LINEAR ALGEBRA ON THE LIE ALGEBRA ON TWO GENERATORS

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ABSTRACT

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In this thesis, we study the free Lie algebra on two generators and a deformation of the free Lie bracket. Our goal is a hands-on derivation of relations which this deformed Lie bracket satisfies. The technical achievement that makes this possible is the identification of a basis for where the relations occur. Using that basis, we verify and extend the calculations found in Schneps (2006). An interesting connection to the Euler polynomials is also discussed.

I would like to dedicate this work to my family and everyone else who helped make writing this thesis possible.

ACKNOWLEDGMENTS

I would like to acknowledge my family, my teachers, and my professors. Thank you for all of your help.

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PREFACE

In this paper, we begin by studying Lie algebras and their derivations. We recall a couple results about derivations, including that the set of derivations on a Lie algebra is itself a Lie algebra.

Then, we specialize to discussing the free Lie algebra on two generators x and y and its derivations. We further our research by focusing on a deformation of the free Lie bracket. This bracket is partially constructed from derivations and has the same underlying vector space as the free Lie algebra on two generators: $\mathbb{L} = \bigoplus \mathbb{L}_n^d$. It also has antisymmetry and the Jacobi identity, and so is itself a Lie bracket. Working with said Lie bracket raises questions of generators and relations, which we then find when $d = 1$ and $d = 2$, where d is the number of y's in the Lie word.

In Schneps (2006), the author shows that the space of relations may be identified with the space of Modular cusp forms. Using this, she calculated these relations for $n \leq 22$. One goal of this thesis is to give a hands-on derivation of these relations which requires no background in Modular forms. The technical achievement which makes this possible is the identification of a basis for $\mathbb{L}_n^2$, which is eventually used to recalculate Schneps's relations. Said basis was found with the help of planar binary trees, which were used to formulate a correspondence. Specifically, we identified a new set of Triples $(n - s - t - 2, s, t)$, which can be mapped to an $xy-$Tree that is then mapped to a specific Lie word. This correspondence helped reveal that triples where $s = 0$ and t is odd map to a basis for $\mathbb{L}_n^2$. Because of our graphical intuition, we call the basis elements odd combs.

Once the basis was determined, we formulated how to write any given Lie word as a linear combination of odd combs. Note that we are slightly abusing notation to refer to both the graphs and their corresponding Lie words as combs. To do this, the main objective was to find how to rewrite even combs in terms of odd combs, which we then accomplished. The resulting coefficients bear a striking similarity to those appearing in the even-indexed Euler polynomials. We conjecture that the coefficients are always the same and verify the conjecture for the first 12 even-indexed Euler polynomials.

Once we are able to write any Lie word in our basis, we are then able to carry out the calculations to verify Schneps's relations, which were mentioned above. Furthermore, we are able to calculate all relations in $\mathbb{L}_n^2$ for $n \leq 26$.

CHAPTER 1 LIE ALGEBRAS AND $\mathbb{L}$

1.1 Lie Algebras and Derivations

We will begin our discussion by defining the main object that we will be working with throughout this paper.

Definition 1.1. *A Lie algebra is a vector space V over $\mathbb{Q}$ along with a bilinear map*

$$V \times V \to V,$$

$$(x, y) \mapsto [x, y]$$

$$\textit{such that } [x, y] = -[y, x], \forall x, y \in V$$

$$\textit{and } [[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \forall x, y, z \in V$$

Within this context, we can now discuss one of the ways we can manipulate the elements of a Lie algebra.

Definition 1.2. *Let L be a Lie algebra. A derivation of a Lie algebra is a linear map $D : L \to L$ such that*

$$D([a, b]) = [D(a), b] + [a, D(b)]$$

for all $a, b \in L$.

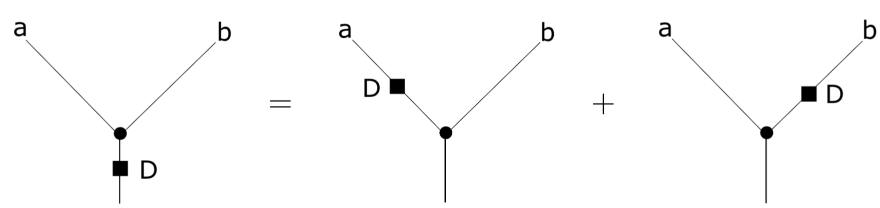


Figure 1.1 Derivation of a Lie Algebra

Lemma 1.1. *For any two derivations D_1 and D_2, $D_1 \circ D_2 - D_2 \circ D_1$ is a derivation.*

Proof: Let $F = D_1 \circ D_2 - D_2 \circ D_1$ and let $a, b \in L$. From Definition 1.2, derivation of a Lie algebra, we get the following:

$$\begin{aligned}
F([a,b]) &= (D_1 \circ D_2 - D_2 \circ D_1)([a,b]) \\
&= D_1([D_2(a), b] + [a, D_2(b)]) - D_2([D_1(a), b] + [a, D_1(b)]) \\
&= [D_1(D_2(a)), b] + [D_2(a), D_1(b)] + [D_1(a), D_2(b)] + [a, D_1(D_2(b))] \\
&\quad -[D_2(D_1(a)), b] - [D_1(a), D_2(b)] - [D_2(a), D_1(b)] - [a, D_2(D_1(b))] \\
&= [D_1(D_2(a)), b] - [D_2(D_1(a)), b] + [a, D_1(D_2(b))] - [a, D_2(D_1(b))] \\
&= [F(a), b] + [a, F(b)].
\end{aligned}$$

Therefore, $D_1 \circ D_2 - D_2 \circ D_1$ is a derivation. □

Theorem 1.1. *The set of derivations on a Lie algebra L is itself a Lie algebra.*

Proof: Firstly, since derivations are closed under scalar multiplication and addition, the derivations on a Lie algebra are a vector space.

Secondly, since $D_1 \circ D_2 - D_2 \circ D_1$ is a derivation, the bilinear map is

$$\begin{aligned}
Der(L) \times Der(L) &\to Der(L), \\
(D_1, D_2) &\mapsto [D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1.
\end{aligned}$$

It's easy to see that the map is antisymmetric, since
$D_1 \circ D_2 - D_2 \circ D_1 = -(D_2 \circ D_1 - D_1 \circ D_2)$.

Next, we need to show that the map has the Jacobi identity. Let D_1, D_2, and D_3 be derivations.

We can then calculate the following: $[[D_1, D_2], D_3] + [[D_2, D_3], D_1] + [[D_3, D_1], D_2]$

$$\begin{aligned}
= \;& [D_1, D_2] \circ D_3 - D_3 \circ [D_1, D_2] + [D_2, D_3] \circ D_1 - D_1 \circ [D_2, D_3] + [D_3, D_1] \circ D_2 \\
& - D_2 \circ [D_3, D_1]
\end{aligned}$$

$$
\begin{aligned}
&= (D_1 \circ D_2 - D_2 \circ D_1) \circ D_3 - D_3 \circ (D_1 \circ D_2 - D_2 \circ D_1) + (D_2 \circ D_3 - D_3 \circ D_2) \circ D_1 \\
&\quad - D_1 \circ (D_2 \circ D_3 - D_3 \circ D_2) + (D_3 \circ D_1 - D_1 \circ D_3) \circ D_2 - D_2 \circ (D_3 \circ D_1 - D_1 \circ D_3) \\
&= (D_1 \circ D_2) \circ D_3 - (D_2 \circ D_1) \circ D_3 - D_3 \circ (D_1 \circ D_2) + D_3 \circ (D_2 \circ D_1) \\
&\quad + (D_2 \circ D_3) \circ D_1 - (D_3 \circ D_2) \circ D_1 - D_1 \circ (D_2 \circ D_3) + D_1 \circ (D_3 \circ D_2) \\
&\quad + (D_3 \circ D_1) \circ D_2 - (D_1 \circ D_3) \circ D_2 - D_2 \circ (D_3 \circ D_1) + D_2 \circ (D_1 \circ D_3) \\
&= 0.
\end{aligned}
$$

The above calculation works because addition is distributive with the composition of functions and because the composition of functions is associative. □

Note that the above proof would work for any associative operation, not just the composition of functions.

1.2 The Free Lie Algebra, $\mathbb{L}$

Definition 1.3. *The free Lie algebra on two generators has the underlying vector space* $\mathbb{L} = \bigoplus \mathbb{L}_n^d$. $\mathbb{L}_n^d$ *is a span of Lie words with two generators,* x *and* y. *Specifically, the elements are of word length* n *where* y *appears* d *times. Note that* $d < n$, *where* $n \geq 1$ *and* $d \geq 0$. *The Lie words in* $\mathbb{L}_n^d$ *are then bracketed together with the bracket* $[-,-]$ *to generate more Lie words. Therefore, we can see that* $[\mathbb{L}_{n_1}^{d_1}, \mathbb{L}_{n_2}^{d_2}] \subseteq \mathbb{L}_{n_1+n_2}^{d_1+d_2}$.

Examples of elements in $\mathbb{L}$ include the following:

$$
\begin{aligned}
x &\in \mathbb{L}_1^0 \\
y &\in \mathbb{L}_1^1 \\
-[y,x] = [x,y] &\in [\mathbb{L}_1^0, \mathbb{L}_1^1] \subseteq \mathbb{L}_2^1 \\
[y,x] = -[x,y] &\in \mathbb{L}_2^1 \\
[x,[y,[[[y,x],x],[x,y]]]] &\in [\mathbb{L}_5^2, \mathbb{L}_2^1] \subseteq \mathbb{L}_7^3
\end{aligned}
$$

$$
\begin{aligned}
[[x,[[x,y],[[y,[[x,y],y]]]]],[[[x,y],x],[y,x]]] &\in [\mathbb{L}_7^4,\mathbb{L}_5^2] \subseteq \mathbb{L}_{12}^6 \\
[x,[x,[x,y]]] &\in \mathbb{L}_4^1 \\
[[x,y],[x,[x,y]]] = -[[x,[x,y]],[x,y]] &\in \mathbb{L}_5^2.
\end{aligned}
$$

Lemma 1.2. *Let $a, b \in \mathbb{L}$. There exists a unique derivation $D : L \to L$ such that $D(x) = a$ and $D(y) = b$.*

Proof: We will induct on word length to prove that a derivation which satisfies $D(x) = a$ and $D(y) = b$ is uniquely determined. Our base case is that the image of a Lie word with length 1 is uniquely determined. This is defined as $D(x) = a$ and $D(y) = b$.

For our induction step, we will assume that the image of all Lie words of length $k \geq 1$ are uniquely defined. We want to prove that the image of all Lie words of length $k+1$ is therefore also uniquely defined.

We know that any Lie word of length greater than or equal to 2 looks like $[f, g]$ such that f and g are of a length greater than or equal to 1. Therefore, if a Lie word L is of length $k+1$, it can also be written in the form $L = [f, g]$. Since both f and g are of a length greater than or equal to 1, f and g must both be of a length less than $k+1$; i.e., f and g must have a word length of at most k. Therefore, when using Definition 1.2, we get $D([f, g]) = [D(f), g] + [f, D(g)]$, where $D(f)$ and $D(g)$ are uniquely defined by assumption. Therefore, the image of a Lie word L of length $k+1$ is uniquely determined.

Therefore, a derivation which satisfies $D(x) = a$ and $D(y) = b$ is uniquely determined. □

Definition 1.4. *For $f \in \mathbb{L}$, define D_f to be the unique derivation such that the following holds:*

$$D_f(x) = 0 \textit{ and } D_f(y) = [y, f]$$

This derivation allows us to deform the free Lie bracket on $\mathbb{L}$ to give the same underlying vector space a new structure. In *On the Poisson bracket on the free Lie algebra in two generators*, the following deformation is the titular Poisson bracket (Schneps, 2006, Page 1).

Definition 1.5. *The deformed Lie bracket has the same underlying vector space as the free Lie algebra. The Lie bracket is the only different aspect, so it is instead denoted with the curly brackets:* $\{-,-\}$. *Specifically, it is given by*

$$\{f,g\} = [f,g] + D_f(g) - D_g(f).$$

Now that we have defined the deformed Lie bracket, we can ask questions about it. For example, how do we know that the curly bracket has antisymmetry and follows the Jacobi identity? This proof can be found Schneps's paper, but we will be demonstrating a more explicit computation (Schneps, 2006, Page 4).

Lemma 1.3. *The deformed Lie bracket has antisymmetry and follows the Jacobi identity.*

Proof: Let f, g, and $h \in \mathbb{L}$.

First, we'll show that $\{f,g\} + \{g,f\} = 0$ with the following calculation:

$$\{f,g\} + \{g,f\} = [f,g] + D_f(g) - D_g(f) + [g,f] + D_g(f) - D_f(g) = 0.$$

Second, we'll show that $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$.

From Definition 1.5, we can calculate the following:

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}$$

$$\begin{aligned}
&= \{[f,g],h\} + \{D_f(g),h\} - \{D_g(f),h\} + \{[g,h],f\} + \{D_g(h),f\} - \{D_h(g),f\} \\
&\quad + \{[h,f],g\} + \{D_h(f),g\} - \{D_f(h),g\} \\
&= [[f,g],h] + D_{[f,g]}(h) - D_h([f,g]) + [D_f(g),h] + D_{D_f(g)}(h) - D_h(D_f(g)) \\
&\quad - [D_g(f),h] - D_{D_g(f)}(h) + D_h(D_g(f)) + [[g,h],f] + D_{[g,h]}(f) - D_f([g,h]) \\
&\quad + [D_g(h),f] + D_{D_g(h)}(f) - D_f(D_g(h)) - [D_h(g),f] - D_{D_h(g)}(f) + D_f(D_h(g)) \\
&\quad + [[h,f],g] + D_{[h,f]}(g) - D_g([h,f]) + [D_h(f),g] + D_{D_h(f)}(g) - D_g(D_h(f)) \\
&\quad - [D_f(h),g] - D_{D_f(h)}(g) + D_g(D_f(h))).
\end{aligned}$$

From the Jacobi identity, we know that $[[f,g],h]+[[g,h],f]+[[h,f],g]=0$. Using that and Definition 1.2, we find that $\{[f,g],h\}+\{[g,h],f\}+\{[h,f],g\}$

$$\begin{aligned}
= \quad & D_{[f,g]}(h)-[D_h(f),g]-[f,D_h(g)]+[D_f(g),h]+D_{D_f(g)}(h)-D_h(D_f(g)) \\
& -[D_g(f),h]-D_{D_g(f)}(h)+D_h(D_g(f))+D_{[g,h]}(f)-[D_f(g),h]-[g,D_f(h)] \\
& +[D_g(h),f]+D_{D_g(h)}(f)-D_f(D_g(h))-[D_h(g),f]-D_{D_h(g)}(f)+D_f(D_h(g)) \\
& +D_{[h,f]}(g)-[D_g(h),f]-[h,D_g(f)]+[D_h(f),g]+D_{D_h(f)}(g)-D_g(D_h(f)) \\
& -[D_f(h),g]-D_{D_f(h)}(g)+D_g(D_f(h)) \\
= \quad & D_{[f,g]}(h)+D_{[g,h]}(f)+D_{[h,f]}(g)+D_{D_f(g)}(h)-D_{D_f(h)}(g)-D_{D_g(f)}(h)+D_{D_g(h)}(f) \\
& -D_{D_h(g)}(f)+D_{D_h(f)}(g)-D_f(D_g(h))+D_f(D_h(g))-D_g(D_h(f))+D_g(D_f(h)) \\
& -D_h(D_f(g))+D_h(D_g(f))-[g,D_f(h)]-[D_f(h),g]-[D_g(f),h]-[h,D_g(f)] \\
& -[f,D_h(g)]-[D_h(g),f].
\end{aligned}$$

Due to antisymmetry, $\{[f,g],h\}+\{[g,h],f\}+\{[h,f],g\}$

$$\begin{aligned}
= \quad & D_{[f,g]}(h)+D_{[g,h]}(f)+D_{[h,f]}(g)+D_{D_f(g)}(h)-D_{D_f(h)}(g)-D_{D_g(f)}(h)+D_{D_g(h)}(f) \\
& -D_{D_h(g)}(f)+D_{D_h(f)}(g)-D_f(D_g(h))+D_f(D_h(g))-D_g(D_h(f))+D_g(D_f(h)) \\
& -D_h(D_f(g))+D_h(D_g(f)).
\end{aligned}$$

Claim: For $f,g\in\mathbb{L}$, $D_{[f,g]}=-D_{D_f(g)}+D_{D_g(f)}+D_f\circ D_g-D_g\circ D_f$

Proof of Claim: First, we know that a sum of derivations is a derivation. Additionally, we know that $D_{[f,g]}$, $D_{D_f(g)}$, and $D_{D_g(f)}$ are derivations. Lemma 1.1 also gives us that $D_f\circ D_g-D_g\circ D_f$ is a derivation. Therefore, $-D_{D_f(g)}+D_{D_g(f)}+D_f\circ D_g-D_g\circ D_f$ is a derivation.

Therefore, to show that $D_{[f,g]}$ is equal to $-D_{D_f(g)}+D_{D_g(f)}+D_f\circ D_g-D_g\circ D_f$, we need to show what both do to x and y. If the results are equal, then, since derivations are unique by Lemma 1.2, the two derivations are equal.

First, we will calculate $D_{[f,g]}$:

$$\begin{aligned} D_{[f,g]}(x) &= 0 \\ D_{[f,g]}(y) &= [y,[f,g]]. \end{aligned}$$

Second, we will calculate $-D_{D_f(g)} + D_{D_g(f)} + D_f \circ D_g - D_g \circ D_f$. Note that, for any $a \in \mathbb{L}$, $D_a(0) = 0$ because the derivation is not being applied anywhere. Therefore, $-D_{D_f(g)}(x) + D_{D_g(f)}(x) + D_f(D_g(x)) - D_g(D_f(x)) = 0$.

Evaluating at y we find that $-D_{D_f(g)}(y) + D_{D_g(f)}(y) + D_f(D_g(y)) - D_g(D_f(y)) =$

$$\begin{aligned} &- [y, D_f(g)] + [y, D_g(f)] + [D_f(y), g] + [y, D_f(g)] - [D_g(y), f] - [y, D_g(f)] \\ &= [[y,f],g] - [[y,g],f] \\ &= -[g,[y,f]] + [f,[y,g]] \\ &= -[g,[y,f]] - [f,[g,y]] = [y,[f,g]] \text{ (by the Jacobi identity).} \end{aligned}$$

Therefore, for $f, g \in \mathbb{L}$, $D_{[f,g]} = -D_{D_f(g)} + D_{D_g(f)} + D_f \circ D_g - D_g \circ D_f$.

Returning to our proof, we still want to show that the equation will equal zero. Using our claim, we can calculate the following: $\{[f,g],h\} + \{[g,h],f\} + \{[h,f],g\}$

$$\begin{aligned} = \; & D_{[f,g]}(h) + D_{[g,h]}(f) + D_{[h,f]}(g) + D_{D_f(g)}(h) - D_{D_f(h)}(g) - D_{D_g(f)}(h) + D_{D_g(h)}(f) \\ & -D_{D_h(g)}(f) + D_{D_h(f)}(g) - D_f(D_g(h)) + D_f(D_h(g)) - D_g(D_h(f)) + D_g(D_f(h)) \\ & -D_h(D_f(g)) + D_h(D_g(f)) \\ = \; & -D_{D_f(g)}(h) + D_{D_g(f)}(h) + D_f(D_g(h)) - D_g(D_f(h)) \\ & +D_{D_f(g)}(h) - D_{D_g(f)}(h) - D_f(D_g(h)) + D_g(D_f(h)) \\ & -D_{D_g(h)}(f) + D_{D_h(g)}(f) + D_g(D_h(f)) - D_h(D_g(f)) \\ & +D_{D_g(h)}(f) - D_{D_h(g)}(f) - D_g(D_h(f)) + D_h(D_g(f)) \\ & -D_{D_h(f)}(g) + D_{D_f(h)}(g) + D_h(D_f(g)) - D_f(D_h(g)) \\ & +D_{D_h(f)}(g) - D_{D_f(h)}(g) - D_h(D_f(g)) + D_f(D_h(g)) \\ = \; & 0 \end{aligned}$$

Therefore, $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.

Therefore, the deformed Lie bracket has antisymmetry and follows the Jacobi identity. $\square$

Note that, when referring to the free Lie bracket, we will use $(\mathbb{L}, [-, -])$. Additionally, when referring to the not free bracket, we will use $(\mathbb{L}, \{-, -\})$.

Since $(\mathbb{L}, \{-, -\})$ is not a free Lie algebra, we might ask what are its generators and relations. We're not going to look for all generators; instead, we'll stick to low values of d. Particularly, we'll restrict ourselves to $d = 1$ and $d = 2$.

If $d = 1$, then the vector space $\mathbb{L}_i^1$ is one dimensional and spanned by $f_i = [x, [..., [x, y]]]$. Examples of these generators include the following:

$$\begin{aligned} f_3 &= [x, [x, y]] \in \mathbb{L}_3^1 \\ f_4 &= [x, [x, [x, y]]] \in \mathbb{L}_4^1 \\ f_5 &= [x, [x, [x, [x, y]]]] \in \mathbb{L}_5^1. \end{aligned}$$

Combined together, these elements form a basis for the subspace over all n.

Therefore, if $d = 2$, then we can bracket them together as $\{f_i, f_j\}$. However, the relations between $\{f_i, f_j\}$ (where i and j vary, but n remains the same) are not immediately apparent, and it is one of the goals of this paper to find some.

Therefore, our question specifically becomes the following: What are the relations between different $\{f_i, f_j\}$ in $\mathbb{L}_n^2$? In Schneps (2006), she gives a correspondence of Modular cusp forms as an answer to this question. However, we want a more down to earth and explicit formula, which is what this thesis will cover.

In order to obtain such a formula, we will need a basis for $\mathbb{L}_n^2$ because that is where our questions live. In order to obtain that basis, we will need to work with Planar Binary Trees.

CHAPTER 2 FROM TREES TO LIE WORDS

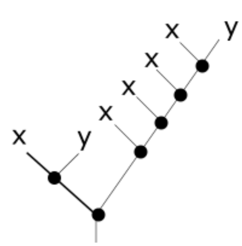
In this chapter, we will ultimately create a correspondence from a new set A_n, which we will call the set of triples, to a collection of planar binary tree graphs, to Lie words in $\mathbb{L}_n^2$. We begin this chapter by describing the specific graphs, which we label T_n^2, that map to the Lie words for which we are eventually going to find a basis. Then, we will outline the set of triples. We will conclude this chapter by specifically defining the aforementioned correspondence.

2.1 Planar Binary Trees

Trees are connected graphs consisting of vertices and edges with no circuits. Binary trees possess three edges at each internal vertex. Therefore, planar binary trees are binary trees embedded in the plane with a choice of L (what goes on the left of a vertex), R (what goes on the right of a vertex), and D (what goes down/below the vertex).

Definition 2.1. *An* <u>XY*-Tree*</u> *is a planar binary tree with external vertices labeled by x and y, except for one root which is drawn at the bottom (i.e., below the lowest internal vertex). We will now define the set of XY-Trees as T_n^d, where n is the total number of external vertices and y appears d times. Note that $d < n$, where $n \geq 1$ and $d \geq 0$.*

XY-Trees can be used to represent Lie words in $\mathbb{L}_n^d$, where each bracket indicates an internal vertex. An example of this relation can be seen in Figure 2.1.



$\mapsto [[x, y], [x, [x, [x, [x, y]]]]] \in \mathbb{L}_7^2$

Figure 2.1 A Tree Mapped to its Lie Word

Definition 2.2. *The* <u>*branching point*</u> *of a tree in T_n^2 is the vertex of greatest height (in the context of the tree visualization) such that both y's are above it.*

The branching points of two trees (including the tree found in Figure 2.1) are shown to be red in Figure 2.2.

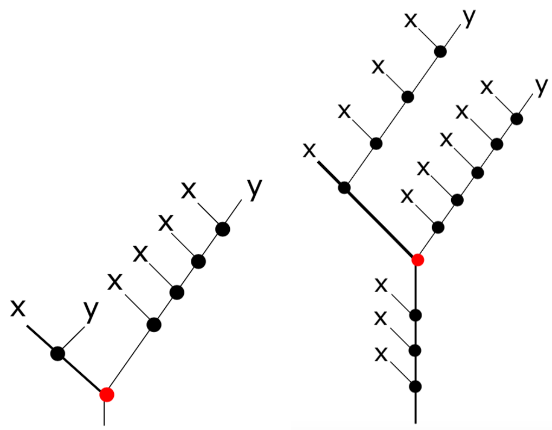
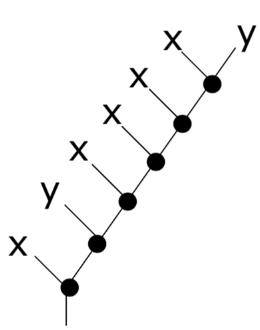


Figure 2.2 Branching Points

Definition 2.3. *<u>(Right) combs</u> in T_n^2 are trees where all of the external edges labeled with an x and one external edge labeled with a y branch off to the left of their internal vertices. In addition, all internal edges and the topmost external edge labeled with a y are therefore branching off to the right of their internal vertices.*

Right combs look like the tree picture on the left hand side of Figure 2.3, where, excluding the two topmost external vertices, each external vertex is labeled with an x or y accordingly.

When mapping a right comb to a Lie word, the Lie word will look like the following: $[-,[-,[-,...,[-,-]...]]]$. An example of this mapping can be seen in Figure 2.3.



$$\mapsto [x,[y,[x,[x,[x,[x,y]]]]]] \in \mathbb{L}_7^2$$

Figure 2.3 A Right Comb Mapped to its Lie Word

2.2 Triples

Now, we will define the set of triples:

$$A_n = \{(r,s,t) | s \leq t, r+s+t = n-2, 0 \leq r, 0 \leq s, \text{and } 1 \leq t\}.$$

Using the branching point, we can define a function $A_n \to T_n^2$ in the following way:

$$(r,s,t) \mapsto T_{(r,s,t)},$$

where r is the number of edges below the branching point, and s and t are the number of edges on the above branches such that $s \leq t$. Additionally, $r+s+t = n-2$, $0 \leq r$, $0 \leq s$, and $1 \leq t$. Another way to think of r, s, and t for the mapping to an XY-Tree is that they represent the number of x's on their respective section of the tree.

In $T_{(r,s,t)}$, the trees will follow the convention that all of the external vertices labeled with an x branch off to the left. Note that we will refer to this as the x on the left convention. This will leave the two external y edges to either branch off to the right of their internal edges or follow Definition 2.3, right combs. Additionally, all s branches will be found on the left side of the graph; i.e., the shorter of the two branches ending with y at the top will be on left side of the branching point.

Visually, the mapping can be seen in Figure 2.4.

$(r,s,t) \mapsto$

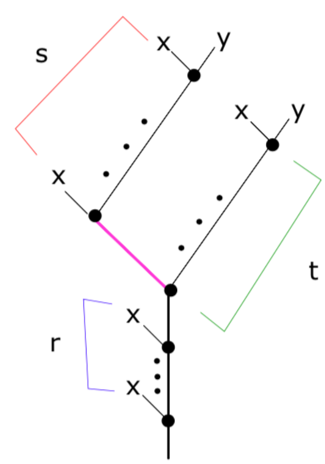


Figure 2.4 Triples to Trees

Every right comb in the image of A_n is of the form $(r,0,t)$ for some r,t.

2.3 Correspondence

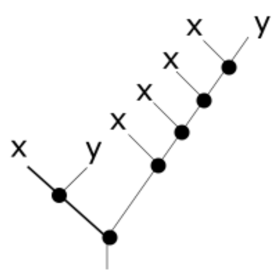
Definition 2.4. *For each (r,s,t) in A_n, there is a corresponding Lie word $[r,s,t] \in \mathbb{L}_n^2$. The following map outlines the connection between all three of the discussed representations via the maps defined above:*

$$\begin{aligned} A_n \quad &\to T_n^2 \quad \to \mathbb{L}_n^2 \\ (r,s,t) \quad &\mapsto T_{(r,s,t)} \quad \mapsto [r,s,t] \end{aligned} \tag{2.3.1}$$

This mapping follows from the map from $A_n \to T_n^2$ discussed in Section 2.2 Triples and the map from $T_n^2 \to \mathbb{L}_n^2$ discussed in Section 2.1 Planar Binary Trees.

Examples of this correspondence can be found in Figures 2.5 and and 2.6.

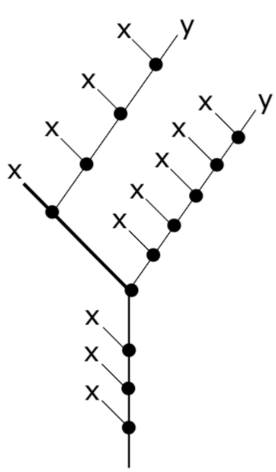
$(0,1,4) \mapsto$



$\mapsto [[x,y],[x,[x,[x,[x,y]]]]] = [0,1,4] \in \mathbb{L}_7^2$

Figure 2.5 Mapping the Triple $(3,4,5)$ to its Corresponding Lie Word

$(3,4,5) \mapsto$



$\mapsto [x,...,[x,y]]]],[x,...,[x,y]...] = [3,4,5] \in \mathbb{L}_{14}^2$

Figure 2.6 The Correspondence of a Triple in A_{14} (Left) is Sent to the XY-Tree $T_{(3,4,5)}$ (Center), Which is Then Sent to its Corresponding Lie Word (Right)

CHAPTER 3 FINDING A BASIS FOR $\mathbb{L}_n^2$

Now that we have constructed Definition 2.4, we will now work our way towards finding a basis. However, before we can determine our basis, we must first determine some sets which span $\mathbb{L}_n^2$.

3.1 Spanning Theorems

Theorem 3.1. *The image of the composite in Equation* 2.3.1 *is a spanning set for* $\mathbb{L}_n^2$.

Proof: It is enough to show that each Lie word is in the span of the image. Therefore, we can take any Lie word and apply the x on the left convention to get a Lie word of the form $\pm[[-,[x,y]],[-,[x,y]]]$, which is in the image of the composite. Therefore, since any Lie word that is the same after the application of the convention is equal except possibly by sign, the above composite spans $\mathbb{L}_n^2$. □

Recalling Definition 2.3, we will now distinguish between the following two types of combs.

Definition 3.1. *<u>Even combs</u> occur if the t of a (r,s,t) triple is an even integer.*

Therefore, even right combs map to $[a,0,2i]$ such that $a+2i=n-2$, $0\leq a$, and $1\leq i$.

Definition 3.2. *<u>Odd combs</u> occur if the t of an (r,s,t) triple is an odd integer.*

Therefore, odd right combs map to $[a,0,2i+1]$ such that $a+2i+1=n-2$, $0\leq a$, and $0\leq i$.

When n is even, there are $\frac{n-2}{2}$ odd combs. Additionally, when n is odd, there are $\frac{n-1}{2}$ odd combs.

Lemma 3.1. $[r,s,s]=0$.

Proof: Anti-symmetry. □

Lemma 3.2. $[r,s,t]-[r+1,s-1,t]+[r,s-1,t+1]=0$.

First, let's build some intuition for this proof by looking at a general (r, s, t) triple, as shown in Figure 2.4. Notice that the s and t branches of $T_{(r,s,t)}$, when taken alone, look like trees in T_i^1. It's obvious that trees in T_i^1 map to Lie words in $\mathbb{L}_i^1$. As mentioned in *Section 1.2 The Free Lie Algebra,* $\mathbb{L}$, if $d = 1$, then the vector space is one dimensional and spanned by $f_i = [x, [..., [x, y]]] \in \mathbb{L}_i^1$. Therefore, the s and t branches could be relabeled as f_{s+1} and f_{t+1}, respectively. This relabeling and the complete mapping as defined in *Section 2.3 Correspondence* can be seen in Figure 3.1.

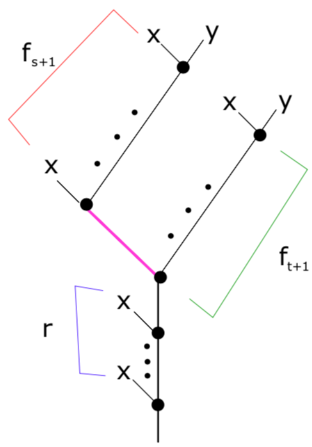
$(r, s, t) \mapsto$  $\mapsto [x, ..., [[x, ...[x, y]], [x, ..., [x, y]]]]$

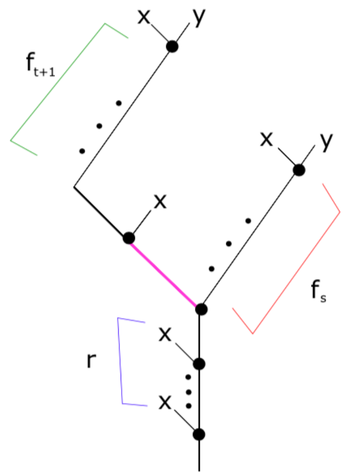
Figure 3.1 Tree Associated to a General (r, s, t)

When applying the Jacobi identity to $T_{(r,s,t)}$, we can think of it as choosing an edge and rotating the three edges above the two vertices of that edge. In this case, our desired results come from when we pick the lowest edge above the branching point which lies on the s branch. In Figure 3.1, this edge is colored pink.

Now, we will perform the first rotation, which will result in the number of x's below the branching point remaining the same, but the values of the s and t branches changing such that the Lie word is $[x, ..., [[f_{t+1}, x], f_s]]$. This can be seen visually in Figure 3.2.

Next, we will perform the second rotation, which will result in the number of x's below the branching point increasing by one and the value of the s branch decreasing by one. The t branch will not change. The resulting Lie word is $[x, ..., [x, [f_s, f_{t+1}]]]$. This can be seen visually in Figure 3.3.

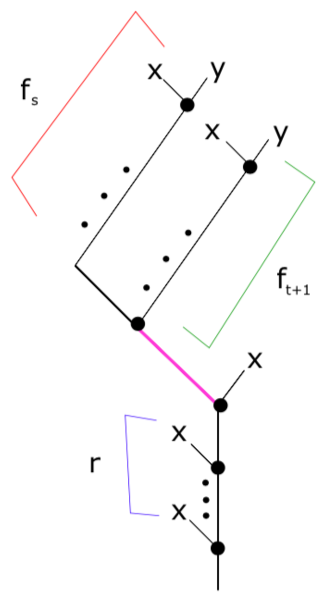
$(r, s-1, t+1) \mapsto$



$\mapsto [x, ..., [[f_{t+1}, x], f_s]]$

Figure 3.2 The First Rotation

$(r+1, s-1, t) \mapsto$



$\mapsto [x, ..., [x, [f_s, f_{t+1}]]]$

Figure 3.3 The Second Rotation

In the final step of building our intuition, if we ignore the r-times that x appears on the left of each the three terms that we found, then we will get an equation which looks like the Jacobi identity.

Now, keeping this intuition in mind, we will more rigorously prove Lemma 3.2.

Proof: First, we will start with a generic $(r, s, t) \in A_n$. Therefore,

$$\begin{aligned}(r, s, t) \mapsto [r, s, t] &= [x, ..., [[x, ...[x, y]], [x, ..., [x, y]]]] \\ &= [x, ..., [f_{s+1}, f_{t+1}]] \\ &= [x, ..., [[x, f_s], f_{t+1}]] \text{ such that there are } r\ x\text{'s on the left.}\end{aligned}$$

Next, we find that

$$\begin{aligned}(r, s-1, t+1) \mapsto [r, s-1, t+1] &= [x, ..., [f_s, [x, f_{t+1}]] \\ &= -[x, ..., [[x, f_{t+1}], f_s]] \\ &= [x, ..., [[f_{t+1}, x], f_s]].\end{aligned}$$

Then, we find that

$$\begin{aligned}-(r+1, s-1, t) \mapsto -[r+1, s-1, t] &= -[x, ..., [x, [f_s, f_{t+1}]]] \\ &= [x, ..., [[f_s, f_{t+1}], x]].\end{aligned}$$

Therefore, by combining the above calculations,

$$\begin{aligned}&(r, s, t) - (r+1, s-1, t) + (r, s-1, t+1) \\ \mapsto\ &[x, ..., [[x, f_s], f_{t+1}]] + [x, ..., [[f_s, f_{t+1}], x]] + [x, ..., [[f_{t+1}, x], f_s]]\end{aligned}$$

Finally, since each of the three terms that we found have r-times that x appears on the left, we can ignore those x's. Therefore, we get the following, which looks like the Jacobi identity:

$$\begin{aligned}[[x, f_s], f_{t+1}] + [[f_s, f_{t+1}], x] + [[f_{t+1}, x], f_s] &= 0 \\ \text{i.e., } [r, s, t] - [r+1, s-1, t] + [r, s-1, t+1] &= 0. \ \square\end{aligned}$$

By abuse of terminology, we refer to both the applicable trees in T_n^2 and their image in $\mathbb{L}_n^2$ as right odd or even combs.

Proposition 3.1. *Right combs span* $\mathbb{L}_n^2$.

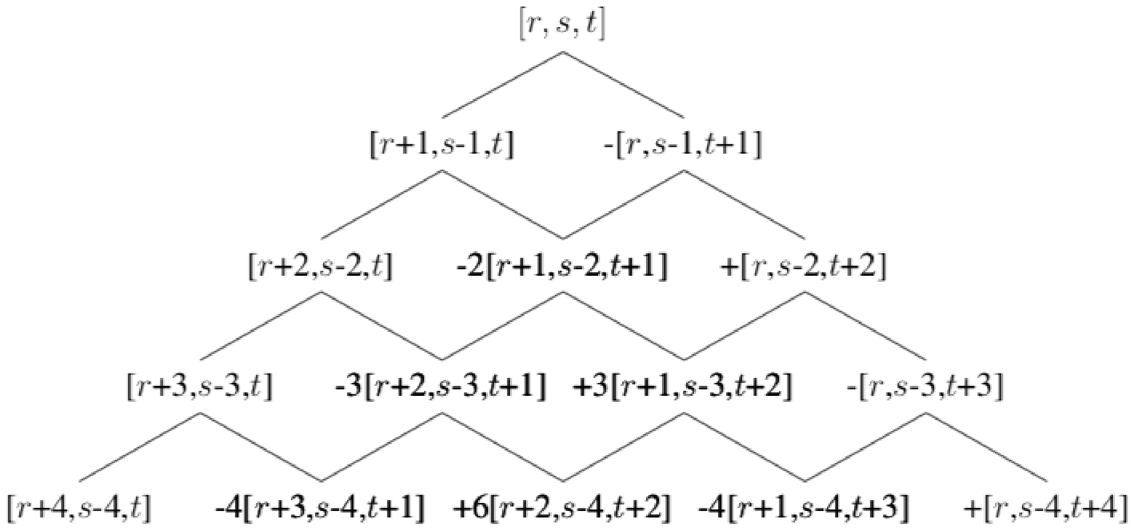
Proof In order to prove that right combs span $\mathbb{L}_n^2$, we need to show that any Lie word $[r, s, t] \in \mathbb{L}_n^2$ can be written as a linear combination of right combs. To do this, we will use Lemma 3.2: $[r, s, t] - [r+1, s-1, t] + [r, s-1, t+1] = 0$.

First, we can move two of the terms to the other side of the equation. This gives us $[r, s, t] = [r+1, s-1, t] - [r, s-1, t+1]$. Therefore, any Lie word $[r, s, t] \in \mathbb{L}_n^2$ such that $s = 1$ has now been written as a linear combination of right combs.

Second, we assume that $s \neq 1$. Therefore, we can continue to apply Lemma 3.2 to the results until we get an equation entirely consisting of right combs (i.e., until $s = 0$). In the case of $s = 2$, the process would look like this:

$$\begin{aligned} [r+1, 2-1, t] &= [r+2, 2-2, t] - [r+1, 2-2, t+2] \\ \text{and } [r, 2-1, t+1] &= [r+1, 2-2, t+1] - [r, 2-2, t+2] \\ \Rightarrow [r, 2, t] &= [r+2, 0, t] - [r+1, 0, t+2] - ([r+1, 0, t+1] - [r, 0, t+2]). \end{aligned}$$

Assuming that $s \neq 0$ after the first two applications of Lemma 3.2, we then continue until we get the following set of equations:



And so on until we get: $[r, s, t]$

$$= \binom{s}{0}[r+s, 0, t] - \binom{s}{1}[r+s-1, 0, t+1] + ... + (-1)^s \binom{s}{s}[r, 0, t+s].$$

Therefore,

$$[r, s, t] = \sum_{k=0}^{s} (-1)^k \binom{s}{k}[r+s-k, 0, t+k]. \tag{3.1.1}$$

Therefore, any Lie word $[r, s, t] \in \mathbb{L}_n^2$ can be written as a linear combination of right combs. In other words, right combs span. $\square$

3.2 Basis Theorem

To simplify the notation further, when dealing with both even and odd combs, the following symbols may be used interchangeably: $[r, 0, t] = R_t$. More generally, in $\mathbb{L}_n^2$, this means that $[n-2-t, 0, t] = R_t$.

For example, the following list includes all of the odd combs in $\mathbb{L}_{12}^2$:

$$\begin{aligned} [9, 0, 1] &= R_1 \\ [7, 0, 3] &= R_3 \\ [5, 0, 5] &= R_5 \end{aligned}$$

$$\begin{aligned}[3,0,7] &= R_7 \\ [1,0,9] &= R_9.\end{aligned}$$

Theorem 3.2. *Odd combs form a basis for $\mathbb{L}_n^2$.*

Proof First, we will show that odd combs are in a span of $\mathbb{L}_n^2$.

According to Lemma 3.1, we know that $[r,s,s]=0$. Therefore, if we combine Lemma 3.1 and the proof of Proposition 3.1, namely Equation 3.1.1, we get the following:

$$\begin{aligned} 0 &= \binom{s}{0}[r+s,0,s]-\binom{s}{1}[r+s-1,0,s+1]+...+(-1)^s\binom{s}{s}[r,0,2s] \\ &= \sum_{k=0}^{s}(-1)^k\binom{s}{k}[r+s-k,0,s+k]\end{aligned}$$

.

Therefore, each even comb is in the span of the combs below it. In other words, we can iterate to write any Lie word as a linear combination of just odd combs.

Second, we know the formula for the dimension from Schneps (2006):

$$\dim(\mathbb{L}_n^d)=\frac{1}{n}\sum_{a|(d,n-d)}\mu(a)\frac{(\frac{n}{a})!}{(\frac{n-d}{a})!(\frac{d}{a})!},$$

where μ denotes the Möbius function.

Therefore, when $d=2$ and n is even, the dimension equation simplifies to:

$$\begin{aligned}\dim(\mathbb{L}_n^2) &= \frac{1}{n}(\mu(1)\frac{n!}{(n-2)!(2)}+\mu(2)\frac{(\frac{n}{2})!}{(\frac{n-2}{2})!} \\ &= \frac{n-1}{2}-\frac{n}{2}\cdot\frac{1}{n} \\ &= \frac{n}{2}-\frac{1}{2}-\frac{1}{2}=\frac{n}{2}-1 \\ &= \frac{n-2}{2}.\end{aligned}$$

Additionally, when $d=2$ and n is odd, the dimension equation simplifies in the following way:

$$\begin{aligned}\dim(\mathbb{L}_n^2) &= \frac{1}{n} \cdot \mu(1) \cdot \frac{n(n-1)}{2} \\ &= \frac{n-1}{2}.\end{aligned}$$

Therefore, since odd combs both span and have the correct number of vectors (as determined by the formula for the dimension found in Schneps's paper), the odd combs are linearly independent.

Therefore, odd combs form a basis. □

CHAPTER 4 REWRITING EVEN COMBS IN TERMS OF ODD

Since we now know that odd combs span $\mathbb{L}_n^2$, we will determine how to write even combs as linear combinations of odd combs. After doing so, we will specifically calculate R_{2i} for $1 \leq i \leq 12$. These calculations bear a striking resemblance to the first twelve even-indexed Euler polynomials, so we conclude this chapter with the introduction of a conjecture relating the two and possible pathways to proving it.

4.1 Calculating Even Combs

We will now inductively rewrite even combs in terms of odd combs using $[r, s, s] = 0 = \sum_{k=0}^{s}(-1)^k \binom{s}{k}[r + s - k, 0, s + k]$ and show that coefficients are in fact integers. Recall that, in $\mathbb{L}_n^2$, $[n - 2 - t, 0, t] = R_t$. Note that R_t depends on n, but we will prove identities which hold for all n.

For our base case, we will let $s = 1$. This gives us that $[r_1, 1, 1] = R_1 - R_2$. Therefore, $R_2 = R_1$.

For our induction step, we will assume that $R_{2k} = a_1R_1 + a_3R_3 + ... + a_{2k-1}R_{2k-1}$ for $k \in \mathbb{N}$ and some $a_1, a_3, ..., a_{2k-1} \in \mathbb{Z}$. Now, we want to show that $R_{2(k+1)} = R_{2k+2} = b_1R_1 + ... + b_{2k+1}R_{2k+1}$ for some $b_1, , ..., b_{2k+1} \in \mathbb{Z}$.

First, let $s = k + 1$. Then,

$$\begin{aligned}
[r, k+1, k+1] &= 0 \\
&= \sum_{n=0}^{k+1}(-1)^n \binom{k+1}{n}[r + (k+1) - n, 0, (k+1) + n] \\
&= R_{k+1} - (k+1)R_{k+2} + ... + (-1)^{k-1}(\frac{1}{2}k(k+1))R_{2k} \\
&\quad +(-1)^k(k+1)R_{2k+1} + (-1)^{k+1}R_{2k+2}
\end{aligned}$$

By assumption, for all $m \leq k$, $R_{2m} = m_1R_1 + m_3R_3 + ... + m_{2m-1}R_{2m-1}$ such that $m_1, m_3, ..., m_{2m-1} \in \mathbb{Z}$. Therefore, for every even comb R_{2m} such that $m \leq k$, we can substitute in the linear combination of odd combs that add up to be R_{2m}.

In our first case, if $k+1=(2l+1)+1$ for some $l \in \mathbb{N}$, then $k+1$ is even. Therefore,

$$0=[R_{2l+2}]-(2l+2)R_{2l+3}+...+(\frac{1}{2}(2l+1)(2l+2))[R_{4l+2}]-(2l+2)R_{4l+3}+R_{4l+4},$$

where each even comb in brackets would be replaced with the linear combination of odd combs as previously described.

Therefore,

$$R_{4l+4}=-[R_{2l+2}]+(2l+2)R_{2l+3}-...-((2l+1)(l+1))[R_{4l+2}]+(2l+2)R_{4l+3}.$$

In other words, if a number k can be written as $k=2l+1$, then R_{2k+2} can be written as a linear combination of odd combs.

In our second case, if $k+1=2l+1$ for some $l \in \mathbb{N}$, then $k+1$ is odd. Therefore,

$$0=R_{2l+1}-(2l+1)[R_{2l+2}]+...-(l(2l+1))[R_{4l}]+(2l+1)R_{4l+1}-R_{4l+2},$$

where each even comb in brackets would be replaced with the linear combination of odd combs as previously described.

Therefore,

$$R_{4l+2}=R_{2l+1}-(2l+1)[R_{2l+2}]+...-(l(2l+1))[R_{4l}]+(2l+1)R_{4l+1}.$$

In other words, if a number k can be written as $k=2l$, then R_{2k+2} can be written as a linear combination of odd combs.

The following consists of the calculations for the first twelve even combs written as a linear combination of odd combs:

$$
\begin{aligned}
[r_1,1,1] &= [r_1+1,0,1]-[r_1,0,2] \\
&\Rightarrow \underline{R_2 = R_1} \\
[r_2,2,2] &= R_2 - 2R_3 + R_4 \\
&= R_1 - 2R_3 + R_4 \\
&\Rightarrow \underline{R_4 = -R_1 + 2R_3} \\
[r_3,3,3] &= R_3 - 3R_4 + 3R_5 - R_6 \\
&= R_3 - 3(-R_1 + 2R_3) + 3R_5 - R_6 \\
&\Rightarrow \underline{R_6 = 3R_1 - 5R_3 + 3R_5} \\
[r_4,4,4] &= R_4 - 4R_5 + 6R_6 - 4R_7 + R_8 \\
&= (-R_1 + 2R_3) - 4R_5 + 6(3R_1 - 5R_3 + 3R_5) - 4R_7 + R_8 \\
&\Rightarrow \underline{R_8 = -17R_1 + 28R_3 - 14R_5 + 4R_7} \\
[r_5,5,5] &= R_5 - 5R_6 + 10R_7 - 10R_8 + 5R_9 - R_{10} \\
&= R_5 - 5(3R_1 - 5R_3 + 3R_5) + 10R_7 - 10(-17R_1 + 28R_3 - 14R_5 + 4R_7) \\
&\quad +5R_9 - R_{10} \\
&\Rightarrow \underline{R_{10} = 155R_1 - 255R_3 + 126R_5 - 30R_7 + 5R_9} \\
[r_6,6,6] &= R_6 - 6R_7 + 15R_8 - 20R_9 + 15R_{10} - 6R_{11} + R_{12} \\
&\Rightarrow \underline{R_{12} = -2073R_1 + 3410R_3 - 1683R_5 + 396R_7 - 55R_9 + 6R_{11}} \\
[r_7,7,7] &= R_7 - 7R_8 + 21R_9 - 35R_{10} + 35R_{11} - 21R_{12} + 7R_{13} - R_{14} \\
&= 38227R_1 - 62881R_3 + 31031R_5 - 7293R_7 + 1001R_9 - 91R_{11} \\
&\quad +7R_{13} - R_{14} \\
&\Rightarrow \underline{R_{14} = 38227R_1 - 62881R_3 + 31031R_5 - 7293R_7 + 1001R_9 - 91R_{11} + 7R_{13}}
\end{aligned}
$$

$$
\begin{aligned}
[r_8,8,8] &= R_8 - 8R_9 + 28R_{10} - 56R_{11} + 70R_{12} - 56R_{13} + 28R_{14} - 8R_{15} + R_{16} \\
&= 929569R_1 - 1529080R_3 + 754572R_5 - 177320R_7 + 24310R_9 - 2184R_{11} \\
&\quad +140R_{13} - 8R_{15} + R_{16} \\
&\Rightarrow \underline{R_{16} = -929569R_1 + 1529080R_3 - 754572R_5 + 177320R_7 - 24310R_9} \\
&\quad \underline{+2184R_{11} - 140R_{13} + 8R_{15}} \\
[r_9,9,9] &= R_9 - 9R_{10} + 36R_{11} - 84R_{12} + 126R_{13} - 126R_{14} + 84R_{15} - 36R_{16} \\
&\quad +9R_{17} - R_{18} \\
&= 28820619R_1 - 47408019R_3 + 23394924R_5 - 5497596R_7 + 753610R_9 \\
&\quad -67626R_{11} + 4284R_{13} - 204R_{15} + 9R_{17} - R_{18} \\
&\Rightarrow \underline{R_{18} = 28820619R_1 - 47408019R_3 + 23394924R_5 - 5497596R_7 + 753610R_9} \\
&\quad \underline{-67626R_{11} + 4284R_{13} - 204R_{15} + 9R_{17}} \\
[r_{10},10,10] &= R_{10} - 10R_{11} + 45R_{12} - 120R_{13} + 210R_{14} - 252R_{15} + 210R_{16} - 120R_{17} \\
&\quad +45R_{18} - 10R_{19} + R_{20} \\
&= 1109652905R_1 - 1825305870R_3 + 900752361R_5 - 211668360R_7 \\
&\quad +29015090R_9 - 2603380R_{11} + 164730R_{13} - 7752R_{15} + 285R_{17} \\
&\quad -10R_{19} + R_{20} \\
&\Rightarrow \underline{R_{20} = -1109652905R_1 + 1825305870R_3 - 900752361R_5 + 211668360R_7} \\
&\quad \underline{-29015090R_9 + 2603380R_{11} - 164730R_{13} + 7752R_{15} - 285R_{17} + 10R_{19}}
\end{aligned}
$$

$$\begin{aligned}
[r_{11}, 11, 11] &= R_{11} - 11R_{12} + 55R_{13} - 165R_{14} + 330R_{15} - 462R_{16} + 462R_{17} - 330R_{18} \\
&\quad +165R_{19} - 55R_{20} + 11R_{21} - R_{22} \\
&= 51943281731R_1 - 85443273685R_3 + 42164565597R_5 \\
&\quad -9908275971R_7 + 1358205310R_9 - 121863378R_{11} + 7710010R_{13} \\
&\quad -362406R_{15} + 13167R_{17} - 385R_{19} + 11R_{21} - R_{22} \\
&\Rightarrow \underline{R_{22} = 51943281731R_1 - 85443273685R_3 + 42164565597R_5} \\
&\quad \underline{-9908275971R_7 + 1358205310R_9 - 121863378R_{11} + 7710010R_{13}} \\
&\quad \underline{-362406R_{15} + 13167R_{17} - 385R_{19} + 11R_{21}} \\
[r_{12}, 12, 12] &= R_{12} - 12R_{13} + 66R_{14} - 220R_{15} + 495R_{16} - 792R_{17} + 924R_{18} - 792R_{19} \\
&\quad +495R_{20} - 220R_{21} + 66R_{22} - 12R_{23} + R_{24} \\
&= 2905151042481R_1 - 4778781919252R_3 + 2358234353706R_5 \\
&\quad -554162862132R_7 + 75963449111R_9 - 6815721192R_{11} + 431208876R_{13} \\
&\quad -20266312R_{15} + 735471R_{17} - 21252R_{19} + 506R_{21} - 12R_{23} + R_{24} \\
&\Rightarrow \underline{R_{24} = -2905151042481R_1 + 4778781919252R_3 - 2358234353706R_5} \\
&\quad \underline{+554162862132R_7 - 75963449111R_9 + 6815721192R_{11} - 431208876R_{13}} \\
&\quad \underline{+20266312R_{15} - 735471R_{17} + 21252R_{19} - 506R_{21} + 12R_{23}}
\end{aligned}$$

4.2 Euler Polynomials

One interesting note is how our calculations thus far relate to the Euler polynomials. Euler polynomials have been studied for hundreds of years. We recall a few details here, and for further explanation refer to Abramowitz and Stegun (1964) and the references therein.

The Euler polynomials are a similar set of polynomials to the Bernoulli polynomials based on a generating function. This is notable in that Schneps's paper notes a connection between these relations and arithmetic properties of the Bernoulli numbers (Schneps, 2006, Page 2).

The generating function for the Bernoulli polynomials is

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

where $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}x^k$ and, for $n \geq 0$, B_k are Bernoulli numbers. The Bernoulli numbers B_n can be calculated with the help of the Bernoulli polynomials; specifically, that relation is $B_n = B_n(0)$.

Meanwhile, the generating function for Euler polynomials is the following:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

where $E_m(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{E_k}{2^k}(x - \frac{1}{2})^{m-k}$ and E_k are the Euler numbers. The Euler numbers are calculated by $\frac{1}{\cosh(t)} = \frac{2}{e^t+e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!}t^n$. Additionally, the Euler numbers are related to a special value of the Euler polynomials: $E_n = 2^n E_n(\frac{1}{2})$.

An explicit formula for the Euler polynomials is given by

$$E_m(x) = \sum_{n=0}^{m} \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x + k)^m. \tag{4.2.1}$$

This formula can then be used to derive the first few Euler polynomials:

$$\begin{aligned}
E_0(x) &= 1 \\
E_1(x) &= x - \frac{1}{2} \\
E_2(x) &= x^2 - x \\
E_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{4} \\
E_4(x) &= x^4 - 2x^3 + x \\
E_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2} \\
E_6(x) &= x^6 - 3x^5 + 5x^3 - 3x.
\end{aligned}$$

Notice that we have already seen something which looks surprisingly similar to E_2, E_4, and E_6 in the previous section of this chapter. Specifically, if we define polynomials $\hat{E}_{2n} = \sum a_i R^i$ where $R_{2n} = \sum_{i \text{ odd}} a_i R_i$, then it's natural to conjecture:

Conjecture 4.1. $(-1)^n E_{2n} = \hat{E}_{2n}$

This conjecture can be seen in the following two equations:

$$\begin{aligned}
[r_1, 1, 1] = R_1 - R_2 \Rightarrow \quad & \hat{E}_2 && = -R^2 + R^1 \\
& -E_2(x) && = -x^2 + x \\
[r_2, 2, 2] = R_1 - 2R_3 + R_4 \Rightarrow \quad & \hat{E}_4 && = R^4 - 2R^3 + R^1 \\
& E_4(x) && = x^4 - 2x^3 + x
\end{aligned}$$

Furthermore, note that this conjecture has been proven in *Section 4.1 Calculating Even Combs* for the cases such that $n \leq 12$. The conjecture is mainly helpful in that it creates a fast way to perform some of the desired calculations for the following section.

Even though we have not yet been able to prove the conjecture for all values of n, we did attempt a few methods. For example, we attempted to find a different formula for the Euler polynomials that more closely matched the formulas used to find $\hat{E}_{2n}$:

$$\begin{aligned}
E_{4a} &= \sum_{k=0}^{2a} (-1)^k \binom{2a}{k} x^{2a+k} - \sum_{k=0}^{\frac{2a}{2}-1} \binom{2a}{2k} E_{2a+2k} \\
&= \sum_{k=0}^{2a} (-1)^k \binom{2a}{k} x^{2a+k} - \sum_{k=0}^{a-1} \binom{2a}{2k} E_{2(a+k)} \qquad (4.2.2) \\
-E_{4a+2} &= \sum_{k=0}^{2a+1} (-1)^k \binom{2a+1}{k} x^{2a+1+k} + \sum_{k=0}^{\frac{2a+1-1}{2}-1} \binom{2a+1}{2k+1} E_{2a+1+2k+1} \\
&= \sum_{k=0}^{2a+1} (-1)^k \binom{2a+1}{k} x^{2a+1+k} + \sum_{k=0}^{a-1} \binom{2a+1}{2k+1} E_{2(a+k+1)} \qquad (4.2.3)
\end{aligned}$$

Equations 4.2.2 and 4.2.3 were formulated after generalizing the pattern found in the following calculations (which are almost exactly the same calculations used to find $\hat{E}_{2n}$):

$$\begin{aligned}
E_2 &= \sum_{k=0}^{1}(-1)^{k+1}\binom{1}{k}x^{1+k} \\
E_4 &= \sum_{k=0}^{2}(-1)^{k}\binom{2}{k}x^{2+k} - \binom{2}{0}E_2 \\
E_6 &= \sum_{k=0}^{3}(-1)^{k+1}\binom{3}{k}x^{3+k} - \binom{3}{1}E_4 \\
E_8 &= \sum_{k=0}^{4}(-1)^{k}\binom{4}{k}x^{4+k} - \binom{4}{0}E_4 - \binom{4}{2}E_6 \\
E_{10} &= \sum_{k=0}^{5}(-1)^{k+1}\binom{5}{k}x^{5+k} - \binom{5}{1}E_6 - \binom{5}{3}E_8 \\
E_{12} &= \sum_{k=0}^{6}(-1)^{k}\binom{6}{k}x^{6+k} - \binom{6}{0}E_6 - \binom{6}{2}E_8 - \binom{6}{4}E_{10}
\end{aligned}$$

In an effort to prove Equations 4.2.2 and 4.2.3, we initially attempted to show that the functions equaled Equation 4.2.1, but we were unable to do so. Additionally, we tried to use the property of Euler polynomials that says the derivative of E_n is equal to nE_{n-1}. Specifically, since we're only working with the even-indexed Euler polynomials, we used the formula $E_n''(x) = n(n-1)E_{n-2}$. Because we're only working with even-indexed Euler polynomials, this formula might be a good way to prove the conjecture.

CHAPTER 5 REVISITING $(\mathbb{L}, \{-,-\})$

We now return to our main goal: to calculate relations for the deformed Lie bracket. In Schneps (2006), the author finds that the dimension of the space of relations in degree n is $\dim S_n(SL_2(\mathbb{Z}))$, the dimension of the space of modular cusp forms. For integers $k \geq 0$,

$$\dim S_{2k}(SL_2(\mathbb{Z})) = \begin{cases} [\frac{k}{6}] & \text{if } k \not\equiv 1 \mod 6, \\ [\frac{k}{6}] - 1 & \text{if } k \equiv 1 \mod 6 \end{cases}$$

see, for example, (Lang, 1995, Page 12).

Using modular cusp forms, Schneps was able to calculate several relations for $n \leq 22$. Now that we have a more hands-on understanding of $\mathbb{L}_n^2$, we can verify her calculations using right combs. Then, we can give relations for larger values of n.

5.1 Schneps's Equations

In Schneps's paper, she finds the following relations in terms of her calculations: For $n = 12, 16, 18, 20, 22$, we have $\dim S_n(SL_2(\mathbb{Z})) = 1$. Therefore, up to scalar multiple, there is exactly one relation for each of these values of n (Schneps, 2006, Page 13). They are given by

$$n = 12 \quad : \quad \{f_3, f_9\} - 3\{f_5, f_7\} = 0 \tag{5.1.1}$$

$$n = 16 \quad : \quad -2\{f_3, f_{13}\} + 7\{f_5, f_{11}\} - 11\{f_7, f_9\} = 0 \tag{5.1.2}$$

$$n = 18 \quad : \quad 8\{f_3, f_{15}\} - 25\{f_5, f_{13}\} + 26\{f_7, f_{11}\} = 0 \tag{5.1.3}$$

$$n = 20 \quad : \quad 3\{f_3, f_{17}\} - 10\{f_5, f_{15}\} + 14\{f_7, f_{13}\} - 13\{f_9, f_{11}\} = 0 \tag{5.1.4}$$

$$n = 22 \quad : \quad 32\{f_3, f_{19}\} - 105\{f_5, f_{17}\} + 136\{f_7, f_{15}\} - 85\{f_9, f_{13}\} = 0. \tag{5.1.5}$$

Note that, for $n = 14$, the $\dim S_n(SL_2(\mathbb{Z})) = 0$, so there are no relations. Therefore, it is our goal to both check to see if these formulas can be recreated following our calculations and to see if we can calculate more formulas.

As defined in Definition 1.5, the deformed Lie bracket is given by

$\{f, g\} = [f, g] + D_f(g) - D_g(f)$. Now, let f and g be the right combs (i.e., generators with one y) f_n and g_m such that n and m are odd, meaning that we now need to determine to what $D_{f_n}(g_m)$ and $D_{g_m}(f_n)$ are equal.

Proposition 5.1. $D_{f_{2i+1}}(f_{2j+1}) = [2j, 0, 2i]$

Proof: Recall from Definition 1.2 that, for derivation D, $D([a, b]) = [D(a), b] + [a, D(b)]$. Additionally, recall from Definition 1.4 that, for $f \in \mathbb{L}$, $D_f(x) = 0$ and $D_f(y) = [y, f]$.

For the sake of clarity in the calculations, let $f_{2i+1} = f_n$ and $f_{2j+1} = g_{2j+1} = g_m$.

Now, we will calculate $D_{f_n}(g_m)$

$$= D_{f_n}([x, g_{2j}])$$

$$= [D_{f_n}(x), g_{2j}] + [x, D_{f_n}(g_{2j})]$$

$$= [x, D_{f_n}(g_{2j})]$$

$$= [x, D_{f_n}([x, g_{2j-1}])]$$

$$= [x, [D_{f_n}(x), g_{2j-1}]] + [x, [x, D_{f_n}(g_{2j-1})]]$$

$$= [x, [x, D_{f_n}(g_{2j-1})]]$$

Repeat this process until we have $2j - 1$ x's to the left of the derivation

$$= [x, [..., [x, D_{f_n}([x, y])]]]$$

$$= [x, [..., [x, [D_{f_n}(x), y]]]] + [x, [..., [x, [x, D_{f_n}(y)]]]]$$

$$= [x, [..., [x, [x, D_{f_n}(y)]]]] \text{ (Note that we have } 2j \text{ } x\text{'s to the left of } D_{f_n}(y))$$

$$= [x, [..., [x, [x, [y, f_n]]]]]$$

$$= [x, [..., [x, [x, [y, [x, [..., [x, y]]]]]]]] \text{ (Note that } f_n \text{ has } 2i \text{ } x\text{'s)}$$

$$= [2j, 0, 2i]. \square$$

Therefore, $\{f_{2i+1}, f_{2j+1}\} = [f_{2i+1}, f_{2j+1}] + D_{f_{2i+1}}(f_{2j+1}) - D_{f_{2j+1}}(f_{2i+1})$, where, for $2i + 1 \leq 2j + 1$, $[f_{2i+1}, f_{2j+1}] = [0, 2i, 2j]$, $D_{f_{2i+1}}(f_{2j+1}) = [2j, 0, 2i]$ and,

$D_{f_{2j+1}}(f_{2i+1}) = [2i, 0, 2j]$; i.e., for $n \leq m$,

$$\{f_n, f_m\} = [0, n-1, m-1] + [m-1, 0, n-1] - [n-1, 0, m-1] \in \mathbb{L}^2_{n+m}.$$

5.2 Calculating Schneps's Equations

With all of that established, we can now calculate the equations given by Schneps.

First, we will calculate Equation 5.1.1: $\{f_3, f_9\} - 3\{f_5, f_7\}$.

Using Proposition 5.1, we know that

$$\{f_3, f_9\} - 3\{f_5, f_7\} = [0, 2, 8] + [8, 0, 2] - [2, 0, 8] - 3([0, 4, 6] + [6, 0, 4] - [4, 0, 6]).$$

Using Equation 3.1.1 and our calculations found in *Section 4.1 Calculating Even Combs*, we will find that

$$\begin{aligned} [0, 2, 8] &= R_8 - 2R_9 + R_{10} = 138R_1 - 227R_3 + 112R_5 - 26R_7 + 3R_9 \\ [2, 0, 8] &= R_8 = -17R_1 + 28R_3 - 14R_5 + 4R_7 \\ [8, 0, 2] &= R_2 = R_1 \end{aligned}$$

and

$$\begin{aligned} [0, 4, 6] &= 56R_1 - 92R_3 + 45R_5 - 10R_7 + R_9 \\ [4, 0, 6] &= R_6 = 3R_1 - 5R_3 + 3R_5 \\ [6, 0, 4] &= R_4 = -R_1 + 2R_3. \end{aligned}$$

Therefore,

$$\begin{aligned} \{f_3, f_9\} &= [0, 2, 8] - [2, 0, 8] + [8, 0, 2] = 156R_1 - 255R_3 + 126R_5 - 30R_7 + 3R_9 \\ \{f_5, f_7\} &= [0, 4, 6] + [6, 0, 4] - [4, 0, 6] = 52R_1 - 85R_3 + 42R_5 - 10R_7 + R_9. \end{aligned}$$

Therefore, $\{f_3, f_9\} - 3\{f_5, f_7\} = 0$.

We will now repeat this process with the remaining four equations.

For Equation 5.1.2, we can find $\{f_3, f_{13}\}$, $\{f_5, f_{11}\}$, and $\{f_7, f_9\}$ through the respective equations below:

$$\begin{aligned}
[0,2,12] &= 36154R_1 - 59471R_3 + 29348R_5 - 6897R_7 + 946R_9 - 85R_{11} + 5R_{13} \\
[2,0,12] &= R_{12} = -2073R_1 + 3410R_3 - 1683R_5 + 396R_7 - 55R_9 + 6R_{11} \\
[12,0,2] &= R_2 = R_1
\end{aligned}$$

$$\begin{aligned}
[0,4,10] &= 25944R_1 - 42676R_3 + 21059R_5 - 4947R_7 + 676R_9 - 59R_{11} + 3R_{13} \\
[4,0,10] &= R_{10} = 155R_1 - 255R_3 + 126R_5 - 30R_7 + 5R_9 \\
[10,0,4] &= R_4 = -R_1 + 2R_3
\end{aligned}$$

$$\begin{aligned}
[0,6,8] &= 9440R_1 - 15528R_3 + 7662R_5 - 1799R_7 + 245R_9 - 21R_{11} + R_{13} \\
[6,0,8] &= R_8 = -17R_1 + 28R_3 - 14R_5 + 4R_7 \\
[8,0,6] &= R_6 = 3R_1 - 5R_3 + 3R_5.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\{f_3, f_{13}\} &= 38228R_1 - 62881R_3 + 31031R_5 - 7293R_7 + 1001R_9 - 91R_{11} + 5R_{13} \\
\{f_5, f_{11}\} &= 25788R_1 - 42419R_3 + 20933R_5 - 4917R_7 + 671R_9 - 59R_{11} + 3R_{13} \\
\{f_7, f_9\} &= 9460R_1 - 15561R_3 + 7679R_5 - 1803R_7 + 245R_9 - 21R_{11} + R_{13}.
\end{aligned}$$

Therefore, $-2\{f_3, f_{13}\} + 7\{f_5, f_{11}\} - 11\{f_7, f_9\} = 0$.

For Equation 5.1.3, we can find that $\{f_3, f_{15}\}$, $\{f_5, f_{13}\}$, and $\{f_7, f_{11}\}$ come from the following calculations:

$$
\begin{aligned}
[0,2,14] &= -891342R_1 + 1466199R_3 - 723541R_5 + 170027R_7 - 23309R_9 + 2093R_{11} \\
&\quad -133R_{13} + 6R_{15} \\
[2,0,14] &= R_{14} = 38227R_1 - 62881R_3 + 31031R_5 - 7293R_7 + 1001R_9 - 91R_{11} + 7R_{13} \\
[14,0,2] &= R_2 = R_1
\end{aligned}
$$

$$
\begin{aligned}
[0,4,12] &= -702280R_1 + 1155204R_3 - 570069R_5 + 133958R_7 - 18359R_9 + 1644R_{11} \\
&\quad -102R_{13} + 4R_{15} \\
[4,0,12] &= R_{12} = -2073R_1 + 3410R_3 - 1683R_5 + 396R_7 - 55R_9 + 6R_{11} \\
[12,0,4] &= R_4 = -R_1 + 2R_3
\end{aligned}
$$

$$
\begin{aligned}
[0,6,10] &= -387104R_1 + 636760R_3 - 314226R_5 + 73835R_7 - 10115R_9 + 903R_{11} \\
&\quad -55R_{13} + 2R_{15} \\
[6,0,10] &= R_{10} = 155R_1 - 255R_3 + 126R_5 - 30R_7 + 5R_9 \\
[10,0,6] &= R_6 = 3R_1 - 5R_3 + 3R_5.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\{f_3, f_{15}\} &= -929568R_1 + 1529080R_3 - 754572R_5 + 177320R_7 - 24310R_9 + 2184R_{11} \\
&\quad -140R_{13} + 6R_{15} \\
\{f_5, f_{13}\} &= -700208R_1 + 1151796R_3 - 568386R_5 + 133562R_7 - 18304R_9 + 1638R_{11} \\
&\quad -102R_{13} + 4R_{15} \\
\{f_7, f_{11}\} &= -387256R_1 + 637010R_3 - 314349R_5 + 73865R_7 - 10120R_9 + 903R_{11} \\
&\quad -55R_{13} + 2R_{15}.
\end{aligned}
$$

Therefore, $8\{f_3, f_{15}\} - 25\{f_5, f_{13}\} + 26\{f_7, f_{11}\} = 0$.

For Equation 5.1.4, we get the following:

$$
\begin{aligned}
[0,2,16] &= 27891050R_1 - 45878939R_3 + 22640352R_5 - 5320276R_7 + 729300R_9 \\
&\quad -65442R_{11} + 4144R_{13} - 196R_{15} + 7R_{17} \\
[2,0,16] &= R_{16} = -929569R_1 + 1529080R_3 - 754572R_5 + 177320R_7 - 24310R_9 \\
&\quad +2184R_{11} - 140R_{13} + 8R_{15} \\
[16,0,2] &= R_2 = R_1
\end{aligned}
$$

$$
\begin{aligned}
[0,4,14] &= 23281432R_1 - 38296420R_3 + 18898523R_5 - 4440969R_7 + 608751R_9 \\
&\quad -54613R_{11} + 3451R_{13} - 160R_{15} + 5R_{17} \\
[4,0,14] &= R_{14} = 38227R_1 - 62881R_3 + 31031R_5 - 7293R_7 + 1001R_9 - 91R_{11} + 7R_{13} \\
[14,0,4] &= R_4 = -R_1 + 2R_3
\end{aligned}
$$

$$
\begin{aligned}
[0,6,12] &= 15448416R_1 - 25411624R_3 + 12540126R_5 - 2946795R_7 + 403920R_9 \\
&\quad -36225R_{11} + 2283R_{13} - 104R_{15} + 3R_{17} \\
[6,0,12] &= R_{12} = -2073R_1 + 3410R_3 - 1683R_5 + 396R_7 - 55R_9 + 6R_{11} \\
[12,0,6] &= R_6 = 3R_1 - 5R_3 + 3R_5
\end{aligned}
$$

$$
\begin{aligned}
[0,8,10] &= 5410688R_1 - 8900224R_3 + 4392080R_5 - 1032088R_7 + 141465R_9 - 12684R_{11} \\
&\quad +798R_{13} - 36R_{15} + R_{17} \\
[8,0,10] &= R_{10} = 155R_1 - 255R_3 + 126R_5 - 30R_7 + 5R_9 \\
[10,0,8] &= R_8 = -17R_1 + 28R_3 - 14R_5 + 4R_7.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\{f_3, f_{17}\} &= 28820620R_1 - 47408019R_3 + 23394924R_5 - 5497596R_7 + 753610R_9 \\
&\quad -67626R_{11} + 4284R_{13} - 204R_{15} + 7R_{17} \\
\{f_5, f_{15}\} &= 23243204R_1 - 38233537R_3 + 18867492R_5 - 4433676R_7 + 607750R_9 \\
&\quad -54522R_{11} + 3444R_{13} - 160R_{15} + 5R_{17} \\
\{f_7, f_{13}\} &= 15450492R_1 - 25415039R_3 + 12541812R_5 - 2947191R_7 + 403975R_9 \\
&\quad -36231R_{11} + 2283R_{13} - 104R_{15} + 3R_{17} \\
\{f_9, f_{11}\} &= 5410516R_1 - 8899941R_3 + 4391940R_5 - 1032054R_7 + 141460R_9 - 12684R_{11} \\
&\quad +798R_{13} - 36R_{15} + R_{17}.
\end{aligned}$$

Therefore, $3\{f_3, f_{17}\} - 10\{f_5, f_{15}\} + 14\{f_7, f_{13}\} - 13\{f_9, f_{11}\} = 0$

Finally, for Equation 5.1.5, we can calculate the following:

$$\begin{aligned}
[0, 2, 18] &= -1080832286R_1 + 1777897851R_3 - 877357437R_5 + 206170764R_7 \\
&\quad -28261480R_9 + 2535754R_{11} - 160446R_{13} + 7548R_{15} - 276R_{17} + 8R_{19} \\
[2, 0, 18] &= R_{18} = 28820619R_1 - 47408019R_3 + 23394924R_5 - 5497596R_7 + 753610R_9 \\
&\quad -67626R_{11} + 4284R_{13} - 204R_{15} + 9R_{17} \\
[18, 0, 2] &= R_2 = R_1
\end{aligned}$$

$$\begin{aligned}
[0, 4, 16] &= -937658760R_1 + 1542386836R_3 - 761137389R_5 + 178860104R_7 \\
&\quad -24517740R_9 + 2199808R_{11} - 139166R_{13} + 6536R_{15} - 235R_{17} + 6R_{19} \\
[4, 0, 16] &= R_{16} = -929569R_1 + 1529080R_3 - 754572R_5 + 177320R_7 - 24310R_9 \\
&\quad +2184R_{11} - 140R_{13} + 8R_{15} \\
[16, 0, 4] &= R_4 = -R_1 + 2R_3
\end{aligned}$$

$$
\begin{aligned}
[0,6,14] &= -691248928R_1 + 1137058904R_3 - 561116050R_5 + 131856927R_7 \\
&\quad -18074589R_9 + 1621659R_{11} - 102563R_{13} + 4806R_{15} - 170R_{17} + 4R_{19} \\
[6,0,14] &= R_{14} = 38227R_1 - 62881R_3 + 31031R_5 - 7293R_7 + 1001R_9 - 91R_{11} + 7R_{13} \\
[14,0,6] &= R_6 = 3R_1 - 5R_3 + 3R_5
\end{aligned}
$$

$$
\begin{aligned}
[0,8,12] &= -366677120R_1 + 603159680R_3 - 297647344R_5 + 69944264R_7 - 9587737R_9 \\
&\quad +860190R_{11} - 54390R_{13} + 2544R_{15} - 89R_{17} + 2R_{19} \\
[8,0,12] &= R_{12} = -2073R_1 + 3410R_3 - 1683R_5 + 396R_7 - 55R_9 + 6R_{11} \\
[12,0,8] &= R_8 = -17R_1 + 28R_3 - 14R_5 + 4R_7.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\{f_3, f_{19}\} &= -1109652904R_1 + 1825305870R_3 - 900752361R_5 + 211668360R_7 \\
&\quad -29015090R_9 + 2603380R_{11} - 164730R_{13} + 7752R_{15} - 285R_{17} + 8R_{19} \\
\{f_5, f_{17}\} &= -936729192R_1 + 1540857758R_3 - 760382817R_5 + 178682784R_7 \\
&\quad -24493430R_9 + 2197624R_{11} - 139026R_{13} + 6528R_{15} - 235R_{17} + 6R_{19} \\
\{f_7, f_{15}\} &= -691287152R_1 + 1137121780R_3 - 561147078R_5 + 131864220R_7 \\
&\quad -18075590R_9 + 1621750R_{11} - 102570R_{13} + 4806R_{15} - 170R_{17} + 4R_{19} \\
\{f_9, f_{13}\} &= -366675064R_1 + 603156298R_3 - 297645675R_5 + 69943872R_7 \\
&\quad -9587682R_9 + 860184R_{11} - 54390R_{13} + 2544R_{15} - 89R_{17} + 2R_{19}.
\end{aligned}
$$

Therefore, $32\{f_3, f_{19}\} - 105\{f_5, f_{17}\} + 136\{f_7, f_{15}\} - 85\{f_9, f_{13}\} = 0$.

5.3 Finding New Relations

Using the dimension equation, we can find that the $\dim S_{26}(SL_2(\mathbb{Z})) = 1$. Therefore, as with Schneps's equations, up to scalar multiple, there is exactly one relation for $n = 26$:

$$1032\{f_3, f_{23}\} - 3395\{f_5, f_{21}\} + 4466\{f_7, f_{19}\} - 3135\{f_9, f_{17}\} + 1292\{f_{11}, f_{15}\} = 0. \quad (5.3.1)$$

Now, we can show that this calculation works with the equations found below:

$$\begin{aligned}
[0,2,22] &= -2853207760750R_1 + 4693338645567R_3 - 2316069788109R_5 \\
&\quad +544254586161R_7 - 74605243801R_9 + 6693857814R_{11} - 423498866R_{13} \\
&\quad +19903906R_{15} - 722304R_{17} + 20867R_{19} - 495R_{21} + 10R_{23} \\
[2,0,22] &= R_{22} = 51943281731R_1 - 85443273685R_3 + 42164565597R_5 - 9908275971R_7 \\
&\quad +1358205310R_9 - 121863378R_{11} + 7710010R_{13} - 362406R_{15} + 13167R_{17} \\
&\quad -385R_{19} + 11R_{21} \\
[22,0,2] &= R_2 = R_1
\end{aligned}$$

$$\begin{aligned}
[0,4,20] &= -2594601005000R_1 + 4267947583012R_3 - 2106147712485R_5 \\
&\quad +494924874666R_7 - 67843232341R_9 + 6087144304R_{11} - 385113546R_{13} \\
&\quad +18099628R_{15} - 656754R_{17} + 18952R_{19} - 444R_{21} + 8R_{23} \\
[4,0,20] &= R_{20} = -1109652905R_1 + 1825305870R_3 - 900752361R_5 + 211668360R_7 \\
&\quad -29015090R_9 + 2603380R_{11} - 164730R_{13} + 7752R_{15} - 285R_{17} + 10R_{19} \\
[20,0,4] &= R_4 = -R_1 + 2R_3
\end{aligned}$$

$$\begin{aligned}
[0,6,18] &= -2142617789472R_1 + 3524464994008R_3 - 1739253760242R_5 \\
&\quad +408708250371R_7 - 56024842201R_9 + 5026753596R_{11} - 318025392R_{13} \\
&\quad +14946298R_{15} - 542232R_{17} + 15621R_{19} - 361R_{21} + 6R_{23} \\
[6,0,18] &= R_{18} = 28820619R_1 - 47408019R_3 + 23394924R_5 - 5497596R_7 \\
&\quad 753610R_9 - 67626R_{11} + 4284R_{13} - 204R_{15} + 9R_{17} \\
[18,0,6] &= R_6 = 3R_1 - 5R_3 + 3R_5
\end{aligned}$$

$$
\begin{aligned}
[0,8,16] &= -1527608809600R_1 + 2512815771520R_3 - 1240024878960R_5 \\
&\quad +291394164776R_7 - 39943679961R_9 + 3583891864R_{11} - 226739884R_{13} \\
&\quad +10655880R_{15} - 386501R_{17} + 11116R_{19} - 254R_{21} + 4R_{23} \\
[8,0,16] &= R_{16} = -929569R_1 + 1529080R_3 - 754572R_5 + 177320R_7 - 24310R_9 \\
&\quad +2184R_{11} - 140R_{13} + 8R_{15} \\
[16,0,8] &= R_8 = -17R_1 + 28R_3 - 14R_5 + 4R_7
\end{aligned}
$$

$$
\begin{aligned}
[0,10,14] &= -794719937024R_1 + 1307261897856R_3 - 645107888320R_5 \\
&\quad +151594275984R_7 - 20780213910R_9 + 1864475711R_{11} - 117958379R_{13} \\
&\quad +5543472R_{15} - 201036R_{17} + 5775R_{19} - 131R_{21} + 2R_{23} \\
[10,0,14] &= R_{14} = 38227R_1 - 62881R_3 + 31031R_5 - 7293R_7 + 1001R_9 - 91R_{11} + 7R_{13} \\
[14,0,10] &= R_{10} = 155R_1 - 255R_3 + 126R_5 - 30R_7 + 5R_9.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\{f_3, f_{23}\} &= -2905151042480R_1 + 4778781919252R_3 - 2358234353706R_5 \\
&\quad +554162862132R_7 - 75963449111R_9 + 6815721192R_{11} - 431208876R_{13} \\
&\quad +20266312R_{15} - 735471R_{17} + 21252R_{19} - 506R_{21} + 10R_{23} \\
\{f_5, f_{21}\} &= -2593491352096R_1 + 4266122277144R_3 - 2105246960124R_5 \\
&\quad +494713206306R_7 - 67814217251R_9 + 6084540924R_{11} - 384948816R_{13} \\
&\quad +18091876R_{15} - 656469R_{17} + 18942R_{19} - 444R_{21} + 8R_{23} \\
\{f_7, f_{19}\} &= -2142646610088R_1 + 3524512402022R_3 - 1739277155163R_5 \\
&\quad +408713747967R_7 - 56025595811R_9 + 5026821222R_{11} - 318029676R_{13} \\
&\quad +14946502R_{15} - 542241R_{17} + 15621R_{19} - 361R_{21} + 6R_{23}
\end{aligned}
$$

$$
\begin{aligned}
\{f_9, f_{17}\} &= -1527607880048R_1 + 2512814242468R_3 - 1240024124402R_5 \\
&\quad +291393987460R_7 - 39943655651R_9 + 3583889680R_{11} - 226739744R_{13} \\
&\quad +10655872R_{15} - 386501R_{17} + 11116R_{19} - 254R_{21} + 4R_{23} \\
\{f_{11}, f_{15}\} &= -794719975096R_1 + 1307261960482R_3 - 645107919225R_5 \\
&\quad +151594283247R_7 - 20780214906R_9 + 1864475802R_{11} - 117958386R_{13} \\
&\quad +5543472R_{15} - 201036R_{17} + 5775R_{19} - 131R_{21} + 2R_{23}.
\end{aligned}
$$

Thus, we have verified Equation 5.3.1.

Now, there exist no other values of n such that $\dim S_n(SL_2(\mathbb{Z})) = 1$. However, we can look at one n where the dimension equals 2. In the case of $n = 24$, $\dim S_{24}(SL_2(\mathbb{Z})) = 2$. We now show that every relation is a linear combination of the two equations listed below:

$$-470\{f_3, f_{21}\} + 1519\{f_5, f_{19}\} - 1862\{f_7, f_{17}\} + 969\{f_9, f_{15}\} + 0\{f_{11}, f_{13}\} = 0 \quad (5.3.2)$$

$$-194\{f_3, f_{21}\} + 605\{f_5, f_{19}\} - 627\{f_7, f_{17}\} + 0\{f_9, f_{15}\} + 646\{f_{11}, f_{13}\} = 0. \quad (5.3.3)$$

The calculations to prove the above functions are shown below:

$$
\begin{aligned}
[0, 2, 20] &= 50833628826R_1 - 83617967815R_3 + 41263813236R_5 - 9696607611R_7 \\
&\quad +1329190220R_9 - 119259998R_{11} + 7545280R_{13} - 354654R_{15} \\
&\quad +12882R_{17} - 375R_{19} + 9R_{21} \\
[2, 0, 20] &= R_{20} = -1109652905R_1 + 1825305870R_3 - 900752361R_5 + 211668360R_7 \\
&\quad -29015090R_9 + 2603380R_{11} - 164730R_{13} + 7752R_{15} - 285R_{17} + 10R_{19} \\
[20, 0, 2] &= R_2 = R_1
\end{aligned}
$$

$$
\begin{aligned}
[0,4,18] &= 45314184920R_1 - 74538846484R_3 + 36783446355R_5 - 8643763407R_7 \\
&\quad +1184868380R_9 - 106310724R_{11} + 6725914R_{13} - 316098R_{15} \\
&\quad +11466R_{17} - 329R_{19} + 7R_{21} \\
[4,0,18] &= R_{18} = 28820619R_1 - 47408019R_3 + 23394924R_5 - 5497596R_7 \\
&\quad +753610R_9 - 67626R_{11} + 4284R_{13} - 204R_{15} + 9R_{17} \\
[18,0,4] &= R_4 = -R_1 + 2R_3
\end{aligned}
$$

$$
\begin{aligned}
[0,6,16] &= 35729867872R_1 - 58773276840R_3 + 29003449470R_5 \\
&\quad -6815537191R_7 + 934258800R_9 - 83824884R_{11} + 5303180R_{13} - 249178R_{15} \\
&\quad +9021R_{17} - 255R_{19} + 5R_{21} \\
[6,0,16] &= R_{16} = -929569R_1 + 1529080R_3 - 754572R_5 + 177320R_7 - 24310R_9 \\
&\quad +2184R_{11} - 140R_{13} + 8R_{15} \\
[16,0,6] &= R_6 = 3R_1 - 5R_3 + 3R_5
\end{aligned}
$$

$$
\begin{aligned}
[0,8,14] &= 22864454016R_1 - 37610519296R_3 + 18560047184R_5 - 4361435944R_7 \\
&\quad +597855811R_9 - 53641497R_{11} + 3393537R_{13} - 159414R_{15} \\
&\quad +5761R_{17} - 161R_{19} + 3R_{21} \\
[8,0,14] &= R_{14} = 38227R_1 - 62881R_3 + 31031R_5 - 7293R_7 + 1001R_9 - 91R_{11} + 7R_{13} \\
[14,0,8] &= R_8 = -17R_1 + 28R_3 - 14R_5 + 4R_7
\end{aligned}
$$

$$
\begin{aligned}
[0,10,12] &= 7867739648R_1 - 12941912960R_3 + 6386577984R_5 - 1500785520R_7 \\
&\quad +205724250R_9 - 18458187R_{11} + 1167705R_{13} - 54846R_{15} \\
&\quad +1980R_{17} - 55R_{19} + R_{21} \\
[10,0,12] &= R_{12} = -2073R_1 + 3410R_3 - 1683R_5 + 396R_7 - 55R_9 + 6R_{11} \\
[12,0,10] &= R_{10} = 155R_1 - 255R_3 + 126R_5 - 30R_7 + 5R_9.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\{f_3, f_{21}\} &= 51943281732R_1 - 85443273685R_3 + 42164565597R_5 - 9908275971R_7 \\
&\quad +1358205310R_9 - 121863378R_{11} + 7710010R_{13} - 362406R_{15} + 13167R_{17} \\
&\quad -385R_{19} + 9R_{21} \\
\{f_5, f_{19}\} &= 45285364300R_1 - 74491438463R_3 + 36760051431R_5 - 8638265811R_7 \\
&\quad +1184114770R_9 - 106243098R_{11} + 6721630R_{13} - 315894R_{15} + 11457R_{17} \\
&\quad -329R_{19} + 7R_{21} \\
\{f_7, f_{17}\} &= 35730797444R_1 - 58774805925R_3 + 29004204045R_5 - 6815714511R_7 \\
&\quad +934283110R_9 - 83827068R_{11} + 5303320R_{13} - 249186R_{15} + 9021R_{17} \\
&\quad -255R_{19} + 5R_{21} \\
\{f_9, f_{15}\} &= 22864415772R_1 - 37610456387R_3 + 18560016139R_5 - 4361428647R_7 \\
&\quad +597854810R_9 - 53641406R_{11} + 3393530R_{13} - 159414R_{15} + 5761R_{17} \\
&\quad -161R_{19} + 3R_{21} \\
\{f_{11}, f_{13}\} &= 7867741876R_1 - 12941916625R_3 + 6386579793R_5 - 1500785946R_7 \\
&\quad +205724310R_9 - 18458193R_{11} + 1167705R_{13} - 54846R_{15} + 1980R_{17} \\
&\quad -55R_{19} + R_{21}.
\end{aligned}$$

Thus, we have verified Equations 5.3.2 and 5.3.3.

Equations 5.3.1, 5.3.2, and 5.3.3 are not the only relations that can be found using the methods described in this thesis. First, choose an even n and find the $\dim S_n(SL_2(\mathbb{Z}))$ to predict the number of equations you will find whose linear combinations will equal every relation for that n. Then, find all odd number pairs (such that both numbers are greater than 1) which add together to equal n. Using those pairs, the next step is to calculate $\{f_3, f_{n-3}\}$, $\{f_5, f_{n-5}\}$, ..., and $\{f_{2k+1}, f_{n-(2k+1)}\}$ using Proposition 5.1, Equation 3.1.1, and the calculations and processes from *Section 4.1 Calculating Even Combs*. Note that proving Conjecture 4.1 will make it easier to find more of the even combs that we rewrite as linear combinations of odd combs; these rewritten combs are needed for calculating relations for $n > 26$. Once we calculate $\{f_{2i+1}, f_{2j+1}\}$ for all $2i+1, 2j+1$ such that $2i+2j+2=n$, we can then input them into a matrix and find the null space of said matrix. The null space gives the desired relations. With all of these formulas and processes, continuing the calculations and finding more relations is straight forward but gets increasingly tedious.

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