THE LAPLACE TRANSFORMATION AND ITS APPLICATION TO THE SOLUTION OF CERTAIN GENERAL LINEAR DIFFERENTIAL EQUATIONS

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CHAPTER I

INTRODUCTION

1.1 Definition of Laplace Transform.

If a suitable function $x(t)$ is multiplied by e^{-pt} and the product is integrated with respect to t from zero to infinity, there results a new function of the parameter p, which is called the Laplace transform of $x(t)$ and is denoted by symbols such as $\bar{x}(p)$, or $L\{x(t)\}$. The opposite procedure, that of obtaining the function $x(t)$ from the transform $\bar{x}(p)$, is called the inverse transform, and is denoted by the symbol $L^{-1}\{\bar{x}(p)\}$.

1.2 Conditions for Existence of Laplace Transform.

The Laplace transform of $x(t)$ exists if $x(t)$ is sectionally continuous in every finite interval in the range $t \geq 0$, and if the function is of exponential order as $t \rightarrow \infty$. These are sufficient conditions.

A function $x(t)$ is sectionally continuous in a finite interval $a \leq t \leq b$, if it is possible to subdivide that interval into a finite number of subintervals in each of which $x(t)$ is continuous and has finite limits as t approaches either end point of the subinterval from the interior. Any discontinuities of such a function in the interval (a,b) are of the type known as ordinary points of discontinuity, where the value of the function makes a finite jump.

A function $x(t)$ is of exponential order as $t \rightarrow \infty$

provided constants M and α exist such that $e^{-\alpha t}|x(t)| < M$ for all t greater than some fixed value of t.

1.3 Some Fundamental Results.

1.31 Transform of nth Order Derivative.

Since the transforms of derivatives play an important part in the application of the Laplace transform to the solution of differential equations, it is necessary to obtain a formula for the transform of the nth order derivative.

Using customary notation,

$$L\left\{\frac{dx}{dt}\right\} = \int_0^\infty e^{-pt} \frac{dx}{dt}\, dt.$$

Integration by parts gives

$$\int_0^\infty e^{-pt} \frac{dx}{dt}\, dt = \Big[e^{-pt} x\Big]_0^\infty + p\int_0^\infty e^{-pt} x\, dt = -x(0) + p\bar{x}(p),$$

where x(0) represents the value of x(t) when t = 0. Also

$$\int_0^\infty e^{-pt} \frac{d^2x}{dt^2}\, dt = \Big[e^{-pt} \frac{dx}{dt}\Big]_0^\infty + p\int_0^\infty e^{-pt} \frac{dx}{dt}\, dt$$
$$= -x'(0) + p\int_0^\infty e^{-pt} \frac{dx}{dt}\, dt,$$

where x'(0) represents the value of $\frac{dx}{dt}$ when t = 0. But $\int_0^\infty e^{-pt} \frac{dx}{dt}\, dt$ has already been found above. Therefore

$$\int_0^\infty e^{-pt} \frac{d^2x}{dt^2}\, dt = -x'(0) - px(0) + p^2\bar{x}(p).$$

In a similar manner of applying successive integration by parts, the nth order derivative will follow. That is;

$$L\left\{\frac{d^nx}{dt^n}\right\} = \int_0^\infty e^{-pt} \frac{d^nx}{dt^n}\, dt$$
$$= p^n\bar{x}(p) - p^{n-1}x(0) - p^{n-2}x'(0) - \cdots - px^{(n-2)}(0) - x^{(n-1)}(0).$$ [1]

1.32 The Shifting Theorem.

A fundamental result useful in extending the table of

[1] H. S. Carslaw and J. C. Jaeger, Operational Methods in Applied Mathematics (second edition; London: Oxford University Press, 1943), pp. 1-2.

transforms is called the shifting theorem. This theorem states that if $\bar{x}(p)$ is the transform of $x(t)$, and a is any constant, then $\bar{x}(p+a)$ is the transform of $e^{-at}\, x(t)$.[2]

Thus, to find the transform of te^{-2t}, first observe that $\frac{1}{p^2}$ is the transform of t. Therefore

$$L\left\{te^{-2t}\right\} = \frac{1}{(p+2)^2}.$$

1.33 The Convolution Theorem.

If it is recognized that a given transform consists of the product of the transforms of two known functions, use can be made of the convolution theorem to find the inverse transform of the product. That is;

$$L^{-1}\left\{\bar{x}_1(p)\ \bar{x}_2(p)\right\} = \int_0^t x_1(t-\tau)\ x_2(\tau)\ d\tau = \int_0^t x_1(\tau)\ x_2(t-\tau)\ d\tau,$$

where the inverse transforms of $\bar{x}_1(p)$ and $\bar{x}_2(p)$ are $x_1(t)$ and $x_2(t)$ respectively.[3]

As an example of the use of this theorem, the inverse transform of $\frac{1}{p^2\,(p-a)}$ is

$$L^{-1}\left\{\frac{1}{p^2}\cdot\frac{1}{p-a}\right\} = \int_0^t (t-\tau)\ e^{a\tau}\ d\tau = \frac{1}{a^2}\,(e^{at}-1-at),$$

since t and e^{at} are the inverse transforms of $\frac{1}{p^2}$ and $\frac{1}{p-a}$ respectively.

1.34 Differentiation Under the Integral Sign.

Another method of extending the table of transforms is by differentiation under the integral sign with respect to a

[2] Carslaw and Jaeger, Operational Methods in Applied Mathematics, p. 6.

[3] Ruel V. Churchill, Modern Operational Mathematics in Engineering (New York: McGraw-Hill Book Company, Inc., 1944), pp. 36-7.

suitable parameter. This procedure is valid provided that the resulting integral is uniformly convergent.[4] Thus, if the result

$$\int_0^\infty e^{-pt} \sin at \, dt = \frac{a}{p^2+a^2} \cdot$$

is given, the transform of t cos at can be found by differentiating both sides with respect to a. Therefore;

$$\int_0^\infty e^{-pt} \, t \cos at \, dt = \frac{p^2-a^2}{(p^2+a^2)^2} \cdot$$

Again, since

$$\int_0^\infty e^{-pt} \frac{dx}{dt} \, dt = p\bar{x}(p)-x(0),$$

the transform of $t \frac{dx}{dt}$ is found by differentiating both sides with respect to p. That is;

$$\int_0^\infty e^{-pt} \, t \frac{dx}{dt} \, dt = -p \frac{d\bar{x}(p)}{dp} - \bar{x}(p).$$

This result indicates the method used to transform differential equations with variable coefficients which are polynomials in the independent variable.

1.35 Integration Under the Integral Sign.

The defining integral of the Laplace transform may be integrated under the integral sign with respect to p, provided that it is uniformly convergent for some interval of p. Thus,

$$\int_p^\infty \bar{x}(p) \, dp = \int_p^\infty \int_0^\infty e^{-pt} \, x(t) \, dt \, dp = \int_0^\infty \frac{x(t)}{t} \, e^{-pt} \, dt.$$

With this result, differential equations which contain variable coefficients of the form t^{-n} can be transformed by repeated

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[4]Ivan S. Sokolnikoff, Advanced Calculus (New York: McGraw-Hill Book Company, Inc., 1939), p. 356.

integration under the integral sign.[5]

1.4 General Methods of Application of Laplace Transform to the Solution of Linear Differential Equations.

1.41 Partial Fractions Method.

When a given differential equation has been transformed and solved for $\bar{x}(p)$, the problem of completing the solution by finding the inverse transforms of the various terms arises since these terms normally become rather complicated.

The most elementary method used and perhaps the most difficult to apply is that of partial fractions. If $\bar{x}(p)$ is of the form $\frac{g(p)}{h(p)}$, where g(p) and h(p) are polynomials in p with h(p) of higher degree than g(p); and h(p) is factorable, write

$$(1.411) \qquad \frac{g(p)}{h(p)} = \frac{A}{p-a} + \frac{B}{p-b} + \cdots + \frac{N}{p-n},$$

where p-a, p-b, ..., p-n are the factors of h(p) and A, B, ..., N are constants to be determined. If any factors of h(p) are irreducible quadratics, then the numerators of these terms must be of the form $Ap + B$. The solution x(t) will follow directly by obtaining the inverse transforms of the terms in (1.411).

1.42 Fourier-Mellin Theorem.

The Fourier-Mellin theorem states that if

$$\bar{x}(p) = \int_0^\infty e^{-pt}\, x(t)\, dt,$$

where the real part of $p>0$,

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[5] Churchill, Modern Operational Mathematics in Engineering, pp. 52-3.

$$(1.421) \qquad x(t) = \frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda t}\, \bar{x}(\lambda)\, d\lambda.$$

In this line integral, p has been replaced by λ, where λ is a complex number such that the real part of $\lambda = \gamma$, a constant greater than the real part of all the singularities of $\bar{x}(\lambda)$.[6] The line integral in (1.421) may be evaluated by the calculus of residues, since it can be shown that the line integral may be replaced by a closed curve of a semicircular nature, and, furthermore, that the integral taken over the arc portion of the closed curve tends to zero as the radius increases indefinitely.[7] But the integral taken around a closed curve is equal to the sum of its residues multiplied by $2\pi i$.[8] Therefore

$$(1.422) \qquad x(t) = \sum \text{Residues of } \bar{x}(\lambda).$$

To find the residues of $\bar{x}(\lambda)$, the poles must first be found. This is done by equating the denominator of $\bar{x}(\lambda)$ to zero and solving for λ. Use is then made of one of the applicable formulae for determining the residue at a pole depending on the order of the pole. These formulae may be found in the literature.[9] Then the solution follows in view of (1.422).

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[6] Carslaw and Jaeger, Operational Methods in Applied Mathematics, p. 71.

[7] Ibid., pp. 75-6.

[8] Ruel V. Churchill, Introduction to Complex Variables and Applications (New York: McGraw-Hill Book Company, Inc., 1948), p. 118.

[9] Ibid., pp. 122-4.

1.43 Heaviside's Partial Fractions Expansion.

A less tedious method of finding the inverse transform when $\bar{x}(p) = \frac{g(p)}{h(p)}$, is called Heaviside's Partial Fractions Expansion. As before g(p) and h(p) are polynomials in p and the degree of h(p) is at least one higher than that of g(p). A further condition for the validity of this expansion is that all the factors of h(p) must be linear and non-repeating. This expansion may be written as a formula as follows;

$$L^{-1}\left\{\frac{g(p)}{h(p)}\right\} = \sum_{n=1}^{m} \frac{g(a_n)}{h'(a_n)} \; e^{a_n t}.$$

The a_n are the poles of $\bar{x}(p)$ including imaginary poles. An example may serve to clarify its use. Let

$$\bar{x}(p) = \frac{p^2+3}{p(p-3)(p+2)}.$$

For the pole at zero, $g(0) = 3$, and since $h'(p) = 3p^2-2p-6$, $h'(0) = -6$. The first term in the solution is therefore $-\frac{1}{2}$. In a similar manner $4/5\ e^{3t}$ and $7/10\ e^{-2t}$ are found as the other terms corresponding to the poles at 3 and -2 respectively. Therefore, $x(t) = -\frac{1}{2} + 4/5\ e^{3t} + 7/10\ e^{-2t}$.

A refinement of the formula is needed when h(p) contains repeated factors. For the case when h(p) contains a linear factor to the power s, write

$$\bar{x}(p) = \frac{g(p)}{h(p)} \equiv \frac{g(p)}{(p-a)^s\ h\ (p)} \equiv \frac{\varphi(p)}{(p-a)^s}.$$

The inverse transform corresponding to the factor $(p-a)^s$ is[10]

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[10] Churchill, <u>Modern Operational Mathematics in Engineering</u>, pp. 48-9.

$$D = e^{at} \sum_{n=1}^{s} \frac{\varphi^{(s-n)}(a)}{(s-n)!} \frac{t^{n-1}}{(n-1)!} .$$

1.5 Advantages and Disadvantages of Laplace Transform Method.

The particular advantage of the Laplace transform method in dealing with linear differential equations with constant coefficients is that the given differential equation transforms into an algebraic equation which may be much easier to solve.

When the differential equation to be solved involves coefficients containing the independent variable, the advantage of the Laplace transform method is much more restricted. In general, if the degree of the independent variable is lower than the order of the derivative in any term, then the order of the corresponding term in the transformed equation will be lowered by their difference.

CHAPTER II

SECOND ORDER EQUATION WITH CONSTANT COEFFICIENTS.

Let it be required to find the general solution of an equation of the type

$$A\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Cx=0, \tag{2.1}$$

where A, B, and C are real constants, $x(0)=a$, and $x'(0)=b$.

With the aid of a table of transforms, transform each term of equation (2.1), obtaining

$$Ap^2\bar{x}(p)-Aap-Ab+Bp\bar{x}(p)-Ba+C\bar{x}(p)=0.$$ [11]

This is an algebraic equation in $\bar{x}(p)$ for which the solution is

$$\bar{x}(p)=\frac{Ab+a\left[Ap+B\right]}{Ap^2+Bp+C}.$$

Since this is of the form $\frac{g(p)}{h(p)}$, and the degree of h(p) is one greater than that of g(p), it satisfies the conditions for Heaviside's partial fractions expansion. Therefore

$$x(t)=\sum_{n=1}^{m}\frac{g(a_n)}{h'(a_n)}e^{a_n t},$$

where the a_n are the simple poles of $\bar{x}(p)$, including imaginary poles. The poles of $\bar{x}(p)$ are

$$a_1=\frac{-B+\sqrt{B^2-4AC}}{2A} \text{ and } a_2=\frac{-B-\sqrt{B^2-4AC}}{2A},$$

which will be simple provided the discriminant does not vanish. Then

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[11] Carslaw and Jaeger, Operational Methods in Applied Mathematics, p. 4, pp. 257-8, pp. 353-6.

$$(2.2)\qquad x(t) = \frac{2Ab + Ba + a\sqrt{B^2-4AC}}{2\sqrt{B^2-4AC}}\, e^{\left(\frac{-B+\sqrt{B^2-4AC}}{2A}\right)t} - \frac{2Ab + Ba - a\sqrt{B^2-4AC}}{2\sqrt{B^2-4AC}}\, e^{\left(\frac{-B-\sqrt{B^2-4AC}}{2A}\right)t}.$$

Thus $x(t)$ may take any one of three different forms depending upon the value of the discriminant.

If $B^2-4AC > 0$ and is not a perfect square, (2.2) becomes

$$x(t) = ke^{(k_1+k_2)t} - k_0\, e^{(k_1-k_2)t},$$

where k, k_0, k_1, and k_2 are real constants, of which k, k_0, and k_2 will contain radicals.

If $B^2-4AC > 0$ and is a perfect square, (2.2) is

$$x(t) = ke^{k_1 t} - k_0 e^{k_2 t},$$

where k, k_0, k_1, and k_2 are real constants.

If $B^2-4AC < 0$, (2.2) becomes

$$(2.3)\qquad x(t) = ke^{(k_1+k_2)t} - k_0\, e^{(k_1-k_2)t},$$

where k_1 is a real constant and k, k_0, and k_2 are complex constants. Using the Euler relationships

$$\sin\alpha t = \frac{e^{i\alpha t} - e^{-i\alpha t}}{2i} \quad\text{and}\quad \cos\alpha t = \frac{e^{i\alpha t} + e^{-i\alpha t}}{2},$$

(2.3) can be expressed as

$$x(t) = ke^{k_1 t}\sin k_2 t + k_0 e^{k_1 t}\cos k_2 t,$$

where k, k_0, k_1, and k_2 are real constants.

When $\overline{x}(p)$ has a pole of order two, that is, $B^2-4AC = 0$,

$$(2.4)\qquad x(t) = \Big[g(p)\Big]_{p=a_n} te^{a_n t} + \Big[\frac{d}{dp}\, g(p)\Big]_{p=a_n} e^{a_n t},$$

where $a_n = a_1 = a_2 = -\frac{B}{2A}$, so that (2.4) becomes[12]

$$x(t) = \left[Ab + \frac{Ba}{2}\right] te^{\left(\frac{-B}{2A}\right)t} + Aae^{\left(\frac{-B}{2A}\right)t} .$$

This can be expressed as

$$x(t) = e^{k_1 t}\left[kt + k_o\right].$$

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[12] J. C. Jaeger, An Introduction to the Laplace Transformation (London: Methuen and Co. Ltd., 1949), p. 12.

CHAPTER III

CERTAIN FIRST ORDER EQUATIONS

Some special types of linear differential equations of the first order will be discussed.

3.1 Variables Separable.

The general form for this type equation is

$$\frac{dx}{dt} = f(x)\ g(t), \tag{3.11}$$

where $x(0) = a$.

In order to obtain the Laplace transformation of the right hand side of (3.11), $f(x)$ must equal x or some constant. For, from the definition of the Laplace transform, $\bar{x}(p)$ is the transform of $x(t)$, a function of t that represents the unknown solution of (3.11). If the explicit relationship between x and t is known, then the solution of (3.11) is known and further investigation is unnecessary.

If $f(x)$ is a constant, it may be taken as unity and then

$$\frac{dx}{dt} = g(t).$$

Applying the Laplace transform to each side gives

$$p\bar{x}(p) - a = G(p),$$

where $G(p)$ is the transform of $g(t)$, (assumed to exist). Solving for $\bar{x}(p)$ gives

$$\bar{x}(p) = \frac{G(p) + a}{p}.$$

The solution can be found by the method of (1.43).

When $f(x) = x$, $g(t)$ must be a function of t such that the transform of $x\, g(t)$ exists. In order to obtain various forms of $g(t)$ that satisfy this condition, different devices are used and it is necessary to demonstrate their validity.

As stated in (1.34), differentiation under the integral sign with respect to a suitable parameter is valid when the resulting integral is uniformly convergent. Thus, the transforms of xt, xt^2, $\cdots$, xt^n can be obtained from the definition of the Laplace transform.

Similarly, if the defining integral of the Laplace transform is uniformly convergent, it may be integrated under the integral sign to obtain the transform of $\frac{x}{t}$. Repeating the process a sufficient number of times will give the transforms of $\frac{x}{t^2}$, $\frac{x}{t^3}$, $\cdots$, $\frac{x}{t^n}$. Thus, $g(t)$ will be taken as a polynomial in t or in t^{-1}.

Now, if $g(t) = t$,

(3.12) $$\frac{dx}{dt} = xt.$$

Applying the Laplace transform to each side gives

$$p\bar{x}(p)-a = -\frac{d\bar{x}(p)}{dp}.$$

This is a linear differential equation of the first order which has $e^{\frac{p^2}{2}}$ as an integrating factor. Hence, it is less easily solved than (3.12).

If $g(t) = t^2$, the transformed equation is

$$p\bar{x}(p)-a = \frac{d^2\bar{x}(p)}{dp},$$

a second order linear equation with variable coefficients.

In general, if $g(t) = t^n$,

(3.13) $$\frac{dx}{dt} = xt^n,$$

and the transformed equation is

$$p\bar{x}(p)-a = (-1)^n \frac{d^n x(p)}{dp^n} .$$

Thus, the order of the transformed equation will be equal to the degree of the independent variable in (3.13). The same result can be extended to the case when g(t) is any general polynomial in t.

When $g(t) = \frac{1}{t}$,

(3.14) $$\frac{dx}{dt} = \frac{x}{t} .$$

Transforming (3.14) gives

$$p\bar{x}(p)-a = \int_p^{\infty} \bar{x}(p)\ dp,$$

which is an integral equation that is less easily solved than (3.14).

If $g(t) = \frac{1}{t^2}$, the transformed equation is

$$p\bar{x}(p)-a = \int_p^{\infty} dp \int_p^{\infty} \bar{x}(p)\ dp,$$

an integral equation which is more complex than the original differential equation.

For the general case when $g(t) = \frac{1}{t^n}$, the transformed equation becomes

$$p\bar{x}(p)-a = \int_p^{\infty} dp \int_p^{\infty} dp \cdots \int_p^{\infty} \bar{x}(p)\ dp,\ (n \text{ times}),$$

which is of little use in solving the given differential equation.

Therefore the Laplace transform method is applicable to only the simplest case of the variables separable type of linear differential equation.

3.2 Linear Differential Equations of the First Order.

The general form for this type equation can be written

$$\frac{dx}{dt} + f(t)\, x = g(t), \qquad (3.21)$$

where $x(0) = a$.

When $f(t) = c$, a constant,

$$\frac{dx}{dt} + cx = g(t).$$

Transforming this equation yields

$$p\overline{x}(p) - a + c\overline{x}(p) = G(p),$$

where $G(p)$ is the transform of $g(t)$. Solving for $\overline{x}(p)$ gives

$$\overline{x}(p) = \frac{G(p) + a}{p + c}.$$

The solution follows from one of the methods of (1.4).

From the discussion of the previous section, if $f(t)$ is any function of t other than a polynomial in powers of t or t^{-1}, the Laplace transform of $xf(t)$ is either very difficult or impossible to find.

When $f(t) = t$, the transformed equation is

$$p\overline{x}(p) - a - \frac{d\overline{x}(p)}{dp} = G(p),$$

that is,

$$\frac{d\overline{x}(p)}{dp} - p\overline{x}(p) = -a - G(p),$$

which is the same type as the original equation.

If $f(t) = t^2$, the transform of (3.21) is

$$p\overline{x}(p) - a + \frac{d^2\overline{x}(p)}{dp^2} = G(p),$$

or

$$\frac{d^2\overline{x}(p)}{dp^2} + p\overline{x}(p) = G(p) + a.$$

This is of higher order than (3.21) and has variable coefficients.

When $f(t)$ is a polynomial in t, the order of the transformed equation is equal to the degree of $f(t)$. Thus the Laplace transform method is useful only when $f(t) = c$, some constant.

CHAPTER IV

SECOND ORDER EQUATION WITH VARIABLE COEFFICIENTS

The Laplace transform method will be applied to the second order linear differential equation of the form

(4.1) $$t^m \frac{d^2x}{dt^2} + A(t-\alpha)^n \frac{dx}{dt} + C(t-\gamma)^r x = 0,$$

where $0 \leq m, n, r \leq 2$; A, C, α, and γ are fixed constants; $x(0) = a$, and $x'(0) = b$.

4.2 Consider the Case When $m = 2$.

The transformed equation becomes

$$p^2 \frac{d^2\bar{x}(p)}{dp^2} + 4p \frac{d\bar{x}(p)}{dp} + 2\bar{x}(p) + A\left(-\frac{d}{dp}\right)^n \left[p\bar{x}(p)\right] - A\alpha n\left(-\frac{d}{dp}\right)^{n-1}\left[p\bar{x}(p)\right]$$
$$+ \frac{n(n-1)A\alpha^2}{2}\left(-\frac{d}{dp}\right)^{n-2}\left[p\bar{x}(p)\right] + (-1)^{n+1} Aa\alpha^n + C(-1)^r \frac{d^r\bar{x}(p)}{dp^r}$$
$$-Cr\gamma(-1)^{r-1}\frac{d^{r-1}\bar{x}(p)}{dp^{r-1}} + \frac{Cr(r-1)\gamma^2}{2}(-1)^{r-2}\frac{d^{r-2}\bar{x}(p)}{dp^{r-2}} = 0.$$

The transformed equation is of the same order as the original equation. The coefficient of the second order term is a polynomial in p of degree two. Thus (4.1) has not been simplified by taking its Laplace transform.

4.3 Consider the Case When $m = 1$.

When $0 < n \leq 2$, the transformed equation becomes

(4.31) $$-p^2 \frac{d\bar{x}(p)}{dp} - 2p\bar{x}(p) + a + A\left(-\frac{d}{dp}\right)^n \left[p\bar{x}(p)\right] - A\alpha n\left(-\frac{d}{dp}\right)^{n-1}\left[px(p)\right]$$
$$+ \frac{n(n-1)A\alpha^2}{2}\left(-\frac{d}{dp}\right)^{n-2}\left[px(p)\right] + (-1)^{n+1} Aa\alpha^n + C(-1)^r \frac{d^r x(p)}{dp^r}$$
$$-Cr\gamma(-1)^{r-1}\frac{d^{r-1}\bar{x}(p)}{dp^{r-1}} + \frac{Cr(r-1)\gamma^2}{2}(-1)^{r-2}\frac{d^{r-2}\bar{x}(p)}{dp^{r-2}} = 0.$$

This equation will be of the second order if $n = 2$. The coefficient of the second order term will be a polynomial in p. Thus, the Laplace transform method would not be useful under these conditions.

When $n = 1$ and $r = 2$, (4.31) becomes

$$C\,\frac{d^2\bar{x}(p)}{dp^2} - (p^2 + Ap - 2C\gamma)\,\frac{d\bar{x}(p)}{dp} - (2p + A\alpha p + A - C\gamma^2)\,\bar{x}(p) + a(A\alpha + 1) = 0.$$

The coefficient of the second order term is now a constant C, so that the transformation has produced a useful result. If $A\alpha = -2$, and $A = C\gamma^2$, the coefficient of the $\bar{x}(p)$ term will be zero. A substitution may then be used to reduce this second order equation to one of the first order.

When $n = 1$ and $r = 1$, (4.31) is

$$\text{(4.32)}\qquad \frac{d\bar{x}(p)}{dp} + \left(\frac{A\alpha p + 2p + A + C\gamma}{p^2 + Ap + C}\right)\bar{x}(p) - \left[\frac{a(A\alpha + 1)}{p^2 + Ap + C}\right] = 0.$$

This is a linear differential equation of the first order. If $A\alpha = -1$, the constant term will vanish and (4.32) can be solved since the variables are separable. If $\alpha = \gamma = 0$, the numerator of the coefficient of $\bar{x}(p)$ will be an exact differential of the denominator. An integrating factor could then be used to solve (4.32).

For the case when $n = 1$ and $r = 0$, (4.31) becomes

$$\text{(4.33)}\qquad \frac{d\bar{x}(p)}{dp} + \left(\frac{A\alpha p + 2p + A - C}{p^2 + Ap}\right)\bar{x}(p) - \left[\frac{a(A\alpha + 1)}{p^2 + Ap}\right] = 0,$$

which is a linear differential equation of the first order. Again, if $A\alpha = -1$, the constant term will vanish and the variables will be separable.

When $n = 0$, transforming (4.1) gives

$$(4.34)\quad -p^2 \frac{d\bar{x}(p)}{dp} - 2p\bar{x}(p) + a + Ap\bar{x}(p) - Aa + C(-1)^r \frac{d^r\bar{x}(p)}{dp^r}$$

$$-Cr\gamma(-1)^{r-1} \frac{d^{r-1}\bar{x}(p)}{dp^{r-1}} + \frac{Cr(r-1)\gamma^2}{2} (-1)^{r-2} \frac{d^{r-2}\bar{x}(p)}{dp^{r-2}} = 0.$$

Equation (4.34) will be of the second order when $r = 2$:

$$(4.35)\quad C\frac{d^2\bar{x}(p)}{dp^2} + (-p^2 + 2C\gamma)\frac{d\bar{x}(p)}{dp} + (Ap - 2p + C\gamma^2)\,\bar{x}(p)$$

$$+ a(1-A) = 0.$$

The coefficient of the second order term here is C as compared to t in (4.1). Thus, the transformation has reduced (4.1) to a simpler form. If $A = 2$, and $\gamma = 0$, the $\bar{x}(p)$ term will vanish and a substitution is possible to reduce (4.35) to a linear differential equation of the first order.

If $r = 1$, (4.34) becomes

$$(4.36)\quad \frac{d\bar{x}(p)}{dp} + \left(\frac{2p - Ap + C\gamma}{p^2 + C}\right)\bar{x}(p) + a\left(\frac{A-1}{p^2 + C}\right) = 0.$$

This is a linear differential equation of the first order. When $A = 1$, the constant term vanishes and (4.36) can be solved since the variables are separable.[13] Also, if $A = 2$ and $\gamma = 0$, the $\bar{x}(p)$ term vanishes and the variables are again separable.

When $r = 0$, equation (4.34) becomes

$$(4.37)\quad \frac{d\bar{x}(p)}{dp} + \left(\frac{2p - Ap - C}{p^2}\right)\bar{x}(p) + a\left(\frac{A-1}{p^2}\right) = 0,$$

which is a linear differential equation of the first order.

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[13] Churchill, _Modern Operational Mathematics in Engineering_, pp. 32-3.

If A = 1, the constant term vanishes, and (4.37) may be solved since the variables are separable.

4.4 Consider the Case When m = 0.

When $0 < n \leq 2$, the transformed equation is

$$(4.41) \quad p^2\bar{x}(p) - pa - b + A\left(-\frac{d}{dp}\right)^n\left[p\bar{x}(p)\right] - A\alpha n\left(-\frac{d}{dp}\right)^{n-1}\left[p\bar{x}(p)\right]$$

$$+ \frac{n(n-1)A\alpha^2}{2}\left(-\frac{d}{dp}\right)^{n-2}\left[p\bar{x}(p)\right] + (-1)^{n+1} Aa\alpha^n + C(-1)^r \frac{d^r\bar{x}(p)}{dp^r}$$

$$-Cr\gamma(-1)^{r-1}\frac{d^{r-1}\bar{x}(p)}{dp^{r-1}} + \frac{Cr(r-1)\gamma^2}{2}(-1)^{r-2}\frac{d^{r-2}\bar{x}(p)}{dp^{r-2}} = 0.$$

This equation will be of the second order if n or r equals 2 or if n = r = 2. When n = 2, the coefficient of the second order term will be a polynomial in p. Thus, the transformation is not useful in simplifying the original equation.

If n = 1, (4.41) will be of the second order when r = 2. The coefficient of the second order term in both the original equation and (4.41) will be a constant. Thus, the Laplace transform method has not simplified (4.1). If r = 1, (4.41) becomes

$$(4.42) \quad \frac{d\bar{x}(p)}{dp} + \left(\frac{-p^2 + A\alpha p + C\gamma + A}{Ap + C}\right)\bar{x}(p) + \left(\frac{pa - A\alpha a + b}{Ap + C}\right) = 0,$$

which is a linear differential equation of the first order. The variables will be separable in (4.42) if a = b = 0. When r = 0, (4.41) is

$$\frac{d\bar{x}(p)}{dp} + \left(\frac{-p^2 + A\alpha p + A - C}{Ap}\right)\bar{x}(p) + \left(\frac{pa + b - A\alpha a}{Ap}\right) = 0.$$

This first order differential equation will have the variables separable when $a = b = 0$.[14]

When $n = 0$, transforming (4.1) gives

$$(4.43)\quad p^2\bar{x}(p) - pa - b + Ap\bar{x}(p) - Aa + C(-1)^r \frac{d^r\bar{x}(p)}{dp^r}$$

$$-Cr\gamma(-1)^{r-1}\frac{d^{r-1}\bar{x}(p)}{dp^{r-1}} + \frac{Cr(r-1)\gamma^2}{2}(-1)^{r-2}\frac{d^{r-2}\bar{x}(p)}{dp^{r-2}} = 0.$$

This will be a second order differential equation when $r = 2$. The coefficients of the second order terms will be constants in both (4.1) and (4.43). Therefore, the transformation has not simplified the original equation. If $r = 1$, (4.43) becomes

$$(4.44)\quad \frac{d\bar{x}(p)}{dp} + \left(\frac{-p^2 - Ap + C\gamma}{C}\right)\bar{x}(p) + \left(\frac{pa + b + Aa}{C}\right) = 0,$$

which is a linear differential equation of the first order. When $a = b = 0$, the variables will be separable in (4.44). Setting $r = 0$ reduces (4.1) to a linear differential equation of the second order with constant coefficients.

It should be emphasized that completing the solution for $\bar{x}(p)$ in any of the above transformed differential equations does not constitute the final solution. After solving for $\bar{x}(p)$, it is necessary to find the inverse transforms of both sides of the equation involving $\bar{x}(p)$. This process is often accomplished by the use of infinite series.

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[14] Churchill, Modern Operational Mathematics in Engineering, pp 31-2.

CHAPTER V

TABULAR SUMMARY FOR $t^m \frac{d^2x}{dt^2} + A(t-\alpha)^n \frac{dx}{dt} + C(t-\gamma)^r x = 0$

m	n	r	Result of Applying Laplace Transform	m	n	r	Result of Applying Laplace Transform
0	0	1	a	1	1	1	a
0	0	2	c	1	1	2	b
0	1	0	a	1	2	1	c
0	2	0	d	1	2	2	c
0	1	1	a	2	0	0	c
0	1	2	c	2	0	1	c
0	2	1	d	2	0	2	c
0	2	2	d	2	1	0	c
1	0	0	a	2	2	0	c
1	0	1	a	2	1	1	c
1	0	2	b	2	1	2	c
1	1	0	a	2	2	1	c
1	2	0	c	2	2	2	c

a-Reduction to first order equation
b-Reduces m by one
c-No simplification in either order or m
d-Increases m by one

CHAPTER VI

TWO EXAMPLES

Consider the second order equation (4.1) with $m = 1$, $n = 1$, and $r = 0$: $t\,\frac{d^2x}{dt^2} + A(t-\alpha)\,\frac{dx}{dt} + Cx = 0$. The transform of this equation is

$$\frac{d\bar{x}(p)}{dp} + \left(\frac{A\alpha p + 2p + A - C}{p^2 + Ap}\right)\bar{x}(p) - \left[\frac{a(A\alpha + 1)}{p^2 + Ap}\right] = 0.$$

When $A\alpha = -1$, the last term is zero so that this becomes

$$\frac{d\bar{x}(p)}{dp} + \left(\frac{A\alpha p + 2p + A - C}{p^2 + Ap}\right)\bar{x}(p) = 0.$$

Let $A = 1$ and $C = 2$ to obtain

$$\frac{d\bar{x}(p)}{dp} + \frac{(p-1)}{(p^2 + p)}\,\bar{x}(p) = 0.$$

Then

$$\frac{d\bar{x}(p)}{dp} = -\frac{(p-1)}{(p^2 + p)}\,dp.$$

Integrating this gives

$$\log \bar{x}(p) = -\tfrac{1}{2}\log(p^2 + p) + 3/2\,\log\left(\frac{p}{p+1}\right),$$

and therefore

$$\bar{x}(p) = \frac{p}{(p+1)^2}.$$

Taking the inverse transforms of both sides by the method of (1.43) gives the solution

$$x(t) = c^{-t}\,(1-t).$$

For the case when $m = n = r = 1$, (4.1) is $t\,\frac{d^2x}{dt^2} + A(t-\alpha)\,\frac{dx}{dt} + C(t-\gamma)\,x = 0$. Transforming this gives

$$\frac{d\bar{x}(p)}{dp} + \left(\frac{A\alpha p + 2p + A + C\gamma}{p^2 + Ap + C}\right)\bar{x}(p) - \left[\frac{a(A\alpha + 1)}{p^2 + Ap + C}\right] = 0.$$

Let $A = C = 2$ and $A\alpha = C\gamma = -1$, and the transformed equation becomes

$$\frac{d\bar{x}(p)}{dp} + \left(\frac{p+1}{p^2 + 2p + 2}\right)\bar{x}(p) = 0,$$

since the constant term vanishes. Separating variables and integrating gives

$$-\log \bar{x}(p) = \tfrac{1}{2}\log(p^2 + 2p + 2) + \log \frac{1}{k},$$

where $\log \frac{1}{k}$ is the constant of integration. Therefore

$$\bar{x}(p) = k(p^2 + 2p + 2)^{-\frac{1}{2}}.$$

This can be expressed as

$$\bar{x}(p) = \frac{k}{p}\left(1 + \frac{2p+2}{p^2}\right)^{-\frac{1}{2}}.$$

Making use of the binomial expansion,

$$\bar{x}(p) = k\left(\frac{1}{p} - \frac{1}{p^2} + \frac{1}{2p^3} + \frac{1}{2p^4} - \frac{39}{24p^5} + \cdots\right).$$

Taking inverse transforms of both sides gives the solution

$$x(t) = k\left(1 - t + \frac{t^2}{4} + \frac{t^3}{12} - \frac{39t^4}{576} + \cdots\right).$$

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