) \in G_n$ as explained after Theorem 2.1.1:

$$\frac{p_n}{\frac{1}{(2\pi n)^{(k-1)/2}\sqrt{p_1...p_k}} e^{-\frac{1}{2}\sum_{j=1}^{k}\frac{nx_j^2}{p_j}}} \to 1;$$

or equivalently:

$$\mathcal{Q} = \frac{n^{\frac{k-1}{2}} p_n}{\frac{1}{(2\pi)^{(k-1)/2}\sqrt{p_1...p_k}} e^{-\frac{1}{2}\sum_{j=1}^{k}\frac{nx_j^2}{p_j}}} \to 1;$$

where: $p_n = p_n(n_1, \ldots, n_k)$ is the probability function of a multinomial distribution $(p_n = \dfrac{n!}{n_1!.n_2! \ldots n_k!} p_1^{n1} \ldots . p_k^{n_k})$.

We will start by rewriting the vector of occurrences $Z_n = (n_1, \ldots, n_k)$ as

$$Z_n = \sum_{j=1}^{n} Y_j, \;\; j = 1, \ldots, n,$$

where the $Y_j, j = 1, \ldots, n$ are k-dimensional vectors; each Y_j represents the outcome of the jth trial, being a vector with a 1 in position l if the l-th event occurs $l = 1, \ldots k$ on that trial and $0's$ in all other positions.

If we denote Y_j by $Y_j = (Y_{j1}, \ldots, Y_{j(k-1)}), j = 1, \ldots n$, then the random variables $Y_j's, j = 1, \ldots, n$, are independent.

Note: For each $j = 1,...,n$ the coordinate random variables in $Y_j = (Y_{j1}, \dots, Y_{jk})$ are not independent since $n_k = n - (n_1 + \dots n_{k-1})$.

We introduce the following notations:

$\tilde{n} = (n_1, \dots, n_{k-1}),\ \ n' = (n_1, \dots, n_k),\ \ \sum_{j=1}^{k} n_j = n;$

$\tilde{p} = (p_1, \dots, p_{k-1}),\ \ p' = (p_1, \dots, p_k),\ \ \sum_{j=1}^{k} p_j = 1;$

$\tilde{t} = (t_1, \dots, t_{k-1}) \in \mathbb{R}^{k-1},\ \ t' = (t_1, \dots, t_k) \in \mathbb{R}^k;$

$\tilde{s} = (s_1, \dots, s_{k-1}) \in \mathbb{R}^{k-1},\ \ s' = (s_1, \dots, s_k) \in \mathbb{R}^k;$

and

$\tilde{x} = (x_1, \dots, x_{k-1})$, where $x_j = \dfrac{n_j - np_j}{n}, j = 1, \dots, k$, so $\sum_{j=1}^{k} x_j = 0$.

The symbol $\langle , \rangle$ denotes the inner product of two vectors.

We will proceed using the Fourier inversion formula using the following well known theorem in [40].

Theorem 2.2.4 *For all $f \in L^1(G)$, the function $\hat{f}$ defined on τ by*

$$\hat{f}(\gamma) = \int_G f(x)(x, \gamma) dx, (\gamma \in \tau)$$

is called the fourier transform of f where τ is the dual group of G with respect to the Haar measure. Then we have

$$f(x) = \int_\tau \widehat{f}(\gamma)(-x, \gamma) \mathrm{d}\gamma.$$

We have two choices for G in our situation. First, let $G = \mathbb{Z}^k$ with the counting measure. Then its dual group is the k-dimensional torus $[-\pi, \pi)^k$ with normalized Haar measure $(2\pi)^{-k}dt$. Choosing the function

$$f(n') = P(Z_n = n')$$

we see that

$$\widehat{f}(t) = \int_{\mathbb{Z}^k} f(n')e^{-i\langle n',t\rangle}\mathrm{d}n' = \phi_{n'} = \phi_{Z_n},$$

the characteristic function of Z_n. Therefore

$$p_n(n') = f(n') = \frac{1}{(2\pi)^k}\int_{[-\pi,\pi)^k} \phi_n(t)e^{-i\langle n',t\rangle}\mathrm{d}t.$$

This can be seen directly using characteristic functions of k-dimensional random variables. Let $p_{X'}$ denote the probability function of the discrete random variable $X' = (X_1, \ldots, X_k)$ with values in $\mathbb{Z}^k$ and $\phi_{X'}$ its characteristic function. Using the definition of the characteristic function we get

$$\phi_{X'}(t') = \sum_{n_1,\ldots,n_k} P(X_1 = n_1, \ldots, X_k = n_k)e^{i\langle t',n'\rangle} \tag{2.2.1}$$

If we multiply (2.2.1) by $\frac{1}{(2\pi)^k}e^{-i\langle t',m'\rangle}$ and integrate over $[-\pi, \pi)^k$ with respect to the normalized Haar measure we get

$$\frac{1}{(2\pi)^k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \phi_{X'}(t')e^{-i\langle t',m'\rangle}\mathrm{d}t'$$

$$= \sum_{n_1,\ldots,n_k} \frac{1}{(2\pi)^k}\underbrace{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}}_{k\ integrals} P(X_1 = n_1, \ldots, X_k = n_k)e^{i\langle t',n'\rangle}e^{-i\langle t',m'\rangle}\mathrm{d}t'$$

$$= \sum_{n_1,\ldots,n_k} \frac{1}{(2\pi)^k}\underbrace{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}}_{k\ integrals} P(X_1 = n_1, \ldots, X_k = n_k)e^{i\langle t',n'-m'\rangle}\mathrm{d}t'$$

But $\int_{-\pi}^{\pi} e^{i\langle t_j, n_j - m_j \rangle} = 0$ if $n_j \neq m_j$ and 1 otherwise.

So

$$\frac{1}{(2\pi)^k} \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{k \text{ integrals}} \phi_{X'}(t) e^{-i\langle t', n' \rangle} \mathrm{d}t' = P(X_1 = n_1, \ldots, X_k = n_k) = p_n.$$

The right hand side is the probability function of X' and therefore

$$p_{X'}(n') = \frac{1}{(2\pi)^k} \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{k \text{ integrals}} \phi_{X'}(t) e^{-i\langle t', n' \rangle} \mathrm{d}t' \tag{2.2.2}$$

It is possible to work in this setup, however it turns out that it is more convenient to use the group $G = \mathbb{Z}^{k-1}$ with the counting measure and its dual group $[-\pi, \pi)^{k-1}$ with the normalized Haar measure $\frac{1}{(2\pi)^{k-1}} \mathrm{d}t$.

Proposition 2.2.5 *The Fourier inversion formula of p_n in $\mathbb{Z}^{k-1}$ is given by*

$$p_n(\widetilde{n}) = \frac{1}{(2\pi)^{k-1}} \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{k-1 \text{ integrals}} \phi_n(\tilde{t}) e^{-i\langle \tilde{n}, \tilde{t} \rangle} \mathrm{d}\tilde{t}. \tag{2.2.3}$$

Proof. Consider the function

$$f(n_1, ..., n_{k-1}) = f(\tilde{n}) = P(Z'_n = (n_1, ..., n_{k-1})),$$

where

$$Z'_n = \sum_{j=1}^{m} Y'_j$$

and

$$Y'_j = (Y_{j,1}, ..., Y_{j,k-1}) \qquad j = 1, ..., m.$$

Then, as before $\widehat{f}(\tilde{t}) = \phi_{Z'_n}(\tilde{t})$, the characteristic function of Z'_n and

$$p_n(n') = P(Z'_n = \widetilde{n}) = \frac{1}{(2\pi)^{k-1}} \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{k-1 \; integrals} \phi_{n'}(\tilde{t}) e^{-i\langle \tilde{n}, \tilde{t} \rangle} \mathrm{d}\tilde{t}. \qquad \square$$

In the proof of Theorem 2.1 we shall work with the Fourier inversion in $k-1$ dimensions.

Lemma 2.2.6

$$\phi_{Z'_n - n\tilde{p}}(t) = \left(E\left(e^{i\langle t, Y'_1 - \tilde{p} \rangle} \right) \right)^n$$

Proof. Since Z'_n is a sum of n independent identically distributed random variable Y'_j with expectation $\tilde{p}$, the lemma follows. $\square$

For the proof of Theorem 2.2.3 we need to estimate $E\left(e^{i\langle t, Y'_1 - \tilde{p}\rangle}\right)$ for $-\pi \leq t < \pi$. We do this by splitting $[-\pi, \pi)$ into three different regions. We take β as in Richter's theorem, so $\frac{1}{3} < \beta < \frac{1}{2}$. The first two lemmas and corollaries hold for the larger regions G_n, hence also apply for the regions G'_n.

Lemma 2.2.7 *For each* $\delta > 0$ *there exists* $0 < \Delta < 1$ *such that*

$$\sup_{\delta < \max_j |t_j| \leq \pi} \left| E\left(e^{i\langle t, Y'_1 - \tilde{p} \rangle} \right) \right| \leq \Delta.$$

Proof. By definition

$$\begin{aligned} E\left(e^{i\langle t, Y'_1 - \tilde{p}\rangle}\right) &= \sum_{j=1}^{k-1} e^{it_j - i\langle t, \tilde{p}\rangle} P(Y_{1,j} = 1) + e^{-i\langle t, \tilde{p}\rangle} P(Y_{1,l} \neq 1, \forall l) \\ &= e^{-i\langle t, \tilde{p}\rangle} \left[\sum_{j=1}^{k-1} p_j e^{it_j} + (1 - p_1 - \ldots - p_{k-1}) \right]. \end{aligned}$$

This complex number has a modulus which is bounded by one and is equal to 1 if and only if each $t_j = 0$. Thus, given $\delta > 0$ and one of the $t_j > \delta$, the modulus of the number is strictly bounded away from 1, where the bound Δ can be taken independent of these $-\pi \leq t \leq \pi$ and is smaller than 1. This proves the claim. $\square$.

Corollary 2.2.8 *Letting* $I_\delta = (-\delta, \delta)^{k-1}$ *we have*

$$\frac{1}{(2\pi)^{k-1}}\left|\int \cdots \int_{I_\delta^c} \phi_n(\tilde{t}) e^{-i\langle\tilde{n},\tilde{t}\rangle} \mathrm{d}\tilde{t}\right| \leq \Delta^n = o\left(C_0 n^{-k+1} e^{-A_0 n^{1-2\beta}}\right).$$

Proof. This follows immediatly from the lemma:

$$\begin{aligned} & \frac{1}{(2\pi)^{k-1}}\left|\int \cdots \int_{I_\delta^c} \phi_n(\tilde{t}) e^{-i\langle\tilde{n},\tilde{t}\rangle} \mathrm{d}\tilde{t}\right| \\ \leq\ & \frac{1}{(2\pi)^{k-1}} \int \cdots \int_{I_\delta^c} \left|E\left(e^{i\langle t, Y_1' - \tilde{p}\rangle}\right)\right| \mathrm{d}\tilde{t} \\ \leq\ & \Delta^n. \square. \end{aligned}$$

We will next use the expansion for $e^{i<\frac{\tilde{s}}{\sqrt{n}},(Y_1-\tilde{p})>}$ to approximate $\phi_{Z_n'-n\tilde{p}}(\frac{\tilde{s}}{\sqrt{n}})$, where $\max_j |\tilde{s}_j| \leq \delta\sqrt{n}$.

Lemma 2.2.9 *For any* $0 < \delta < \frac{2}{3k}$ *there exists* $0 \leq q < 1$ *such that as* $\max_j |\tilde{s}_j| \leq \delta\sqrt{n}$, *it follows that*

$$|\phi_{Z_n'-n\tilde{p}}(\frac{\tilde{s}}{\sqrt{n}})| \leq e^{-\frac{1-q}{2}E(\langle\tilde{s},(Y_1-\tilde{p})\rangle)^2}.$$

Proof. Using the definition of the complex exponential function we have

$$\begin{aligned} e^{i\langle\frac{\tilde{s}}{\sqrt{n}},(Y_1-\tilde{p})\rangle} &= 1 + i\langle\frac{\tilde{s}}{\sqrt{n}},(Y_1-\tilde{p})\rangle - \frac{1}{2}(\langle\frac{\tilde{s}}{\sqrt{n}},(Y_1-\tilde{p})\rangle)^2 + \ldots \\ &= \sum_{l=0}^{\infty} \frac{i^l}{l!}\langle\frac{\tilde{s}}{\sqrt{n}},(Y_1-\tilde{p})\rangle^l. \end{aligned}$$

Taking expectation and observing that $E\langle\frac{\tilde{s}}{\sqrt{n}},(Y_1-\tilde{p})\rangle = 0$ we derive

$$Ee^{i<\frac{\tilde{s}}{\sqrt{n}},(Y_1-\tilde{p})>} = 1 - \sum_{l=2}^{\infty} \frac{i^{l-2}}{l!} E\langle\frac{\tilde{s}}{\sqrt{n}},(Y_1-\tilde{p})\rangle^l.$$

Now observe that for $\max_j |s_j| \le \delta\sqrt{n}$

$$\begin{aligned}
|\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle| &= \sum_{j=1}^{k-1} \frac{s_j}{\sqrt{n}}(Y_{1,j} - p_j) \\
&= \sum_{j=1}^{k-1} \frac{s_j Y_{1,j}}{\sqrt{n}} - \sum_{j=1}^{k-1} \frac{s_j p_j}{\sqrt{n}} \\
&\le k\delta.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left| \sum_{l=3}^{\infty} \frac{i^l}{l!} E\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle^l \right| \\
\le\ & \sum_{l=3}^{\infty} \frac{1}{l!} E \left| \langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle^l \right| \\
\le\ & E\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle^2 \sum_{l=3}^{\infty} \frac{1}{l!}(k\delta)^{l-2} \\
\le\ & \frac{1}{6(1 - k\delta)} E\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle^2.
\end{aligned}$$

Because $\delta < \frac{2}{3k}$

$$\left| \sum_{l=3}^{\infty} \frac{i^{l-2}}{l!} E\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle^l \right| \le \frac{q}{2} E\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle^2,$$

where $q < 1$.

It follows now that

$$\begin{aligned}
E(e^{i\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle}) &= 1 + iE\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle - \frac{1}{2} E(\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle)^2 \\
&\quad + \frac{i^3}{3!} E(\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle)^3 + \ldots,
\end{aligned}$$

and moreover, since $E\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle = 0$,

$$\left| E(e^{i\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1-\tilde{p})\rangle}) \right| \leq 1 - \frac{1-q}{2} E(< \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) >)^2.$$

Applying the inequality $1 - x \leq e^{-x}$ we arrive at

$$\begin{aligned} \left| \left(E(e^{i\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1-\tilde{p})\rangle}) \right)^n \right| &\leq e^{-n\frac{1-q}{2}E(\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1-\tilde{p})\rangle)^2} \\ &= e^{-\frac{1-q}{2}E(\langle \tilde{s}, (Y_1-\tilde{p})\rangle)^2}. \end{aligned}$$

This finishes the proof of the lemma. □.

Corollary 2.2.10 *Let* $\delta < \frac{2}{3k}$, $\alpha > \frac{1-2\beta}{2}$, $I_\alpha = [-n^\alpha, n^\alpha]^{k-1}$ *and* $I_{\delta\sqrt{n}} = (-\delta\sqrt{n}, \delta\sqrt{n})^{k-1}$. *Then*

$$\begin{aligned} &\left| \frac{1}{(2\pi)^{k-1}} \int \cdots \int_{I_{\delta\sqrt{n}} \setminus I_\alpha} e^{-i\langle \tilde{s}, \frac{\tilde{n}-n\tilde{p}}{\sqrt{n}}\rangle} \left(E e^{i\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1-\tilde{p})\rangle} \right)^n \mathrm{d}\tilde{(s)} \right| \\ &= o\left(C_0 e^{-A_0 n^{1-2\beta}} \right). \end{aligned}$$

Proof. First note that by the next lemma (Lemma 2.7) it follows that there is a constant $c > 0$ such that

$$E\langle \tilde{s}, Y_1 - \tilde{p}\rangle^2 \geq c n^{2\alpha},$$

whenever $\tilde{s} \in I_\alpha^c$. Therefore, using the previous lemma as well,

$$\begin{aligned} &\left| \frac{1}{(2\pi)^{k-1}} \int \cdots \int_{I_\delta \setminus I_\alpha} e^{-i\langle \tilde{s}, \frac{\tilde{n}-n\tilde{p}}{\sqrt{n}}\rangle} \left(E e^{i\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1-\tilde{p})\rangle} \right)^n \mathrm{d}\tilde{(s)} \right| \\ \leq\ & \frac{1}{(2\pi)^{k-1}} \int \cdots \int_{I_\delta \setminus I_\alpha} e^{-\frac{(1-q)c}{2} n^{2\alpha}} \mathrm{d}\tilde{(s)} \\ \leq\ & e^{-\frac{(1-q)c}{2} n^{2\alpha}} = o\left(C_0 n^{-(k-1)/2} e^{-A_0 n^{1-2\beta}} \right). \square. \end{aligned}$$

Using Corollaries 2.2.8 and 2.2.10 we have reduced the estimation problem to

$$\frac{1}{(2\pi)^{k-1}} \int \cdots \int_{I_\alpha} e^{-i\langle \tilde{s}, \frac{\tilde{n}-n\tilde{p}}{\sqrt{n}}\rangle} \left(E e^{i\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1-\tilde{p})\rangle} \right)^n \mathrm{d}\tilde{(s)}.$$

This will be used below.

In a second step we compute the covariance structure of Y_1 and hence thae covariance of the limiting normal density function. This is needed to represent the denominator as a Fourier integral.
We compute $E\langle\tilde{s}, Y_1 - \tilde{p}\rangle^2$ and show the proposition below

Proposition 2.2.11 *Let* $\tilde{s} \in \mathbb{R}^{k-1}$. *Then*

$$E < \tilde{s}, Y_1 - \tilde{p} >^2 = \sum_{l=1}^{k-1} s_l^2 p_l(1-p_l) - \sum_{l \neq l'} s_l s_{l'} p_l p_{l'}.$$

Moreover,

$$\tilde{s} \mapsto e^{\frac{-1}{2}\sum_{l=1}^{k-1} s_l^2 p_l(1-p_l) - \sum_{l\neq l'} s_l s_{l'} p_l p_{l'}}$$

is the characteristic function of a $(k-1)$*-multivariate normal distribution with mean* 0 *and covariance matrix*

$$\Sigma = \begin{pmatrix} p_1(1-p_1) & -p_1p_2 & \dots & -p_1p_{k-1} \\ -p_1p_2 & p_2(1-p_2) & \dots & -p_2p_{k-1} \\ & & \vdots & \\ -p_1p_{k-1} & -p_2p_{k-1} & \dots & p_{k-1}(1-p_{k-1}) \end{pmatrix}$$

The determinant of the covariance matrix is $|\Sigma| = p_1p_2\dots p_k$.

Proof. The first equality follows easily since for $\tilde{s} = (s_1, ..., s_{k-1}) \in \mathbb{R}^{k-1}$ we have

$$\begin{aligned} E < \tilde{s}, Y_1 - \tilde{p} >^2 &= E\left(\sum_{l=1}^{k-1} s_l(y_{1l} - p_l)\right)^2 \\ &= E\sum_{ll'} s_l s_{l'} E(Y_{1l} - p_l)(Y_{1l'} - p_{l'}) \\ &= \sum_{l=1}^{k-1} s_l^2 var(Y_{1l}) + \sum_{l\neq l'} s_l s_{l'} E(Y_{1l}Y_{1l'} - Y_{1l}p_{l'} - Y_{1l'}p_l + p_l p_{l'}). \end{aligned}$$

Next

$$\Sigma = \begin{pmatrix} p_1(1-p_1) & -p_1p_2 & -p_1p3 & \dots & -p_1p_{k-2} & -p_1p_{k-1} \\ -p_1p_2 & p_2(1-p_2) & -p_2p_3 & \dots & -p_2p_{k-2} & -p_2p_{k-1} \\ -p_1p_3 & -p_2p_3 & p_3(1-p_3) & \dots & -p_3p_{k-2} & -p_3p_{k-1} \\ & & \vdots & & & \\ -p_1p_{k-2} & -p_2p_{k-2} & -p_3p_{k-2} & \dots & p_{k-2}(1-p_{k-2}) & -p_{k-2}p_{k-1} \\ -p_1p_{k-1} & -p_2p_{k-1} & -p_3p_{k-1} & \dots & -p_{k-2}p_{k-1} & p_{k-1}(1-p_{k-1}) \end{pmatrix}.$$

$|\Sigma| = p_1 \times p_2 \times \ldots \times p_{k-1} \times |\Sigma'|$ where

$$\Sigma' = \begin{pmatrix} c_1 & c_2 & c_3 & \dots & c_{k-2} & c_{k-1} \\ 1-p_1 & -p_2 & -p3 & \dots & -p_{k-2} & -p_{k-1} \\ -p_1 & 1-p_2 & -p_3 & \dots & -p_{k-2} & -p_{k-1} \\ -p_1 & -p_2 & 1-p_3 & \dots & -p_{k-2} & -p_{k-1} \\ & & \vdots & & & \\ -p_1 & -p_2 & -p_3 & \dots & 1-p_{k-2} & -p_{k-1} \\ -p_1 & -p_2 & -p_3 & \dots & -p_{k-2} & 1-p_{k-1} \end{pmatrix}.$$

We will denote the rows in Σ' as $l_1, l_2, \ldots, l_{k-2}, l_{k-1}$ and proceed as follows

$$\begin{pmatrix} & c_1 & c_2 & c_3 & \dots & c_{k-2} & c_{k-1} \\ l_1 - l_2 & 1 & -1 & 0 & \dots & 0 & 0 \\ l_2 - l_3 & 0 & 1 & -1 & \dots & 0 & 0 \\ l_3 - l_4 & 0 & 0 & 1 & -1 & \dots & 0 \\ & & \vdots & & & & \\ l_{k-2} - l_{k-1} & 0 & 0 & 0 & \dots & 1 & -1 \\ l_{k-1} & -p_1 & -p_2 & -p_3 & \dots & -p_{k-2} & 1-p_{k-1} \end{pmatrix}$$

$$\begin{pmatrix} c_1 & c_2 + c_1 = c_2' & c_3 & \dots & c_{k-2} & c_{k-1} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \dots & 1 & -1 \\ -p_1 & -p_2 - p_1 & -p_3 & \dots & -p_{k-2} & 1-p_{k-1} \end{pmatrix}$$

$$\begin{pmatrix} c_1 & c_2' & c_3 + c_2' = c_3' & \dots & c_{k-2} & c_{k-1} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \dots & 1 & -1 \\ -p_1 & -p_2 - p_1 & -p_3 - p_2 - p_1 & \dots & -p_{k-2} & 1 - p_{k-1} \end{pmatrix}$$

$$\begin{pmatrix} c_1 & c_2' & c_3 + c_2' = c_3' & \dots & c_{k-2} + c_{k-3}' = c_{k-2}' & c_{k-1} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & & \dots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \dots & 1 & -1 \\ -p_1 & -p_2 - p_1 & -p_3 - p_2 - p_1 & \dots & -p_{k-2} - \dots - p_2 - p_1 & 1 - p_{k-1} \end{pmatrix}$$

$$\begin{pmatrix} c_1 & c_2' & c_3' & \dots & c_{k-2}' & c_{k-1} + c_{k-2}' = c_{k-1}' \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & & \dots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \\ -p_1 & -p_2 - p_1 & -p_3 - p_2 - p_1 & \dots & -\sum_{j=1}^{k-2} p_j & 1 - \sum_{j=1}^{k-1} p_j \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & & \dots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \\ -p_1 & -p_2 - p_1 & -p_3 - p_2 - p_1 & \dots & -\sum_{j=1}^{k-2} p_j & p_k \end{pmatrix}$$

The determinant of Σ' is equal to the determinant of the above triangular matrix, so $Det(\Sigma') = 1 \times 1 \times \ldots \times p_k = p_k$.

Finally $Det(\Sigma) = p_1 p_2 \dots p_{k-1} \times det(\Sigma') = p_1 p_2 \dots p_{k-1} p_k$.

We will now show that $-\dfrac{1}{2}\tilde{z}'\Sigma^{-1}\tilde{z} = -\dfrac{1}{2}\sum_{j=1}^{k}\frac{z_j^2}{p_j} = -\dfrac{1}{2}\sum_{j=1}^{k}\frac{mx_j^2}{p_j}$

$$\sum_{j=1}^{k}\frac{z_j^2}{p_j} = \frac{z_1^2}{p_1} + \frac{z_2^2}{p_2} + \dots + \frac{z_{k-1}^2}{p_{k-1}} + \frac{\sum_{j=1}^{k-1} z_j^2}{1 - \sum_{j=1}^{k-1} p_j}$$

$$= \frac{z_1^2}{p_1}\left(\frac{1}{p_1} + \frac{1}{p_k}\right) + \dots + \frac{z_{k-1}^2}{p_{k-1}}\left(\frac{1}{p_{k-1}} + \frac{1}{p_k}\right) + \sum_{j \neq j'}\frac{z_j z_{j'}}{p_k}.$$

So

$$\sum_{j=1}^{k}\frac{z_j^2}{p_j} = \tilde{z}'\Delta\tilde{z},$$

Where

$$\Delta = \begin{pmatrix} \frac{1}{p_1} + \frac{1}{p_k} & \frac{1}{p_k} & \dots & \frac{1}{p_k} \\ \frac{1}{p_k} & \frac{1}{p_2} + \frac{1}{p_k} & \dots & \frac{1}{p_k} \\ & & \vdots & \\ \frac{1}{p_k} & \frac{1}{p_k} & \dots & \frac{1}{p_{k-1}} + \frac{1}{p_k} \end{pmatrix}$$

We need to prove that $\Delta = \Sigma^{-1}$ or equivalently $\Delta\Sigma = I$, where I denotes the $(k-1)$ identity matrix. Note that

$$\Delta\Sigma = \begin{pmatrix} \frac{1}{p_1} + \frac{1}{p_k} & \frac{1}{p_k} & \dots & \frac{1}{p_k} \\ \frac{1}{p_k} & \frac{1}{p_2} + \frac{1}{p_k} & \dots & \frac{1}{p_k} \\ & & \vdots & \\ \frac{1}{p_k} & \frac{1}{p_k} & \dots & \frac{1}{p_{k-1}} + \frac{1}{p_k} \end{pmatrix} \begin{pmatrix} p_1(1-p_1) & -p_1p_2 & \dots & -p_1p_{k-1} \\ -p_1p_2 & p_2(1-p_2) & \dots & -p_2p_{k-1} \\ & & \vdots & \\ -p_1p_{k-1} & -p_2p_{k-1} & \dots & p_{k-1}(1-p_{k-1}) \end{pmatrix}$$

with $\Delta = (\delta_{ij})_{i,j=1,\dots,k-1}$ where $\delta_{ii} = \frac{1}{p_i} + \frac{1}{p_n}$ and $\delta_{ij} = \frac{1}{p_n}$ for $i \neq j$

and $\Sigma = (\sigma_{ij})_{i,j=1,\dots,k-1}$ where $\sigma_{ii} = p_i(1-p_i)$ and $\sigma_{ij} = -p_ip_j$ for $i \neq j$.

Denote $\Delta\Sigma$ by $P = (p_{ij})_{i,j=1,\dots,k-1}$, where $p_{ij} = \sum_{r=1}^{k-1}\delta_{ir}\sigma_{rj}$.

for $i = j$, $p_{ii} = 1$ because

$$p_{ii} = \sum_{r=1}^{k-1} \delta_{ir}\sigma_{ri} = \delta_{ii}\sigma_{ii} + \sum_{r\neq i} \delta_{ir}\sigma_{ri} = \left(\frac{1}{p_i} + \frac{1}{p_k}\right)(p_i(1-p_i)) + \sum_{r\neq i} \frac{1}{p_k}(-p_r p_i)$$

$$= 1 - p_i + \frac{p_i(1-p_i)}{p_k} - \frac{p_i}{p_k}\sum_{r\neq i} p_r = 1 - p_i + \frac{p_i(1-p_i)}{p_k} - \frac{p_i}{p_k}(1 - p_i - p_k)$$

$$= 1 - p_i + \frac{p_i(1-p_i)}{p_k} - \frac{p_i(1-p_i)}{p_k} + \frac{p_i p_k}{p_k} = 1 - p_i + p_i = 1$$

for $i \neq j$, $p_{ij} = 0$ because

$$p_{ij} = \sum_{r=1}^{k-1} \delta_{ir}\sigma_{rj} = \delta_{ii}\sigma_{ij} + \delta_{ij}\sigma_{jj} + \sum_{r\neq i,j} \delta_{ir}\sigma_{rj}$$

$$= \left(\frac{1}{p_i} + \frac{1}{p_k}\right)(-p_i p_j) + \frac{1}{p_k} p_j(1-p_j) + \sum_{r\neq i,j} \frac{1}{p_k}(-p_r p_j)$$

$$= -p_j - \frac{p_i p_j}{p_k} + \frac{p_j(1-p_j)}{p_k} - \frac{p_j}{p_k}(1 - p_j - p_i - p_k)$$

$$= -p_j - \frac{p_i p_j}{p_k} + \frac{p_j(1-p_j)}{p_k} - \frac{p_j(1-p_j)}{p_k} + \frac{p_j p_i}{p_k} + \frac{p_j p_k}{p_k} = 0$$

Therefore $P = I$ and $\Delta = \Sigma^{-1}$. $\square$.

In a last step we prove Theorem [**?**], using the previous two steps.
We will split the work for the proof of the main theorem into two parts, one for the numerator and one for the denominator of $\mathcal{Q}$. For both expressions, we will proceed in the same fashion, that can be summarized as follows:

- Use the fourier inversion formula, where the bounds of integration are $[-\pi, \pi)$.
- Make a change of variables for which the integration bounds become $[-\sqrt{n}\pi, \sqrt{n}\pi)$.

- Split the integration in three parts: one in $(-\sqrt{n}\pi, -n^{\alpha})$, then in $(-n^{\alpha}, n^{\alpha})$ and finally $(n^{\alpha}, \sqrt{n}\pi)$ where $\alpha < 1/2$. The first and last region can be treated by Corollaries 2.2.8 and 2.2.10.

- Get an estimate of the integral over each of these intervals as $n \to \infty$.

Starting with the denominator, we will only recall for now lemma 2.2.2 that

$$\frac{1}{denominator} \sim O\left(C_0^{-1} n^{(k-1)/2} e^{A_0 n^{1-2\beta}}\right) \quad x \in G_n,$$

and from remark (p 32) that

$$\frac{1}{denominator} \sim O\left(C_0^{-1} n^{(k-1)/2} e^{n \frac{A}{2} \sum_j p_j^{-1}}\right) \quad x \in G'_n.$$

For the numerator we will start by using the Fourier inversion formula (2.2.3). Then for $\delta < max_j |t_j| < \pi$, split the integral in I_δ and I_δ^c.

For $\tilde{t} \in I_\delta^c$ we have from corollary 2.2.8 that the $numerator \sim o\left(C_0 n^{-(k-1)} e^{-A_0 n^{1-2\beta}}\right)$ and therefore

$$\mathcal{Q} = numerator \times \frac{1}{denominator} \sim o\left(C_0 n^{-(k-1)/2} e^{-A_0 n^{1-2\beta}} \times C_0^{-1} n^{(k-1)/2} e^{A_0 n^{1-2\beta}}\right)$$

Hence $\mathcal{Q} \sim o(1)$ which is of order less than one and goes to zero uniformly in $x \in I_\delta^c$. We will continue with the numerator in the region I_δ.

We multiply (2.2.3) by $n^{\frac{k-1}{2}}$

$$n^{\frac{k-1}{2}} p_n = n^{\frac{k-1}{2}} \frac{1}{(2\pi)^{k-1}} \underbrace{\int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta}}_{k-1 \ integrals} e^{-i\langle \tilde{t}, \tilde{n} \rangle} \phi_n(\tilde{t}) d\tilde{t} \tag{2.2.4}$$

where $\phi_n(\tilde{t})$ is the characteristic function of $\sum_{j=1}^{n} Y_j$. Replace $\phi_n(\tilde{t})$ by $E e^{i\langle \tilde{t}, \sum_{j=1}^{n} Y_j \rangle}$ which yields

$$n^{\frac{k-1}{2}}p_n = n^{\frac{k-1}{2}}\frac{1}{(2\pi)^{k-1}}\underbrace{\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta}}_{k-1\ integrals} e^{-i\langle\tilde{t},\tilde{n}\rangle}Ee^{i\langle\tilde{t},\sum_{j=1}^{n}Y_j\rangle}\mathrm{d}\tilde{t}$$

Multiplying above by $e^{i\langle\tilde{t},n\tilde{p}\rangle}e^{-i\langle\tilde{t},n\tilde{p}\rangle} = 1$ gives

$$n^{\frac{k-1}{2}}p_n = n^{\frac{k-1}{2}}\frac{1}{(2\pi)^{k-1}}\underbrace{\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta}}_{k-1\ integrals} e^{-i\langle\tilde{t},\tilde{n}\rangle}E(e^{i\langle\tilde{t},\sum Y_i\rangle})e^{i\langle\tilde{t},n\tilde{p}\rangle}e^{-i\langle\tilde{t},n\tilde{p}\rangle}\mathrm{d}\tilde{t}$$

$$= n^{\frac{k-1}{2}}\frac{1}{(2\pi)^{k-1}}\underbrace{\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta}}_{k-1\ integrals} e^{-i\langle\tilde{t},\tilde{n}-n\tilde{p}\rangle}Ee^{i<\tilde{t},\sum(Y_i-\tilde{p})>}\mathrm{d}\tilde{t}.$$

But $Ee^{i\langle\tilde{t},\sum(Y_i-\tilde{p})\rangle}$ is the characteristic function of $\sum(Y_i-\tilde{p}) = Z'_n - n\tilde{p}$, so from lemma 2.2.6

$$n^{\frac{k-1}{2}}p_n = n^{\frac{k-1}{2}}\frac{1}{(2\pi)^{k-1}}\underbrace{\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta}}_{k-1\ integrals} e^{-i\langle\tilde{t},\tilde{n}-n\tilde{p}\rangle}\phi_{Z'_n-n\tilde{p}}(t)\mathrm{d}\tilde{t}$$

$$= n^{\frac{k-1}{2}}\frac{1}{(2\pi)^{k-1}}\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta} e^{-i\langle\tilde{t},\tilde{n}-n\tilde{p}\rangle}\left(E\left(e^{i\langle t,Y'_1-\tilde{p}\rangle}\right)\right)^n\mathrm{d}\tilde{t}. \tag{2.2.5}$$

Using the change of variables $s_l = \sqrt{n}\,t_l$ for $l = 1,\ldots,k-1$,

equation (2.2.5) becomes

$$n^{\frac{k-1}{2}}p_n = \frac{1}{(2\pi)^{k-1}}\int\cdots\int_{I_{\delta\sqrt{n}}} e^{-i\langle\tilde{s},\frac{\tilde{n}-n\tilde{p}}{\sqrt{n}}\rangle}(Ee^{i\langle\frac{\tilde{s}}{\sqrt{n}},(Y_1-\tilde{p})\rangle})^n\mathrm{d}\tilde{s} \tag{2.2.6}$$

Splitting the integral in (2.2.6) in I_α and $I_{\delta\sqrt{n}}\setminus I_\alpha$ with $I_\alpha = (-n^\alpha, n^\alpha)^{k-1}$ and $I_{\delta\sqrt{n}} = (-\delta\sqrt{n},\delta\sqrt{n})^{k-1}$ and using corollary 2.2.10, we get that in $I_{\delta\sqrt{n}}\setminus I_\alpha$, $\mathcal{Q}$ is of

$o(1)$ uniformly in $x \in G_n$ as before, so also on G'_n.

Therefore the proof of the main theorem is reduced to the integral over I_α.

For the numerator it is left to estimate the following integral

$$n^{\frac{k-1}{2}} p_n = \frac{1}{(2\pi)^{k-1}} \int \dots \int_{I_\alpha} e^{-i\langle \tilde{s}, \frac{\tilde{n}-n\tilde{p}}{\sqrt{n}} \rangle} (E e^{i\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle})^n \mathrm{d}\tilde{s} \tag{2.2.7}$$

To do so we will continue by using Taylor's expansion for $e^{i\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle}$ yielding

$$e^{i\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle} = 1 + i\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle - \frac{1}{2}(\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle)^2 +$$

$$\frac{i^3}{3!}(\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle)^3 + \dots$$

Taking the expectation

$$E(e^{i\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle}) = 1 + iE\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle - \frac{1}{2}E(\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle)^2 +$$

$$\frac{i^3}{3!}E(\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle)^3 + \dots$$

But $E\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle = 0$, and equation (2.2.7) becomes

$$n^{\frac{k-1}{2}} p_n = \frac{1}{(2\pi)^{k-1}} \int \dots \int_{I_\alpha} e^{-i\langle \tilde{s}, \frac{\tilde{n}-n\tilde{p}}{\sqrt{n}} \rangle} [1 - \frac{1}{2}E(\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle)^2 + \frac{i^3}{3!}E(\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle)^3$$

$$+ \dots]^n \mathrm{d}\tilde{s}.$$

Lemma 2.2.12 *Let $0 < x_n \leq \sqrt{qn}$, where $0 < q < 1$, and $y_n \in \mathbb{C}$ satisfying $|y| = o(n^{-\eta})$ for some $\eta > 0$. Then* $\left(1 - \frac{x}{n} + y\right)^n \sim e^{-x}\left(1 + o(\frac{x^2}{n} + n^{-\eta})\right)$,

Proof. We need to show $\dfrac{\left(1 - \dfrac{x}{n} + y\right)^n}{e^{-x}} - 1 = o(\frac{x^2}{n} + n^{-\eta})$

or equivalently, $\left(1 - \dfrac{x}{n} + y\right)^n e^x - 1 = o(\frac{x^2}{n} + n^{-\eta})$

Define the function $f(y) = \left(1 - \dfrac{x}{n} + y\right)^n e^x - 1.$

We know for some z in the ball $B(1 - \frac{x}{n}, y)$,

$$|f(y) - f(0)| \leq |f'(z)||y|$$

and by the triangular inequality

$$|f(y)| < |f'(z)||y| + |f(0)|.$$

Note that

$$f(0) = \left(1 - \frac{x}{n}\right)^n e^x - 1$$

and that

$$f'(z) = n\left(1 - \frac{x}{n} + z\right)^{n-1} e^x - 1$$

is bounded. Hence we need to estimate $f(0)$.

In order to do so, let

$$g(x) = \left(1 - \frac{x}{n}\right)^n e^x.$$

We now consider $\log(g(x))$:

$$\begin{aligned}
\log(g(x)) &= \log\left[\left(1-\frac{x}{n}\right)^n e^x\right] \\
&= n\log\left(1-\frac{x}{n}\right)+x \\
&= n\left(-\sum_{j=1}^{\infty}\frac{x^j}{n^j}\frac{1}{j}\right)+x \\
&= -\sum_{j=2}^{\infty}\frac{x^j}{n^{j-1}}\frac{1}{j} \\
&= \frac{x^2}{n}\left(-\sum_{j=0}^{\infty}\frac{x^j}{n^{j-1}}\frac{1}{(j+2)}\right)
\end{aligned}$$

$$= K(x)\frac{x^2}{n}.$$

Therefore

$$g(x) = \exp\left\{K(x)\frac{x^2}{n}\right\}.$$

Note that $|K(x)| \leq K$ for some constant K since $\frac{x^2}{n} < 1$, and hence,

$$\begin{aligned}
|f(0)| &= |g(x)-1| = \left|\exp\left\{K(x)\frac{x^2}{n}\right\}-1\right| \\
&\leq |1+K\frac{x^2}{n}+\frac{1}{2}K\frac{x^4}{n^2}+\ldots-1| \\
&< K'\frac{x^2}{n}
\end{aligned}$$

for some constant k', since $\dfrac{x^2}{n} < 1$. $\Box$.

We apply this lemma to the integral over I_{n^α}.

We know $s_j < n^\alpha$ So

$$|x| = \left|E(\langle\tilde{s},(Y_1-\tilde{p})\rangle)^2\right| < ((k-1)n^\alpha)^2 = (k-1)^2 n^{2\alpha}$$

where $x \in \mathbb{R}$ and $\alpha < \frac{1}{2}$. If we suppose also that $\alpha < \frac{1}{6}$ we get $2\alpha < \frac{1}{3}$ and $\frac{|x|}{n} < n^{2\alpha-1} < 1$, so using the Taylor expansion for log in this case is appropriate.

Recall that α was chosen to be greater than $\frac{1-2\beta}{2}$ and since $\frac{1}{3} < \beta < \frac{1}{2}$ then $\frac{1-2\beta}{2} < \frac{1}{6}$, and therefore, α can be chosen appropriately.

It follows that $\frac{x}{n}$ is the expression

$\frac{1}{2}E(\langle\frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle)^2$ *i.e* $x = \frac{1}{2}E(\langle\tilde{s}, (Y_1 - \tilde{p})\rangle)^2$.

Define $y \in \mathbb{C}$ by

$$y = \frac{i^3}{3!}E(\langle\frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle)^3 + \ldots = \sum_{l=3} \frac{i^l}{l! n^{l/2}} E(< \tilde{s}, (Y_1 - \tilde{p}) >)^l.$$

So we need to estimate

$$n^{\frac{k-1}{2}} p_n = \frac{1}{(2\pi)^{k-1}} \int \ldots \int_{I_\alpha} e^{-i\langle\tilde{s}, \frac{\tilde{n}-n\tilde{p}}{\sqrt{m}}\rangle} [1 - \frac{x}{m} + y]^n \mathrm{d}\tilde{s}. \tag{2.2.8}$$

The order of $|f(0)|$ is that of $\frac{|x^2|}{n} \sim o(n^{4\alpha-1}) = o(n^{-\eta})$ where $\eta > \frac{1}{3}$.

Recall that $|f'(z)||y| = \left|n\left(1 - \frac{x}{n} + z\right)^{n-1} e^x\right| |y| = O(|y|)$,

and

$$y = \sum_{l=3}^{\infty} \frac{i^l}{l!} E(\langle\frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle)^l.$$

Hence

$$|y| < \sum_{l=3}^{\infty} \frac{1}{l!} \left|E(\langle\frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p})\rangle)^l\right|.$$

But $s_j < n^{\alpha}$ so $\frac{s_j}{\sqrt{n}} < n^{\alpha-\frac{1}{2}}$.

Therefore,

$$\left| E(\langle \frac{\tilde{s}}{\sqrt{n}}, (Y_1 - \tilde{p}) \rangle)^l \right| < \left((k-1) n^{\alpha - \frac{1}{2}} \right)^l,$$

since $\alpha - \frac{1}{2} < 0$ and $l \geq 3$, $n^{l(\alpha-\frac{1}{2})} < n^{3(\alpha-\frac{1}{2})} = n^{3\alpha-\frac{3}{2}}$.
So

$$|y| < n^{3\alpha-\frac{3}{2}} \sum_{l=3} \frac{1}{l!}(k-1)^l < n^{3\alpha-\frac{3}{2}} e^{k-1}.$$

It follows that $|f'(z)||y| < Constant\, n^{-\zeta}$, for some $\zeta > 1$, since α is smaller that $\frac{1}{6}$.

So $|f(y)| \sim (n^{-\eta}), \eta > 1/3$.

We denote $z_j = \frac{\tilde{n} - n\tilde{p}}{\sqrt{n}}, j = 1, \dots, n$ and use lemma 2.2.12. Equation (2.2.8) becomes:

$$n^{\frac{k-1}{2}} p_n = \frac{1}{(2\pi)^{k-1}} \int \dots \int_{I_\alpha} e^{-i\langle \tilde{s}, \tilde{z} \rangle} (1 + o(n^{-\eta})) e^{-x} \mathrm{d}\tilde{s}$$

$$n^{\frac{k-1}{2}} p_n = \frac{1}{(2\pi)^{k-1}} \int \dots \int_{I_\alpha} e^{-i\langle \tilde{s}, \tilde{z} \rangle} e^{\frac{-1}{2} E\langle \tilde{s}, Y_1 - \tilde{p} \rangle^2} (1 + o(n^{-\eta})) \mathrm{d}\tilde{s} \tag{2.2.9}$$

And as we already proved in proposition 2.2.11, equation (2.2.9) becomes

$$\frac{1}{(2\pi)^{k-1}} \int \dots \int_{I_\alpha} e^{-i<\tilde{s},\tilde{z}>} \phi_{\tilde{z}}(\tilde{s}) (1 + o(n^{-\eta})) \mathrm{d}\tilde{s}. \tag{2.2.10}$$

Using the fact that $\phi_{\tilde{z}}$ is real we obtain

$$\left| \int \dots \int_{I_\alpha} e^{-i<\tilde{s},\tilde{z}>} \phi_{\tilde{z}}(\tilde{s}) o(n^{-\eta}) d\tilde{s} \right| < C n^{-\eta},$$

and hence

$$\frac{1}{(2\pi)^{k-1}} \int \dots \int_{I_\alpha} e^{-i<\tilde{s},\tilde{z}>} \phi_{\tilde{z}}(\tilde{s}) \mathrm{d}\tilde{s} + o(n^{-\eta}). \tag{2.2.11}$$

We now go back to the denominator of $\mathcal{Q}$.

Now $\frac{1}{(2\pi)^{k/2}\sqrt{p_1 \ldots p_k}} e^{-\frac{1}{2}\sum_{j=1}^k \frac{nx_j^2}{p_j}}$ is the probability density function of a multivariate normal random variable with mean $\tilde{0}$ and the same covariance matrix Σ as in proposition 2.2.11. Therefore applying the Fourier inversion formula (2.2.1) and restricting the integration over I_α (which is asymptotic to the integral over $[-\pi\sqrt{n}, \pi\sqrt{n}]^{k-1}$ by Corollaries 2.2.10 and 2.2.8) we get

$$\frac{1}{(2\pi)^{k/2}\sqrt{p_1 \ldots p_k}} e^{-\frac{1}{2}\sum_{j=1}^k \frac{nx_j^2}{p_j}} \sim \frac{1}{(2\pi)^{k-1}} \int \ldots \int_{I_\alpha} e^{-i\langle \tilde{s}, \tilde{z}\rangle} \phi_{\tilde{z}}(\tilde{s}) \mathrm{d}\tilde{s}. \qquad (2.2.12)$$

Therefore, we have the following for $\mathcal{Q}$ in I_α:

$$\begin{aligned} \frac{n^{\frac{k-1}{2}} p_n}{\frac{1}{(2\pi)^{k/2}\sqrt{p_1 \ldots p_k}} e^{-\frac{1}{2}\sum_{j=1}^k \frac{mx_j^2}{p_j}}} &\sim \frac{\int \ldots \int_{I_\alpha} e^{-i<\tilde{s},\tilde{z}>} \phi_{\tilde{z}}(\tilde{s}) \mathrm{d}\tilde{s} + o(n^{-\eta})}{\int \ldots \int_{I_\alpha} e^{-i<\tilde{s},\tilde{z}>} \phi_{\tilde{z}}(\tilde{s}) \mathrm{d}\tilde{s}} \\ &= 1 + \frac{o(n^{-\eta})}{\int \ldots \int_{I_\alpha} e^{-i<\tilde{s},\tilde{z}>} \phi_{\tilde{z}}(\tilde{s}) d\tilde{s}}. \end{aligned}$$

By Remark (p 32) we have

$$\sup_{(x_1,\ldots,x_k)\in G'_n} \left| (2\pi n)^{(k-1)/2} (p_1 \cdot \ldots \cdot p_k)^{1/2} e^{\frac{1}{2}\sum_{j=1}^k \frac{nx_j^2}{p_j}} \right| \leq C_0^{-1} m^{(k-1)/2} n^{\frac{A}{2}\sum_{j=1}^k \frac{1}{p_j}},$$

where

$$C_0^{-1} = (2\pi)^{(k-1)/2} (p_1 \cdot \ldots \cdot p_k)^{1/2}$$

and

$$A \sum_{j=1}^k \frac{1}{p_j} < 2\eta.$$

The proof of the theorem is complete. □.

Chapter 3

Perturbation Theory of Linear Operators

In this chapter, we will study the Richter type local limit theorem for weakly dependent random variables. The weak dependence originates from dynamical systems and the perturbation theory of the Perron-Frobenius operators allows to express the Fourier transform in terms of perturbed eigenvalues. It follows that the sequence of jumps will be considered as a stochastic stationary sequence of the form $\tilde{f} \circ T, \tilde{f} \circ T^2, \ldots, \tilde{f} \circ T^n$, T being the transformation and $\tilde{f}$ a function in $\mathbb{R}^k$.

We will start by introducing the mathematics behind the model.

3.1 Maps of the Interval

3.1.1 Dynamical systems

Here we use the notation of a dynamical system obtained from a map $T : \Omega \to \Omega$, where Ω is a set. We call (Ω, T) a *dynamical system.*

Definition 3.1.1 *Let* (Ω, T) *be a dynamical system and let* $(\Omega, \mathbb{F}, \mathbb{P})$ *be a probability space. Then* T *is said to be probability (or measure) preserving if* $\mathbb{P}(T^{-1}A) = \mathbb{P}(A)$

for all $A \in \mathbb{F}$.

Remark. An $\mathbb{R}$-valued stationary stochastic sequence $X_1, X_2, \ldots$ defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ is, in general, generated by a dynamical system and a measurable function: $X_n = f \circ T^n$ where T is a probability preserving transformation of Ω, and $f : \Omega \to \mathbb{R}$ is measurable.

3.1.2 Basics of Banach spaces

Recall that a *normed* space is a vector space X in which a function $|| \ ||$ is defined and satisfies the following conditions:

(i) For $u \in X$, $||u|| \geq 0; ||u|| = 0$ if and only if $u = 0$.

(ii) $||\alpha u|| = |\alpha|||u||$ for $u \in X$ and $\alpha \in \mathbb{C}$.

(iii) $||u + v|| \leq ||u|| + ||v||$ for $u, v \in X$.

Any function $||u||$ defined for all $u \in X$ and satisfying these conditions is called a *norm*.

A sequence u_n is called a Cauchy sequence if $u_n \to u$ implies $||u_n - u_m|| \to 0, m, n \to \infty$. A normed space in which every Cauchy sequence has a limit is said to be *complete*.

A complete normed space X is called a *Banach space*. We write $(X, || \ ||_X)$.

Let X and Y be two vector spaces. A map $T : X \to Y$ is a function whose domain is X and whose range is contained in Y; that is for every $x \in X$, the map T assigns an element $T(x) \in Y$.

A *linear map* or *linear operator* T between normed spaces X and Y is a map T: $X \to Y$ such that

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

for all $\alpha, \beta \in \mathbb{C}$ and all $x, y \in X$.

The linear operator $T : X \to Y$ is said to be bounded if there exists some $k > 0$ such that

$$||Tx|| \leq k||x||.$$

We denote by $\mathbb{B}(X,Y)$ the set of all bounded operators on X to Y and $\mathbb{B}(X,X) = \mathbb{B}(X)$ the set of all bounded operators from X to itself.

The product TS of two linear operators T and S is defined by

$$(TS)u = T(Su)$$

for all $u \in X$ where X is the domain space of S provided that the domain space of T is Y.

We write $TT = T^2, TTT = T^3$ and so on.

The identity operator is denoted by I and is defined by $Iu = u$ for every $u \in X$.

If $T \in \mathbb{B}(X)$ is nonsingular, the inverse T^{-1} exists and belongs to $\mathbb{B}(X)$ and is defined by

$$T^{-1}T = TT^{-1} = I.$$

A *projection* in an arbitrary linear space X is a linear operator T such that $T^2 = T$.

Two linear operators T and S in an arbitrary linear space X are said to be *orthogonal* if $(TS)u = 0$ for every $u \in X$.

A complex function f on a Banach space X is said to be analytic if its derivative f' exists and is continuous on X.

A scalar-valued linear map from a linear space X to $\mathbb{R}$ is called a *linear functional* or *linear form* on X.

The space of all bounded linear functionals on X, $\mathbb{B}(X, \mathbb{C})$, is denoted by X^* and called the dual or adjoint space of X.

Since $\mathbb{C}$ is complete, X^* is a Banach space.

Consider an operator T from X to Y. The operator T^* from Y^* to X^* is called the *adjoint* or *dual* of T if

$$g(Tu) = (T^*g)(u), u \in X, g \in Y^*.$$

One can find the definitions above in [1, 24, 13, 25].

Remark. Let m be a probability measure and L^p_m the Banach space of functions f with $\int |f|^p dm < \infty$, $1 \leq p \leq \infty$. Let $T : L^p_m \to \mathbb{C}$ be a linear operator. Then the dual space is L^q_m for $1 \leq p < \infty$ and $p^{-1} + q^{-1} = 1$ and it holds that

$$\int T(f) \cdot \overline{g} dm = \int f \cdot \overline{T^* g} dm$$

for all $f \in L^p_m$ and $g \in L^q_m$. For $p = \infty$ the dual space is the space of signed measures which contains the space L^1_m canonically. If a measure is S-preserving for some map $S : \Omega \to \Omega$ then it defines a linear operator $T : L^p_m \to \mathbb{C}$ in a canonical way:

$$Tf(\omega) = f(S(\omega)).$$

We then have

$$T^* m = m \tag{3.1.1}$$

since for $f \in L^\infty_m$ and $g \in L^1_m$ the measure gdm lies in the dual of L^∞_m and

$$\int fT^*(g)dm = \int T(f)gdgm.$$

In particular, if $g = 1$,

$$\int f \mathrm{d}T^* m = \int T f \mathrm{d}m. \tag{3.1.2}$$

3.1.3 Spectral theory

Let X be a finite dimensional complex Banach space and T a linear operator on $\mathbb{B}$. Let I denote the identity operator, interpreted as T^0.

Definition 3.1.2 *The spectrum $\sigma(t)$ of an operator T in a Banach space is the set of complex numbers λ such that $(\lambda I - T)^{-1}$ does not exist or is not continuous [13].*

Definition 3.1.3 *The resolvent set $\rho(T)$ of T is the set of complex numbers λ, for which $(\lambda I - T)^{-1}$ exists as a bounded operator with domain X. The spectrum $\sigma(T)$ is the complement of $\rho(T)$. The function $R(\lambda, T) = (\lambda I - T)^{-1}$, defined on $\rho(T)$, is called the resolvent function of T, or simply the resolvent of T. $R(\lambda, T)$ is analytic in $\rho(T)$ [13].*

Definition 3.1.4 *The quantity*

$$r(T) = sup|\sigma(T)| = \lim_{n\to\infty} \sqrt[n]{|T^n|}$$

is called the spectral radius of T.

Definition 3.1.5 *Let $T \in \mathbb{B}(X)$. A complex number λ is called an eigenvalue of T if there is a non-zero vector $u \in X$ such that*

$$Tu = \lambda u,$$

where u is called an eigenvector of T associated with the eigenvalue λ [24].

For more about Banach spaces, linear operators and their spectral theory one can also refer to [23, 5, 49, 21, 25, 52]

3.2 The Perron-Frobenius Operator

We take the setup as in [39].

Definition 3.2.1 *Consider a map T from I to I, where $I = [0,1]$. Denote m the Lebesgue measure and L^1_m the space of integrable functions on $[0,1]$ with respect to m. We consider a finite or countable partition of I given by points $a_i \in I$, where $I_j = (a_{j-1}, a_j)$ is an open interval satisfying*

1. *The restriction of T on I_j is strictly monotone and is expanding as a C^2 map on $\overline{I_j}$ where $\overline{I_j}$ is the closure of I_j.*

2. *There exists a finite family of disjoint intervals such that each image $T(I_j)\}$ is a union of sets from this finite family. This property is called the finite range property.*

3. *There exists an n such that $\gamma = \inf_{x \in I} |(T^n(x))'| > 1$.*

It is known that Lebesgue measure m on $[0,1]$ is non-singular (i.e. $m(T^{-1}(A) = 0$ iff $m(A) = 0$). The Perron-Frobenius operator associated with T is the operator Φ from L^1_m in L^1_m defined as follows

$$\int_0^1 \Phi f \cdot g \mathrm{d}m = \int_0^1 f \cdot g \circ T \mathrm{d}m \tag{3.2.1}$$

where $f \in L^1_m$ and $g \in L^\infty_m$, L^∞_m being the space of all bounded measurable functions.

One immediately calculates that $\Phi^* m = m$, i.e. Lebesgue measure is an eigenmeasure for the eigenvalue 1 of the dual operator.

Remark. The system $([0,1], T)$ is a dynamical system.

3.2.1 Properties

1. From elementary integration theory one immediately infers that Φ can be written in the form

$$\Phi f(x) = \sum_{Ty=x} f(y) e^{\phi(y)}. \tag{3.2.2}$$

For $x \in I$ and for some function $\phi : I \to \mathbb{R}$ called a potential. In this special case one finds

$$\phi(y) = -\log |T'(y)| < 0.$$

This means that, if $x \in X_j = T(\overline{I_j})$, $\overline{I_j}$ being the closure of I_j, then $\exists\, y \in I_j$ with $Ty = x$.

2. Let $\mathcal{N}$ and $\mathcal{L}$ be two Banach spaces such that $\mathcal{N} \subset \mathcal{L}$, with respective norms $||\,||_{\mathcal{N}}$ and $||\,||_{\mathcal{L}}$. Due to a theorem by Ionescu-Tulcea and Marinescu [20], if the application T verifies the conditions of definition 3.2.1 then Φ has only a finite number of eigen values of modulus 1: $\lambda_1, \ldots, \lambda_p$. The corresponding eigen vectors $E_i = \{f \in \mathcal{L} : \phi f = \lambda_i f\}$ are of finite dimension and are included in $\mathcal{N}$,

3. This can be seen by letting $\Phi h = h$ where the measure $\mu = hm$ is invariant by T i.e if $\int_0^1 f \circ T \mathrm{d}\mu = \int_0^1 f \mathrm{d}\mu$, for all $f \in L^1_m$.

To see this let $\Phi h = h$ then the measure $\mu = hm$ is T-invariant.

Indeed, using 3.1.1 and 3.1.2 we have

$$\int_0^1 f \circ T \mathrm{d}\mu = \int_0^1 f \circ T h \mathrm{d}m = \int_0^1 f \circ T h \mathrm{d}\Phi^* m = \int_0^1 \Phi(f \circ T h) \mathrm{d}m$$

$$= \int_0^1 \sum_{Ty=x} f(Ty) h(y) e^{\phi(y)} \mathrm{d}m = \int_0^1 f(x) \Phi(h(x)) \mathrm{d}m = \int_0^1 f h \mathrm{d}m = \int_0^1 f \mathrm{d}\mu. \qquad \square$$

The converse can be shown similarly.

4. Since Lebesgue measure is an eigenmeasure for the eigenvalue 1, the oprerator Φ has the eigenvalue 1 as well. By the theorem of Ionescu-Tulcea and Marinescu [20] there exists an eigenvector h for the eigenvalue 1, such that $\Phi h = h$ and $\mu = hm$ will be T-invariant.

5. It follows from the discussion so far that (see Theoreme 1 in [39]) the operator Φ^n can be written as:

$$\Phi^n = \sum_{i=1}^{p} \lambda_i^n \phi_i + \psi^n, n \geq 1,$$

where ϕ_i are the projections on the eigenvectors E_i, $||\phi_i||_{\mathcal{L}} \leq 1||$ and ψ is an operator on L^1_m such that $sup_{n\geq 1}||\psi^n||_{\mathcal{L}} < \infty$. Also, ϕ_i and ψ are orthogonal.

6. If the map T verifies the conditions of definition 3.2.1, it is known that T is weakly mixing. This implies that Φ satisfies the properties cited above, and moreover $p = 1$ [39].

7. Note that Φ is a bounded operator [39].

3.2.2 The adjoint operator

Continuing with the same set up as above (see [39]) we define the adjoint operator of T with respect to the invariant measure μ as follows

Definition 3.2.2 *The adjoint operator of T restricted to L^1_μ is defined by*

$$Pf = \frac{\Phi(fh)}{h}. \tag{3.2.3}$$

For the readers's convenience, we include the proofs of the following folklore statements, proposition 3.2.3 and lemma 3.2.4.

Proposition 3.2.3 *P is an operator on L^1_μ and it is the Perron-Frobenius operator associated with T with respect to the new measure μ.*

Proof. Using definition 3.2.2 and equation 3.2.1,

$$\int_0^1 Pf\cdot g\mathrm{d}\mu = \int_0^1 \frac{\Phi(fh)}{h}\cdot gh\mathrm{d}m = \int_0^1 fh\cdot g\circ T\mathrm{d}m = \int_0^1 f\cdot g\circ Th\mathrm{d}m = \int_0^1 f\cdot g\circ T\mathrm{d}\mu \qquad \square$$

Lemma 3.2.4 *P also can be expressed explicitly as $Pf(x) = \sum_{Ty=x} f(y)e^{\phi'(y)}$ where $\phi'(y)$ is the cohomologous potential*

$$\phi'(y) = \phi(y) + \log h(y) - \log h(T(y)).$$

Proof. Using 3.2.2,

$$Pf(x) = \frac{\Phi(fh)(x)}{h(x)} = \frac{\sum_{Ty=x} f(y)h(y)e^{\phi(y)}}{h(Ty)} = \sum_{Ty=x} f(y)e^{\phi'(y)},$$

Where $\phi'(y) = \phi(y) + \log h(y) - \log(h(Ty))$. $\square$.

Later we will denote ϕ' by ϕ for simplicity of notation.

Note: Like Φ^n, we have $P^n = \mu + Q^n$, where Q is an operator with spectral radius $\rho(Q) < 1$ [39], since μ is weakly mixing.

In addition to this, P satisfies the same properties (1-7) as Φ [39].

3.3 Characteristic Function Operators

In his paper[39], Rousseau-Egele gave a series of lemmas and propositions describing the operator P_f and its perturbation theory for $t \in \mathcal{R}$ and $f \in \mathcal{N}$ to give an expression of the characteristic function of $S_n f$ as n goes to infinity. In this section we will extend the work for $t \in \mathbb{R}^d$ and $\tilde{f} \in \mathcal{N}^d$.

Definition 3.3.1 *Let $\mathcal{N}$ be a Banach space. Let $\tilde{f} = (f_1, \ldots, f_d)$ be an $\mathbb{R}^d$-valued function $\in \mathcal{N}^d$ and $\tilde{t} = (t_1, \ldots, t_d) \in \mathcal{R}^d$, $d \in \mathbb{N}$.*

We define the operator $P_{\tilde{f}}(i\tilde{t})$ on L_m^∞ by

$$P_{\tilde{f}}(i\tilde{t})g = P(\exp(i\langle \tilde{t}, \tilde{f}\rangle)g), \tag{3.3.1}$$

where $g \in L_m^\infty$ and $\langle , \rangle$ denotes the inner product.

Lemma 3.3.2 *Let $S_n\tilde{f} = \sum_{k=0}^{n-1} \tilde{f} \circ T^k, n \geq 1$ and $S_0\tilde{f} = \tilde{0}$.*

For every $\tilde{t} \in \mathbb{R}^d$,

$$P_{\tilde{f}}^n(i\tilde{t})\mathbb{1} = P^n(\exp(i\langle \tilde{t}, S_n\tilde{f}\rangle)\mathbb{1}), \;\; n \geq 0. \tag{3.3.2}$$

Proof. We will proceed by induction

$$P_{\tilde{f}}^2(i\tilde{t})\mathbb{1} = P_{\tilde{f}}(P_{\tilde{f}}(i\tilde{t})\mathbb{1}) = P_{\tilde{f}}(P(e^{i\langle \tilde{t}, \tilde{f}\rangle}) \cdot \mathbb{1}) = P_{\tilde{f}}(\sum_{Ty=x} e^{i\langle \tilde{t}(y), \tilde{f}(y)\rangle} e^{\phi(y)})$$

$$= P(\sum_{Ty=x} e^{i\langle \tilde{t}(y), \tilde{f}(y)\rangle} e^{\phi(y)} e^{i\langle \tilde{t}, \tilde{f}(z)\rangle}) = \sum_{Tz=y} \sum_{Ty=x} e^{i\langle \tilde{t}(y), \tilde{f}(y)\rangle} e^{\phi(y)} e^{i\langle \tilde{t}, \tilde{f}(z)\rangle} e^{\phi(z)}$$

$$= \sum_{T^2z=x} e^{i\langle \tilde{t}, \tilde{f}(z) + \tilde{f}(Tz)\rangle} e^{\phi(z)+\phi(Tz)} = P^2(e^{i\langle \tilde{t}, S_2\tilde{f}\rangle}).$$

We suppose $P_{\tilde{f}}^{n-1}(i\tilde{t})\mathbb{1} = P^{n-1}(e^{i\langle \tilde{t}, S_n\tilde{f}\rangle}), n \geq 0$

$$P_{\tilde{f}}^{n}(i\tilde{t})\mathbb{1} = P_{\tilde{f}}(P_{\tilde{f}}^{n-1}(i\tilde{t})\mathbb{1}) = P_{\tilde{f}}(P^{n-1}(e^{i\langle \tilde{t}, S_{n-1}\tilde{f}\rangle}) \cdot \mathbb{1})$$

$$P_{\tilde{f}}\left(\sum_{T^{n-1}z=x} e^{i\langle \tilde{t}(z), S_{n-1}\tilde{f}(z)\rangle} e^{\phi(z)+\phi(T^2z)+\ldots+\phi(T^{n-1}z)}\right)$$

$$= P\left(\sum_{T^{n-1}z=x} e^{i\langle \tilde{t}(z), S_{n-1}\tilde{f}(z)\rangle} e^{\phi(z)+\phi(T^2z)+\ldots+\phi(T^{n-1}z)} e^{i\langle \tilde{t}(y), \tilde{f}(y)\rangle}\right)$$

$$= \sum_{tx=y}\sum_{T^{n-1}z=x} e^{i<\tilde{t}(z), S_{n-1}\tilde{f}(z)>} e^{\phi(z)+\phi(T^2z)+\ldots+\phi(T^{n-1}z)} e^{i\langle \tilde{t}(y), \tilde{f}(y)\rangle} e^{\phi(y)}$$

$$= \sum_{T^nz=y} e^{i\langle \tilde{t}(z), S_{n-1}\tilde{f}(z)+\tilde{f}(T^nz)\rangle} e^{\phi(z)+\phi(T^2z)+\ldots+\phi(T^{n-1}z)+\phi(T^nz)}$$

$$P_{\tilde{f}}^{n}(i\tilde{t})\mathbb{1} = P^n(e^{i\langle \tilde{t}, S_n\tilde{f}\rangle} \cdot \mathbb{1}). \qquad \square.$$

Remark. Due to equality 3.3.2, $P_{\tilde{f}}$ is called the characteristic function operator.

Indeed

$$\int_0^1 P_{\tilde{f}}^{n}(i\tilde{t})\mathbb{1}\mathrm{d}\mu = \int_0^1 P^n(e^{i<\tilde{t}, S_n\tilde{f}>} \cdot \mathbb{1})\mathrm{d}\mu$$

But by the definition of P we have $\int_0^1 Pf \cdot g\mathrm{d}\mu = \int_0^1 f \cdot g \circ T\mathrm{d}\mu$ which means

$\int_0^1 Pf \cdot \mathbb{1}\mathrm{d}\mu = \int_0^1 f \cdot \mathbb{1} \circ T\mathrm{d}\mu = \int_0^1 f\mathrm{d}\mu$ and $\int_0^1 P^n f \cdot \mathbb{1}\mathrm{d}\mu = \int_0^1 f\mathrm{d}\mu$.

Therefore,

$$\int_0^1 P_{\tilde{f}}^{n}(i\tilde{t})\mathbb{1}\mathrm{d}\mu = \int_0^1 P^n(e^{i\langle \tilde{t}, S_n\tilde{f}\rangle} \cdot \mathbb{1})\mathrm{d}\mu = \int_0^1 e^{i\langle \tilde{t}, S_n\tilde{f}\rangle}\mathrm{d}\mu \qquad (3.3.3)$$

which is the expression of the characteristic function of $S_n\tilde{f} = \sum_{k=0}^{n-1} \tilde{f} \circ T^k$.

3.4 Perturbation Theory of $P_{\tilde{f}}$

In this section we continue the extension of the work in [39] for $t \in \mathcal{R}^d$ and $\tilde{f} \in \mathcal{N}^d$and give an expression of the characteristic function of $S_n\tilde{f}$ for large n. The first proposition is an extension of Proposition 4 in [39] to the multidimensional case.

Proposition 3.4.1 *There exists a real number $\alpha > 0$ such that if $|\tilde{t}| < \alpha$*

1. *for all $g \in \mathcal{N}$ and $n \geq 1$*

$$P^n_{\tilde{f}}(i\tilde{t})(g) = (\lambda(i\tilde{t}))^n N_1(i\tilde{t})(g) + P^n_2(i\tilde{t})(g) \tag{3.4.1}$$

where

   (a) $\lambda(i\tilde{t})$ is the unique eigenvalue of highest module of $P^n_{\tilde{f}}(i\tilde{t})(g)$ and $|\lambda(i\tilde{t})| > (2 + \rho(Q))/3$

   (b) $N_1(i\tilde{t})$ is the projection on the eigen sub-space of dimension 1 corresponding to $\lambda(i\tilde{t})$

   (c) $P_2(i\tilde{t})$ is an operator on $\mathcal{N}$ with spectral radius $\rho(P_2(i\tilde{t})) \leq \theta|\tilde{t}|$, fore some $\theta < 1$.

   (d) and $P_2(i\tilde{t})E_{\tilde{t}} = 0$.

2. *The maps $\tilde{t} \to \lambda(i\tilde{t})$, $\tilde{t} \to N_1(i\tilde{t})$ and $\tilde{t} \to P_2(i\tilde{t})$ are analytic.*

3. *$||P^n_2(i\tilde{t})\mathbb{1}||_{\mathcal{N}} \leq C|\tilde{t}|\theta^n$ where C is a positive constant and $\theta < 1$*

Proof. For part 1, see [1] section 4.

Part 2 follows from the perturbation theory of $P_{\tilde{f}}$ since $t \to P_{\tilde{f}}$ is analytic.

And part 3 is the same as in [39]. $\square$.

We will use this proposition to find an explicit expression of the characteristic function of $S_n\tilde{f}$.

First notice that using lemma 3.3.2, equation (3.3.3) and proposition 3.4.1 with $g = \mathbb{1}$, we have

$$\int_0^1 e^{i\langle \tilde{t}, S_n\tilde{f}\rangle} \mathrm{d}\mu = (\lambda(i\tilde{t}))^n \int_0^1 N_1(i\tilde{t})\mathbb{1}\mathrm{d}\mu + \int_0^1 P_2^n(i\tilde{t})\mathbb{1}\mathrm{d}\mu \tag{3.4.2}$$

We will then use the Taylor expansion for $\lambda(i\tilde{t})$.

Lemma 3.4.2

$$\frac{\partial \lambda}{\partial t_i} |_{\tilde{t}=\tilde{0}} = \mu(f_i), \ \ i = 1, \dots, d$$

Proof. For n sufficiently large, equation (3.4.2) becomes

$$\int_0^1 e^{i\langle \tilde{t}/n, S_n\tilde{f}\rangle} \mathrm{d}\mu = (\lambda(i\tilde{t}/n))^n \int_0^1 N_1(i\tilde{t}/n)\mathbb{1}\mathrm{d}\mu + \int_0^1 P_2^n(i\tilde{t}/n)\mathbb{1}\mathrm{d}\mu. \tag{3.4.3}$$

We use Taylor expansion for operator N_1 as in [13].

$$N_1(i\tilde{t}/n) = N_1(\tilde{0}) + \frac{i}{n}\sum_{j=1}^{d} t_i \frac{\partial N_1}{\partial t_i} |_{\tilde{t}=\tilde{0}} - \frac{1}{2n^2} \sum_{1\le i,l\le d} \frac{\partial^2 N_1}{\partial t_i \partial t_l} |_{\tilde{t}=\tilde{0}} + \frac{t^2}{n^2}\overline{N}_1(i\tilde{t}/n), \tag{3.4.4}$$

where $N_1(\tilde{0}) = \mu$ and $\overline{N}_1(i\tilde{t}/n)$, $\frac{\partial N_1}{\partial t_i}$ and $\frac{\partial^2 N_1}{\partial t_i \partial t_l}$ are bounded operators.

Therefore we get

$$\lim_{n\to\infty} \int_0^1 N_1(i\tilde{t}/n)\mathbb{1}\mathrm{d}\tilde{t} = N_1(0) = 1$$

For P_2 we have by proposition 3.4.1 part 1(c),

$$\left|\int_0^1 P_2(i\tilde{t}/n)\mathbb{1}\mathrm{d}\mu\right| \leq \left|\left|P_2(i\tilde{t}/n)\mathbb{1}\right|\right|_{\mathcal{N}} \leq \frac{1}{n}C(|\tilde{t}|)\theta^n.$$

Because $\theta < 1$, $\int_0^1 P_2(i\tilde{t}/n)\mathbb{1}\mathrm{d}\mu \sim O(\frac{1}{n})$.

We will now use the Taylor expansion for $\lambda(i\tilde{t}/n)$ with remainder [13] that exists due to the analycity of $\lambda(i\tilde{t})$ by proposition 3.4.1 part 2 and since for t_i small enough we have $\frac{1}{n}\left|i\sum_{j=1}^{d} t_i\frac{\partial\lambda}{\partial t_i}\ |_{\tilde{t}=\tilde{0}}\right| < 1$. Hence,

$$\begin{aligned}\lambda(i\tilde{t}/n) &= 1+\frac{i}{n}\sum_{j=1}^{d} t_i\frac{\partial\lambda}{\partial t_i}\ |_{\tilde{t}=\tilde{0}} - \frac{1}{2n^2}\sum_{1\leq i,l\leq d}\frac{\partial^2\lambda}{\partial t_i\partial t_l}\ |_{\tilde{t}=\tilde{0}} + \frac{t^2}{n^2}\overline{\lambda}(i\tilde{t}/n)\\ &= 1+\frac{1}{n}\left(i\sum_{j=1}^{d} t_i\frac{\partial\lambda}{\partial t_i}\ |_{\tilde{t}=\tilde{0}} - \frac{1}{2n}\sum_{1\leq i,l\leq d}\frac{\partial^2\lambda}{\partial t_i\partial t_l}\ |_{\tilde{t}=\tilde{0}} + \frac{t^2}{n}\overline{\lambda}(i\tilde{t}/n)\right)\end{aligned}$$

where $\overline{\lambda}$ is bounded. Thus,

$$S_n\tilde{f} = \sum_{k=0}^{n-1}\tilde{f}\circ T^k = \left(\sum_{k=0}^{n-1} f_1\circ T^k,\ldots,\sum_{k=0}^{n-1} f_d\circ T^k\right).$$

Denoting $S_nf_i = \sum_{k=0}^{n-1} f_i\circ T^k$ for $i = 1,\ldots,d$ and $S_n\tilde{f} = (S_nf_i)_{i=1,\ldots,d}$

And using ergodic theory as in [15],

$$\lim_{n\to\infty}\frac{1}{n}S_nf_i = \mu(f_i),\ \ i = 1,\ldots,d.$$

So we get

$$\left(\lambda(\frac{i\tilde{t}}{n})\right)^n = \int_0^1 e^{i\langle\frac{\tilde{t}}{n},S_n\tilde{f}\rangle}\mathrm{d}\mu + o(1). \tag{3.4.5}$$

The LHS in equation (3.4.5) is

$$1 + i\sum_{j=1}^{d} t_i \frac{\partial \lambda}{\partial t_i} \mid_{\tilde{t}=\tilde{0}} + o(1)$$

and the RHS is

$$1 + \sum_{j=1}^{d} t_i \int_0^1 \frac{1}{n} S_n f_i \mathrm{d}\mu + o(1).$$

So we have

$$1 + i\sum_{j=1}^{d} t_i \frac{\partial \lambda}{\partial t_i} \mid_{\tilde{t}=\tilde{0}} + o(1) = 1 + \sum_{j=1}^{d} t_i \int_0^1 \frac{1}{n} S_n f_i \mathrm{d}\mu + o(1)$$

and

$$\frac{\partial \lambda}{\partial t_i} \mid_{\tilde{t}=\tilde{0}} = \mu(f_i), \;\; i = 1, \ldots, d.$$

Without loss of generality and for the simplicity of calculations, we will suppose that $\mu(\tilde{f}) = \tilde{0}$. (i.e $\mu(f_i) = 0, i = 1, \ldots, d$)

We note equation (3.4.4) as a separate corollary.

Corollary 3.4.3

$$N_1(i\tilde{t}) = N_1(\tilde{0}) + O(\tilde{t}).$$

Lemma 3.4.4

$$\begin{aligned} \frac{\partial^2 \lambda}{\partial t_i \partial t_l}\Big|_{\tilde{t}=\tilde{0}} &= \lim_{n\to\infty} \int_0^1 \left(S_n f_i/\sqrt{n}\right)\left(S_n f_l/\sqrt{n}\right) \mathrm{d}\mu, \;\; 1 \le i, l \le d \\ &= \lim_{n\to\infty} E\left(S_n f_i/\sqrt{n}\right)\left(S_n f_l/\sqrt{n}\right), \;\; 1 \le i, l \le d \end{aligned}$$

To prove lemma 3.4.4 we need the following remark.

Remark. $P_2^n(i\tilde{t}/\sqrt{n}) \to 0$

Proof. Notice that

$$\frac{\partial^2}{\partial t_i \partial t_l}\left\{\int_0^1 e^{i\langle \tilde{t}/\sqrt{n}, S_n\tilde{f}\rangle} \mathrm{d}\mu\right\}_{|\tilde{t}=\tilde{0}} = -\int_0^1 \left(S_n f_i/\sqrt{n}\right)\left(S_n f_l/\sqrt{n}\right) \mathrm{d}\mu.$$

Then repeating the calculations for lemma 3.4.2 with $\frac{\partial \lambda}{\partial t_i} |_{\tilde{t}=\tilde{0}} = 0, \;\; i = 1, \ldots, d$, and replacing $i\tilde{t}/n$ by $i\tilde{t}/\sqrt{n}$ we get for the LHS of (3.4.5)

$$\left(1 - \frac{1}{2n}\sum_{1\le i,l\le d} t_i t_l \frac{\partial^2 \lambda}{\partial t_i \partial t_l}|_{\tilde{t}=\tilde{0}}\right)^n = 1 - \frac{1}{2}\sum_{1\le i,l\le d} t_i t_l \frac{\partial^2 \lambda}{\partial t_i \partial t_l}|_{\tilde{t}=\tilde{0}} + o(1)$$

and the RHS

$$\begin{aligned}\int_0^1 \left(1 - \frac{1}{2}\sum_{1\le i,l\le d} t_i t_l \frac{1}{n} S_n f_i S_n f_l\right) \mathrm{d}\mu &= 1 - \frac{1}{2}\sum_{1\le i,l\le d} t_i t_l \frac{1}{n}\int_0^1 S_n f_i S_n f_l \mathrm{d}\mu \\ &= 1 - \frac{1}{2}\sum_{1\le i,l\le d} t_i t_l E\frac{1}{n} S_n f_i S_n f_l + o(1).\end{aligned}$$

LHS=RHS leads to the statement of the lemma. □.

Lemma 3.4.5 *Let* $\Sigma = \left(lim_{n\to\infty} \int_0^1 (S_n f_i/\sqrt{n})(S_n f_l/\sqrt{n})\, \mathrm{d}\mu\right)_{1\le i,l\le d}$.

Σ *is positive definite.*

Proof. Σ is clearly non-negative definite. We will assume that it is positive definite in order to consider it as a covariance matrix. □.

Lemma 3.4.6

$$lim_{n\to\infty}\int_0^1 P_{\tilde f}^n(i\tilde t/\sqrt{n})(1)\mathrm{d}\mu = \exp\left(-\frac{1}{2}\tilde t'\Sigma\tilde t\right)$$

Proof. The statement of the lemma is evident. □.

3.5 Richter Type's Theorem for Dependent Random Variables

In Chapter 2 we found a new proof for the Richter-type local limit theorem that was used in finding the limit of the CRR option's price where the vector of occurrences in the n independent trials of the random variable X was assumed to follow a multinomial distribution. In this section, the n trials of X are not supposed to be independent and no distribution is assumed. However, we will still denote the vector of occurrences by Z_n and its probability mass function by p_n, and all other notations we used in Chapter 2 will remain the same.
We will start by using the Fourier Inversion formula (2.3) of p_n,

$$p_n(\widetilde{n}) = \frac{1}{(2\pi)^{k-1}}\underbrace{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}}_{k-1\ integrals}\phi_n(\tilde t)e^{-i\langle\tilde n,\tilde t\rangle}\mathrm{d}\tilde t, \tag{3.5.1}$$

where ϕ_n is the characteristic function of p_n. The following lemma substitutes lemma 2.3 and gives the expression of this characteristic function using remark (p 64) and proposition 3.4.1.

Lemma 3.5.1

$$\begin{aligned}\phi_{Z_n'-n\tilde p}(t) &= \int_0^1 P_{\tilde f}^n(i\tilde t)\mathbb{1}\mathrm{d}\mu\\ &= (\lambda(i\tilde t))^n\int_0^1 N_1(i\tilde t)\mathbb{1}\mathrm{d}\mu + \int_0^1 P_2^n(i\tilde t)\mathbb{1}\mathrm{d}\mu\\ &= (\lambda(i\tilde t))^n\int_0^1 N_1(i\tilde t)\mathbb{1}\mathrm{d}\mu + Ko(\theta^n),\end{aligned}$$

where $\theta < 1$*, and* $\lim_{n\to\infty} \int_0^1 N_1(i\tilde{t})\mathbb{1}\mathrm{d}\mu = 1$*. Ysing Corollary 3.4.3,*

$$\phi_{Z'_n - n\tilde{p}}(t) = (\lambda(i\tilde{t}))^n(1 + O(\tilde{t})) + o(\theta^n). \tag{3.5.2}$$

Before substituting $\phi_{Z'_n - n\tilde{p}}(\tilde{t})$ by the expression (3.5.2) in equation (3.5.1), note that $\int_{-\pi}^{\pi} K\theta^n)e^{-i\langle\tilde{n},\tilde{t}\rangle}\mathrm{d}\tilde{t} \to 0$ exponentially. Therefore, equation (3.5.1) becomes

$$p_n(\widetilde{n}) = \frac{1}{(2\pi)^{k-1}} \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{k-1 \ integrals} [(\lambda(i\tilde{t}))^n(1 + O(\tilde{t})) + o(\theta^n)]e^{-i\langle\tilde{n}-n\tilde{p},\tilde{t}\rangle}\mathrm{d}\tilde{t}. \tag{3.5.3}$$

Multiplying equation (3.5.3) by $n^{(k-1)/2}$,

$$\begin{aligned} n^{(k-1)/2}p_n(\widetilde{n}) &= \frac{n^{(k-1)/2}}{(2\pi)^{k-1}} \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{k-1 \ integrals} (\lambda(i\tilde{t}))^n e^{-i\langle\tilde{n},\tilde{t}\rangle}\mathrm{d}\tilde{t} \qquad (3.5.4) \\ &+ \frac{n^{(k-1)/2}}{(2\pi)^{k-1}} \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{k-1 \ integrals} [(\lambda(i\tilde{t}))^n(O(\tilde{t})) + o(\theta^n)]e^{-i\langle\tilde{n}-n\tilde{p},\tilde{t}\rangle}\mathrm{d}\tilde{t}. \end{aligned}$$

Using the same change of variables $s_l = \sqrt{n}t_l$, for $l = 1, \ldots k-1$, equation (3.5.4) becomes

$$\begin{aligned} n^{\frac{k-1}{2}}p_n &= \frac{1}{(2\pi)^{k-1}} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \cdots \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} e^{-i\langle\tilde{s},\frac{\tilde{n}-n\tilde{p}}{\sqrt{n}}\rangle}(\lambda(i\tilde{s}/\sqrt{n}))^n\mathrm{d}\tilde{s} \qquad (3.5.5) \\ &+ \frac{1}{(2\pi)^{k-1}} \underbrace{\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \cdots \int_{-\pi\sqrt{n}}^{\pi} \sqrt{n}}_{k-1 \ integrals}[(\lambda(i\widetilde{s/\sqrt{n}}))^n(O(\tilde{s}/\sqrt{n})) + o(\theta^n)]e^{-i\langle\tilde{s},\frac{\tilde{n}-n\tilde{p}}{\sqrt{n}}\rangle}\mathrm{d}\tilde{s} \end{aligned}$$

We then use the Taylor expansion for $\lambda\left(\frac{i\tilde{s}}{\sqrt{n}}\right)$ and use lemmas 3.4.2 and 3.4.4 to get as in Chapter 2 that

$$n^{\frac{k-1}{2}}p_n = \frac{1}{(2\pi)^{k-1}}\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}}\cdots\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} e^{-i\langle\tilde{s},\frac{\tilde{n}-n\tilde{p}}{\sqrt{n}}\rangle}e^{(-\frac{1}{2}\tilde{s}'\Sigma\tilde{s})}\mathrm{d}\tilde{s} \tag{3.5.6}$$

$$+ \frac{1}{(2\pi)^{k-1}}\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}}\cdots\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} e^{-i\langle\tilde{s},\frac{\tilde{n}-n\tilde{p}}{\sqrt{n}}\rangle}e^{(-\frac{1}{2}\tilde{s}'\Sigma\tilde{s})}O(\tilde{s}/\sqrt{n})\mathrm{d}\tilde{s} \tag{3.5.7}$$

$$+ \frac{1}{(2\pi)^{k-1}}\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}}\cdots\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} e^{-i\langle\tilde{s},\frac{\tilde{n}-n\tilde{p}}{\sqrt{n}}\rangle}o(\theta^n)\mathrm{d}\tilde{s}. \tag{3.5.8}$$

We denote $\dfrac{\tilde{n}-n\tilde{p}}{\sqrt{n}}$ by $\tilde{z}$ as in Chapter 2.

We know that $e^{(-\frac{1}{2}\tilde{s}'\Sigma\tilde{s})}$ is the characteristic of a (k-1) multivariate normal distribution with mean $\tilde{0}$ and covariance matrix Σ. So the limit of the integral in equation (3.5.6) is the probability density function of the same multivariate normal at $\tilde{z}$

$$\frac{1}{(2\pi)^{(k-1)/2}|\Sigma|^{1/2}}\exp\left\{-\frac{1}{2}\tilde{z}'\Sigma^{-1}\tilde{z}\right\}$$

It is then immediate to see that for $z \in G_n'$ (as in Chapter 2),

$$\left|\frac{\frac{1}{(2\pi)^{k-1}}\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}}\cdots\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} e^{-i\langle\tilde{s},\frac{\tilde{n}-n\tilde{p}}{\sqrt{n}}\rangle}e^{(-\frac{1}{2}\tilde{s}'\Sigma\tilde{s})}o(\tilde{s}/\sqrt{n})\mathrm{d}\tilde{s}}{\frac{1}{(2\pi)^{(k-1)/2}|\Sigma|^{1/2}}\exp\left\{-\frac{1}{2}\tilde{z}'\Sigma^{-1}\tilde{z}\right\}}\right| \to 0$$

and

$$\left|\frac{\frac{1}{(2\pi)^{k-1}}\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}}\cdots\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} e^{-i\langle\tilde{s},\frac{\tilde{n}-n\tilde{p}}{\sqrt{n}}\rangle}o(\theta^n)\mathrm{d}\tilde{s}}{\frac{1}{(2\pi)^{(k-1)/2}|\Sigma|^{1/2}}\exp\left\{-\frac{1}{2}\tilde{z}'\Sigma^{-1}\tilde{z}\right\}}\right| \to 0$$

which means we have

$$n^{(k-1)/2}p_n(n_1,\ldots,n_k) \sim \frac{1}{(2\pi)^{(k-1)/2}|\Sigma|^{1/2}}\exp\left\{-\frac{1}{2}\tilde{z}'\Sigma^{-1}\tilde{z}\right\}$$

We now state in the following theorem.

Theorem 3.5.2 *Let $T : [0,1] \to [0,1]$ be a piecewise monotone and expanding map of the unit interval, having the finite range property and the absolutely continuous invariant probability μ. Let f be a measurable function admitting k different values $\beta_1, \dots \beta_k$ with probabilities $\mu(f = \beta_j) = p_j,\ j = 1, \dots k$. Let $X_n = f \circ T^n$ denote the stationary sequence generated by f and T. Let Z_n be the vector of occurrences of these k possible outcomes in n iterations of f. Let G_n denote the region of points $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ for which $|x_j| \leq \sqrt{\frac{A \log n}{n}}$ for $j + 1, \dots, k$, where $A > 0$:*

As $(x_1, \dots, x_k) = \frac{1}{n}(n_1 - np_1, \dots, n_k - np_k) \in G_n$ and $n \to \infty$ there exists a $(k-1)-$multivariate normal distribution with mean $\tilde{0}$ and covariance matrix Σ such that

$$\mu(Z_n = (n_1, ..., n_k)) = p_n(n_1, \dots, n_k) \sim \frac{1}{(2\pi n)^{(k-1)/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}\tilde{x}'\Sigma^{-1}\tilde{x}\right\}$$

Remark. Recall that in Chapter 2, we wrote the vector of occurrences $Z_n = (n_1, \dots, n_k)$ as $\sum_{j=1}^{k-1} Y_j$ where the $Y_j's, j = 1, \dots, n$ are k-dimensional vectors; each Y_j represents the outcome of the jth trial which is a vector with a 1 in position l if the l-th event occurs, $l = 1, \dots k$, on that trial and $0's$ in all other positions. Then take $Y_j = (Y_{j1}, \dots, Y_{j(k-1)}), j = 1, \dots n$. Since $n_k = n - (n_1 + \dots n_{k-1})$, the $Y_j's$ are independent.

Here we write our vector of occurrences as

$$(n_1, \dots, n_k) = S_n\tilde{f} = \sum_{j=0}^{n-1} \tilde{f} \circ T^j = \sum_{j=1}^{n} \tilde{f} \circ T^{j-1}, \qquad \tilde{f} \in \mathbb{R}^k.$$

Note that $\tilde{f} = (f_1, \dots, f_k)$ where each $f_l, l = 1, \dots, k$, takes the value 1 in position l if the l^{th} event occurs and 0 otherwise, just like the $Y_j's$, and the dependence between the n trials is described by $T^j, j = 1, \dots, n$.

So we have

$$Y_1 = \tilde{f}$$

$$Y_2 = \tilde{f} \circ T$$

$$Y_3 = \tilde{f} \circ T^2$$

$$\vdots$$

$$Y_n = \tilde{f} \circ T^{n-1}$$

and $\sum_{j=1}^{n} Y_j = n$. Because $n = n-(n_1+\ldots+n_{k-1})$, we work with $\tilde{f} = (f_1, \ldots, f_{k-1})$.

Chapter 4

Generalized CRR Option Price Formula with Dependent Jumps

In this chapter we will give the expression of the arbitrage price of a European call option written on one share of stock, when the stock price follows the generalized CRR stock pricing model with dependent jumps. We will also give the asymptotic behavior of this expresion.

In the generalized CRR setting the jumps of the price of the stock were described by the sequence $(X_t)_{t \leq T}$. Conditioning on the values that $X_{T-m}, \ldots, X_T$ for $m = 0, \ldots, T$ yield the conditional generalized CRR option price formula.

The assumptions on $X_{T-m}, \ldots, X_T$ have an important role in finding the expression of the option price. Since the generalized CRR option pricing model, as introduced in [22], is the generalization of the binomial model, it is straightforward to think about the multinomial distribution.

In her Ph. D dissertation [22], Kan imposed on the random variables $X_t, \quad t = 1, \ldots, T$, to be independent and identically distributed, in order to the vector of occurrences of the values that those n independent variables can take, follows the multinomial distribution, and she introduced the multinomial model, that we described in Chapter 1.

As is known, stock prices change because of the economics of market forces, supply and demand in the market determines stock price. There are many theories that try to explain the way stock prices move the way they do. Unfortunately, there is no one theory that can explain everything.

Even if one cannot explain in well defined terms the numerous intricate situations that determine the price changes, one can see that those situations cause the changes in the stock price one day obviously depend on the changes that occurred the day before.

In the following section we consider the case where the random variables $X_t, t \leq T$ are dependent, but still take values in the multinomiality set $\mathbb{C}_k$. We have chosen this approach because of its relevance to the stock market.

4.1 Generalized CRR Option Price Formula with Dependent Jumps

We consider a finite probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and a map $S : [0,1] \to [0,1]$ verifying the conditions of definition 3.2.1. We assume $X_{T_l} = f \circ S^{T-l}, \quad l = T-m, \ldots, T$, where f is a measurable function admitting k different values $\beta_1, \ldots \beta_k$ with probabilities $\mu(f = \beta_j) = p_j, \;\; j = 1, \ldots k$.

We will start with the generalized CRR option price model derived by Kan [22].

Recall from chapter 1 that in the generalized conditional CRR model, the stock price, or risky asset, is modelled as a strictly positive discrete-time process $S = (S_t)_{t \leq T}$ defined as follows

$$S_t = S_{t-1} X_{t-1} \nu_{t-1}, \qquad \forall t \le T \tag{4.1.1}$$

where $\{\nu_t\}_{t\le T^*}$ are Bernouilli random variables taking values u and d with probabilities p and $1-p$ and $\{X_t\}_{t\le T}$ describes the jumps of upward and downward movements and is assumed to be independent on $\{\nu_t\}_{t\le T}$.

The bond or risk-free asset price process is given by

$$(1+r)^t, \qquad \forall t \le T^* \tag{4.1.2}$$

where $\hat{r} = 1 + r$, r is any positive real number.

The expression of the generalized CRR option price is

$$C_{T-m} = S_{T-m} \sum_{j=0}^{m} \left(\sum_{J \in I_{j,m}(S_{T-m})} \overline{P}(J^{(j)}, T) - \frac{K}{\hat{r}^m} \sum_{J \in I_{j,m}(S_{T-m})} P(J^{(j)}, T) \right) \tag{4.1.3}$$

where

$$p_{T-k} := \frac{\hat{r} - \xi^d_{T-k}}{\xi^u_{T-k} - \xi^d_{T-k}}, \;\; \overline{p}_{T-k} = \frac{\xi^u_{T-k}}{\hat{r}} p_{T-k}$$

$$q_{T-k} = 1 - p_{T-k},, \;\; \overline{q}_{T-k} = 1 - \overline{p}_{T-k}$$

$$\overline{P}(J^{(j)}, T) = \prod_{k \in J} \overline{p}_{T-k} \prod_{k \notin J} \overline{q}_{T-k} \tag{4.1.4}$$

and

$$P(J^{(j)}, T) = \prod_{k \in J} p_{T-k} \prod_{k \notin J} q_{T-k}. \tag{4.1.5}$$

The set $\Gamma_m = \{1, \ldots, m\}$. For any fixed $m \in \mathbb{N}$ and $j \in \Gamma_m$, $I_{j,m}$ denotes the following

$$I_{j,m}(x) = \left\{ J \subset \Gamma_m, |J| = j, x \prod_{k \in J} \xi^u_{T-k} \prod_{k \in \overline{J}} \xi^d_{T-k} > K \right\} \tag{4.1.6}$$

where $|J|$ stands for the cardinality of the set J and $\overline{J}$ stands for the complement of J.

Remark In finding the corresponding generalized CRR option pricing formula (4.1.3), Kan [22] used the backward induction technique with respect to m. This method consists in selecting a portfolio $\phi_{T-m-1} = (\alpha_{T-m-1}, \beta_{T-m-1})$ for the period $[T-m-1, T-m)$ in a way that the portfolio's wealth, as defined in definition 1.2.2, at time $T-m$ replicates the option's payoff at time T, C_T, starting with the last period before expiration which means with $m = 0$. And as was shown, the assumption needed was that of absence of arbitrage, that can be represented in the market as $\mathbb{P}\{\xi^d_t < 1 + r < \xi^u_t\} = 1, \ \forall t \leq T$. For the full proof, one can refer to [22] pp. 20-23, 26-29. The same setup with the same assumptions was used by Musiela and Rutkowski in [36] pp.43-46 for the binomial model.

Note that C_{T-m} as in (4.1.3) depends on the random sequence $X_{T-m}, \ldots, X_t$.

We now consider the expected value of C_{T-m} in (4.1.3) and call this expectation the arbitrage price,

$$\tilde{C}_{T-m} = E(C_{T-m}). \tag{4.1.7}$$

For integrating C_{T-m} we need the integrals $E(P(J^{(j)}, T)$ and $E(\overline{P}(J^{(j)}, T)$; therefore, we need to compute (4.1.8) and (4.1.9), where $\forall \quad z = (z_{T-m}, \ldots, z_T) \in \mathbb{R}^m$

$$\tilde{C}^{(1)}_{T-m}(j) = \underbrace{\int \cdots \int}_{m \ integrals} \sum_{J \in I_{j,m}(S_{T-m}, z)} \overline{P}(J^{(j)}, T, z) dF^{X_{T-m}, \ldots, X_T}(z) \tag{4.1.8}$$

and

$$\tilde{C}^{(2)}_{T-m}(j) = \underbrace{\int \cdots \int}_{m\ integrals} \sum_{J \in I_{j,m}(S_{T-m},z)} P(J^{(j)}, T, z) dF^{X_{T-m},\dots,X_T}(z) \tag{4.1.9}$$

where the set

$$I_{j,m}(x,z) = \left\{ J \subset \{1,\dots,m\}, |J| = j : x \prod_{k\in J} u z_{T-k} \prod_{k \in \overline{J}} d z_{T-k} > K \right\} \tag{4.1.10}$$

is a realization of the random set (4.1.6).

Also, $F^{X_{T-m},\dots,X_T}(z) = F_{T-m,\dots,T}(z)$ is the joint distribution function of $X_{T-m},\dots,X_T$.

Equation (4.1.7) becomes

$$\tilde{C}_{T-m} = S_{T-m} \sum_{j=0}^{m} C^{(1)}_{T-m}(j) - \frac{K}{\hat{r}^m} \sum_{j=0}^{m} C^{(2)}_{T-m}(j). \tag{4.1.11}$$

Remark As one can see, obtaining the option price from equation (4.1.11) depends on the sequence of random variables $(X_t)_{t\leq T}$ and the assumptions imposed on it.

Note that $\tilde{C}_{T-m}$ in (4.1.11) is still random as S_{T-m} is random. However, for $m = T$, S_0 is assumed to be constant, thus $\tilde{C}_0$ is computable as a non random number.

The important point is that (4.1.8) and (4.1.9) has to be computed and this is where we make a contribution in this thesis, in the case where $X_{T-m},\dots,X_T$ are dependent and take finitely many values.

Let $N_{c_1},\dots,N_{c_k}$ denote the occurrences of $c_1,\dots,c_k$ among $f\circ T^l, \quad l = T-m,\dots,T$.

For every $z = (z_{T-m}, \dots, z_T) \in \mathbb{R}^m$, we have

$$\overline{P}(J^{(j)}, T, z) = \prod_{k \in J} \overline{p}_{T-k}(z_{T-k}) \prod_{k \notin J} \overline{q}_{T-k}(z_{T-k})$$

and

$$P(J^{(j)}, T, z) = \prod_{k \in J} p_{T-k}(z_{T-k}) \prod_{k \notin J} q_{T-k}(z_{T-k})$$

where

$$p_{T-k}(z_{T-k}) := \frac{\hat{r} - dz_{T-k}}{uz_{T-k} - dz_{T-k}}, \;\; \overline{p}_{T-k}(z_{T-k}) = \frac{uz_{T-k}}{\hat{r}} p_{T-k}(z_{T-k}),$$

$$q_{T-k}(z_{T-k}) = 1 - p_{T-k}(z_{T-k}), \;\; \overline{q}_{T-k}(z_{T-k}) = 1 - \overline{p}_{T-k}(z_{T-k})$$

We fix z which is a sequence $c_{l_{T-m}}, \dots, c_{l_T}$, then

$$p_{T-l}(c_{j_{T-l}}) = \frac{\hat{r} - dc_{j_{T-l}}}{c_{j_{T-l}}(u-d)}, \qquad \overline{p}_{T-l}(c_{j_{T-l}}) = \frac{uc_{j_{T-l}}}{\hat{r}} p_{T-l}(c_{j_{T-l}}),$$

$$q_{T-l}(c_{j_{T-l}}) = 1 - p_{T-l}(c_{j_{T-l}}), \qquad \overline{q}_{T-l}(c_{j_{T-l}}) = 1 - \overline{p}_{T-l}(c_{j_{T-l}})$$

Let $J \subset \{1, \dots, m\}$. Let m_1 be the number of occurrences of c_1 when $l \in J$ and m_1' the number of occurrences of c_1 when $l \notin J$.

Next recall that the random set

$$I_{j,m}(x, z) = \left\{ J \subset \{1, \dots, m\}, |J| = j : x \prod_{k \in J} uz_{T-k} \prod_{k \in \overline{J}} dz_{T-k} > K \right\}$$

is independent of the order of the values $z_{T-m}, \dots, z_T$

since

$$x \prod_{k \in J} u z_{T-k} \prod_{k \in \overline{J}} d z_{T-k} > K \tag{4.1.12}$$

means that

$$x u^{|J|} d^{|\overline{J}|} c_1^{m_1+m_1'} \cdot \ldots \cdot c_k^{m_k+m_k'} > K.$$

We define the set O of all points z_{T-m} where we have (m_1+m_1') occurrences of $c_1, \ldots,$ (m_k+m_k') occurrences of c_k.

On the set O for all sets J with $|J| = j$, condition 4.1.12 holds or does not hold, so $I_{(j,m)}(x,z)$ is independent of the order of $z_{T-m}, \ldots, z_T$ and only depends on the values of the random variables $N_{c_1}, \ldots, N_{c_k}$ alone; that is,

$$N_{c_1} = m_1 + m_1', \ldots, N_{c_k} = m_k + m_k'.$$

Let

$$I(m, j, M_1, \ldots, M_k) = \left\{ \begin{array}{l} (m_1, \ldots, m_{2k}) : m_i \in \tau_m, i = 1, \ldots, 2k; \\ m_1 + \ldots + m_{2k} = m : m_1 + \ldots + m_k = j; \\ M_1 = m_1 + m_1', \ldots, M_k = m_k + m_k' \end{array} \right\}$$

.

Let a be the random number of the smallest index j with

$$u^j d^{m-j} c_1^{N_{c_1}} \cdot \ldots \cdot c_k^{N_{c_k}} > K.$$

This means that the sum over $J \in I_{(j,m)}(S_{T-m}, z)$ equals the sum over $I(m, j, N_{c_1}, \ldots, N_{c_k})$.

Integral (4.1.9) becomes

$$\begin{aligned} \tilde{C}_{T-m}^{(2)}(j) &= \underbrace{\int \cdots \int}_{m \ integrals} \sum_{J \in I_{j,m}(S_{T-m},z)} P(J^{(j)}, T, z) dF^{X_{T-m},\ldots,X_T}(z) \qquad (4.1.13) \\ &= E \sum_{I(m,j,N_{c_1},\ldots,N_{c_k})} P(J^{(j)}, T, X_{T-m}, \ldots, X_T). \end{aligned}$$

Defining the set $\mathcal{C}_l = \{z : c_l \text{ occurs } M_l \text{ times}\}$, we get

$$\begin{aligned}
\tilde{C}^{(2)}_{T-m}(j) &= \sum_{\{M_1,\ldots,M_k\}} \int_{\mathcal{C}_l} \sum_{I(m,j,M_1,\ldots,M_k)} P(J^{(j)}, T, z) dF^{X_{T-m},\ldots,X_T}(z) \\
&= \sum_{\{M_1,\ldots,M_k; \sum M_j = m\}} \sum_{I(m,j,M_1,\ldots,M_k)} P(J^{(j)}, T, M_1, \ldots, M_k) \mathbf{P}(N_{c_1} = M_1, \ldots, N_{c_k} = M_k).
\end{aligned}$$

With the same calculations, (4.1.8) becomes

$$\begin{aligned}
\tilde{C}^{(1)}_{T-m}(j) &= \underbrace{\int \cdots \int}_{m \ integrals} \sum_{J \in I_{j,m}(S_{T-m}, z)} \overline{P}(J^{(j)}, T, z) dF^{X_{T-m},\ldots,X_T}(z) \qquad (4.1.14) \\
&= E \sum_{I(m,j,N_{c_1},\ldots,N_{c_k})} \overline{P}(J^{(j)}, T, X_{T-m}, \ldots, X_T) \\
&= \sum_{\{M_1,\ldots,M_k\}} \int_{\mathcal{C}_l} \sum_{I(m,j,M_1,\ldots,M_k)} \overline{P}(J^{(j)}, T, z) dF^{X_{T-m},\ldots,X_T}(z) \\
&= \sum_{\{M_1,\ldots,M_k; \sum M_j = m\}} \sum_{I(m,j,M_1,\ldots,M_k)} \overline{P}(J^{(j)}, T, M_1, \ldots, M_k) \mathbf{P}(N_{c_1} = M_1, \ldots, N_{c_k} = M_k).
\end{aligned}$$

Proposition 4.1.1 *The generalized CRR option price with dependent jumps is given by the following formula:*

$$\begin{aligned}
\hat{C}_{T-m} &= S_{T-m} \sum_{j=0}^{m} \sum_{\{M_1,\ldots,M_k; \sum M_j = m\}} \sum_{I(m,j,N_{c_1},\ldots,N_{c_k})} \overline{P}(J^{(j)}, T, z) \mathbf{P}(N_{c_1} = m_1, \ldots, N_{c_k} = m_k) \\
&- \frac{K}{\hat{r}^m} \sum_{j=0}^{m} \sum_{\{M_1,\ldots,M_k; \sum M_j = m\}} \sum_{I(m,j,N_{c_1},\ldots,N_{c_k})} P(J^{(j)}, T, z) \mathbf{P}(N_{c_1} = m_1, \ldots, N_{c_k} = m_k).
\end{aligned}$$

(4.1.15)

4.2 Limit of the Generalized CRR Option Price Formula with Dependent Jumps

We assume that there exists a set $\mathbb{C}_{k,n} = \{c_{1,n}, \dots, c_{k,n}\}$ such that $x_n \in \mathbb{C}_{k,n}$.

We introduce the following notations. Let

$$x_i = \frac{m_i - mp_i}{\sqrt{m}} \quad i = 1, \dots, 2k$$

and

$$\overline{x}_i = \frac{m_i - m\overline{p}_i}{\sqrt{m}} \quad i = 1, \dots, 2k$$

where $\sum_{i=1}^{2k} p_i = 1, \sum_{i=1}^{2k} m_i = m$ so $\sum_{i=1}^{2k} x_i = \sum_{i=1}^{2k} \overline{x}_i = 0$

We define the set

$$\mathbb{X} = \left\{ x = (x_1, \dots, x_k) : |x_i| \le \sqrt{\frac{A \log m}{m}}, i = 1, \dots, k \right\}$$

where A is a positive constant.

Theorem 4.2.1 *For $x \in \mathbb{X}, i = 1, \dots, k$ as $n \to \infty$ the following asymptotic holds true*

$$\begin{aligned} \hat{C}_{T-m} \sim \; & S_{T-m} \sum_{j=0}^{m} \sum_{\{M_1,\dots,M_k; \sum M_j = m\}} \sum_{I(m,j,N_{c_1},\dots,N_{c_k})} \overline{P}(J^{(j)}, T) \frac{1}{(2\pi m)^{\frac{k-1}{2}} |\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2} x' \Sigma^{-1} x \right\} \\ - \; & \frac{K}{\hat{r}^m} \sum_{j=0}^{m} \sum_{\{M_1,\dots,M_k; \sum M_j = m\}} \sum_{I(m,j,N_{c_1},\dots,N_{c_k})} P(J^{(j)}, T) \frac{1}{(2\pi m)^{\frac{k-1}{2}} |\Sigma|^{(k-1)/2}} \exp\left\{ -\frac{1}{2} x' \Sigma^{-1} x \right\}, \end{aligned} \tag{4.2.1}$$

where Σ is the covariance matrix of a zero mean $(k-1)$-normal distribution as in Chapter 3.

Proof. Due to theorem 3.5.2 the statement of the theorem is the true one. □.

4.3 Outline

We rewrite equation 4.1.1 in the following fashion

$$\begin{aligned}
\hat{C}_{T-m} &= k^m \sum_{j=0}^{m} \sum_{\{M_1,\dots,M_k\}} \sum_{I(m,j,M_1,\dots,M_k)} (S_{T-m}\overline{P}(J^{(j)},T) - \frac{K}{\hat{r}^m} P(J^{(j)},T)) \\
&\times \mathbf{P}(N_{c_1} = M_1, \dots, N_{c_k} = M_k) \\
&= k^m \sum_{j=0}^{m} \sum_{\{M_1,\dots,M_k\}} \sum_{I(m,j,M_1,\dots,M_k)} \left(S_{T-m}\overline{P}(J^{(j)},T) - \frac{K}{\hat{r}^m} P(J^{(j)},T) \right) \frac{m!}{M_1!.\dots.M_k!} k^{-m} \\
&\times \frac{\mathbf{P}(N_{c_1} = M_1, \dots, N_{c_k} = M_k)}{k^{-m}\dfrac{m!}{M_1!.\dots M_k!}} \\
&= k^m \left(S_{T-m} \sum_{j=0}^{m} \sum_{I(m,j)} M_{2k}(m,\overline{p}) - \frac{K}{\hat{r}^m} \sum_{j=0}^{m} \sum_{I(m,j)} M_{2k}(m,p) \right) \frac{P(N_{c_1} = M_1, \dots, N_{c_k} = M_k)}{k^{-m}\dfrac{m!}{M_1!.\dots M_k!}} \\
&= \tilde{C}_{T-m} \frac{P(N_{c_1} = M_1, \dots, N_{c_k} = M_k)}{k^{-m}\dfrac{m!}{M_1!.\dots M_k!}},
\end{aligned}$$

where

$$p_1(c_1) = \frac{1}{k}\frac{\hat{r} - c_1 d}{c_1(u-d)}, \dots, p_k(c_k) = \frac{1}{k}\frac{\hat{r} - c_k d}{c_k(u-d)},$$

$$p_{k+1}(c_1) = \frac{1}{k}\frac{c_1 u - \hat{r}}{c_1(u-d)}, \dots, p_{2k}(c_k) = \frac{1}{k}\frac{c_k u - \hat{r}}{c_k(u-d)},$$

$$\overline{p}_1 = \frac{c_1 u}{\hat{r}} p_1, \dots, \overline{p}_k = \frac{c_k u}{\hat{r}} p_k, \;\; \overline{p}_{k+1} = \frac{c_1 d}{\hat{r}} p_{k+1}, \dots, \overline{p}_{2k} = \frac{c_k d}{\hat{r}} p_{2k},$$

$$M_{2k}(m,\overline{p}) = \frac{m!}{m_1! \dots m_{2k}!} \overline{p}_1^{m_1} \dots \overline{p}_{2k}^{m_{2k}}, \tag{4.3.1}$$

and

$$M_{2k}(m,p) = \frac{m!}{m_1! \dots m_{2k}!} p_1^{m_1} \dots p_{2k}^{m_{2k}}. \tag{4.3.2}$$

- Kan [22] used the following asymptotic procedure:
  - Taking n of the form $n = 2^s$, s is a natural number
  - Then dividing the interval $[0, T]$, for $T > 0$ into n equal subintervals I_j of length $\Delta_n = \frac{T}{n}$ for $j = 0, \dots, n-1$.
  - Then find the asymptotic value of the European call option price $\hat{C}_{T-m}$ for any $T > 0$ and $m \in [0, T]$ when the number of periods increases as n goes to infinity which means the size of the time steps goes to zero.

  and showed

  $$\tilde{C}_{T-m} \to S_t \Phi(f_1(S_t, T-t)) - Ke^{-r(T-t)} \Phi(f_2(S_t, T-t)), as \quad n \to \infty$$

  uniformly where

  $$f_1(s,t) = \frac{\ln \frac{s}{K} + (T-t) \frac{\ln c_1 \dots c_k + k}{k} \left(\frac{r}{k} \frac{\ln c_1 \dots c_k + k}{\prod_{i=1}^{k} (\ln c_i + 1)} + \frac{\sigma^2}{2} \frac{\ln c_1 \dots c_k + k}{k} \right)}{\sigma \frac{\ln c_1 \dots c_k + k}{k} \sqrt{T-t}}$$

  $$f_2(s,t) = f_(s,t) - \sigma \frac{\ln c_1 \dots c_k + k}{k} \sqrt{T-t}$$

  and Φ stands for the standard Gaussian cumulative distribution function $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du, \forall \quad x \in \mathbb{R}$

- Due to Richter's local limit theorem we have

  $$k^{-m} \frac{m!}{M_1! \dots M_k!} \sim \frac{1}{(2\pi m)^{\frac{k-1}{2}} k^{-k/2}} \exp \left\{ -\frac{m}{2k} \sum_{j=1}^{k} x_j^2 \right\}.$$

- Due to theorem 3.5.2

  $$\mathbf{P}(N_{c_1} = M_1, \dots, N_{c_k} = M_k) \sim \frac{1}{(2\pi m)^{\frac{k-1}{2}} |\Sigma|^{(k-1)/2}} \exp \left\{ -\frac{1}{2} x' \Sigma^{-1} x \right\}.$$

Using the same asymptotic procedure as in [22], the following can be shown:

Theorem 4.3.1 *Let X be a random variable which can assume k different values $\beta_1, \dots \beta_k$ with probabilities $P(X = \beta_j) = \frac{1}{k}$, $j = 1, \dots k$. Let Z_n be the vector of occurrences of these k possible outcomes in n dependent trials of X.*

Let

$$\mathbb{X} = \left\{ x = (x_1, \dots, x_k) : |x_i| \le \sqrt{\frac{A \log m}{m}}, i = 1, \dots, k \right\}.$$

As $(x_1, \dots, x_k) = \dfrac{1}{m}(m_1 - mp_1, \dots, m_k - mp_k) \in \mathbb{X}$ *and* $m \to \infty$,

there exists a $(k-1)$-normal distribution with zero mean and covariance matrix Σ such that:

If condition $(\mathcal{C})$ holds true

$$\frac{\frac{1}{|\Sigma|^{(k-1)/2}} \exp\left\{-\frac{1}{2}x'\Sigma^{-1}x\right\}}{k^{-k/2} \exp\left\{-\frac{m}{2k} \sum_{j=1}^{k} x_j^2\right\}} \to 1 \qquad (\mathcal{C})$$

uniformly for $x \in \mathbb{X}$, then the following convergence is valid

$$\lim_{n\to\infty} \hat{C}_{T-m} = S_t \Phi(f_1(S_t, T-t)) - Ke^{-r(T-t)} \Phi(f_2(S_t, T-t)), as \quad n \to \infty$$

where

$$f_1(s,t) = \frac{\ln \frac{s}{K} + (T-t)\frac{\ln c_1 \dots c_k + k}{k} \left(\frac{r}{k} \frac{\ln c_1 \dots c_k + k}{\prod_{i=1}^{k}(\ln c_i + 1)} + \frac{\sigma^2}{2} \frac{\ln c_1 \dots c_k + k}{k} \right)}{\sigma \frac{\ln c_1 \dots c_k + k}{k} \sqrt{T-t}}$$

$$f_2(s,t) = f_(s,t) - \sigma \frac{\ln c_1 \dots c_k + k}{k} \sqrt{T-t}$$

and Φ *stands for the standard Gaussian cumulative distribution function* $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du, \forall \quad x \in \mathbb{R}$.

In this chapter we showed that the option price for a European call option written on one share of stock, when the stock price follows the generalized CRR model with dependent jumps as in equation (4.1.1) where the $X_t's$, representing the jumps in the stock price are dependent, has a similar expression as in the independent case. Using theorem 3.5.2, the limit of the option price also is similar as the one in the independent case.

Bibliography

[1] Aaronson J., Denker M., *Local Limit Theorems for Partial Sums of Stationary Sequences Generated by Gibbs-Markov Maps*. journal. (year)

[2] Bachelier, L., *Théorie de la Speculation* . Annales scientifiques de l' É.N.S, 3ème serie, tome 17, 21-86. (1900)

[3] Bernstein P.L., *Capital Ideas: The Improbable Origins of Modern Wall Street.* Free Press. (1993)

[4] Bingham N.H., Kies R., *Rist Neutral Valuation: Pricing and Hedging of Financial Derivatives*. (editor). (year)

[5] Birman M.S., Solomjak M.Z, *Spectral theory of self-adjoint operators in hilbert space*. D. Reidel publishing company. (1987)

[6] Black, F., Scholes, M., *The Valuation of option contracts and a test of market efficiency*. Journal of Finance 27, 399-417. (1972)

[7] Breiman L., *Probability*. Addison-Wesley, Reading, Mass. (1968)

[8] Chesney, M., Scott, L. *Pricing European currency options: a comparison of the modified Black-Scholes model and a random variance model.* J. Finan. Quant. Anal. 24, pp. 267-284. (1989)

[9] Chung S-L., Shif P-T., *Generalized Cox Ross Rubinstein Binomial Models.* Journal of Management Science, 53,3, pp. 508-520. (year)

[10] Clark E., Ghosh D., *Arbitrage, Hedging and speculation*. Praeger. (2004)

[11] Cox, J.C., Ross, S., Rubinstein, M., *The Valuation of options for alternative stochastic processes*. Journal of Financial economics 3, 145-166. (1976)

[12] Cuthbertson K., Nitzche D., *Financial Engineering. Derivatives and risk management*. JOHN WILEY and SONS. (2001)

[13] Dunford N., Schwartz J.T., *Linear Operators*. editor. (1976)

[14] Edward R.E., *Fourier Series a Modern Introduction vol.1 second edition.* Springer-Verlag. (1979)

[15] Li T.Y., Yorke J.A., *Ergodic Transformation from an interval into himself.* Trans. Amer. Math. Soc. 235. pp. 183-192

[16] Forsyth P.A., *An Introduction to Computational Finance*. Departememt of Computer Science, Waterloo Ontario. (2003)

[17] Galai, D. *Tests of market efficiency of the Chicago Board Options Exchange.* J. Business 50, pp. 167-197. (1977)

[18] Hong Q. X., *Multidimensional Local Limit Theorem of Some Non-Uniformly Hyperbolic Systems*. Acta Mathematica Sinica, English Series 25,4, pp. 565-580. (2009)

[19] Hull, J. C. *Options, Futures, and Other Derivatives. 3rd ed.* Prentice- Hall, Englewood Cliffs (New Jersey). (1997)

[20] Ionescu T. Marinescu G., *Theorie ergodique pour des classes d'opeŕations non compltement continues.* Ann. Maths. 47, pp. 140-147

[21] Istratescu. V, *Introduction to Linear Operator Theory*. MARCEL DEKKER, INC. (1981)

[22] Kan, N., *The CRR Option Pricing model and its Black-Scholes Type Limit*. Ph. D dissertation, Gotingen, Germany. (2000)

[23] Kato T., *A Short Introduction to Perturbation Theory for Linear Operatots.* Springer-Verlag. (1980)

[24] Kato T., *Perturbation Theory for Linear Operators*. Springer-Verlag. (1966)

[25] King's College, London, *Functional Analysis- Gently Done.* Mathematics Departement

[26] Kolb R.W., Overdahl J.A., *Financial Derivatives, 3rd Edition.* John Wiley and Sons, Inc. (2003)

[27] Kolb R.W., Overdahl J.A., *Futures, options and swaps.* John Wiley and Sons, Inc. (2003)

[28] *real options.* Kulatilaka A., Harvard business school press. (1999)

[29] Lasser R., *introduction to fourier series.* Marcel Dekker, Inc. (1996)

[30] Lehmann, B.N., *The Legacy of Fischer Black.* Oxford university press, (2005)

[31] Madan, D., Milne, F., Shefrin, H. *The multinomial option pricing model and its Brownian and Poisson limits.* Rev. Finan. Stud. 2, 251-265. (1989)

[32] Mehrling P., *Fischer Black and the Revolutionary Idea of Finance.* Wiley, (2005)

[33] Mellino A., Turnbull S.M. *Pricing foreign currency options with stochastic volatility.* J. Econometrics 45, 239-265. (1990)

[34] Mel'nikov A.V., *Financial Markets Stochastic Analysis and the Pricing of Derivative Securities.* American Mathematical Society. (1999)

[35] Merton, R.C., *Option pricing when underlying stock returns are discontinuous.* Journal of Financial Economics, 3, 125-144. (1976)

[36] Musiela, M., Rutkowski,M., *Martingale Methods in Financial Modelling.* Springer. (1998)

[37] Pliska, S. R., *Introduction to mathematical finance: discrete time models.* Blackcwell Publishers Ltd, Oxford, UK. (1997)

[38] Richter, H. Wahrscheinlichkeitstheorie. Springer-Verlag. Berlin- Goettingen-Heidelberg. (1956)

[39] Rousseau-Egele J., *Un Theoreme de la Limite Locale pour une Classe de Transformations Dilatantes et Monotones par Morceaux*. The Annals of Probability, 11,3, pp.772-788. (1983)

[40] Rudin, W., *Functional Analysis*. McGraw hill company. (1921)

[41] Samuelson, P.A., *Rational Theory of Warrant Pricing. In the Random Character of Stock Market Prices.* Ed. P. Cootner, 15, 506-532. Cambridge, MIT press. (1964)

[42] Scott, L.O. *Option pricing when the variance changes randomly: theory, estimation, and an application.* J. Finan. Quant. Anal. 22, pp. 419-438. (1987)

[43] Scott, L.O. *Random-variance option pricing: empirical tests of the model and delta-sigma hedging.* Adv. in Futures Options Res. 5, pp. 113-135 (1991)

[44] Schachermayer, W. *Introduction to the Mathematics of Financial Markets.* Lectures on probability theory and statistics, Saint-Flour 2000, Lecture Notes in Math.,1816 Springer Verlag. (2000)

[45] Sharpe, W.F., *Investments*. Prentice hall, Englewood cliffs, NJ. (1978)

[46] Shiryaev, A. N., *Essentials of Stochastic Finance: Facts, Models, Theory*. World Scientific Publishing Co. Pte. Ltd. (1999)

[47] Sullivan, E., Weithers, T., *Louis Bachelier: The Father of Modern Option Pricing Theory*. The Journal of Economic Education, Vol. 22, No. 2, pp. 165-171. (1991)

[48] Thomsett,M.C., *The Options Trading Body of Knowledge: The Definitive Source for Information About the Options Industry*. Pearson education, New Jersey, 2nd Edition. (2009)

[49] Weidmann J. *lin op in hilbert spaces*. Springer -Verlag. (1980)

[50] Wiggins, J.B. *Option values under stochastic volatility: theory and empirical estimates.* J. Finan. Econom. 19, pp. 351-372. (1987)

[51] Wilde I.F., *Vector Analysis-Gently Done. Topological Vector Spaces*. Lecture notes, Departement of Mathematics, King's College, London (1986)

[52] Zhu K., *Operator Theory in Function Spaces*. MARCEL DEKKER, INC. (1990)