USING HILBERT SPACE THEORY AND QUANTUM MECHANICS TO EXAMINE THE ZEROS OF THE RIEMANN-ZETA FUNCTION

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ABSTRACT

Kit Chan, Advisor

In this thesis, I examine Hilbert Space Theory and the basic postulates of quantum mechanics and use these two different areas to look for zeros of the Riemann Zeta function. In particular, after stating some preliminaries of Hilbert Space Theory, I will examine how these ideas appear in quantum theory. As an illustrative example, I place a particle in the famous infinite square well to see these ideas in action. Finally, having discussed and explored quantum mechanics I will expound on the Riemann Zeta Function and the Riemann Hypothesis. I will then discuss various conjectures, that are a potential way to solve the famous Riemann Hypothesis via an unknown Hermitian operator on an unknown space. As Hamiltonian operators are Hermitian operators, we have a bridge into physics to take advantage of in solving this problem.

If I were to awaken after having slept for a thousand years, my first question would be: has the Riemann Hypothesis been proven?

– David Hilbert

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PREFACE

In this thesis we examine Hilbert space theory, quantum mechanics, and the Riemann Zeta function and how they can amalgamate into one solid foundation for trying to solve the Riemann Hypothesis. We begin with some introductory material on Hilbert space material which in turn provides the tools to tour quantum mechanics. Quantum mechanics is an interesting area as it is no longer a deterministic area of physics but rather a statistic one. We derive the famous Schrödinger Equation and work through a common example. Then, we introduce the Riemann Zeta function and the Riemann Hypothesis. This function has baffled mathematicians and even physicists for the past century. The Riemann Zeta function has countless connections to many areas of mathematics and physics, the biggest one is prime numbers. Solving the Riemann Hypothesis, which states all the non-trivial zeros of the Riemann Zeta function lie on the vertical line $s = \frac{1}{2} + it$ in the complex plane, would have far reaching consequences. These three separate areas of mathematics and physics unite in attempts to provide a solution to the Riemann Hypothesis. Finally, we conclude by discussing and examining some of this research.

CHAPTER 1 HILBERT SPACE THEORY

1.1 Introduction

In this section, we explore Hilbert space theory, which is the underlying mathematics of quantum mechanics. It is essentially a generalization of linear algebra. We utilize [8] to build the foundations, though utilizing [14], [7], or [13] would be a more rigorous of an approach. We will slowly build up from a vector space to a Hilbert space. Once we have established Hilbert spaces we will move to operator theory in Hilbert spaces.

1.2 Vector Space

Definition 1.2.1. *A set V is said to be a vector space (or linear space) over a field F, typically $\mathbb{C}$ or $\mathbb{R}$, is an Abelian group under addition (denoted by $+$) and, if for each $a \in F$ and $v \in V$, there is an element av in V such that the following conditions hold for all a,b in F and all u,v in V.*

1. $a(v+u) = av + au$
2. $(a+b)v = av + bv$
3. $a(bv) = (ab)v$
4. $1v = v$

Or more simply, the set V is closed under scalar multiplication and vector addition. There are numerous examples of vector spaces such as: $\mathbb{R}^N$, $\mathbb{C}^N$, and the set of entire complex functions. Notice $\mathbb{R}^N$ and $\mathbb{C}^N$ are of finite dimension N while the set of entire functions is of infinite dimension. We say that a vector space is a complex vector space if F is $\mathbb{C}$.

1.3 Inner Product Space

Inner Product spaces are sometimes called Pre-Hilbert spaces.

Definition 1.3.1. *Let E be a complex vector space. A mapping $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{C}$ is called an inner product in E if for any $x, y, z \in E$ and $a, b \in \mathbb{C}$ the following conditions are satisfied:*

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$

2. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

3. $\langle x, x\rangle \geq 0$

4. $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

From properties 1 and 2 we can obtain that

$$\langle x, ay + bz\rangle = \overline{\langle ay + bz, x\rangle} = \overline{a\langle y, x\rangle} + \overline{b\langle z, x\rangle} = \bar{a}\langle x, y\rangle + \bar{b}\langle x, z\rangle .$$

We will use the fact that the inner product is linear in the first component and antilinear in the second component; some mathematicians and often physicists will define this in the opposite manner, the fact remains that one component is linear and the other is antilinear. A vector space with an inner product is called an Inner Product Space. The vector spaces $\mathbb{C}$ and $\mathbb{C}^n$ are familiar inner product spaces. Another common inner product space is $C\big([a, b]\big)$, which is the space of continuous functions on the interval $[a, b]$ with inner product given by:

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

and $L^2(\mathbb{R})$ which are the square-integrable functions on the real line, which we will be using extensively, with inner product given by:

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx.$$

Every inner product space is also a normed space with norm defined by $||x|| = \sqrt{\langle x, x\rangle}$. This is well defined since $\langle x, x\rangle \in \mathbb{R}^+ \cup \{0\}$, $||x|| = 0$ if and only if $x = 0$, $||\lambda x|| = |\lambda|||x||$, and the triangle inequality is true. However, to see the triangle inequality, we need to make use of the Cauchy-Schwarz Inequality.

Theorem 1.3.2 (Cauchy-Schwarz Inequality)**.** *For all x and y in an inner product space we have,*

$$|\langle x, y\rangle| \leq ||x||\,||y||.$$

Equality holds if and only if x and y are linearly dependent.

As there are numerous proofs for this inequality it is skipped.

Theorem 1.3.3 (Triangle Inequality)**.** *For any two elements x and y in an inner product space we have,*

$$||x + y|| \leq ||x|| + ||y||.$$

Proof.

$$||x + y||^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + 2\Re\big(\langle x, y\rangle\big) + \langle y, y\rangle.$$

$$\langle x, x\rangle + 2\Re\big(\langle x, y\rangle\big) + \langle y, y\rangle \leq \langle x, x\rangle + 2|\langle x, y\rangle| + \langle y, y\rangle.$$

Now by the Cauchy-Schwarz Inequality and definition of the norm, we have

$$\langle x, x\rangle + 2|\langle x, y\rangle| + \langle y, y\rangle \leq ||x||^2 + 2||x||\,||y|| + ||y||^2 = (||x|| + ||y||)^2.$$

Upon taking a square root, we have the desired result. □

One excellent result which will be used often, will be that of orthogonality.

Definition 1.3.4. *Two vectors x and y in an inner product space are called orthogonal, denoted $x \perp y$ if $\langle x, y\rangle = 0$.*

We conclude this section with the inner product version of the Pythagorean Theorem.

Theorem 1.3.5. *For any pair of orthogonal vectors, we have that*

$$||x + y||^2 = ||x||^2 + ||y||^2.$$

Proof.

$$||x+y||^2 = ||x||^2 + \langle x, y\rangle + \langle y, x\rangle + ||y||^2.$$

Now since $x \perp y$,

$$||x+y||^2 = ||x||^2 + ||y||^2,$$

which concludes the proof. □

1.4 Hilbert Space

Near the turn of the 20th century, David Hilbert and John von Neumann laid the ground work for the theory of Hilbert spaces. The addition of completeness to an inner product space will turn it into a Hilbert space.

Definition 1.4.1. *We say a space E is complete if every Cauchy sequence converges to an element of E. That is, for any sequence $\{x_n\}$, the relation*

$$\lim_{n,m\to\infty} ||x_n - x_m|| = 0,$$

defines a unique limit x in E such that

$$\lim_{n\to\infty} ||x - x_n|| = 0.$$

Definition 1.4.2 (Hilbert Space)**.** *A complete inner product space is a Hilbert space.*

Since $\mathbb{C}$ and $\mathbb{C}^n$ are complete, they are common examples of Hilbert spaces we use often. While $C\big([a,b]\big)$ is an inner product space, it is not a Hilbert space. There exists many Cauchy sequences of continuous functions which do not converge to a continuous function. However, $L^2(\mathbb{R})$ and $L^2\big([a,b]\big)$ are Hilbert spaces.

1.5 Operators on Hilbert Space

Often we might take an ”action” on an element in a Hilbert space and transform it to another element, these ”actions” are called operators.

Definition 1.5.1. *We say that an operator T is linear if $T(ax+by) = aT(x)+bT(y)$ for all x and y in H and scalars a and b.*

Many operators in calculus are in fact linear, the differential and integral operators are great examples.

Definition 1.5.2. *We say that a linear operator A is bounded in H if there exists $c > 0$ such that for all $x \in H$, we have that $||Ax|| \leq c||x||$.*

Definition 1.5.3. *Let A be a bounded operator on a Hilbert space H. The operator $A^* : H \to H$ defined by $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in H$ is called the adjoint operator of A.*

There are several basic properties of adjoint operators:

1. $(A+B)^* = A^* + B^*$,
2. $(\alpha A)^* = \bar{\alpha}A^*$,
3. $I^* = I$, and
4. $(AB)^* = B^*A^*$.

A particularly useful operator is a self-adjoint operator.

Definition 1.5.4. *If $A = A^*$, then A is called self-adjoint. That is, $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all x and y in H.*

This type of operator is often called hermitian. Finally, there are several other types of operators as well.

Definition 1.5.5. *Let A be an operator defined on a vector subspace of E. An operator B defined on $R(A)$ is called the inverse of A if $ABx = x$ and $BAx = x$ for all $x \in R(a)$ and for all $x \in D(A)$. An operator which has an inverse is called invertible.*

It is worth noting inverses are unique.

Definition 1.5.6. *A bounded operator T is called a normal operator if it commutes with its adjoint. That is, $TT^* = T^*T$.*

Definition 1.5.7. *A bounded operator T on a Hilbert space H is called an isometric operator if $||Tx|| = ||x||$ for all $x \in H$.*

Definition 1.5.8. *A bounded operator T on a Hilbert space H is called a unitary operator if $T^*T = TT^* = I$ on H.*

Definition 1.5.9. *An operator A is called positive if it is self-adjoint and $\langle Ax, x\rangle \geq 0$ for all x in H.*

Definition 1.5.10. *Let S be a closed subspace of a Hilbert space H. The operator P on H defined by $Px = y$ if $x = y + z$, where $y \in S$ and $z \in S^{\perp}$ is called the orthogonal projection operator onto S, or simply projection onto S.The vector y is called the projection of x onto S. Projection onto a subspace S will be usually denoted by P_S.*

Definition 1.5.11. *An operator T is idempotent if $T^2 = T$.*

Definition 1.5.12. *An operator A on a Hilbert space H is called a compact operator if, for every bounded sequence $\{x_n\}$ in H, the sequence $\{Ax_n\}$ contains a convergent subsequence.*

Definition 1.5.13. *An operator A defined in a normed space E is called densely defined if its domain is a dense subset of E, that is $cl\big(D(A)\big) = E$.*

Definition 1.5.14. *A densely defined operator A in a Hilbert space H is said to be symmetric if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all x and y in $D(A)$.*

To conclude this section on bounded operators, we present some useful theorems which will be used later on.

Theorem 1.5.15. *All eigenvalues of a self-adjoint operator on a Hilbert space are real.*

Proof. Let λ be an eigenvalue of a self-adjoint operator, that is $A = A^*$, and $u \neq 0$ be its corresponding eigenvector. So, we have

$$\lambda \langle u, u \rangle = \langle \lambda u, u \rangle = \langle Au, u \rangle = \langle u, Au \rangle = \langle u, \lambda u \rangle = \overline{\lambda} \langle u, u \rangle .$$

Since $||u|| \neq 0$, this gives us that $\lambda = \overline{\lambda}$ which can only happen if λ is real. Thus for a self-adjoint operator the eigenvalues are real. □

Theorem 1.5.16. *A linear operator T is bounded if and only if it is continuous.*

Proof. If $T = 0$, then the proof is trivial. Let $T \neq 0$, then $||T|| \neq 0$. We first assume that T is bounded, that is for some $c > 0$, $||Tx|| \leq c||x||$. Let $\varepsilon > 0$ be given. Pick $\delta = \frac{\epsilon}{||T||}$. Thus,

$$||Tx - Tx_0|| = ||T(x - x_0)|| \leq ||T||\,||x - x_0|| \leq \delta||T|| = \varepsilon,$$

if $||x - x_0|| < \delta$. Hence T is continuous. Now conversely, assume T is continuous. This means that, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $||x - x_0|| \leq \delta$ then $||Tx - Tx_0|| \leq \varepsilon$. Take $y \neq 0$ in the domain of T and set $x = x_0 + \frac{\delta}{||y||}y$, so $x - x_0 = \frac{\delta}{||y||}y$.

$$||x - x_0|| = \Big|\Big|\frac{\delta y}{||y||}\Big|\Big| = ||\delta|| \cdot 1 = \delta.$$

Since $\frac{y}{||y||}$ is a unit vector. Next,

$$||Tx - Tx_0|| = ||T(x - x_0)|| = ||T\big(\frac{\delta y}{||y||}\big)|| = \frac{\delta}{||y||}||Ty|| \leq \varepsilon.$$

Thus,

$$||Ty|| \leq \frac{\varepsilon}{\delta}||y||.$$

Hence we see that T is bounded, with $c = \frac{\varepsilon}{\delta}$ □

Theorem 1.5.17. *Eigenvectors corresponding to distinct eigenvalues of a self-adjoint or unitary operator are orthogonal.*

Proof. For a self-adjoint operator A, let u_1 and u_2 be eigenvectors with distinct λ_1 and λ_2. By an earlier theorem, the eigenvalues are real.

$$\lambda_1 \langle u_1, u_2 \rangle = \langle \lambda_1 u_1, u_2 \rangle = \langle Au_1, u_2 \rangle = \langle u_1, Au_2 \rangle = \langle u_1, \lambda_2 u_2 \rangle = \lambda_2 \langle u_1, u_2 \rangle .$$

So, we have that $(\lambda_1 - \lambda_2) \langle u_1, u_2 \rangle = 0$ which, since the eigenvalues are distinct, we have that $\langle u_1, u_2 \rangle = 0$. Thus the eigenvectors are orthogonal. Now suppose that A is unitary. Let $\lambda_1 \neq \lambda_2$, we have that $\lambda_1 \bar{\lambda}_2 \neq 1$ because if $\lambda_1 \bar{\lambda}_2 = 1$, then $\lambda_2 = \lambda_2 \lambda_1 \bar{\lambda}_2 = \lambda_1 \cdot 1 = \lambda_1$.Thus,

$$\lambda_1 \bar{\lambda}_2 \langle u_1, u_2 \rangle = \langle \lambda_1 u_1, \lambda_2 u_2 \rangle = \langle Au_1, Au_2 \rangle = \langle u_1, A^* Au_2 \rangle = \langle u_1, u_2 \rangle .$$

So, we have that $(\lambda_1 \bar{\lambda}_2 - 1) \langle u_1, u_2 \rangle = 0$, hence $\langle u_1, u_2 \rangle = 0$. Thus the eigenvectors are orthogonal.

□

One thing all these operators are assuming is boundedness. Boundedness is key to many results; unfortunately, many operators are not bounded. Luckily, many results remain the same, though the details get a bit more technical.

1.6 Unbounded Operators

Since many operators are not bounded in L^2-spaces we need to discuss this and the complications that arise. The derivative $\frac{d}{dx}$ and multiplication by x of a function $f(x)$ are unbounded operators. We utilize [11] to explore these technicalities. To deal with this issue, we must limit our domain to one on which the operator is bounded. We introduce our first definition, note the domain.

Definition 1.6.1. *An unbounded operator A on a Hilbert space H is a linear map from a dense subspace $Dom(A) \subset H$ into H.*

Nothing is stopping us from having $Dom(A) = H$ and A being bounded though. To define the adjoint operator, we again must be careful about the domain.

Definition 1.6.2. *For an unbounded operator A on H, the adjoint A^* of A is defined as follows. A vector $\phi \in H$ belongs to the domain $Dom(A^*)$ of A^* if the linear functional $\langle \phi, A\cdot \rangle$ defined on $Dom(A)$ is bounded. For $\phi \in Dom(A^*)$, let $A^*\phi$ be the unique vector χ such that $\langle \chi, \psi \rangle = \langle \phi, A\psi \rangle$ for all $\psi \in Dom(A)$.*

Now since $\langle \phi, A\cdot \rangle$ is bounded and since $Dom(A)$ is dense, we have by the Bounded Linear Transformation Theorem that $\langle \phi, A\cdot \rangle$ has a unique bounded extension to all of H. Finally, since we have this bounded extension we can use the Riesz Representation Theorem to get χ. To further refine on this unbounded notion of adjointness, we need to distinguish between symmetric operators and self-adjoint operators.

Definition 1.6.3. *An unbounded operator A on H is symmetric if $\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$ for all ϕ and ψ in $Dom(A)$. The operator A is self-adjoint if $Dom(A^*) = Dom(A)$ and $A^*\phi = A\phi$ for all ϕ in $Dom(A)$.*

In this unbounded case, we say an operator is self-adjoint if $A = A^*$ and they have the same domain. Clearly, a self-adjoint operator is symmetric, but this is not necessarily true in reverse.

Theorem 1.6.4. *Suppose A is a symmetric operator on H and suppose λ is an eigenvector for A, meaning that $A\psi = \lambda\psi$ for some nonzero $\psi \in Dom(A)$. Then $\lambda \in \mathbb{R}$.*

Proof. The proof is identical to that of Theorem 1.5.15. □

With the mathematical tools we describe in this chapter, we are able to investigate partial differential equations and quantum mechanics with a renewed and powerful perspective.

CHAPTER 2 QUANTUM MECHANICS

2.1 Introduction to the Quantum World

The playground for quantum mechanics are Hilbert spaces. Having a firm grasp on Hilbert spaces and how operators tie in, we aim to integrate that understanding into the realm of quantum mechanics. We cover the postulates of quantum mechanics, many of which relate the physical quantum world to abstract mathematical Hilbert spaces. After giving an introduction of quantum mechanics, we derive the famous Schrödinger Equation, which governs quantum mechanics. We then conclude by working through a famous example, the particle in an infinite square well. See [11], [8], and [10] for a more in-depth look at the quantum world.

2.2 The Postulates of Quantum Mechanics

In this section, we present the basic, fundamental principles of quantum mechanics. The exact number and order of these postulates vary from author to author or textbook to textbook, however, they all fundamentally state the same information.

Postulate 2.2.1 (1 - The State Vector)**.** *Every possible state of a given system corresponds to a separable Hilbert Space over* $\mathbb{C}$*. There is a one-to-one correspondence between nonzero state (of the system) vectors, and its scalar multiple, to nonzero vectors, and its scalar multiple, in the Hilbert Space.*

This postulate is vital as we see that wavefunctions live in Hilbert space. We denote the state vector as $\psi(x)$ or in Dirac notation: $\langle x|\psi\rangle$ or $|\psi(x)\rangle$. The scalar multiple condition can be nullified via a normalization condition, that is:

$$\int \overline{\psi(x)}\psi(x)\,dx = \int |\psi(x)|^2\,dx = \langle\psi|\psi\rangle = 1.$$

Since our state vectors or wavefunctions live in Hilbert space, we will reintroduce an inner product using Dirac notation.

Definition 2.2.2 (Inner Product)**.** *Given $|\phi\rangle$ and $|\psi\rangle$, the inner product is,*

$$\langle\phi|\psi\rangle = \int \overline{\phi(x)}\psi(x)\,dx.$$

Of course we have conjugate symmetry, $\langle\phi|\psi\rangle = \overline{\langle\psi|\phi\rangle}$. We still have a sense of orthogonality as well, two state vectors are orthogonal if $\langle\phi|\psi\rangle = 0$.

Definition 2.2.3 (Orthonormal Set (Of State Vectors))**.** *A set $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$, ... is orthonormal if:*

$$\langle\psi_i|\psi_j\rangle = \delta_{ij} = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{otherwise} \end{cases}.$$

Since Hilbert spaces are in themselves vector spaces we have then any element can be represented as a linear combination of the basis elements.

Definition 2.2.4 (Linear Combination/Quantum Superposition)**.** *A state vector $|\psi\rangle$ can be represented as:*

$$|\psi\rangle = \sum_i c_i\,|\psi_i\rangle$$

where $c_i = \langle\psi_i|\psi\rangle$, which is sometimes an infinite series and convergence of such a series must be considered.

All that can possibly be known comes from $\Psi(x,t)$.

Postulate 2.2.5 (2 - Observable Operators and Their Values)**.** *1. To every physical observable (position, momentum, energy, etc) there corresponds a Hermitian operator A, which has a complete set of orthonormal eigenvectors $\{\psi_n\}$ with corresponding $\{\lambda_n\}$ such that $A\psi_n = \lambda_n\psi_n$. Conversely, every such operator in the Hilbert space corresponds to some physical observable.*

2. The only possible values of a physical observable are the various eigenvalues.

This is where we get the "quantum" for quantum mechanics, the observables are quantized taking only this discrete set of eigenvalues. Further, the fact that A is Hermitian guarantees the fact that our eigenvalues are real values which makes sense. It would be strange to measure complex values for velocity or momentum. In fact, in an earlier section it was shown that the eigenvalues of a Hermitian operator are real.

Definition 2.2.6 (Commutator)**.** *Given two operators A and B we define the commutator as*

$$[A, B] = AB - BA.$$

So if $[A, B] = 0$ then A and B commute. Similarly, we have the anticommutator,

Definition 2.2.7 (Anticommutator)**.** *Given two operators A and B we define the anticommutator as*

$$\{A, B\} = AB + BA.$$

Physically, if the commutator is not 0, the value is associated with the magnitude of the unavoidable disturbances between the two measurements A and B. Here a measurement is a well-defined operation which yields a single real number with no error or ambiguity. One big axiom of quantum mechanics is that a measurement disturbs the system.

This leads us to a version of the famous Heisenberg Uncertainty Principle, given the position operator $x = x$ and the momentum operator $p = -i\hbar\frac{\partial}{\partial x}$, we can examine the commutator and some state ψ.

$$[x, p]\psi = (xp - xp)\psi = -i\hbar x\frac{\partial \psi}{\partial x} + ih\frac{\partial}{\partial x}(x\psi).$$

$$= -i\hbar x\frac{\partial \psi}{\partial x} + ih\psi + i\hbar x\frac{\partial \psi}{\partial x} = i\hbar\psi.$$

Thus, $[x, p] = i\hbar$. Mathematically we see that the position and momentum operators do not commute and physically we can not simultaneously know both the momentum and position. This leads us to our next definition.

Definition 2.2.8 (Complementary)**.** *Two observables are complementary if the corresponding operators do not commute.*

We just saw that position and momentum are complementary since the operators do not commute. Mathematically we have that the simultaneous eigenstates of x and p do not exist.

Postulate 2.2.9 (3 - Correspondence Principle)**.** *A quantum observable corresponding to a dynamical variable is obtained by replacing the canonical variable in classical mechanics by its corresponding quantum mechanical operator.*

Below is a brief table of the one-dimensional classical observable and the quantum operator. To extend to a higher dimension replace the partial derivative with the del operator.

	Classical Quantity	Quantum Operator
Position	x	x
(Linear) Momentum	$m\frac{d}{dt}$	$-i\hbar\frac{\partial}{\partial x}$
Potential Energy	$V(r,t)$	$V(r,t)$
Kinetic Energy T	$\frac{p^2}{2m}$	$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

Table 2.1 Table of Classical and Quantum Operators

However, there are quantum observables such as spin that have no classical analogue.

Postulate 2.2.10 (4 - Quantization)**.** *Every pair of canonically conjugate observable operators satisfies the following Heisenberg commutation relations:* $[q_m, q_n] = 0 = [p_m, p_n]$ *and* $[q_m, p_n] = i\hbar\delta_{mn}$*, where* q_m *is the observable operator corresponding to the generalized coordinates, and* p_m *is the momentum corresponding to the generalized momentum.*

We have already seen an example of this in one-dimension, the Heisenberg Uncertainty Principle: $[p, x] = i\hbar$.

One main way of looking at quantum mechanics is through a statistic approach. The remaining postulates have more to do with the statistical approach.

Definition 2.2.11 (Expectation Value)**.** *The expectation value* $\langle A\rangle$ *of an observable operator* A *in the state* $\psi(x)$ *of a physical system is defined by*

$$\langle A\rangle = \frac{\langle\psi|A\psi\rangle}{\langle\psi|\psi\rangle}.$$

Or simply, $\langle A\rangle = \langle\psi|A\psi\rangle$ *if* ψ *is normalized.*

We can think of the expectation value as roughly the average value to be taken of that quantum measurement. This may not necessarily mean most probable measurement though. We have a similar analogue comparable to standard deviation.

Definition 2.2.12 (Root-Mean-Square Deviation)**.** *The root-mean-square deviation* (∇A) *is defined by the square root of the expectation value of* $(A - \langle A\rangle)^2$ *in the state* ψ *in which* $\langle A\rangle$ *is computed.*

Then since A is Hermitian and $\langle A \rangle$ is real, observe:

$$(\nabla A)^2 = \langle (A - \langle A \rangle)^2 \rangle = \langle \psi | (A - \langle A \rangle)^2 \psi \rangle = \langle (A - \langle A \rangle)\psi | (A - \langle A \rangle)\psi \rangle$$

$$= ||(A - \langle A \rangle)\psi||^2.$$

Theorem 2.2.13. *(Root Mean Square Deviation Calculations)*

1. $(\nabla A)^2 = \langle A^2 \rangle - \langle A \rangle^2$
2. $\langle A^2 \rangle = ||A\psi||^2$

Proof. For 1, we use linearity of the inner product, the fact that $\langle A \rangle$ is real, and that ψ is normalized.

$$(\nabla A)^2 = \langle (A - \langle A \rangle)^2 \rangle = \langle \psi | (A - \langle A \rangle)^2 \psi \rangle = \langle \psi | (A^2 - 2A \langle A \rangle + \langle A \rangle^2)\psi \rangle$$

$$= \langle \psi | A^2 \psi \rangle - 2 \langle \psi | A \langle A \rangle \psi \rangle + \langle \psi | \langle A \rangle^2 \psi \rangle$$

$$= \langle \psi | A^2 \psi \rangle - 2 \langle A \rangle \langle \psi | A\psi \rangle + \langle A \rangle^2 \langle \psi | \psi \rangle$$

$$= \langle A^2 \rangle - 2 \langle A \rangle \langle A \rangle + \langle A \rangle^2 = \langle A^2 \rangle - 2 \langle A \rangle^2 + \langle A \rangle^2$$

$$= \langle A^2 \rangle - \langle A \rangle^2 .$$

For 2, we use the fact A is Hermitian.

$$\langle A^2 \rangle = \langle \psi | A^2 \psi \rangle = \langle A\psi | A\psi \rangle = ||A\psi||^2.$$

□

Theorem 2.2.14. *A necessary and sufficient condition for a physical system to be in an eigenstate of an observable A is $\nabla A = 0 = \sqrt{\langle (A - \langle A \rangle)^2 \rangle}$.*

Proof. For necessity, suppose $A\psi = \lambda\psi$ for some $\lambda \in \mathbb{R}$.

$$\langle A \rangle = \langle \psi | A\psi \rangle = \langle \psi | \lambda\psi \rangle = \lambda \langle \psi | \psi \rangle = \lambda.$$

$$\langle A^2 \rangle = \langle \psi | A^2\psi \rangle = \langle \psi | \lambda A\psi \rangle = \langle \psi | \lambda^2\psi \rangle = \lambda^2 \langle \psi | \psi \rangle = \lambda^2.$$

For sufficiency, suppose $\nabla A = 0$.

$$0 = (\nabla A)^2 = ||(A - \langle A \rangle)^2\psi||^2.$$

Since we are in a normed space, the norm is only 0 if and only if the element is zero hence,

$$(A - \langle A \rangle)\psi = 0 \implies A\psi - \langle A \rangle \psi = 0$$

or

$$A\psi = \langle A \rangle \psi.$$

Thus ψ is an eigenvector with eigenvalue $\langle A \rangle$. □

Postulate 2.2.15 (5 - Outcome of Quantum Measurement)**.** *If an observable operator A has eigenbasis $\{\psi_n\}$ with the corresponding $\{\lambda_n\}$, then the probability that a measurement will yield the eigenvalue λ_n of A of the system in the normalized state $\psi(x)$ is $P(\lambda_n) = |\langle \psi_n | \psi \rangle|^2$.*

There are a few consequences of this postulate. First, it is impossible to predict with absolute certainty the outcome of a measurement that is made on a quantum system in a completely defined state. Quantum mechanics is statistical in nature while classical mechanics is deterministic in nature. Second, when in state $\psi(x)$, the value obtained in measurement of an observable operator A is a random variable with a probability distribution whose mean value is associated with a large number of measurements in this particular state. Finally, if two repeated measurements of the same observable are performed on a given state, the results will not necessarily be the same. This is in contrary to classical mechanics.

If the state of the system is in $\psi_n(x)$, a pure eigenvector of A, with corresponding λ_n, then

$$\langle A \rangle = \frac{\langle \psi_n | A\psi_n \rangle}{\langle \psi_n | \psi_n \rangle} = \frac{\langle \psi_n | \lambda_n \psi_n \rangle}{\langle \psi_n | \psi_n \rangle} = \lambda_n.$$

Thus, the state is a pure eigenstate, we will always measure the outcome to be λ_n. Thus we see the expected value of an eigenstate is the corresponding eigenvalue. If $\psi(x)$ is not an eigenvector then it can be expressed as a linear combination of the eigenvectors, since the eigenvectors are a complete orthonormal set.

$$\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) = \sum_{n=1}^{\infty} \langle \psi_n | \psi \rangle \, \psi_n(x).$$

Postulate 2.2.16. *(6 - Hamiltonian Operator and Schrödinger's Equation)*

1. *Hamiltonian Operator:*

   For every physical system there exists a linear Hermitian operator H, the Hamiltonian operator which represents the observable operator corresponding to the total energy of the state vector of the system.

2. *Schrödinger's Equation:*

   If a physical system is not disturbed by any experiment, the Hamiltonian operator H determines the time development of the state vector of the system $\Psi(\vec{r}, t)$ through the partial differential equation

   $$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi(\vec{r}, t).$$

Theorem 2.2.17. *If $\hat{A}$ and $\hat{B}$ are Hermitian operators, then for any state vector,*

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}\Big|\frac{1}{i}\langle [\hat{A}, \hat{B}] \rangle \Big|.$$

Proof. Denote: $\Delta\hat{A} = ||\psi_1||$ and $\Delta\hat{B} = ||\psi_2||$, where $\psi_1 = \big(\hat{A} - \langle \hat{A} \rangle\big)\psi$ and $\psi_2 = \big(\hat{B} - \langle \hat{B} \rangle\big)\psi$.

Now by the Cauchy-Schwarz's Inequality,

$$\Delta\hat{A}\Delta\hat{B} = ||\psi_1||||\psi_2|| \geq |\langle\psi_1|\psi_2\rangle| \geq |\Im\langle\psi_1|\psi_2\rangle|.$$

Now using $\Im(z) = \frac{1}{2}(z - \bar{z})$ for complex numbers.

$$|\Im\langle\psi_1|\psi_2\rangle| = \Big|\frac{1}{2i}\big(\langle\psi_1|\psi_2\rangle - \overline{\langle\psi_1|\psi_2\rangle}\big)\Big| = \Big|\frac{1}{2i}\big(\langle\psi_1|\psi_2\rangle - \langle\psi_2|\psi_1\rangle\big)\Big|$$

$$= \Big|\frac{1}{2i}\big(\langle\big(\hat{A} - \langle\hat{A}\rangle\big)\psi|\big(\hat{B} - \langle\hat{B}\rangle\big)\psi\rangle - \langle\big(\hat{B} - \langle\hat{B}\rangle\big)\psi|\big(\hat{A} - \langle\hat{A}\rangle\big)\psi\rangle\big)\Big|$$

$$= \Big|\frac{1}{2i}\big[\langle\psi|\big(AB - A\langle B\rangle - \langle A\rangle B + \langle A\rangle\langle B\rangle\big)\psi\rangle - \langle\psi|\big(BA - B\langle A\rangle - \langle B\rangle A + \langle B\rangle\langle A\rangle\big)\psi\rangle\big]\Big|$$

$$= \Big|\frac{1}{2i}\langle\psi|\big(AB - BA\big)\psi\rangle\Big| = \Big|\frac{1}{2i}\langle\psi|\big[A, B\big]\psi\rangle\Big| = \Big|\frac{1}{2i}\langle\psi|\big[A, B\big]\psi\rangle\Big| = \frac{1}{2}\Big|\frac{1}{i}\langle\psi|\big[A, B\big]\psi\rangle\Big|.$$

Thus,

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}\Big|\frac{1}{i}\langle[A, B]\rangle\Big|.$$

□

2.3 Schrödinger Equation

In this section a derivation of the Schrödinger Equation is given. Often, even in physics literature, the derivation of this famous equation is skipped and presented as fact or postulate. However, using some modern physics and Maxwell's Equations in free space we can derive it. We start with Maxwell's equations.

$$\nabla \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}.$$

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}.$$

$$\nabla \cdot \vec{E} = 0.$$

$$\nabla \cdot \vec{B} = 0.$$

We start by using the first equation and applying the del operator to both sides:

$$\nabla \times \big(\nabla \times \vec{E}\big) = \nabla \times \Big(- \frac{\partial \vec{B}}{\partial t}\Big).$$

On the left hand side of this equation we use the true triple vector product order for non-commutative vectors and $\nabla \cdot \vec{E} = 0$.

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - (\nabla \cdot \nabla)\vec{E} = 0 - \nabla^2 \vec{E} = -\nabla^2 \vec{E}.$$

Now on the right hand side of the original equation, we can take the derivative and constant out of the cross product. Next, we can substitute Maxwell's second equation in and take the derivative.

$$\nabla \times \Big(- \frac{\partial \vec{B}}{\partial t}\Big) = -\frac{\partial}{\partial t}\big(\nabla \times \vec{B}\big) = -\frac{\partial}{\partial t}\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}.$$

Now, by equating the left and the right side and multiplying by -1 we obtain the electromagnetic wave equation in three-dimensions.

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}.$$

In one-dimension this becomes:

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}.$$

One can easily verify that $E(x,t) = E_0 e^{i(kx-\omega t)}$ is a solution.

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E_0} e^{i(kx-\omega t)} = 0.$$

$$\left(- k^2 + \frac{\omega^2}{c^2}\right)\vec{E_0} e^{i(kx-\omega t)} = 0.$$

Upon taking a non-trivial amplitude for the wave we find the dispersion relation of light in free space:

$$k = \pm\frac{\omega}{c}.$$

Now recall that Einstein says that $\varepsilon = \hbar\omega$ and De Broglie says that $p = \hbar k$. We can rewrite our solution as:

$$E(x,t) = E_0 e^{\frac{i}{\hbar}(px-\varepsilon t)}.$$

Again, we can verify this is a solution to the wave equation and find that:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_0 e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0$$

$$-\frac{1}{\hbar^2}\left(p^2 - \frac{\varepsilon}{c^2}\right)\vec{E}_0 e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0.$$

Taking the non-trivial solution again we can find the massless relativistic energy $\varepsilon^2 = p^2c^2$. This is comforting as electromagnetic waves, light, is massless. Let us now reverse engineer this solution for a particle with mass. We now go to relativistic energy with mass $\varepsilon^2 = p^2c^2 + m^2c^4$. Since we are no longer dealing with electromagnetic waves, let us now call our function a *wave* function, Ψ, (as opposed to E) that is a function that came from a wave equation.

$$-\frac{1}{\hbar^2}\left(p^2 - \frac{\varepsilon}{c^2} + m^2c^2\right)\Psi_0 e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0.$$

Still reverse engineering this equation we get:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\right)\Psi_0 e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0$$

With an analogy to Electromagnetics, recall that number of protons, S, where μ_0 is the permeability of free space constant is given by:

$$S = \frac{1}{c\mu_0}E^2.$$

We demand that our wave function be normalizable, that is the particle must be found somewhere.

$$\int_{-\infty}^{\infty} \overline{\Psi}\Psi \, dx = 1.$$

Now simply using this normalization condition, we obtain the Klein-Gordon equation for a free particle in one-dimension, which is a relativistic equation.

$$\frac{\partial^2\Psi}{\partial x^2} - \frac{m^2c^2}{\hbar^2}\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}.$$

Now going back into three-dimensions is easy, the partial simply becomes a Laplacian.

$$\nabla^2\Psi - \frac{m^2c^2}{\hbar^2}\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}.$$

This is the Klein-Gordon equation in three-dimensions, which will describe any spinless particle, such as the Higgs Boson. The Klein-Gordon Equation describes orbital momentum. From here, let us make a non-relativistic equation. Using Einstein's energy relation, a Taylor series expansion, and recalling that kinetic energy is $T = \frac{p^2}{2m}$:

$$\varepsilon = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}}$$

$$\approx mc^2\left(1 + \frac{p^2}{2m^2c^2}\right) = mc^2 + \frac{p^2}{2m} = mc^2 + T.$$

Using this approximation into our wave function we find:

$$\Psi = \Psi_0 e^{\frac{i}{\hbar}(px-\varepsilon t)} = \Psi_0 e^{\frac{i}{\hbar}(px-mc^2t-Tt)} = \Psi_0 e^{-\frac{i}{\hbar}mc^2t} e^{\frac{i}{\hbar}(px-Tt)}.$$

Now because we are assuming non-relativistic velocities, i.e. $mv \ll mc$ and $p^2 \ll m^2c^2$ observe that the first exponential oscillates quickly and the second one slowly, let us call the slow one ϕ. Thus, $\Psi = \Psi_0 e^{-\frac{i}{\hbar}mc^2t}\phi$. Let us take the second partial with respect to t and then substitute this into the one-dimensional Klein-Gordon Equation.

$$\frac{\partial\Psi}{\partial t} = -\frac{i}{\hbar}mc^2 e^{-\frac{i}{\hbar}mc^2t}\phi + e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\phi}{\partial t}.$$

$$\frac{\partial^2\Psi}{\partial t^2} = \left(-\frac{m^2c^4}{\hbar^2}e^{-\frac{i}{\hbar}mc^2t}\phi - \frac{i}{\hbar}mc^2e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\phi}{\partial t}\right) + \left(-\frac{i}{\hbar}mc^2e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\phi}{\partial t} + e^{-\frac{i}{\hbar}mc^2t}\frac{\partial^2\phi}{\partial t^2}\right)$$

$$= \left(-\frac{m^2c^4}{\hbar^2}e^{-\frac{i}{\hbar}mc^2t}\phi - \frac{2i}{\hbar}mc^2e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\phi}{\partial t}\right) + e^{-\frac{i}{\hbar}mc^2t}\frac{\partial^2\phi}{\partial t^2}.$$

The term in the parentheses is very large and the second term is very small, so neglecting the second term and using these partials in the Klein-Gordon equation we obtain

$$\frac{\partial^2\Psi}{\partial x^2} - \frac{m^2c^2}{\hbar^2}\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{1}{c^2}\left(-\frac{m^2c^4}{\hbar^2}e^{-\frac{i}{\hbar}mc^2t}\phi - \frac{2i}{\hbar}mc^2e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\phi}{\partial t}\right).$$

Putting Ψ into this and rearranging we obtain:

$$e^{-\frac{i}{\hbar}mc^2t}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{2im}{\hbar}\frac{\partial\phi}{\partial t}\right) = 0.$$

The exponential will never be equal to 0, so divide by it and then rearranging we obtain:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{2im}{\hbar}\frac{\partial\phi}{\partial t} = 0$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} = i\hbar\frac{\partial\phi}{\partial t}.$$

This is the one-dimensional Schödinger Equation for a free particle; that is, it has zero potential energy. We can again generalize to three-dimensions by replacing the 2nd partial of ϕ with respect to x with a Laplacian. Notice, though that $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ is the total energy of a particle, and as it has no potential it is pure kinetic energy. If we instead make this the total energy for a nonzero potential we get $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)$. Putting this into the previous equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + V(x,t)\phi = i\hbar\frac{\partial\phi}{\partial t}.$$

Generalizing this to three-dimensions we obtain the time-dependent Schödinger Equation:

$$\frac{-\hbar^2}{2m}\nabla^2\phi + V(x,y,z,t)\phi = i\hbar\frac{\partial\phi}{\partial t}.$$

For historical reasons ϕ has been replaced with Ψ, so many people are used to seeing the time-dependent Schödinger Equation as:

$$\boxed{\frac{-\hbar^2}{2m}\nabla^2\Psi + V(x,y,z,t)\Psi = i\hbar\frac{\partial\Psi}{\partial t}.}$$

Finally, I end with giving the one-dimensional time-independent Schrödinger Equation, as this is the equation to solve to give us the basis for the quantum mechanical system we are dealing with.

$$\boxed{\frac{-\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V(x)\Psi = E\Psi.}$$

2.4 Example: Infinite Square Well

In this section we consider the prototypical example to examine some quantum theory, the infinite square well, sometimes called a particle in a box. This simple one-dimensional theoretical scenario will be plenty to get us started. First, consider the one-dimensional Schrödinger Equation,

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial\psi}{\partial t} = E\psi.$$

We will consider

$$V(x) = \begin{cases} 0, & 0 \leq x \leq L \\ C, & \text{otherwise} \end{cases},$$

with a finite C we expect $\psi(x)$ to decay very quickly to zero outside of the box, outside of $0 \leq x \leq L$ that is. When we take the limit as $C \to \infty$, we obtain:

$$V(x) = \begin{cases} 0, & 0 \leq x \leq L \\ \infty, & \text{otherwise} \end{cases},$$

here the particle will not be able to escape the box. The particle will never have enough energy to be outside the box. Thus, $\psi(x) \equiv 0$ outside the box; this is the potential function we will use.

Since $\psi(x) \equiv 0$ outside the box, we only need to use $V(x) = 0$ inside the box to get solve the Schrödinger Equation.

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi.$$

At this point, physically we expect that the energy levels will be quantized and could impose E_n and ψ_n here but, I will wait and show that this appears naturally through mathematics. We now have a standard second order differential equation that can be solved quite easily.

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi.$$

Let $k = \frac{\sqrt{2mE}}{\hbar}$ so our equation reduces to:

$$\frac{d^2\psi}{dx^2} = -k^2\psi,$$

which has the general solution, (also the simple harmonic oscillator solution):

$$\psi(x) = A\sin(kx) + B\cos(kx).$$

To determine A and B we need boundary conditions. We require $\psi(x)$ to be continuous, thus on the edges of our box, $\psi(0) = \psi(L) = 0$. Using $\psi(0) = 0$, we get that $B = 0$. Using $\psi(L) = 0$ we get:

$$\psi(L) = A\sin\big(\frac{\sqrt{2mE}L}{\hbar}\big).$$

Taking $A = 0$ give the trivial solution, so taking $A \neq 0$ we require:

$$\frac{\sqrt{2mE}L}{\hbar} = n\pi,$$

for integer n. If $n = 0$ we simply get the trivial solution again, so $n \neq 0$. If n is a negative integer we can simply absorb the negative into the constant A since sine is an odd function. Thus n takes

the value of positive integers. We can now solve explicitly for our now quantized energy levels:

$$E_n = \frac{n^2\hbar^2\pi^2}{2mL^2} = n^2 E_1,$$

notice the only variable is n, so all our energy levels are multiples of E_1, which is the ground state. Thus we have our eigenstates:

$$\psi_n(x) = A\sin\left(\frac{n\pi x}{L}\right).$$

Even with our boundary conditions we were still unable to determine A, we require that our wave function be normalized.

$$1 = \int_0^L \psi_n^2(x)\,dx = \int_0^L A^2\sin^2\left(\frac{n\pi x}{L}\right)dx.$$

Routine integration gives us $A = \sqrt{\frac{2}{L}}$, thus our normalized eigenstates or eigenvectors or stationary states are:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right),$$

with $n \in \mathbb{Z}^+$. Below are the first four eigenstates.

There are some properties to be observed here. First, notice that the eigenstates are alternatingly even and odd functions, this is due to that fact that our potential V is symmetric. Another observation is that for each energy level we go up, a new nodal point is introduced. Another remarkable fact is that each eigenstate is mutually orthogonal; that is, $\langle\psi_m, \psi_n\rangle = \delta_{mn}$. Furthermore, the eigenstates are a complete basis, so we have that any function $f(x)$ can be given by $\sum_{n=1}^{\infty} c_n\psi_n$. Finally, putting these last two observations together we see that the eigenstates form an orthonormal basis for $L^2([0, L])$. While H for $H\psi_n = \lambda_n\psi_n$ is unbounded, we have that H is a symmetric operator, thus $\lambda_n \in \mathbb{R}$, which we see. To get the time-evolution of the system we append an

Figure 2.1 Eigenstates $n = 1$ and $n = 2$

Figure 2.2 Eigenstates $n = 3$ and $n = 4$

exponential factor to our eigenstate.

$$\psi_n(x,t) = \sqrt{\frac{2}{L}} \sin\big(\frac{n\pi x}{L}\big) e^{\frac{-iE_n t}{\hbar}} = \sqrt{\frac{2}{L}} \sin\big(\frac{n\pi x}{L}\big) e^{\frac{-in^2\pi^2\hbar t}{2mL^2}}.$$

To get the most general solution, we take advantage of the completeness of the basis, thus the most general solution is a linear combination of the eigenstates.

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x,t).$$

We will now consider $\psi_1(x) = \sqrt{\frac{2}{L}} \sin\big(\frac{\pi x}{L}\big)$, and examine some expectation values $\langle A \rangle = \langle \psi_1 | A \psi_1 \rangle$. First, $A = x$, the position operator:

$$\langle x \rangle = \langle \psi_1 | x \psi_1 \rangle = \int_0^L \frac{2}{L} x \sin^2\big(\frac{\pi x}{L}\big)\, dx = \frac{L}{2}.$$

This should not be surprising as this is the halfway point! Likewise we find its square,

$$\langle x^2 \rangle = \langle \psi_1 | x^2 \psi_1 \rangle = \int_0^L \frac{2}{L} x^2 \sin^2\big(\frac{\pi x}{L}\big)\, dx = \frac{1}{6}\Big(2 - \frac{3}{\pi^2}\Big) L^2.$$

We now get the expectation value for momentum and its square.

$$\langle p^2 \rangle = \langle \psi_1 | p^2 \psi_1 \rangle = \hbar^2 \int_0^L \frac{2}{L} \sin\big(\frac{\pi x}{L}\big) \sin\big(\frac{\pi x}{L}\big) \frac{\pi^2}{L^2}\, dx = \frac{\hbar^2 \pi^2}{L^2}$$

Now recall our standard deviation or "uncertainty" in an operator,

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}.$$

We compute Δx and Δp now.

$$\Delta x = \sqrt{\frac{1}{6}\Big(2 - \frac{3}{\pi^2}\Big) L^2 - \frac{L^2}{4}} = \sqrt{\frac{(\pi^2 - 6) L^2}{12\pi^2}} = \frac{\sqrt{\pi^2 - 6}\, L}{2\sqrt{3}\pi}.$$

$$\Delta p = \sqrt{\frac{\hbar^2\pi^2}{L^2} - 0^2} = \frac{\hbar\pi}{L}.$$

We now have all the ingredients to verify the Heisenberg Uncertainty Principle! Recall $\Delta x \Delta p \geq \frac{\hbar}{2}$.

$$\frac{\sqrt{\pi^2 - 6}\, L}{2\sqrt{3}\pi} \frac{\hbar\pi}{L} = \frac{\sqrt{\pi^2 - 6}\, \hbar}{2\sqrt{3}} \approx 0.567\hbar \geq \frac{\hbar}{2}.$$

As expected we can only be so certain about these expectation values.

CHAPTER 3 RIEMANN ZETA FUNCTION AND RIEMANN HYPOTHESIS

3.1 Introduction

In this chapter, we will explore some various properties of the famous Riemann Zeta Function, the Riemann Hypothesis, and some ideas involving the previous two chapters of mathematical and physical ideas in attempts to solve it. For an examination and historical perspective of Riemann's Zeta function we examine Edwards's This text *Riemann's Zeta Function* [9], provides valuable insight into the original paper by Riemann and much of its development.

3.2 Riemann Zeta Function and Some Elementary Properties

The Riemann Zeta Function is one of the most studied functions in mathematics right now and has been for well over the past century.

Definition 3.2.1 (Riemann Zeta Function)**.** *Let $s = \sigma + i\tau$ be a complex number such that $\Re(s) > 1$, thus the Riemann Zeta Function is:*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Euler had first introduced it for real numbers, namely positive integers, and later Riemann had altered it to work for complex numbers as well. Euler had found some useful identities such as:

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6},$$

$$\zeta(4) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}, \text{ and}$$

$$\zeta(1) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$$

Another wonderful formula that Euler had found was his product formula.

Theorem 3.2.2 (Euler Product Formula)**.**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Proof. We first examine each individual term of this product.

$$\left(1 - \frac{1}{p_i^s}\right)^{-1} = \frac{1}{1 - \frac{1}{p_i^s}} = 1 + \frac{1}{(p_i^s)^1} + \frac{1}{(p_i^s)^2} + \frac{1}{(p_i^s)^3} + \cdots$$

This is a geometric series which converges, since $\frac{1}{p_i} < 1$. Upon multiplying each of these terms out, we get terms that take the form:

$$\frac{1}{\left(p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4} \cdots p_r^{n_r}\right)^s} = \frac{1}{n}$$

for some integers n_i and some integer r. By the fundamental theorem of arithmetic, we see that these are just unique factorizations of integers n. Hence,

$$\prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

□

We now present a second proof using a different method.

Proof.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots$$

Now we subtract the second line from the first.

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \cdots$$

$$\frac{1}{3^s}\big(1-\frac{1}{2^s}\big)\zeta(s)=\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\cdots$$

We now subtract this second line from the first.

$$\big(1-\frac{1}{3^s}\big)\big(1-\frac{1}{2^s}\big)\zeta(s)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\cdots$$

We keep repeating this process for all the prime numbers, this is a sieving process. We are left with:

$$\cdots\big(1-\frac{1}{7^s}\big)\big(1-\frac{1}{5^s}\big)\big(1-\frac{1}{3^s}\big)\big(1-\frac{1}{2^s}\big)\zeta(s)=1$$

$$\zeta(s)=\frac{1}{\big(1-\frac{1}{2^s}\big)\big(1-\frac{1}{3^s}\big)\big(1-\frac{1}{5^s}\big)\big(1-\frac{1}{7^s}\big)\cdots}=\prod_p\big(1-\frac{1}{p^s}\big)^{-1}.$$

Thus,

$$\prod_p\big(1-\frac{1}{p^s}\big)^{-1}=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

□

Riemann takes Euler's Product Formula and extends it to a complex variable s. To do so, requires the factorial functional and its generalization. The factorial functional is $n!=n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$, to generalize this we use an integral:

$$n!=\int_0^{\infty}e^{-x}x^n\,dx,$$

which utilizing Gauss's formula, this works for all complex s such that $\Re(s)>-1$,

$$\Pi(s)=\int_0^{\infty}e^{-x}x^s\,dx.$$

Unfortunately, due to Legendre the more famous Gamma Function is typically used which shifts the argument over, using this following relation $\Gamma(s)=\Pi(s-1)$ we obtain,

$$\Gamma(s)=(s-1)!=\int_0^{\infty}e^{-x}x^{s-1}\,dx,$$

for $\Re(s) > 0$. To derive the Riemann Zeta Function, Riemann takes $\Pi(s-1)$ and substitutes nx for x,

$$\Pi(s-1) = \int_0^\infty e^{-x} x^{s-1}\, dx \implies \frac{\Pi(s-1)}{n^s} = \int_0^\infty e^{-nx} x^{s-1}\, dx$$

this is valid where $s > 0$. Now summing over n, utilizing $\sum r^{-n} = (r-1)^{-1}$, and the Monotone Convergence Theorem ([6]) we obtain:

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \Pi(s-1) \sum_{n=1}^{\infty} \frac{1}{n^s}$$

From here, we again utilize the $\Gamma(s) = \Pi(s-1)$ relation and state the Riemann Zeta Function as it appears above.

Definition 3.2.3 (Riemann Zeta Function)**.** *Let $s = \sigma + i\tau$ be a complex number such that $\Re(s) > 1$, we define the Riemann Zeta Function to be:*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx.$$

Finally, to get the Riemann Zeta function to be defined on the entire complex plane except for a simple pole at $s = 1$, we use Riemann's Functional Equation which relates the values at s and $1 - s$. Using Riemann's notation we have:

$$\zeta(s) = \Pi(-s)(2\pi)^{s-1} 2 \sin\left(\frac{s\pi}{2}\right) \zeta(1-s).$$

However, again today it is typically stated using the Gamma Function.

Definition 3.2.4 (Riemann Zeta Functional Equation)**.**

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s) \zeta(1-s).$$

Finally we can start looking at some particular values of the Riemann Zeta Function.

With the Functional Equation in mind we can now start looking for zeroes of the Riemann Zeta

s	0	1	2	3	4	$-2n$	-1	-3	∞
$\zeta(s)$	$\frac{1}{2}$	∞	$\frac{\pi^2}{6}$	≈ 1.202 (Apery's Constant)	$\frac{\pi^4}{90}$	0	$-\frac{1}{12}$	120	1

Table 3.1 Values of the Riemann Zeta Function

Function. When $s = -2n$ where n is a positive integer, we have a trivial zero. This is due to the sine term being zero in the Functional Equation. However, where can other zeroes be?

3.3 Riemann Hypothesis & Applications

For well over a century the question of where all the non-trivial zeros of the Riemann Zeta function has baffled mathematicians. The Clay Institute of Mathematics has offered a one million dollar reward to someone who can provide a complete and accurate proof of the hypothesis. As previously stated, the Riemann Zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which is an absolutely convergent series for $\Re(s) > 1$. This can be analytically extended to $\mathbb{C}$ and per Riemann extends to a meromorphic function with only a simple pole at $s = 1$. It satisfies the Functional Equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s)\zeta(1-s).$$

From [19], we have that since $\zeta(s) \neq 0$ for all s such that $\Re(s) > 1$ and the functional equation all the non-trivial zeros of the Riemann Zeta Function lie between $\Re(s) = 0$ and $\Re(s) = 1$ this is known as the critical strip. However, Riemann finds it "very likely" that all non-trivial zeros lie on the vertical line $s = \frac{1}{2} + it$, where t is a real number. This is perhaps the most famous problem in all of analytic number theory. [16]

Theorem 3.3.1 (Riemann Hypothesis)**.** *The non-trivial zeros of the Riemann Zeta Function* $\zeta(s)$ *have real part equal to* $\frac{1}{2}$.

That is, all the non-trivial zeros are on the vertical line $s = \frac{1}{2} + it$, where t is a real number. This has been checked for the first $10,000,000,000$ solutions! Yet, no concrete proof has been

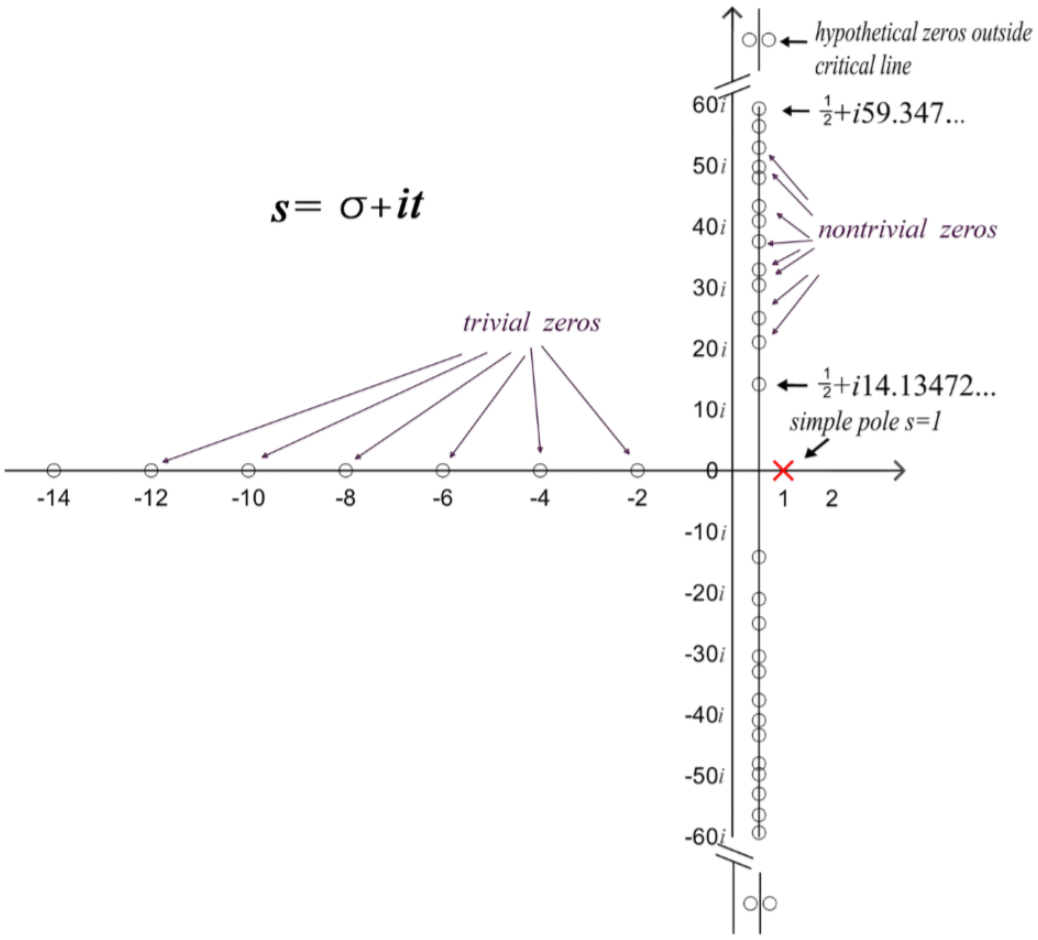

Figure 3.1 A graph of the Complex Plane showing the location of the zeros of Riemann Zeta Function and the pole of $\zeta(s)$ at $s = 1$. (Source: [19])

provided. We also have that if s is a zero of the Zeta function then so is $\bar{s}$, this just goes to show zeros can still come in complex conjugate pairs. Due to the functional equation, we have that if s is a zero then so are $1 - s$ and $1 - \bar{s}$.

Solving the Riemann hypothesis, would lead to many far reaching consequences. Notably, the connection to the distribution of prime numbers. Since,

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

we have an immediate connection to the prime numbers. There are some slight modifications to make to this relation. Taking the log of both sides of the previous equality, and then using the log

series $\log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$ we obtain:

$$\log(\zeta(s)) = \sum_p \Big(\sum_n \frac{1}{n} p^{-ns} \Big).$$

Since this is an absolutely convergent series the order of summation does not matter. Rewriting this as a Stieltjes integral, we obtain:

$$\log(\zeta(s)) = \int_0^\infty x^{-s}\, dJ(x),$$

where

$$J(x) = \frac{1}{2}\Big(\sum_{p^n < n} \frac{1}{n} + \sum_{p^n \leq n} \frac{1}{n} \Big).$$

Integration by parts yields,

$$\log(\zeta(s)) = s \int_0^\infty J(x) x^{-s-1}\, dx,$$

since $J(x) = 0$ for $x < 2$ and since $J(x) < x$ for all x. From here we introduce the prime counting function $\pi(x)$, it is simply the count of all the primes less than x. There is a connection to $\pi(x)$ and $J(x)$, since the number of prime nth powers p^n less than x is $\pi(x^{\frac{1}{n}})$, it follows

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots .$$

Using the Möbius inversion formula one finds:

$$\pi(x) = J(x) - \frac{1}{2}J(x^{\frac{1}{2}}) - \frac{1}{3}J(x^{\frac{1}{3}}) - \frac{1}{5}J(x^{\frac{1}{5}}) + \frac{1}{6}J(x^{\frac{1}{6}}) + \cdots + \frac{\mu(n)}{n}J(x^{\frac{1}{n}}) + \cdots , \quad (3.3.2)$$

where $\mu(n) = 0$ if n is divisible by a prime square, $\mu(n) = 1$ is n is a product of an even number of distinct primes, and $\mu(n) = -1$ is n is a product of an odd number of distinct primes. Using an

analytic formula for $J(x)$,

$$J(x) = Li(x) - \sum_{p} Li(x^p) - \ln(2) + \int_x^{\infty} \frac{dt}{t(t^2-1)\ln(t)}$$

and noting this is a finite series for any fixed x we obtain an analytic formula for $\pi(x)$.

3.4 A Brief History of $\pi(x)$

There has been a long interest in $\pi(x)$ for mathematicians. In the late 18th century, Gauss and Legendre had both independently formulated that:

Theorem 3.4.1 (Prime Number Theorem)**.**

$$\pi(x) \sim \frac{x}{\ln(x)},$$

which more precisely means that

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln(x)}} = 1.$$

This is known as the Prime Number Theorem, this gives an estimate for the count of prime numbers. Later on, Gauss had found a better result.

$$\pi(x) \sim Li(x) = \int_2^x \frac{dt}{\ln(t)}.$$

Finally, Riemann has been able to come up with an exact formula of it, (3.3.2)!

$$\pi(x) = J(x) - \frac{1}{2}J(x^{\frac{1}{2}}) - \frac{1}{3}J(x^{\frac{1}{3}}) - \frac{1}{5}J(x^{\frac{1}{5}}) + \frac{1}{6}J(x^{\frac{1}{6}}) + \cdots + \frac{\mu(n)}{n}J(x^{\frac{1}{n}}) + \cdots$$

Combining this with what $J(x)$ is and dropping periodic and constant terms, we obtain that an

x	$\pi(x)$	Gauss	Legendre	Riemann
1000	168	178	172	168.4
10000	1229	1246	1231	1226.9
100000	9592	9630	9588	9587.4
1000000	78498	78628	78534	78527.4
10000000	664579	664918	665138	664667.4
100000000	5761455	5762209	5769341	5761551.9
1000000000	50847534	50849235	50917519	50847455.4
10000000000	455052511	455055614	455743004	455050683.3

Table 3.2 A table showing the estimates of the various prime counting function estimates. (Source: https://primes.utm.edu/howmany.html)

exceptional and fairly simple approximation is:

$$\pi(x) \sim Li(x) - \frac{1}{2}Li(x^{\frac{1}{2}}) - \frac{1}{3}Li(x^{\frac{1}{3}}) - \frac{1}{5}Li(x^{\frac{1}{5}}) + \frac{1}{6}Li(x^{\frac{1}{6}}) - \frac{1}{7}Li(x^{\frac{1}{7}}) - \cdots .$$

In which the first term here is basically Gauss's estimate. Adding more terms gives only a better approximation out performing Gauss's estimation.

Besides just the distribution of prime numbers the Riemann Hypothesis also gives insight into some other areas as well. Being connected to the primes, it is able to inform us about the gaps between primes as well. A practical modern day application would be encryption for credit cards, debit cards, banks, cybersecurity, and similar assets. There is also the Generalized Riemann Hypothesis as well which deals with L-functions.

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ is a Dirichlet character which is a multiplicative function. There exists a positive integer k such that $\chi(n) = \chi(n+k)$ for all n, if $\gcd(n,k) > 1$ then $\chi(n) = 0$, and if $\gcd(n,k) = 1$ then $\chi(n) \neq 0$. By analytic continuation this can be extended to a meromorphic function defined on $\mathbb{C}$.The Generalized Riemann Hypothesis states that, for every Dirichlet character χ and every complex number s with $L(\chi, s) = 0$, if s is not a negative real number, then the real part of s is $\frac{1}{2}$, just like the Riemann Hypothesis. The case where $\chi(n) = 1$ for all n is the normal Riemann

Hypothesis.

3.5 Attempted Proofs of the Riemann Hypothesis

Over the past century there has been countless attempts to solve the Riemann Hypothesis. None of which have been successful at explaining the Riemann Hypothesis. Any knowledge we could find about the distribution of primes could be very helpful in finding a solution. See [18] for many proposed proofs all of which are inaccurate, incomplete, or not rigorous enough to justify the Riemann Hypothesis. The most straight forward way seems to use analytic number theory. Though perhaps a geometric argument can be made. However, there is no concrete way or approach to solving this famous problem, though, the operator theoretic method seems very promising. We explore this method in the next chapter.

CHAPTER 4 BERRY-KEATING CONJECTURE AND RECENT PROGRESS

4.1 Introduction

In this section we discuss the operator theoretical method of solving the fabled Riemann Hypothesis. We start with a dissection of the initial thoughts into the idea from David Hilbert and George Pólya. We then discuss the Berry-Keating conjecture, which seems to be the most promising way to tackle the Riemann Hypothesis from this angle. Germán Sierra does similar work to Berry and Keating and expands more. Bender, Brody, and Muller propose another conjecture as well, similar to the Berry-Keating conjecture. Since all of these conjectures involve operators, it seems natural that quantum mechanics appears.

4.2 Hilbert-Pólya Conjecture

Dubbed the most promising way to solve the Riemann Hypothesis, the origin goes back all the way to the 1910's.

Theorem 4.2.1 (Hilbert-Póyla Conjecture)**.** *The eigenvalues of a self-adjoint linear operator (that is, a hermitian operator) correspond to the non-trivial zeroes of the Riemann Zeta Function.*

In 1982, Andrew Odlyzko wrote a letterer to George Pólya about the physical basis the Riemann Hypothesis and the conjecture associated to Pólya himself and David Hilbert. Pólya answered in his old age and told Odlyzko that while he was in Germany in the early 1910's he was asked by Edmund Landau for a physical reason that the Riemann Hypothesis should be true, and suggested that this would be the case if the imaginary parts, say of the non-trivial zeros of the Riemann zeta function corresponded to eigenvalues of an unbounded and unknown self adjoint operator.[1] Some hopeful confirmation of this was finally given in the early 1950's by Selberg with his trace formula which was quite similar to what Riemann's explicit formula.[16] This trace formula presents a duality between the eigenvalues of the Laplacian acting on a Riemann surface of constant negative curvature (another source states compact instead of constant negative curvature [1]) to the length spectrum of the surface geodesics.[16] Another hopeful confirmation of this

came in 1973 when Hugh Montgomery who, assumed the Riemann Hypothesis to be true, showed that the Riemann zeros are distributed according to the Gaussian Unitary Ensemble statistics of random matrix models. [16] Further, Montgomery investigated and found that the statistical distribution of the zeros on the critical line has a certain property, which is called Montgomerys pair correlation conjecture- "The Riemann zeros tend not to cluster too closely together, but to repel". [1] In 1972, this was brought to Freeman Dyson, a founder of random matrix theory, who realized that this statistical distribution appeared to be the same as the pair correlation distributions for the eigenvalues of random and "very big/large" Hermitian matrices of size $N \times N$. [1] We finally start to get to see the connection to physics now, as these Hermitian matrices are connected to Hamiltonians. The eigenstates of these Hamiltonian matrices, for example various energy levels of an atomic nucleus or a particle in an infinite square well, satisfy these statistics. Subsequent work has strongly borne out the connection between the distribution of the zeros of the Riemann Zeta function and the eigenvalues of a random Hermitian matrix drawn from the theory of the Gaussian Unitary Ensemble, and both are now believed to obey the same statistics.[1] Thus the Hilbert-Pólya conjecture connects the Riemann Hypothesis to quantum mechanics quite well, though no solution has been found using this connection. However, more recent progress has looked for this operator at a semi-classical level in physics.

4.3 Berry-Keating

In 1999, Jon Keating and Michael Berry made strides forward in understand this connection and looking for a "Riemann" operator which would give the heights t_n of the non-trivial zeros. They propose this operator is a quantization of the classical Hamiltonian XP, where P is the momentum operator and X is the position operator. Thus if their conjecture is true, the simplest Hermitian operator at the quantum level corresponding to XP would be

$$H = \frac{1}{2}\big(xp + px\big) = -i\big(x\frac{d}{dx} + \frac{1}{2}\big).$$

[2] This seems to be a relatively simple operator, it is a multiple of the anticommutator of the position and momentum operators. While they have not been able to find this operator nor the space it should act on, these have found many properties it should have. These properties are: [3]

1. H has a classical counterpart (the "Riemann dynamics"), corresponding to a Hamiltonian flow, or a symplectic transformation, in a phase space.

2. The Riemann dynamics is chaotic, that is, unstable and bounded.

3. The Riemann dynamics does not have time-reversal symmetry. They note the recent discovery of modified statistics of the low zeros for the ensemble of Dirichlet L-functions, associated with a symplectic structure.

4. The Riemann dynamics is homogeneously unstable.

5. The classical periodic orbits of the Riemann dynamics have periods that are independent of "energy" t, and given by multiples of logarithms of prime numbers. In terms of symbolic dynamics, the Riemann dynamics is peculiar, and resembles Chinese: each primitive orbit is labelled by its own symbol (the prime p) in contrast to the usual situation where periodic orbits can be represented as words made of letters in a finite alphabet.

6. The Maslov phases associated with the orbits are also peculiar: they are all π. The result appears paradoxical in view of the relation between these phases and the winding numbers of the stable and unstable manifolds associated with periodic orbits , but finds an explanation in a scheme of Connes.

7. The Riemann dynamics possesses complex periodic orbits (instantons) whose periods are multiples of $i\pi$.

8. For the Riemann operator, leading-order semi-classical mechanics is exact: as in the case of the Selberg trace formula, $(\frac{1}{2} + it)$ is a product over classical periodic orbits, without corrections.

9. The Riemann dynamics is quasi-one-dimensional. There are two indications of this. First, the number of zeros less than t increases as $t\log(t)$; for a D-dimensional scaling system, with energy parameter $\alpha(E)$ proportional to $\frac{1}{h}$, the number of energy levels increases as $\alpha(E)^D$. Second, the presence of the factor $p^{-\frac{m}{2}}$ in the counting function fluctuation formula, rather than the determinant in the more general Gutzwiller formula, suggests that there is a single expanding direction and no contracting direction.

10. The functional equation for $\zeta(s)$ resembles the corresponding relation a consequence of hermiticity for the quantum spectral determinant.

The XP operator seems to satisfy most of these properties. It is only unstable but not bounded, primes do not appear in any obvious way, and they have no explanation of the 6th property. It is evident that XP is simply a canonically rotated version of the inverted harmonic oscillator p^2-x^2, which in turn is a complexified version of the usual harmonic oscillator p^2+x^2.

Berry and Keating propose that the eigenvalues would be a continuum of emission lines while Connes thought they would be more like absorption lines.

4.4 Sierra

This brings us to 2007 [16] where Germán Sierra settles the debate between Connes and Berry and Keating. Sierra starts from a quantized version of $H=xp$ and adds a non-local interaction which depends on two potentials. These potentials realize, at the quantum level, the semi-classical regularization of $H=xp$ proposed by Berry and Keating. The classical Hamiltonian $H=xp$ is unbounded, so one should not expect a discrete spectrum at the quantum level. Sierra found that the potentials whose resonances approach the smooth Riemann zeros asymptotically and in the classical limit the potentials reproduce the Berry-Keating semi-classical regularization.

Furthermore, in 2019 Sierra proposes an interferometer that may yield an experimental observation of the Riemann zeros. [17] Sierra continues to work to provide a physical realization of the Riemann zeros. The main steps in this approach are [17]:

1. Spectral realization of Connes's xp model using the Landau model of an electron in a mag-

netic eld and electrostatic potential,

2. Construction of modied quantum xp models whose spectra reproduce, in average, the behavior of the zeros,

3. Reformulation of the $x(p + \frac{1}{p})$model as a relativistic theory of a massive Dirac fermion in a region of Rindler space-time,

4. Inclusion of the prime numbers into the massless Dirac equation by means of delta function potentials acting as moving mirrors that, in the limit where they become semi-transparent, leads to a spectral realization of the zeros,

5. A route for proving the Riemann Hypothesis, and

6. Proposal of an interferometer that may provide an experimental observation of the zeros of the Riemann zeta function and other Dirichlet L-functions.

Having a way to experimentally realize the Riemann zeros just goes to show how interconnected the Riemann Hypothesis and quantum mechanics really are.

4.5 Bender-Brody-Müller

In 2017, Carl Bender, Dorje Brody, and Markus Müller propose a Hamiltonian that may work and is consistent with the Berry-Keating Conjecture. Their proposed Hamiltonian is [2]:

$$H = \frac{1}{1 - e^{-ip}}(xp + px)(1 - e^{-ip}).$$

Their main findings are as follows:

1. The non-Hermitian Hamiltonian H in formally satisfies the conditions of the Hilbert-Pólya conjecture. That is, if the eigenfunctions of H are required to satisfy the boundary condition $\psi_n(0) = 0$ for all n, then the eigenvalues $\{E_n\}$ have the property that $\left\{\frac{1}{2}(1 - iE_n)\right\}$ are the nontrivial zeros of the Riemann zeta function.

2. The Hamiltonian H reduces to the classical Hamiltonian $H = 2xp$ when x and p commute, in agreement with the Berry-Keating conjecture. They derive the corresponding boundary condition that leads to the quantization of the Berry-Keating Hamiltonian $h^{BK} = xp + px$.

3. Although H is not Hermitian, iH is PT symmetric; that is, iH is invariant under parity-time reflection (in the sense to be defined), which means that the eigenvalues of iH are either real or else occur in complex-conjugate pairs. If iH has maximally broken PT symmetry, that is, if all of its eigenvalues are pure imaginary complex-conjugate pairs then the eigenvalues of H are real, then the Riemann hypothesis follows.

4. While H is not Hermitian (symmetric) with respect to the conventional L^2 inner product, they introduce an alternative inner product such that $\langle H\phi, \psi\rangle = \langle \phi, H\psi\rangle$ for all $\phi(x)$ and $\psi(x)$ belonging to the linear span of the eigenstates of H.

5. If the Riemann hypothesis is correct, then the eigenvalues of H are non-degenerate, and conversely if there are nontrivial roots of $\zeta(s)$ for which $\Re(z) \neq \frac{1}{2}$ then the corresponding eigenvalues and eigenstates are both degenerate.

Essentially their findings are a further refinement of the Berry-Keating conjecture and if they could make their argument more rigorous, they would have the solution to the Riemann Hypothesis. They seem to have discovered the Hamiltonian, whose classical limit is $2xp$, whose eigenvalues correspond to the non-trivial zeros of the Riemann Zeta function. So while they have potentially discovered the operator they are still unsure of the domain of H. Looking more into the domain and the self-adjointness of the operator may lead to a solution of the Riemann Hypothesis.

4.6 Quantum Physics

As one can tell there seems to be a very intimate connection between the Riemann Hypothesis and physics; in fact, a physicist might be the one to solve the Riemann Hypothesis.[18] In this section, I would like to flush some of these connections out.

First, there is a fictitious, fermionic, many-body system based on the complex zeros of the Riemann zeta function- it is called the Riemannium. [12] In search of Hilbert's and Pólya's operator,

which is still not known, this system has come about. This operator may be viewed as the quantization of a hypothetical classical dynamical system, which is aligning with the view of Berry and Keating. When the eigenvalues are interpreted at the quantum level, the statistics properties indicate this system is chaotic and has no time-reversal symmetry; this matches again with Berry and Keating. Thus, this connection between chaotic dynamics and the Riemann Zeta function is established. This connection results in breakthroughs in Fermi systems and quantum chaotic motion. In terms of finding a solution, time-periodic dynamical evolutions seem to be promising.

Second, there many connections between the Riemann Hypothesis and physics, beyond just quantum mechanics. In this brief paragraph I will provide some highlights of [15]. There are connections to classical mechanics, nuclear physics, condensed matter physics, and statistical physics. In studying chaotic billiard boards, a potential to the Riemann Hypothesis this way has come about. With billiard boards with two holes in the boundary, primes eventually enter and the connection to the Riemann Hypothesis follows, with a probability function depending uniquely on $\zeta(s)$. In the quantum world there are two models consider a scattering state model and a bound state model. In the scattering state model, on a surface with negative curvature, the eigenvalues take form: $\frac{1}{2} + i\rho_n$ which is the form of the non-trivial zeros. Bound state models follow from random matrix theory.

Third, there are some observations from [1]. The Riemann Zeta function $\zeta(s)$ is important in physics and chemistry since the energy levels look similar.

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

While the energy levels look like:

$$E_n = -13.6eV\frac{1}{n^2}.$$

The energy separation between two different energy levels n and m is given by:

$$\Delta E(n,m) = -13.6eV\Big(\frac{1}{n^2} - \frac{1}{m^2}\Big).$$

The fact that $\zeta(2)$ is finite implies that the energy level separation of the hydrogen atom in the Böhr level tends to zero and that the sum of all the possible energy levels in the hydrogen atom is nite since $\zeta(2)$ is nite. $\zeta(2)$ corresponds to the hydrogen atom. In fact, our particle in a box looks similar to $\zeta(-2)$. Further, $\zeta(-1)$ corresponds to the the harmonic oscillator $E_n = \big(n + \frac{1}{2}\big)\hbar\omega$. In general one could ask, does:

$$E_n = C\frac{1}{n^s}$$

have any physical significance, where C is some constant?

Finally, what I deem astonishing is there is a Hamiltonian which gives the trivial zeros of the Riemann Zeta Function.[19] This Hamiltonian is constructed from the harmonic oscillator:

$$H_{triv} = \frac{d^2}{dx^2} - x^2 + 1.$$

The energy levels are given by:

$$E_n\big(n + \frac{1}{2}\big)\hbar\omega = (2n+1)\frac{\hbar\omega}{2},$$

where $n \in \mathbb{N}$. This to me clearly indicates proof of concept and I think is by far the most promising way to solve the famous Riemann Hypothesis.

4.7 Summary of Current Works

As one can tell, there seems to be exceptional promise in the solving the Riemann Hypothesis via the Hilbert-Pólya conjecture. Further refinement by Berry, Keating, Sierra, Bender, Brody, Muller, and countless others continue advance this approach with new ideas. Berry and Keating have established the main properties to look for in this "Riemann" operator-H:

1. H has a classical counterpart. In addition, the dynamics contain a scaling symmetry. As a consequence, the trajectories are the same at all energy scales.

2. The Riemann dynamics is chaotic and unstable.

3. The dynamics lack time-reversal symmetry.

4. The dynamics is quasi one-dimensional, because for a generic d dimensional scaling system the number of energy eigenvalues increases as $\approx E^d$ while for $\zeta(s)$ the number of zeros $T < N(T) \approx T\ln(T) < T^2$.

Clearly, there is an intimate and deep connection with these refinements and physics. Advancements in either one can in turn have an effect on the other. I think future research into quantum systems could easily pave the way for a solution of the Riemann Hypothesis, since there is an operator for the trivial zeros, it would make sense there is an operator for the non-trivial zeros.

CHAPTER 5 CONCLUDING REMARKS

In conclusion, I believe the most promising way to solve the Riemann Hypothesis is this operator theoretical method utilizing quantum mechanics. It requires extensive knowledge of Hilbert space theory and functional analysis, quantum mechanics and statistical mechanics, and a command of Riemann's Zeta function. I think further research into a finding a physical system related would serve fruitful.

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