UNIVERSALITY OF COMPOSITION OPERATORS ITH CONFORMAL MAP ON THE UPPER HALF PLANE

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ABSTRACT

Kit Chan, Advisor

The main theme of this dissertation is the dynamical behavior of composition operators on the Fréchet space $H(\mathbb{P})$ of holomorphic functions on the upper half-plane $\mathbb{P}$. In this dissertation, we prove a new version of the Seidel and Walsh Theorem [21] for the Fréchet space $H(\mathbb{P})$. Indeed, we obtain a necessary and sufficient condition for the sequence of linear fractional transformations σ_n such that the sequence of composition operators $\{C_{\sigma_n}\}_{n=1}^{\infty}$ for the Fréchet space $H(\mathbb{P})$ is universal. For that, we use the Riemann Mapping Theorem to transfer dynamical results on the space $H(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$ to the space of holomorphic functions $H(\mathbb{P})$. Furthermore, we generalize our first result by proving equivalent conditions for a sequence of composition operators in the space $H(\mathbb{D})$ to be universal.

Consequently, taking the point of view that hypercyclicity is a special case of universality, we obtain a new criterion for a linear fractional transformation σ so that C_σ is hypercyclic on $H(\mathbb{P})$. Indeed, we provide necessary and sufficient conditions in terms of the coefficients a, b, c, d of a linear fractional transformation $\sigma(z) = \frac{az+b}{cz+d}$ so that C_σ is hypercyclic on $H(\mathbb{P})$. Moreover, we use this result to derive a necessary and sufficient condition in terms of α and θ so that C_φ is hypercyclic on $H(\mathbb{D})$ where $\varphi(z) = e^{\theta}\frac{z-\alpha}{1-\alpha z}$ is a linear fractional transformation defined on $\mathbb{D}$.

Motivated by the Denjoy-Wolff Theorem [23, p. 78], we further work on the conformal map of the upper half-plane $\mathbb{P}$ is to make a connection between the hypercyclicity and the limit of the iterations σ^n. In particular, we give a complete characterization for the limit point of σ^n in the extended boundary $\partial_\infty\mathbb{P} = \partial\mathbb{P} \cup \{\infty\}$. Similarly, we provide an analogous result for the unit disk $\mathbb{D}$.

Finally, we obtain a new universal criterion in the space $H(\Omega)$ of holomorphic functions on a bounded simply connected region Ω that is not the whole complex plane $\mathbb{C}$. We show that a sequence of composition operators $\{C_{\sigma_n}\}_{n=1}^{\infty}$ on $H(\Omega)$ is universal if and only if there are a boundary limit point $w \in \partial\Omega$ and a subsequence $\{\sigma_{n_k}\}_k$ of $\{\sigma_n\}_n$ such that $\sigma_{n_k} \to w$ uniformly on compact

subsets of Ω. Our last result extends a result of Grosse-Erdmann, and Manguillot in a particular case when the complement $\mathbb{C} \setminus \Omega$ of Ω has a nonempty interior.

To my parents Aminah and Abdulmohsen Almohammedali.

To my husband Hussain Almutair.

To Abdullah, Afnan, Fatimah, Mohammed, Shahad, and Aya.

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CHAPTER 1 UNIVERSAL COMPOSITION OPERATORS

1.1 Introduction

1.1.1 Introduction to Universal Composition Operators

Let X be a separable metrizable topological vector space, and for each integer $n \geq 1$ let $T_n : X \to X$ be a continuous linear map. The sequence $\{T_n\}_n$ is said to be *universal* if there is a vector x of X such that the set $\{T_n x : n \geqslant 1\}$ is dense in X. Such a vector x is called a *universal vector* of $\{T_n\}_n$. In the case that T_n is the n-th power T^n of a continuous linear map $T : X \to X$, then T is said to be *hypercyclic*, and we call x a *hypercyclic vector* of T; that is, the orbit orb$(T, x) = \{T^n x : n \geqslant 1\}$ is dense in X. In our setting, X is the Fŕechet space $H(G)$ of holomorphic functions on a region G in the complex plane $\mathbb{C}$, and T_n is a composition operator $C_{\phi_n} : H(G) \to H(G)$ defined by $C_{\phi_n}(f) = f \circ \phi_n$, where $\phi_n : G \to G$ is a conformal map. In the Fŕechet space $H(G)$ a sequence $\{f_n\}_{n=1}^{\infty}$ in $H(G)$ converges to f in $H(G)$ if and only if $f_n \to f$ uniformly on every compact subset of G; see [3]. This dissertation is dedicated to study the dynamical behavior of the sequence $\{C_{\phi_n}\}_{n\geqslant 1}$ in the space of holomorphic functions $H(G)$ when G is the upper half plane $\mathbb{P}$ or the unit disk $\mathbb{D}$.

The following discussion is based on the Shapiro's note [22]. If U and V are open subsets of $\mathbb{C}$ and $\phi : U \to V$ is a holomorphic map (not necessarily one to one or onto), then ϕ induces a composition operator $C_\phi : H(V) \to H(U)$ defined by $C_\phi f = f \circ \phi$ where $f \in H(V)$. Now if we take the region G to be the unit disk $\mathbb{D}$, then the Riemann Mapping Theorem [3, p. 160] allows us to transfer the dynamic behavior about composition operator on $H(\mathbb{D})$ to $H(G)$, where G is any simply connected planar region that is not $\mathbb{C}$. In fact, the Riemann Mapping Theorem gives us a bijective holomorphic map ψ taking $\mathbb{D}$ onto G. Therefore, the corresponding composition operator C_ψ is an isomorphism of $H(G)$ onto $H(\mathbb{D})$. If σ is a self map of G, then $\phi = \psi^{-1} \circ \sigma \circ \psi$ is holomorphically conjugate to σ (see Definition 1.2.4) and ϕ is a holomorphic self map of $\mathbb{D}$. Since

σ and ϕ are conjugate of each other, the operator

$$C_\phi = C_{\psi^{-1}\circ\sigma\circ\psi} = C_\psi \circ C_\sigma \circ (C_\psi)^{-1}$$

is similar to $C_\sigma : H(G) \to H(G)$. In this dissertation, we make use of the isomorphism between these Fréchet spaces $H(G)$ and $H(\mathbb{D})$.

1.1.2 Classical Results

The study of composition operators has a long history, utilizing methods from different topics in analysis, such as functional analysis, operator theory, measure theory, and analytic function theory. Composition operators have been studied by many authors on various spaces of analytic functions. For general references on the theory of composition operators, see the well-known books by Cowen and MacCluer [4], Shapiro [23] and Erdmann and Manguillot [9].

In 1929 Birkhoff [2] proved that there exists an entire function $g(z)$ such that for any arbitrary entire function $h(z)$, there exists a sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that

$$\lim_{k\to\infty} g(z+n_k) = h(z)$$

uniformly on compact subsets of $\mathbb{C}$. In other words, the translation operators $T_n : H(G) \to H(G)$ defined by $T_n(f) = f(z+n)$ form a universal sequence of operators.

After that, many authors have worked on this topic, on the space $H(G)$ on a region G particularly when $G = \mathbb{C}$ or $\mathbb{D}$ for self mappings $\varphi : G \to G$ which may not be one-to-one or onto on the domain G; see [8].

In 1941 Seidel and Walsh [21] proved a result analogous to Birkhoff's Theorem for the space of holomorphic functions on the unit disk $\mathbb{D}$. They showed that for any sequence $\{b_n\}_{n=0}^{\infty} \subset \mathbb{D}$ with $b_n \to 1$, there exists a function g in $H(\mathbb{D})$ such that for any function h in $H(\mathbb{D})$, there exists a subsequence $\{b_{n_k}\}_{k=0}^{\infty}$ with

$$\lim_{n\to\infty} g\left(\frac{b_{n_k} - z}{1 - \overline{b_{n_k}} z}\right) = h(z)$$

Later in 1955 Heins [7] showed for any sequence $\{b_n\}_{n=1}^{\infty}$ in the unit disk $\mathbb{D}$ with $b_n \to 1$ there exists a Blaschke product B such that any holomorphic function in $\mathbb{D}$ that is bounded by 1 can be locally uniformly approximated by functions of the form $B \circ \phi_{n_k}$, where $\phi_{n_k}(z) = \frac{b_{n_k} - z}{1 - \overline{b_{n_k}} z}$. Also, in 1976 Luh [1] proved that in the space $H(\mathbb{C})$ of entire functions, for any sequence $\{b_n\}_{n=0}^{\infty}$ in $\mathbb{C}$ with $\lim_{n\to\infty} b_n = \infty$, there exists an holomorphic function g in $H(\mathbb{C})$ such that for any compact set K with property that $\mathbb{C} \setminus K$ is connected and for every function h holomorphic on $\mathbb{C}$ there exists subsequence $\{b_{n_k}\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} g(z + b_{n_k}) = h(z)$ uniformly on K.

In 1987 Gethner and Shapiro [6] discovered a sufficient condition for a sequence of continuous linear maps on a Fréchet space to be universal. This condition for the Banach spaces case was first discovered by Carol Kitai in her thesis [12], but she never published it. This condition is now known as the Universality Criterion, which can be applied to spaces of holomorphic functions. Thus Universality Criterion gives a unified proof of universality for composition operators, including the theorems of Birkhoff, Seidel and Walsh and others. More related results on compositions operators can be found in [8].

1.1.3 An Outline of the Dissertation

The organization of this dissertation is as follows. In Chapter 1, we cover few definitions related to linear fractional transformations. Then we introduce well-studied concepts of classification of linear fractional transformations that we need in the dissertation.

In Chapter 2, we present a few classical results on universality, including the universality criterion and the Seidel and Walsh Theorem. In addition, we obtain straightforward observations and results related to the Seidel and Walsh Theorem.

In Chapter 3, we continue the work of Seidel and Walsh [21] who proved a universality result for $C_{\tau_n}\colon H(\mathbb{D}) \to H(\mathbb{D})$ where τ_n is a sequence of non-Euclidean translation on the unit disk $\mathbb{D}$. Fur-thermore, we make a connection between a sequence of universal composition operators in $H(\mathbb{D})$ and a sequence of composition operators in $H(\mathbb{P})$; see Theorem 3.2.6. Then in Corollary 3.2.7 we obtain a complete characterization of a sequence of conformal maps that produce a sequence of universal composition operators $\{C_{\sigma_n}\}_{n\geqslant 1}$ on $H(\mathbb{P})$. We conclude Chapter 3 with a few examples of universal composition operators.

In Chapter 4, we study the relation between a linear fractional transformation $\sigma(z) = \frac{az+b}{cz+d}$ and its coefficient matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We first prove basic properties of the matrix and its eigenvalues in Lemma 4.2.1 and Lemma 4.2.2. In Proposition 4.2.3 and Theorem 4.2.5 we provide necessary and sufficient conditions for a sequence of composition operators $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ to be hypercyclic, in terms of the matrix A of the conformal map σ. To conclude the chapter, we provide a few examples to illustrate the main result in Theorem 4.2.5.

In Chapter 5, we first prove Lemma 5.1.6 and Lemma 5.1.10 that give us equivalent conditions in terms of eigenvalues, fixed points, and the coefficients of linear fractional map σ. Motivated by Denjoy-Wolff Theorem [23, p. 78], we study a limiting behavior of iterations of a linear fractional transformation σ defined on the upper half-plane $\mathbb{P}$. Using these conditions, we give a specific characterization of its limit point for different classes of linear fractional transformations in Theorem 5.2.6.

In Chapter 6, we provide a complete characterization of the linear fractional transformation $\phi(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$ so that the composition operator C_ϕ is hypercyclic on $H(\mathbb{D})$; see Theorem 6.2.1. To be more precise, we derive necessary and sufficient conditions for a linear fractional self-map ϕ on $\mathbb{D}$ in terms of α and θ so that C_ϕ is hypercyclic. At the end of this chapter, we provide a numerical example and a series of corollaries.

In Chapter 7, we investigate universal composition operators in the setting of the Fréchet space $H(G)$ of space of holomorphic functions for any simply connected region G in the complex plane $\mathbb{C}$, when its complement $\mathbb{C}\setminus G$ has a nonempty interior. We obtain a new criterion for σ_n on G so that $\{C_{\sigma_n}\}_{n=1}^{\infty}$ is universal in $H(G)$; see Corollary 7.2.3, Theorem 7.2.2 and Theorem 7.2.4.

1.2 Linear Fractional Transformations

The universality of a sequence of composition operators $\{C_{\phi_n}\}_{n=1}^{\infty}$ is studied with the properties of the symbols ϕ_n. The following proposition is well known and demonstrates the connection between the hypercyclicity of a composition operator C_ϕ with its inducing map ϕ.

Proposition 1.2.1. *(Shapiro [22]) If C_ϕ is hypercyclic on $H(\mathbb{D})$ then ϕ is an univalent (that is one to one holomorphic function) and has no fixed point in $\mathbb{D}$.*

Proof. By way of contradiction, assume ϕ has a fixed point $r \in \mathbb{D}$, (that is, $\phi(r) = r$). Let $f \in H(\mathbb{D})$. Any function in the orbit orb(C_ϕ, f) is in the form $C_\phi^n f = f \circ \phi^n$, where $\phi^n = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{\text{n-times}}$ and $n \geqslant 0$. Hence by induction, we have

$$\begin{aligned} C_\phi^{n+1} f(r) = f \circ \phi^{n+1}(r) &= f \circ \phi^n(\phi(r)) \\ &= f(\phi^n(r)) \\ &= f(r). \end{aligned}$$

Hence $\{C_\phi^n f : n \geqslant 1\}$ is not dense in $H(\mathbb{D})$. Thus, C_ϕ is not hypercyclic.

Suppose ϕ is not univalent, so there exists distinct point $r,\ s \in \mathbb{D}$ with $\phi(r) = \phi(s)$. Then if $f \in H(\mathbb{D})$, then each function in the orbit orb(C_ϕ, f) we have by induction for $n \geqslant 0$,

$$\begin{aligned} C_\phi^{n+1} f(r) = f(\phi^{n+1}(r)) = f(\phi^n(\phi(r))) &= f(\phi^n(\phi(s))) \\ &= f(\phi^{n+1}(s)) \\ &= C_\phi^{n+1} f(s) \end{aligned}$$

Hence, $f \circ \phi^n(r) = f \circ \phi^n(s)$ for all $n \geq 0$. Therefore, orb(C_ϕ, f) can not be dense. □

It is clear that in this proof we may replace the unit disk $\mathbb{D}$ with any open set G since we did not use any specific characteristic of the unit disk $\mathbb{D}$.

The main goal of this section is to study the set of linear fractional transformations on the complex plane $\mathbb{C}$, and more precisely on the upper half plane $\mathbb{P}$ and the unit disk $\mathbb{D}$. We state basics definitions and some facts of the linear fractional transformations.

Definition 1.2.2. *([26, p. 20]) A map of the form* $\phi(z) = \frac{az+b}{cz+d}$ *is called a linear fractional transformation. If* $a, b, c, d \in \mathbb{C}$ *and if* $ad - bc \neq 0$ *then* ϕ *is a Möbius transformation.*
In the case that $ad - bc = 1$ *then* ϕ *is in the standard form or a normalized transformation.*
In the case that $c \neq 0$*, this definition extends to the whole Riemann sphere* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ *by defining* $f(\frac{-d}{c}) = \infty, f(\infty) = \frac{a}{c}$*.*
In the case that $c = 0$ *and* $a \neq 0$ *we define* $f(\infty) = \infty$*.*

The following definition gives us a description of a linear fractional transformation for $\mathbb{P}$; see [26, p. 20].

Definition 1.2.3. *Let* $a, b, c, d \in \mathbb{R}$ *be such that* $ad - bc = 1$ *we define the map*

$$\phi(z) = \frac{az+b}{cz+d}$$

as a form of linear fractional transformation $\phi : H(\mathbb{P}) \to H(\mathbb{P})$*. This function* ϕ *is called a Möbius transformation of* $\mathbb{P}$*, or a conformal map of* $\mathbb{P}$*.*

One may ask when two linear fractional transformations are conjugate? The answer is in the below definition.

Definition 1.2.4. *([20]) Two linear fractional transformations* f *and* g *are said to be*

- *conjugate if there exists a linear fractional transformation* h *such that the diagram*

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{g} & \hat{\mathbb{C}} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \end{array} \tag{1.2.5}$$

  is commutative; that is; $g = h^{-1} \circ f \circ h$*.*

- *topologically conjugate if there exists a homeomorphism $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $g = h^{-1} \circ f \circ h$ (a mapping h is a homeomorphism if h and h^{-1} are continuous bijections).*

Many of linear fractional transformations are important in our study. The first ones are the linear fractional transformations of the disk $\mathbb{D}$,

$$\phi(z) = e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z}$$

where $\alpha \in \mathbb{D}$ and $\theta \in [0, 2\pi]$. If $\theta = \pi$, then $\phi(z) = \frac{z-\alpha}{1-\overline{\alpha} z}$ and these maps are self -inverse, that is $\phi = \phi^{-1}$. Also it is important to know that ϕ maps the open unit disk $\mathbb{D}$ to itself and the boundary of the unit disk $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ to itself. These linear fractional transformations are useful because they take the point α to point 0 and 0 to $-\alpha$.

The second one is the conformal automorphism which is called a Cayley transform $\psi : \mathbb{P} \to \mathbb{D}$ that conformally maps the upper half-plane $\mathbb{P}$ to the unit disk $\mathbb{D}$ by:

$$\psi(z) = \frac{z - i}{z + i}, \quad \text{where } z \in \mathbb{P}.$$

its inverse is given by:

$$\psi^{-1}(z) = i \frac{1 + z}{1 - z}, \quad \text{where } z \in \mathbb{D}.$$

Note that the Cayley transform maps the boundary to the boundary, that is the extended boundary $\partial_\infty \mathbb{P} = \mathbb{R} \cup \{\infty\}$ to $\partial\mathbb{D}$. Consequently, much of the work on $H(\mathbb{P})$ uses the results on $H(\mathbb{D})$, utilizing the function ψ; see [15], [14, p. 20].

1.3 Classification of Linear Fractional Transformations

In this section, we give a brief account on the classification of linear fractional transforms on $\mathbb{P}$. This brief discussion is from Shapiro [23], and more facts, results, and definitions can be found in Walkden [25].

Definition 1.3.1. *(Shapiro [23]) If $\phi(z) = \frac{az+b}{cz+d}$ is a linear fractional transformation in the stan-*

dard form, then we define $T(\phi) = \pm(a+d)$ *to be the trace of* ϕ*. where the ambiguous sign is intended to signal the "plus-minus" ambiguity in the standard form representation of T.*

It is not hard to see that classification of linear fractional transformations can be done based in two view points:

(a) The number of its fixed points.

To determine the fixed points of a linear fractional transformation, set

$$z = \phi(z) = \frac{az+b}{cz+d}, \quad with \;\; ad-bc=1.$$

Thus

$$z(cz+d) = az+b.$$

This implies

$$cz^2 + (d-a)z - b = 0. \tag{1.3.2}$$

Hence, if α and β the roots of equation (1.3.2) we get the following

$$\begin{aligned}
\alpha, \beta &= \frac{-(d-a) \pm \sqrt{(d-a)^2 + 4cb}}{2c} \\
&= \frac{a-d \pm \sqrt{(a+d)^2 - (a+d)^2 + (d-a)^2 + 4cb}}{2c} \\
&= \frac{a-d \pm \sqrt{(a+d)^2 - (4ad-4cb)}}{2c}, \qquad \text{where} \quad ad-bc=1 \\
&= \frac{a-d \pm \sqrt{T^2-4}}{2c}. \qquad (1.3.3)
\end{aligned}$$

From this equation we can see the types of fixed points.

Type 1. If $c \neq 0, T \neq \pm 2$, then we get two finite distinct fixed points $\alpha \neq \beta$.

Type 2. If $c \neq 0, T = \pm 2$, then we get only one finite point $\alpha = \beta = \frac{a-d}{2c}$.

Type 3. If $c = 0, T \neq \pm 2$ then we get two fixed points and one of them is $\beta = \infty$ and the other one is a finite fixed point α. To see this, since $c = 0$ then $ad = 1$ this implies that $a = \frac{1}{d}$. Thus

$$\phi(z) = \frac{az}{d} + \frac{b}{d} = a^2 z + ba$$

is a linear transformation. Setting $z = a^2 z + ba$, we obtain the following two fixed points:

$$\alpha = \frac{ab}{1 - a^2}\ ,\ \beta = \infty.$$

Type 4. If $c = 0$ and $T = \pm 2$, then we get only one fixed point which will be ∞. To see this, since $c = 0$ implies $ad = 1$, and $T = a + d = \pm 2$, we have $a = \pm 1$. Hence, $\phi(z) = z \pm b$. Note that $\phi(\infty) = \infty$.

We conclude that ϕ has a unique fixed point in $\hat{\mathbb{C}}$ if and only if $|T(\phi)| = 2$ and ϕ has two fixed points if and only if $|T(\phi)| \neq 2$.

The next question is how we classify these transformations? In order to see that we need to write the linear fractional transformation in the form as explain in the below theorems.

Theorem 1.3.4. *(Kaur [11]) If a linear fractional transformation*

$$\omega = \phi(z) = \frac{az + b}{cz + d}, \quad where \ \ ad - bc = 1,$$

has two distinct fixed points α and β, then the transformation takes the form

$$\frac{\omega - \alpha}{\omega - \beta} = k \left(\frac{z - \alpha}{z - \beta} \right).$$

Proof. The results is obvious in the case that $\beta = \infty$. It remains to prove the result for the case that $\alpha, \beta \in \mathbb{C}$. Since α, β are the roots of the equation $(cz)^2 + (d - a)z - b = 0$, this implies that

$$c\alpha^2 + (d-a)\alpha - b = 0 \text{ and } c\beta^2 + (d-a)\beta - b = 0.$$

Thus,

$$c\alpha^2 - a\alpha = b - d\alpha \text{ and } c\beta^2 - a\beta = b - d\beta. \tag{1.3.5}$$

Consider

$$\begin{aligned}
\frac{\omega-\alpha}{\omega-\beta} &= \frac{\frac{az+b}{cz+d}-\alpha}{\frac{az+b}{cz+d}-\beta} \\
&= \frac{az+b-\alpha cz-\alpha d}{az+b-\beta cz-d\beta} \\
&= \frac{(a-\alpha c)z+(b-\alpha d)}{(a-\beta c)z+(b-d\beta)} \\
&= \frac{(a-\alpha c)z+c\alpha^2-a\alpha}{(a-\beta c)z+c\beta^2-a\beta}, \text{ by } (1.3.5)
\end{aligned}$$

$$\begin{aligned}
&= \frac{(a-\alpha c)z+\alpha(c\alpha-a)}{(a-\beta c)z+\beta(c\beta-a)} \\
&= \frac{(a-\alpha c)(z-\alpha)}{(a-\beta c)(z-\beta)} \\
&= k\frac{(z-\alpha)}{(z-\beta)}.
\end{aligned}$$

where

$$k = \frac{a-\alpha c}{a-\beta c} = \frac{T-\sqrt{T^2-4}}{T+\sqrt{T^2-4}}, \text{ by } (1.3.3).$$

□

Theorem 1.3.6. *([11]) If a linear fractional transformation $\omega = \phi(z) = \frac{az+b}{cz+d}$ has only one* *finite fixed point say $\alpha \in \mathbb{C}$, then the transformation takes the form*

$$\frac{1}{\omega - \alpha} = \frac{1}{z - \alpha} + k.$$

Proof. Put $\omega = z$; that is,

$$z = \frac{az+b}{cz+d}, \quad \text{which implies} \quad \alpha = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c},$$

where

$$(a-d)^2 + 4bc = 0.$$

This implies that

$$\alpha = \frac{a-d}{2c}.$$

Thus

$$z\alpha c = a - d.$$

Hence

$$d = a - 2\alpha c. \tag{1.3.7}$$

Again α is a root of equation $cz^2 + (d-a)z - b = 0$ in (1.3.2). This implies

$$c\alpha^2 + (d-a)\alpha - b = 0.$$

Thus

$$c\alpha^2 - a\alpha = b - d\alpha. \tag{1.3.8}$$

Now,

$$\frac{1}{\omega - \alpha} = \frac{1}{\frac{az+b}{cz+d} - \alpha}$$

$$
\begin{aligned}
&= \frac{cz+d}{az+b-\alpha cz-\alpha d}\\
&= \frac{cz+d}{(a-\alpha c)z+(b-\alpha d)}\\
&= \frac{cz+a-2\alpha c}{(a-\alpha c)z+(b-\alpha d)} \qquad \text{from} \quad (1.3.7).
\end{aligned}
$$

Hence by (1.3.8),

$$
\begin{aligned}
\frac{1}{\omega-\alpha} &= \frac{cz+a-\alpha c-\alpha c}{(a-\alpha c)z+(c\alpha^2-a\alpha)}\\
&= \frac{(cz-\alpha c)+(a-\alpha c)}{(a-\alpha c)z+\alpha(c\alpha-a)}\\
&= \frac{c(z-\alpha)+(a-\alpha c)}{(a-\alpha c)(z-\alpha)}\\
&= \frac{c}{\alpha-\alpha c}+\frac{1}{z-\alpha}.
\end{aligned}
$$

Now by taking

$$k = \frac{c}{a-\alpha c},$$

we get

$$\frac{1}{\omega-\alpha} = k + \frac{1}{z-\alpha}.$$

□

In the case of one fixed point, the transformation is called parabolic. We summarize the classification of linear fractional transformations by its multiplier k in the following definition:

Definition 1.3.9. *[15, p. 42] In the case of two distinct fixed points of the linear fractional transformation, the multiplier k is given by $k = \frac{T-\sqrt{T^2-4}}{T+\sqrt{T^2-4}}$. We say that:*

(1) A transformation is hyperbolic if $k > 0$.

(2) A transformation is elliptic if $k = e^{i\alpha}, \alpha \neq 0, |k| = 1$.

(3) A transformation is loxodramic if $k = ae^{i\alpha}$ where $a \neq 1, \alpha \neq 0$ and α, β are both real numbers $a > 0$. In other words, T is neither elliptic nor parabolic.

(b) Classification by the trace of a matrix representing the linear fractional transformation. Let A be a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The trace of A is $T = a + d$.

Theorem 1.3.10. *([23]) Suppose ϕ is a linear fractional transformation that is not the identity, then ϕ is loxodramic if and only if its trace $T(\phi)$ is not real. If $T(\phi)$ is real, then ϕ is:*

- *Hyperbolic if $|T(\phi)| > 2$ (where $k = \frac{a-b}{a+b}, k$ real$, k > 0, k \neq 0)$.*
- *Parabolic if $|T(\phi)| = 2$ (where $k = 1, \alpha = \beta)$.*
- *Elliptic if $|T(\phi)| < 2$ (where $k = \frac{a-ib}{a+ib},\ |k| = 1)$.*

Lemma 1.3.11. *([25, p. 61]) A linear fractional transformation $\phi(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ is a parabolic if and only if it is conjugate to a translation.*

Proof. Suppose ϕ is parabolic and has a unique fixed point at ξ. Let $h(z) = \frac{1}{z-\xi}$ be a linear fractional transformation that maps ξ to ∞. Then $h \circ \phi \circ h^{-1}$ is a linear fractional transformation with a unique fixed point at ∞ because $h \circ \phi \circ h^{-1}(\infty) = h(\phi(\xi)) = h(\xi) = \infty$.
We claim that $h \circ \phi \circ h^{-1}$ is a translation. Write

$$h \circ \phi \circ h^{-1}(z) = \frac{az+b}{cz+d}.$$

Since ∞ is a fixed point we must have that $c = 0$ and so we can write

$$h \circ \phi \ \circ h^{-1}(z) = \frac{az}{d} + \frac{b}{d},$$

and thus has a fixed point at $\frac{b}{d-a}$ because

$$\begin{aligned} h \circ \phi \circ h^{-1}(\frac{b}{d-a}) &= \frac{a(\frac{b}{d-a})}{d} + \frac{b}{d} \\ &= \frac{1}{d}\left[\frac{ab}{d-a} + \frac{b(d-a)}{d-a}\right] \\ &= \frac{b}{d-a}. \end{aligned}$$

Since $h \circ \phi \ \circ h^{-1}$ has only one fixed point at ∞ we must have that $d = a$. Thus,

$$h \circ \phi \circ h^{-1} : z \to z + \frac{b}{d}.$$

Conversely, assume that ϕ is conjugate to a translation, that is for some map h we have that

$$h \circ \phi \circ h^{-1} : z \to z + b,$$

for some $b \in \mathbb{R} \setminus \{0\}$. But this has a unique fixed point ∞ and it is therefore a parabolic. □

Lemma 1.3.12. *([25, p. 62]) A linear fractional transformation $\phi(z) = \frac{az+b}{cz+d}$ is a hyperbolic if and only if conjugate to a dilation.*

Proof. If ϕ is conjugate to a dilation of the form $\psi(z) = kz$. Thus there is a map h such that $h \circ \phi \circ h^{-1} = \psi$ then this map clearly has precisely two fixed points 0 and ∞ and so does ϕ. Therefore, ϕ is hyperbolic.

If $\phi(z)$ fixes 0 and ∞ then we claim it is a dilation. To show that, write $\phi(z) = \frac{az+b}{cz+d}$. Since ∞ is a fixed point we must have $c = 0$. Also since 0 is fixed point we have $b = 0$. Hence, $\phi(z) = \frac{a}{d}z$.

More generally, suppose $\phi(z)$ is hyperbolic with exactly two fixed points ξ_1, ξ_2. First suppose that $\xi_1 = \infty$ and $\xi_2 \in \mathbb{R}$. Let $h(z) = z - \xi_2$. Then the conjugate map $h \circ \phi \circ h^{-1}$ has fixed points 0 and

∞,

$$\begin{aligned} h \circ \phi \circ h^{-1}(0) &= h \circ \phi(\xi_2) = h(\xi_2) = 0; \\ h \circ \phi \circ h^{-1}(\infty) &= h(\phi(\infty)) = h(\infty) = \infty. \end{aligned}$$

Thus $h \circ \phi \circ h^{-1}$ is a dilation by the above.

Finally, assume that ϕ has two real fixed points ξ_1, ξ_2. We may assume that $\xi_1 < \xi_2$. Let h be the transformation $f(z) = \frac{z-\xi_2}{z-\xi_1}$ This is a linear fractional map and the conjugate map $h \circ \phi \circ h^{-1}$ has fixed points 0 and ∞. Hence, it is a dilation by our argument above. $\square$

In conclusion, since we aim to study universal composition operator on space $H(\mathbb{P})$, we will focus in the rest of this dissertation on linear fractional transformations that have no fixed point in $\mathbb{P}$; that in case a linear fractional transformation is hyperbolic or parabolic. We give more facts about that in Chapter 5.

CHAPTER 2 UNIVERSALITY RESULTS FOR THE UNIT DISK $\mathbb{D}$

2.1 A Sufficient Condition for Universality

Suppose X is a separable Fréchet space and $T : X \to X$ is a continuous linear operator. Let d a translation invariant metric d makes X a separable complete metric space. For $x \in X$, we denote the quantity $\|x\| = d(x, 0)$. In this section we state Gethner and Shapiro's condition and provide their proof. But first we need to mention to the following useful proposition.

Proposition 2.1.1. *(Gethner and Shapiro [6]) If T has a universal vector, then it has a dense G_δ set of universal vectors.*

Proof. Fix a countable dense subset $\{y_j\}$ of X. For positive integers N, j, and k, set

$$F = F(j.N, k) = \{x \in X : \|T^n x - y_j\| < \frac{1}{k} \text{ for some } n \geqslant N\} = \bigcup_{n \geq N} T^{-n} B(y_j, \frac{1}{k})$$

By the continuity of T, each $F(j, N, k)$ is open. The set of T-universal vectors is the set

$$\bigcap_{j\ N\ k} F(j, N, k),$$

which is therefor a G_δ subset of X. If x is a universal vector, then so is every member of the dense orbit orb$(T, x) = \{T^n x; n \geqslant 1\}$. □

Theorem 2.1.2. *(Gethner and Shpiro [6]) Suppose T is a continuous linear operator on a separable Fréchet space X. Suppose there exist a dense subset D of X and a right inverse S for T (TS= identity on X) such that $\|T^n x\| \to 0$ and $\|S^n x\| \to 0$ for every $x \in D$. Then X has T-universal vectors.*

Proof. By Baire's Theorem [19, p. 42] it is enough to prove that each of the G_δ sets $F = F(j.N, k)$ defined in the proof of Proposition 2.1.1 is dense in X.

To see this, fix $F = F(j, N, k)$, and for ease of notation write $\epsilon = \frac{1}{k}$ and $y = y_j$. Fix z in X

and $\delta > 0$. We must find an $x \in F$ lying within δ of z. Since D is dense in X. We can choose y_0 and z_0 in D with $\|z - z_0\| < \frac{\delta}{2}$, and $\|y - y_0\| < \frac{\epsilon}{2}$. Since the sequence T^n and S^n converge pointwise to zero on D, we may choose a positive integer n such that simultaneously $\|T^n z_0\| < \frac{\epsilon}{2}$, and $\|S^n y_0\| < \frac{\delta}{2}$. Write $x = S^n y_0 + z_0$. Then

$$\|x - z\| \leqslant \|x - z_0\| + \|z_0 - z\| = \|S^n y_0\| + \|z_0 - z\| < \frac{\delta}{2} + \frac{\delta}{2},$$

and so $\|x - z\| < \delta$, as desired. Moreover, since TS is the identity map on X, so is $T^n S^n$. Thus

$$\|T^n x - y\| = \|T^n S^n y_0 - y + T^n z_0\| \leqslant \|y_0 - y\| + \|T^n z_0\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and so $x \in F$, and the proof is complete. □

From this sufficient condition, we can get a unified proof of universality for many operators. We now move to the next section, where we use this condition to prove theorem of Seidel and Walsh.

2.2 The Seidel and Walsh Theorem

In fact the proof of Proposition 2.1.1 gives more as we see that in the following remark which shows that the sequence $\{T^{n_j} x : j \geq 0\}$ is dense in X for a dense G_δ subset of $x's$ where $\{n_j : j \geq 0\}$ any fixed subsequence of positive integers with $n_j \nearrow \infty$.

Remark 2.2.1. *[6, p. 283] Let $(T_k)_k$ be a sequence of continuous linear operators on X and D be a dense subset of X such that $T_k \to 0$ pointwise on D. Further assume that each operator T_k has a right inverse S_k for each $k \geqslant 1$, and $S_k \to 0$ pointwise on D. Then for a dense G_δ set of vectors $x \in X$ we get that the set $\{T_k x : k \geqslant 0\}$ is dense in X.*

One application of this final form of Gethner and Shapiro's condition is the Seidel and Walsh Theorem [21]. Recall that if $H(\mathbb{D})$ is the set of holomorphic functions on the unit disc $\mathbb{D}$ topologized by uniform convergence on compact subsets, then $H(\mathbb{D})$ is a separable Fréchet space. Now we prove the following theorem.

Theorem 2.2.2. *(Seidel and Walsh [21]) Suppose* $\{\alpha_n\}_n$ *is a sequence of points in the open unit* $\mathbb{D}$ *with* $\lim_{n\to\infty}\alpha_n = 1$*. Let* ϕ_n *be the linear fractional transformation of* $\mathbb{D}$ *defined by:*

$$\phi_n(z) = \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \qquad \textit{where } z \in \mathbb{D} \ \ , and \quad n \geqslant 1.$$

If $C_n : H(\mathbb{D}) \to H(\mathbb{D})$ *is given by* $C_n f = f \circ \phi_n$*, then* $\{C_n\}_{n=1}^{\infty}$ *is universal.*

Proof. (Gather and Shapiro [6]) Note that each ϕ_n is its own inverse, by putting $C_n = C_n^{-1}$ the condition $TS = I$ in Theorem 2.1.2 is satisfied. We define a functions $f_{m,k} : \mathbb{D} \to \mathbb{D}$ by

$$f_{m,k}(z) = z^m \frac{1 - z^k}{1 + z^k}, \qquad \text{where } \ z \in \mathbb{D}, \ m \geqslant 0, \ \text{and } k > 0.$$

Let $D = \text{span}\{f_{m,k}(z) : z \in \mathbb{D}, \ m \geqslant 0, \ k > 0\}$. Then for fixed m, as $k \to \infty$ the sequence $f_{m,k}$ converges to the function z^m uniformly on compact subset of $\mathbb{D}$, so that the linear span D of such functions is dense in $H(\mathbb{D})$. Now, since ϕ_n converges uniformly on compact subset of $\mathbb{D}$ to the constant function 1 as $\alpha_n \to 1$, and each function $f_{m,k}$ is holomorphic on a neighborhood of 1 and vanishes at 1, we get that $C_n f_{m,k} \to 0$ uniformly on compact subsets as $n \to \infty$. Thus $C_n \to 0$ on D, and so by Remark 2.2.1, $\{C_n\}_n$ is universal.

□

From the Seiled and Walsh Theorem we know that if $\{b_n\}_{=1}^{\infty} \subset \mathbb{D}$ and $b_n \to 1 \in \mathbb{D}$, then there is $g \in H(\mathbb{D})$ such that $\{g \circ \psi_n : n \geqslant 1\}$ is dense in $H(\mathbb{D})$, where

$$\psi_n(z) = \frac{b_n - z}{1 - \bar{b}z}.$$

Now let $e^{i\theta} \in \partial\mathbb{D}$, and $f(z) = g(e^{-i\theta}z)$, and $\varphi_n(z) = e^{i\theta}\frac{b_n - z}{1 - \bar{b}z}$. Then

$$\begin{aligned} f \circ \varphi_n(z) &= f\left(e^{-i\theta}e^{i\theta}\frac{b_n - z}{1 - \bar{b}z}\right) \\ &= g \circ \psi_n(z). \end{aligned}$$

Thus $\{f \circ \varphi_n : n \geqslant 1\}$ is dense in $H(\mathbb{D})$.

Form above remark we are ready to prove the following Corollary 4.2.2 which we consider it as a general case of Seidel and Walsh Theorem 2.2.2. There is no doubt that this Corollary 2.2.4 is well-known but we can not locate a proper reference in the literature.

Corollary 2.2.3. *Let* $\{a_n\}_{n=1}^{\infty} \subset \mathbb{D}$ *with* $|a_n| \to 1$*, and* $\varphi_n(z) = \frac{a_n - z}{1 - \overline{a_n} z}$ *then there is a function* $f \in H(\mathbb{D})$ *such that* $\{f \circ \varphi_n : n \geqslant 1\}$ *is dense in* $H(\mathbb{D})$.

Proof. Since $\overline{\mathbb{D}}$ is compact, there is a subsequence of $\{a_n\}_n$, still denote by $\{a_n\}_n$ such that $a_n \to \beta \in \partial\mathbb{D}$. Let $\beta = e^{i\theta}$, thus $e^{-i\theta} a_n \to e^{-i\theta}\beta = 1$. Let

$$\varphi_n(z) = \frac{a_n - z}{1 - \overline{a_n} z},$$

which can be rewritten as

$$\varphi_n(z) = e^{i\theta} \frac{e^{-i\theta} a_n - e^{-i\theta} z}{1 - \overline{a_n e^{-i\theta}} e^{-i\theta} z}.$$

Let $b_n = e^{-i\theta} a_n$ and so $b_n \to 1$. Hence if we let

$$\psi_n(z) = \frac{b_n - z}{1 - \overline{b_n} z}.$$

Then by the Seidel and Walsh Theorem, there is a function $g \in H(\mathbb{D})$ such that $\{g \circ \psi_n(z) : n \geqslant 1\}$ is dense in $H(\mathbb{D})$. By our remark above there is a function $f \in H(\mathbb{D})$ such that if

$$\phi_n(z) = e^{i\theta} \psi_n(z)$$

then $\{f \circ \phi_n(z) : n \geqslant 1\}$ is dense in $H(\mathbb{D})$. That is, $\{f \circ \phi_n(e^{-i\theta} z) : n \geqslant 1\}$ is dense in $H(\mathbb{D})$. Now our statement follows from the observation that

$$\phi_n(e^{-i\theta} z) = e^{i\theta} \frac{e^{-i\theta} a_n - e^{-i\theta} z}{1 - \overline{e^{-i\theta} a_n} e^{-i\theta} z} = \frac{a_n - z}{1 - \overline{a_n} z} = \varphi_n(z),$$

which completes the proof. $\square$

We now prove the following lemma.

Lemma 2.2.4. *Let* $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{D}$ *and* $\{\theta_n\}_{n=1}^{\infty} \subset [-\pi, \pi]$. *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be defined by*

$$\varphi_n(z) = \frac{\alpha_n - z}{1 - \overline{\alpha_n} z},$$

and $\phi_n(z) : \mathbb{D} \to \mathbb{D}$ *be defined by*

$$\phi_n(z) = e^{i\theta_n} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z}.$$

Then the sequence $\{C_{\varphi_n}\}_{n=1}^{\infty}$ *is universal if and only if the sequence* $\{C_{\phi_n}\}_{n=1}^{\infty}$ *is universal.*

Proof. Assume that $\{C_{\varphi_n}\}_{n=1}^{\infty}$ is universal. Since every subsequence $\{\alpha_{n_k}\}_k$ is contained in the compact set $\overline{\mathbb{D}}$. It must have a convergent subsequence. If every subsequence converges to a point α inside $\mathbb{D}$, then for any function f in $H(\mathbb{D})$, $\{f \circ \varphi_n\}_{n \geqslant 1}$ can only converge to the function of the form $f\left(\frac{\alpha - z}{1 - \overline{\alpha} z}\right)$, which takes α to $f(0)$. Thus $\{f \circ \varphi_n : n \geqslant 1\}$ can not be dense in $H(\mathbb{D})$. Hence we must have a subsequence $\{n_k\}_k$ such that $|\alpha_{n_k}| \to 1$. By the compactness of $[-\pi, \pi]$, there is a further subsequence of $\{n_k\}_K$, still denote by $\{n_k\}_k$ such that $\theta_{n_k} \to \theta$ for some $\theta \in [-\pi, \pi]$.
Let

$$f_k(z) = e^{i\theta} \frac{z - \alpha_{n_k}}{1 - \overline{\alpha_{n_k}} z}.$$

Since

$$|\alpha_{n_k}| \to 1, \text{ the sequence } C_{f_k} : H(\mathbb{D}) \to H(\mathbb{D})$$

is universal. Since $\theta_{n_k} \to \theta$, and $\phi_{n_k}(z) = \frac{e^{i\theta_{n_k}}}{e^{i\theta}} f_k(z)$ we see that $\{C_{\phi_{n_k}}\}_k$ is universal, and hence $\{C_{\phi_n}\}_n$ is universal.

Conversely, if $\{C_{\varphi_n}\}_n$ is universal, then by the compactness of $[-\pi, \pi]$, we have a subsequence $\{\theta_{n_k}\}_k$ of $\{\theta_n\}_n$ that converges to a value $\theta \in [-\pi, \pi]$. By repeating the above argument, we prove that $\{C_{\varphi_n}\}_n$ is universal. □

We now move to Chapter 3, where we state Theorem 3.2.6 that gives equivalent conditions that connect universality of $\{C_{\phi_n}\}_n$ on space $H(\mathbb{D})$ with universality of $\{C_{\sigma_n}\}_n$ on space $H(\mathbb{P})$. Later in Chapter 6, we use the above Lemma and Theorem 3.2.6 to study the limit point of the iterations of a linear fractional transformation φ defined on the unit disk $\mathbb{D}$. In particular, We obtain the formula for the limit point in terms of α and θ.

CHAPTER 3 UNIVERSALITY RESULTS FOR THE UPPER HALF PLANE $\mathbb{P}$

3.1 Introduction

Birkhoff [2] proved the universality of the family $\{C_{\tau_a}\}_{a\in\mathbb{C}}$ of composition operators

$$C_{\tau_a} : H(\mathbb{C}) \to H(\mathbb{C}), \text{ defined by } C_{\tau_a}(f) = f \circ \tau_a,$$

where $\tau_a(z) = z + a$, with $a \in \mathbb{C}$ and $a \neq 0$, is indeed a linear fractional transformation or more precisely a translation. After that, Seidel and Walsh [21] established an analogue of the Birkhoff theorem for the unit disk $\mathbb{D}$. As a continuation of that work, in this chapter we prove a parallel version of the Seidel and Walsh Theorem for the upper half plane $\mathbb{P}$. Indeed, we derive a new necessary and sufficient condition for a linear fractional transformation $\sigma_n(z) = \frac{a_n z+b_n}{c_n z+d_n}$ with $a_n, b_n, c_n, d_n \in \mathbb{R}$ and $a_n d_n - b_n c_n = 1$ such that the sequence of composition operators $\{C_{\sigma_n}\}_n$ is universal on $H(\mathbb{P})$.

If $\phi(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$ with $\alpha \in \mathbb{D}$ and $\theta \in [0, \pi]$ is a linear fractional transformation that takes $\mathbb{D}$ onto itself and $\psi(z) = \frac{z-i}{z+i}$ is a conformal map that takes $\mathbb{P}$ onto $\mathbb{D}$, then $\sigma = \psi^{-1} \circ \phi \circ \psi$ is holomorphically conjugate to ϕ and it is a holomorphic self map of $\mathbb{P}$. We summarize in the below diagram.

$$\mathbb{P} \xrightarrow{\psi} \mathbb{D} \xrightarrow{\varphi} \mathbb{D} \xrightarrow{\psi^{-1}} \mathbb{P}, \qquad \mathbb{P} \xrightarrow{\sigma=\psi^{-1}\circ\varphi\circ\psi} \mathbb{P} \tag{3.1.1}$$

Since σ and ϕ are conjugate to each other, then the corresponding operator

$$C_\sigma = C_{\psi^{-1}\circ\phi\circ\psi} = C_\psi \circ C_\phi \circ (C_\psi)^{-1}$$

is defined on $H(\mathbb{P})$ and is similar to C_ϕ which is defined on $H(\mathbb{D})$; see [22, p. 23]. Using this fact we establish our new results in the next section.

3.2 A Necessary and Sufficient Condition for Universality on $H(\mathbb{P})$

In order to prove our main result we first need to prove the following lemma in which we construct a formula for $e^{i\theta}$ and α in terms of a, b, c, d the coefficients of the conformal self map on the upper half-plane $\mathbb{P}$.

Lemma 3.2.1. *Let* $a, b, c, d \in \mathbb{R}$ *with* $ad - bc = 1$. *Let* $\phi(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$ *with* $|\alpha| < 1$ *and* $\theta \in [0, 2\pi]$, *and* $\psi(z) = \frac{z-i}{z+i}$. *If*

$$\psi^{-1} \circ \phi \circ \psi(z) = \frac{az+b}{cz+d},$$

then

$$e^{i\theta} = \frac{(c-b) + i(a+d)}{(b-c) + i(a+d)} \qquad \text{and} \qquad \alpha = \frac{(b+c) + i(a-d)}{(b-c) - i(a+d)}.$$

Proof. Note that

$$\psi^{-1}(z) = i\frac{1+z}{1-z}.$$

Set

$$\psi^{-1} \circ \phi \circ \psi(z) = \frac{az+b}{cz+d}, \quad \text{where} \quad a, b, c, d \in \mathbb{R}.$$

Thus,

$$\begin{aligned}
\psi^{-1} \circ \phi(z) &= \frac{a\psi^{-1}(z) + b}{c\psi^{-1}(z) + d} \\
&= \frac{ai(\frac{1+z}{1-z}) + b}{ci(\frac{1+z}{1-z}) + d} \\
&= \frac{ai + aiz + b - bz}{ci + ciz + d - dz}.
\end{aligned}$$

Hence

$$\phi(z) = \psi\left(\frac{ai + aiz + b - bz}{ci + ciz + d - dz}\right)$$

$$= \frac{\left(\frac{ai+aiz+b-bz}{ci+ciz+d-dz}\right)-i}{\left(\frac{ai+aiz+b-bz}{ci+ciz+d-dz}\right)+i}.$$

Thus

$$\begin{aligned}\phi(z) &= \frac{(ai+aiz+b-bz)-i(ci+ciz+d-dz)}{(ai+aiz+b-bz)+i(ci+ciz+d-dz)} \\ &= \frac{(ai-b+c+id)z+(ai+b+c-id)}{(ai-b-c-id)z+(ai+b-c+id)} \\ &= \frac{[(c-b)+i(a+d)]z+[(b+c)+i(a-d)]}{[-(b+c)+i(a-d)]z+[(b-c)+i(a+d)]}.\end{aligned}$$

Hence we get

$$\phi(z) = \frac{(c-d)+i(a+d)}{(b-c)+i(a+d)}\frac{z+\frac{(b+c)+i(a-d)}{(c-b)+i(a+d)}}{\frac{-(b+c)+i(a-d)}{(b-c)+i(a+d)}z+1}. \tag{3.2.2}$$

Since

$$\frac{(c-b)+i(a+d)}{(b-c)+i(a+d)}^{2} = \frac{(c-b)^2+(a+b)^2}{(b-c)^2+(a+d)^2} = 1,$$

we let

$$e^{i\theta} = \frac{(c-b)+i(a+d)}{(b-c)+i(a+d)},$$

for some $\theta \in [0, 2\pi]$.

Let

$$\alpha = -\frac{(b+c)+i(a-d)}{(c-b)+i(a+d)}.$$

Hence

$$\overline{\alpha} = -\frac{(b+c)-i(a-d)}{(c-b)-i(a+d)}$$

$$= \frac{-(b+c)+i(a-d)}{(c-b)-i(a+d)}$$

$$= -\frac{-(b+c)+i(a-d)}{(b-c)+i(a+d)}.$$

Hence we can rewrite equation (3.2.2) as

$$\phi(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}.$$

It remains to show $|\alpha| < 1$.

$$\begin{aligned}
|\alpha|^2 &= \frac{(b+c)^2+(a-d)^2}{(c-b)^2+(a+d)^2} \\
&= \frac{a^2+b^2+c^2+d^2+2(bc-ad)}{a^2+b^2+c^2+d^2+2(ad-bc)} \\
&= \frac{a^2+b^2+c^2+d^2-2}{a^2+b^2+c^2+d^2+2} < 1.
\end{aligned} \tag{3.2.3}$$

□

We can now state and prove our main result. The following proposition provides us with a necessary and sufficient condition for a composition operator acting on Fréchet space $H(\mathbb{P})$ of holomorphic functions for the upper half-plane to have a universal vector. By using the formula of $e^{i\theta}$ and α in Lemma 3.2.1 we obtain our condition, which is analogous to the condition of Seidel and Walsh Theorem 2.2.2 for the universality of composition operator on the unit disk.

Proposition 3.2.4. *Let* $a_n, b_n, c_n, d_n \in \mathbb{R}$ *with* $a_n d_n - b_n c_n = 1$.
Let

$$\sigma_n(z) = \frac{a_n z + b_n}{c_n z + d_n},$$

and

$$C_{\sigma_n} : H(\mathbb{P}) \to H(\mathbb{P}) \quad \textit{be defined by}$$

$$C_{\sigma_n} f = f \circ \sigma_n.$$

The sequence $\{C_{\sigma_n}\}_n$ *is universal if and only if*

$$\limsup_{n\to\infty} (|a_n| + |b_n| + |c_n| + |d_n|) = \infty.$$

Proof. In view of Lemma 3.2.1, let $\alpha_n \in \mathbb{D}$ and $\phi_n(z) = e^{i\theta_n} \frac{z-\alpha_n}{1-\bar{\alpha}z}$, where

$$e^{i\theta_n} = \frac{(c_n - b_n) + i(a_n + d_n)}{(b_n - c_n) + i(a_n + d_n)}$$

and

$$\alpha_n = \frac{(b_n + c_n) + i(a_n - d_n)}{(b_n - c_n) - i(a_n + d_n)}.$$

Hence

$$C_{\sigma_n} = C_\psi \circ C_{\phi_n} \circ C_\psi^{-1}.$$

For any $f \in H(\mathbb{P})$,

$$C_\psi^{-1} \circ C_{\sigma_n}(f) = C_{\phi_n} \circ C_\psi^{-1}(f) = C_{\phi_n}(f \circ \psi^{-1}).$$

By the continuity of the bijection C_ψ^{-1}, it takes a dense set to a dense set. Thus a vector f is a universal for C_{σ_n} if and only if $f \circ \psi^{-1}$is a universal vector for C_{ϕ_n}. We now proceed to establish the $\limsup$ condition in the theorem. We first observe that $\limsup |\alpha_n| = 1$ if and only if there is a subsequence $\{\alpha_{n_k}\}_k$ such that $|\alpha_{n_k}| \to 1$. Since $\overline{\mathbb{D}}$ is compact, there is a further subsequence, still denoted by $\{\alpha_{n_k}\}_k$, such that $\alpha_{n_k} \to \alpha$ in $\partial\mathbb{D}$. The sequence $\{\theta_{n_k}\}_k$ is in the compact interval $[0, 2\pi]$. By picking a subsequence of $\{\theta_{n_k}\}_k$ if necessary, we can assume $\theta_{n_k} \to \theta \in [0, 2\pi]$. Hence by the theorem of Seidel and Walsh, we have that $\{C_{\sigma_n}\}_n$ is universal if and only if $\limsup |\alpha_n| =$

1. By the inequality (3.2.3) we see that $\limsup|\alpha_n| = 1$ if and only if

$$\limsup_{n\to\infty}\left(|a_n| + |b_n| + |c_n| + |d_n|\right) = \infty.$$

This completes our proof. □

The above proposition can be generalized a step further by making a connection between a universal sequence of composition operators in the unit disk and a universal sequence of composition operators in the upper half-plane. In fact, we can establish more related results in the following Theorem, keeping in mind the following diagram:

$$\mathbb{P} \xrightarrow{\psi(z)=\frac{z-i}{z+i}} \mathbb{D} \xrightarrow[\alpha_n\in\mathbb{D}]{\phi(z)=\frac{\alpha_n-z}{1-\overline{\alpha_n}z}} \mathbb{D} \xrightarrow{\psi^{-1}(z)=i\frac{1+z}{1-z}} \mathbb{P} \tag{3.2.5}$$

Theorem 3.2.6. *For $n \geq 1$, let $\alpha \in \mathbb{D}$, and $\phi_n(z) = \frac{\alpha_n - z}{1-\overline{\alpha_n}z}$ be linear fractional transformations on $\mathbb{D}$. Suppose*

$$\sigma_n(z) = \psi^{-1}\circ\phi_n\circ\psi(z) = \frac{a_n z + b_n}{c_n z + d_n},$$

where a_n, b_n, c_n, $d_n \in \mathbb{R}$ with $a_n d_n - b_n c_n = 1$, are linear fractional transformations on $\mathbb{P}$. The following six statements are equivalent:

(1) $|\alpha_n| \to 1$.

(2) The sequence of composition operators $C_{\phi_n} : H(\mathbb{D}) \to H(\mathbb{D})$ is universal.

(3) There are a point $e^{i\theta} \in \partial\mathbb{D}$ and a subsequence $\{\phi_{n_k}\}_k$ such that $\phi_{n_k} \to e^{i\theta}$ uniformly on compact subsets of $\mathbb{D}$.

(4) $\limsup|a_n| + |b_n| + |c_n| + |d_n| = \infty$.

(5) There are a point $\zeta \in \partial_\infty\mathbb{P} = \mathbb{R}\cup\{\infty\}$ and a subsequence $\{\sigma_{n_k}\}_k$ of $\{\sigma_n\}_n$ such that $\sigma_{n_k}(z) \to \zeta$ uniformly on compact subsets of $\mathbb{P}$.

(6) The sequence of composition operators $C_{\sigma_n} : H(\mathbb{P}) \to H(\mathbb{P})$, where $n \geqslant 1$ is universal.

Proof. The equivalence of statements (1) and (2) is from Corollary 2.2.3. From Proposition 3.2.4, statements $(1), (2), (4)$ and (6) are equivalent. It remains to show statement (1) is equivalent to statement (3) and statement (3) is equivalent to statement (5).

To show (1) implies (3): Suppose $|\alpha_n| \to 1$. Then there is a subsequence $\{\alpha_{n_k}\}_k$ such that

$$|\alpha_{n_k}| \to e^{i\theta}.$$

Note that

$$\begin{aligned}
\phi_{n_k}(z) &= \frac{\alpha_{n_k} - z}{1 - \overline{\alpha_{n_k}} z} \\
&= \frac{\overline{\alpha_{n_k}}}{\overline{\alpha_{n_k}}} \frac{\alpha_{n_k} - z}{1 - \overline{\alpha_{n_k}} z} \\
&= \frac{1}{\overline{\alpha_{n_k}}} \frac{|\alpha_{n_k}|^2 - \overline{\alpha_{n_k}} z}{1 - \overline{\alpha_{n_k}} z} \\
&= \frac{1}{\overline{\alpha_{n_k}}} \frac{|\alpha_{n_k}|^2 - 1 + 1 - \overline{\alpha_{n_k}} z}{1 - \overline{\alpha_{n_k}} z} \\
&= \frac{1}{\overline{\alpha_{n_k}}} \frac{|\alpha_{n_k}|^2 - 1}{1 - \overline{\alpha_{n_k}} z} + \frac{1}{\overline{\alpha_{n_k}}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\phi_{n_k}(z) - e^{i\theta} &= \frac{1}{\overline{\alpha_{n_k}}} \frac{|\alpha_{n_k}|^2 - 1}{1 - \overline{\alpha_{n_k}} z} + \frac{1}{\overline{\alpha_{n_k}}} - e^{i\theta} \\
&\leqslant \frac{1}{\overline{\alpha_{n_k}}} \frac{1 - |\alpha_{n_k}|^2}{1 - R} + \frac{1}{\overline{\alpha_{n_k}}} - e^{i\theta} \to 0 \quad (\text{if } |z| \leqslant R),
\end{aligned}$$

uniformly on $R\overline{\mathbb{D}}$.

To show (3) implies (1): Suppose $\phi_n(z) \to e^{i\theta}$ uniformly on compact subsets of $\mathbb{D}$. Then

$$|\phi_n(z)| \to 1,$$

and hence

$$
\begin{aligned}
(1-|\phi_n(z)|^2) &= (1-|\phi_n(z)|)(1+|\phi_n(z)|) \\
&\leqslant 2(1-|\phi_n(z)|) \to 0,
\end{aligned}
$$

uniformly on compact subsets of $\mathbb{D}$. Now

$$
\begin{aligned}
1-|\phi_n(z)|^2 &= 1-\frac{(\alpha_n - z)(\overline{\alpha_n}-\overline{z})}{(1-\overline{\alpha_n}z)(1-\alpha_n\overline{z})} \\
&= \frac{1-\alpha_n\overline{z}-\overline{\alpha_n}z+|\alpha_n|^2|z|^2-|\alpha_n|^2+\alpha_n\overline{z}+\overline{\alpha_n}z-|z|^2}{(1-\overline{\alpha_n}z)(1-\alpha_n\overline{z})} \\
&= \frac{1-|\alpha_n|^2+|\alpha_n|^2|z|^2-|z|^2}{|1-\overline{\alpha_n}z|^2} \\
&= \frac{(1-|\alpha_n|^2)(1-|z|^2)}{|1-\overline{\alpha_n}z|^2} \\
&\geqslant \frac{(1-|\alpha_n|^2)(1-R^2)}{(1+R)^2} \quad \text{if} \quad z \in R\overline{\mathbb{D}} \\
&= \frac{1-R}{1+R}(1-|\alpha_n|^2).
\end{aligned}
$$

Hence $|\alpha_n|^2 \to 1$, which establishes (1).

To show statement (3) implies statement (5) we first make a claim of an easy fact.

Claim. *For all θ in $\mathbb{R}$ with $e^{i\theta} \neq 1$, $i\frac{1+e^{i\theta}}{1-e^{i\theta}} \in \mathbb{R}$.*

Proof of Claim. It suffices to show Re $\frac{1+e^{i\theta}}{1-e^{i\theta}} = 0$. To see that:

$$
\begin{aligned}
2\,\text{Re}\,\frac{1+e^{i\theta}}{1-e^{i\theta}} &= \frac{1+e^{i\theta}}{1-e^{i\theta}} + \frac{1+e^{-i\theta}}{1-e^{-i\theta}} \\
&= \frac{1-e^{-i\theta}+e^{i\theta}-1+1-e^{i\theta}+e^{-i\theta}-1}{|1-e^{i\theta}|^2} \\
&= 0.
\end{aligned}
$$

This completes the proof of the claim. □

Now suppose $\lim \phi_n(z) = e^{i\theta}$ uniformly on compact subsets of $\mathbb{D}$. Note that ψ and ψ^{-1} are continuous, they take compact sets to compact sets. Recall $\sigma_n(z) = \lim \psi^{-1} \circ \phi_n \circ \psi(z)$. Therefore,

$$\begin{aligned}
\lim \sigma_n(z) &= \lim \psi^{-1} \circ \phi_n(\psi(z)) \\
&= \psi^{-1}(\lim \phi_n(\psi(z))) \\
&= \begin{cases} \infty & \text{if } e^{i\theta} = 1 \\ i\frac{1+e^{i\theta}}{1-e^{i\theta}} & \text{if } e^{i\theta} \neq 1 \end{cases} \qquad \in \mathbb{R} \cup \{\infty\}, \quad \text{by our claim.}
\end{aligned}$$

To show statement (5) implies statement (3): Suppose there is a point $\rho \in \mathbb{R} \cup \{\infty\}$ such that $\sigma_n(z) \to \rho$ uniformly on compact subsets of $\mathbb{D}$. Since $\phi_n(z) = \psi \circ \sigma_n \circ \psi^{-1}(z)$ and ψ and ψ^{-1} takes compact sets to compact sets, we have

$$\begin{aligned}
\lim \phi_n(z) &= \lim \psi \circ \sigma_n \circ \psi^{-1}(z) \\
&= \psi(\lim \sigma_n(\psi^{-1}(z))) \\
&= \begin{cases} 1 & \text{if } \lim \sigma_n(z) = \infty \\ \frac{x-i}{x+i} & \text{if } \lim \sigma_n(z) = x \in \mathbb{R} \end{cases} \qquad \in \partial\mathbb{D},
\end{aligned}$$

which concludes our proof. □

The equivalence of Statements $(2), (3), (5), (6)$ is generalized in Theorem 7.2.4 to a simply connected region G whose complements has a nonempty interior. We now take some steps similar to Lemma 3.2.1 in the converse direction to construct coefficients a_n, b_n, c_n, d_n of the conformal maps σ_n that produce a sequence of universal composition operators C_{σ_n} on the upper half-plane. As an easy consequence of Theorem 3.2.6, we have the following corollary.

Corollary 3.2.7. *Let* $\{\mu_n\}_n$ *and* $\{\rho_n\}_n$ *be two sequences of positive numbers.*

(1) If $\mu_n \to 0$*, then there exists a function* $F(z)$ *in* $H(\mathbb{P})$ *such that* $\{F(\mu_n)\}_n$ *is dense in* $H(\mathbb{P})$*.*

(2) If $\rho_n \to \infty$*, then there exists a function* $G(z)$ *in* $H(\mathbb{P})$ *such that* $\{G(\rho_n)\}_n$ *is dense in* $H(\mathbb{P})$*.*

Proof. Both statements follow easily from the observation that if $\sigma_n(z) = a_n z = \frac{\sqrt{a_n} z}{\frac{1}{\sqrt{a_n}}}$ then $\sqrt{a_n}\frac{1}{\sqrt{a_n}} = 1$ and so σ_n is normalized. Now both statements follow easily from Statement (5) of Theorem 3.2.6. □

3.3 Examples

In the end of this chapter we provide a few examples of conformal maps that produce universal composition operators on the upper half plane $\mathbb{P}$ and satisfy our necessary and sufficient conditions of universality in Theorem 3.2.6.

Example 3.3.1. *Let* $\sigma(z) = n - \frac{1}{z}$ *that is* $a_n = n, b_n = -1, c_n = 1, d_n = 0$*, with* $a_n d_n - b_n c_n = 1$*. These conformal maps produce a universal sequence of composition operators on the upper half plane. Here the coefficients satisfying our condition* $\limsup(|a_n| + |b_n| + |c_n| + |d_n|) = \infty$*.*

Example 3.3.2. *Let* $\sigma(z) = n^2 z$ *where* $a_n = n, b_n = 0, c_n = 0, d_n = \frac{1}{n}$*, and* $a_n d_n - b_n c_n = 1$*. This is a sequence of dilations that produce a universal sequence of composition operators on the upper half plane. Note that the coefficients satisfy our condition* $\limsup(|a_n| + |b_n| + |c_n| + |d_n|) = \infty$*.*

Example 3.3.3. *Take* $\sigma_n(z) = z + n$ *with* $a_n d_n - b_n c_n = 1$*, where* $a_n = 1$*,* $b_n = n$*,* $c_n = 0$*,* $d_n = 1$ *so that* $\limsup(|a_n| + |b_n| + |c_n| + |d_n|) = \infty$*. Hence this is a sequence of translations that produce a universal sequence of composition operators on the upper half plane.*

CHAPTER 4 AN EQUIVALENT CONDITION FOR A COMPOSITION OPERATOR ON $H(\mathbb{P})$ TO BE HYPERCYCLIC

4.1 Introduction

It is very useful to take a step back to return to study linear fractional transformations using their coefficient matrices. It is known that similarity induces an equivalence relation on the set of all $n \times n$ square matrices. This similarity divides the set of all $n \times n$ square matrices into disjoint equivalence classes. From the fact that all matrices in an equivalence class are similar, and matrices in different classes are not similar, we utilize similar matrices to share many intrinsic properties of the same class of matrices, for more details, see [10, p. 57],[5, p. 249], [17, p. 26]. In this chapter, our main result is Theorem 4.2.5. For that we investigate the properties of a linear fractional transformation that induces a hypercyclic composition operator on $H(\mathbb{P})$. Recall that there is a connection between the function theoretic properties of linear fractional transformation ϕ and behavior of C_ϕ on $H(\mathbb{P})$ as we mentioned in Section 1.1.1. In this section, we introduce some basic concepts from linear algebra that help us study the properties of linear fractional transformations on $\mathbb{P}$.

To begin, we can identify the transformation

$$\phi(z) = \frac{az+b}{cz+d} \tag{4.1.1}$$

with its coefficient matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{4.1.2}$$

This identification between a linear fractional transformation on $\mathbb{P}$ and its coefficient matrix is useful because of the simple facts given in the following Lemma.

Lemma 4.1.3. *(1) if $r \neq 0$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $r \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are the same linear fractional transforma-*

tion.

(2) Let $\sigma(z) = \frac{az+b}{cz+d}$ *and* $\rho(z) = \frac{a^{'}z+b^{'}}{c^{'}z+d^{'}}$ *be two linear fractional transformations, and let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad and \;\; B = \begin{bmatrix} a^{'} & b^{'} \\ c^{'} & d^{'} \end{bmatrix}$$

be the coefficient matrices. Then

$$\sigma \circ \rho(z) = \frac{\alpha z + \beta}{\gamma z + \delta},$$

whose coefficient matrix is given by

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = AB.$$

Proof. Statement (1) follows directly from the observation:

$$\frac{az+b}{cz+d} = \frac{raz+rb}{rcz+rd}.$$

Statement (2) follows directly the observation:

$$\begin{aligned} \sigma \circ \rho(z) = \frac{a\rho(z)+b}{c\rho(z)+d} \quad &= \quad \frac{a\left(\frac{a^{'}z+b^{'}}{c^{'}z+d^{'}}\right)+b}{c\left(\frac{a^{'}z+b^{'}}{c^{'}z+d^{'}}\right)+d} \\ &= \quad \frac{(aa^{'}+bc^{'})z+(ab^{'}+bd^{'})}{(ca^{'}+dc^{'})z+(cb^{'}+dd^{'})}. \end{aligned}$$

We now compute AB to cheek the result.

$$\begin{aligned} AB &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \\ &= \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix} \\ &= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}. \end{aligned}$$

□

We now refer to some definitions from linear algebra; the interested reader may see [10]. Since we are motivated to study the set of linear fractional transformations on $\mathbb{P}$. We restrict our statements to only 2×2 square matrices.

Definition 4.1.4. *We define the set of matrices*

$$\text{SL}(2, \mathbb{R}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ a, b, c, d \in \mathbb{R}, \ \det A = 1 \right\}$$

to be the special linear group of $\mathbb{R}^2$.

Definition 4.1.5. *([10]) We call a matrix of the form* $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ *a diagonal matrix.*

Definition 4.1.6. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If there is* $\lambda \in \mathbb{C}$ *and a nonzero vector* $x \in \mathbb{C}^2$ *with* $x \neq 0$ *satisfying the equation*

$$Ax = \lambda x, \tag{4.1.7}$$

then λ *is called an eigenvalue of* A *and* x *is called an eigenvector of* A *associated with* λ.

Now the question is: how many eigenvalues does a 2×2 square complex matrix A have? According to Equation 4.1.7 λ is an eigenvalue if and only if $\det(A - \lambda I) = 0$. Thus A has at most two complex eigenvalues.

Definition 4.1.8. *We define a characteristic polynomial of matrix A as*

$$P_A(x) = \det(A - xI) \tag{4.1.9}$$

and we called the equation $P_A(x) = 0$ a characteristic equation of A.

Definition 4.1.10. *Let A is a 2×2 square matrix. The multiplicity of an eigenvalue λ of A is its multiplicity as a zero of the characteristic polynomial $P_A(x)$.*

Note that the eigenvalues of A are the same as the zeroes of the characteristic polynomial of A, counting multiplicities. For more concepts from linear algebra to illustrate the similarity relation in matrices; see [10, p. 164],

Definition 4.1.11 (Jordan block)**.** *An $m \times m$ upper triangular matrix $B(\lambda, m)$ is called a Jordan block provided all m diagonal entries are the same eigenvalue λ and all super-diagonal entries are 1; that is,*

$$B(\lambda, m) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{bmatrix}. \textit{ Thus}, B(\lambda, 1) = [\lambda], B(\lambda, 2) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Definition 4.1.12 (Jordan Form)**.** *Given an $n \times n$ matrix A, a Jordan form J for A is a block diagonal matrix $J = diag(B(\lambda_1, m_1), B(\Lambda_2, m_2), \dots, B(\lambda_k, m_k))$, where λ_1, ..., λ_k are eigenvalues of A and $m_1 + \cdots + m_k = n$. In other words, it is a direct sum of Jordan blocks $J = B(\lambda_1, m_1) \oplus B(\lambda_2, m_2) \oplus \cdots \oplus B(\lambda_k, m_k)$, where $m_1 + m_2 + \ + m_k = n$.*

The relation $A = PJP^{-1}$, where P is an invertible matrix, is called a Jordan decomposition of A. The invertible matrix P is called the matrix of generalized eigenvectors of A. In this case we say that the matrix A and matrix J are similar. Every complex matrix is similar to one Jordan matrix as we state that without proof in the following theorem.

Theorem 4.1.13. *([10, p. 167]) Every $n \times n$ complex matrix A has a Jordan decomposition $A = PJP^{-1}$. If A is real and has only real eigenvalues, then P can be chosen to be real.*

We conclude this section with Theorem 4.1.14 which gives a matrix of a linear transformation which rotates all vectors through an angle of θ.

Theorem 4.1.14. *([13]) Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by rotating vectors through an angle of θ. Then the matrix $R(\theta)$ of R_θ is given by*

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

4.2 Equivalent Conditions for Hypercyclicity on $H(\mathbb{P})$

Before we prove our main result of this chapter, Theorem 4.2.5 which characterizes all conformal maps σ on $\mathbb{P}$ that produce a hypercyclic composition operator $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$. We give some basic properties of the coefficient matrix A of a linear fractional transformation σ and its eigenvalues in Lemma 4.2.1, and Lemma 4.2.2. Moreover, in Proposition 4.2.3 we utilize our result in Proposition 2.1.1 to provide a necessary and sufficient condition in terms of the eigenvalues of A so that the composition operator C_σ is hypercyclic on $H(\mathbb{P})$. Consequently with the notation in Proposition 4.2.3 we state Corollary 4.2.4, which restates the necessary and sufficient condition in a different way.

Lemma 4.2.1. *Let $a, b, c, d \in \mathbb{R}$ and*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $ad - bc = 1$. Then we have the following

(1) If A has one real eigenvalue λ with multiplicity 2, then $\lambda = \pm 1$.

(2) If A has two distinct real eigenvalues $\lambda_1 < \lambda_2$, then $\lambda_1 < 1 < \lambda_2$.

(3) If A has two complex eigenvalues λ and $\overline{\lambda} \notin \mathbb{R}$. then $|\lambda| = 1$.

Proof. The characteristic equation of matrix A is

$$\begin{aligned} 0 = \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} &= \lambda^2 - (a+d)\lambda + (ad-bc) \\ &= \lambda^2 - (a+d)\lambda + 1. \end{aligned}$$

Suppose the two roots of the characteristic equation are λ_1 and λ_2. Then the characteristic equation becomes:

$$\begin{aligned} \lambda^2 - (a+d)\lambda + 1 &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2. \end{aligned}$$

Thus $\lambda_1\lambda_2 = 1$, from which statements (1), (2) and (3) follow immediately. □

Lemma 4.2.2. *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $a, b, c, d \in \mathbb{R}$, and $\det A = ad - bc = 1$. Suppose A has a real eigenvalue λ with multiplicity 2. Then the following statements are equivalent.

(1) A is diagonalizable.

(2) $A = \pm I$.

(3) A has two linearly independent eigenvectors.

(4) $b = c = 0$.

Proof. Clearly statements (1) and (3) are equivalent. Suppose statement (1) holds true, and if λ_1, λ_2 are two eigenvalues, then $\lambda_1\lambda_2 = \det A = 1$. Since $\lambda_1 = \lambda_2 \in \mathbb{R}$ by our hypothesis, thus either $\lambda_1 = \lambda_2 = 1$ or $\lambda_1 = \lambda_2 = -1$.

Now we show statement (1) implies statement (2). Thus, statement (1) implies that there is an invertible matrix P such that, $P^{-1}AP = \pm I$.

Hence, $A = P(\pm I)P^{-1} = \pm PP^{-1} = \pm I$. Therefore statement (2) holds. Clearly statement (2) implies statement (4). It remains to show statement (4) implies statement (1).

If $b = c = 0$ then

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},$$

and so A is diogonalizable. □

Proposition 4.2.3. *Let $\sigma(z) = \frac{az+b}{cz+d}$ be a linear fractional transformation with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be the coefficient matrix of σ. Then $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ is hypercyclic if and only if A is not $\pm I$ and A has real eigenvalues.

Proof. The characteristic equation of matrix A is

$$\begin{aligned} 0 &= \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - (a+d)\lambda + 1. \end{aligned}$$

Let the two eigenvalues of A be λ_1 and λ_2. Then the characteristic equation becomes

$$(\lambda-\lambda_1)(\lambda-\lambda_2)=\lambda^2-(\lambda_1+\lambda_2)\lambda+\lambda_1\lambda_2=\lambda^2-(a+b)\lambda+1=0,$$

and so $\lambda_1\lambda_2=\det A=ad-bc=1$. We now proceed our discussion with different possibilities of λ_1 and λ_2 and use the sup-norm notation for a 2×2 matrix:

For a matrix $B=\begin{bmatrix} w & x \\ y & z \end{bmatrix}$, we let $\|B\|_\infty=\sup(|w|,|x|,|y|,|z|)$.

Case(A): λ_1, $\lambda_1\notin\mathbb{R}$. Since $\det A=1$, we have an invertible matrix P so that $P^{-1}AP$ is the rotation $R(\theta)$ of some angle θ:

$$P^{-1}AP=R(\theta)=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Raising both sides of the equation to the nth power, we get

$$P^{-1}A^nP=R(n\theta)=\begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}.$$

Since $A^n=PR(n\theta)P^{-1}$, we have $\|A^n\|_\infty<\infty$. So that $C_\sigma : H(\mathbb{P})\to H(\mathbb{P})$ is not hypercyclic, by Theorem 3.2.6.

Case(B): $\lambda_1,\lambda_2\in\mathbb{R}$. Since $\lambda_1\lambda_2=1$, we have the following subcases for real eigenvalues λ_1,λ_2.

Case(i): $\lambda_1\neq\lambda_2$, without loss of generality, assume $\lambda_2<1<\lambda_2$. Hence A is diagonalizable and so there exists an invertible matrix P such that

$$P^{-1}AP=\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Thus

$$P^{-1}A^nP=\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix},$$

and so

$$\|P^{-1}A^nP\|_\infty \to \infty.$$

Hence $\|A^n\|_\infty \to \infty$, and thus $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ is hypercyclic, by Theorem 3.2.6.

Case (ii): $\lambda_1 = \lambda_2 = \pm 1$. With out loss of generality we need only discuss the case $\lambda_1 = \lambda_2 = 1$. The other case of $\lambda_1 = \lambda_2 = -1$ follows from the exact same argument. Suppose A has two linearly independent vectors then A is diagonalizable and so there exists an invertible matrix P such that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

and hence $A = PIP^{-1} = I$. Thus $\sigma(z) = z$ and $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ is not hypercyclic.

Suppose A does not have two linearly independent vectors for the eigenvalue $\lambda = 1$ with multiplicity 2. Thus A is similar to a Jordon block; see [24, Lecture. 28, p. 4]

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

In other words, there exists an invertible P such that

$$P^{-1}AP = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Hence,

$$P^{-1}A^nP = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix},$$

and so

$$\|P^{-1}A^nP\|_\infty \to \infty.$$

Thus

$$A^n = P \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} P^{-1} \text{ and so } \|A^n\|_\infty \to \infty.$$

Consequently, $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ is hypercyclic, by Theorem 3.2.6. □

The above proposition can be rephrased as the following corollary.

Corollary 4.2.4. *With the notation in Proposition 4.2.3, $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ is hypercyclic if and only if exactly one the following statements holds true:*

1) A has two distinct real eigenvalues.

2) A has an eigenvalue $\lambda = 1$ with multiplicity 2 or an eigenvalue $\lambda = -1$ with multiplicity 2, and the corresponding eigenspace ker$(A - \lambda I)$ has dimension 1.

We now continue our work in Proposition 4.2.3 to state a characterization of $\sigma(z) = \frac{az+b}{cz+d}$ on $\mathbb{P}$ in terms of the coefficient a, b, c, d, for the composition operator to be hypercyclic.

Theorem 4.2.5. *Let $\sigma(z) = \frac{az+b}{cz+d}$ be a linear fractional transformation with a, b, c, $d \in \mathbb{R}$ and $ad - bc = 1$. Then $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ is hypercyclic if and only if one of the following two conditions hold;*

1) $|a + d| > 2$.

2) $|a + d| = 2$ and at least one of b and c is nonzero.

Proof. Let A be the coefficient matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The characteristic equation of A is given by

$$\begin{aligned} 0 = \det(A - \lambda I) &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - (a+d)\lambda + 1. \end{aligned} \tag{4.2.6}$$

Thus A has two distinct real eigenvalues if and only if the determinant $(a+d)^2 - 4 > 0$; that is $|a+d| > 2$, in which case $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ is hypercyclic, by Proposition 4.2.3 . Furthermore A has an eigenvalue λ with multiplicity 2 if and only if $(a+b)^2 - 4 = 0$; that is, $|a+d| = 2$. In this case, by (4.2.6) we also have

$$\begin{aligned} \lambda^2 &= \text{product of two eigenvalues} \qquad (4.2.7) \\ &= \det A = 1. \end{aligned}$$

Thus $\lambda = 1, -1$. Thus for either case of λ, the coefficient matrix A has two linearly independent eigenvectors if and only if there exists an invertible matrix P such that

$$P^{-1}AP = \pm I;$$

that is

$$A = \pm PIP^{-1} = \pm I.$$

Thus A has an eigenvalue $\lambda = 1$ with multiplicity 2 or $\lambda = -1$ with multiplicity 2 and $A \neq \pm I$ if and only if $|a+b| = 2$ and at least one of b and c is nonzero, by the Lemma 4.2.2. $\square$

Theorem 3.2.6 provides a necessary and sufficient condition for a sequence $C_{\sigma_n} : H(\mathbb{P}) \to H(\mathbb{P})$ to be universal. Theorem 4.2.5 provides a necessary and sufficient condition for an operator $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ to be hypercyclic. These two results together complete the picture for the dynamics for composition operators on $H(\mathbb{P})$.

4.3 Examples

To conclude this chapter, we illustrate our results in Section 4.2 with the following examples.

Example 4.3.1. *Take*

$$A = \begin{bmatrix} 2 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} (\textit{clearly} \;\; ad - bc = 1).$$

$$\begin{aligned} \frac{az+b}{cz+d} = \frac{2z}{z+\frac{1}{2}} = z \quad & \textit{if and only if} \quad 2z = z^2 + \frac{1}{2}z, \\ & \textit{if and only if} \quad z(2z - \frac{3}{2}) = 0, \\ & \textit{if and only if} \quad z = 0 \;\; \textit{or} \;\; z = \frac{3}{2}. \end{aligned}$$

Thus

$$\begin{bmatrix} 2 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 2 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Note that from Theorem 4.2.5 or Corollary 4.2.4 we conclude that C_σ is hypercyclic, where σ is the conformal map represented by matrix A.

Example 4.3.2. *Take*

$$A = \begin{bmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} (\textit{clearly} \;\; ad - bc = 1).$$

$$\frac{az+b}{cz+d} = \frac{2z+1}{\frac{1}{2}} = 4z + 2 = z \;\; \textit{if and only if} \;\; z = \frac{-2}{3}$$

Thus,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{-2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{-2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{-2}{3} \\ 1 \end{bmatrix},$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

From Theorem 4.2.5 or Corollary 4.2.4 we conclude that C_σ is hypercyclic, where σ is the conformal map represented by matrix A.

Example 4.3.3. *Take*

$$A = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} (\textit{clearly}\ \ ad - bc = -3 - (-4) = 1).$$

$$\begin{aligned} \frac{az+b}{cz+d} = \frac{3z+2}{-2z-1} = z \quad & \textit{if and only if} \quad 3z + 2 = -2z^2 - z, \\ & \textit{if and only if} \quad (z+1)^2 = 0, \\ & \textit{if and only if} \quad z = -1. \end{aligned}$$

$$0 = \det \begin{bmatrix} 3-\lambda & 2 \\ -2 & -1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \ \ \textit{if and only if} \ \ \lambda = 1.$$

$$0 = (A - \lambda I) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3-1 & 2 \\ -2 & -1-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \ \ \textit{if and only if} \ \ x + y = 0.$$

Therefore the eigenspace is the span of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, *so that $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ is hypercyclic, by Corollary 4.2.4.*

Example 4.3.4. *Take*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (\textit{clearly} \;\; ad - bc = 1),$$

and so

$$\frac{az+b}{cz+d} = z + 1.$$

Thus there is no z *such that*

$$\frac{az+b}{cz+d} = z.$$

Clearly $(A - \lambda I) = 0$ *if and only if* $(1-\lambda)^2 = 0$ *and so* $\lambda = 1$ *is an eigenvalue with multiplicity* 2*.*

$$0 = (A - \lambda I) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

if and only if $y = 0$*.*

Thus eigenspace is the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$*, so that* $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ *is hypercyclic, by Corollary 4.2.4.*

Example 4.3.5. *Take*

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} (\textit{clearly} \;\; ad - bc = 1).$$

$$\frac{az+b}{cz+d} = \frac{-1}{z+1} = z \;\; \textit{if and only if} \;\; z^2 + z + 1 = 0 \;\; \textit{if and only if} \;\; z = \frac{-1 \pm i\sqrt{3}}{2}.$$

$$0 = \det \begin{bmatrix} -\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda + 1 \;\; \textit{if and only if} \;\; \lambda = \frac{1 \pm i\sqrt{3}}{2}.$$

The matrix A *has no real eigenvalue. Therefor according to Theorem 4.2.5, the operator* C_σ : $H(\mathbb{P}) \to H(\mathbb{P})$ *is not hypercyclic.*

Example 4.3.6. *Take*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\textit{clearly} \;\; ad - bc = 1),$$

and

$$\sigma(z) = \frac{az+b}{cz+d} = \frac{z+0}{0z+1} = z.$$

Thus $\sigma(z) = z$ *for all* $z \in \mathbb{C}$. *The matrix* A *has eigenvalue* $\lambda = 1$ *with multiplicity* 2. *Now* A *has two linearly independent eigenvectors*

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} and \;\; \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus, $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ *is not hypercyclic, by Corollary 4.2.4.*

CHAPTER 5 FIXED POINTS OF A CONFORMAL MAP ON UPPER HALF PLANE

5.1 Introduction

In this chapter, we continue to study the linear dynamics of linear fractional transformations $\sigma : \mathbb{P} \to \mathbb{P}$ on the upper half-plane $\mathbb{P}$, with a focus on the fixed points of σ. To begin, we state Proposition 5.1.1 without proof.

Proposition 5.1.1. *([26, p. 57]) Let $\sigma(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ be a linear fractional transformation on $\mathbb{P}$ and suppose that σ is not the identity. Then σ has either:*

(i) two distinct fixed points in $\mathbb{R}$ and none in $\mathbb{P}$;

(ii) one fixed point in $\mathbb{R} \cup \{\infty\}$ and none in $\mathbb{P}$;

(iii) no fixed points in $\mathbb{R}$ and one in $\mathbb{P}$.

Definition 5.1.2. *([26]) Let σ be a linear fractional transformation of $\mathbb{P}$. We say that*

(i) σ is hyperbolic if it has two distinct fixed points in $\mathbb{R}$ and none in $\mathbb{P}$,

(ii) σ is parabolic if it has one fixed point in $\mathbb{R} \cup \{\infty\}$ and none in $\mathbb{P}$,

(iii) σ is elliptic if it has one fixed point in $\mathbb{P}$ and none in $\mathbb{R}$.

Now, we prove an auxiliary result which is used in the proof of Lemma 5.1.6 and Lemma 5.1.10. The next few results give us equivalent conditions that make a connection between eigenvalues, fixed points, and the coefficients of the linear fractional map σ.

Lemma 5.1.3. *Let $\sigma(z) = \frac{az+b}{cz+d}$ be a linear fractional transformation with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Let $z_\circ \in \mathbb{C}$ and*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be the coefficient matrix σ*, then* $\sigma(z_\circ) = z_\circ$ *if and only if* $\begin{bmatrix} z_\circ \\ 1 \end{bmatrix}$ *is an eigenvector of* A *corresponding to a nonzero eigenvalue.*

Remark 5.1.4. *Since the product of two eigenvalues* = $\det A = ad - bc \neq 0$*,* A *has no zero eigenvalue.*

Proof of Lemma. Suppose

$$A \begin{bmatrix} z_\circ \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} z_\circ \\ 1 \end{bmatrix}$$

for some $\lambda \neq 0$. That is,

$$\begin{cases} az_\circ + b = \lambda z_\circ \\ cz_\circ + d = \lambda. \end{cases}$$

Hence

$$\frac{az_\circ + b}{cz_\circ + d} = \frac{\lambda z_\circ}{\lambda} = z_\circ,$$

or equivalently,

$$\sigma(z_\circ) = z_\circ.$$

Conversely Suppose $\sigma(z_\circ) = z_\circ$

Case(i): If $\sigma(z_\circ) = z_\circ$ with $z_\circ = 0$. Hence $\sigma(0) = 0$, which implies $\frac{b}{d} = 0$. That is $b = 0$ and $d \neq 0$ (because $ad - bc = 1$). Hence

$$A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}.$$

Thus

$$A \begin{bmatrix} z_\circ \\ 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ d \end{bmatrix} = d \begin{bmatrix} z_\circ \\ 1 \end{bmatrix},$$

Hence

$$\begin{bmatrix} z_\circ \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding the eigenvalue $d \neq 0$.

Case(ii): $\sigma(z_\circ) = z_\circ$ with $z_\circ \neq 0$. That is,

$$\frac{az_\circ + b}{cz_\circ + d} = z_\circ \neq 0.$$

Thus, we let

$$\lambda = \frac{az_\circ + b}{z_\circ} \text{ and } \mu = cz_\circ + d \neq 0.$$

Hence we have

$$\begin{cases} az_\circ + b = \lambda z_\circ \\ cz_\circ + d = \mu \neq 0. \end{cases} \tag{5.1.5}$$

Thus,

$$z_\circ = \frac{az_\circ + b}{cz_\circ + d} = \frac{\lambda z_\circ}{\mu}.$$

Therefore

$$\lambda = \mu \neq 0.$$

So $(5.1.5)$ becomes

$$\begin{cases} az_\circ + b = \lambda z_\circ \\ cz_\circ + d = \lambda. \end{cases}$$

That is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z_\circ \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} z_\circ \\ 1 \end{bmatrix},$$

with $\lambda \neq 0$. which completes the proof. $\square$

The following two lemmas give us equivalent conditions that make a connection between eigenvalues, fixed points, and the coefficients of the linear fractional map σ.

Lemma 5.1.6. *Let* $\sigma(z) = \frac{az+b}{cz+d}$ *be a linear fractional transformation with* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$*, and let* A *be coefficient matrix given by*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The following statements are equivalent:

(1) A *has distinct real eigenvalues.*

(2) σ *has two distinct fixed points in* $\mathbb{R}$*.*

(3) $|a + d| > 2$*.*

(4) σ *is hyperbolic with its fixed point in* $\mathbb{R}$*.*

Proof. We first show statements (1) and (3) are equivalent. Expanding the characteristic equation $\det(A - \lambda I) = 0$, we get

$$\lambda^2 - (a + b)\lambda + (ad - bc) = \lambda^2 - (a + d)\lambda + 1 = 0.$$

Thus A has two distinct real eigenvalues if and only if the determinant $(a + d)^2 - 4 > 0$; that is $|a + d| > 2$.

Secondly we show statements (2) and (3) are equivalent. We observe that $\sigma(z) = z$ if and only if $\frac{az+b}{cz+d} = z$ or equivalently $cz^2 + (d - a)z - b = 0$. which is a quadratic equation having the following solutions:

$$\begin{aligned} z_{1,2} &= \frac{(a-d) \pm \sqrt{(d-a)^2 + 4cb}}{2c} \\ &= \frac{(a-d) \pm \sqrt{d^2 - 2ad + a^2 + 4cb}}{2c} \end{aligned}$$

$$= \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c},$$

so that the equation has two distinct real solutions if and only if $(a+d)^2 > 4$. Finally, it is trivial that statement (4) equivalent to statements (1), (2) and (3) by Definition 5.1.2. □

Remark 5.1.7. *Using the quadratic formula, we remark that the two distinct eigenvalues of A in Lemma 5.1.6 are*

$$\frac{(a+d) \pm \sqrt{(a+d)^2 - 4}}{2}. \tag{5.1.8}$$

Using the quadratic formula, we see that the two distinct fixed points of σ in Lemma 5.1.6 are

$$\frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}. \tag{5.1.9}$$

Lemma 5.1.10. *Let σ and A be given as in Lemma 5.1.6. The following statements are equivalent:*

(1) A has one real eigenvalue with multiplicity 2.

(2) σ has at most one fixed point in $\mathbb{R}$.

(3) $|a+d| = 2$.

(4) σ is parabolic or without any fixed point.

Proof. Our proof follows from the same argument as in proof of the Lemma 5.1.6 with slight modification. Clearly A has one real eigenvalue with multiplicity 2 if and only if $|a+d| = 2$. Also A has precisely one real fixed point if and only if $|a+d| = 2$ and $c \neq 0$. In addition, from the equation (5.1.9) we see that σ has a fixed point at ∞ if and only if $a = d = \pm 1$ and $c = 0$. □

5.2 Iterations of a Linear Fractional on $\mathbb{P}$

The n-th iterate $\sigma^n = \sigma \circ \sigma \circ \ldots \sigma$ of a map σ from some set X to itself is the composition of σ with itself n number of times. If σ and ϕ are two maps satisfying, $\sigma = \psi^{-1} \circ \phi \circ \psi$ then $\sigma^n = \psi^{-1} \circ \phi^n \circ \psi$ where σ, ψ, and ϕ defined as we mentioned in Section 1.1.1. Hence, the dynamics of the map ϕ follows dynamics of σ and vice versa. Among this line, let us recall the following statement in [23].

Theorem 5.2.1. *(Denjoy-Wolff [23, p. 78]) If $\phi : \mathbb{D} \to \mathbb{D}$ is an holomorphic map with no fixed point in $\mathbb{D}$. Then there exists a point $z_\circ \in \partial\mathbb{D}$ such that $\phi^n \to z_\circ$ uniformly on compact subsets of $\mathbb{D}$.*

In the following propositions, we study a limiting behavior of iteration of representatives a linear fractional transformation σ defined on the upper half-plane $\mathbb{P}$ in the different conjugacy classes (see 4.1). Further, we give a specific characterization of its limit points for each class.

Proposition 5.2.2. *For σ and A in Lemma 5.1.6, let α, $\beta \in \mathbb{R}$ be two distinct eigenvalues of A with $0 < \beta < \alpha$. Let $z_\alpha, z_\beta \in \mathbb{R}$ such that*

$$\begin{bmatrix} z_\alpha \\ 1 \end{bmatrix}, \begin{bmatrix} z_\beta \\ 1 \end{bmatrix}$$

are eigenvectors of A corresponding to the eigenvalues α and β respectively according to Lemma 5.1.3 and Lemma 4.2.2. Then for any $z \in \mathbb{C}$ with $z \neq z_\beta$, we have $\underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{n\text{-times}}(z) = \sigma^n(z) \to z_\alpha$ uniformly on compact subsets of $\mathbb{C}$.

Proof. For any $z \neq z_\beta$, let c_α and $c_\beta \in \mathbb{C}$ such that

$$\begin{bmatrix} z \\ 1 \end{bmatrix} = c_\alpha \begin{bmatrix} z_\alpha \\ 1 \end{bmatrix} + c_\beta \begin{bmatrix} z_\beta \\ 1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} z_\alpha \\ 1 \end{bmatrix}, \begin{bmatrix} z_\beta \\ 1 \end{bmatrix}$$

are linearly independent, $c_\alpha \neq 0$. Observe that

$$A^n \begin{bmatrix} z \\ 1 \end{bmatrix} = c_\alpha A^n \begin{bmatrix} z_\alpha \\ 1 \end{bmatrix} + c_\beta A^n \begin{bmatrix} z_\beta \\ 1 \end{bmatrix}$$

$$= \alpha^n c_\alpha \begin{bmatrix} z_\alpha \\ 1 \end{bmatrix} + \beta^n c_\beta \begin{bmatrix} z_\beta \\ 1 \end{bmatrix}.$$

Thus

$$\frac{1}{\alpha^n} A^n \begin{bmatrix} z \\ 1 \end{bmatrix} = c_\alpha \begin{bmatrix} z_\alpha \\ 1 \end{bmatrix} + \left(\frac{\beta}{\alpha}\right)^n c_\beta \begin{bmatrix} z_\beta \\ 1 \end{bmatrix},$$

from which it follows that

$$\lim_{n\to\infty} \frac{1}{\alpha^n} A^n \begin{bmatrix} z \\ 1 \end{bmatrix} = c_\alpha \begin{bmatrix} z_\alpha \\ 1 \end{bmatrix}, \text{ because } 0 < \beta < \alpha.$$

Thus if we let

$$A^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix},$$

then by Lemma 4.1.3,

$$\sigma^n(z) = \frac{a_n z + b_n}{c_n z + d_n} = \frac{\frac{a_n z + b_n}{\alpha^n}}{\frac{c_n z + b_n}{\alpha^n}} \to \frac{c_\alpha z_\alpha}{c_\alpha} = z_\alpha.$$

□

Note that in the above proposition we excluded the possibility that one of the eigenvectors is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in which case we cannot normalize this as $\begin{bmatrix} z_\alpha \\ 1 \end{bmatrix}$ and $\begin{bmatrix} z_\beta \\ 1 \end{bmatrix}$. Now using our results in Corollary 4.2.4, Lemma 5.1.3 and Lemma 4.2.2, we have the following result.

Proposition 5.2.3. *Let $\sigma(z) = \frac{az+b}{cz+d}$ be a linear fractional transformation with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Suppose A is the coefficient matrix of σ and A is not $\pm I$, and A has a real eigenvalue with multiplicity 2. If σ has a fixed point $z_\circ \in \mathbb{R}$, then $\sigma^n(z) \to z_\circ$ for all $z \in \mathbb{C}$.*

Proof. By Corollary 4.2.4, there exists an invertible 2×2 matrix P such that

$$P^{-1}AP = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

where λ is the eigenvalue with $\lambda = -1$ or $\lambda = +1$. By Lemma 5.1.3, $\begin{bmatrix} z_\circ \\ 1 \end{bmatrix}$ is an eigenvector corresponding to λ. Since

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

we now claim that we may assume P has the property that

$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} z_\circ \\ 1 \end{bmatrix}.$$

To see that let

$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_\circ \\ y_\circ \end{bmatrix}.$$

Hence we can write

$$P^{-1}AP \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$\begin{aligned} P^{-1}A \begin{bmatrix} x_\circ \\ y_\circ \end{bmatrix} &= \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \\ A \begin{bmatrix} x_\circ \\ y_\circ \end{bmatrix} &= \lambda P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_\circ \\ y_\circ \end{bmatrix}. \end{aligned}$$

Thus $\begin{bmatrix} x_\circ \\ y_\circ \end{bmatrix}$ is an eigenvector of A.

Since by Lemma 4.2.2, dim $\ker(A - \lambda I) = 1$, we have

$$\begin{bmatrix} x_\circ \\ y_\circ \end{bmatrix} = t \begin{bmatrix} z_\circ \\ 1 \end{bmatrix} \quad \text{for some nonzero} \quad t \in \mathbb{R}.$$

By dividing P by t if necessarily we can assume

$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_\circ \\ y_\circ \end{bmatrix} = \begin{bmatrix} z_\circ \\ 1 \end{bmatrix}.$$

Let $z \in \mathbb{C}$ with $z \neq z_\circ$, then

$$\begin{bmatrix} z_\circ \\ 1 \end{bmatrix} \notin \text{span} \begin{bmatrix} z_\circ \\ 1 \end{bmatrix}.$$

Thus by the invertibility of P,

$$P^{-1} \begin{bmatrix} z \\ 1 \end{bmatrix} \notin \text{span } P^{-1} \begin{bmatrix} z_\circ \\ 1 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence

$$P^{-1} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

for some $x, y \in \mathbb{R}$ with $y \neq 0$.

Note that for $n \geq 1$,

$$P^{-1} A^n P = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix},$$

and so

$$\begin{aligned}
P^{-1}A^nP\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix};\\
P^{-1}A^n\begin{bmatrix} z \\ 1 \end{bmatrix} &= \begin{bmatrix} \lambda^n x + n\lambda^{n-1}y \\ \lambda^n y \end{bmatrix} = \lambda^n\begin{bmatrix} x \\ y \end{bmatrix} + n\lambda^{n-1}y\begin{bmatrix} 1 \\ 0 \end{bmatrix};\\
A^n\begin{bmatrix} z \\ 1 \end{bmatrix} &= \lambda^n P\begin{bmatrix} x \\ y \end{bmatrix} + n\lambda^{n-1}yP\begin{bmatrix} 1 \\ 0 \end{bmatrix}\\
&= \lambda^n\begin{bmatrix} z \\ 1 \end{bmatrix} + n\lambda^{n-1}y\begin{bmatrix} z_\circ \\ 1 \end{bmatrix}.
\end{aligned}$$

Hence by Lemma 4.1.3, we have, as $n \to \infty$,

$$\begin{aligned}
\sigma^n(z) &= \frac{\lambda^n z + n\lambda^{n-1}yz_\circ}{\lambda^n + n\lambda^{n-1}y}\\
&= \frac{\lambda z + nyz_\circ}{\lambda + ny} \to z_\circ \text{ (because } y \neq 0).
\end{aligned}$$

□

Note that Example 4.3.3 illustrates Proposition 5.2.3. As a continuation of our discussion with Proposition 5.2.3, we now proceed to discuss the case when A has no real fixed point.

Proposition 5.2.4. *Let $\sigma(z) = \frac{az+b}{cz+d}$ be a linear fractional transformation with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Suppose A is the coefficient matrix of $\sigma$and A is not $\pm I$, and A has a real eigenvalue with multiplicity 2. If A has no real fixed point, then for all $z \in \mathbb{C}$ we have $\sigma^n(z) \to \infty$.*

Proof. By the proof of Lemma 5.1.10. $a = d = \pm 1$ and $c = 0$. Thus $b \neq 0$ and so

$$\sigma(z) = \frac{az + b}{d} = z + b,$$

and hence

$$\sigma^n(z) = z + nb \to \infty.$$

□

In order to prove our new result in Theorem 5.2.6, we first need to prove the following lemma which gives us a complete description of eigenvalues and eigenvectors of a linear fractional map σ on the $\mathbb{P}$ in terms of its coefficients.

Lemma 5.2.5. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$*, where* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$ *and* $c \neq 0$*. Suppose* A *has* 2 *distinct eigenvalues*

$$\lambda_1 = \frac{(a+d) + \sqrt{(a+d)^2 - 4}}{2} \quad \textit{and} \quad \lambda_2 = \frac{(a+d) - \sqrt{(a+d)^2 - 4}}{2}.$$

If

$$x_1 = \frac{(a-d) + \sqrt{(a+d)^2 - 4}}{2c} \quad \textit{and} \quad x_2 = \frac{(a-d) - \sqrt{(a+d)^2 - 4}}{2c}$$

are two distinct fixed points of σ *(see Remark 5.1.7). Then*

$$A \begin{bmatrix} x_1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ 1 \end{bmatrix} \textit{ and } A \begin{bmatrix} x_2 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} x_2 \\ 1 \end{bmatrix}.$$

Proof. Suppose $c \neq 0$. Then,

$$\begin{aligned}
(A - \lambda_1 I) \begin{bmatrix} x_1 \\ 1 \end{bmatrix} &= \begin{bmatrix} a - \frac{(a+d)+\sqrt{(a+d)^2-4}}{2} & b \\ c & d - \frac{(a+d)+\sqrt{(a+d)^2-4}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{(a-d)-\sqrt{(a+d)^2-4}}{2}\right) x_1 + b \\ cx_1 + d - \frac{(a+d)+\sqrt{(a+d)^2-4}}{2} \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} \frac{\left((a-d)-\sqrt{(a+d)^2-4}\right)\left((a-d)+\sqrt{(a+d)^2-4}\right)}{4c} + b \\ \frac{(a-d)+\sqrt{(a+d)^2-4}}{2} + d - \frac{(a+d)+\sqrt{(a+d)^2-4}}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{(a-d)^2-(a+d)^2+4}{4c} + b \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-4ad+4}{4c} + b \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-4bc}{4c} + b \\ 0 \end{bmatrix} \quad \text{(use } ad-bc=1\text{)} \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(A-\lambda_2 I)\begin{bmatrix} x_2 \\ 1 \end{bmatrix} &= \begin{bmatrix} a - \frac{(a+d)-\sqrt{(a+d)^2-4}}{2} & b \\ c & d - \frac{(a+d)-\sqrt{(a+d)^2+4}}{2} \end{bmatrix} \begin{bmatrix} x_2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{(a-d)+\sqrt{(a+d)^2-4}}{2}\right)x_2 + b \\ cx_2 + d - \frac{(a+d)-\sqrt{(a+d)^2+4}}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{((a-d)+\sqrt{(a+d)^2-4})((a-d)-\sqrt{(a+d)^2-4})}{4c} + b \\ \frac{(a-d)-\sqrt{(a+d)^2-4}}{2} + d - \frac{(a+d)-\sqrt{(a+d)^2+4}}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{(a-d)^2-(a+d)^2+4}{4c} + b \\ 0 \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} \frac{-4ad+4}{4c} + b \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-4bc}{4c} + b \\ 0 \end{bmatrix} \quad \text{(use } ad - bc = 1\text{)}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This completes the proof. □

Now, we are ready to prove Theorem 5.2.6 which provides a complete characterization of the limit point $z_\circ$ for the iteration σ^n on $\mathbb{P}$.

Theorem 5.2.6. *Let $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ and $\sigma(z) = \frac{az+b}{cz+d}$. If the coefficient matrix A has two distinct real eigenvalues, then $\sigma^n(z) \to z_\circ$ uniformly on compact subsets, where*

$$z_\circ = \begin{cases} \frac{(a-d)+\sqrt{(a+d)^2-4}}{2c} & \text{if } \ c \neq 0. \\ \frac{b}{d-a} & \text{if } \ c = 0 \ \text{ and } \ 0 < \frac{a}{d} < 1 \\ \infty \ \text{ if } \ c = 0 \ \text{ and } \ 0 < \frac{d}{a} < 1. \end{cases}$$

Proof. The eigenvalues λ_1 and λ_2 in Lemma 5.2.5 satisfy

$$\lambda_2 = \frac{(a+d) - \sqrt{(a+d)^2 - 4}}{2} < \frac{(a+d) + \sqrt{(a+d)^2 - 4}}{2} = \lambda_1.$$

Proposition 5.2.2 implies that if $c \neq 0$, then by Lemma 5.2.5

$$\sigma^n(z) \to x_1 = \frac{(a-d) + \sqrt{(a+d)^2 - 4}}{2c}.$$

If $c = 0$, then

$$\sigma(z) = \frac{az+b}{d} \text{ and } d \neq 0$$

(because $ad - bc = ad = 1$) and so

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

has 2 distinct eigenvalues a and d. Since $ad = 1$, either a and d are both positive or both negative. we must have either case (A): where $0 < \frac{a}{d} < 1$, or case (B): where $0 < \frac{d}{a} < 1$. In either case, let

$$b_n = b(a^{n-1} + a^{n-2}d + a^{n-3}d^2 + \cdots + ad^{n-2} + d^{n-1}) = b\frac{d^n - a^n}{d-a},$$

and so

$$A^n = \begin{bmatrix} a^n & b_n \\ 0 & d^n \end{bmatrix}$$

and

$$\begin{aligned} \sigma^n(z) &= \frac{a^n z + b\frac{d^n - a^n}{d-a}}{d^n} = \left(\frac{a}{d}\right)^n z + \frac{b}{d-a}\left(1 - \left(\frac{a}{d}\right)^n\right) \\ &= \left(\frac{a}{d}\right)^n \left(z - \frac{b}{d-a}\right) + \frac{b}{d-a}. \end{aligned}$$

Case (A): $c = 0$ and $0 < \frac{a}{d} < 1$. Then

$$\sigma^n(z) \to \frac{b}{d-a} \text{ for all z} \in \mathbb{P}.$$

Note

$$\sigma(\frac{b}{d-a}) = \frac{b}{d-a} \text{ but } \frac{b}{d-a} \notin \mathbb{P}.$$

Case (B): $c = 0$ and $0 < \frac{d}{a} < 1$. Then

$$\sigma^n(z) \to \infty \ \text{ for all } \ z \in \mathbb{P}.$$

□

Theorem 5.2.6 studies three cases in which C_σ is hypercyclic. We now proceed to study the remaining two cases in the following theorem.

Theorem 5.2.7. *Let $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ and $\sigma(z) = \frac{az+b}{cz+d}$. If the coefficient matrix A has a real eigenvalue λ with multiplicity 2, then $\sigma^n(z) \to z_\circ$ uniformly on compact subsets, where*

$$z_\circ = \begin{cases} \frac{a-d}{2c} & \text{if } \ c \neq 0 \\ \infty & \text{if } \ c = 0. \end{cases}$$

Proof. Suppose A has a real eigenvalue λ with multiplicity 2. Then by Lemma 5.1.10

$$(a+d)^2 = 4 \ \text{ and } \ \lambda^2 = \det A = ad - bc = 1,$$

and so,

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4}}{2} = \frac{a+d}{2} = \pm 1. \tag{5.2.8}$$

Case(i): Assume $c \neq 0$. Then σ has a real fixed point $z_\circ$. Indeed

$$A \begin{bmatrix} z_\circ \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} z_\circ \\ 1 \end{bmatrix}$$

if and only if

$$\sigma(z_\circ) = z_\circ.$$

That is,

$$\frac{az_\circ + b}{cz_\circ + d} = z_\circ.$$

Hence,

$$z_\circ = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c} = \frac{a-d}{2c} \text{ because } (a+d)^2 = 4.$$

Thus by Theorem 4.2.5,

$$C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$$

is hypercyclic. By Proposition 5.2.3,

$$\sigma^n(z) \to z_\circ = \frac{a-d}{2c}.$$

Case(ii): Assume that $c = 0$. In order for $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ to be hypercylic, we must have $b \neq 0$. (by Theorem 4.2.5).

By equation (5.2.8) we see that

$$a + d = \pm 2. \tag{5.2.9}$$

Since $ad - bc = 1$, we must have

$$ad = 1. \tag{5.2.10}$$

Thus by (5.2.9) and (5.2.10), we get $a = d = \pm 1$. Hence,

$$\sigma(z) = \frac{az+b}{cz+d} = \frac{az+b}{d} = z \pm b.$$

Since if b were 0 then $\sigma(z) = z$ and C_σ is not hypercyclic. Thus $\sigma(z) = z \pm b$ and so σ has no real fixed point. Indeed for all $z \in \mathbb{C}$, we have $\sigma^n(z) = z \pm nb \to \infty$. This is Proposition 5.2.4. $\square$

The above two theorems tell us how the iterates $\sigma^n(z)$ of $\sigma(z)$ on $\mathbb{P}$ converges uniformly on compact subsets of $\mathbb{P}$, in all different cases when $C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$ is hypercyclic. These results illustrate Theorem 3.2.6, statement (5) and (6) in the special case of hypercyclicity.

CHAPTER 6 AN EQUIVALENT CONDITION FOR A COMPOSITION OPERATOR ON $H(\mathbb{D})$ TO BE HYPERCYCLIC

6.1 Introduction

Seidel and Walsh obtained a sufficient condition for a sequence $\{\phi_n\}_{n=1}^{\infty}$ of linear fractional self maps on the unit disk $\mathbb{D}$ such that the sequence $\{C_{\phi_n}\}_{n=1}^{\infty}$ is universal on $H(\mathbb{D})$; see Theorem 2.2.2. In our Proposition 3.2.4 we obtain a necessary and sufficient condition for a sequence of linear fractional transformations $\{\sigma_n\}_{n=1}^{\infty}$ on the upper half plane $\mathbb{P}$ such that $\{C_{\sigma_n}\}_{n=1}^{\infty}$ is universal on $H(\mathbb{P})$. Then in Theorem 4.2.5 we obtain a complete characterization of the linear fractional transformation σ so that the composition operator C_σ is hypercyclic on $H(\mathbb{P})$. In this chapter we use our criterion in Theorem 4.2.5 of hypercyclicity for composition operator C_σ on the space $H(\mathbb{P})$ to derive a criterion for hypercyclicity on the space $H(\mathbb{D})$. Indeed, we provide a complete characterization of the linear fractional transformation ϕ so that the composition operator C_ϕ is hypercyclic on $H(\mathbb{D})$. Certainly, not every conformal map $\phi : \mathbb{D} \to \mathbb{D}$ induces a hypercyclic operator $C_\phi : H(\mathbb{P}) \to H(\mathbb{P})$. For instance if $\phi_\alpha(z) = \frac{\alpha - z}{1-\overline{\alpha}z}$ where $\alpha \in \mathbb{D}$, then $\phi^2(z) = z$ and hence C_{ϕ_α} can not be hypercyclic.

6.2 Equivalent Conditions for Hypercyclicity on $H(\mathbb{D})$

In this section we state our main result, Theorem 6.2.1, for this chapter and we prove it by utilizing Theorem 4.2.5. Moreover, as a continuation of Theorem 3.2.6 and Theorem 6.2.1, we provide Corollary 6.2.6 which gives a precise formula for the point $\beta \in \partial\mathbb{D}$ that $\varphi^n(z) \to \beta$.

Theorem 6.2.1. *Let $\alpha \in \mathbb{D}$ and $\theta \in [-\pi, \pi]$, let*

$$\varphi(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}.$$

The operator

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$$

is hypercyclic if and only if one of the following two conditions hold :-

(i) $|\cos\frac{\theta}{2}| > \sqrt{1-|\alpha|^2}$, *or*

(ii) $|\cos\frac{\theta}{2}| = \sqrt{1-|\alpha|^2}$, *and*

$$\begin{bmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\\ \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\end{bmatrix}\begin{bmatrix}\text{Im }\alpha\\ 1\end{bmatrix} \neq \big(-\sin\left(\frac{\theta}{2}\right)\text{ Re }\alpha\big)\begin{bmatrix}1\\ 1\end{bmatrix}.$$

Proof. We first recall the following mappings

$$\mathbb{P} \xrightarrow{\psi} \mathbb{D} \xrightarrow{\varphi} \mathbb{D} \xrightarrow{\psi^{-1}} \mathbb{P} \qquad \sigma=\psi^{-1}\circ\varphi\circ\psi \tag{6.2.2}$$

where $\psi(z) = \frac{z-i}{z+i}$, $\psi^{-1}(z) = i\frac{1+z}{1-z}$, $\varphi(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}$, with $\alpha \in \mathbb{D}$, $\theta \in [-\pi,\pi]$ and $\sigma(z) = \frac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$. We need to compute $\psi^{-1}\circ\varphi\circ\psi(z)$. By Lemma 4.1.3, we have

$$\begin{aligned}
\psi^{-1}\circ\varphi\circ\psi(z) &= \begin{bmatrix} i & i\\ -1 & 1\end{bmatrix}\begin{bmatrix} e^{i\theta} & -\alpha e^{i\theta}\\ -\overline{\alpha} & 1\end{bmatrix}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}\\
&= \begin{bmatrix} i & i\\ -1 & 1\end{bmatrix}\begin{bmatrix} e^{i\theta}(1-\alpha) & -ie^{i\theta}(1+\alpha)\\ 1-\overline{\alpha} & i(1+\overline{\alpha})\end{bmatrix}\\
&= \begin{bmatrix} ie^{i\theta}(1-\alpha)+i(1-\overline{\alpha}) & e^{i\theta}(1+\alpha)-(1+\overline{\alpha})\\ -e^{i\theta}(1-\alpha)+(1-\overline{\alpha}) & ie^{i\theta}(1+\alpha)+i(1+\overline{\alpha})\end{bmatrix}\\
&= e^{i\frac{\theta}{2}}\begin{bmatrix} ie^{i\frac{\theta}{2}}(1-\alpha)+ie^{-i\frac{\theta}{2}}(1-\overline{\alpha}) & e^{i\frac{\theta}{2}}(1+\alpha)-e^{-i\frac{\theta}{2}}(1+\overline{\alpha})\\ -e^{\frac{i\theta}{2}}(1-\alpha)+e^{-i\frac{\theta}{2}}(1-\overline{\alpha}) & ie^{i\frac{\theta}{2}}(1+\alpha)+ie^{-i\frac{\theta}{2}}(1+\overline{\alpha})\end{bmatrix}
\end{aligned}$$

$$= \quad e^{i\frac{\theta}{2}} \begin{bmatrix} 2i \text{ Re } \left(e^{i\frac{\theta}{2}}(1-\alpha)\right) & 2i \text{ Im } \left(e^{i\frac{\theta}{2}}(1+\alpha)\right) \\ -2i \text{ Im } \left(e^{i\frac{\theta}{2}}(1-\alpha)\right) & 2i \text{ Re } \left(e^{i\frac{\theta}{2}}(1+\alpha)\right) \end{bmatrix}$$

$$= \quad 2ie^{i\theta} \begin{bmatrix} \text{Re } \left(e^{i\frac{\theta}{2}}(1-\alpha)\right) & \text{Im } \left(e^{i\frac{\theta}{2}}(1+\alpha)\right) \\ \text{-Im } \left(e^{i\frac{\theta}{2}}(1-\alpha)\right) & \text{Re } \left(e^{i\frac{\theta}{2}}(1+\alpha)\right) \end{bmatrix}.$$

Hence we have, by Lemma 4.1.3,

$$\psi^{-1} \circ \varphi \circ \psi(z) = \frac{\text{Re } \left(e^{i\frac{\theta}{2}}(1-\alpha)\right)z + \text{Im } \left(e^{i\frac{\theta}{2}}(1+\alpha)\right)}{\text{-Im } \left(e^{i\frac{\theta}{2}}(1-\alpha)\right)z + \text{Re } \left(e^{i\frac{\theta}{2}}(1+\alpha)\right)}.$$

To normalize the above linear fractional transformation, let $u = \cos\frac{\theta}{2}$ and $v = \sin\frac{\theta}{2}$, and hence $e^{i\frac{\theta}{2}} = u + iv$ and let

$$\alpha = x + iy \ \ \text{(therefore } \ |\alpha|^2 = x^2 + y^2 < 1),$$

and we compute

$$\text{Re } \left(e^{i\frac{\theta}{2}}(1-\alpha)\right) \text{ Re } \left(e^{i\frac{\theta}{2}}(1+\alpha)\right) + \text{Im } \left(e^{i\frac{\theta}{2}}(1+\alpha)\right) \text{ Im } \left(e^{i\frac{\theta}{2}}(1-\alpha)\right)$$

$$\begin{aligned}
&= \text{Re } \left[(u+iv)\big((1-x)-iy\big)\right] \text{ Re } \left[(u+iv)\big((1+x)+iy\big)\right] \\
&+ \text{Im } \left[(u+iv)\big((1+x)+iy\big)\right] \text{ Im } \left[(u+iv)\big((1-x)-iy\big)\right] \\
&= \big(u(1-x)+vy\big)\big(u(1+x)-vy\big) + \big(uy+v(1+x)\big)\big(-uy+v(1-x)\big) \\
&= \big(u^2-(ux-vy)^2\big) + \big(v^2-(uy+vx)^2\big) \\
&= u^2 - u^2x^2 + 2uxvy - v^2y^2 + v^2 - u^2y^2 - 2uxvy - v^2x^2
\end{aligned}$$

$$= \quad u^2 + v^2 - u^2(x^2 + y^2) - v^2(x^2 + y^2)$$

$$= \quad 1 - (u^2 + v^2)(x^2 + y^2)$$

$$= \quad 1 - (x^2 + y^2) = 1 - |\alpha|^2.$$

Thus

$$\psi^{-1} \circ \varphi \circ \psi(z) = \frac{az + b}{cz + d},$$

where

$$\begin{cases} a = \frac{1}{\sqrt{1-|\alpha|^2}} \text{ Re } \left(e^{i\frac{\theta}{2}}(1-\alpha)\right) = \frac{1}{\sqrt{1-|\alpha|^2}}\big(u(1-x) + vy\big) \\ b = \frac{1}{\sqrt{1-|\alpha|^2}} \text{ Im } \left(e^{i\frac{\theta}{2}}(1+\alpha)\right) = \frac{1}{\sqrt{1-|\alpha|^2}}\big(uy + v(1+x)\big) \\ c = \frac{-1}{\sqrt{1-|\alpha|^2}} \text{ Im } \left(e^{i\frac{\theta}{2}}(1-\alpha)\right) = \frac{1}{\sqrt{1-|\alpha|^2}}\big(uy - v(1-x)\big) \\ d = \frac{1}{\sqrt{1-|\alpha|^2}} \text{ Re } \left(e^{i\frac{\theta}{2}}(1+\alpha)\right) = \frac{1}{\sqrt{1-|\alpha|^2}}\big(u(1+x) - vy\big). \end{cases} \tag{6.2.3}$$

Hence we have $a, b, c, d \in \mathbb{R}$ and the above computations show $ad - bc = 1$. Via similarity between

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$$

and

$$C_{\psi^{-1}\circ\varphi\circ\psi} : H(\mathbb{P}) \to H(\mathbb{P}),$$

we use theorem 4.2.5, to conclude that C_φ is hypercyclic if and only if either

(1) $|a + d| > 2$ or

(2) $|a + d| = 2$, and at least one of b and c is nonzero.

Now,

$$\begin{aligned} |a+d| &= \frac{1}{\sqrt{1-|\alpha|^2}}\ u(1-x)+vy+u(1+x)-vy \\ &= \frac{|2u|}{\sqrt{1-|\alpha|^2}} = \frac{2\cos\frac{\theta}{2}}{\sqrt{1-|\alpha|^2}}. \end{aligned}$$

Thus statement (1) of above is equivalent to

$$\frac{2\cos\frac{\theta}{2}}{\sqrt{1-|\alpha|^2}} > 2;$$

That is,

$$\cos\frac{\theta}{2} \ > \sqrt{1-|\alpha|^2}.$$

Furthermore, statement (2) above is equivalent to

$$\cos\frac{\theta}{2} \ = \sqrt{1-|\alpha|^2}.$$

and at least one of b and c is nonzero.

The second half of above condition is equivalent to

$$uy+v(1+x)\neq 0 \ \text{ or } \ uy-v(1-x)\neq 0;$$

that is,

$$\begin{bmatrix} u & v \\ u & -v \end{bmatrix}\begin{bmatrix} y \\ 1 \end{bmatrix} \neq (-vx)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\square$

Note that by Theorem 6.2.1, we have conditions on θ and α for

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$$

to be hypercyclic, where

$$\varphi(z) = e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z}.$$

By Lemma 2.2.4 and Theorem 3.2.6, there are a point $z_\circ \in \partial\mathbb{D}$ and a subsequence $\{n_k\}_k$ such that

$$\varphi^{n_k}(z) \to z_\circ.$$

Note that $\psi^{-1} \circ \varphi^n \circ \psi = \sigma^n$, as in the diagram given by 6.2.2.

Remark 6.2.4. *Since C_φ is hypercyclic if and only if C_σ is hypercyclic the following statements are equivalent:*

(i) $\sigma^n(z) \to z_\circ$.

(ii) $\psi^{-1} \circ \varphi^n \circ \psi(z) \to z_\circ$.

(iii) $\varphi^n \circ \psi(z) \to \psi(z_\circ)$.

In the following proposition, we obtain the formula for $z_\circ$ in terms of α and θ. For that we continue to use the notation in the proof of Theorem 6.2.1. That is, $\alpha = x + iy$ and $u = \cos\frac{\theta}{2}$ and $v = \sin\frac{\theta}{2}$, and

$$\psi^{-1} \circ \varphi \circ \psi(z) = \frac{az + b}{cz + d} = \sigma(z),$$

where

$$\begin{cases} a = \frac{1}{\sqrt{1-|\alpha|^2}}\big(u(1-x) + vy\big) \\ b = \frac{1}{\sqrt{1-|\alpha|^2}}\big(uy + v(1+x)\big) \\ c = \frac{1}{\sqrt{1-|\alpha|^2}}\big(uy - v(1-x)\big) \\ d = \frac{1}{\sqrt{1-|\alpha|^2}}\big(u(1+x) - vy\big) \end{cases} \tag{6.2.5}$$

satisfy $ad - bc = 1$.

As a continuation work of our results of the Theorem 6.2.1 and Theorem 5.2.6 we have the following proposition.

Proposition 6.2.6. *Let* $\varphi = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}$, *where* $\alpha = x+iy \in \mathbb{D}$ *and* $\theta \in [-\pi, \pi]$. *Let* $u = \cos\frac{\theta}{2}$ *and* $v = \sin\frac{\theta}{2}$. *Then* C_φ *is hypercyclic if and only if one of the following five statements holds true:*

(i) $u^2 + |\alpha|^2 > 1$, *and* $uy \neq v(1-x)$. *In this case,*

$$\varphi^n(z) \to \frac{vy - ux + \sqrt{u^2+|\alpha|^2-1} - i(uy+vx-v)}{vy - ux + \sqrt{u^2+|\alpha|^2-1} + i(uy+vx-v)} \tag{6.2.7}$$

uniformly on compact subsets of $\mathbb{D}$.

(ii) $u^2 + |\alpha|^2 > 1$, *and* $uy = v(1-x)$ *and* $0 < \frac{u(1-x)+vy}{u(1+x)-vy} < 1$. *In this case,*

$$\varphi^n(z) \to \frac{uy + v(1+x) - i2(ux-vy)}{uy + v(1+x) + i2(ux-vy)} \tag{6.2.8}$$

uniformly on compact subsets of $\mathbb{D}$.

(iii) $u^2 + |\alpha|^2 > 1$, *and* $uy = v(1-x)$ *and* $0 < \frac{u(1+x)-vy}{u(1-x)+vy} < 1$. *In this case,*

$$\varphi^n(z) \to 1 \quad \text{uniformly on compact subsets of } \mathbb{D}. \tag{6.2.9}$$

(iv) $u^2 + |\alpha|^2 = 1$, *and* $uy \neq v(1-x)$, *In this case,*

$$\varphi^n(z) \to \frac{(vy-ux) - i(uy+vx-v)}{(vy-ux) + i(uy+vx-v)} \tag{6.2.10}$$

uniformly on compact subsets of $\mathbb{D}$.

(v) $u^2 + |\alpha|^2 = 1$, *and* $uy = v(1-x)$ *and* $uy + v(1+x) \neq 0$. *In this case,*

$$\varphi^n(z) \to 1 \quad \text{uniformly on compact subsets of } \mathbb{D}. \tag{6.2.11}$$

Proof. To prove our proposition we use our results in Proposition 5.2.6 and Remark 5.2.7 to find $z_\circ \in \partial_\infty \mathbb{P} = \mathbb{R} \cup \{\infty\}$ such that $\sigma^n(z) \to z_\circ$. Then use Remark 6.2.4 to find $\psi(z_\circ)$. Before we provide a proof, we first observe that by equations (6.2.5)

$$\begin{aligned}(a+d)^2 - 4 &= \frac{4u^2}{1-|\alpha|^2} - 4 \\ &= \frac{4(u^2 + |\alpha|^2 - 1)}{1-|\alpha|^2}.\end{aligned}$$

Thus $(a+d)^2 - 4 > 0$ if and only if $u^2 + |\alpha|^2 > 1$

Case (i): $u^2 + |\alpha|^2 > 1$ and $uy \neq v(1-x)$. Thus by equation (6.2.5) $(a+d)^2 - 4 > 0$ and $c \neq 0$ and by Theorem 5.2.6, for all $z \in \mathbb{P}$,

$$\sigma^n(z) \to \frac{(a-d) + \sqrt{(a+d)^2 - 4}}{2c}.$$

Let

$$z_\circ = \frac{(a-d) + \sqrt{(a+d)^2 - 4}}{2c} = \frac{(vy - ux) + \sqrt{u^2 + |\alpha|^2 - 1}}{uy - v(1-x)}.$$

Hence by Remark 6.2.4 (ii), we have

$$\varphi^n(z) \to \psi(z_\circ) = \frac{vy - ux + \sqrt{u^2 + |\alpha|^2 - 1} - i(uy + vx - v)}{vy - ux + \sqrt{u^2 + |\alpha|^2 - 1} + i(uy + vx - v)}$$

uniformly on compact subsets of $\mathbb{D}$.

Case (ii): $u^2 + |\alpha|^2 > 1$ and $uy = v(1-x)$ and $0 < \frac{u(1-x)+vy}{u(1+x)-vy} < 1$.

In this case $(a+d)^2 - 4 > 0$ and by equations (6.2.5)

$$c = \frac{1}{\sqrt{1-|\alpha|^2}}\big(uy - v(1-x)\big) = 0, \tag{6.2.12}$$

and also

$$\frac{a}{d} = \frac{u(1-x) + vy}{u(1+x) - vy}; \tag{6.2.13}$$

that is, $a < d$. Since in this case $ad - bc = ad = 1$ we must have $0 < a < 1 < d$.

By our discussion in Case(A) in the proof of Proposition 5.2.6, we have

$$\sigma^n(z) \to \frac{b}{d-a} = \frac{uy + v(1+x)}{2(ux - vy)}$$

uniformly on compact subsets of $\mathbb{D}$. Hence by Remark 6.2.4 (ii),

$$\varphi^n(z) \to \psi(\frac{b}{d-a}) = \frac{uy + v(1+x) - i2(ux - vy)}{uy + v(1+x) + i2(ux - vy)}.$$

Case (iii): $u^2 + |\alpha^2| > 1$ and $uy = v(1-x)$ and $0 < \frac{u(1+x)-vy}{u(1-x)+vy} < 1$.

In this case, $(a+d)^2 - 4 > 0$ and by (6.2.12) and (6.2.13), $c = 0$ and $\frac{d}{a} = \frac{u(1+x)-vy}{u(1-x)+vy}$ satisfying $0 < \frac{d}{a} < 1$. By our discussion in case (B) in the proof of Theorem 5.2.6, we have

$$\sigma^n(z) \to \infty \ \text{ for all } z \in \mathbb{P}.$$

Hence by Remark 6.2.4 (ii),

$$\varphi^n(z) \to \psi(\infty) = 1.$$

Case (iv): $u^2 + |\alpha|^2 = 1$ and $uy - v(1-x) \neq 0$.

Thus $(a+d)^2 - 4 = 0$, and by equations (6.2.5)

$$c = \frac{1}{\sqrt{1-|\alpha|^2}}(uy - v(1-x)) \neq 0.$$

Hence by Theorem 5.2.7 and equations (6.2.5) we get:

$$\sigma^n(z) \to \frac{a-d}{2c} = \frac{vy - ux}{uy + vx - v}.$$

Let

$$z_\circ = \frac{vy - ux}{uy + vx - v}.$$

Thus for all $z \in \mathbb{D}$, by Remark 6.2.4 (ii)

$$\varphi^n(z) \to \psi(z_\circ) = \frac{(vy - ux) - i(uy + vx - v)}{(vy - ux) + i(uy + vx - v)}.$$

Case (v): $u^2 + |\alpha|^2 = 1$ and $uy - v(1 - x) = 0$, Thus $(a + d)^2 - 4 = 0$, and by equation (6.2.5) we get:

$$c = \frac{1}{\sqrt{1 - |\alpha|^2}}(uy - v(1 - x)) = 0,$$

and

$$b = \frac{1}{\sqrt{1 - |\alpha|^2}}(uy + v(1 + x)) \neq 0.$$

Hence by Theorem 5.2.7

$$\sigma^n(z) \to \infty.$$

Thus for all $z \in \mathbb{D}$, by Remark 6.2.4 (ii)

$$\varphi^n(z) \to \psi(\infty) = 1.$$

This completes the proof. □

To conclude this section we remark that the expression in (6.2.7) of Corollary 6.2.6 works for the cases (iv) and (v) as well. That is, (6.2.7) reduces to (6.2.10) and (6.2.11) in various cases (iv) and (v).

6.3 Examples

In this section, we provide a numerical example in which we apply Theorem 6.2.1. In addition, we obtain a series of corollaries which we consider them as general examples in various cases of Theorem 6.2.1.

Example 6.3.1. *Let* $\theta \in (0, \pi)$ *such that,* $u = \cos\frac{\theta}{2} = \frac{2}{\sqrt{5}}$ *and* $v = \sin\frac{\theta}{2} = \frac{1}{\sqrt{5}}$. *Let* $\alpha = x + iy = \frac{1}{5} + i\frac{2}{5} \in \mathbb{D}$ *(because* $|\alpha|^2 = \frac{1}{5}$*).*

Hence,

$$\frac{1}{\sqrt{1-|\alpha|^2}} = \frac{1}{\sqrt{1-\frac{1}{5}}} = \frac{\sqrt{5}}{2}.$$

By equations in (6.2.5)*,*

$$\begin{aligned} a &= \frac{\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}(1-\frac{1}{5}) + \frac{1}{\sqrt{5}}\frac{2}{5}\right) = 1; \\ b &= \frac{\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}\frac{2}{5} + \frac{1}{\sqrt{5}}(1+\frac{1}{5})\right) = 1; \\ c &= \frac{\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}\frac{2}{5} - \frac{1}{\sqrt{5}}(1-\frac{1}{5})\right) = 0; \\ d &= \frac{\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}(1+\frac{1}{5}) - \frac{1}{\sqrt{5}}\frac{2}{5}\right) = 1. \end{aligned}$$

Thus $ad - bc = 1$ *if*

$$\sigma(z) = \frac{az+b}{cz+d} = z+1,$$

then

$$C_\sigma : H(\mathbb{P}) \to H(\mathbb{P})$$

is hypercyclic by Proposition 4.2.3. Furthermore, if $\theta \in (0, \pi)$ *such that* $\tan\frac{\theta}{2} = \frac{1}{2}$ *and* $\alpha = \frac{1}{5} + \frac{2}{5}i$*, and if*

$$\varphi(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z},$$

then

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$$

is hypercyclic, by statement (ii) *of Theorem 6.2.1.*

As a corollary of the Theorem 6.2.1 we have the following six results:

Corollary 6.3.2. *For any $\alpha \in \mathbb{D} \setminus \{0\}$, there exists $\theta \in (-\pi, \pi)$ such that if*

$$\varphi(z) = e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z},$$

then

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$$

is hypercyclic.

Proof. Choose θ in $(-\pi, \pi)$ such that $\cos\frac{\theta}{2} > \sqrt{1 - |\alpha|^2}$. Our result follows from the Theorem 6.2.1. □

Corollary 6.3.3. *For any $\alpha \in \mathbb{D} \setminus \{0\}$, if $\varphi(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$ then*

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$$

is hypercyclic.

Proof. Choose $\theta = 0$, and so

$$\cos\frac{\theta}{2} = 1 > \sqrt{1 - |\alpha|^2}.$$

Our result follows from Theorem 6.2.1. □

Corollary 6.3.4. *For any $\alpha \in \mathbb{D} \setminus \{0\}$, there exists $\theta \in (-\pi, \pi)$ such that if*

$$\varphi(z) = e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z},$$

then

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$$

is not hypercyclic.

Proof. Choose $\theta \in (-\pi, \pi)$ such that

$$\cos\frac{\theta}{2} < \sqrt{1-|\alpha|^2}.$$

Our result follows from Theorem 6.2.1. □

Corollary 6.3.5. *For any $\theta \in (-\pi, \pi)$, there exists $\alpha \in \mathbb{D} \setminus \{0\}$, such that if*

$$\varphi(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z},$$

then

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$$

is not hypercyclic.

Proof. Choose $\alpha \in \mathbb{D} \setminus \{0\}$ such that $|\cos\frac{\theta}{2}| < \sqrt{1-|\alpha|^2}$. Then we get our result from Theorem 6.2.1. □

Corollary 6.3.6. *For any $\theta \in (-\pi, \pi)$, there exists $\alpha \in \mathbb{D} \setminus \{0\}$, such that if*

$$\varphi(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z},$$

then

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$$

is hypercyclic.

Proof. Choose $\alpha \in \mathbb{D}\setminus\{0\}$ such that $|\cos\frac{\theta}{2}| > \sqrt{1-|\alpha|^2}$. Apply our result in Theorem 6.2.1. □

Corollary 6.3.7. *For any $\alpha \in \mathbb{D} \setminus \{0\}$, if*

$$\varphi(z) = \frac{\alpha - z}{1-\overline{\alpha}z},$$

then

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$$

is not hypercyclic.

Proof. Choose $\theta = \pi$ and so $\cos\frac{\theta}{2} = 0$. Apply Theorem 6.2.1. $\square$

It is easy to see that the conformal map $\varphi(z) = \frac{\alpha - z}{1-\overline{\alpha}z}$ in Corollary 6.3.7, where $\alpha \in \mathbb{D}$ does not induce a hypercyclic composition operator $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ without using Theorem 6.2.1. This is because $\varphi^2(z) = \varphi \circ \varphi(z) = z$.

CHAPTER 7 CONCLUSION

7.1 Introduction

In this chapter, we investigate universal composition operators in the setting of the Fréchet space $H(\Omega)$ of holomorphic functions on a simply connected region Ω in the complex plane. We obtain a necessary and sufficient condition for a sequence of composition operators $C_{\sigma_n} : H(\Omega) \to H(\Omega)$ with conformal maps $\sigma_n : \Omega \to \Omega$ to be universal. Specifically, we prove that the sequence $C_{\sigma_n} : H(\Omega) \to H(\Omega)$ is universal if and only if there is a point w in $\partial\Omega$ and a subsequence $\{\sigma_{n_k}\}_k$ of $\{\sigma_n\}_n$ such that $\sigma_{n_k} \to w$ uniformly on compact subset of Ω. Our result extends a result of Grosse-Erdmann and Manguillot [9, p. 116] who proved in 2011 equivalent conditions for a composition operator C_ϕ to be hypercyclic on the simply connected region Ω. Since hypercyclicity is a special case of universality our result extends theirs in the case that int$(\mathbb{C}\backslash G) \neq \emptyset$; see Theorem 7.2.4.

Before we prove our result in this chapter we first introduce some definitions and then state Theorem 7.2.1 from the literature [22, 9].

Definition 7.1.1. *Suppose $T : X \to X$ and $S : Y \to Y$ are mappings of metric space X, and $V : X \to Y$ is a continuous map from X onto Y for which $V \circ T = S \circ V$. In this case we call S a factor of T, and T an extension of S. If $V(X)$ is just dense in Y we say T is quasiconjugate to S.*

Definition 7.1.2. *A continuous map $T : X \to X$ on a complete, separable metric space X is transitive if and only if for every pair U, V of nonempty open subsets of X there is a non-negative integer n such that $T^{-n}(U) \cap V \neq \emptyset$.*

Definition 7.1.3. *(a) A point $x \in X$ is periodic for T if there is a positive integer n such that $T^n x = x$ and we called the least such positive integer n the period of x.*

(b) We say a mapping T of a metric space X is chaotic if it is transitive and has a dense set of periodic points.

Definition 7.1.4. *A continuous operator T on the metric space X is called mixing if for any pair U, V of nonempty open subsets of X, there exists some $N \geqslant 0$ such that*

$$T^n(U) \cap V = \emptyset \text{ for all } n \geqslant N.$$

Definition 7.1.5. *Let G be a region in $\mathbb{C}$ and $\phi_n : G \to G$ be holomorphic maps for $n \geqslant 1$. Then the sequence $\{\phi_n\}_n$ is called a run-away sequence if for any compact subset $K \subset G$, there is some integer N such that $\phi^N(K) \cap K = \emptyset$.*

Let $Aut(\Omega)$ be the set of all automorphisms on Ω; that is, the set of all bijective holomorphic maps $f : \Omega \to \Omega$. These maps are also called conformal maps. Now we move to the next section where we prove Corollary 7.2.3.

7.2 Equivalent Conditions for Hypercyclicity on $H(\Omega)$

We begin this section by stating a result of Grosse-Erdmann and Manguillot.

Theorem 7.2.1. *([9, p. 116]) Let Ω be a simply connected domain and $\varphi \in Aut(\Omega)$. Then the following conditions are equivalent:*

(i) C_φ *is hypercyclic;*

(ii) C_φ *is mixing;*

(iii) C_φ *is chaotic;*

(iv) $(\varphi^n)_n$ *is a ran-away sequence;*

(v) φ *has no fixed point in* Ω*;*

(vi) C_φ *is quasiconjgate to a Birkhoff operator.*

Theorem 7.2.2. *Let Ω, G be two bounded simply connected regions. Let $\psi : G \to \Omega$ be a conformal map and $\varphi_n : \Omega \to \Omega$ and $\sigma_n : G \to G$ be two sequences of conformal maps satisfying*

$$\sigma_n = \psi^{-1} \circ \varphi_n \circ \psi.$$

Suppose there exists a point $\rho \in \partial\Omega$ such that $\varphi_n \to \rho$ uniformly on compact subsets of Ω. Then there exist a subsequence $\{\sigma_{n_k}\}_k$ and $w \in \partial G$ such that $\sigma_{n_k} \to w$ uniformly on compact subset of G.

Proof. Since G is bounded, the sequence $\sigma_n : G \to G$ is a normal family, by Montel's Theorem; see Conway [3, p. 153]. Thus there is a subsequence, still denoted by $\{\sigma_n\}_n$, and a holomorphic function $f : G \to G$ such that $\sigma_n \to f$ uniformly on compact subsets of G; see Conway [3, p. 152] and Conway [3, Definition 1.14. p. 146]. For any point $z_\circ$ in G, the sequence

$$\sigma_n(z_\circ) = \psi^{-1} \circ \varphi_n \circ \psi(z_\circ)$$

is in G. Thus $\{\sigma_n(z_\circ)\}_n$ has a convergent subsequence $\{\sigma_{n_k}(z_\circ)\}_k$ in the compact set $\overline{G}$. Hence there exists $w_\circ \in \overline{G}$ such that

$$\sigma_{n_k}(z_\circ) \to w_\circ.$$

Claim. $w_\circ \in \partial G$.

Proof of Claim. By way of contradiction, suppose $w_\circ \in G$. Since $\psi(z_\circ) \in \Omega$ and $\{\varphi_n\}_n$ converges uniformly on compact subsets of Ω, we have

$$\varphi_{n_k}(\psi(z_\circ)) = \psi \circ \sigma_{n_k}(z_\circ) \to \psi(w_\circ) \in \Omega,$$

which contradicts the hypothesis that $\{\varphi_n\}_n$ converges uniformly on compact subsets to a point $\rho \in \partial\Omega$. This completes the proof of the claim. □

Since $\sigma_n \to f$ uniformly on compact subsets of G, we have

$$\sigma_n(z) \to f(z), \quad \text{for all } z \text{ in } G.$$

By our claim $f(z) \in \partial G$. Hence the range of the holomorphic function f is not an open set. By The Open Mapping Theorem (see Conway [3, p. 99]), f is a constant function. By the claim

$f(z) \equiv w_\circ$ for some point $w_\circ \in \partial G$. That is $\sigma(z) \to w_\circ$ uniformly on compact subsets of G. □

Now we prove a universality result in the Corollary below, in the line of our focus in this dissertation.

Corollary 7.2.3. *Let G be a bounded simply connected region and $\sigma_n : G \to G$ be a sequence of conformal maps. The sequence $C_{\sigma_n} : H(G) \to H(G)$ is universal if and only if there are a point w in ∂G and a subsequence $\{\sigma_{n_k}\}_k$ of $\{\sigma_n\}$ such that $\sigma_{n_k} \to w$ uniformly on compact subsets of G.*

Proof. Suppose $C_{\sigma_n} : H(G) \to H(G)$ is universal. By taking $\Omega = \mathbb{D}$ in Theorem 7.2.2, we see that the sequence of composition operators induce by

$$\varphi_n = \psi \circ \sigma_n \circ \psi^{-1} : \mathbb{D} \to \mathbb{D}$$

is universal. Thus by Theorem 3.2.6, there is $\rho \in \partial\mathbb{D}$ such that $\varphi_n(z) \to \rho$ uniformly on compact subsets of $\mathbb{D}$. Hence by Theorem 7.2.2, there are a subsequence $\{\sigma_{n_k}\}_k$ of $\{\sigma_n\}_n$ and a point $w_\circ \in \partial G$ such that $\sigma_{n_k}(w) \to w_\circ$ uniformly on compact subsets of G.

Conversely, suppose there are a point $w_\circ \in \partial G$ and a subseqeuce $\{\sigma_{n_k}\}_k$ of $\{\sigma_n\}_n$ such that $\sigma_{n_k}(w) \to w_\circ$ uniformly on compact subsets of G. Thus by Theorem 7.2.2, there are a subsequence $\{\varphi_{n_k}\}_k$ of $\{\varphi_n\}_n$ and a point $\rho \in \partial\mathbb{D}$ such that $\varphi_{n_k}(z) \to \rho$ uniformly on compact subsets of $\mathbb{D}$. Thus by Theorem 3.2.6, $C_{\varphi_n} : H(\mathbb{D}) \to H(\mathbb{D})$ is universal. Hence $C_{\sigma_n} = C_\psi \circ C_{\varphi_n} \circ C_{\psi^{-1}}$ is universal on $H(G)$. □

The result of the above Corollary indeed holds true for some simply connected regions which are not bounded. To be precise, we provide the following Theorem to generalize the result in Corollary 7.2.3.

Theorem 7.2.4. *Let G be a simply connected region with either $G = \mathbb{C}$ or int$(\mathbb{C} \setminus G) \neq \emptyset$. Let $\sigma_n : G \to G$ be a sequence of conformal maps. The sequence $C_{\sigma_n} : H(G) \to H(G)$ is universal if* *and only if there are a point ρ in the extended boundary $\partial_\infty G$ and a subsequence $\{\sigma_{n_k}\}_k$ of $\{\sigma_n\}_n$ such that $\sigma_{n_k} \to \rho$ uniformly on compact subsets of G.*

Proof. The result for the case when G is bounded follows from Corollary 7.2.3. For unbounded G, we separate our argument into two cases of G according to the hypothesis of the theorem. Case (i): $G = \mathbb{C}$. In this case, we use the results of Montes-Rodriguez [16, Theorem. 2.2 and 3.1]: A sequence of conformal maps $\sigma_n : \mathbb{C} \to \mathbb{C}$ is universal if and only if $\{\sigma_n\}_n$ is a run-away sequence; that is, for any compact subset K of $\mathbb{C}$, there is a positive integer n such that $K \cap \sigma_n(K) = \emptyset$.

To finish the proof for the case that $G = \mathbb{C}$, we claim that for some subsequence $\{\sigma_{n_k}\}_k$, $\sigma_{n_k} \to \infty$ uniformly on compact subsets of $\mathbb{C}$ if and only if $\{\sigma_n\}_n$ is run-away. To see that, one simply observe that if K is a compact subset, then there is a positive R such that $K \subset R\overline{\mathbb{D}}$. Hence if $r > R$ and if $\{\sigma_n\}_n$ is run-away then there is a positive integer N such that $\sigma_N(K) \subset \sigma_N(r\overline{D})$, which has a nonempty intersection with $r\overline{\mathbb{D}}$, and so $\sigma_N(z) > r$ for all z in K. Thus there is a subsequence $\{\sigma_{n_k}\}_k$ so that $\sigma_{n_k} \to \infty$ uniformly on compact subsets of $\mathbb{C}$.

Conversely, suppose there is a some subsequence $\{\sigma_{n_k}\}_k$ such that $\sigma_{n_k} \to \infty$ uniformly on compact subsets of $\mathbb{C}$. Then for any compact subset K, there is a number $R > 0$ such that $K \subset R\overline{\mathbb{D}}$, and there is a σ_{n_k} such that $\sigma_{n_k}(z) > R + 1$ for all $z \in K$. Hence, $K \cap \sigma_{n_k}(K) = \emptyset$.

Case (ii): G is an unbounded simply connected region with $\text{int}(\mathbb{C} \setminus G) \neq \emptyset$. Let $\alpha \in \mathbb{C}$ such that the open ball $B(\alpha, r) \subset \text{int}(\mathbb{C} \setminus G)$. Hence, the function $\psi(z) = \frac{1}{z-\alpha}$ takes G one to one, onto a bounded simply connected region Ω, with $\psi(\infty) = 0$ and $\Omega \subset R^{-1}\overline{\mathbb{D}}$. Let $\varphi_n : \Omega \to \Omega$ be given by

$$\varphi_n = \psi \circ \sigma_n \circ \psi^{-1}.$$

So the sequence $C_{\varphi_n} : H(\Omega) \to H(\Omega)$ is universal if and only if the sequence $C_{\sigma_n} : H(G) \to H(G)$ is universal. Note ψ takes the extended boundary $\partial_\infty G = G \cup \{\infty\}$ one to one, onto $\partial\Omega$, and ψ is continuous at every point in $\partial_\infty G$. Suppose there is a point $w \in \partial\Omega$ and a subsequence $\{\varphi_{n_k}\}_k$ such that $\varphi_{n_k}(z) \to w$ uniformly on compact subsets of Ω. Then by the continuity of ψ^{-1}

at w, we have

$$\sigma_{n_k}(z) = \psi^{-1} \circ \varphi_{n_k} \circ \psi(z) \to \psi^{-1}(w)$$

uniformly on compact subsets of G. Our theorem now follows directly from Corollary 7.2.3. □

We remark that in the case that $G = \mathbb{C}$, conformal maps $\sigma : \mathbb{C} \to \mathbb{C}$ are well known to be in the form $\sigma(z) = az + b$, where $a, b \in \mathbb{C}$ with $a \neq 0$, but we do not need this specific form in the above proof.

At first glance, one may think that the above theorem should hold true for all simply connected regions G. However, due to the complexity of boundary points that G may have, we have not been able to determine whether that is correct. One evidence for us to focus on the case int$(\mathbb{C} \setminus \Omega) = \emptyset$ may come from the relatively simpler structure of its boundary points; see Rudin [18, Remark. 14.20 (c)].

To conclude our discussion above we raise the following question.

Question 7.2.5. *Does the conclusion of Theorem 7.2.4 continue to hold true for any simply connected region G?*

We conclude the whole dissertation with the following observation. To illustrate Theorem 7.2.1, we now provide an example in the case that the simply connected region is the open unit disk $\mathbb{D}$.

Example 7.2.6. *Let $\alpha \in \mathbb{D}$ and ϕ is a linear fractional transformation on $\mathbb{D}$ defined by*

$$\phi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Then

$$\phi_\alpha^2(z) = \phi_\alpha \circ \phi_\alpha(z) = \frac{\alpha - \frac{\alpha - z}{1 - \overline{\alpha} z}}{1 - \frac{\alpha - z}{1 - \overline{\alpha} z}}$$

$$=\frac{\alpha-|\alpha|^2z-\alpha+z}{1-\overline{\alpha}z-|\alpha|^2+\overline{\alpha}z}=z.$$

Now take $\alpha=\frac{1}{2}$*. To see where* ϕ_α *has a fixed point, set* $\phi_\alpha(z)=\phi_{\frac{1}{2}}(z)=z$*. So that*

$$\begin{aligned}\frac{(\frac{1}{2})-z}{1-(\frac{1}{2})z}&=z;\\ \frac{1}{2}-z&=z-\frac{1}{2}z^2;\\ \frac{1}{2}z^2-2z+\frac{1}{2}&=0;\\ z^2-4z+1&=0.\end{aligned}$$

Hence,

$$z=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}.$$

Therefore $\phi_{\frac{1}{2}}$ *has a fixed point* $2-\sqrt{3}$ *in* $\mathbb{D}$ *and* $\phi^2_{\frac{1}{2}}(z)=z$*. Thus,* $C^2_{\phi_{\frac{1}{2}}}=$ *identity and hence* $C^{2n}_{\phi_{\frac{1}{2}}}=$ *identity for* $n\geqslant 1$*, and* $C^{2n+1}_{\phi_{\frac{1}{2}}}=C_{\phi_{\frac{1}{2}}}$*. Thus* $C_{\phi_{\frac{1}{2}}}$ *is not hypercyclic and* $\phi_{\frac{1}{2}}$ *has a fixed point in* $\mathbb{D}$*.*

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