BOUND GRAPHS RESULTING FROM CERTAIN DIGRAPHS AND PARTIALLY ORDERED SETS

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ABSTRACT

The competition graph of a digraph was first defined in 1968 by Cohen in the study of ecosystems. The competition graph essentially relates any two species which have a common prey.

In this work, a competition-common enemy graph of a digraph is defined and studied. As the term suggests, it relates any two species which have a common prey and a common enemy. Results analogous to those found for competition graphs are obtained.

Since strict double bound graphs are competition-common enemy graphs of transitive, acyclic digraphs, these graphs, as well as double bound graphs in general, are investigated. Some characterizations of double bound graphs analogous to those of upper bound graphs found by Myers in 1982 are proved.

Characterizing digraphs whose competition graph is interval is a problem that has been studied by Lundgren and Maybee, and Roberts and Steif, among others. This problem is addressed in a special sense: partially ordered sets whose upper bound graph is interval.

ACKNOWLEDGEMENT

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PREFACE

It will be helpful to summarize in this short section the basic definitions and notation which will be used throughout this work. All sets considered are finite.

A **graph** $G = (V,E)$ is a nonempty set V and a set E of two element subsets of V. V is called the **vertex set** of G and elements of V are referred to as **vertices.** E is called the **edge set** of G and elements of E are called **edges.** For two vertices x and y, if $\{x,y\} \in E$, then $\{x,y\}$ will be denoted hereafter as xy or yx. Moreover, if $xy \in E$, then x and y are said to be **adjacent.** The set of all vertices adjacent to x is denoted $Adj(x)$. The reader should be aware that at times $V(G)$ and $E(G)$ will be used to refer to the vertex set and edge set of G, respectively.

An **induced subgraph** of a graph $G = (V,E)$ is a graph $H = (V(H),E(H))$ where $V(H)$ is a nonempty subset of V and $xy \in E(H)$ if and only if $x,y \in V(H)$ and $xy \in E$. Henceforth, the mention of subgraph will be taken to be an induced subgraph. A **complete subgraph** of a graph $G = (V,E)$ is a subgraph $C \subseteq V$ such that $xy \in E$ for all $x,y \in C$ with $x \neq y$.

A **clique** of a graph $G = (V,E)$ is a maximal complete subgraph of G with respect to set inclusion. An **independent** set of a graph G is a subgraph $I \subseteq V$ such that $xy \notin E$ for all $x,y \in I$.

A family $\mathcal{C} = \{C_1,C_2,\ldots,C_n\}$ of subgraphs of a graph $G = (V,E)$ is said to **edge cover** G if for each $xy \in E$, there exists $C_i \in \mathcal{C}$ such that

$x,y \in V(C_i)$. If $\mathcal{C}$ is a family of cliques that edge cover G then $\mathcal{C}$ is called an **edge clique cover** of G. If $\mathcal{C}$ is an edge clique cover of G and there is no other edge clique cover of G with smaller cardinality, then $|\mathcal{C}|$ is the size of the smallest edge clique cover and this number is denoted $\theta_e(G)$.

A **path** of a graph G is a finite sequence $v_0,v_1,\ldots,v_n$ of vertices of G such that $v_iv_{i+1} \in E$ for each $i = 0,1,\ldots,n-1$. The graph G is said to be **connected** if there exists a path $v = v_0,v_1,\ldots,v_n = u$ between any two vertices, u and v, of G.

The **complement** of a graph $G = (V,E)$ is the graph $G = (V,E)$, where $xy \in \bar{E}$ if and only if $x,y \in V$ and $xy \notin E$. K_n will denote a graph $G = (V,E)$ where $|V| = n$ and $xy \in E$ for all $x \neq y \in V$. A **simple cycle** is a sequence of distinct vertices $v_1,v_2,\ldots,v_n,v_1$ with $v_{i-1}v_i \in E$ for $i = 2,3,\ldots,n$ and $v_1v_n \in E$. A **chordless cycle** is a simple cycle $v_1,v_2,\ldots,v_n,v_1$ such that $v_iv_j \notin E$ for i and j differing by more than 1 mod n. C_n will denote the chordless cycle on n vertices.

A **partial order** will be defined as an irreflexive, transitive binary relation on a nonempty set P and a **partially ordered set** (poset) is a nonempty set together with a partial order defined on it. Let $(P,<)$ be a poset. Then two elements $x \neq y \in P$ are **comparable** if and only if $x < y$ or $y < x$. Otherwise, x and y are not comparable and this is denoted $x||y$. $Q \subseteq P$ is a **chain** if and only if any two elements of Q are comparable. Q is an **antichain** if and only if no two elements are comparable. The **length** of a poset $(P,<)$ is the maximum cardinality of a chain in P. The **height** of a poset $(P,<)$ is the length minus one. An element $x \in P$ is **maximal (minimal)** if and only if there does not exist a $y \in P$ with $x < y$ ($y < x$). The **converse** of a poset $(P,<)$ is the poset $(P,<^*)$ where $x <^* y$ in $(P,<^*)$ if and only if $y < x$ in $(P,<)$. If $(P,<)$ is a poset and a graph $G = (P,E)$ is defined from $(P,<)$ by some edge rule determined by the relation $<$ on P, then we say P **realizes** G. A **connected poset** will be a poset whose Hasse diagram is a connected graph.

A **digraph** $D = (V,A)$ is a nonempty set V and a set A of ordered pairs from V. V is called the vertex set of D and elements of V are referred to as vertices. A is the edge set (or arc set) of D and elements of A are called arcs. For two vertices x and y, if $(x,y) \in A$, then henceforth (x,y) will be denoted $\overrightarrow{xy}$.

Let $D = (V,A)$ be a digraph. Define the graph $G = (V,E)$ where $xy \in E$ if and only if $x \neq y \in V$ and $\overrightarrow{xz}$, $\overrightarrow{yz} \in A$ for some $z \in V$. G is called the **competition graph** of D. Competition graphs were introduced by Cohen [1] and studied extensively in [2]. For a digraph $D = (V,A)$, define the graph $G = (V,E)$ where $xy \in E$ if and only if $x \neq y \in V$ and $\overrightarrow{wx}$, $\overrightarrow{wy}$, $\overrightarrow{xz}$, $\overrightarrow{yz} \in A$ for some $w,z \in V$. G is called the **competition-common enemy** (CCE) graph of D. Chapter 1 deals with CCE graphs and results analogous to those found for competition graphs are proved.

The reader should refer to [5] or [6] for the definition of any other graph theoretic notation or terms found in this paper.

CHAPTER 1 COMPETITION-COMMON ENEMY GRAPHS

In 1968 Cohen [1] introduced competition graphs associated with food web models of ecosystems. In this chapter analogous results for competition-common enemy graphs are studied.

Section 1.1 The Double Competition Number

In 1978 Roberts [14] observed that starting with any graph G, a competition graph is obtained by adding sufficiently many isolated vertices to G. Following this observation, it was natural for him to define k(G), the competition number of G, to be the smallest integer k such that $G \cup I_k$ is a competition graph of an acyclic digraph, where I_k is a set of isolated vertices added to G.

Now analogously define dk(G), the double competition number of G, to be the smallest integer k such that $G \cup I_k$ is a CCE graph of an acyclic digraph, where I_k is again a set of k isolated vertices added to G. Note that dk(G) is well-defined since given any graph G, a CCE graph arising from an acyclic digraph can be constructed as follows.

For each edge $\alpha = xy$ in G, add a pair of isolated vertices $\{x_\alpha, y_\alpha\}$ to G. Then define the digraph D such that $V(D) = V(G) \cup \{x_\alpha, y_\alpha : \alpha \in E(G)\}$ and with arcs from the endpoints x and y of α to vertex y_α and from x_α to endpoints x and y of α. See figure 1.

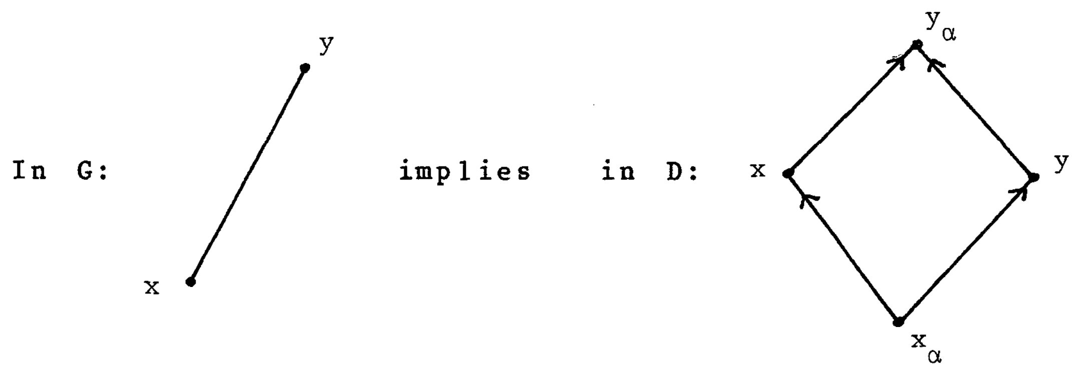


figure 1

The above construction gives an upper bound for dk(G), namely $2 \cdot |E(G)|$. But this can easily be improved. Note that for each edge $\alpha = xy$ in G, there must be a common prey and a common enemy in D. First find k(G) = k, which adds k isolated vertices to G. If D is the digraph with competition graph $G \cup I_k$, then endpoints of each edge in G have a common prey in D. A common enemy results by adding one more isolated vertex to G with arcs in D from it to each original vertex of G. Hence,

$$dk(G) \leq k(G) + 1.$$

For any graph without isolated vertices a lower bound for dk(G) is immediate. As Harary, Norman, and Cartwright proved in [7], if D is any acyclic digraph and $|V(D)| = n$, then integers $1, 2, \ldots, n$ can be assigned to the vertices of D such that every arc goes from a lower number to a higher one. Thus, v_1 has no incoming arcs and v_n has no outgoing arcs. The CCE graph G of D clearly has at least two isolated vertices, namely v_1 and v_n, and thus for graphs without isolated vertices $dk(G) \geq 2$.

The above observations lead to the following two propositions.

Proposition 1. For $G = K_n$, $n \geq 2$, $dk(G) = 2$.

Proof: Let $V(G) = \{v_1, v_2, \dots, v_n\}$. Since G has no isolated vertices, $dk(G) \geq 2$. Add the isolated vertices v_0, v_{n+1} to G and define D such that $V(D) = V(G) \cup \{v_0, v_{n+1}\}$ with $E(D) = \{\overrightarrow{v_0 v_i}, \overrightarrow{v_i v_{n+1}} : i=1,2,\dots,n\}$. D is illustrated in figure 2. Clearly D is acyclic with CCE graph $G \cup I_2$, where $I_2 = \{v_0, v_{n+1}\}$.///

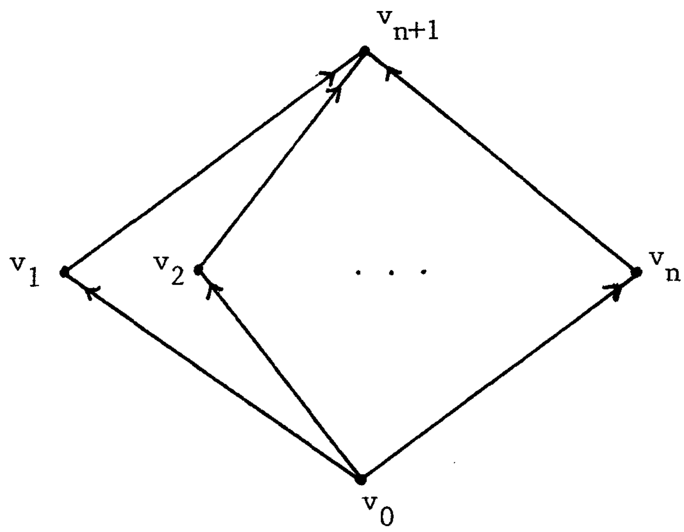
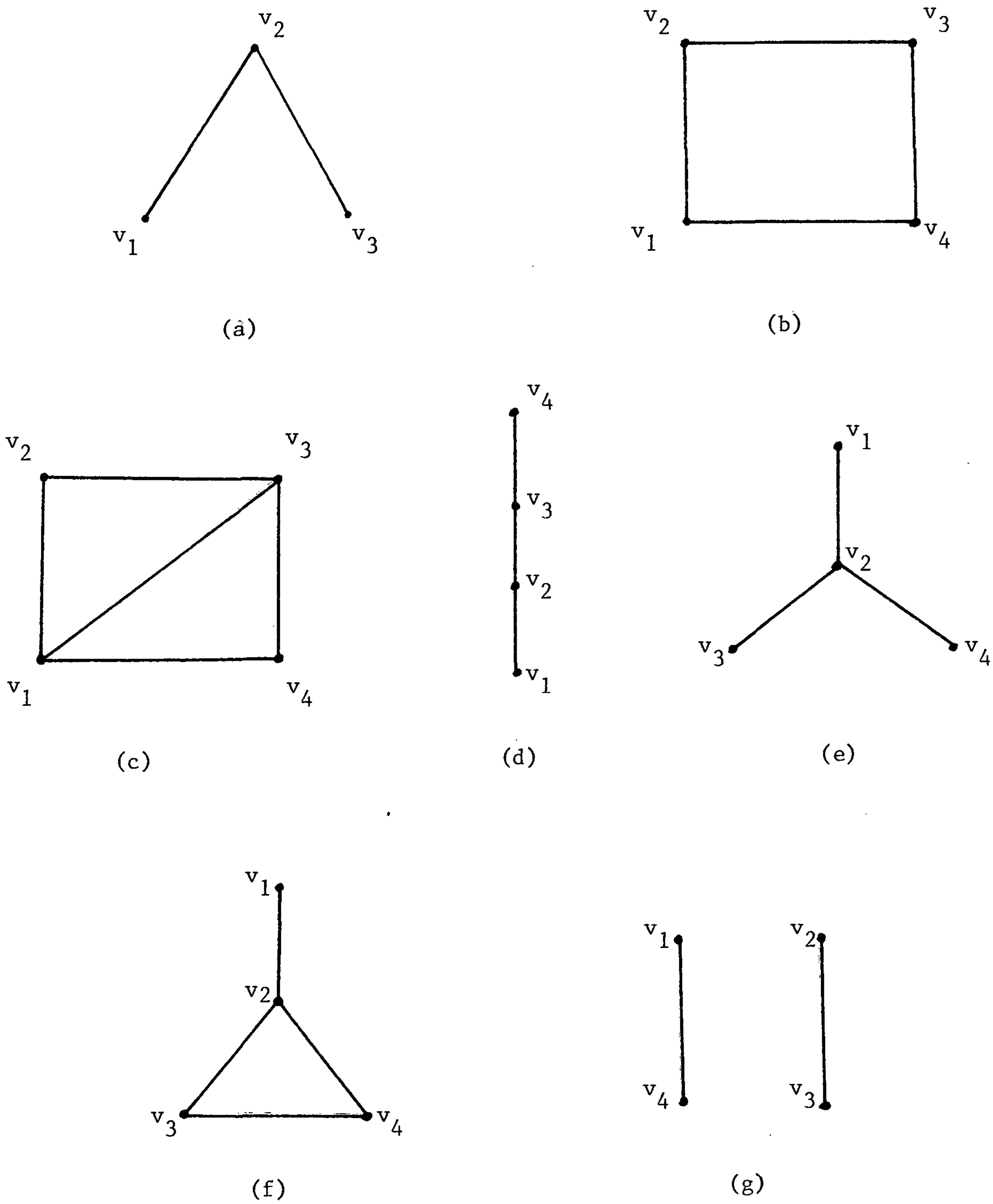


figure 2

Proposition 2. If G is a graph without isolated vertices and $|V(G)| \leq 4$, then $dk(G) = 2$.

Proof: For $G = K_2$, K_3, or K_4 the result follows from Proposition 1. Otherwise G is one of the graphs (a) - (g) in figure 3.

Since G has no isolated vertices, $dk(G) \geq 2$. Add isolated vertices v_0 and v_{n+1} to G. The digraphs in (a) - (g) of figure 4 have CCE graph $G \cup I_2$ ($I_2 = \{v_0, v_{n+1}\}$), with G the graph in (a) - (g) of figure 3 respectively.///


v_2
v_1
v_3
(a)
v_2
v_3
v_1
v_4
(b)
v_2
v_3
v_1
v_4
(c)
v_4
v_3
v_2
v_1
(d)
v_1
v_2
v_3
v_4
(e)
v_1
v_2
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(f)
v_1
v_2
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(g)

figure 3

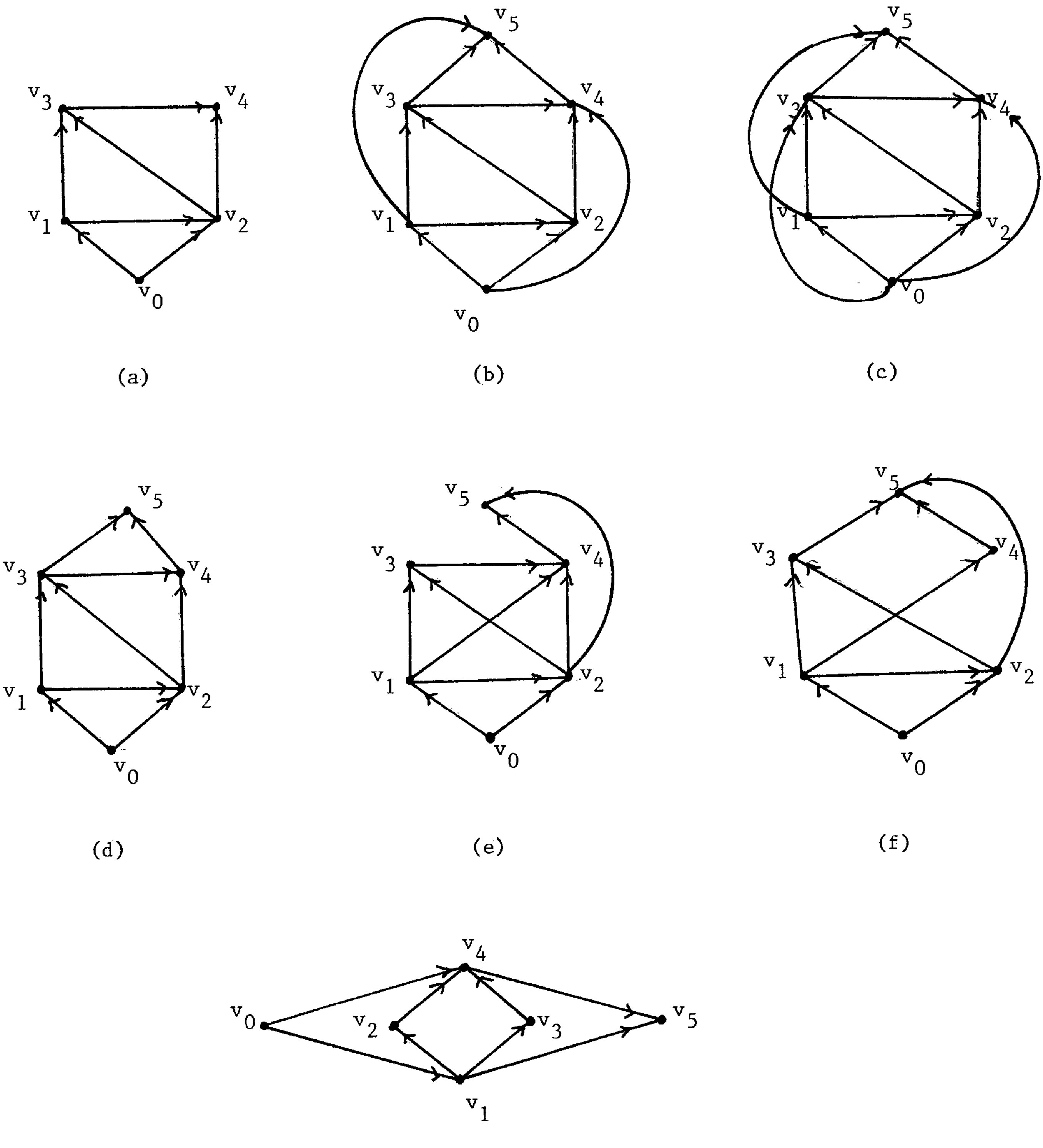


figure 4

Notice that $dk(C_4) = 2$ follows from Proposition 2. Roberts [14] proved that if $G = C_n$, $n > 3$, then $k(G) = 2$. Thus, C_4 is an example where $dk(G) < k(G) + 1$. In general, if $G = C_n$, $n \geq 4$, then $dk(G) = k(G)$. This follows directly from the next theorem. However, $k(C_3) = 1$ while $dk(C_3) = 2$.

Theorem 1. For $G = C_n$, $n \geq 3$, $dk(G) = 2$.

Proof: Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of G such that $E(G) = \{v_i v_{i+1}, v_1 v_n : i = 1,2,\dots,n-1\}$. Since G has no isolated vertices, $dk(G) \geq 2$. Add isolated vertices v_0 and v_{n+1} to G. Define D to be the digraph with $V(D) = V(G) \cup \{v_0, v_{n+1}\}$ and $E(D)=\{\overrightarrow{v_0 v_n}, \overrightarrow{v_1 v_{n+1}}\} \cup \{\overrightarrow{v_i v_{i+1}} : i=0,1,\dots,n\} \cup \{\overrightarrow{v_i v_{i+2}} : i=0,1,\dots,n-1\}$. D is shown in figure 5. The reader can check that D is acyclic and has CCE graph $G \cup I_2$, where $I_2 = \{v_0, v_{n+1}\}$.///

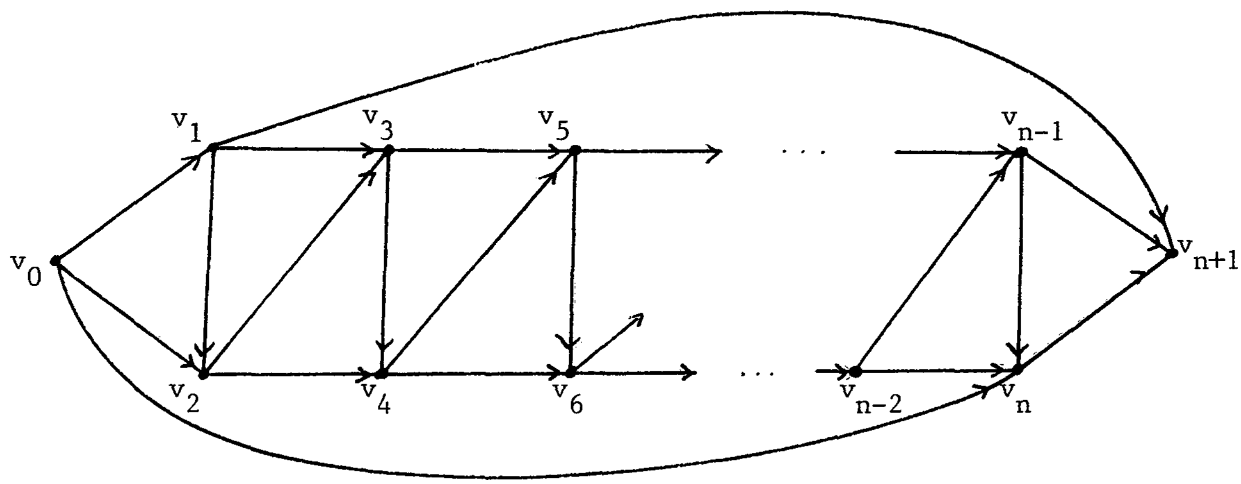


figure 5

Corollary 1. For $G = P_n$, $n \geq 2$, $dk(G) = 2$.

Proof: Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of G such that $E(G) = \{v_i v_{i+1} : i=1,2,\ldots,n-1\}$. Since G has no isolated vertices, $dk(G) \geq 2$. Add isolated vertices v_0 and v_{n+1} to G. Define D to be the digraph with $V(D) = V(G) \cup \{v_0, v_{n+1}\}$ and $E(D) = \{\overrightarrow{v_i v_{i+1}} : i=0,1,\ldots,n\} \cup \{\overrightarrow{v_i v_{i+2}} : i=0,1,\ldots,n-1\}$. D is shown in figure 6. Clearly D is acyclic and has CCE graph $G \cup I_2$, where $I_2 = \{v_0, v_{n+1}\}$.///

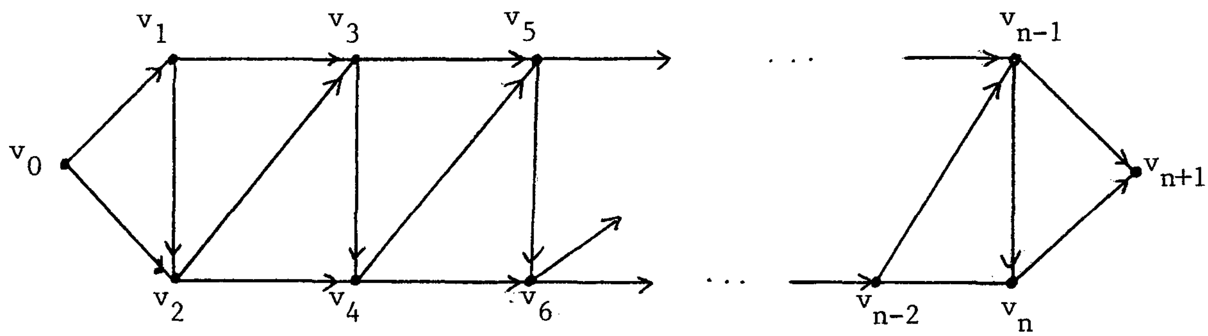


figure 6

It is clear that if $G = C_n \cup I_1$, $n \geq 3$, then $dk(G) = 1$;

if $G = C_n \cup I_2$, $n \geq 3$, then $dk(G) = 0$;

if $G = P_n \cup I_1$, $n \geq 2$, then $dk(G) = 1$;

and if $G = P_n \cup I_2$, $n \geq 2$, then $dk(G) = 0$.

The following theorem of Dutton and Brigham [4] will prove useful.

Theorem 2. G is a competition graph of an acyclic digraph if and only if G has an edge cover by complete subgraphs $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ and a labelling of vertices $v_1, v_2, \ldots, v_n$ such that $v_i \in C_j$ implies $i < j$.

In Theorem 2 complete subgraphs are allowed to be empty or singleton sets.

Theorem 3. Let $G = (V,E)$, $|V| = n$. If G is a competition graph, then $dk(G) \leq 1$.

Proof: Let $\{C_1, C_2, \ldots, C_n\}$ be the edge cover and $v_1, v_2, \ldots, v_n$ be the vertices such that $v_i \in C_j$ implies $i < j$ as given by Theorem 2. Note that G has at least one isolated vertex, namely v_n. Add an isolated vertex v_0 to G and define D as $V(D) = V \cup \{v_0\}$ and $E(D) = \{\overrightarrow{v_i v_j}: v_i \in C_j, j=1,\ldots,n\} \cup \{\overrightarrow{v_0 v_i}: v_i \in V\}$. Clearly D is acyclic and elements of any C_j have common prey v_j and common enemy v_0. Thus, an edge α in G implies a common enemy and common prey in D for endpoints of α.

Now suppose $\overrightarrow{v_i v_b}, \overrightarrow{v_j v_b}, \overrightarrow{v_a v_i}, \overrightarrow{v_a v_j} \in E(D)$, $a < i < j < b$. By definition of arcs in D, $\overrightarrow{v_i v_b} \in E(D)$ implies $v_i = v_0$ or $v_i \in C_b$. If $v_i = v_0$, then $\overrightarrow{v_a v_i}$ is not defined, for there does not exist an a such that $\overrightarrow{v_a v_0} \in E(D)$. It follows that $v_i \in C_b$. Likewise, $\overrightarrow{v_j v_b}$ implies $v_j \in C_b$. Hence, $v_i v_j \in E(G)$. Thus, $G \cup \{v_0\}$ is the CCE graph of D.///

Example 1 below shows that it may actually be the case that $dk(G) = 0$ if G is a competition graph with more than one isolated vertex.

Example 1. Consider the graph $G = C_4 \cup I_2$. It follows from Roberts [14] that G is a competition graph. Moreover, G is also a CCE graph by the remarks following Corollary 1. Hence, dk(G) = 0.

The following corollary is immediate from Theorem 3.

Corollary 2. dk(G) = k(G) + 1 if one of the following holds:

(i) k(G) = 0 and G has exactly one isolated vertex

(ii) k(G) = 1 and G has no isolated vertices.

The following theorem of Roberts [14] is used to obtain an example where dk(G) < k(G).

Theorem 4. If G is connected, $|V(G)| > 1$, and G has no triangles, then $k(G) = |E(G)| - |V(G)| + 2$.

Note that it follows from Theorem 4 and Corollary 1 that if $G = P_n$, $n \geq 2$, dk(G) = k(G) + 1.

Example 2. Let G be the graph in figure 7(a). G satisfies the conditions of Theorem 4 and thus k(G) = 6-5+2=3. However, the digraph shown in figure 7(b) has CCE graph $G \cup I_2$, where $I_2 = \{v_0, v_6\}$. Thus, dk(G) < k(G).

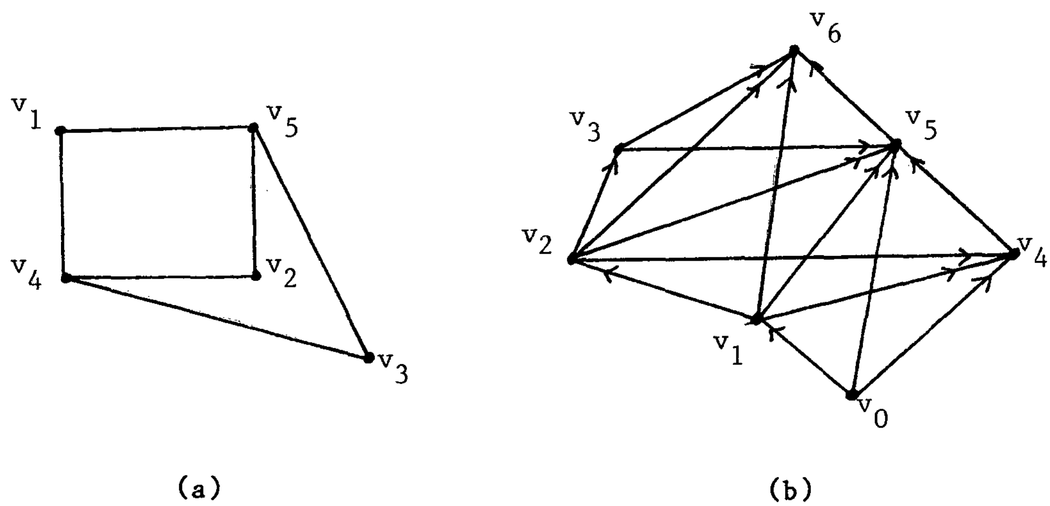


figure 7

The following corollaries are immediate from Theorem 3 and Corollaries 3 and 4 of Roberts [14].

Corollary 3. Every chordal graph G has dk(G) $\leq$ 2.

Corollary 4. Every interval graph G has dk(G) $\leq$ 2.

From a corollary of Roberts [14], it follows dk(G) = 2 for a tree G.

Corollary 5. If G is a tree, then dk(G) = 2.

I have not yet found a graph for which dk(G) > 2. In searching for such a graph G, I have looked mainly at connected graphs without triangles. For these graphs, the largest complete subgraph is an edge and hence, each edge needs a distinct pair associated with it as a common enemy and common prey in D. Moreover, for these graphs $k(G) = |E(G)| - |V(G)| + 2$.

The following graphs in (a) - (e) of figure 8 are examples where $k(G) = 3,4,5,6$ and 7, respectively. The digraphs in (a)-(e) of figure 9 have CCE graph $G \cup I_2$, G the graph in (a)-(e) of figure 8, respectively.

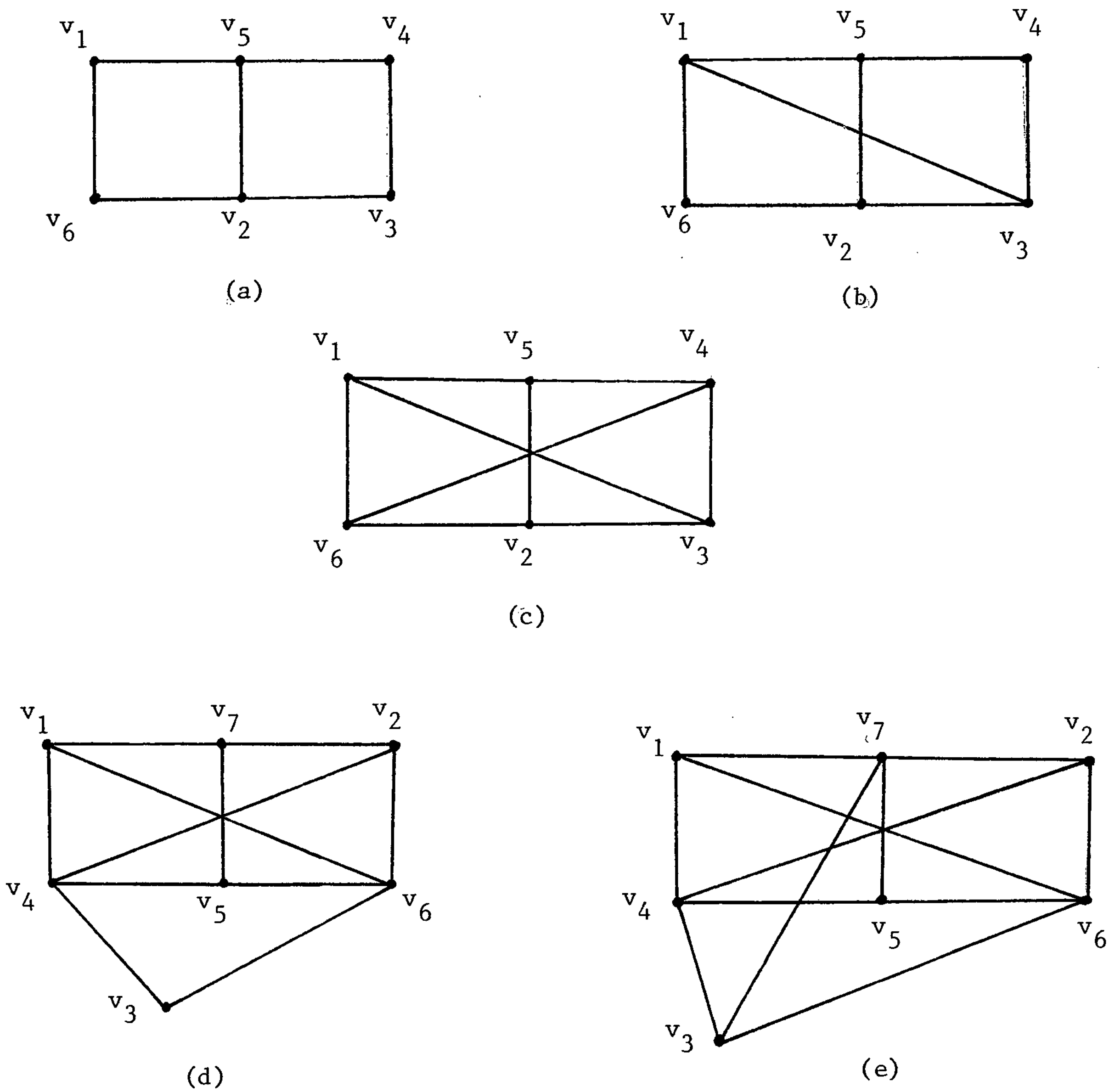
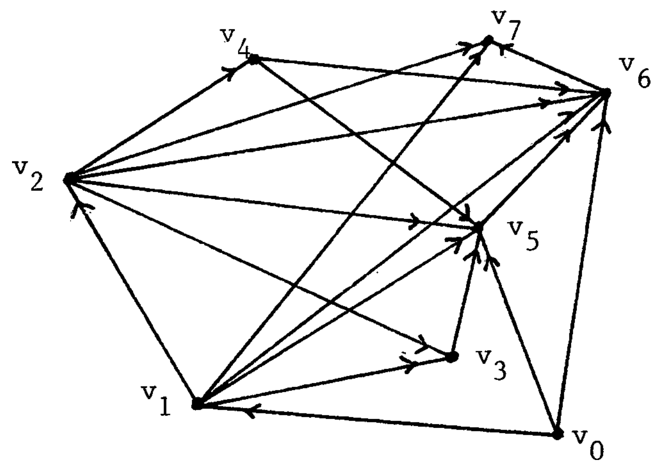


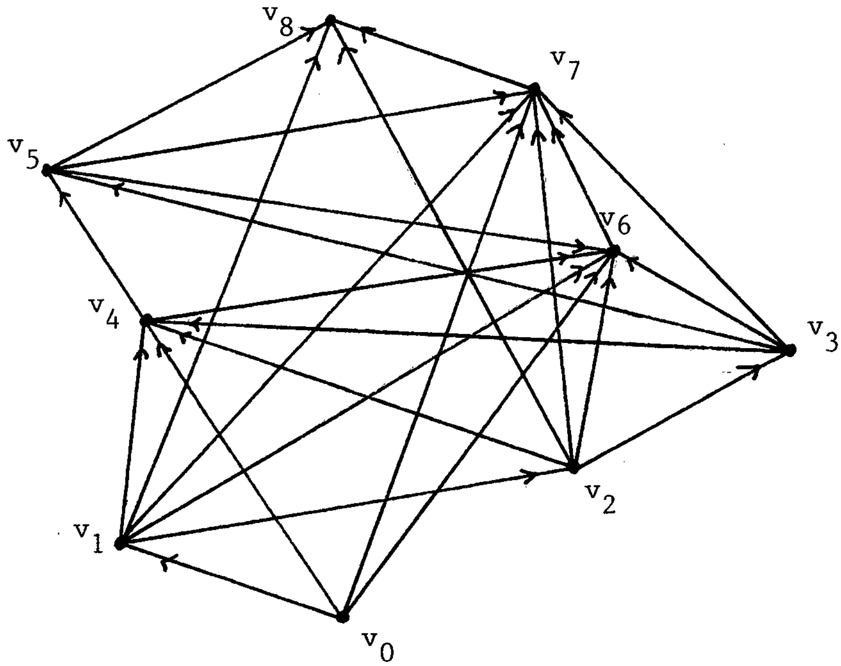
figure 8



(a)

(b) Add $\overrightarrow{v_0v_3}$ to the digraph in (a).

(c) Add $\overrightarrow{v_4v_7}$ to the digraph in (b).



(d)

(e) Add $\overrightarrow{v_3v_8}$ to the digraph in (d).

figure 9

Section 1.2 Competion Number and Double Competion Number for Transitive Acyclic Digraphs

<u>Definition 1</u>. Let $k_t(G)$, the **transitive competition number**, be the smallest integer k such that $G \cup I_k$ is a competition graph of a transitive acyclic digraph, where I_k is a set of k isolated vertices added to G.

Define $dk_t(G)$, the **transitive double competition number,** to be the smallest integer k such that $G \cup I_k$ is a CCE graph of a transitive acyclic digraph, where I_k is a set of k isolated vertices added to G.

In this section, graphs with $k_t(G) = k$ ($k \geq 0$) are characterized and examples are given of graphs G with $dk_t(G) > 2$. First, recall some definitions.

Let X be a nonempty set with a partial order < defined on it. Define the following graphs associated with the poset (X,<). The **upper bound graph** (UB-graph) is the graph $U = (X,E(U))$ where $xy \in E(U)$ if and only if $x \neq y$ and there exists an $m \in X$ such that $x,y \leq m$. We say a graph G is a UB-graph if there exists a poset whose upper bound graph is isomorphic to G. The **strict upper bound graph** corresponding to the poset (X,<) has vertex set X and an edge between $x \neq y$ in X if and only if there exists $m \in X$ such that $x,y < m$.

The **double bound graph** (DB-graph) of a poset (X,<) is the graph $D = (X,E(D))$ where $xy \in E(D)$ if and only if $x \neq y$ and there exist $m,n \in X$ such that $n \leq x,y \leq m$. We say that a graph G is a DB-graph if there exists a poset whose double bound graph is isomorphic to G. The **strict double bound graph** corresponding to the poset (X,<) has vertex set X and an edge between $x \neq y$ in X if and only if there exist $m,n \in X$ such that $n < x,y < m$.

As observed by Roberts [15], the strict upper bound graph of a poset $(X,<)$ is the competition graph of the transitive, acyclic digraph corresponding to the partial order. ($x < y$ in $(X,<)$ corresponds to $\overrightarrow{xy}$ in the digraph.) Similarly, the strict double bound graph of a poset $(X,<)$ is the competition-common enemy graph of the transitive, acyclic digraph corresponding to the partial order. With these observations, the following two facts are evident.

Fact 1. A graph G is a strict UB-graph if and only if $k_t(G) = 0$.

Fact 2. A graph G is a strict DB-graph if and only if $dk_t(G) = 0$.

Strict UB-graphs have been characterized by McMorris and Zaslavsky [11]. The characterization is given below in Theorem 5.

Theorem 5. The graph $G = (V,E)$ is a strict UB-graph if and only if there exists a family $\mathcal{C} = \{C_1,C_2,\ldots,C_m\}$ of cliques that edge covers G and $V = C_1 \cup C_2 \cup \ldots \cup C_m \cup \overline{K}_n$ for some $n \geq m$, where $\overline{K}_n$ has no vertices in common with any C_i.

Theorem 5 above will be used to determine $k_t(G)$ for any graph G. Let $\mathcal{C} = \{C_1,C_2,\ldots,C_m\}$ be an edge clique cover for a graph G such that $\Theta_e(G) = m$. Then G must have at least m isolated vertices to be a strict UB-graph.

Theorem 6. Let $G = (V,E)$ be a graph, and $\mathcal{C} = \{C_1,\ldots,C_m\}$ be an edge clique cover of G where $\theta_e(G) = m$, and let t be the number of isolated vertices of G. Then $k_t(G) = 0$ if and only if $t \geq m$ and $k_t(G) = k$ $(k > 0)$ if and only if $m-t = k$.

Proof: Suppose $k_t(G) = 0$. Fact 1 implies that G is a strict UB-graph and it follows from Theorem 5 that $t \geq m$. If $k_t(G) = k$ $(k > 0)$, then $G \cup I_k$ is a strict UB-graph. It then follows from Theorem 5 that $t + k \geq m$ or $m - t \leq k$. But by definition of $k_t(G)$, k is the smallest integer such that $G \cup I_k$ is a competition graph of a transitive, acyclic digraph. Thus, $m - t = k$.

The converse follows immediately from the fact that $\theta_e(G) = m$.///

In [11], strict DB-graphs were mentioned but not characterized. A characterization for strict DB-graphs might lead to a characterization of graphs for which $dk_t(G) = k$, $k > 0$. However, Theorem 7 below shows that any graph G with a sufficient number of isolated vertices is a strict DB-graph.

Theorem 7. Let $G = (V,E)$ be a graph. If there exists a family $\mathcal{C} = \{C_1,\ldots,C_m\}$ of cliques that edge covers G and $V = C_1 \cup \ldots \cup C_m \cup \overline{K}_n$, where $n \geq m + 1$ and $\overline{K}_n$ has no vertices in common with any C_i, then G is a strict DB-graph.

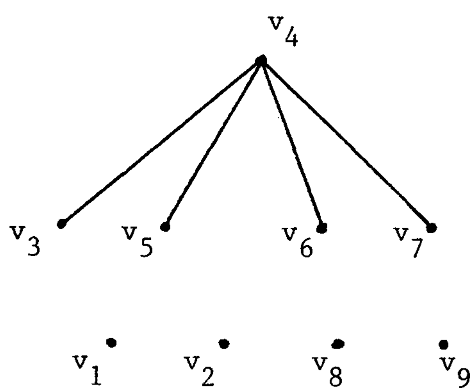
Proof: Assign exactly one vertex $a_i \in \overline{K}_n$ to each clique $C_i \in \mathcal{C}$ and define the poset $(V,<)$ as follows.

For each a_i, define $v < a_i$ for all $v \in C_i$. There is at least one vertex $a \in \overline{K}_n$ remaining. For each i, set $a < v$, for all

$v \in C_i$ (i=1,2,...,m). Clearly, G is the strict DB-graph of (V,<).///

Example 3 illustrates that $n \geq m + 1$ is not necessary for G to be a strict DB-graph, where n and m are as stated in Theorem 7.

Example 3. Let G be the graph in figure 10(a), with $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ where $C_1 = \{v_3, v_4\}$, $C_2 = \{v_4, v_5\}$, $C_3 = \{v_4, v_6\}$, and $C_4 = \{v_4, v_7\}$. $\overline{K}_n = \{v_1, v_2, v_8, v_9\}$ and so m = n = 4. But G is the strict DB-graph of the poset in figure 10(b).



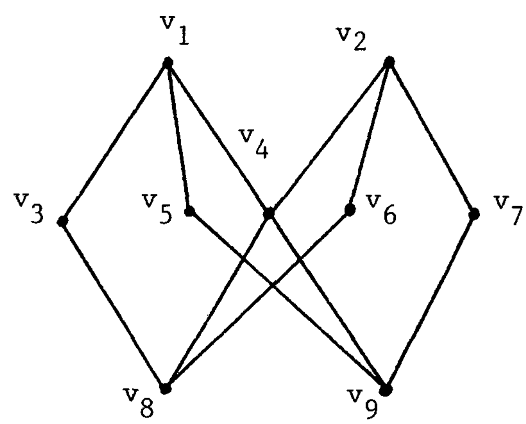
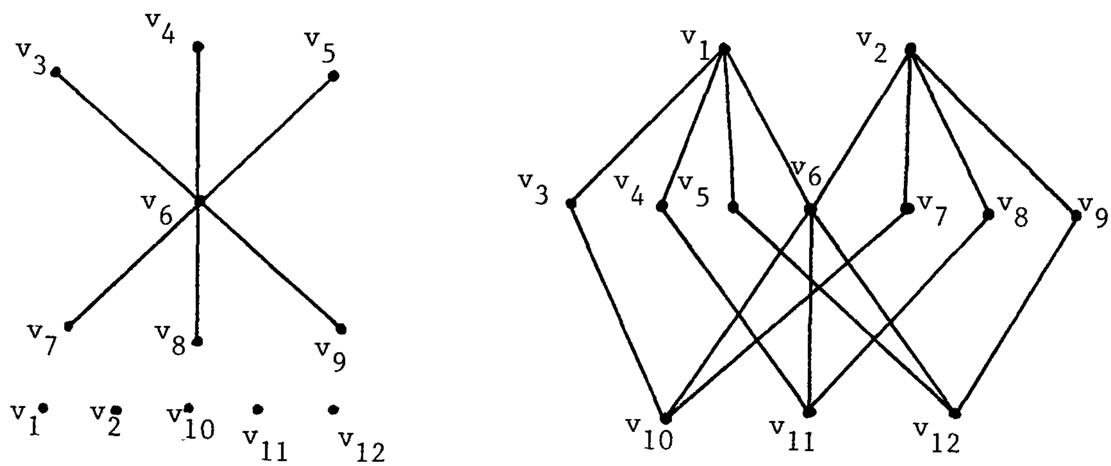


figure 10

In fact, n may be <u>strictly</u> less than m.

Example 4. Let G be the graph in figure 11(a), with $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ where $C_1 = \{v_3, v_6\}$, $C_2 = \{v_4, v_6\}$, $C_3 = \{v_5, v_6\}$, $C_4 = \{v_6, v_7\}$, $C_5 = \{v_6, v_8\}$, and $C_6 = \{v_6, v_9\}$. $\overline{K}_n = \{v_1, v_2, v_{10}, v_{11}, v_{12}\}$ and so $n = 5 < 6 = m$. But G is the strict DB-graph of the poset in figure 11(b).



(a) (b)

figure 11

Let G be a strict DB-graph and $\mathcal{C} = \{C_1,\ldots C_m\}$ be an edge clique cover for G such that $\theta_e(G) = m$. Let $(P,<)$ be any poset realizing G. Throughout the remainder of this section let M be the set of nonisolated maximal elements in $(P,<)$ and N be the set of nonisolated minimal elements. Then it must be the case that $|M|\cdot|N| \geq m$.

Theorem 8. If $G = (V,E)$ is a strict DB-graph, then there exists a family $\mathcal{C} = \{C_1,\ldots,C_m\}$ of cliques that edge covers G and $V = C_1 \cup \ldots \cup C_m \cup \overline{K}_n$, where $\overline{K}_n$ has no vertices in common with any C_i. Furthermore,

(i) $n^2 - 1 \geq 4m$, if n is odd

(ii) $n^2 \geq 4m$, if n is even.

Proof: Recall that given an integer n, if x and y are integers whose sum is n and whose product is a maximum, then

(i) $x = (n-1)/2$, $y = (n+1)/2$, if n is odd

and (ii) $x = y = n/2$, if n is even.

Let G be a strict DB-graph and $(V,<)$ be a poset with strict DB-graph isomorphic to G. Let M be the set of nonisolated maximal elements of V, and let N be the set of nonisolated miminal elements of V. For convenience, let I be the set of isolated vertices of (V,<). For each $x \in M$, $y \in N$ with $y < x$, define $C(x,y) = \{v \in V : y < v < x\}$. Then note that some C(x,y)'s may be empty or singleton sets. However, the collection of these C(x,y)'s clearly edge cover G. Let $\mathcal{C}$ be the collection of those C(x,y) which are maximal complete subgraphs in G. Since no element of M or N is contained in any C(x,y), $M \cup N$ is a set of isolated vertices of G. It follows that $\bar{K}_n = M \cup N \cup I$. Let $x = |M|$, $y = |N|$. Then $x + y \leq n$ and $x \cdot y \geq m$.

<u>Case 1</u>. n is odd.

Then $(n+1)/2 + (n-1)/2 = n$ and $[(n-1)/2] \cdot [(n+1)/2]$ gives the maximum product. Since $x + y \leq n$, it follows $m \leq x \cdot y \leq [(n+1)/2] \cdot [(n-1)/2] = (n^2-1)/4$.

<u>Case 2</u>. n is even.

Then $n/2 + n/2 = n$ and $(n/2)\cdot(n/2)$ gives the maximum product. Since $x + y \leq n$, $m \leq x \cdot y \leq (n/2)\cdot(n/2) \leq n^2/4$.///

Theorems 7 and 8 give upper and lower bounds, respectively, for $dk_t(G)$. Let G be a graph and $\mathcal{C} = \{C_1,\ldots,C_m\}$ be a family of cliques that edge covers G and such that $\theta_e(G) = m$. Then $n \leq dk_t(G) \leq m + 1$, where n is the smallest integer such that $n^2 \geq 4m$. If G′ is the graph in example 3 or 4 without the isolated vertices, then $dk_t(G') = 4$ or 5, respectively. These are cases where the lower bound n is achieved. Example 5 below will be used later to give a graph where the upper bound is assumed.

Example 5. Let G be the graph in figure 12 with $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ where $C_1 = \{v_3, v_4, v_5, v_6\}$, $C_2 = \{v_7, v_8, v_9\}$, $C_3 = \{v_5, v_7, v_8\}$, and $C_4 = \{v_3, v_5, v_6, v_7\}$. Note that $V = C_1 \cup C_2 \cup C_3 \cup C_4 \cup \overline{K}_4$, $n = 4$, $m = 4$, $\theta_e(G) = 4$, and $n^2 = 16 \geq 16 = 4m$. If $(P,<)$ were a poset realizing G, then $|M| \cdot |N| \geq 4$. It follows that $|M| = |N| = 2$. Without loss of generality, set $M = \{v_1, v_2\}$ and $N = \{v_{10}, v_{11}\}$. For G to be a strict DB-graph, to each clique $C_i \in \mathcal{C}$ there must correspond a unique pair $(m,n) \in M \times N$ such that $n < v < m$ for all $v \in C_i$. The possible pairs to correspond to a $C_i \in \mathcal{C}$ are $\{(v_1, v_{10}), (v_1, v_{11}), (v_2, v_{10}), (v_2, v_{11})\}$. The reader can check by exhaustion that in all cases it will happen that $v_7 < v_1, v_2$ and $v_{10}, v_{11} < v_7$ and thus, the DB-graph of any poset would have $v_4 v_7$ as an edge which G does not. Hence, G is not a strict DB-graph.

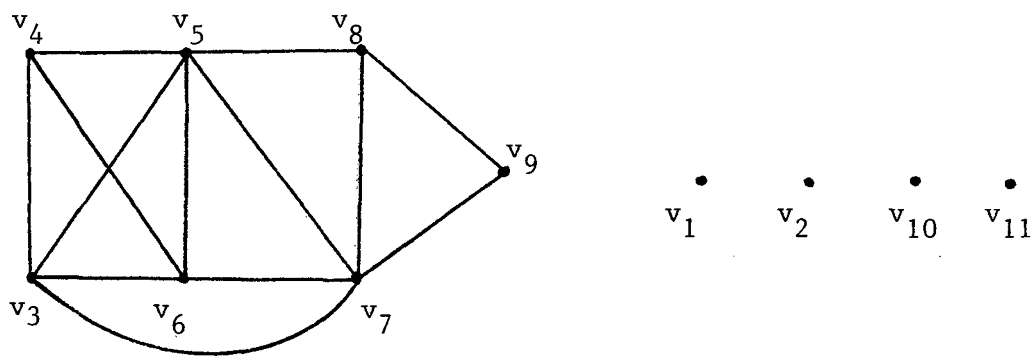


figure 12

In each of examples 3, 4 and 5 above, if $G' = G - \{v \in V : v \text{ is an isolated vertex}\}$, then G' would be an example of a graph for which $dk_t(G) > 2$. Recall that an example for which dk(G) > 2 was not found. The following is yet one more case for which $dk_t(G) > 2$. In a sense, it is the "minimal" such example.

Example 6. Let G be the graph in figure 13(a). An edge clique cover for G is $\mathcal{C} = \{C_1, C_2\}$ where $C_1 = \{v_1, v_2\}$ and $C_2 = \{v_2, v_3\}$. Using Theorem 7, if we add 3 isolated vertices to G, $G \cup I_3$ is surely a strict DB-graph. Theorem 8 implies that if n is the number of isolated vertices added to G, then for G to be a strict DB-graph, the smallest n could possibly be is 3. It follows that $dk_t(G) = 3$ and a transitive, acyclic digraph with CCE graph $G \cup I_3$ is shown in figure 13(b).

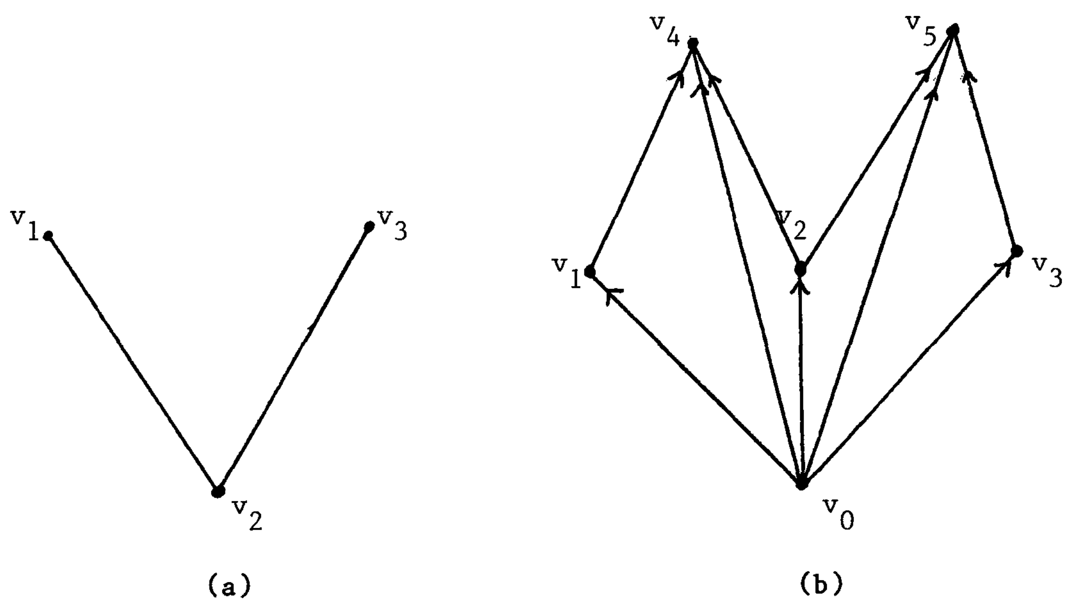


figure 13

Example 7 below gives a graph G′ for which $dk_t(G) = m + 1$, where $\theta_e(G') = m$. Note that the graph in example 6 has $dk_t(G) = m + 1$. Hence, both are graphs for which the upper bound on $dk_t(G)$ is achieved.

Example 7. Let G' be the graph in example 5 without the isolated vertices. Then $\theta_e(G') = 4$ and $\mathcal{C} = \{C_1, V_2, C_3, C_4\}$, where C_1, C_2, C_3, and C_4 are as in example 5, is an edge clique cover of G'. The transitive, acyclic digraph in figure 14 has CCE graph $G' \cup I_5$, where $I_5 = \{v_1, v_2, v_{10}, v_{11}, v_{12}\}$. Since the graph in example 5 with 4 isolated vertices is not a strict DB-graph, it follows that $dk_t(G') > 4$. Thus, $dk_t(G') = 5 = m + 1$.

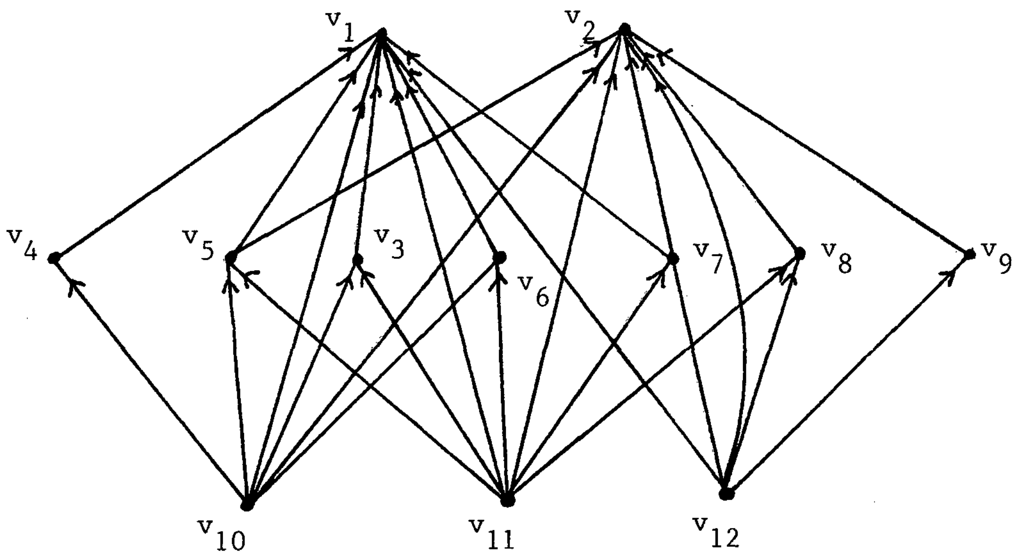


figure 14

The reader should be aware that transitivity played a key role in obtaining graphs G for which $dk_t(G) > 2$.

Section 1.3 CCE Graphs of Various Digraphs

In this section, graphs G for which $dk(G) = 0$ are characterized, that is, graphs which arise as competition-common enemy graphs of acyclic digraphs. Graphs which are CCE graphs of arbitrary digraphs without loops and digraphs in general (loops allowed) are also characterized.

The following theorem is analogous to a result by Dutton and Brigham [4].

Theorem 9. $G = (V,E)$, $|V| = n$, is the CCE graph of an acyclic digraph if and only if G has an edge cover by complete subgraphs $\mathcal{C} = \{C_{ij} : \text{where } 1 \leq i < j \leq n\}$ and a labelling of the vertices $v_1, v_2, \ldots, v_n$ such that the following hold:

(i) $v_k \in C_{ij}$ implies $i<k<j$,

and (ii) For any i and j, define

$$I_i = \bigcup_{b'>i} C_{ib'} \bigcup \{v_b : v_i \in C_{ab},\ a<i<b\}$$

$$J_j = \bigcup_{c'<j} C_{c'j} \bigcup \{v_c : v_j \in C_{cd},\ c<j<d\}$$

If $|I_i \cap J_j| > 1$, then $I_i \cap J_j = C_{ij}$.

Proof: Assume G is the CCE graph of an acyclic digraph D. Then since D is acyclic, the vertices can be labelled $v_1, v_2, \ldots, v_n$ such that $\overrightarrow{v_i v_j} \in E(D)$ implies $i < j$. Define $C_{ij} = \{v_k : \overrightarrow{v_i v_k}, \overrightarrow{v_k v_j} \in E(D)\}$. Let $\mathcal{C}$ be the collection of nonempty, nonsingleton C_{ij}'s. Clearly $\mathcal{C}$ is an edge cover for G and the C_{ij} are complete subgraphs of G. Moreover, condition (i) is satisfied because of the choice of the labelling of vertices in D. It remains to show condition (ii) holds.

Fix i and j and let I_i and J_j be sets as defined in the Theorem. Since $C_{ij} \subseteq I_i$ and $C_{ij} \subseteq J_j$, clearly $C_{ij} \subseteq I_i \cap J_j$. Now assume $|I_i \cap J_j| > 1$ and let $v_k \in I_i \cap J_j$. There are 4 cases which are possible and it follows easily in each case that $I_i \cap J_j \subseteq C_{ij}$ which proves (ii).

Conversely, let G have an edge cover $\mathcal{C} = \{C_{ij} : \text{where } 1 \leq i < j \leq n\}$

by complete subgraphs of G and a labelling of the vertices $v_1, v_2, \ldots, v_n$ such that (i) and (ii) hold. (Observe that G must have at least two isolated vertices, namely, v_1 and v_n by condition (i)). Define D as follows: $V(D) = V$ and $\overrightarrow{v_i v_k}, \overrightarrow{v_k v_j} \in E(D)$, for all $v_k \in C_{ij}$. Condition (i) guarantees that D is acyclic. Since endpoints of every edge in G are contained in some C_{ij} (the C_{ij}'s edge cover G), there is a common prey v_j and a common enemy v_i for vertices of an edge.

Now suppose $\overrightarrow{v_i v_k}, \overrightarrow{v_i v_l}, \overrightarrow{v_k v_j}, \overrightarrow{v_l v_j} \in E(D)$. We show $v_k v_l \in E(G)$. Since $\overrightarrow{v_i v_k}, \overrightarrow{v_i v_l} \in E(D)$, it follows $\{v_k, v_l\} \subseteq I_i$. Likewise, $\overrightarrow{v_k v_j}, \overrightarrow{v_l v_j} \in E(D)$ implies $\{v_k, v_l\} \subseteq J_j$. Thus, $\{v_k, v_l\} \subseteq I_i \cap J_j = C_{ij}$ and this implies $v_k v_l \in E(G)$. Hence, G is the CCE graph of D.///

Note that condition (i) of Theorem 9 ensures that the digraph D constructed as in the proof above will be acyclic. Modifying condition (i) as in Corollary 6 below gives a characterization for CCE graphs of digraphs without loops. This Corollary is analogous to a result by Roberts and Steif [16].

Corollary 6. $G = (V,E)$, $|V| = n$, is the CCE graph of a digraph without loops if and only if G has an edge cover $\mathcal{C} = \{C_{ij} : i,j \in \{1,2,\ldots,n\}\}$ by complete subgraphs and a labelling of the vertices $v_1, v_2, \ldots, v_n$ such that the following hold:

(i) $v_i, v_j \notin C_{ij}$,

and (ii) For any i and j, define

$$I_i = \bigcup_{b'} C_{ib'} \bigcup \{v_b : v_i \in C_{ab}\}$$

$$J_j = \bigcup_{c'} C_{c'j} \bigcup \{v_c : v_j \in C_{cd}\}$$

Then $I_i \cap J_j = C_{ij}$, if $|I_i \cap J_j| > 1$.

Proof: Assume G is the CCE graph of a digraph D without loops. Define for $i \neq j$, $C_{ij} = \{v_k : \overrightarrow{v_i v_k}, \overrightarrow{v_k v_j} \in E(D)\}$. If $\mathcal{C} = \{C_{ij}: |C_{ij}| > 1\}$, then clearly $\mathcal{C}$ edge covers G and the $C_{ij} \in \mathcal{C}$ are complete subgraphs of G. Since D has no loops, $v_i, v_j \notin C_{ij}$ and (i) holds. Condition (ii) is checked similar to that in Theorem 9 above.

The converse follows analogously to the proof of Theorem 9.///

Observe that condition (i) does not exclude the possibility of C_{ii} being a complete subgraph for consider the digraph D in figure 15. D has no loops and has CCE graph G shown in figure 16. Taking $\mathcal{C} = \{C_{11}\}$, where $C_{11} = \{v_2, v_3\}$, the conditions of the Corollary are satisfied.

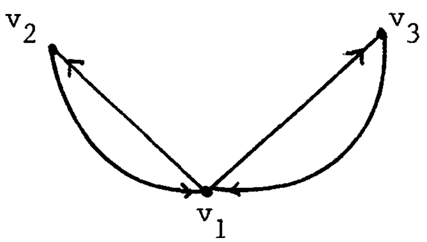


figure 15

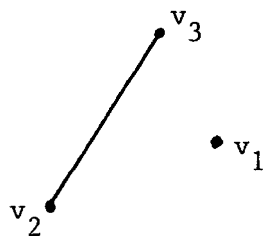


figure 16

Example 8 below shows that different digraphs can be constructed (as in the proof of Corollary 6) with the same CCE graph if different labels are chosen for the complete subgraphs in the edge cover, so long as conditions (i) and (ii) of Corollary 6 are satisfied.

Example 8. Let G be the graph in figure 17. Choose $\mathcal{C} = \{C_{12}, C_{14}, C_{34}, C_{25}, C_{52}, C_{56}\}$, where $C_{12} = \{v_3, v_6\}$,

$C_{14} = C_{34} = \{v_2,v_5\}$, $C_{25} = \{v_3,v_4\}$, $C_{52} = \{v_1,v_6\}$ and $C_{56} = \{v_1,v_4\}$. The reader can check that the conditions of Corollary 6 are satisfied. Constructing the digraph as in the proof of the corollary, the digraph in figure 18 is obtained.

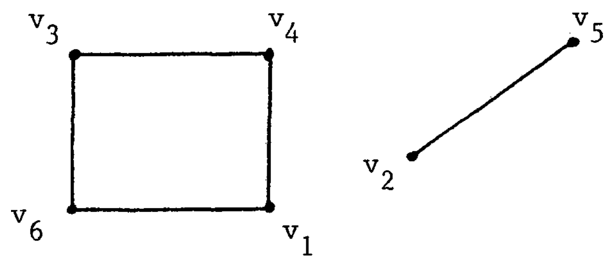


figure 17

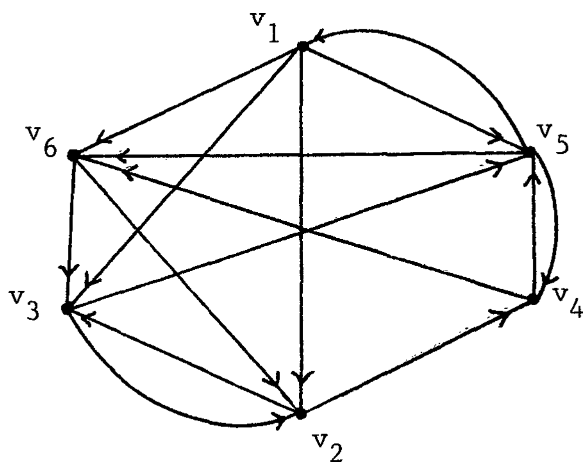


figure 18

Now let $\mathcal{C} = \{C_{16},C_{32},C_{34},C_{52},C_{54}\}$, where $C_{16} = \{v_2,v_5\}$, $C_{32} = \{v_1,v_4\}$, $C_{34} = \{v_1,v_6\}$, $C_{52} = \{v_3,v_4\}$ and $C_{54} = \{v_3,v_6\}$. The conditions of Corollary 6 are again satisfied but this time the digraph in figure 19 is obtained.

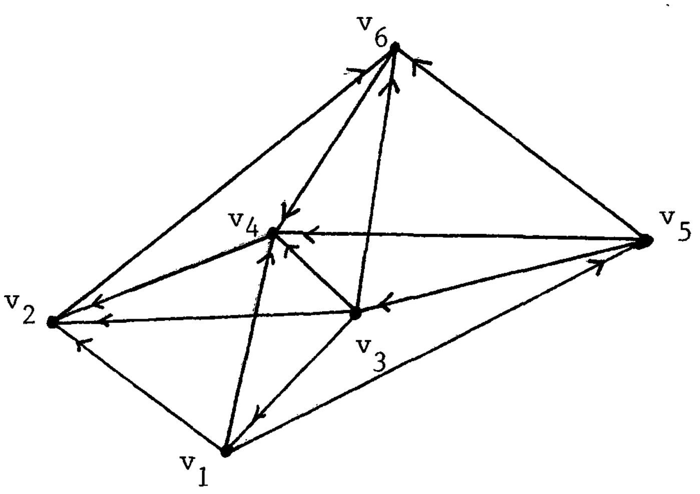


figure 19

A characterization for CCE graphs of arbitrary digraphs, loops allowed, follows immediately from Theorem 9 and Corollary 6.

Corollary 7. $G = (V,E)$, $|V| = n$, is the CCE graph of an arbitrary digraph (loops allowed) if and only if G has an edge cover by complete subgraphs, $\mathcal{C} = \{C_{ij} : i,j \in \{1,\ldots,n\}\}$ and a labelling of the vertices $v_1, v_2, \ldots, v_n$ such that for any $i,j \in \{1,2,\ldots,n\}$, define

$$I_i = (\bigcup_{b'} C_{ib'}) \bigcup \{v_b : v_i \in C_{ab}\}$$

$$J_j = (\bigcup_{c'} C_{c'j}) \bigcup \{v_c : v_j \in C_{cd}\}$$

if $|I_i \cap J_j| > 1$, then $I_i \cap J_j = C_{ij}$.///

A reflexive digraph, a digraph with loops on all the vertices, is such that its CCE graph includes all edges of the digraph (considered undirected in the CCE graph), with possible additional edges. Adding one condition to Corollary 7, a characterization of CCE graphs of reflexive digraphs follows.

Corollary 8. $G = (V,E)$, $|V| = n$, is the CCE graph of a reflexive digraph if and only if there exists a set of complete subgraphs $\mathcal{C} = \{C_{ij} : i,j \in \{1,2,\ldots,n\}\}$ that edge covers G and a labelling of the vertices $v_1, v_2, \ldots, v_n$ such that the following hold:

(i) For all $v_i \in V$, $v_i \in (\bigcup_b C_{ib}) \cup (\bigcup_a C_{ai})$,

and (ii) For any $i,j \in \{1,2,\ldots,n\}$ define

$$I_i = (\bigcup_{b'} C_{ib'}) \cup \{v_b : v_i \in C_{ab}\}$$

$$J_j = (\bigcup_{c'} C_{c'j}) \cup \{v_c : v_j \in C_{cd}\}$$

Then $I_i \cap J_j = C_{ij}$ if $|I_i \cap J_j| > 1$.///

Example 9. The graph G in figure 20 is the CCE graph of each of the digraphs (a) and (b) of figure 21. Notice that the digraph in figure 21(b) is just an orientation of the edges in G with the addition of loops on all the vertices.

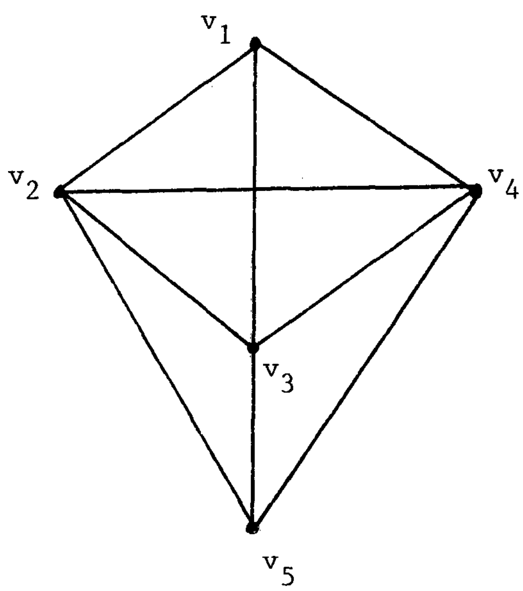
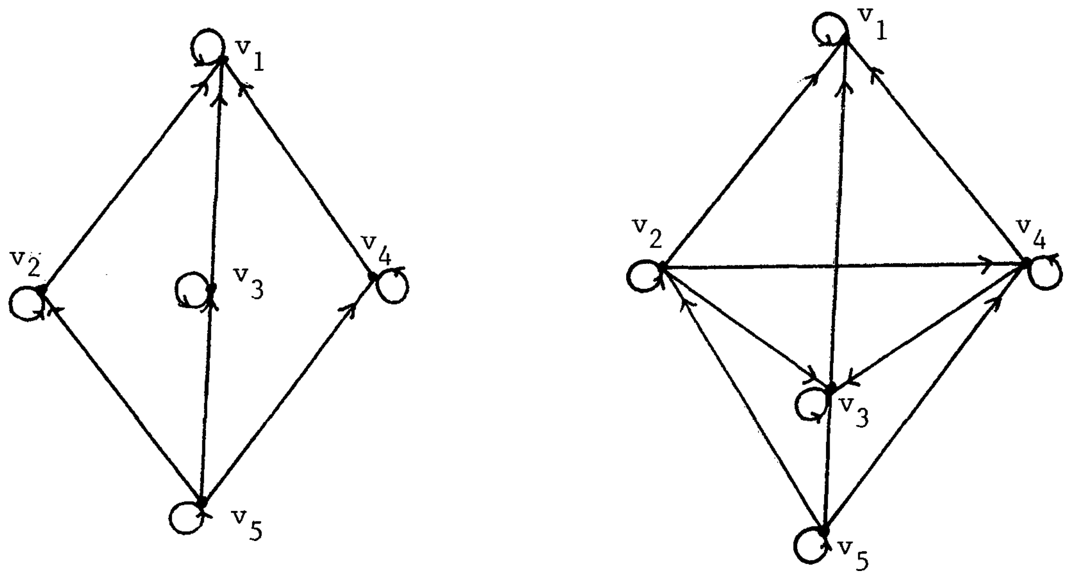


figure 20



(a) (b)

figure 21

The reader may have noticed that the digraph in figure 21(b) of Example 9 is the digraph of figure 21(a) with the addition of an orientation on the remaining edges of G. Given a reflexive digraph D and its CCE graph G, is it always possible to give an orientation to the edges of G such that the edges that appear in both D and G have the same orientation? Example 10 shows that the answer to this question is no.

<u>**Example 10**</u>. The digraph in figure 22(a) has for its CCE graph the graph in figure 22(b). The reader can check that if the edges in G which appear in D are given the orientation of those in D, then the remaining edges in G cannot be oriented so that after adding loops to all vertices it becomes a reflexive digraph with CCE graph G.

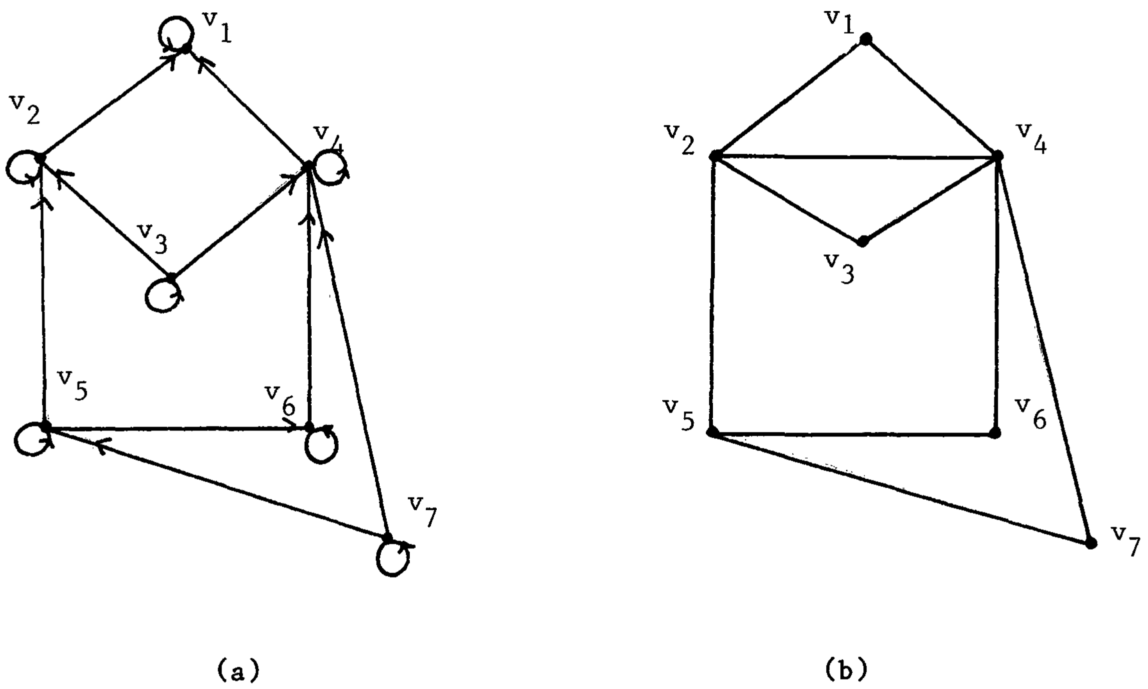


figure 22

However, the edges of G can be oriented such that adding loops to all the vertices yields a reflexive digraph with CCE graph G of figure 22(b). Such a digraph is shown in figure 23.

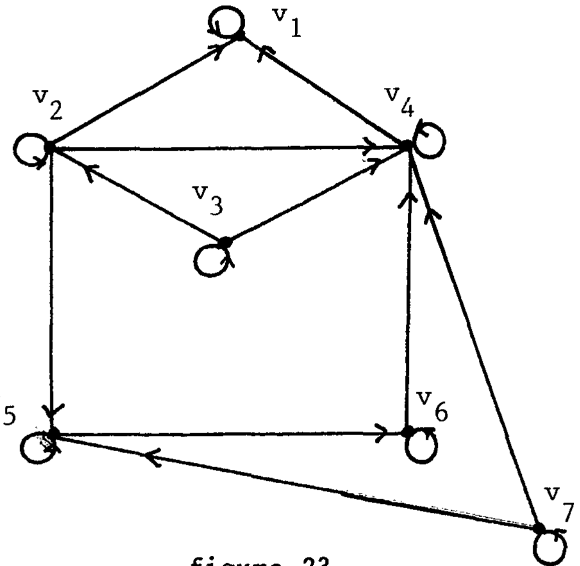


figure 23

Given a graph G, is it possible to orient the edges of G so that adding loops to all the vertices yields a reflexive digraph with CCE graph G? This remains an interesting open problem. A few elementary observations can be made at this time.

Let D be a digraph. Define D to be **class-C_4** if and only if D has an induced subdigraph isomorphic to one of the digraphs of figure 24.

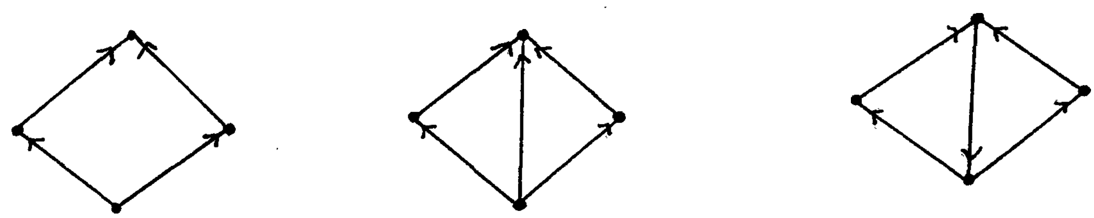


figure 24

Observe that any graph G for which there exists an orientation G' of the edges in G such that G' is not class-C_4 is a CCE graph of a reflexive digraph. The following statements now follow from this observation.

Any graph G with no cycle of length 4 is a CCE graph of a reflexive digraph.

Every tree is the CCE graph of a reflexive digraph.

Note also that $G = K_n$, $n \geq 2$, is a CCE graph of a reflexive digraph.

CHAPTER 2 RESULTS FOR DOUBLE BOUND GRAPHS

Myers [12] characterized UB-graphs of interval orders and semi-orders. In this chapter, some analogous results for DB-graphs are proved. All posets considered will be connected.

Section 2.1 Uniqueness of the DB-clique Cover

Let $G = (V,E)$ be a graph, M and N disjoint independent subsets of V, and $v \in V - (M \cup N)$. Define the following sets as in [3]:

$U(v) = \{x \in M : xv \in E\}$

$L(v) = \{y \in N : yv \in E\}$

and let $u(v) = |U(v)|$, $l(v) = |L(v)|$.

Theorem 10 stated below is proved in [3] and will be referred to throughout this chapter.

<u>Theorem</u> <u>10</u>. A graph $G = (V,E)$ is a DB-graph if and only if there exist a family of cliques $\mathcal{C} = \{C_1,C_2,\ldots,C_n\}$ and disjoint, independent subsets M and N of V such that

(i) $\mathcal{C}$ edge covers G,

(ii) For each C_i, there exist $x_i \in M$, $y_i \in N$ such that

$\{x_i,y_i\} \subseteq C_i$ and $\{x_i,y_i\} \nsubseteq C_j$ for any $j \neq i$, and

(iii) For each $v \in V - (M \cup N)$,

$u(v) \cdot l(v)$ equals the number of cliques of $\mathcal{C}$ containing v.

Furthermore, $\mathcal{C}$ is the unique, minimal edge covering family of cliques in G.

Let G = (V,E) be a DB-graph. Throughout this chapter, M and N will represent sets defined as in Theorem 10 and C(x,y) will denote a clique in G defined by $C(x,y) = \{z : xz, yz \in E\}$, where $x \in M$, $y \in N$ and $xy \in E$. This notation follows that used by McMorris and Zaslavsky [11] and is also found in [3].

Corollary 9 and Lemma 1, which follow, are analogous to the results for UB-graphs proved by Myers in [12].

Corollary 9. Let G = (V,E) be a DB-graph, $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ a family of cliques of G, and M and N disjoint, independent subsets of V satisfying (i)-(iii) of Theorem 10. Then there exists a partial order < on V realizing G that has M for its set of maximal elements and N for its set of minimal elements.///

The Corollary is a direct result of the proof of Theorem 10, but the definition of the partial order < on V will be stated here for convenience.

For each $C_i \in \mathcal{C}$, let $\{x_i, y_i\}$ be a fixed set given by (ii) of Theorem 10. Define the partial order < on V by setting $y_i < x_i$ and $y_i < z < x_i$ for each $z \in C_i - \{x_i, y_i\}$ with no other comparabilities.

Any poset constructed as above will be referred to as a **canonical poset** realizing G.

Lemma 1. Let $G = (V,E)$ be a DB-graph and $\mathcal{C}$, M and N be sets satisfying (i)-(iii) of Theorem 10. Let $(V,<)$ be a poset realizing G, X the set of maximal elements of $(V,<)$ and Y the set of minimal elements. Then there exists a one-to-one correspondence between the set $\mathcal{P} = \{(x,y) : x \in X,\ y \in Y, \text{ and } y < x\}$ and $\mathcal{C}$ such that the pair (x_i,y_i) associated with clique C_i is such that $\{x_i,y_i\} \subseteq C_i$ and $\{x_i,y_i\} \nsubseteq C_j$ for any $j \neq i$, for each $i = 1,2,\ldots,n$.

Proof: Let $(x,y) \in \mathcal{P}$. Since $(V,<)$ realizes G and $y < x$ in $(V,<)$, it follows $xy \in E$ and hence $\{x,y\} \subseteq C_i$ for some $C_i \in \mathcal{C}$ since $\mathcal{C}$ edge covers G. Now suppose $\{x,y\} \subseteq C_i \cap C_j$ for some $i,j = 1,2,\ldots,n$, where $i \neq j$. Choose $\{x_i,y_i\} \subseteq C_i$ and $\{x_i,y_i\} \nsubseteq C_k$ for $k \neq i$ and $\{x_j,y_j\} \subseteq C_j$ and $\{x_j,y_j\} \nsubseteq C_k$ for $k \neq j$, where $x_i,x_j \in M$ and $y_i,y_j \in N$.

Since $\{x,y\} \subseteq C_i \cap C_j$, it follows that xx_i, yx_i, xy_i, yy_i, xx_j, yx_j, xy_j, $yy_j \in E$. But $(V,<)$ realizes G so xx_i, $xx_j \in E$ imply that there exist a, b, c, $d \in V$ such that $a \leq x_i,x \leq b$ and $c \leq x_j,x \leq d$. Maximality of x in $(V,<)$ then implies that $x_i,x_j \leq x$. Similarly x_iy, $x_jy \in E$ imply that $y \leq x_i,x_j$. It follows that $x_ix_j \in E$ and an analogous argument can be used to show $y_iy_j \in E$. However, this contradicts the fact that $x_i,x_j \in M$ and $y_i,y_j \in N$ where M and N are independent. Thus, every pair $(x,y) \in \mathcal{P}$ belongs to exactly one clique of $\mathcal{C}$.

Suppose there exist two pairs (x_1,y_1), $(x_2,y_2) \in \mathcal{P}$ which belong to the same clique $C_i \in \mathcal{C}$. Then x_1x_2, $y_1y_2 \in E$. This implies there exist $a,b,c,d \in V$ such that $a \leq x_1,x_2 \leq b$ and $c \leq y_1,y_2 \leq d$. The maximality of x_1 and x_2 in $(V,<)$ and the minimality of y_1 and y_2 in $(V,<)$ implies $x_1 = b = x_2$ and $y_1 = c = y_2$. Thus no two distinct pair $(x,y) \in \mathcal{P}$ belong to the same clique of $\mathcal{C}$.

It remains to show that every clique $C_i \in \mathcal{C}$ contains an element $(x,y) \in \mathcal{P}$. To achieve this, let $C_i \in \mathcal{C}$ be arbitrary and choose $\{x_i,y_i\} \subseteq C_i$ and $\{x_i,y_i\} \nsubseteq C_k$ for $k \neq i$ where $x_i \in M$, $y_i \in N$. Thus, $x_iy_i \in E$ and since $(V,<)$ realizes G there is a maximal element x and a minimal element y in $(V,<)$ such that $y \leq x_i, y_i \leq x$. This implies that x and y are contained in a clique which contains x_i and y_i. Since $\{x_i,y_i\} \subseteq C_i$ and $\{x_i,y_i\} \nsubseteq C_k$ for $k \neq i$, it follows that $\{x,y\} \subseteq C_i$.///

Recall that for $G = (V,E)$ a graph and $v \in V$ the set of neighbors of v is $N(v) = \{v\} \cup \{u \in V : uv \in E\}$. Observe that if $G = (V,E)$ is a DB-graph with a family $\mathcal{C} = \{C_1,C_2,\ldots,C_n\}$ of cliques and sets M and N satisfying (i)-(iii) of Theorem 10, and if $\{x_i,y_i\} \subseteq C_i$ and $\{x_i,y_i\} \nsubseteq C_j$ for any $j \neq i$, then $N(x_i) \cap N(y_i) = C_i$. This observation, together with Corollary 9 and Theorem 10, leads to the next theorem.

Theorem 11. If $G = (V,E)$ is a DB-graph, then the family of cliques which satisfies the conditions of Theorem 10 is the unique such family.

Proof: Suppose there exist two such families $\mathcal{C} = \{C_1,C_2,\ldots,C_n\}$ and $\mathcal{B} = \{B_1,B_2,\ldots,B_m\}$ both satisfying the conditions of Theorem 10. Let M and N be disjoint, independent subsets of V such that for each $C_i \in \mathcal{C}$, there exist $x_i \in M$ and $y_i \in N$ such that $\{x_i,y_i\} \subseteq C_i$ and $\{x_i,y_i\} \nsubseteq C_j$ for any $j \neq i$ and such that (iii) of Theorem 10 holds.

By Corollary 9, there exists a partial order $<$ on V realizing G with M the set of maximal elements and N the set of minimal elements. Note that there are n distinct pairs (x_i,y_i) with $x_i \in M$, $y_i \in N$ and such that $y_i < x_i$.

By Lemma 1, there exists a one-to-one correspondence between the pairs (x_i,y_i), where $y_i < x_i$ in $(V,<)$, and $\mathcal{B}$ such that for each $i = 1,2,\ldots,n$ there exists $j(i) = 1,2,\ldots,m$ with $\{x_i,y_i\} \subseteq B_{j(i)}$ and $\{x_i,y_i\} \nsubseteq B_k$ for any $k \neq j(i)$. It follows that $n = m$. Moreover, since $\{x_i,y_i\} \subseteq C_i$ and $\{x_i,y_i\} \nsubseteq C_k$ for any $k \neq i$ and from the observation made prior to the statement of the Theorem, $B_{j(i)} = N(x_i) \cap N(y_i) = C_i$ for all $i = 1,2,\ldots,n$. Hence, $\mathcal{C}$ and $\mathcal{B}$ are the same family of cliques.///

In regard to the above Theorem, if $G = (V,E)$ is a DB-graph and $\mathcal{C} = \{C_1,C_2,\ldots,C_n\}$ is the unique family of cliques satisfying the conditions of Theorem 10, then $\mathcal{C}$ will be called the **DB-clique cover** of G.

Definition 2. Let $G = (V,E)$ be a DB-graph of a poset $(V,<)$. G is said to be a **unique DB-graph** if and only if $(V,<)$ and the converse of $(V,<)$, $(V,<^*)$, are the only posets realizing G.

Observe that a connected graph $G = (V,E)$ is a unique DB-graph of a height-1 poset if and only if G is bipartite. That is, no graph with a cycle of odd length is a DB-graph of a height-1 poset.

A characterization for unique DB-graphs in general is somewhat difficult. One must characterize those DB-graphs for which there exists exactly one pair of subsets $M,N \subseteq V$ satisfying the conditions of Theorem 10. Since every canonical poset realizing a given DB-graph G is height-1 or height-2, it is necessary that there do not exist elements $u,v \in V-(M \cup N)$ such that $L(u) \subseteq L(v)$ and $U(v) \subseteq U(u)$, for sets M and N satisfying the conditions of Theorem 10. This condition guarantees that any poset realizing G is at most height-2, which leads to the following theorem.

Theorem 12. Let $G = (V,E)$ be a DB-graph and $\mathcal{C}$ be the DB-clique cover of G. Then every poset realizing G is height-2 if and only if

(i) There exists $C_i \in \mathcal{C}$ such that $|C_i| \geq 3$,

and (ii) For all sets M and N satisfying the conditions of Theorem 10, there does not exist elements $u,v \in V-(M \cup N)$ such that $L(u) \subseteq L(v)$ and $U(v) \subseteq U(u)$.

Proof: Assume every poset realizing G is height-2 and let $(V,<)$ be such a poset. Since $(V,<)$ is height-2, $(V,<)$ contains a chain of length 3 which implies G has a clique of order at least 3. Hence, (i) holds. Suppose (ii) fails. Construct a height-3 poset $(V,<')$ by setting $u <' v$ and all other comparabilities the same as in $(V,<)$. Then $(V,<')$ realizes G but is not height-2. Thus, (ii) holds.

Conversely, let G be a DB-graph satisfying (i) and (ii) and let $(V,<)$ be any poset realizing G. Since G has a clique of order 3 or more, $(V,<)$ is at least height-2. If $(V,<)$ is height-n, $n \geq 3$, then there exists a chain in $(V,<)$, say $x_{n+1} < x_n < \ldots < x_1$. If M is the set of maximal elements in V and N is the set of minimal elements, M and N are sets satisfying the conditions of Theorem 10. But $L(x_3) \subseteq L(x_2)$ and $U(x_2) \subseteq U(x_3)$, where $x_2, x_3 \in V-(M \cup N)$, which contradicts (ii). Thus, no such chain exists in $(V,<)$ and $(V,<)$ must be height-2.///

As noted above, to characterize DB-graphs which are unique, one must first examine DB-graphs for which exactly one pair of subsets

$M,N \subseteq V$ exist satisfying the conditions of Theorem 10. To find such DB-graphs is not an easy task. Example 11 below illustrates the difficulty in finding all sets M and N satisfying the conditions of Theorem 10. If an easier method could be found to obtain all such sets M and N, then the uniqueness problem for DB-graphs might be answered.

Example 11. Consider the graph G in figure 25. $\mathscr{C} = \{C_1, C_2, C_3, C_4\}$ where $C_1 = \{a,b,c,f\}$, $C_2 = \{a,b,g,e\}$, $C_3 = \{a,d,e,i\}$ and $C_4 = \{a,c,d,h\}$ is an edge clique cover of G. To find sets M and N satisfying the conditions of Theorem 10, first list for each clique C_i all possible pairs $\{x,y\}$ such that $\{x,y\} \subseteq C_i$ and $\{x,y\} \not\subseteq C_j$ for all $j \neq i$. This gives four distinct pairs for each clique C_1, C_2, C_3 and C_4 which results in 4^4 possibilities for the sets M and N. However, if vertex a is used in a particular pair $\{x,y\}$, it alone must form either set M or set N since every vertex is adjacent to vertex a and M and N must be independent sets. This leaves no choice for the other set N (or M). It must consist of the vertices $\{f,g,h,i\}$. Let $M = \{a\}$ and $N = \{f,g,h,i\}$. The conditions of Theorem 10 hold and the corresponding canonical poset is shown in figure 26. If vertex a is not used, 3^3 possibilities remain for sets M and N, but only one of these results in sets M and N satisfying conditions (ii) and (iii) of Theorem 10. The canonical poset is given in figure 27. Observe that every poset realizing G is height-2.

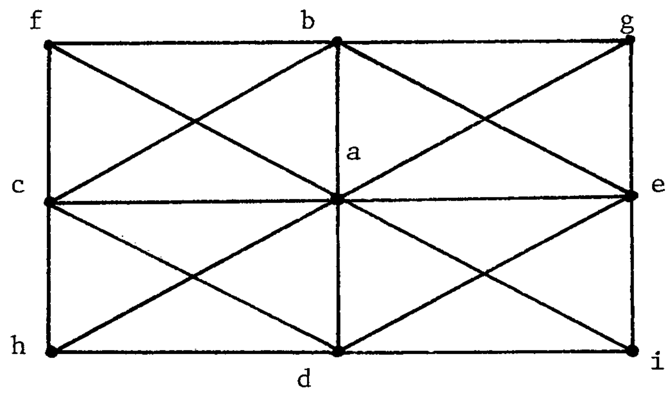


figure 25

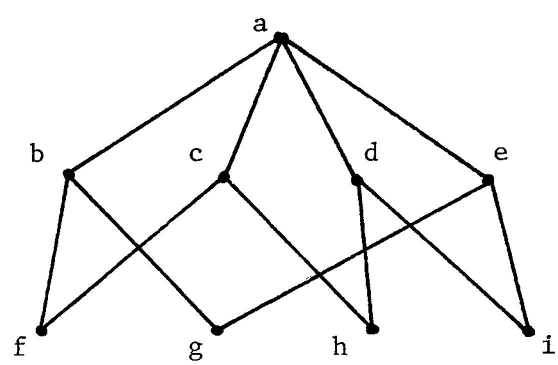


figure 26

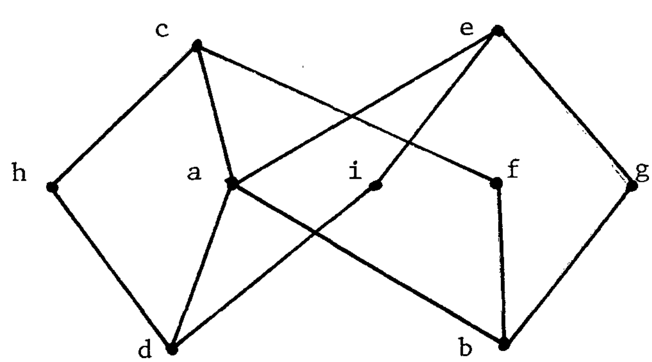


figure 27

Section 2.2 DB-graphs of Interval Orders

Let $(V,<)$ be a poset. $(V,<)$ is an **interval order** if and only if for $x,y,z,w \in V$, if $x < y$ and $z < w$ then $x < w$ or $z < y$.

<u>Definition 3</u>. Let $(V,<)$ be a poset and $v \in V$. Define the set of **lower holdings** of v to be the set $LH(v) = \{u \in V: u < v\}$ and the set of **upper holdings** of v to be the set $UH(v) = \{u \in V: v < u\}$. Denote $\mathcal{LH}(V) = \{LH(v): v \in V\}$ and $\mathcal{UH}(V) = \{UH(v): v \in V\}$.

Rabinovitch [13] proved that for a poset $(V,<)$, the following are equivalent:

(i) $(V,<)$ is an interval order,

(ii) $(\mathcal{LH}(V),\subseteq)$ is a chain,

(iii) $(\mathcal{UH}(V),\subseteq)$ is a chain.

Let $G = (V,E)$ be a DB-graph and M and N subsets of V satisfying the conditions of Theorem 10. Recall that $L(v) = \{y \in N: vy \in E\}$ and $U(v) = \{x \in M: xv \in E\}$ for $v \in V-(M \cup N)$. Modify this slightly and define $L'(v) = \{y \in N: vy \in E\}$ for $v \in V-N$ and $U'(v) = \{x \in M: vx \in E\}$ for $v \in V-M$. Denote $\mathcal{L}(V) = \{L'(v): v \in V-N\}$ and $\mathcal{U}(V) = \{U'(v): v \in V-M\}$.

<u>Theorem 13</u>. Let $G = (V,E)$ be a DB-graph of a height-1 poset. Then the poset is an interval order if and only if $\mathcal{L}(V)$ (or $\mathcal{U}(V)$) forms a chain with respect to set inclusion.

<u>Proof:</u> Let $G = (V,E)$ be a DB-graph of a height-1 poset which is an interval order and let $(V,<)$ be the unique poset realizing G. Thus, if N and M are subsets of V satisfying the conditions of Theorem 10, then N is the set of minimal elements of $(V,<)$ and M is

the set of maximal elements of $(V,<)$. Suppose $(\mathcal{L}(V),\subseteq)$ is not a chain. Then there exists distinct $u,v \in V-N$ such that $L'(u) \nsubseteq L'(v)$ and $L'(v) \nsubseteq L'(u)$. Let $a \in L'(u)-L'(v)$ and $b \in L'(v)-L'(u)$. By definition, $a,b \in N$ and au, $bv \in E$ but av, $bu \notin E$. It follows that $a < u$ and $b < v$ but $b \nless u$ and $a \nless v$ in $(V,<)$.

A similar argument shows $(\mathcal{U}(V),\subseteq)$ is a chain.

Conversely, suppose $G = (V,E)$ is a DB-graph of a height-1 poset $(V,<)$, where N is the set of minimal elements of $(V,<)$ and M is the set of maximal elements. Suppose $(\mathcal{L}(V),\subseteq)$ is a chain and let $x,y,z,w \in V$ such that $x < y$ and $z < w$. Since $(V,<)$ is height-1, $x,z \in N$. Then $L'(y) \subseteq L'(w)$ or $L'(w) \subseteq L'(y)$. Without loss of generality, assume $L'(y) \subseteq L'(w)$. This implies $x \in L'(w)$ and so $xw \in E$. Hence, $x < w$ and $(V,<)$ is an interval order.///

Example 12. The graph in figure 28 is a DB-graph of a height-1 poset which is an interval order, while the graph in figure 29 is a DB-graph of a height-1 poset which is not an interval order. Note from this observation and Theorem 10 that every C_{2n}, $n \geq 3$, is a DB-graph whose realizing poset is not an interval order. Hence, any bipartite graph with an induced cycle of length greater than or equal to 6 is not a DB-graph of an interval order.

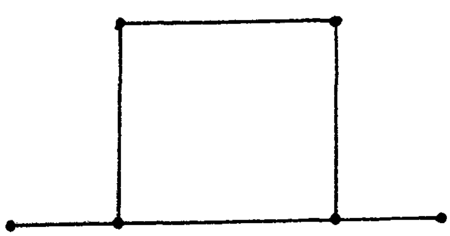


figure 28

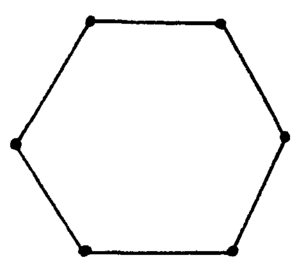


figure 29

For convenience, the characterization in [12] for UB-graphs of interval orders is stated below.

Theorem 14. Let G = (V,E) be a connected UB-graph with UB-clique cover $\mathcal{C}$ where $|\mathcal{C}| = n$. Then every partial order realizing G is an interval order if and only if the following two conditions hold:

(i) $|V| - (\bigcap_{C \in \mathcal{C}} C)| \leq n + 1$,

and (ii) $|C| \leq 4$ for all $C \in \mathcal{C}$.

Unlike the above characterization for UB-graphs of interval orders, there is no restriction on clique order for DB-graphs of interval orders, as example 13 illustrates.

Example 13. Let (V,<) be the poset shown in figure 30. Observe that (V,<) is an interval order. In addition, notice that the DB-graph of (V,<) will have cliques of order 3,4,5,6 and 7. If $M = \{x_1, x_2, x_3, x_4, x_5\}$ and $N = \{y_1, y_2, y_3, y_4, y_5\}$, then

$$L'(a) \subseteq L'(b) \subseteq L'(c) \subseteq L'(d) \subseteq L'(e) \quad \text{and}$$

$$U'(a) \subseteq U'(b) \subseteq U'(c) \subseteq U'(d) \subseteq U'(e).$$

In a similar manner, one can construct posets which are interval orders and for which the DB-graph contains a clique of whatever order one desires.

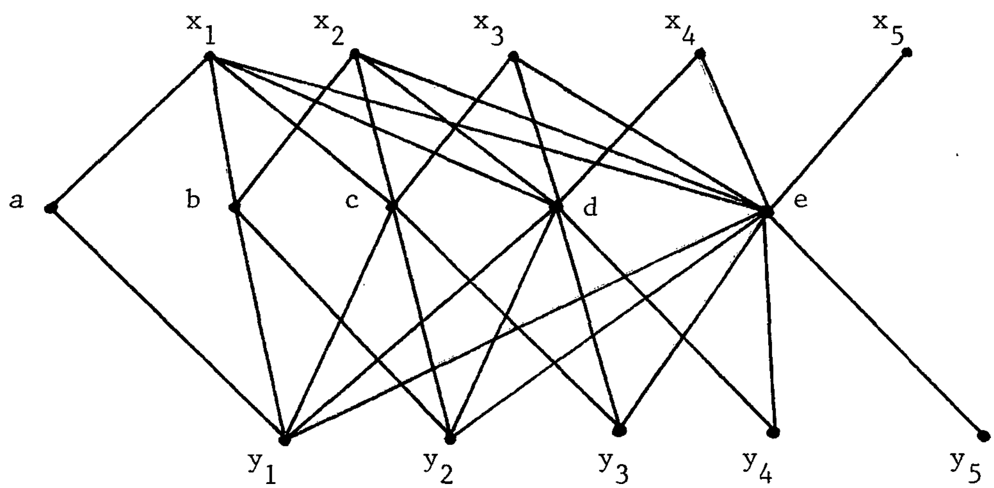


figure 30

Observe that if (V,<) is a poset with DB-graph G, and if M and N are the sets of maximal and mimimal elements, respectively, then M and N satisfy the conditions of Theorem 10. This follows from the proof of Theorem 10 and will be used in the next theorem.

__Theorem 15.__ Let G = (V,E) be a DB-graph such that every poset realizing G is at most height-2 and let $\mathcal{C}$ be the DB-clique cover for G. Then every poset realizing G is an interval order if and only if

(i) For all sets M and N satisfying the conditions of Theorem 10, $(\mathcal{L}(V),\subseteq)$ and $(\mathcal{U}(V),\subseteq)$ are chains,

and (ii) For any two cliques $C(x_i,y_i)$, $C(x_j,y_j) \in \mathcal{C}$ with $|C(x_i,y_i)|$, $|C(x_j,y_j)| \geq 3$, it follows that x_iy_j and $x_jy_i \in E$.

__Proof:__ Let G be a DB-graph such that every poset realizing G is at most height-2 and an interval order. Let $\mathcal{C}$ be the DB-clique cover for G and let M and N be sets as in Theorem 10. If (V,<) is a poset realizing G which is height-1, then Theorem 13 implies that (i) holds and (ii) is vacuously true since no cliques of order greater than or equal 3 exist in G. Thus, assume any poset realizing G is height-2.

Suppose (i) fails. Without loss of generality, assume $(\mathcal{L}(V),\subseteq)$ is not a chain. Then for some $x,y \in V-N$. $L'(x) \nsubseteq L'(y)$ and $L'(y) \nsubseteq L'(x)$. Let $a \in L'(x)-L'(y)$ and $b \in L'(y)-L'(x)$. Note that by definition $a,b \in N$. If $(V,<)$ is a canonical poset of G with M the set of maximal elements and N the set of minimal elements, then $a < x$ and $b < y$ in $(V,<)$. Furthermore, by assumption, $a \not< y$ and $b \not< x$. Also, x and y are not comparable in $(V,<)$ for otherwise $L'(y) \subseteq L'(x)$ or $L'(x) \subseteq L'(y)$ would hold. This implies $(V,<)$ is not an interval order, a contradiction. A similar argument shows $(\mathcal{U}(V),\subseteq)$ is a chain, and thus (i) holds.

Since every realizing poset of G is height-2 (by assumption), G has a clique of order three or more by Theorem 12. Let $C(x_i,y_i)$, $C(x_j,y_j)$ be cliques in $\mathcal{C}$ of order greater than or equal to 3. Then there exist elements $a,b \in V-(M \cup N)$ such that $a \in C(x_i,y_i)$ and $b \in C(x_j,y_j)$. Let $(V,<)$ be a canonical poset of G. If $a = b$, then $y_j < a < x_i$ and $y_i < a < x_j$ in $(V,<)$. This implies $x_iy_j, x_jy_i \in E$. So assume $a \neq b$. Since (i) holds, $U'(a) \subseteq U'(b)$ or $U'(b) \subseteq U'(a)$. Without loss of generality, assume $U'(a) \subseteq U'(b)$. Then $x_i \in U'(a)$ implies $x_i \in U'(b)$ and $b < x_i$ in $(V,<)$. Also by (i), $L'(a) \subseteq L'(b)$ or $L'(b) \subseteq L'(a)$. If $L'(a) \subseteq L'(b)$, then $y_i \in L'(a)$ implies $y_i \in L'(b)$ and $y_i < b$ in $(V,<)$. It follows that $y_i < b < x_i$; $y_j < b < x_j$ and thus $x_iy_j, x_jy_i \in E$. So assume $L'(b) \subseteq L'(a)$. Then $y_j < a$ in $(V,<)$ and $x_ix_j \in E$. Figure 31 illustrates this.

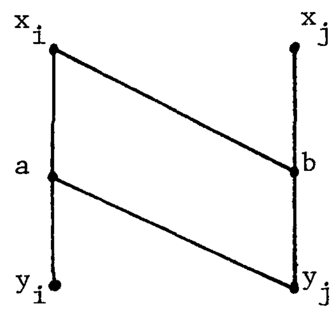
In (V,<): 

figure 31

Now if $x_j y_i \notin E$, then x_j does not cover a or y_i and b does not cover y_i in (V,<). Note that a and b are not comparable since every realizing poset of G is height-2. But then $y_i < a$; $b < x_j$; $y_i < x_j$, and $b < a$ which implies (V,<) is not an interval order. It follows that $x_j y_i \in E$ and (ii) holds.

Conversely, let G be a DB-graph such that every realizing poset of G is at most height-2 and assume (i) and (ii) hold. Let (V,<) be a poset realizing G, M the set of maximal elements in (V,<) and N the set of minimal elements. Let $x,y,z,w \in V$ with $y < x$ and $w < z$.

If (V,<) is height-1, then (i) implies that (V,<) is an interval order. So assume that (V,<) is height-2.

Case 1: $x,z \in M$.

By (i), $U'(y) \subseteq U'(w)$ or $U'(w) \subseteq U'(y)$ which implies $y < x$ or $w < x$.

Case 2: $y,w \in N$.

Follows as in Case 1.

Case 3: $x \in M$, $w \in N$, $y,z \notin M \cup N$.

Then $y \in C(x,y_i)$ for some $y_i \in N$ and $z \in C(x_j,w)$ for some

$x_j \in M$. By (ii) we have that xw, $x_j y_i \in E$. Since $x \in M$, $w \in N$ and $xw \in E$, it follows that $w < x$ in $(V,<)$.

Case 4: $z \in M$, $y \in N$, $x,w \notin M \cup N$.

Follows as in Case 3.

Since every poset realizing G is height-2, this exhausts all possibilities for x,y,z,w and in all cases $(V,<)$ is an interval order.///

The next theorem characterizes DB-graphs of arbitrary interval orders.

Theorem 16. Let G be a DB-graph and $\mathcal{C}$ the DB-clique cover for G. Then every realizing poset of G is an interval order if and only if for all sets M and N as in Theorem 10, the following hold:

(i) $(\mathcal{L}(V),\subseteq)$ and $(\mathcal{U}(V),\subseteq)$ are chains,

(ii) For any two cliques $C(x_i,y_i)$, $C(x_j,y_j) \in \mathcal{C}$ with $|C(x_i,y_i)|$, $|C(x_j,y_j)| \geq 3$, it follows that $x_i y_j$ and $x_j y_i \in E$.

(iii) There does not exist distinct $x,y,z,w \in V-(M \cup N)$ such that $L'(y) \subseteq L'(x)$, $U'(x) \subseteq U'(y)$ and $L'(w) \subseteq L'(z)$, $U'(z) \subseteq U'(w)$,

and (iv) For $x,y \in V-(M \cup N)$ with $L'(x) \subseteq L'(y)$, $U'(y) \subseteq U'(x)$, and for all $w \in M \cup N$,

(a) if $w \in C_i$, $|C_i| \geq 3$ for some i, then xw or $yw \in E$

and (b) if $w \in C_i$ and $C_i = \{w,z\}$, then xz,xw,yz or $yw \in E$.

Proof: Assume every poset realizing G is an interval order. Since a canonical poset of G is height-1 or height-2, (i) and (ii) hold by Theorems 14 and 15 above.

Let M and N be sets as in Theorem 10 and suppose (iii) fails. Let

$x,y,z,w \in V-(M \cup N)$ with x,y,z, and w distinct such that $L'(y) \subseteq L'(x)$, $U'(x) \subseteq U'(y)$, $L'(w) \subseteq L'(z)$ and $U'(z) \subseteq U'(w)$. Let $(V,<)$ be the corresponding canonical poset of G with the additional comparabilities $y < x$ and $w < z$. Since $x,y,z,w \notin M \cup N$, we have $x \not< z$, $y \not< z$, $z \not< x$ and $w \not< x$. Thus, $(V,<)$ realizes G but is not an interval order, so (iii) holds.

To prove (iv)(a), assume $x,y \in V-(M \cup N)$ with $L'(x) \subseteq L'(y)$ and $U'(y) \subseteq U'(x)$. Let $w \in M \cup N$ and $w \in C_j$ where $|C_j| \geq 3$ for some $C_j \in \mathcal{C}$. Without loss of generality, assume $w \in M$ and $C_j = C(w,y_j)$. Since $L'(y) \subseteq L'(x)$ and $U'(x) \subseteq U'(y)$, there exists a $C(x_i,y_i) \in \mathcal{C}$ such that $\{x,y\} \subseteq C(x_i,y_i)$. Consider the corresponding canonical poset $(V,<)$ with the added comparability $x < y$. If $x_i = w$, (iv)(a) holds. So assume $x_i \neq w$.

<u>Case 1</u>: $y_i = y_j$.

Let $z \in C(w,y_j)$, $z \neq w,y_j$. Such an element exists since $|C(w,y_j)| \geq 3$. If $z = x$ or $z = y$, (iv)(a) holds. Thus, assume $z \neq x,y$. By (i), $U'(y) \subseteq U'(z)$ or $U'(z) \subseteq U'(y)$. If $U'(z) \subseteq U'(y)$, then $w \in U'(y)$. Hence, yw, $xw \in E$ and (iv)(a) holds. So assume $U'(y) \subseteq U'(z)$. See figure 32.

In $(V,<)$:

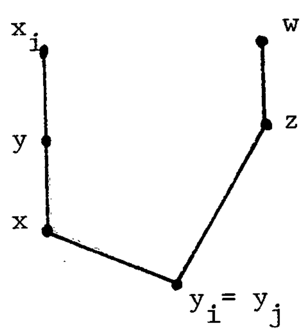


figure 32

If xw or yw$\notin$ E, these are the only comparabilities among x_i, y, x, y_i, z and w. But then (V,<) is not an interval order.

<u>Case</u> <u>2</u>: $y_i \neq y_j$.

Suppose xw,yw$\notin$ E. Then $U'(y) \subseteq U'(z)$, because $U'(z) \subseteq U'(y)$ implies yw$\in$ E. There two subcases.

Subcase (I): $L'(z) \subseteq L'(x)$.

This implies $y_j \in L'(x)$ which implies $y_j < x$ in (V,<). It follows that $x_i y_j \in E$. From (ii), $wy_i \in E$, so w covers y, x or y_i in (V,<). But if xw, yw $\notin$ E, w must cover y_i and these are the only comparabilities among x_i, x, y, w, z, y_i, and y_j. See figure 33.

In (V,<):

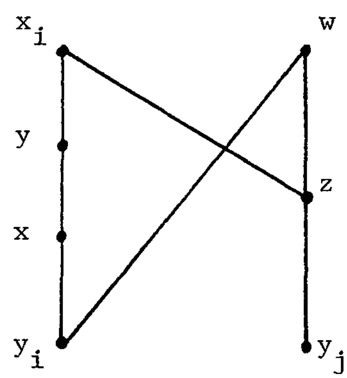


figure 33

But then (V,<) is not an interval order.

Subcase (II): $L'(x) \subseteq L'(z)$.

This implies $y_i \in L'(z)$ and thus $y_i < z$ in (V,<). See figure 34.

In (V,<):

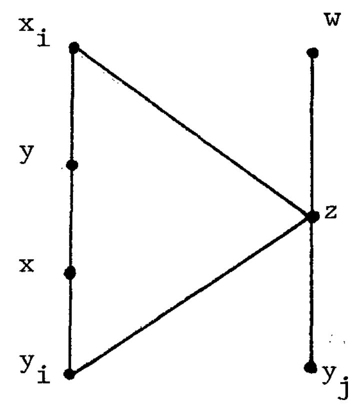


figure 34

If xw, yw $\notin$ E, these are the only comparabilities among x_i, y, x, z, w and y_i. But then (V,<) is not an interval order.

Hence, in all cases (iv)(a) holds.

To prove (iv)(b), again assume x, y $\in$ V−(M $\cup$ N) with $L'(x) \subseteq L'(y)$ and $U'(y) \subseteq U'(x)$. Assume w $\in$ M $\cup$ N and for some C $\in \mathcal{C}$, C = {w,z}. Since $L'(x) \subseteq L'(y)$ and $U('y) \subseteq U'(x)$ and x,y $\notin$ M $\cup$ N, we have $\{x,y\} \subseteq C(x_i,y_i)$ for some $C(x_i,y_i) \in \mathcal{C}$. Without loss of generality, assume w $\in$ M and z $\in$ N. Let (V,<) be the corresponding canonical poset of G with the added comparability x < y so that $y_i < x < y < x_i$ in (V,<). If z = y_i, then xz, yz $\in$ E and (iv)(b) holds. Similarly if w = x_i, then xw, yw $\in$ E. So assume w $\neq x_i$, z $\neq y_i$. By (i), $U'(y) \subseteq U'(z)$ or $U'(z) \subseteq U'(y)$. If $U'(z) \subseteq U'(y)$ then w $\in U'(y)$. Thus, xw, yw $\in$ E. So assume $U'(y) \subseteq U'(z)$. Also by (i), $L'(x) \subseteq L'(w)$ or $L'(w) \subseteq L'(x)$. If $L'(w) \subseteq L'(x)$, then z $\in L'(x)$ and xz, yz $\in$ E. So assume $L'(x) \subseteq L'(w)$. See figure 35.

In $(V,<)$:

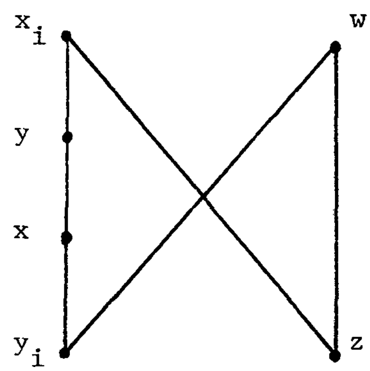


figure 35

If xw, yw, xz and $yz \notin E$, then $(V,<)$ is not an interval order, a contradiction. Hence, (iv)(b) holds.

Conversely, let G be a DB-graph with DB-clique cover $\mathcal{C}$ and such that (i)-(iv) hold. Let $(V,<)$ be any poset realizing G with M the set of maximal elements and N the set of minimal elements. Let $x,y,z,w \in V$ such that $y < x$ and $w < z$.

Case 1: $x,z \in M$.

By (i), $U'(y) \subseteq U'(w)$ or $U'(w) \subseteq U'(y)$. This implies $y < z$ or $w < x$.

Case 2: $y,w \in N$.

Analogous to Case 1.

Case 3: $x,y,z,w \notin M \cup N$.

Then $y < x$ implies $L'(y) \subseteq L'(x)$ and $U'(x) \subseteq U'(y)$, and $w < z$ implies $L'(w) \subseteq L'(z)$ and $U'(z) \subseteq U'(w)$. But this contradicts (iii), so this case is not possible.

Case 4: $y \in N$, $x \in M$, $w,z \notin M \cup N$.

Let $w,z \in C(x_i,y_i)$ for some $C(x_i,y_i) \in \mathcal{C}$. If $y_i = y$, then $y < z$. Similarly, if $x_i = x$, then $w < x$. Thus, assume $y_i \neq y$ and

$x_i \neq x$. By (iv), wx, zx, wy or $zy \in E$.

If $wx \in E$, then $x \in U'(w)$; hence, $w < x$.

If $zx \in E$, then $x \in U'(z) \subseteq U'(w)$; hence, $w < x$.

If $wy \in E$, then $y \in L'(w) \subseteq L'(z)$; hence $y < z$.

If $zy \in E$, then $y \in L'(z)$; hence, $y < z$.

Case 5 $x \in M$, $y,z,w \notin M \cup N$.

Let $y \in C(x,y_i)$, $z,w \in C(x_j,y_j)$. Note that $L'(w) \subseteq L'(z)$ and $U'(z) \subseteq U'(w)$ and $x \notin C(x,y_i)$, where $|C(x,y_i)| \geq 3$. Thus, by (iv)(a), xz or $xw \in E$. If $xz \in E$, then $x \in U'(z)$ which implies $w < z < x$. If $xw \in E$, then $x \in U'(w)$ and $w < x$.

Case 6: $y \in N$, $x,z,w \notin M \cup N$.

Analogous to Case 5.

Case 7: $x \in M$, $w \in N$, $y,z \notin M \cup N$.

Let $y \in C(x,y_i)$, $z \in C(x_j,w)$. If $x = x_j$, clearly $w < x$. Likewise, if $y_i = w$, then $w < x$. So assume $x \neq x_j$, $y_i \neq w$. Then $y_i < y < x$ and $w < z < x_j$ are chains in $(V,<)$. From (ii) it follows that $w < x$.

Any other possibility is analogous to one of the cases above. In any case, $w < x$ or $y < z$ and thus $(V,<)$ is an interval order.///

Example 14 illustrates that G can be a DB-graph of a height-2 poset which is an interval order and also the DB-graph of a height-3 poset which is not an interval order.

Example 14. Let G be the graph in figure 36. The reader can check that G is the DB-graph of both posets P_1 and P_2 shown in figure 36. Let $M = \{a,b\}$, $N = \{c,d\}$. Although P_1 is an interval order, since G fails to satisfy (iv)(a), define the poset P_2 with the same comparabilities of P_1 and the added comparability $y < x$ so that P_2 realizes G but is not an interval order.

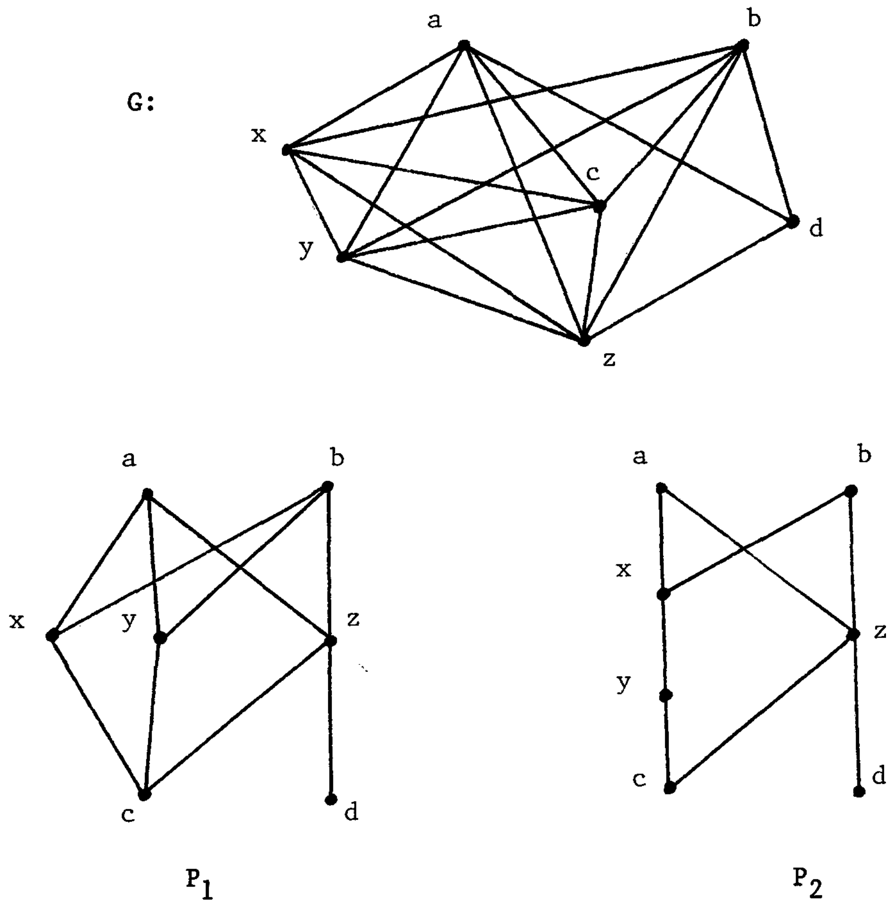


figure 36

Corollary 10. Let G be a DB-graph and $\mathcal{C}$ the DB-clique cover of G. Then there exists a poset realizing G which is an interval order if and only if for some M and N of Theorem 10 the following hold:

(i) $(\mathcal{L}(V), \subseteq)$ and $(\mathcal{U}(V), \subseteq)$ are chains.

and (ii) For any cliques $C(x_i,y_i)$, $C(x_j,y_j) \in \mathcal{C}$ of order ≥ 3, it follows that $x_i y_j$ and $x_j y_i \in E$.

Proof: Clearly if $(V,<)$ is an interval order which realizes G. then (i) and (ii) hold by Theorem 16 above.

Conversely, if G is a DB-graph such that (i) and (ii) hold for some M and N of Theorem 10, then the corresponding canonical poset which has M as its set of maximal elements and N as its set of minimal elements is an interval order by Theorems 14 and 15.///

Section 2.3 DB-graphs of Strict Weak orders and Semi-orders

A **strict weak order** is a poset $(V,<)$ such that for distinct $x,y,z \in V$, if $x < y$ then $x < z$ or $z < y$.

Let $(V,<)$ be a partial order. Then $(V,<)$ is a **semi-order** if and only if

(i) For $x,y,z,w \in V$, $x < y$ and $z < w$ implies $x < w$ or $z < y$

and (ii) For $x,y,z,w \in V$, $x < y$ and $y < z$ implies $x < w$ or $w < z$.

Clearly every semi-order is an interval order.

Note that if $G = (V,E)$ is a DB-graph of a height-1 poset $(V,<)$, then $(V,<)$ is a strict weak order if and only if G is a complete bipartite graph.

Proposition 3. Let $G = (V,E)$ be a DB-graph with DB-clique cover $\mathfrak{C}$ and such that every realizing poset of G is at most height-2. Then every poset realizing G is a strict weak order if and only if for all sets M and N of Theorem 10 the following hold:

(i) $|\mathfrak{C}| = |M|\cdot|N|$,

and (ii) For all $v \in V-(M \cup N)$ it follows that $M \cup N \subseteq Adj(v)$.

Proof: If G is realized by a height-1 poset then G is a complete bipartite graph which implies (i); moreover, (ii) is vacuously true. So assume every poset realizing G is height-2 and a strict weak order. Let M and N be subsets of V satisfying the conditions of Theorem 10 and choose $C(x,y) \in \mathcal{C}$ with $|C(x,y)| \geq 3$. Such a clique exists because G is realized by a height-2 poset. Let $z \in C(x,y)$ with $z \neq x,y$. Let $C(u,w)$ be any other clique in $\mathcal{C}$, and suppose u or $w \notin Adj(z)$. Construct the corresponding canonical poset $(V,<)$ for M and N where $x,u \in M$ and $y,w \in N$. Then $(V,<)$ has one of the posets in figure 37 as a subposet. In any case, $(V,<)$ is not a strict weak order. Hence, $u,w \in Adj(v)$ and since $C(u,w)$ was arbitrary, (ii) holds. Condition (i) now follows from (ii) since we assumed every realizing poset is height-2.

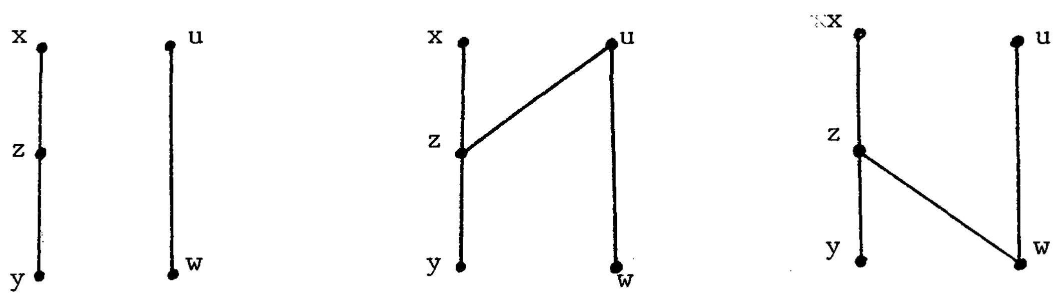


figure 37

Conversely, assume (i) and (ii) hold. Let $(V,<)$ be any poset realizing G and let $x,y,z \in V$ with $x < y$. Condition (i) implies that G is connected and thus z is not an isolated vertex. Since $(V,<)$ is at most height-2, either $x \in M$ or $y \in N$. If $x \in M$ and $z \notin M \cup N$, then (ii) implies that $xz \in E$ and hence $z < x$ in $(V,<)$. If $x \in M$ and $z \in M \cup N$, then (i) implies that either $xz \in E$ if $z \in N$, from which it follows that $z < x$, or that $yz \in E$ if $z \in M$, from which we have that $y < z$.

A similar argument is used if $y \in N$. In any case, $(V,<)$ is a strict weak order.///

Theorem 17. Let $G = (V,E)$ be a DB-graph with DB-clique cover $\mathcal{C}$. Then every realizing poset is a strict weak order if and only if for all sets M and N of Theorem 10 the following hold:

(i) $|\mathcal{C}| = |M|\cdot|N|$

(ii) For all $v \in V-(M \cup N)$ it follows that $M \cup N \subseteq Adj(v)$,

and (iii) There does not exist a $C \in \mathcal{C}$ such that $|C| \geq 5$.

Proof: Conditions (i) and (ii) follow from Proposition 3 above. It remains to prove that (iii) holds. Let M and N be any subsets of V satisfying the conditions of Theorem 10 and suppose $C(x,y) \in \mathcal{C}$ such that $|C(x,y)| \geq 5$. Let $a,b,c \in C(x,y)$ be distinct with $a,b,c \notin (M \cup N)$. Construct the corresponding canonical poset for M and N and add the comparability $a < b$ and denote this poset by $(V,<)$. Then $(V,<)$ has the poset in figure 38 as a subposet, which implies that $(V,<)$ is not a strict weak order.

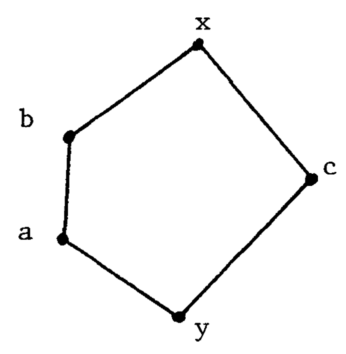


figure 38

Conversely, assume (i)-(iii) hold and let $(V,<)$ be any poset realizing G and choose $a,b,c \in (V,<)$ with $a < b$.

<u>Case 1</u>: $b \in M$ and $c \notin (M \cup N)$.

Then condition (ii) implies that $c < b$.

<u>Case 2</u>: $a \in N$ and $c \notin (M \cup N)$.

Then condition (ii) implies that $a < c$.

<u>Case 3</u>: $b \in M$ and $c \in (M \cup N)$.

Then if $c \in N$, (i) implies that $c < b$.

If $c \in M$ and $y \in N$ then (i) implies that $a < c$.

Finally, if $c \in M$ and $y \notin N$, then (ii) implies that $a < c$.

<u>Case 4</u>: $a \in N$ and $c \in (M \cup N)$.

It follows analogously to Case 3.

<u>Case 5</u>: $a,b \notin (M \cup N)$ and $c \in (M \cup N)$.

If $c \in M$, then (ii) implies that $a < b < c$.

If $c \in N$, then (ii) implies that $c < a < b$.

<u>Case 6</u>: $a,b,c \notin (M \cup N)$.

Then (ii) implies that there exists a clique $C \in \mathcal{C}$ such that $|C| \geq 5$ which contradicts (iii).

Hence, in any case, $(V,<)$ is a strict weak order.///

<u>Corollary 11</u>. Let G be a DB-graph with DB-clique cover $\mathcal{C}$. Then there exists a poset realizing G which is a strict weak order if and only if there exist sets M and N of Theorem 10 such that the following hold:

(i) $|\mathcal{C}| = |M|\cdot|N|$,

and (ii) For all $v \in V-(M \cup N)$ it follows that $M \cup N \subseteq Adj(v)$.///

Observe that every height-1 poset which is an interval order is a semi-order, so G is a DB-graph of a height-1 poset which is a semi-order if and only if G is bipartite and $(\mathcal{L}(V), \subseteq)$ is a chain.

Theorem 18. Let G = (V,E) be a DB-graph such that every realizing poset is at most height-2. Let $\mathcal{C}$ be the DB-clique cover of G. Then every poset realizing G is a semi-order if and only if for all sets M and N of Theorem 10 the following hold:

(i) $(\mathcal{L}(V), \subseteq)$ and $(\mathcal{U}(V), \subseteq)$ are chains,

(ii) For all $C(x_i,y_i)$, $C(x_j,y_j) \in \mathcal{C}$, it follows that x_iy_j, $x_jy_i \in E$,

and (iii) For all $C(x_i,y_i)$, $C(x_j,y_j) \in \mathcal{C}$ such that $|Cx_i,y_i)|$, $|C(x_j,y_j)| \geq 3$, if $x \in C(x_i,y_i)$ and $x \neq x_i,y_i$, then $xx_j \in E$ or $xy_j \in E$.

Proof: Assume G is a DB-graph with DB-clique cover $\mathcal{C}$ and that every poset realizing G is a semi-order and at most height-2. Let (V,<) be such a poset realizing G with M and N the sets of maximal elements and minimal elements, respectively. Then since (V,<) is a semi-order, it is also an interval order and (i) holds.

If every $C(x_i,y_i) \in \mathcal{C}$ has cardinality 2, then (V,<) is height-1 and (i) implies (ii). Thus, assume $|C(x_i,y_i)| \geq 3$ or $|C(x_j,y_j)| \geq 3$.

Case 1: $|C(x_i,y_i)| \geq 3$, $|C(x_j,y_j)| = 2$.

Let $z \in C(x_i,y_i)$, $z \neq x_i,y_i$. Then $y_i < z < x_i$ in (V,<) and since (i) holds, $U'(z) \subseteq U'(y_j)$ or $U'(y_j) \subseteq U'(z)$ and $L'(z) \subseteq L'(x_j)$ or $L'(x_j) \subseteq L'(z)$. There are 4 subcases.

Subcase (I): $U'(z) \subseteq U'(y_j)$ and $L'(z) \subseteq L'(x_j)$.

Then $y_j < x_i$ in (V,<) and $x_iy_j \in E$. Also $y_i < x_j$ in (V,<)

and thus $x_jy_i \in E$.

Subcase (II): $U'(z) \subseteq U'(y_j)$ and $L'(x_j) \subseteq L'(z)$.

Then $y_j < x_i$ in $(V,<)$ and $x_iy_j \in E$. Moreover, $L'(x_j) \subseteq L'(z)$ implies $y_j < z < x_i$ in $(V,<)$. Consider $y_i < z < x_i$ and x_j. Since $(V,<)$ is a semi-order, $y_i < x_j$ or $x_j < x_i$. Since x_j and $x_i \in M$, x_i and x_j are not comparable, which implies $y_i < x_j$ in $(V,<)$ and hence $x_jy_i \in E$.

Subcase (III): $U'(y_j) \subseteq U'(z)$ and $L'(z) \subseteq L'(x_j)$.

Then $z < x_j$ in $(V,<)$ and since $y_i < z$, it follows that $x_jy_i \in E$. Now consider $y_i < z < x_i$ and y_j. Since $(V,<)$ is a semi-order, $y_j < x_i$ or $y_i < y_j$. But $y_i, y_j \in N$ and so are not comparable in $(V,<)$ which implies $y_j < x_i$ and $x_iy_j \in E$.

Subcase (IV): $U'(y_j) \subseteq U'(z)$ and $L'(x_j) \subseteq L'(z)$.

Then $z < x_j$ and $y_i < z$ which imply $y_i < x_j$. Thus, $x_jy_i \in E$. Also $L'(x_j) \subseteq L'(z)$ implies $y_j < z$. But $z < x_i$ implies $y_j < x_i$ and thus $x_iy_j \in E$.

<u>Case</u> <u>2</u>: $|C(x_i,y_i)|$, $|C(x_j,y_j)| \geq 3$.

Since $(V,<)$ is a semi-order, $(V,<)$ is an interval order and condition (ii) follows from Theorem 15.

Thus, in all cases (ii) holds.

To prove (iii), let $C(x_i,y_i)$, $C(x_j,y_j) \in \mathcal{C}$ such that $|C(x_i,y_i)|$, $|C(x_j,y_j)| \geq 3$ and $x \in C(x_i,y_i)$, $x \neq x_i,y_i$. Since $|C(x_j,y_j)| \geq 3$, there exists $w \in C(x_j,y_j)$ with $w \neq x_j,y_j$. If $x = w$, clearly (iii) holds. Thus, assume $x \neq w$. Since (i) holds, $U'(x) \subseteq U'(w)$ or $U'(w) \subseteq U'(x)$. If $U'(w) \subseteq U'(x)$, then $xx_j \in E$. So assume $U'(x) \subseteq U'(w)$. This implies $w < x_i$ in $(V,<)$ and $wx_i \in E$. Also by (i), $L'(x) \subseteq L'(w)$ or

$L'(w) \subseteq L'(x)$. If $L'(w) \subseteq L'(x)$, then it follows $xy_j \in E$. So assume $L'(x) \subseteq L'(w)$. See figure 39.

In $(V,<)$:

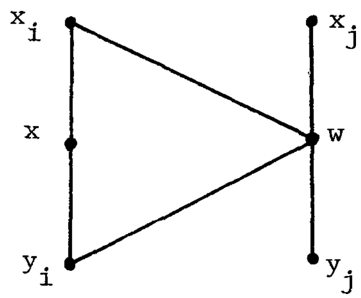


figure 39

Now consider $y_j < w < x_j$ and x. Since $(V,<)$ is a semi-order, $y_j < x$ or $x < x_j$ which implies xx_j or $xy_j \in E$.

Conversely, let G be a DB-graph with DB-clique cover $\mathcal{C}$ and such that every realizing poset is at most height-2, and conditions (i)-(iii) hold for all set M and N of Theorem 10. Let $(V,<)$ be a poset realizing G and let M and N be the sets of maximal elements and minimal elements, respectively. If $(V,<)$ is height-1, (i) implies that $(V,<)$ is a semi-order by the observation made prior to the statement of the Theorem. Thus, assume $(V,<)$ is height-2 and let $x,y,z,w \in V$ with $z < y < x$. The goal is to show $z < w$ or $w < x$. Since $(V,<)$ is height-2, $x \in M$ and $z \in N$.

Case 1: $w \in M$.

Since $(V,<)$ is connected, there exists $y_j \in N$ such that $y_j < w$. Consider $C(x.z)$, $C(w,y_j)$. By (ii), $wz \in E$ which implies $z < w$.

Case 2: $w \in N$.

Analogous to Case 1.

Case 3: $w \notin M \cup N$.

If $w \in C(x,z)$, then $w < x$ and $z < w$. So assume $w \notin C(x,z)$. Then there exists $C(x_i,y_i)$ such that $w \in C(x_i,y_i)$ which implies $y_i < w < x_i$ in $(V,<)$. If $x = x_i$, then $w < x$ and if $z = y_i$ then $z < w$. So assume $x \neq x_i$, $z \neq y_i$. See figure 40.

In $(V,<)$:

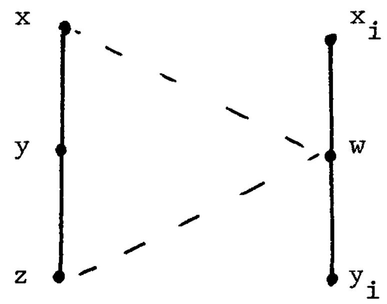


figure 40

Since $|C(x,z)|$ and $|C(x_i,y_i)| \geq 3$, and $w \in C(x_i,y_i)$, where $w \neq x_i,y_i$, condition (iii) implies wx or $wz \in E$. It follows that $w < x$ or $z < w$ in $(V,<)$.

Hence, in all cases, $w < x$ or $z < w$ which implies $(V,<)$ is a semi-order.///

The following corollary is immediate from the Theorem.

Corollary 12. Let G be a DB-graph and $\mathcal{C}$ the DB-clique cover of G. Then there exists a poset realizing G which is a semi-order if and only if there exist sets M and N of Theorem 10 such that conditions (i)-(iii) of Theorem 18 above hold.///

CHAPTER 3 POSETS WITH INTERVAL UB-GRAPHS

Cohen [2] discovered that most competition graphs arising from food webs are interval graphs. This discovery prompted others to characterize those digraphs which have interval competition graphs. Lundgren and Maybee [9] give a characterization in terms of a competition cover. In view of the forbidden subgraph characterization for interval graphs Steif [17] has proved that a forbidden sink induced subdigraph list exists, but no one has yet found this list.

In Section 1.2 it was observed that competition graphs of transitive acyclic digraphs are strict UB-graphs. If one can derive results for posets which have interval UB-graphs, analogous results may possibly be attained for competition graphs. This approach was suggested in [9] and is attributed to McMorris.

Section 3.1 Immediate Results

In 1964, Gilmore and Hoffman [5] proved that a graph G is interval if and only if the cliques of G can be linearly ordered such that for every vertex $x \in G$, the cliques containing x occur consecutively.

One can make an analogous statement for posets which have interval UB-graphs. First, let $(P,<)$ be a poset and $M = \{m_1, m_2, \ldots, m_k\}$ be the set of nonisolated maximal elements in

(P,<). Observe that the set of all elements less than or equal to m_i, for each i, $1 \leq i \leq k$, induces a clique in the UB-graph. If $x \in P-M$ and $M(x) = \{m_i \in M: x < m_i\}$, then the Gilmore and Hoffman result for posets translates to Theorem 19.

Theorem 19. A poset (P,<) with M the set of nonisolated maximal elements of (P,<), has an interval UB-graph if and only if there exists a linear ordering of $M = \{m_1, m_2, \ldots, m_k\}$ such that for every $x \in P-M$, $M(x) = \{m_i, m_{i+1}, \ldots, m_{i+j}\}$ for some $i > 0$ and $j \geq 0$.///

Recall the forbidden subgraph characterization of interval graphs by Lekkerkerker and Boland [8].

A graph G is interval if and only if it does not contain the graphs G_1–G_5 of figure 41 as induced subgraphs.

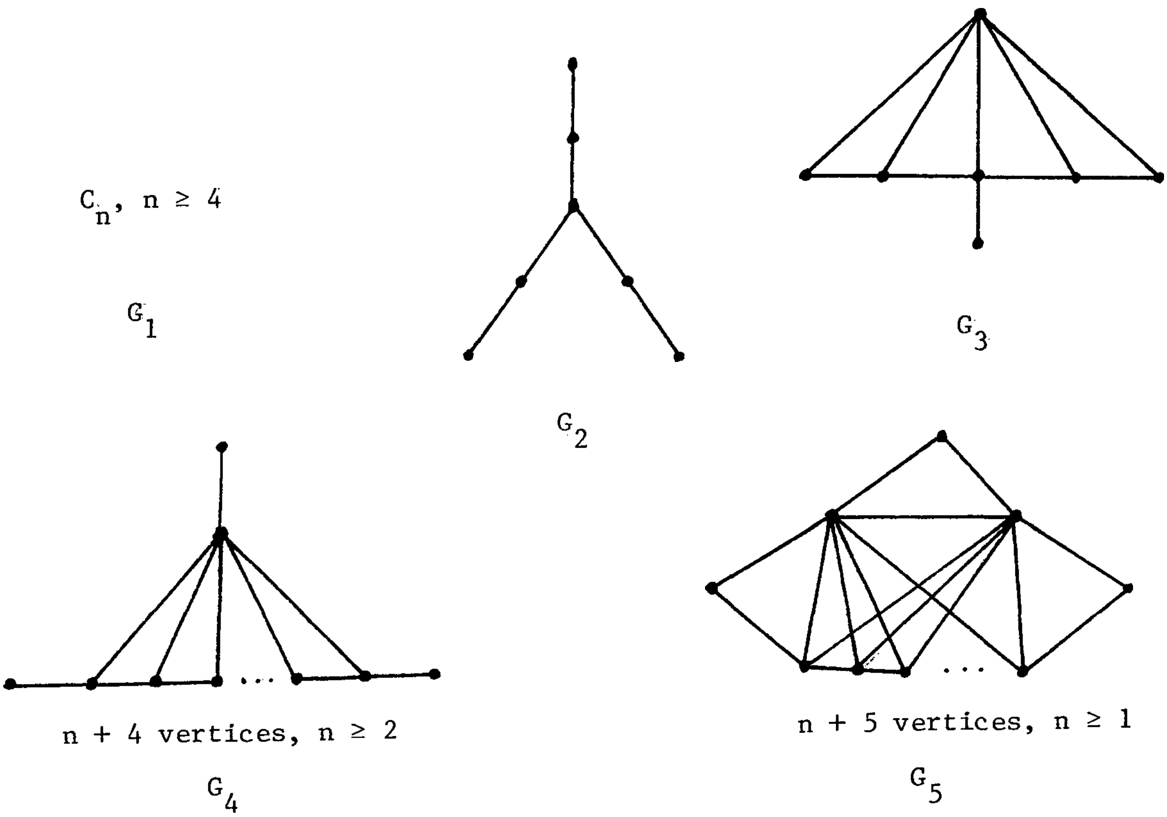


figure 41

The graphs G_1-G_5 in figure 41 will be referred to throughout the remainder of this chapter.

We next consider various posets with interval UB-graphs. Let $(P,<)$ be a poset. $(P,<)$ is said to be a **tree poset** if and only if for all distinct $x,y,z \in P$, if $x,y < z$ then $x < y$ or $y < x$.

Let $(P,<)$ be a poset. Define the **comparability graph** $G = (P,E)$ of P where $xy \in E$ if and only if $x < y$ or $y < x$. The following is a result of the work done by Wolk in [18] and [19].

Theorem 20. A graph $G = (V,E)$ is the comparability graph of a tree poset if and only if G contains no induced subgraph isomorphic to C_4 or P_4.

Observe that if $(P,<)$ is a tree poset, then the UB-graph of $(P,<)$ is the comparability graph of $(P,<)$. (The UB-graph is also the DB-graph of a tree poset.)

Corollary 13. The UB-graph G of a tree poset is an interval graph.

Proof: Let $(P,<)$ be a tree poset and let G be the UB-graph of $(P,<)$. From Theorem 20 above, G has no induced C_4 or P_4. Since each of the graphs in figure 41 has either an induced C_4 or P_4, it follows that G is an interval graph.///

Theorem 21. If $(P,<)$ is an interval order, then the UB-graph of $(P,<)$ is an interval graph.

Proof: Let M be the set of nonisolated maximal elements of $(P,<)$ and let the elements of P-M be labelled $v_1,v_2,\ldots,v_k$ such that

$UH(v_1) \subseteq UH(v_2) \subseteq \ldots \subseteq UH(v_k)$. Such a labelling is possible because (P,<) is an interval order which implies ($\mathcal{UH}$ (P),$\subseteq$) is a chain by [13]. But since $M(v_i) \subseteq UH(v_i)$ for all $i = 1,\ldots,k$ and $M(v_i) \subseteq M$, it follows that $M(v_1) \subseteq M(v_2) \subseteq \ldots \subseteq M(v_k)$. Set

$M(v_1) = \{m_{11}, m_{12}, \ldots, m_{1n(1)}\}$.

$M(v_2) = \{m_{11}, m_{12}, \ldots, m_{1n(1)}, m_{21}, m_{22}, \ldots, m_{2n(2)}\}$,

.
.
.

and $M(v_k) = \{m_{11}, \ldots, m_{1n(1)}, m_{21}, \ldots, m_{2n(2)}, \ldots, m_{k1}, \ldots, m_{kn(k)}\}$.

Then, $m_{11}, m_{12}, \ldots, m_{1n(1)}, m_{21}, \ldots, m_{2n(2)}, \ldots, m_{k1}, \ldots, m_{kn(k)}$ is a desired ordering of the elements in M such that for each $x \in P-M$, the maximal elements in M(x) occur consecutively. It follows from Theorem 19 that the UB-graph of (P,<) is interval.///

Corollary 14. If (P,<) is a semi-order, then the UB-graph of (P,<) is an interval graph.///

Section 3.2 A Characterization for the Canonical Poset of (P,<)

In 1982, Steif [17] showed that no forbidden induced subdigraph characterization was possible for digraphs with interval competition graphs. It also follows that no forbidden subposet characterization exists for posets with interval UB-graphs, for suppose such a characterization did exist with (Q,<) a forbidden subposet. Let (P,<) be the poset (Q,<) with an added vertex m and added comparabilities $q < m$ for each $q \in Q$. (P,<) obviously has subposet (Q,<) but its UB-graph is complete and therefore interval.

However, some results were obtained for special subdigraphs, called sink induced subdigraphs. (See [9] and [17] for specific details.) Defining a special subposet similar to a sink induced subdigraph does allow one to obtain some analogous results.

__Definition 4.__ Let $(P,<)$ be a poset and $Q \subseteq P$. Then Q is a **m-subposet** of $(P,<)$ if and only if Q is a subposet with the additional property that if $x,y \in Q$ and $x,y < m$ for some $m \in P$, then there exists $m' \in Q$ with $x,y < m'$.

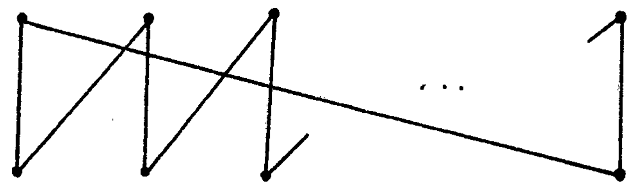
__Definition 5.__ Let $(P,<)$ be a poset and let M be the set of maximal elements of $(P,<)$. The **canonical (sub)poset of (P,<)** is the height-1 poset $(P,<')$ where $x <' y$ if and only if $y \in M$, $x \notin M$ and $x < y$ in $(P,<)$.

Observe that if G is a UB-graph, then all posets realizing G have isomorphic canonical posets.

The graphs G_1-G_5 of figure 41 and the posets P_1-P_5 of figure 42 below will be referred to in the five lemmas which follow.

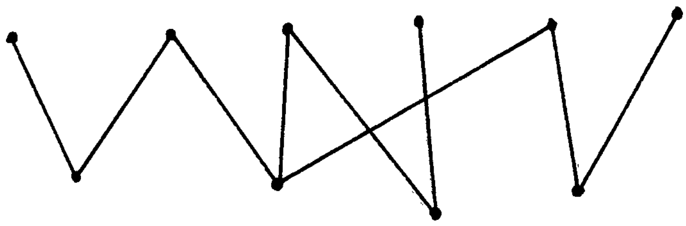
As an aid in determining the posets P_4 and P_5 for various n, figure 43 illustrates the posets P_4 and P_5 for the first three values of n.

P_1:

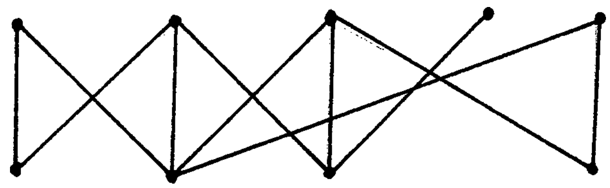


2n vertices, $n \geq 4$

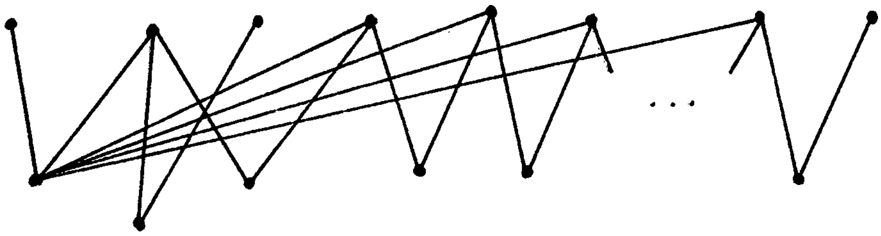
P_2:



P_3:

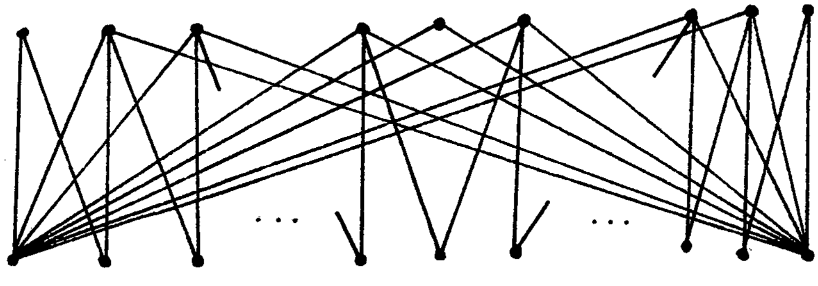


P_4:



$5 + 2(n-1)$ vertices, $n \geq 2$

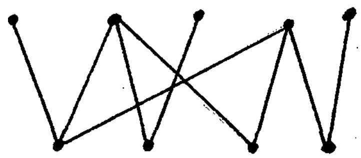
P_5:



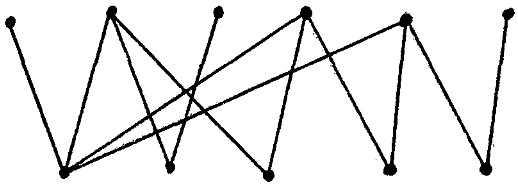
$4 + 2n$ vertices, $n \geq 1$

figure 42

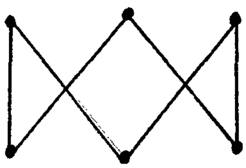
P_4 with n = 2



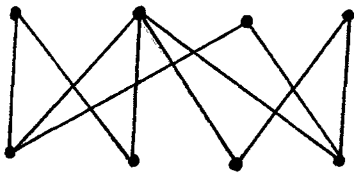
P_4 with n = 3



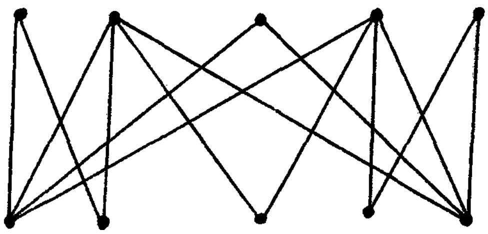
P_4 with n = 4



P_5 with n = 1



P_5 with n = 2



P_5 with n = 3

figure 43

In the following five lemmas, let G = (V,E) be a UB-graph and let (V,<) be the canonical poset realizing G.

Lemma 2. C_n, $n \geq 4$, is an induced subgraph of G if and only P_1 is a m-subposet of (V,<).

Proof: Suppose that C_n, $n \geq 4$, is an induced subgraph of G.

Label the vertices of C_n, as $v_1, v_2, \ldots, v_n$ such that $v_i v_{i+1} \in E$ for $i = 1, \ldots, n$, where $v_{n+1} = v_1$. Since G is a UB-graph, for any edge $v_i v_{i+1} \in E$, there exists a $a_{i+1} \in V$ such that $v_i, v_{i+1} < a_{i+1}$ and $v_j \nless a_{i+1}$ for all $j \neq i, i+1$ in $(V,<)$. This follows from Theorem 1 of [11]. Hence, $(V,<)$ contains the poset in figure 44 as a subposet Q. (Note that Q is isomorphic to P_1.)

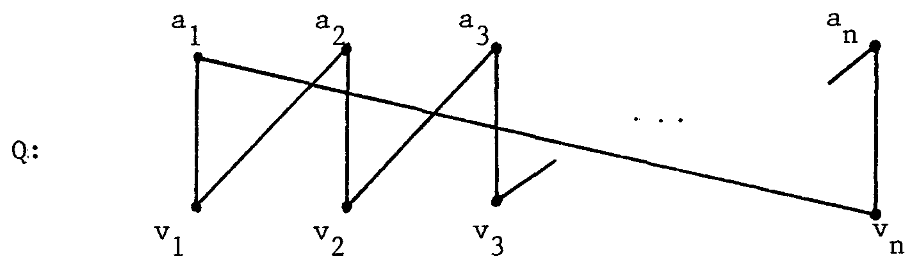


figure 44

But Q is also a m-subposet of $(V,<)$ for the existence of any $m \in V$ with $v_i, v_j < m$ but $v_i, v_j \nless m'$ for all $m' \in Q$ would yield a chord in C_n.

Conversely, if $(V,<)$ contains P_1 as a m-subposet, then no v_i, v_j are below the same maximal element m in $(V,<)$ unless both are below some $m' \in P_1$. This implies that edges among $\{v_1, v_2, \ldots, v_n\}$ arise from $(V,<)$ if and only if they arise from P_1. Moreover, the UB-graph of P_1 is C_n, $n \geq 4$. Thus G has C_n, $n \geq 4$, as an induced subgraph.///

Lemma 3. G has G_2 as an induced subgraph if and only if $(V,<)$ has P_2 as a m-subposet.

Proof: Suppose G_2 is an induced subgraph of G. Label the vertices of G_2 as $v_1, v_2, \ldots, v_7$ such that $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_3v_6, v_6v_7\}$. Since G is a UB-graph, Theorem 1 of McMorris and Zaslavsky [11] implies that there exist $a_1, a_2, \ldots, a_6 \in V$ such that $v_2 < a_1$ and $v_1 \leq a_1$, $v_2, v_3 < a_2$, $v_3, v_4 < a_3$, $v_4 < a_4$ and $v_5 \leq a_4$, $v_3, v_6 < a_5$ and $v_7 < a_6$ and $v_6 \leq a_6$ with no other comparabilities between $v_1, v_2, \ldots, v_7$ and $a_1, a_2, \ldots, a_6$. Thus, $(V,<)$ contains the poset in figure 45 as a subposet Q. (Q is isomorphic to P_2.)

Q:

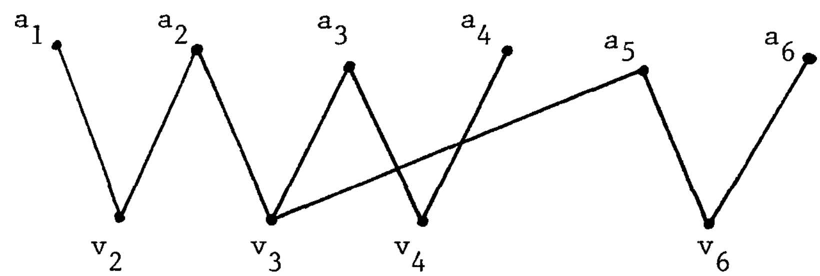


figure 45

Moreover, Q is a m-subposet because the existence of any $m \in V$ with $v_i, v_j < m$ but $v_i, v_j \nless m'$ for all $m' \in Q$ would yield an edge which is not in G_2.

Conversely, if $(V,<)$ contains P_2 as a m-subposet, then edges among $\{v_2, v_3, v_4, v_6\}$ arise form $(V,<)$ if and only if they do from P_2. Thus, the UB-graph of P_2 is an induced of G and contains G_2 as an induced subgraph. Hence, G contains G_2 as an induced subgraph.///

Lemma 4. G has G_3 as an induced subgraph if and only if $(V,<)$ contains P_3 as a m-subposet.

Proof: Suppose G_3 is an induced subgraph of G and let $v_1, v_2, \ldots, v_7$ be a labelling of the vertices such that v_1v_2, v_1v_3, v_2v_3, v_1v_4, v_3v_4, v_1v_5, v_4v_5, v_1v_6, v_5v_6, $v_4v_7 \in E$. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ $(m \geq 5)$ be the UB-clique cover of G and

$a_1, a_2, \ldots, a_m \in V$ such that $a_i \in C_i - (\bigcup_{j \neq i} C_j)$, for all $i = 1,2,\ldots,m$. Assume the labelling of G_3 is such that $C_1 \cap \{v_1,\ldots,v_7\} = \{v_1,v_2,v_3\}$, $C_2 \cap \{v_1,\ldots,v_7\} = \{v_1,v_3,v_4\}$, $C_3 \cap \{v_1,\ldots,v_7\} = \{v_1,v_4,v_5\}$, $C_4 \cap \{v_1,\ldots,v_7\} = \{v_1,v_5,v_6\}$ and $C_5 \cap \{v_1,\ldots,v_7\} = \{v_4,v_7\}$. Then $(V,<)$ must have the poset in figure 46 as a subposet Q. (Note that it is possible for $a_1 = v_2$, $a_4 = v_6$ and $a_5 = v_7$ because a_1, a_4 and a_5 are contained in only one clique of G_3. This conclusion follows from Theorem 1 of [11].)

Q:

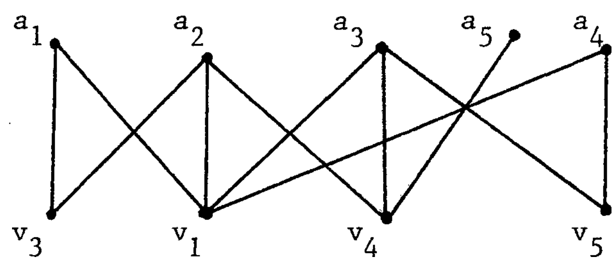


figure 46

Q is isomorphic to P_3 and is in fact a m-subposet by an argument similar to that used in Lemma 3 above.

Conversely, if $(V,<)$ contains P_3 as a m-subposet, then any two elements of $\{v_1,v_3,v_4,v_5\}$ are below the same maximal element in $(V,<)$ if and only if both are below a same maximal element in P_3. Thus, the UB-graph of P_3 is an induced subgraph of G. It follows that G_3 is an induced subgraph of G since it is an induced subgraph of the UB-graph of P_3.///

Lemma 5. G has G_4 (for some $n \geq 2$) as an induced subgraph if and only if $(V,<)$ has P_4 (same n) as a m-subposet.

Proof: Fix $n \geq 2$ and suppose G has G_4 as an induced subgraph. Assume G_4 is labelled as in figure 47.

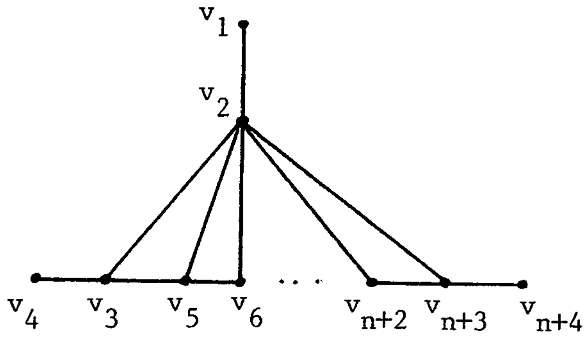


figure 47

Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ $(m \geq n+2)$ be the UB-clique cover for G and $a_1, a_2, \ldots, a_m \in V$ such that $a_i \in C_i - (\bigcup_{j \neq i} C_j)$ for all $i = 1, \ldots, m$. Assume the labelling of G is such that $\{v_1, v_2\} \subseteq C_1$, $\{v_2, v_3, v_5\} \subseteq C_2$, $\{v_3, v_4\} \subseteq C_3$ and:

if $n = 2$, then $\{v_{n+3}, v_{n+4}\} \subseteq C_4$;

if $n = 3$, then $\{v_2, v_{n+2}, v_{n+3}\} \subseteq C_4$, $\{v_{n+3}, v_{n+4}\} \subseteq C_5$;

.
.
.

if $n = k$, then $\{v_2, v_5, v_6\} \subseteq C_4$, $\{v_2, v_6, v_7\} \subseteq C_5$, ... , $\{v_2, v_{n+2}, v_{n+3}\} \subseteq C_{n+1}$ and $\{v_{k+3}, v_{k+4}\} \subseteq C_{n+2}$. Then $(V, <)$ has the poset in figure 48 as a subposet Q. (Q is isomorphic to P_4.)

Q:

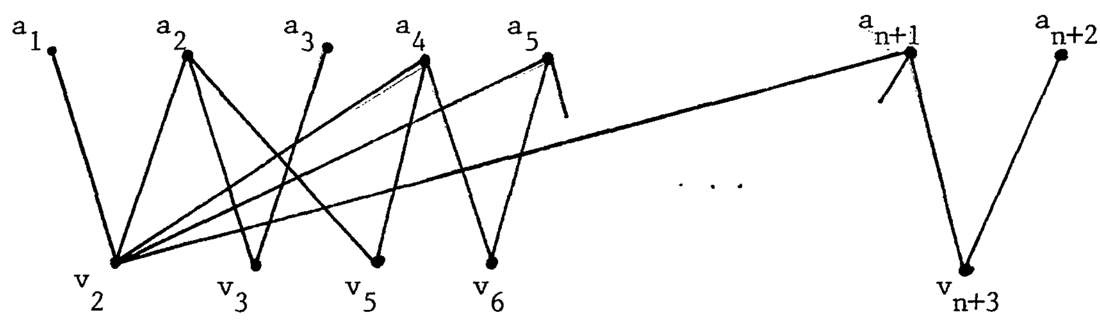


figure 48

Q is also a m-subposet of (V,<) by an argument similar to that used previously.

Conversely, if (V,<) contains P_4 (for a fixed n) as a m-subposet, then any two elements of $\{v_2, v_3, v_5, v_6, v_7, \ldots, v_{n+3}\}$ are below the same maximal element in (V,<) if and only if both are below a same maximal element in P_4. Thus, the UB-graph of P_4 is an induced subgraph of G. It follows that G_4 is an induced subgraph of G since it is an induced subgraph of the UB-graph of P_4.///

__Lemma 6.__ G has G_5 (for some fixed $n \geq 1$) as an induced subgraph if and only if (V,<) has P_5 (same n) as a m-subposet.

__Proof__: Fix $n \geq 1$ and suppose G has G_5 as an induced subgraph. Assume the vertices of G_5 are labelled as in figure 49.

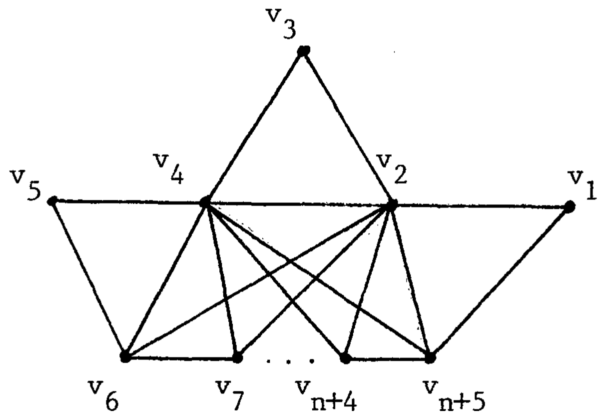


figure 49

Let $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ ($m \geq n+2$) be the UB-clique cover of G and $a_1, a_2, \ldots, a_m \in V$ such that $a_i \in C_i - (\bigcup_{j \neq i} C_j)$ for all $i = 1, 2, \ldots, m$.

<u>If n = 1</u>: Assume the labelling of G is such that $\{v_1, v_2, v_6\} \subseteq C_1$, $\{v_2, v_3, v_4\} \subseteq C_2$, $\{v_4, v_5, v_6\} \subseteq C_3$. Then $(V, <)$ has the poset in figure 50 as a subposet Q. (Q is isomorphic to P_5 with $n = 1$.) Note that it is possible for $a_1 = v_1$, $a_2 = v_3$, or $a_3 = v_5$.

Q:

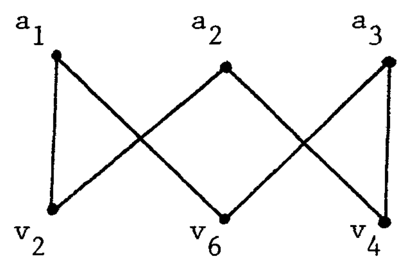


figure 50

Q is also a m-subposet of $(V, <)$ for the existence of a $m \in V$ with $v_i, v_j < m$ but $v_i, v_j \not< m'$ for all $m' \in Q$ would yield an edge which is not in G_5.

<u>If n = 2</u>: Assume the labelling of G is such that $\{v_1, v_2, v_7\} \subseteq C_1$, $\{v_2, v_4, v_6, v_7\} \subseteq C_2$, $\{v_2, v_3, v_4\} \subseteq C_3$, and $\{v_4, v_5, v_6\} \subseteq C_4$. Then $(V, <)$ has

the poset in figure 51 as a subposet Q and Q is isomorphic to G_5 with $n = 2$. Again, Q is also a m-subposet of $(V,<)$ by an argument similar to that above.

Q:

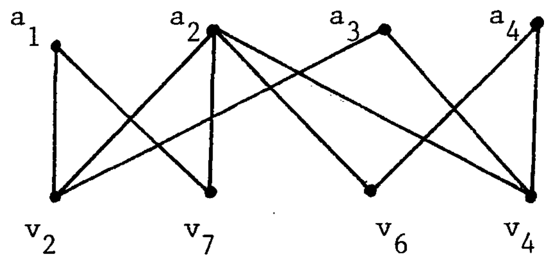


figure 51

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.
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<u>If $n = k$, $k > 2$ and odd</u>: Assume that the labelling of G is such that $\{v_1, v_2, v_{k+5}\} \subseteq C_1$, $\{v_2, v_4, v_{k+4}, v_{k+5}\} \subseteq C_2$, $\{v_2, v_4, v_{k+3}, v_{k+4}\} \subseteq C_3, \ldots,$ $\{v_2, v_4, v_{(k+11)/2}, v_{(k+13)/2}\} \subseteq C_{(k+1)/2}$, $\{v_2, v_3, v_4\} \subseteq C_{(k+3)/2}$, $\{v_2, v_4, v_{(k+9)/2}, v_{(k+11)/2}\} \subseteq C_{(k+5)/2}$, $\{v_2, v_4, v_{(k+7)/2}, v_{(k+9)/2}\} \subseteq C_{(k+7)/2}$, $\ldots$, $\{v_4, v_5, v_6\} \subseteq C_{k+2}$. Then $(V,<)$ has the poset in figure 52 as a subposet Q and Q is isomorphic to P_5 with $n = k$.

Q:

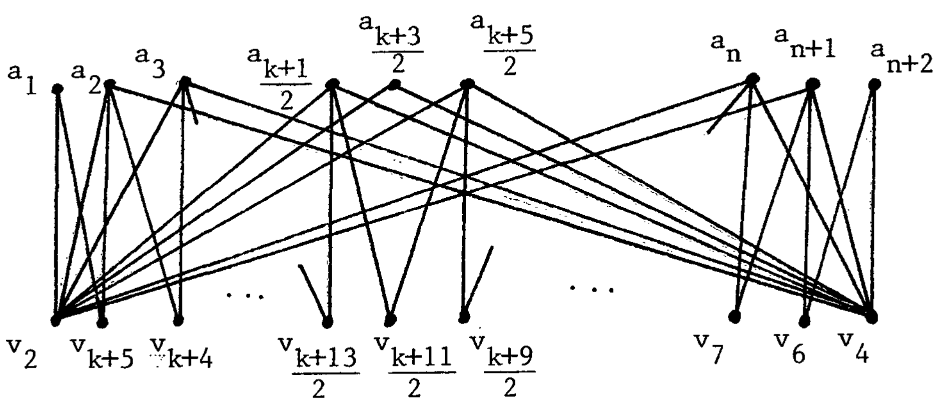


figure 52

Q is also a m-subposet of (V,<) by the usual argument.

If $n = k$, $k \geq 2$ and even: Assume the labelling of G is such that $\{v_1, v_2, v_{k+5}\} \subseteq C_1$, $\{v_2, v_4, v_{k+4}, v_{k+5}\} \subseteq C_2$, $\{v_2, v_4, v_{k+3}, v_{k+4}\} \subseteq C_3$, ... , $\{v_2, v_4, v_{(k+10)/2}, v_{(k+12)/2}\} \subseteq C_{k/2}$, $\{v_2, v_3, v_4\} \subseteq C_{(k+2)/2}$, $\{v_2, v_4, v_{(k+10)/2}, v_{(k+8)/2}\} \subseteq C_{(k+4)/2}$, $\{v_2, v_4, v_{(k+8)/2}, v_{(k+6)/2}\} \subseteq C_{(k+6)/2}$, ... , $\{v_4, v_5, v_6\} \subseteq C_{k+2}$. Then (V,<) has the poset in figure 53 as a subposet Q and Q is isomorphic to P_5, with n = k.

Q:

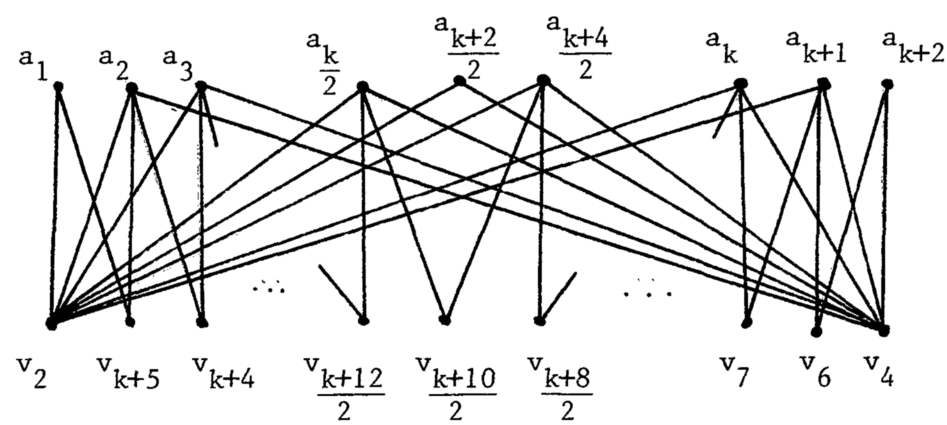


figure 53

Again, Q is a m-subposet of (V,<) by the usual argument.

Conversely, if (V,<) contains P_5 (for some n) as a m-subposet, then an argument similar to the ones in the previous four lemmas imply that G contains G_5 (same n) as an induced subgraph.///

The five lemmas now yield Theorem 22.

Theorem 22. Let (V,<) be a poset and (V,<′) the canonical poset of (V,<). Then the UB-graph G of (V,<) is an interval graph if and only if (V,<′) does not contain any of the posets in figure 42 as a m-subposet.///

Section 3.3 Posets with Interval UB-graphs

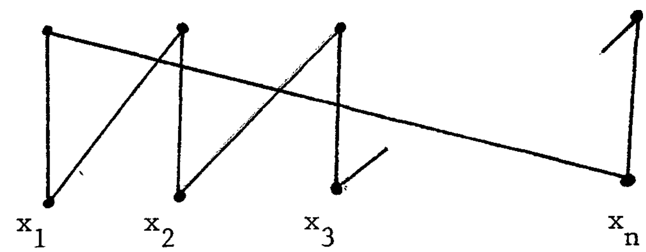
In this section, lemmas analogous to those in Section 3.2 are proved for a general poset $(P,<)$, resulting in a characterization of posets with interval UB-graphs.

Let $(P,<)$ be a poset and M the set of nonisolated maximal elements of P. Define $P(x) = \{y \in P: x < y\}$ for $x \in P-M$.

Definition 6. Let $(P,<)$ be a poset and $x_1,x_2,\dots,x_k \in P-M$. Then the subposet $(I,<)$ of P, where $I = \{x_1,x_2,\dots,x_k\} \cup (\bigcup P(x_i))$ is called the **order ideal (or upset) of P generated by $x_1,x_2,\dots,x_k$** and is denoted $I(x_1,x_2,\dots,x_k)$.

The graphs G_1-G_5 of figure 41 will be referred to in the following lemmas. Also, in the five lemmas which follow, let $G = (V,E)$ be a UB-graph and $(V,<)$ be any poset realizing G, with M the set of maximal elements of $(V,<)$.

Lemma 7. C_n, $n \geq 4$, is an induced subgraph of G if and only if there exists an antichain $\{x_1,x_2,\dots,x_n\} \subseteq V-M$ such that $I(x_1,x_2,\dots,x_n)$ contains the poset in figure 54 as a m-subposet.



(2n vertices)

figure 54

<u>Proof</u>: Suppose C_n, $n \geq 4$, is an induced subgraph of G. Label the vertices of C_n as $x_1,x_2,\ldots,x_n$ where $x_ix_{i+1} \in E$ for all $i = 1,\ldots,n$ and $v_{n+1} = v_1$. It follows from Theorem 1 of [11] that $x_1,x_2,\ldots,x_n \notin M$, so consider the set $\{x_1,x_2,\ldots,x_n\} \subseteq P-M$. If $x_i < x_j$ for some $i \neq j \in \{1,\ldots,n\}$, then $P(x_j) \subseteq P(x_i)$. It follows $Adj(x_j)-\{x_i\} \subseteq Adj(x_i)-\{x_j\}$, with $x_ix_j \in E$. Thus, x_ix_j is an edge in C_n which implies $j = i+1$. But for such an i and j the adjacency relationship is not possible. Hence, $\{x_1,x_2,\ldots,x_n\}$ is an antichain in $(V,<)$.

Since G is a UB-graph, any two endpoints of an edge in G (hence in C_n) have a common upper bound (thus a common maximal element) in $(V,<)$. It easily follows that $I(x_1,x_2,\ldots,x_n)$ contains the poset in figure 54 as a subposet Q. Moreover, Q is a m-subposet, for if there exists $x_i,x_j \in Q$ such that $x_i,x_j < m$ in $(V,<)$, then $j = i+1$ and hence there is a $m' \in Q$ such that $x_i,x_j < m'$.

Conversely, suppose there exists an antichain $\{x_1,x_2,\ldots,x_n\}$, $n \geq 4$, in $(V,<)$ such that the poset in figure 54 is a m-subposet of $I(x_1,\ldots,x_n)$. Then no two x_i,x_j have common upper bound in $(V,<)$ unless they do in the m-subposet. Hence, the UB-graph G contains C_n, $n \geq 4$, as an induced subgraph.///

Recall that a graph $G = (V,E)$ is **chordal** if and only if every cycle of length greater than or equal to four has a chord. (Equivalently, G does not contain an induced subgraph isomorphic to C_n, $n > 3$.)

The following corollary is an immediate consequence of Lemma 7.

Corollary 15. A poset $(P,<)$ has UB-graph that is chordal if and only if there does not exist an antichain $\{x_1,x_2,\ldots,x_n\} \subseteq P-M$ $(n \geq 4)$ such that $I(x_1,\ldots,x_n)$ contains the poset in figure 54 as a m-subposet.///

Lemma 8. G has G_2 as an induced subgraph if and only if there exists an antichain $\{x_1,x_2,x_3,x_4\} \subseteq V-M$ such that $I(x_1,x_2,x_3,x_4)$ contains the poset in figure 55 as a m-subposet.

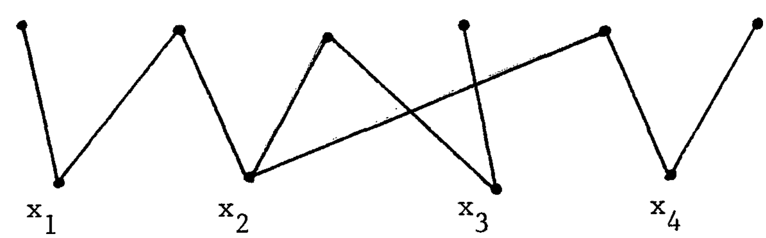


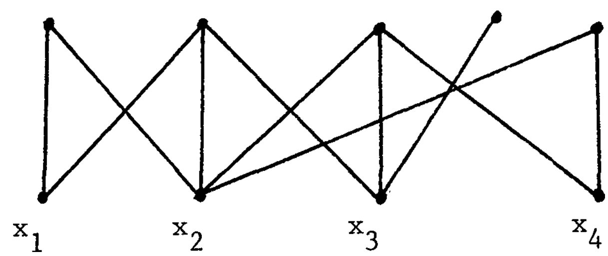
figure 55

Proof: Suppose G_2 is an induced subgraph of G with the vertices of G_2 labelled $x_1,x_2,\ldots,x_7$ such that x_1x_5, x_1x_2, x_2x_3, x_3x_6, x_2x_4, $x_4x_7 \in E$. It follows from the Theorem 1 of McMorris and Zaslavsky [11] that $x_1,x_x,x_3,x_4 \notin M$. So consider the set $\{x_1,x_2,x_3,x_4\} \subseteq V-M$. If $x_i < x_j$, $i \neq j \in \{1,\ldots,4\}$, then $P(x_j) \subseteq P(x_i)$ in $(P,<)$ which implies $x_ix_j \in E(G_2)$ and $Adj(x_j)-\{x_i\} \subseteq Adj(x_i)-\{x_j\}$ in G_2. But this is impossible for $i \neq j \in \{1,\ldots,4\}$. Thus, $\{x_1,x_2,x_3,x_4\}$ is an antichain in $(V,<)$.

The remainder of the proof follows analogously to that for Lemma 3. ///

Lemma 9. G has G_3 as an induced subgraph if and only if there exists an antichain $\{x_1,x_3,x_4\} \subseteq V-M$ and an element $x_2 \in V-M$ such that $x_2 || x_3$ and $I(x_1,x_2,x_3,x_4)$ contains one of the posets in figure 56 as a m-subposet.

(i)



(ii) The poset in (i) with the addition of any combination of the following comparabilities: $x_2 < x_1$; $x_2 < x_4$. That is, any one of the following three posets:

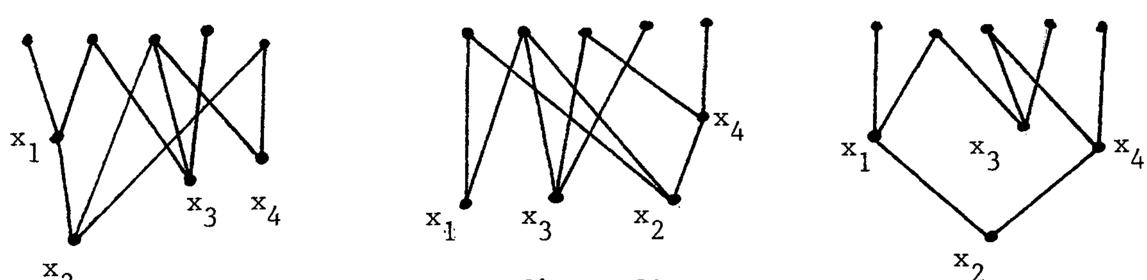


figure 56

Proof: Suppose G_3 is an induced subgraph of G with the vertices of G_3 labelled as in figure 57. It follows from Theorem 1 of [11] that $x_1,x_2,x_3,x_4 \notin M$. Thus, consider the set $\{x_1,x_3,x_4\} \subseteq V-M$. If any two elements x_i,x_j of the set $\{x_1,x_3,x_4\}$ are comparable in $(V,<)$, say $x_i < x_j$, then $P(x_j) \subseteq P(x_i)$ which would imply that $Adj(x_j)-\{x_i\} \subseteq Adj(x_i)$. But this is impossible for distinct $x_i,x_j \in \{x_1,x_3,x_4\}$. It follows that $\{x_1,x_3,x_4\}$ is an antichain of $(V,<)$. A similar argument can be used to show $x_2 || x_3$ in $(V,<)$.

The remainder of the proof follows similarly to that of Lemma 4 with an additional argument similar to that above for the various comparabilities for x_2.///

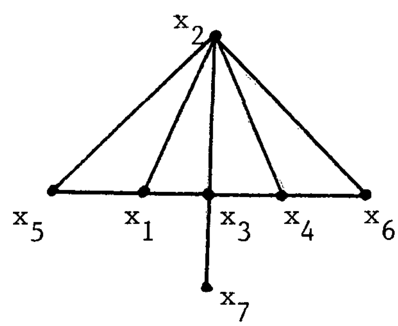
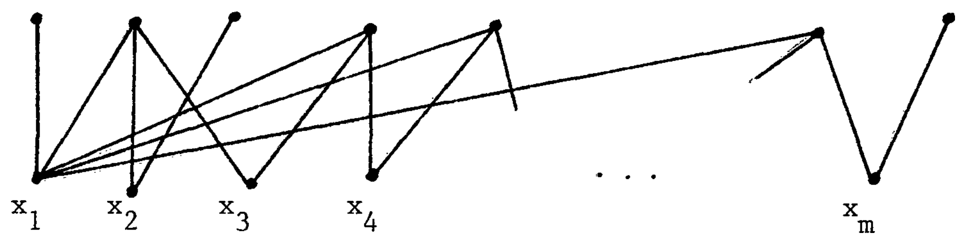


figure 57

Lemma 10. G has G_4 (for some $n \geq 2$) as an induced subgraph if and only if there exists an antichain $\{x_2,x_3,\ldots,x_{m-1},x_m\} \subseteq V-M$ and an element $x_1 \in V-M$ such that $x_1 || x_2$, $x_1 || x_m$ and $I(x_1,x_2,\ldots,x_m)$ contains one of the posets in figure 58 as a m-subposet.

Proof: Suppose G has G_4 as an induced subgraph with the vertices of G_4 labelled as in figure 59. It follows from Theorem 1 of [11] that $x_1,x_2,\ldots,x_m \notin M$. Consider the set $\{x_2,x_3,\ldots,x_m\} \subseteq V-M$. If any two distinct elements $x_i,x_j \in \{x_2,x_3,\ldots.x_m\}$ are comparable in (V,<), say $x_i < x_j$, then $P(x_j) \subseteq P(x_i)$ which implies that $x_ix_j \in E$ and $Adj(x_j)-\{x_i\} \subseteq Adj(x_i)$. But this is impossible for any $x_i,x_j \in \{x_2,x_3,\ldots,x_m\}$. A similar argument is used to show $x_1 || x_2$ and $x_1 || x_m$.

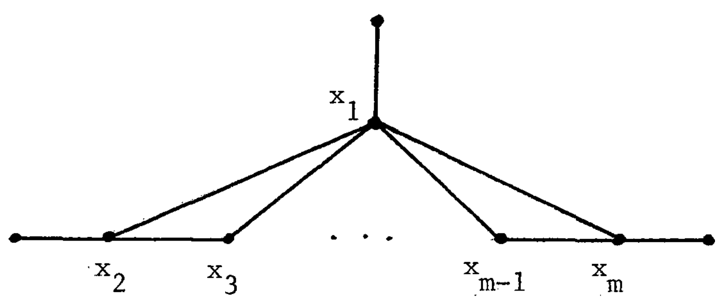
The remainder of the proof is similar to that for Lemma 5 with an additional argument similar to that above for the various comparabilities for x_1.///

(i) 

5 + 2(n-1) vertices, ($n \geq 2$)

(ii) The poset in (i) with the addition of any combination of the following comparabilities: $x_1 < x_3$; $x_1 < x_4$;...; $x_1 < x_{m-1}$. There are 2^{m-3} combinations and hence, 2^{m-3} posets.

figure 58

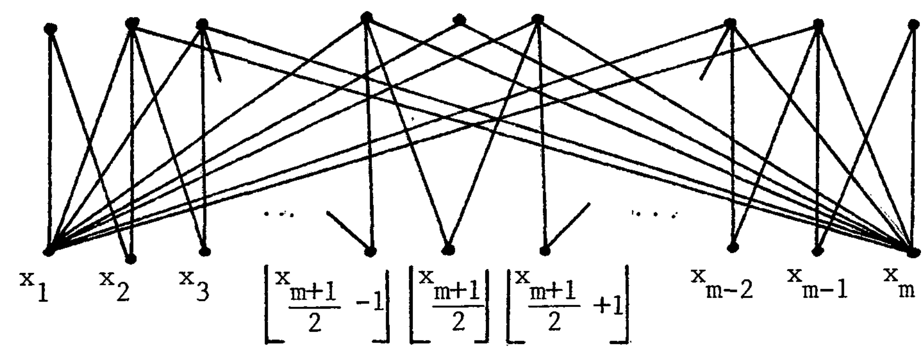


n + 4 vertices, ($n \geq 2$)

figure 59

Lemma 11. G has G_5 (for some $n \geq 1$) as an induced subgraph if and only if there exists an antichain $\{x_2,x_3,\ldots,x_{m-1}\} \subseteq V-M$ and two vertices $x_1,x_m \in V-M$ such that $x_1 || x_m$, $x_1 || x_{m-1}$, $x_m || x_2$ and $I(x_1,x_2,\ldots,x_{m-1},x_m)$ contains one of the posets in figure 60 as a m-subposet.

(i)



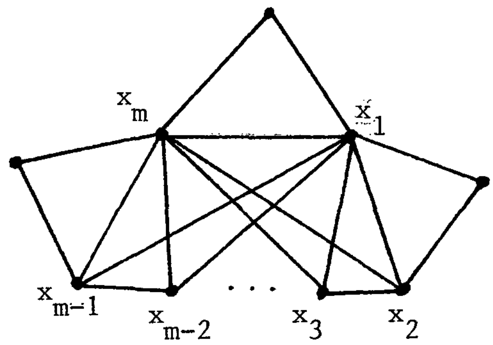
4 + 2n vertices, ($n \geq 1$)

(ii) The poset in (i) with the addition of any combination of the following comparabilities: $x_1 < x_2$; $x_1 < x_3$; ...; $x_1 < x_{m-2}$; $x_m < x_3$; $x_m < x_4$; ... ;$x_m < x_{m-1}$. There are 2^{2m-6} combinations and hence 2^{2m-6} posets.

figure 60

Proof: Suppose G has G_5 (for some $n \geq 1$) as an induced subgraph with the vertices of G_5 labelled as in figure 61. It follows from Theorem 1 of [11] that $x_1, x_2, \ldots, x_m \in V-M$. So consider the set $\{x_2, x_3, \ldots, x_{m-1}\} \subseteq V-M$. If any two distinct $x_i, x_j \in \{x_2, x_3, \ldots, x_{m-1}\}$ are comparable in $(V,<)$, say $x_i < x_j$, then $P(x_j) \subseteq P(x_i)$. This implies $x_i x_j \in E$ and $Adj(x_j)-\{x_i\} \subseteq Adj(x_i)$. But this is impossible for $x_i, x_j \in \{x_2, x_3, \ldots, x_{m-1}\}$. Hence, $\{x_2, x_3, \ldots, x_{m-1}\}$ is an antichain in $(V,<)$. By a similar argument, it can be shown that $x_1 || x_m$, $x_1 || x_{m-1}$ and $x_m || x_2$.

The remainder of the proof follows analogously to that for Lemma 6 with an additional argument similar to that used above for the various comparabilities for x_1 and x_m.///



n + 5 vertices, ($n \geq 1$)

figure 61

The following theorem is a direct consequence of the preceding five lemmas.

Theorem 23. A poset (P,<) has an interval UB-graph if and only if P does not contain a set $\{x_1,x_2,\ldots,x_m\}$, satisfying the conditions of Lemma 7,8,9,10, or 11, such that $I(x_1,x_2,\ldots,x_m)$ contains a poset in figure 54,55,56,58, or 60, respectively, as a m-subposet.///

CHAPTER 4 OPEN PROBLEMS AND FUTURE WORK

Open problems and future work on the material which has been presented are summarized below.

In Section 1.1 no graph G was found for which dk(G) > 2. This problem remains open. Does such a graph exist or is it true that for all graphs G, dk(G) $\leq$ 2? If the latter is the case, what does this mean in terms of ecosystems?

Using the characterization for strict UB-graphs, Theorem 6 of Chapter 1 determines $k_t(G)$ for any graph G. However, only upper and lower bounds were obtained for $dk_t(G)$ since strict DB-graphs have not been characterized. Characterizing strict DB-graphs is not a straightforward result of the characterization for strict UB-graphs as one might at first suppose.

Section 1.3 closed with the question of whether, given a graph G, it is possible to orient the edges of G so that adding loops to all the vertices yields a reflexive digraph with CCE graph G. This remains an interesting open problem.

Uniqueness of DB-graphs and/or CCE graphs has not been addressed in general. It was observed that bipartite graphs are unique DB-graphs of height-1 posets. From the work done in Chapter 2, it should be obvious that no DB-graph of a poset of height-n, n $\geq$ 3, is unique. Thus, it is necessary that a DB-graph be realized by at most a height-2 poset if it is to be unique, and hence, the condition that there do not exist elements $u, v \in V-(M \cup N)$ such that $L'(u) \subseteq L'(v)$ and $U'(v) \subseteq U'(u)$, for sets M and N of Theorem 10 is necessary for uniqueness of DB-graphs, but not sufficient. Conditions for sufficiency remain open problems. However, uniqueness results for UB-graphs have been proved by McMorris and Myers [10].

In Chapter 3, a list of forbidden m-subposets was obtained for posets with interval UB-graphs. In 1982, Steif [17] proved that a forbidden sink induced subdigraph list exists for (acyclic) digraphs with interval competition graphs. Lundgren and Maybee [9] point out that to find such a list appears to be a difficult problem and they give an example to illustrate the difficulty. They suggest finding some modification of the notion of sink induced subdigraph. In view of Theorem 22 and the definition of m-subposet, perhaps redefining a <u>sink induced subdigraph</u> H of a digraph D as an induced subdigraph with the additional property: if $x, y \in H$ and $\overrightarrow{xz}$, $\overrightarrow{yz} \in E(D)$, then there exists a $w \in H$ such that $\overrightarrow{xw}$, $\overrightarrow{yw} \in E(D)$, may be a more useful definition than the original definition by Steif [17]. With this definition, the problem Lundgren and Maybee encounter in their example [9] is eliminated.

BIBLIOGRAPHY

[1] Cohen, J. E.; Interval graphs and food webs: a finding and a problem, RAND Corporation Document 17696-PR, Santa Monica, CA (1968).

[2] Cohen, J. E.; **Food Webs and Niche Space**, Princeton University Press, Princeton, NJ (1978).

[3] Diny, D. A.; The double bound graph of a partially ordered set, to appear in **Journal of Combinatorics, System and Information Sciences.**

[4] Dutton, R. D. and Brigham, R. C.; A characterization of competition graphs, **Discrete Applied Mathematics**, 6(1983), 315 317.

[5] Golumbic, M.; **Algorithmic graph theory and perfect graphs**, Academic Press, New York, NY (1980), 172.

[6] Harary, F.; **Graph theory**, Addison-Wesley, Reading, Mass. (1969).

[7] Harary, F., Norman, R. Z., and Cartwright, D.; **Structural models: an introduction to the theory of directed graphs**, John Wiley & Sons, Inc., New York (1965).

[8] Lekkerkerker, C. B. and Boland, J. C.; Representation of a finite graph by a set of intervals on the real line, **Fund. Math.**, 51(1962), 45-64.

[9] Lundgren, J. R. and Maybee, J. S.; Food webs with interval competition graphs, to appear in **Proceedings of the First Colorado Symposium on Graph Theory**, F. Harary and J. Maybee, eds., Wiley, NY.

[10] McMorris, F. R. and Myers, G. T.; Some uniqueness results for upper bound graphs, **Discrete Mathematics**, 44(1983), 321-323.

[11] McMorris, F. R. and Zaslavsky, T.; Bound graphs of a partially ordered set, **Journal of Combinatorics, Information & System Sciences**, 7(1982), 134-138.

[12] Myers, G.; Ph.D. Dissertation, Bowling Green State University, Bowling Green, OH (1982).

[13] Rabinovitch, I.; Ph.D. Dissertation, Dartmouth College, Hanover, NH (1973).

[14] Roberts, F. S.; Food webs, competition graphs, and the boxicity of ecological phase space, Rutgers University, New Brunswick, NJ (1978).

[15] Roberts, F. S.; Applications of edge coverings by cliques, Dept. of Math., Rutgers University (June 1983).

[16] Roberts, F. R. and Steif, J. E.; A characterization of competition graphs of arbitrary digraphs, to appear in **Discrete Applied Math.**

[17] Steif, J. E.; Frame dimension, generalized competition graphs, and forbidden sublist characterizations, Henry Rutgers Thesis, Dept. of Math., Rutgers Univer., New Brunswick, NJ (1982).

[18] Wolk, E. S.; The comparability graph of a tree, **Proc. Amer. Math. Soc.**, 13(1962), 789-795.

[19] Wolk, E. S.; A note on the comparability graph of a tree, **Proc. Amer. Math. Soc.**, 16(1965), 17-20.