EVERY PURE QUASINORMAL OPERATOR HAS A SUPERCYCLIC ADJOINT

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ABSTRACT

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We prove that every pure quasinormal operator $T : H \to H$ on a separable, infinite-dimensional, complex Hilbert space H has a supercyclic adjoint (see Theorem 3.3.2 and Corollary 3.3.12). It follows that if an operator has a pure quasinormal extension then the operator has a supercyclic adjoint. Our result improves a result of Wogen [52] who proved in 1978 that every pure quasinormal operator has a cyclic adjoint.

Feldman [26] proved in 1998 that every pure subnormal operator has a cyclic adjoint. Continuing with our result, it implies in particular that every pure subnormal operator having a pure quasinormal extension has a supercyclic adjoint (see Corollary 3.3.15). Hence improving Feldman's result in this special case.

Indeed, we show that the adjoint T^* of every pure quasinormal operator T is unitarily equivalent to an operator of the form $Q : \bigoplus_{i=0}^{\infty} L^2(\mu) \to \bigoplus_{i=0}^{\infty} L^2(\mu)$ defined by $Q(f_0, f_1, f_2, \ldots) = (A_1 f_1\,, A_2 f_2\,, A_3 f_3\,, \ldots)$ for all vectors $(f_0, f_1, f_2, \ldots) \in \bigoplus_{i=0}^{\infty} L^2(\mu)$, where each $A_n : L^2(\mu) \to L^2(\mu)$ is a left multiplication operator M_{φ_n} with symbol $\varphi_n \in L^\infty(\mu)$ satisfying $\varphi_n \neq 0$ a.e. We constructively obtain a supercyclic vector for the operator Q and this then yields our result by the fact that unitary equivalence preserves supercyclicity. Furthermore, we prove that the adjoint T^* of a pure quasinormal operator $T : H \to H$ is hypercyclic precisely when T is bounded below by a scalar $\alpha > 1$ (see Theorem 2.6.4 and Corollary 2.6.8).

To my wife Diane, and my children Jonathan (Jojo) and Owen

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CHAPTER 1 INTRODUCTION AND PRELIMINARIES

1.1 Introduction

This dissertation is devoted to the study of the orbits of the adjoints of pure quasinormal operators and pure subnormal operators. Quasinormal operators, which were introduced by Brown in [14], form a class of operators which is larger than the class of normal operators but smaller than the class of subnormal operators. An operator is a continuous linear transformation $T : H \to L$ of Hilbert spaces H and L. The operator T is called *an isometry* if it preserves the inner product. A linear transformation $T : H \to L$ is called *an isomorphism*, *or a unitary operator,* if it is a surjective isometry. When $H = L$, an operator $A : H \to H$ on a Hilbert space H is called *a normal operator* if A commutes with its adjoint A^*; that is, $AA^* = A^*A$. An operator $A : H \to H$ is unitary if and only if $A^*A = AA^* = I$, the identity operator. Thus every unitary operator $A : H \to H$ is normal. An operator $A : H \to H$ on a Hilbert space H is called *a quasinormal operator* if A commutes with A^*A; that is, $A(A^*A) = (A^*A)A$, or $AA^*A = A^*A^2$. It is clear that every normal operator is quasinormal, the converse is obviously false. If, for instance, we take A to be a nonunitary isometry, then A^*A is the identity I and therefore A commutes with A^*A. Since A is not unitary, A is not normal. A concrete example is the unilateral forward shift which we will define later in the dissertation.

Subnormal operators were introduced by Halmos in [31]. These are operators which have normal extensions. More precisely, an operator $S : H \to H$ on a Hilbert space H is *subnormal* if there exists a normal operator N on a Hilbert space K that contains H as a closed subspace such that the subspace H is invariant under the operator N, and the restriction of N to H coincides with S.

In 1976, Deddens and Wogen [52] raised the question which subnormal operators have cyclic adjoints. Sarason answered this question by proving that for any subnormal operator S, the adjoint S^* of S is cyclic if S is an isometry [32, Problem 160]. Also, Bram [13] showed that if a subnormal operator S has a cyclic minimal normal extension, then the adjoint S^* is cyclic. In 1998, Feldman [26] proved that every pure subnormal operator has a cyclic adjoint by using a method of comparing operators.

In 1978, Wogen [52] proved that every pure quasinormal operator has a cyclic adjoint by proving a more general result for the class of operators each of which has a triangular matrix representation. It follows that if an operator has a pure quasinormal extension, then the adjoint of the operator is cyclic.

In this dissertation, we improve Wogen's result. More precisely, we prove that every pure quasinormal operator has a supercyclic adjoint. It turns out that if an operator has a pure quasinormal extension, then the adjoint of the operator is supercyclic. Since every supercyclic operator is cyclic, our result improves Wogen's result.

In the rest of this chapter, we give the basic definitions and mathematical ideas of hypercyclicity, supercyclicity, and cyclicity that we explore in the next chapters. In addition we present examples and briefly discuss their properties, hypercyclicity criterion, and some existing results in hypercyclicity.

In Chapter 2, we study hypercyclic properties of operators on Hilbert spaces. We develop the notions of a pure operator and polar decomposition which are key to the understanding of Brown's Theorem [21, p. 135]. In section 2.6 of this Chapter, we subsequently consider pure quasinormal operators which are bounded below by 1, and we use Brown's Theorem and the hypercyclicity criterion to show that these operators have indeed hypercyclic adjoints.

In Chapter 3, we show that any pure quasinormal operator has a supercyclic adjoint. It is our intention here to construct such a supercyclic vector. To reach our goal, we define a backward shifting operator on a Hilbert space of direct sum of countably infinitely many copies of square integrable functions with a σ-finite measure, and of which we are able to construct a supercyclic vector using the norm-topology of the space. Using Brown's Theorem, it turns out that the backward shifting operator reduces to the adjoint of the pure quasinormal operator. This phenomenon may be explained by the result of Hilden and Wallen [34], who showed that every unilateral weighted backward shift is indeed supercyclic. In addition, we present results involving the construction and we show how they relate to general operators, and to subnormal operators in particular.

Finally, in Chapter 4, we present more existing results in the theory of subnormal operators and we present an improvement in a special case of a result by Feldman [26], who proved that every pure subnormal operator has a cyclic adjoint.

1.2 Hypercyclic Operators

In this section, we consider continuous linear operators $T : X \to X$ on a Fréchet space X. We recall that a Fréchet space is a vector space X with a metric d that satisfies the following:

(i) scalar multiplication and vector addition are continuous with respect to d,

(ii) d is translation invariant, meaning $d(x+z, y+z) = d(x, y)$ for all $x, y, z \in X$, and

(iii) the metric space (X, d) is complete.

If X is a separable Fréchet space, and $T : X \to X$ is a continuous linear operator, then we can consider the powers T^n of T where $n = 0, 1, 2, 3, \ldots$. Applying these powers on a vector x of X gives rise to the concept of the orbit of x under T.

Definition 1.2.1. *Let X be a separable Fréchet space, and $T : X \to X$ be an operator, that is, a continuous linear transformation. For a vector $x \in X$, we define its orbit under T to be the set*

$$\mathrm{Orb}(T, x) = \{x, Tx, T^2x, \ldots\}.$$

Since $T(\mathrm{Orb}(T, x)) \subset \mathrm{Orb}(T, x)$, by the continuity of T we have $T(\overline{\mathrm{Orb}(T, x)}) \subset \overline{\mathrm{Orb}(T, x)}$. Thus, the closure $\overline{\mathrm{Orb}(T, x)}$ of Orb(T, x) is the smallest T-invariant closed subset of X containing x.

We are primarily interested in operators with a dense orbit, or an orbit whose scalar multiples generate a dense subset. For that we need more definitions.

Definition 1.2.2. *A continuous linear operator $T : X \to X$ is said to be hypercyclic if there exists $x \in X$ such that* $\text{Orb}(T, x)$ *is dense in X. In this case, x is called a hypercyclic vector for T. The set of all hypercyclic vectors for T is denoted by $\mathcal{HC}(T)$.*

Definition 1.2.3. *A continuous linear operator $T : X \to X$ is said to be topologically transitive, if for every pair U, V of nonempty open sets, there exists $n \geq 0$, such that $T^n(U) \cap V \neq \emptyset$.*

From definition 1.2.3, we see that if $T : X \to X$ is continuous and invertible, than T is topologically transitive if and only if T^{-1} is.

There is a strong relationship between the notion of hypercyclicity and the notion of topological transitivity. It relates topological transitivity of a continuous linear transformation to the property of having a dense set of hypercyclic vectors.

Indeed, assume T is topologically transitive, and as usual let $\mathcal{HC}(T)$ denote the set of all hypercyclic vectors for T. Since the Frèchet space X is separable, let $\{x_j : j \geq 1\}$ be a countable dense subset of X. Then the open balls $B(x_j, \frac{1}{m})$ where $m, j \geq 1$ form a countable base for the metric topology of X. We denote the countable base by $(U_k)_{k \geq 1}$. Hence, a vector $x \in \mathcal{HC}(T)$ if and only if for any $k \geq 1$, there exists $n \geq 0$ such that $T^n x \in U_k$. Thus,

$$\mathcal{HC}(T) = \bigcap_{k=1}^{\infty} \bigcup_{n=0}^{\infty} T^{-n}(U_k).$$

By the continuity of T, each set $\bigcup\limits_{n=0}^{\infty} T^{-n}(U_k)$, where $k \geq 1$, is open. By the topological transitivity of T, each set $\bigcup\limits_{n=0}^{\infty} T^{-n}(U_k)$, where $k \geq 1$, is dense. Thus, for each positive integer $k \geq 1$, the union $\bigcup\limits_{n=0}^{\infty} T^{-n}(U_k)$ is an open dense set. By the Baire Category Theorem, we conclude that $\mathcal{HC}(T)$ is a dense G_δ-set.

This is summarized in the following result obtained first in 1920 by Birkhoff [8] in context of maps on compact subsets of $\mathbb{R}^n$, and in 1982 by Kitai [36] in a Banach space setting.

Theorem 1.2.4. (Birkhoff Transitivity Theorem). *Let $T : X \to X$ be an operator on the Fréchet space X. Then T is hypercyclic if and only if it is topologically transitive. In this case, the set*

$\mathcal{HC}(T)$, is a dense G_δ-set.

The next result is an immediate consequence of Birkhoff's Transitivity Theorem.

Corollary 1.2.5. *An invertible operator T is hypercyclic if and only if its inverse T^{-1} is hypercyclic.*

Theorem 1.2.4 implies that the set

$$\mathcal{HC}(T) = \{x \in X : \mathrm{Orb}\,(T, x) \; is \; dense \; in \; X\}$$

of hypercyclic vectors for a given operator T is empty if T is not hypercyclic; it is dense as long as T has hypercyclic vectors. Concerning its algebraic structure, we have the following well-known result.

Theorem 1.2.6. *If T is a hypercyclic operator on the Fréchet space X, then*

$$X = \mathcal{HC}(T) + \mathcal{HC}(T).$$

This means that every vector $x \in X$ can be written as the sum of two hypercyclic vectors.

There exist other forms of hypercyclicity which appear to have similar properties to those of hypercyclic operators. We give their explicit definitions in what follows.

Definition 1.2.7. *Let $T : X \to X$ be a continous linear operator on X.*

(i) A vector $x \in X$ is called cyclic for T if the linear span of its orbit,

$$\mathrm{span}\{T^n x : n \geq 0\}$$

is dense in X.

(ii) A vector $x \in X$ is called supercyclic for T if its projective orbit,

$$\{\lambda T^n x : n \geq 0, \lambda \in \mathbb{C}\}$$

is dense in X.

A *cyclic operator* is an operator that has a cyclic vector. Similarly, a *supercyclic operator* is an operator that has a supercyclic vector. Thus every hypercyclic operator is supercyclic, and every supercyclic operator is cyclic. Note that if x is a vector in X, then

$$\begin{aligned} \text{span Orb}(T,x) &= \text{the linear span } of\ \{x, Tx, T^2x, T^3x, \dots\} \\ &= \{a_0x + a_1Tx + a_2T^2x + \dots + a_nT^nx : a_0, a_1, a_2, \dots a_n \in \mathbb{C}\} \\ &= \{p(T)x : p \text{ is a polynomial}\}. \end{aligned}$$

Thus, the closure $\overline{\text{span Orb}(T,x)}$ of span Orb(T,x) is the smallest closed T-invariant subspace of X that contains x. It is true that both the collections of cyclic and supercyclic vectors for an operator T are also G_δ-sets [12, 48], and that the set of supercyclic vectors is dense as long as it is non-empty. However, the collection of cyclic vectors need not be dense [29, Section 3.2].

1.3 Universality

Definition 1.3.1. *Let X be an infinite-dimensional, separable, complex Fréchet space. A sequence of operators $\{T_n : X \to X | n \geq 1\}$ is said to be universal if there is a vector x in X such that the set $\{T_nx : n \geq 1\}$ is dense in X. Such a vector x is called a universal vector of the sequence $\{T_n\}$.*

Note the special case that when each member T_n of a universal sequence is the nth power T^n of a particular operator $T : X \to X$, then T is hypercyclic. In general, a universal sequence of operators need not have a dense set of universal vectors. Thus, if (T_n) is universal, it may not happen that (T_n) has a dense set of universal vectors [29].

Let $H(G)$ be the vector space of all analytic functions on a region G in the complexe plane $\mathbb{C}$. Given the topology of compact convergence, $H(G)$ becomes a Fréchet space, in which a sequence $\{f_n\} \subset H(G)$ converges to f if and only if $f_n \to f$ uniformly on compact subsets of G.

There are no hypercyclic operators in finite dimensional spaces (See Corollary 1.3.8 below).

Nevertheless, the property of hypercyclicity in infinite-dimensional spaces is not a rare phenomenon. For instance, when the region $G = \mathbb{C}$, one of the very first examples of universality on the space $H(\mathbb{C})$ was obtained by Birkhoff [9] as described in the following subsection, where we provide examples of hypercyclic operators.

1.3.1 Examples

The first examples of hypercyclic operators were found by G.D. Birkhoff [9] in 1929, G.R. MacLane [39] in 1952 and S. Rolewicz [44] in 1969. It turns out that hypercyclicity occurs quite often, and that many natural and well-known operators are hypercyclic.

Example 1.3.2. (Birkhoff [9]). *On the space $H(\mathbb{C})$, we consider the translation operator*

$$T_a f(z) = f(z + a), \text{ where } a \neq 0.$$

Then T_a is hypercyclic.

Example 1.3.3. (MacLane [39]). *The differentiation operator*

$$D : f \mapsto f'$$

on $H(\mathbb{C})$, is hypercyclic.

Example 1.3.4. (Rolewicz [44]). *On the space l^p, $p \in [1, \infty)$, we consider the scalar multiple of the backward shift B given by*

$$T = \lambda B : (x_1, x_2, x_3, \ldots) \mapsto \lambda(x_2, x_3, x_4, \ldots).$$

Then T is hypercyclic precisely when $|\lambda| > 1$.

Example 1.3.5. (Seidel and Walsh [47]). *Let $\mathbb{D}$ be the open unit disk. If $\{a_n\}$ is a sequence of points in $\mathbb{D}$ with $a_n \to 1$, and if*

$$\phi_n(z) = \frac{a_n - z}{1 - \overline{a_n} z},$$

then the sequence of non-Euclidean translations

$$T_n : f(z) \mapsto f \circ \phi_n(z)$$

is universal.

Linear operators on finite-dimensional Hilbert spaces, however, are never hypercyclic, as the following proposition, due to Kitai [36], shows.

Proposition 1.3.6. *If $T : H \to H$ is a hypercyclic operator on a Hilbert space H, then its adjoint $T^* : H \to H$ has no eigenvalues.*

Proposition 1.3.6 remains true if the operator T is defined on any Banach space X. In fact, for a Banach space X, its dual space X^* is defined to be the space of all continuous linear functionals on X. If $T : X \to X$ is an operator, then its *dual operator* $T^* : X^* \to X^*$, is defined by

$$T^*(f)(x) = f(Tx)$$

for all $f \in X^*$ and $x \in X$.

Remark 1.3.7. *The dual operator defined above on a Banach space should not be confused with the adjoint T^* of an operator T defined on a Hilbert space. In general, the adjoint of $T : X \to Y$ is the unique operator $T^* : Y \to X$, $y \mapsto T^*y$, such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in X$ and $y \in Y$.*

Corollary 1.3.8. *There are no hypercyclic operators on a finite-dimensional Banach space.*

1.3.2 Non-hypercyclic Operators on Banach Spaces

As illustrated above, there are hypercyclic operators on certain infinite-dimensional spaces. It is evident that such spaces must be separable, and that any hypercyclic operator must have norm greater than 1. In fact, if $\|T\| \leq 1$, then $\|T^n f\| \leq \|f\|$ for all f and $n \geq 0$. Thus T cannot be hypercyclic in this case. The following definitions can also be found in [30].

Definition 1.3.9. *Let X be a Banach space and $T : X \to X$ a bounded linear operator on X. Then T is called*

(i) a contraction if $\|T\| \leq 1$,

(ii) quasinilpotent if $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$, and

(iii) power bounded if $\sup_{n\geq 0} \|T^n\| < \infty$.

The following result can be found in the book of Grosse-Erdmann and Manguillot [30].

Theorem 1.3.10. *If T is a power bounded operator on the Banach space X, then T fails to be hypercyclic.*

Corollary 1.3.11. *No contraction and no quasinilpotent operator is hypercyclic.*

We already know that there are no hypercyclic operators on finite-dimensional spaces. This immediately extends to finite-rank operators, that is, linear operators $T : X \to X$ with dim ran $T < \infty$.

Proposition 1.3.12. *No finite-rank operator is hypercyclic.*

Definition 1.3.13. *An operator $T : X \to X$ is called compact if the image of the closed unit ball of X has compact closure.*

The following Theorem is due to Kitai [36, Theorem 4.2].

Theorem 1.3.14. (Kitai [36]). *No hypercyclic compact operator exists on a Banach space.*

However, compact perturbations of the identity, that is, operators of the form $I + K$ with K compact can be hypercyclic [30, Example 8.4, p. 218].

Definition 1.3.15. *An operator $T : X \to X$ is called power compact if some power T^n, $n \geq 1$, is compact.*

By Ansari's Theorem (See Theorem 1.4.3 below), an operator T can only be hypercyclic if T^n is, which is impossible if T^n is compact. The following result is an immediate consequence.

Proposition 1.3.16. *No power-compact operator is hypercyclic on a Banach space.*

We have noted above that finite-rank operators cannot be hypercyclic. The same is true for finite-rank perturbations of multiples of the identity I, that is, operators of the form $\lambda I + F$ where $\lambda \in \mathbb{C}$ and F is a finite-rank operator.

Proposition 1.3.17. *No finite-rank perturbation of a multiple of the identity is hypercyclic.*

After having presented all the fundamental notions and several examples of hypercyclic operators, we provide in the next subsection, sufficient conditions for hypercyclicity that yield at one unified approach the four results by Birkhoff, Maclane, Rolewicz, and Seidel and Walsh.

1.3.3 Criteria for Hypercyclicity

The first criterion is due to Kitai [36]. The second, which is a refinement of the first, and known as the Universality criterion, is due to Gethner and Shapiro [28].

Theorem 1.3.18. (Hypercyclicity's criterion [36]). *Let X be a separable infinite dimensional Banach space and let $T : X \to X$ be a bounded linear operator. Then T is hypercyclic if there exists a mapping $S : X \to X$ and a dense subset E of X such that*

(i) TS *is the identity* I*,*

(ii) $T^n x \to 0$ *for all* $x \in E$*, and*

(iii) $S^n x \to 0$ *for all* $x \in E$*.*

Theorem 1.3.19. (Gethner and Shapiro [28]). *Let $\{T_n : X \to X | n \geq 1\}$ be a sequence of continuous linear operators on a Fréchet space X. Then $\{T_n\}$ is universal if there exist dense subsets D_1 and D_2 of X, and a sequence of mappings $\{S_n : X \to X | n \geq 1\}$ such that*

(i) $T_n S_n$ *is the identity I,*

(ii) $T_n x \to 0$ *for each* $x \in D_1$*, and*

(iii) $S_n x \to 0$ *for each* $x \in D_2$*.*

A slight modification of the Gethner and Shapiro's criterion is due to Bès [6]. We state it here.

Theorem 1.3.20. (Bès [6]). *Let T be an operator on the Fréchet space X. Suppose there exists dense subsets X_0, Y_0 of X, a subsequence (n_k) of (n), and a sequence of maps $S_{n_k} : Y_0 \to X$ satisfying the following,*

(i) $T^{n_k}x \to 0$ *for all* $x \in X_0$,

(ii) $S_{n_k}y \to 0$ *for all* $y \in Y_0$, *and*

(iii) $TS_{n_k}y \to 0$ *for all* $y \in Y_0$.

Then T is hypercyclic.

1.4 Some Results in Hypercyclicity

In this section, we state some known results about hypercyclicity. We start by stating the following Theorem by Ansari [4] which guarantees the existence of hypercyclic operators on separable, infinite-dimensional Banach spaces.

Theorem 1.4.1. (Ansari [4]). *For every separable, infinite-dimensional Banach space X, there is a hypercyclic operator T on X.*

Note however there is no hypercyclic operator on a finite-dimensional space, nor on a non-separable Banach space.

When T is hypercyclic on a separable complex Banach space X, $T - \lambda I$ where I is the identity on X has dense range for any complex scalar λ. This is the key to the proof of the following theorem.

Theorem 1.4.2. (Herrero [33], Bourdon [11], Bès [7], Wengenroth [50]). *If X is a hypercyclic operator on a Fréchet space X and $x \in X$ is hypercyclic vector for T, then*

$$\{p(T)x : p \text{ is a polynomial } \} \setminus \{0\}$$

is a dense set of hypercyclic vectors. In particular, any hypercyclic operator admits a dense invariant subspace, consisting, except for 0*, of hypercyclic vectors.*

As an immediate corollary of Theorem 1.4.2, we get that $\mathcal{H}C(T)$ is connected. Thus, for a hypercyclic operator T, $\mathcal{H}C(T)$ is a connected, dense G_δ-set, containing a dense linear subspace consisting, except for 0, of hypercyclic vectors.

For an operator T, if some power T^p is hypercyclic, then it is clear that T is also hypercyclic. The following result due to Ansari, says that also the converse is true.

Theorem 1.4.3. (Ansari [2]). *Let* T *be an operator on the Fréchet space* X*. Then for every* $p \in \mathbb{N}$*,*

$$\mathcal{H}C(T) = \mathcal{H}C(T^p).$$

In particular, if T *is hypercyclic then so is every power* T^p*.*

We conclude this section by mentioning the Invariant Subspace Problem which will be useful later. It is a fascinating problem in operator theory and asks the following: "Must every bounded linear operator $T : H \to H$ on a separable, infinite dimensional Hilbert space H have a nontrivial closed invariant subspace?"

Per Enflo [25] was the first to give a counter-example to this problem in Banach space l^1. Later Charles Read [43] constructed an operator on l^1 such that all the non-zero vectors are hypercyclic, providing a counter-example to the invariant subspace problem in the class of Banach spaces. We state the Theorem here.

Theorem 1.4.4. (Read [43]). *There is an operator* T *on* l^1 *with no nontrivial closed invariant subset. That is every nonzero vector* x *has the property that* $\overline{\mathrm{Orb}(T, x)} = l^1$.

The problem whether such an operator exists on a separable Hilbert space is still open.

CHAPTER 2 HYPERCYCLIC PROPERTIES OF OPERATORS ON HILBERT SPACES

2.1 Introduction

In this Chapter, we study hypercyclic properties of operators on Hilbert spaces. We subsequently study the notions a pure operator, and in section 2.6 of this Chapter, we present new results on the hypercyclicity of the adjoint of a pure quasinormal operator bounded below by a scalar $\alpha > 1$.

In what follows, H is a complex infinite-dimensional Hilbert space. The norm of the space H, and any other normed spaces herein, will be denoted by $\| \cdot \|$. A bounded linear operator $T : H \to H$ is sometimes called an operator. The collection of all bounded linear operators on H is denoted by $B(H)$.

Definition 2.1.1. *Let $T : H \to H$ be a bounded linear operator on H and let M be a linear subspace of H. Then M is called*

(i) nontrivial if it is closed in H and is neither the whole space nor the zero-subspace.

(ii) an invariant subspace of T if $TM \subseteq M$.

(iii) a hyperinvariant subspace of T if it is invariant for every operator on H that commutes with T.

(iv) a reducing subspace of T if $TM \subseteq M$ and $TM^{\perp} \subseteq M^{\perp}$. (Equivalently, M is reducing if and only if $TM \subseteq M$ and $T^{*}M \subseteq M$).

Definition 2.1.2. *An operator $T : H \to H$ is called*

(i) normal if it commutes with its adjoint, that is, $TT^{*} = T^{*}T$.

(ii) quasinormal if T and $T^{*}T$ commute, that is, $TT^{*}T = T^{*}T^{2}$.

(iii) subnormal if there exists a Hilbert space K containing H as a closed subspace and a normal operator $N \in B(K)$ such that $NH \subset H$ and $T = N|_H$. That is, T is subnormal if it has a normal extension.

It follows from the definition that any normal operator is quasinormal. The converse however is not true (See Example 2.1.6 below). Furthermore, every quasinormal operator is subnormal. Again, the converse is not true. The fact that quasinormal operators are subnormal involves polar decomposition of an operator and it will be treated later in the chapter.

Definition 2.1.3. *Let $T : H \to L$ be a linear transformation of Hilbert spaces.*

(i) T is called an isometry if $\langle Tf, Tg\rangle = \langle f, g\rangle$ for all $f, g \in H$.

(ii) T is called an isomorphism if T is a surjective isometry of H onto L.

Remark 2.1.4. *In this dissertation, an isomorphism $T : H \to L$ is also called unitary. When $H = L$, the unitary operator $T : H \to H$ equivalently satisfies $T^*T = TT^* = I$, the identity.*

Definition 2.1.5. *Let $T : H \to H$ and $S : H \to H$ be bounded linear operators on the Hilbert space H. We say that T and S are unitarily equivalent if there exists a unitary operator $U : H \to H$ such that $UTU^* = S$.*

Example 2.1.6. *For $p = 2$, consider $S : l^p \to l^p$, defined by $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Then S is an isometry that is not surjective. Moreover, $\|S\| = 1$ and $S^*S = I$.*

Hence the operator S in Example 2.1.6 is an isometry which is not an isomorphism. It is called the *unilateral forward shift*. It is useful as counter-example in many cases. For instance, the unilateral forward shift is quasinormal but not normal.

2.1.1 Positive Operators

An important class of normal operators are the self-adjoint operators, that is, operators T satisfying $T = T^*$. The following Lemma which can also be found in [21, Proposition 2.12, p. 33] is useful.

Lemma 2.1.7. *Let H be a complex Hilbert space and $A : H \to H$ a bounded linear operator. Then A is self-adjoint if and only if $\langle Ah, h\rangle \in \mathbb{R}$ for all $h \in H$.*

Proof. Clearly if $A^* = A$ then

$$\langle Ah, h\rangle = \langle h, A^*h\rangle = \langle h, Ah\rangle = \overline{\langle Ah, h\rangle}.$$

That is $\langle Ah, h\rangle \in \mathbb{R}$.

Conversely, assume $\langle Ah, h\rangle \in \mathbb{R}$ for all $h \in H$. Let $\alpha \in \mathbb{C}$ and $f, g \in H$. Then $\langle A(h + \alpha g), h + \alpha g\rangle \in \mathbb{R}$. That is

$$\langle Ah, h\rangle + \alpha\langle Ag, h\rangle + \overline{\alpha}\langle Ah, g\rangle + |\alpha|^2\langle Ag, g\rangle \in \mathbb{R}.$$

Since $\langle Ah, h\rangle$ and $\langle Ag, g\rangle \in \mathbb{R}$, we have

$$\alpha\langle Ag, h\rangle + \overline{\alpha}\langle Ah, g\rangle \in \mathbb{R}.$$

It follows that $\alpha\langle Ag, h\rangle + \overline{\alpha}\langle Ah, g\rangle$ is a complex conjugate of itself. Hence,

$$\alpha\langle Ag, h\rangle + \overline{\alpha}\langle Ah, g\rangle = \overline{\alpha}\langle h, Ag\rangle + \alpha\langle g, Ah\rangle.$$

Putting $\alpha = 1$ and $\alpha = i$ respectively, we get

$$\langle Ag, h\rangle + \langle Ah, g\rangle = \langle A^*h, g\rangle + \langle A^*g, h\rangle \tag{2.1.8}$$

and

$$\langle Ag, h\rangle - \langle Ah, g\rangle = -\langle A^*h, g\rangle + \langle A^*g, h\rangle. \tag{2.1.9}$$

Adding (2.1.8) and (2.1.9) gives

$$\langle Ag, h\rangle = \langle A^*g, h\rangle$$

for all $g, h \in H$. Hence $A = A^*$ and this completes the proof. □

Definition 2.1.10. *An operator A on a Hilbert space H is called positive and we write $A \geq 0$ if*

$$\langle Ax, x \rangle \geq 0$$

for all $x \in H$.

Definition 2.1.11. *If A, B are operators on a Hilbert space H, we write $A \geq B$ if $A - B \geq 0$.*

By Lemma 2.1.7, it follows that every positive operator on a complex Hilbert space is self-adjoint.

Definition 2.1.12. *An operator $T : H \to H$ is called hyponormal if $T^*T - TT^* \geq 0$, that is, $\|Tf\| \geq \|T^*f\|$ for all $f \in H$.*

Every subnormal operator is hyponormal. The converse is not true unless we are on a finite-dimensional Hilbert space [32, Solution 202 or Problem 203]. The fact that subnormal operators are hyponormal involves the notion of minimal normal extension of an operator that we will explore in the next section.

Beforehand, we complete this section by recalling the Spectral Theorem [21, p. 272] and some of its consequences.

Theorem 2.1.13. *(The Spectral Theorem). If N is a normal operator on a Hilbert space H, then there exists a measure space (X, Ω, μ) and a function Φ in $L^\infty(X, \Omega, \mu)$ such that N is unitarily equivalent to the multiplication operator*

$$M_\Phi : L^2(X, \Omega, \mu) \to L^2(X, \Omega, \mu)$$

defined by $M_\Phi f = \Phi f$ for all $f \in L^2(X, \Omega, \mu)$.

Corollary 2.1.14. *Suppose $N : H \to H$ is normal and it is unitarily equivalent to $M_\Phi : L^2(\mu) \to L^2(\mu)$. Then* *(i)* N^* *is unitarily equivalent to* $M_{\overline{\Phi}}$*;*

(ii) N *is self-adjoint if and only if* Φ *is real-valued a.e.;*

(iii) N *is positive if and only if* $\Phi \geq 0$ *a.e.;*

(iv) N *is unitary if and only if* $|\Phi| = 1$ *a.e.;*

(v) N *is invertible if and only if* $\Phi^{-1} \in L^\infty(\mu)$*.*

Corollary 2.1.15. *If* $N : H \to H$ *is positive and invertible with* $N \geq I$ *then* $N^{-1} : H \to H$ *is positive and invertible with* $N^{-1} \leq I$*.*

2.2 Pure Operators and the Minimal Normal Extension

We recall that a closed subspace M of a Hilbert space H is a reducing subspace of an operator $T \in B(H)$ if both M and its orthogonal complement $M^\perp$ are invariant for T. In what follows we define the idea of a pure operator. For that, we need the following proposition which can be found in the book of Conway [23, Proposition 2.1, p. 127].

Proposition 2.2.1. *If* $A \in B(H)$*, then there exists a reducing subspace* H_0 *for* A *such that*

(i) $A_0 = A|_{H_0}$ *is normal;*

(ii) $A_1 = A|_{H_0^\perp}$ *has no reducing subspace on which it is normal.*

Proof.

(i) Let $\mathcal{Q}$ be the collection of pairs $(M, A|_M)$ where M is a reducing closed subspace for A and $A|_M$ is normal.

Partially order $\mathcal{Q}$ by set inclusion: $(M, A|_M) \leq (N, A|_N)$ if M is a closed subspace of N. Suppose $\mathcal{C} = \{(M_\alpha, A|_{M_\alpha})\}$ is a chain in $\mathcal{Q}$.

Let

$$N_0 = \bigcup_{\left(M_\alpha, A|_{M_\alpha}\right) \in \mathcal{C}} M_\alpha.$$

Therefore $AN_0 \subseteq N_0$ and $A^*N_0 \subseteq N_0$. That is, $(N_0, A|_{N_0})$ is an upper bound for $\mathcal{C}$ (i.e. $(M_\alpha, A|_{M_\alpha}) \leq (N_0, A|_{N_0})$ for all α). Zorn's Lemma implies that $\mathcal{Q}$ has a maximal element $(H_0, A|_{H_0})$. Hence H_0 is a reducing subspace and $A|_{H_0}$ is normal.

(ii) Suppose L is a closed subspace of $H_0^\perp$ that reduces A such that $A|_L$ is normal. Then $H_0 \oplus L$ reduces A and $A|_{H_0 \oplus L}$ is normal, contradicting the maximality of H_0. The proof is complete. □

Definition 2.2.2. *An operator $A : H \to H$ is pure if it has no reducing subspaces on which it is normal.*

The following Corollary is an immediate consequence of Proposition 2.2.1.

Corollary 2.2.3. *An operator $A \in B(H)$ is pure if for any reducing subspace H_0 such that $A|_{H_0}$ is normal, then H_0 is the zero-subspace.*

We recall that a subnormal operator is an operator which has a normal extension. However this normal extension is never unique. Indeed, if $S \in B(H)$ is subnormal and $N \in B(K)$ is a normal extension of S, it is not difficult to see that $N \oplus M \in B(K \oplus L)$ is also a normal extension of S for any normal operator $M \in B(L)$. The following definition univocally gives a normal extension in the sens of Corollary 2.2.8 below.

Definition 2.2.4. *Let $S : H \to H$ be subnormal and $N : K \to K$ a normal extension of S. Then N is a minimal normal extension (mne) of S if K has no proper subspace that contains H and reduces N. In other words, $N \in B(K)$ is called a mne of $S \in B(H)$ if whenever M reduces N and $H \subset M$, it follows that $M = K$.*

Thus, for instance, if $N : K \to K$ is a normal extension of a subnormal operator $S : H \to H$ and $M : L \to L$ is a normal operator, then $N \oplus M : K \oplus L \to K \oplus L$ cannot be a mne of S because $H \subset K \oplus 0$, $K \oplus 0$ is a proper subspace of $K \oplus L$, and $K \oplus 0$ reduces $N \oplus M$.

The next proposition which can be found in [23, Proposition 2.4, p. 128] suggests that minimal normal extension can be characterized in a very useful way.

Proposition 2.2.5. *Let $S : H \to H$ be a subnormal operator and $N : K \to K$ a normal extension of S. Then N is a mne of S if and only if*

$$K = \overline{\text{span}\{N^{*n} f : n \geq 0, f \in H\}}.$$

Proof. Suppose N is a mne of S. Let $L = \overline{\text{span}\{N^{*n} f : n \geq 0, f \in H\}}$.
Clearly $H \subseteq L$. Moreover, because N is normal, we have $N^* L \subseteq L$ and $NL \subseteq L$. Hence, L is a reducing subspace for N that contains H. Since N is a mne of S, we must have $K = L$.

Conversely, assume $K = L$. Suppose M is a reducing subspace of N and $H \subseteq M$. Then $N^{*n} f \in M$ for all $f \in H$ and for all $n \geq 0$. Hence $K = L \subseteq M$ and so N is a mne of S. This completes the proof. □

These minimal normal extensions however are unique up to isomorphism as shown in the next proposition.

Proposition 2.2.6. *Let $S_1 : H_1 \to H_1$, $S_2 : H_2 \to H_2$ be two subnormal operators and $N_1 : K_1 \to K_1$ and $N_2 : K_2 \to K_2$ be two mne of S_1, S_2 respectively. Suppose there exists an isomorphism $U : H_1 \to H_2$ such that $U S_1 U^* = S_2$. Then there exists an isomorphism $V : K_1 \to K_2$ such that*

(i) $V|_{H_1} = U$;

(ii) $V N_1 V^* = N_2$.

Proof. Define V on K_1 by

$$V N_1^{*n} f_1 = N_2^{*n} U f_1$$

for all $f_1 \in H_1$ and for all $n \geq 0$. We claim that V is well-defined. To see that,
suppose $f_1, \ldots, f_m \in H_1$ and $n_1, \ldots, n_m \geq 0$.
Then

$$\left\| \sum_{k=1}^{m} N_2^{*n_k} U f_k \right\|^2 = \left\langle \sum_{k=1}^{m} N_2^{*n_k} U f_k, \sum_{j=1}^{m} N_2^{*n_j} U f_j \right\rangle = \sum_{k.j=1}^{m} \left\langle N_2^{*n_k} U f_k, N_2^{*n_j} U f_j \right\rangle$$

$$= \sum_{k,j=1}^{m} \langle N_2^{n_j} U f_k, N_2^{n_k} U f_j \rangle,$$

by the normality of N_2 on the third equality. Furthermore, since $US_1U^* = S_2$, we have $US_1^nU^* = S_2^n$ and so

$$\begin{aligned}
\sum_{k,j=1}^{m} \langle N_2^{n_j} U f_k, N_2^{n_k} U f_j \rangle &= \sum_{k,j=1}^{m} \langle U N_1^{n_j} f_k, U N_1^{n_k} f_j \rangle \\
&= \sum_{k,j=1}^{m} \langle N_1^{n_j} f_k, N_1^{n_k} f_j \rangle \;\; (U \; isomorphism) \\
&= \sum_{k,j=1}^{m} \langle N_1^{*n_k} f_k, N_1^{*n_j} f_j \rangle \;\; (N_1 \; normal) \\
&= \left\langle \sum_{k=1}^{m} N_1^{*n_k} f_k, \sum_{j=1}^{m} N_1^{*n_j} f_j \right\rangle \\
&= \left\| \sum_{k=1}^{m} N_1^{*n_k} f_k \right\|^2 .
\end{aligned}$$

This shows that

$$V\left(\sum_{k=1}^{m} N_1^{*n_k} f_k\right) = \sum_{k=1}^{m} N_2^{*n_k} U f_k \tag{2.2.7}$$

is a well-defined operator and an isometry from a linear subspace of K_1 into K_2.

By proposition 2.2.5 and the fact N_1 and N_2 are mne of S_1 and S_2, V is densely defined and has dense range. So V extends to an isomorphism $V : K_1 \to K_2$. By taking $n_k = 0$ in (2.2.7) yields $V|_{H_1} = U$. Now for $f \in H_1$ and $n \geq 0$,

$$VN_1N_1^{*n}f = VN_1^{*n}S_1f = N_2^{*n}US_1f = N_2^{*n}S_2Uf = N_2N_2^{*n}Uf = N_2VN_1^{*n}f.$$

That is, $VN_1V^* = N_2$. This completes the proof. □

The following Corollary is an immediate consequence of Proposition 2.2.6.

Corollary 2.2.8. *If S is any subnormal operator and N_1 and N_2 are mne of S then N_1 and N_2 are unitarily equivalent.*

Because of Corollary 2.2.8, it follows that if $S \in B(H)$ is a subnormal operator, then it is permissible to speak of the minimal normal extension $N \in B(K)$ of S and we write N = mne (S). Nevertheless, the explicit determination of the mne of a subnormal operator can be a non-trivial problem [32]; any special case where that extension is accessible is worth looking at. For a trivial case, consider the unilateral forward shift which is subnormal. It is not difficult to see that its mne is the bilateral shift.

2.3 Matrix Representation

The content of this section can be found in the books of Conway [23], [24]. We assume that $S : H \to H$ is a subnormal operator and $N : K \to K$ the minimal normal extension of S.

If we write $K = H \oplus H^{\perp}$, then N has the 2×2 matrix representation with respect to the decomposition $K = H \oplus H^{\perp}$ in the form

$$N = \begin{pmatrix} S & X \\ 0 & T^* \end{pmatrix}, \tag{2.3.1}$$

where $S : H \to H$, $X : H^{\perp} \to H$, $0 : H \to H^{\perp}$, and $T^* : H^{\perp} \to H^{\perp}$ are operators.
The 0 appearing in (2.3.1) since $NH \subseteq H$. Indeed if $h \in H$, then $Nh \in H$ and $Nh = Sh$ because

$$\begin{pmatrix} S & X \\ 0 & T^* \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix} = \begin{pmatrix} Sh \\ 0 \end{pmatrix}.$$

If the matrix for N^* is written relative to the same decomposition, $K = H \oplus H^{\perp}$, it is

$$N^* = \begin{pmatrix} S^* & 0 \\ X^* & T \end{pmatrix},$$

with $S^* : H \to H$, $0 : H^{\perp} \to H$, $X^* : H \to H^{\perp}$, and $T : H^{\perp} \to H^{\perp}$.

However, if the matrix of N^* is written relative to the decomposition $K = H^{\perp} \oplus H$, it takes the form

$$N^* = \begin{pmatrix} T & X^* \\ 0 & S^* \end{pmatrix},$$

where $T : H^\perp \to H^\perp$, $X^* : H \to H^\perp$, $0 : H^\perp \to H$, and $S^* : H \to H$. Note that if $h \in H^\perp$ then $N^*h \in H^\perp$ and $N^*h = Th$ since

$$\begin{pmatrix} T & X^* \\ 0 & S^* \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix} = \begin{pmatrix} Th \\ 0 \end{pmatrix}.$$

From this, it follows that the operator T appearing in (2.3.1) is subnormal and N^* is a normal extension of T. It's now natural to ask the following question. Is N^* the minimal normal extension of T?

The next proposition (see [24, Proposition 2.10, p. 40]) answers this question, and in fact, it shows that purity and minimality are related concepts in operator theory.

Proposition 2.3.2. *With the notation above, the following statements are equivalents.*

(i) S *is pure.*

(ii) N^* *is the minimal normal extension of* T.

Proof. Let $L = \overline{\text{span}\{N^n g : g \in H^\perp \text{ and } n \geq 0\}}$. By Proposition 2.2.5, $N^* = \text{mne}\,(T)$ if and only if $L = K$. Now it is clear that L reduces N and $L^\perp \subseteq H$ since $H^\perp \subseteq L$. Hence, $L^\perp$ reduces N and $N|_{L^\perp} = S|_{L^\perp}$ is normal. If S is pure, then $0 = H \cap L^\perp$ by Corollary 2.2.3. So, $L = K$ and $N^* = \text{mne}\,(T)$.

Conversely, assume that S is not pure; so there is a proper subspace M of H such that M reduces N. Thus, $M^\perp$ reduces N^* and $H^\perp \subseteq M^\perp$. Thus $N^*|_{M^\perp}$ is a normal extension of $N^*|_{H^\perp} = T$. This completes the proof. □

Let us recall that an operator $A : H \to H$ is hyponormal if and only if $\langle A^*A - AA^*h, h\rangle \geq 0$ for all $h \in H$. Keeping in mind the notation above, the next result which can also be found in the book of Conway [23] tells us that subnormal operators are hyponormal.

Proposition 2.3.3. *Every subnormal operator $S : H \to H$ is hyponormal.*

Proof. Let N= mne (S) and write

$$N = \begin{pmatrix} S & X \\ 0 & T^* \end{pmatrix}$$

its matrix representation with respect to the decomposition $K = H \oplus H^{\perp}$.

Then

$$N^*N = \begin{pmatrix} S^* & 0 \\ X^* & T \end{pmatrix} \begin{pmatrix} S & X \\ 0 & T^* \end{pmatrix} = \begin{pmatrix} S^*S & S^*X \\ X^*S & X^*X + TT^* \end{pmatrix}$$

and

$$NN^* = \begin{pmatrix} S & X \\ 0 & T^* \end{pmatrix} \begin{pmatrix} S^* & 0 \\ X^* & T \end{pmatrix} = \begin{pmatrix} SS^* + XX^* & XT \\ T^*X^* & T^*T \end{pmatrix}.$$

Since N is normal, $N^*N = NN^*$. Hence $S^*S = SS^* + XX^*$,

i.e. $S^*S - SS^* = XX^*$.

Thus if $h \in H$, we obtain

$$\langle S^*S - SS^*h, h\rangle = \langle XX^*h, h\rangle = \langle X^*h, X^*h\rangle = \|X^*h\|^2 \geq 0.$$

Hence S is hyponormal which completes the proof. □

2.4 Nonsupercyclicity of the Hyponormal Operators

In 1982, Kitai [36] proved that hyponormal operators cannot be hypercyclic. It turns out that normal, quasinormal and subnormal operators cannot have hypercyclic vectors. Few years later Chan and Sanders asked in [18] whether there exist a weakly hypercyclic hyponormal operator on a Hilbert space. Sanders [46] gave a negative answer to the question. Concerning supercyclicity of the operators, Hilden and Wallen [34] proved earlier that normal operators do not have supercyclic vectors. A stronger result was found in 1997 by Bourdon [10], who proved that hyponormal operators are not supercyclic. Because of the strength of this result, and given that hyponormal operators form a larger class of quasinormal and subnormal operators, we provide in this section its proof.

The following three results as in Bourdon [10] are preliminary.

Proposition 2.4.1. *Let $T : H \to H$ be a hyponormal operator. Then, for every $f \in H$ and $n \geq 1$,*

$$\|T^n f\|^2 \leq \|T^{n+1} f\| \|T^{n-1} f\|. \tag{2.4.2}$$

Proof. For $f \in H$ and $n \geq 1$,

$$\begin{aligned} \|T^n f\|^2 &= \langle T^n f, T^n f\rangle \\ &= \langle T^* T^n f, T^{n-1}\rangle \\ &\leq \|T^* T^n f\| \|T^{n-1} f\| \\ &\leq \|T^{n+1} f\| \|T^{n-1} f\|, \text{by the hyponormality of T.} \end{aligned}$$

This completes the proof. □

Note that if Tf is non-zero, we have by proposition 2.4.1,
$\|Tf\|^2 \leq \|T^2 f\| \|f\| \leq \|T\| \|Tf\| \|f\|$. So, $\|Tf\| \leq \|T\| \|f\|$. Since $Tf \neq 0$, we must have $f \neq 0$. Also, since $0 < \|T^2 f\| \leq \|T\|^2 \|f\|$, as above, we get $\|T^2 f\| \neq 0$. Continuing in this fashion, we get $f \neq 0, Tf \neq 0, T^2 f \neq 0, \ldots, T^n f \neq 0$. That is all elements in Orb(T, f) must be non-zero. Hence, if T is hyponormal and Tf is non-zero for some vector $f \in H$, then all elements in Orb(T, f) must be non-zero.

Corollary 2.4.3. *Let $T : H \to H$ be a hyponormal operator. If $Tf \neq 0$ for some $f \in H$, then the sequence $\left(\frac{\|T^{n+1} f\|}{\|T^n f\|}\right)$ is increasing.*

Proof. Let $a_n = \frac{\|T^{n+1}f\|}{\|T^n f\|}$. So, $a_{n+1} = \frac{\|T^{n+2}f\|}{T^{n+1}f}$. We have

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\|T^{n+2}f\|}{\|T^{n+1}f\|}\frac{\|T^n f\|}{\|T^{n+1}f\|} \\
&= \frac{\|T^{n+2}f\|\|T^n f\|}{\|T^{n+1}f\|^2} \\
&\geq \frac{\|T^{n+2}f\|\|T^n f\|}{\|T^{n+2}f\|\|T^n f\|}, \text{by } (2.4.2) \\
&= 1
\end{aligned}$$

Thus, $a_{n+1} \geq a_n$ for all $n \geq 1$ implies (a_n) is increasing. □

It turns out that if $f \notin \ker T$ then the sequence $\left(\frac{\|T^{n+1}f\|}{\|T^n f\|}\right)$ is bounded above by $\|T\|$, and hence it converges whenever T is an hyponormal operator by the previous Corollary.

Proposition 2.4.4. *Let $T : H \to H$ be a hyponormal operator. If there exists $f \in H$ such that $\|Tf\| \geq \|f\|$, then $(\|T^n f\|)$ is an increasing sequence.*

Proof. For each $n \geq 0$, let $a_n = \|T^n f\|$. If $f = 0$, then the sequence (a_n) is constant (each term zero) so that the proposition holds. Suppose that f is non-zero. Then, applying the hypothesis of the proposition, we have $a_1 \geq a_o > 0$. Now, observe that if $a_n \geq a_{n-1} > 0$ for some $n \geq 1$ then using (2.4.2), we get

$$a_{n+1} - a_n \geq \frac{(a_n)^2}{a_{n-1}} - a_n = \frac{a_n(a_n - a_{n-1})}{a_{n-1}} \geq 0.$$

Thus, by induction, this completes the proof. □

Let us recall that an operator T on a complex Hilbert space H is supercyclic if there is a vector $h \in H$ such that $\{cT^n h : c \in \mathbb{C} \text{ and } n = 0, 1, 2, \ldots\}$ is dense in H. As noted before, one can verify that supercyclic operators must have dense range. Because all powers of a dense range operator will map dense sets to dense sets, if h is a supercyclic vector for T and c is a non-zero scalar, then $cT^n h$ will also be supercyclic for T for any non-negative integer n. Thus, the collection of supercylic vectors for a given supercyclic operator T on H will always be dense in H.

Theorem 2.4.5. (Bourdon [10]). *Let $T : H \to H$ be a hyponormal operator on the Hilbert space H. Then T cannot be supercyclic.*

Proof. Suppose that T is supercyclic. Set $A = \frac{T}{\|T\|}$ so that A is hyponormal, supercyclic, and has norm 1.

Claim: A cannot be an isometry.

Proof of claim: If A were an isometry, it would have to be onto because supercyclic operators have dense range. In fact the range of a linear isometry $U : H \to H'$ between Hilbert spaces H and H' must always be closed; ie. $\overline{\text{Ran}\, U} = \text{Ran}\, U$. To see this, let (Ux_n) be a Cauchy sequence in $U(H)$. So $\|Ux_n - Ux_m\| \to 0$ as $n, m \to \infty$. Since U is an isometry, $\|x_n - x_m\| = \|Ux_n - Ux_m\|$ so that (x_n) is a Cauchy sequence in H. Since H is complete, $x_n \to x$ for some $x \in H$ and hence $Ux_n \to Ux$. This shows that $\text{Ran}\, U$ is complete, and since a complete subspace of normed space must be closed, $\text{Ran}\, U$ is closed. Now if the range of U is also dense then $H' = \overline{\text{Ran}\, U} = \text{Ran}\, U$. It follows that Ran $U = H'$ and so U is onto.

Hence A is linear, onto, and isometry between Hilbert spaces means that A is unitary. But every unitary operator is normal and a normal operator cannot be supercyclic by a result of Hilden and Wallen [34], a contradiction. □

Thus, since A is not an isometry, we may assume that there is a vector g in H such that

$$\|Ag\| < \|g\|.$$

Let α be a scalar of modulus greater than 1 such that

$$\|\alpha Ag\| < \|g\|. \tag{2.4.6}$$

Let $S = \alpha A$. Then S is supercyclic, hyponormal, and satisfies

$$\|Sg\| = \|\alpha Ag\| < \|g\| \tag{2.4.7}$$

by (2.4.6). Also, we have

$$\|S\| = \|\alpha A\| = |\alpha|\|A\| > 1.$$

Since $\|S\| > 1$ and the set of supercyclic vectors for S is dense in H, there is a supercyclic vector $f \in H$ such that $\|Sf\| > \|f\|$ (ie., Sf is outside a closed ball). Since S is supercyclic, there is a subsequence (n_j) of the sequence of non-negative integers and a sequence (c_j) of scalars such that

$$(c_j S^{n_j} f) \to g. \tag{2.4.8}$$

By continuity, we have

$$(c_j S^{n_j+1} f) \to Sg. \tag{2.4.9}$$

However, because S is hyponormal and $\|Sf\| > \|f\|$, proposition 2.4.4 implies that

$$\|S^{n_j+1} f\| \geq \|S^{n_j} f\| \tag{2.4.10}$$

for every j. Thus,

$$\begin{aligned} \|Sg\| &= \lim_{j\to\infty} \|c_j S^{n_j+1} f\|, \text{ by } (2.4.9) \\ &\geq \lim_{j\to\infty} \|c_j S^{n_j} f\|, \text{ by } (2.4.10) \\ &= \|g\|, \end{aligned}$$

a contradiction. Hence T cannot be supercyclic. This completes the proof. □

2.5 Polar Decomposition of a Hilbert Space Operator

If $\lambda \in \mathbb{C}$, then its polar decomposition is $\lambda = |\lambda| e^{i\theta}$ for some θ. Except for the number 0, this polar decomposition is unique. In this section, we show that there is an analog for operators. It is particularly useful in proving results about quasinormal operators. References for polar decomposition of operators can be found in [21], [23], and [32].

Definition 2.5.1. *Let $A : H \to H$ be a bounded linear operator on the Hilbert space H. The* *modulus of A, denoted by $|A|$, is the unique positive operator $S : H \to H$ such that $S^2 = A^*A$, that is, $|A| = (A^*A)^{1/2}$ (The positive square root of A^*A).*

Definition 2.5.2. *An operator $W \in B(H)$ is called a partial isometry if $\|Wh\| = \|h\|$ for all $h \in (\ker W)^\perp$. The space $(\ker W)^\perp$ is called the initial space of W and the space $\operatorname{ran} W$ is called the final space of W.*

The following Theorem can be found in the book of Conway [21].

Theorem 2.5.3. (Polar decomposition). *If $A \in B(H)$, then there is a partial isometry W with $(\ker A)^\perp$ as its initial space and $\overline{\operatorname{ran} A}$ as its final space such that $A = W|A|$. Moreover, if $A = UP$ where $P \geq 0$ and U is a partial isometry with $\ker U = \ker P$, then $P = |A|$ and $U = W$.*

Proof. If $h \in H$, then

$$\|Ah\|^2 = \langle Ah, Ah\rangle = \langle A^*Ah, h\rangle = \left\langle |A|^2 h, h\right\rangle = \langle |A|h, |A|h\rangle = \||A|h\|^2.$$

Thus,

$$\|Ah\| = \||A|h\|. \tag{2.5.4}$$

In particular, we have $\ker A = \ker |A|$.

If $W_0 : \operatorname{ran} |A| \to \operatorname{ran} A$ is defined by

$$W_o(|A|h) = Ah, \tag{2.5.5}$$

then (2.5.4) implies that W_0 is a well-defined isometry. Thus W_0 extends to an isometry $F : (\ker A)^\perp \to \overline{\text{ran } A}$.

Now let $G = \overline{\text{ran } |A|} = \ker A^\perp$. So we have $H = G \oplus \ker A$, the orthogonal direct sum. For $x = y + z$, where $y \in \overline{\operatorname{ran} |A|}$ and $z \in \ker A$, let $W(x) = F(y)$, i.e., $W = FP_G$, where P_G is the orthogonal projection onto G. If $z \in \ker A$, then clearly $W(z) = 0$.

Also, if $x \in H$ then $x = y + z$, where $y \in \overline{\text{ran } |A|}$, $z \in \ker A$ and we have

$$(W|A|)(x) = (W|A|)(y + z) = W(|A|(y + z)) = W(|A|y) = F(P_G|A|y).$$

Note that $|A|(y) \in \text{ran } |A| \subset \overline{\text{ran } |A|}$. Hence

$$F(P_G|A|y) = F(|A|(y)) = W_o(|A|y) = Ax,$$

implying that $W|A| = A$ on H.

To prove uniqueness, suppose that $A = UP$, where U is a partial isometry, P is positive, and $\ker U = \ker P$. It follows that $A^* = PU^*$ and hence $A^*A = PU^*UP$.
We claim that U^*U is the projection onto the initial space of U, $\ker U^\perp$. To see this, let $M = \ker U^\perp$ and let E be the projection from H onto M. If $f \in M$, then

$$\langle U^*Uf, f\rangle = \|Uf\|^2 = \|f\|^2 = \langle Ef, f\rangle .$$

If $f \perp M$, i.e., $f \in (\ker U^\perp)^\perp = \ker U$, then

$$\langle U^*Uf, f\rangle = 0 = \langle Ef, f\rangle .$$

It follows that $\langle U^*Uf, f\rangle = \langle Ef, f\rangle$ for all f in H, and this implies that $U^*U = E$.

Thus, $A^*A = PEP = P^2$(because E is the projection onto $\ker U^\perp = \ker P^\perp = \overline{\text{ran } P}$).
By the uniqueness of the positive square root, $P = |A|$.
Next, since $A = W|A|$, we have

$$W|A|x = Ax = U|A|x.$$

However,

$$|A|x \in \overline{\text{ran } |A|} = \ker |A|^\perp = \ker A^\perp.$$

So W and U agree on a dense subset of their common initial space. This means that $W = U$. The proof is complete. □

Definition 2.5.6. *The representation $A = UP$ as the product of the unique operators U and P satisfying the conditions of Theorem 2.5.3 is called the polar decomposition of A.*

Corollary 2.5.7. *If $A \in B(H)$ and A has polar decomposition $A = UP$, then A is quasinormal if and only if $UP = PU$.*

Proof. Write $A = UP$. By the uniqueness of the polar decomposition, we know that $A^*A = P^2$. If $UP = PU$, then

$$A(A^*A) = AP^2 = UPP^2 = PUP^2 = P^2UP = P^2A = (A^*A)A.$$

Therefore, A is quasinormal.

Conversely, if A is quasinormal, then A commutes with $A^*A = P^2$. By the functional calculus for positive operators, A and P commute. Hence, if $y = Px$ is in the range of P, then

$$(UP - PU)y = (UP - PU)Px = UPPx - PUPx = APx - PAx = 0,$$

so that $(UP - PU)$ annihilates ran P. Since $\ker P = \ker U$, it is trivial that $(UP - PU)$ annihilates $\ker P$. Hence $UP - PU = 0$ and this completes the proof. □

The following result also found in [32] applies polar decomposition of operators.

Proposition 2.5.8. *Every quasinormal operator $A : H \to H$ is subnormal.*

Proof. We consider two cases.

Case 1: Suppose $\ker A = \{0\}$. If $A = UP$, where $P = |A|$, is the polar decomposition, then U must be an isometry. Let $E = UU^*$. So E is a projection and $(1 - E)U = U^*(1 - E) = 0$. An operator on $H \oplus H$ is given by a two-by-two matrix whose entries are operators on H.

So define operators V and B on $H \oplus H = K$ by

$$V = \begin{pmatrix} U & (1-E) \\ 0 & U^* \end{pmatrix}, \quad B = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix},$$

and let $N = VB$. Since $UP = PU$, we have $U^*P = PU^*$. In particular, V is unitary, B is positive, V and B commute, and therefore N is normal. But

$$N = \begin{pmatrix} A & (1-E)P \\ 0 & U^*P \end{pmatrix} = \begin{pmatrix} A & (1-E)P \\ 0 & A^* \end{pmatrix}$$

implies that N leaves $H = H \oplus 0$ invariant and $N|_H = A$. Thus, N is a normal extension of A.

Case 2: Suppose $\ker A \neq \{0\}$. Here $\ker A = \ker P = L \subseteq \ker A^*$, since $A^* = PU^* = U^*P$. Let $A_1 = A|_{L^\perp}$. So $A = 0 \oplus A_1$ on $L \oplus L^\perp = H$. Hence on $L \oplus L^\perp = H$, $A^*A = 0 \oplus A_1^*A_1$ and this implies that A_1 is quasinormal. Since $\ker A_1 = \{0\}$, case 1 implies that A_1 is subnormal. Hence A is subnormal and this completes the proof. □

2.6 Hypercyclicity of the Adjoint of Pure Quasinormal Operators Bounded from Below

The content of this section is part of the paper by Chan and Phanzu [20]. It contains new results on the hypercyclicity of the adjoint of a pure quasinormal operator. We show that any pure quasinormal operator bounded below by 1 has a hypercyclic adjoint. We begin by stating the following basic Theorem of Brown which can be found in the book of Conway [23, p. 135]. It gives a necessary and sufficient condition for an operator on a complex Hilbert space to be pure quasinormal. Precisely, it characterizes any pure quasinormal operator on a complex Hilbert space as an operator that is unitarily equivalent to a forward shifting operator given in the form of an infinite strictly lower triangular matrix having on its sub-diagonal a positive operator A with $\ker A = \{0\}$.

Theorem 2.6.1. (Brown). *An operator S on H is pure quasinormal if and only if there is a positive operator A on a Hilbert space L with* $\ker A = \{0\}$ *such that S is unitarily equivalent to*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ A & 0 & 0 & 0 & \dots \\ 0 & A & 0 & 0 & \dots \\ 0 & 0 & A & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \end{pmatrix}$$

on $L \oplus L \oplus ....$

We recall that if L is a Hilbert space, then the direct sum H of countably infinitely many copies of L is a Hilbert space given by

$$\begin{aligned} H = \bigoplus_{n=1}^{\infty} L &= L \oplus L \oplus L \oplus \dots \\ &= \left\{ h = (h_1, h_2, h_3, \dots) : \text{ each } h_n \in L \text{ and } \|h\|^2 = \sum_{n=1}^{\infty} \|h_n\|^2 < \infty \right\}, \end{aligned}$$

and its inner product $\langle ., . \rangle$ is given by

$$\langle f, h \rangle = \sum_{n=1}^{\infty} \langle f_n, h_n \rangle,$$

for all vectors $f = (f_1, f_2, \cdots)$ and $h = (h_1, h_2, \cdots)$ in H.

In the main Theorem of this section, which is Theorem 2.6.4, we are concerned with the operator of the form $T : \bigoplus_{n=1}^{\infty} L \to \bigoplus_{n=1}^{\infty} L$ given by

$$T\tilde{h} = T(h_1, h_2, h_3, \dots) = (0,\ A_1 h_1,\ A_2 h_2,\ A_3 h_3, \dots), \tag{2.6.2}$$

for all vectors $\tilde{h} = (h_1, h_2, h_3, \dots) \in \bigoplus_{n=1}^{\infty} L$, where L is a Hilbert space of positive dimension and

$\{A_n : L \to L | n \geq 1\}$ is a sequence of invertible self-adjoint bounded linear operators. When this operator is bounded from below by 1, we will see that the adjoint T^* of the operator T is hypercyclic. Before we prove the Theorem, we need the following Lemma which provides us with a necessary and sufficient condition for the operator T to be bounded from below. Observe that here the operators A_n in the sequence $\{A_n : L \to L | n \geq 1\}$ need not be self-adjoint or invertible.

Lemma 2.6.3. *Let L be a Hilbert space and $\{A_n : L \to L | n \geq 1\}$ be a sequence of uniformly bounded linear operators on L. Suppose $H = \bigoplus_{n=1}^{\infty} L$ and a bounded linear operator $T : H \to H$ is given by*

$$T\tilde{h} = T(h_1, h_2, h_3, \ldots) = (0,\ A_1 h_1,\ A_2 h_2,\ A_3 h_3, \ldots),$$

for all vectors $\tilde{h} = (h_1, h_2, h_3, \ldots) \in H$. Let $\alpha > 0$. Then $\|T\tilde{h}\| \geq \alpha \|\tilde{h}\|$ for all vectors $\tilde{h}$ in H if and only if $\|A_n g\| \geq \alpha \|g\|$ for all vectors $g \in L$.

Proof. Suppose $\|T\tilde{h}\| \geq \alpha\|\tilde{h}\|$ for all vectors $\tilde{h}$ in H, and $n \geq 1$ and g is a vector in L. Then let $\tilde{f} = (f_1, f_2, \ldots)$ be a vector in H defined by $f_n = g$, and $f_m = 0$ whenever $m \neq n$. Thus the inequality $\|T\tilde{f}\| \geq \alpha\|\tilde{f}\|$ is exactly the same as $\|A_n g\| \geq \alpha\|g\|$.

Conversely, suppose $\|A_n g\| \geq \alpha\|g\|$ for all vectors g in L. Then for any vectors $\tilde{h} = (h_1, h_2, h_3, \ldots)$ in H, $\|T\tilde{h}\|^2 = \|(0,\ A_1 h_1,\ A_2 h_2,\ \ldots)\|^2 = \sum \|A_n h_n\|^2 \geq \sum \alpha^2 \|h_n\|^2 = \alpha^2 \|\tilde{h}\|^2$. This completes the proof. □

We are now ready to prove the main Theorem.

Theorem 2.6.4. *Let L be a separable Hilbert space and $H = \bigoplus_{n=1}^{\infty} L$ be the direct sum of infinitely countably many copies of L. Define an operator $T : H \to H$ by*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ A_1 & 0 & 0 & 0 & \ldots \\ 0 & A_2 & 0 & 0 & \ldots \\ 0 & 0 & A_3 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \end{pmatrix},$$

where each operator $A_n : L \to L$ is an invertible self-adjoint bounded linear operator. If there exists $\alpha > 1$ such that $\|T\tilde{h}\| \geq \alpha\|\tilde{h}\|$ for all $\tilde{h}$ in H, then the adjoint T^* of T is hypercyclic on H.

Proof. We prove the Theorem using the Hypercyclicity Criterion. Observe first that if $\tilde{f} = (f_1, f_2, f_3, f_4, \ldots)$ is a vector in H, then

$$\begin{aligned} T\tilde{f} &= \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ A_1 & 0 & 0 & 0 & \ldots \\ 0 & A_2 & 0 & 0 & \ldots \\ 0 & 0 & A_3 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \end{pmatrix} \\ &= (0,\ A_1f_1,\ A_2f_2,\ A_3f_3,\ A_4f_4, \ldots), \end{aligned}$$

which is the operator given in (2.6.2).

Now let $\tilde{f} = (f_1, f_2, f_3, f_4, \ldots)$, $\tilde{g} = (g_1, g_2, g_3, g_4, \ldots)$ be two vectors in H. We have

$$\begin{aligned} \langle T\tilde{f}, \tilde{g}\rangle &= \langle (0,\ A_1f_1,\ A_2f_2,\ A_3f_3,\ A_4f_4, \ldots), (g_1, g_2, g_3, g_4, \ldots)\rangle \\ &= \sum_{j=1}^{\infty} \langle A_jf_j, g_{j+1}\rangle \\ &= \sum_{j=1}^{\infty} \langle f_j, A_j^* g_{j+1}\rangle. \end{aligned}$$

Since for all $n \geq 1$ each operator $A_n : L \to L$ is self-adjoint, we get that

$$\begin{aligned} \sum_{j=1}^{\infty} \langle f_j, A_j^* g_{j+1}\rangle &= \sum_{j=1}^{\infty} \langle f_j, A_j g_{j+1}\rangle \\ &= \langle (f_1, f_2, f_3, f_4, \ldots), (A_1g_2, A_2g_3, A_3g_4, A_4g_5, \ldots)\rangle \\ &= \langle \tilde{f}, T^*\tilde{g}\rangle. \end{aligned}$$

Hence the adjoint T^* of T is an operator given by

$$T^*\tilde{f} = T^*(f_1, f_2, f_3, f_4, \ldots) = (A_1 f_2,\ A_2 f_3,\ A_3 f_4,\ A_4 f_5, \ldots)$$

for all vectors $\tilde{f} = (f_1, f_2, f_3, f_4, \ldots) \in H$.

In the matrix form,

$$T^* = \begin{pmatrix} 0 & A_1 & 0 & 0 & \ldots \\ 0 & 0 & A_2 & 0 & \ldots \\ 0 & 0 & 0 & A_3 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \end{pmatrix}.$$

Let Q be an operator on H of the form

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ A_1^{-1} & 0 & 0 & 0 & \ldots \\ 0 & A_2^{-1} & 0 & 0 & \ldots \\ 0 & 0 & A_3^{-1} & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \end{pmatrix}.$$

Then

$$T^*Q = \begin{pmatrix} 0 & A_1 & 0 & 0 & \ldots \\ 0 & 0 & A_2 & 0 & \ldots \\ 0 & 0 & 0 & A_3 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ A_1^{-1} & 0 & 0 & 0 & \ldots \\ 0 & A_2^{-1} & 0 & 0 & \ldots \\ 0 & 0 & A_3^{-1} & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \end{pmatrix} = I,$$

the identity matrix.

Now let

$$E = \{(x_1, x_2, \ldots, x_k, 0, 0, 0, \ldots) \in H |\ \text{each } x_j \in L \text{ and } k \geq 1\}$$

be a subset of H.

Let $\tilde{h} = (h_1, h_2, h_3, ...)$ be any vector in H and so $\|\tilde{h}\|^2 = \sum_{n=1}^{\infty} \|h_n\|^2 < \infty$. Thus, for any $\varepsilon > 0$, there exists integer $N \geq 1$ such that $\sum_{n=N+1}^{\infty} \|h_n\|^2 < \varepsilon^2$.

Hence, taking $x = (h_1, h_2, \ldots, h_N, 0, 0, 0, \ldots)$, we have that $x \in E$, and so

$$\|x - h\|^2 = \sum_{n=N+1}^{\infty} \|h_n\|^2 < \varepsilon^2$$

which shows that the set E is dense in H.

Next, we show that $T^{*n}h \to 0$ and $Q^n h \to 0$ for all $h \in E$. First, observe that if $h = (h_1, h_2, \ldots, h_k, 0, 0, 0, \ldots) \in E$, then $T^*h = (A_1h_2,\ A_2h_3,\ A_3h_4, \ldots,\ A_{k-1}h_k, 0, 0, 0, \ldots)$, $T^{*2}h = (A_1A_2h_3,\ A_2A_3h_4,\ A_3A_4h_5, \ldots,\ A_{k-2}A_{k-1}h_k, 0, 0, 0, \ldots)$, and so inductively, for $0 < n \leq k-1$, we get

$$T^{*n}h = (\underbrace{A_1 \ldots A_n h_{n+1},\ A_2 \ldots A_{n+1} h_{n+2}, \ldots,\ A_{k-n}A_{k-n+1} \ldots A_{k-1}h_k}_{k-n \text{ terms}}, 0, 0, 0, \ldots).$$

Thus, it turns out that

$$T^{*k}h = (0, 0, 0, \ldots).$$

Hence, we conclude for any vector $h \in E$, there is a positive integer N such that $N \geq k$ implies

$$T^{*N}h = 0.$$

It follows that $T^{*n}h \to 0$ for all $h \in E$.

Now let $\tilde{h} = (h_1, h_2, h_3, \ldots)$ be a vector in H. Then, $Q\tilde{h} = Q(h_1, h_2, h_3, \ldots)$ $= (0,\ A_1^{-1}h_1,\ A_2^{-1}h_2,\ A_3^{-1}h_3, \ldots)$, and $Q^2\tilde{h} = (0, 0,\ A_2^{-1}A_1^{-1}h_1,\ A_3^{-1}A_2^{-1}h_2, \ldots)$, and $Q^3\tilde{h} = (0, 0, 0,\ A_3^{-1}A_2^{-1}A_1^{-1}h_1,\ A_4^{-1}A_3^{-1}A_2^{-1}h_2, \ldots)$. Thus, inductively, for all $n \geq 1$,

$$Q^n\tilde{h} = (\overbrace{0, \ldots, 0}^{n \text{ zeros}},\ A_n^{-1}A_{n-1}^{-1}\ldots A_1^{-1}h_1,\ A_{n+1}^{-1}A_n^{-1}\ldots A_2^{-1}h_2, \ldots).$$

Hence, if $h = (h_1, h_2, \ldots, h_k, 0, 0, 0, \ldots)$ is in E, then

$$Q^n h = (0, \ldots, 0, A_n^{-1}A_{n-1}^{-1}\ldots A_1^{-1}h_1, A_{n+1}^{-1}A_n^{-1}\ldots A_2^{-1}h_2, \ldots,$$

$$A_{n+k-1}^{-1}A_{n+k-2}^{-1}\ldots A_k^{-1}h_k, 0, 0, 0, \ldots).$$

Since, by hypothesis, there exists $\alpha > 1$ such that $\|Th\| \geq \alpha\|h\|$ for all h in H, we know by Lemma 2.6.3 this is the same as $\|A_n g\| \geq \alpha\|g\|$ for all $g \in L$.

Fix $j \in \{1, 2, 3, \ldots\}$ and let $g \in L$. Since each operator A_j is invertible,

$$\|g\| = \|A_j(A_j^{-1}g)\| \geq \alpha\|A_j^{-1}g\|.$$

Hence

$$\|A_j^{-1}g\| \leq \frac{1}{\alpha}\|g\|$$

and

$$\|A_{j+1}^{-1}A_j^{-1}g\| \leq \frac{1}{\alpha}\|A_j^{-1}g\| \leq \left(\frac{1}{\alpha}\right)^2\|g\|.$$

Continuing in this fashion, we see that for any positive integer l such that $l \geq j$,

$$\|A_{l+j-1}^{-1}A_{l+j-2}^{-1}\ldots A_j^{-1}g\| \leq \left(\frac{1}{\alpha}\right)^l\|g\| \tag{2.6.5}$$

for all $g \in L$.

Hence,

$$\|Q^n h\|^2 = \|(0, \ldots, 0, A_n^{-1}A_{n-1}^{-1}\ldots A_1^{-1}h_1, A_{n+1}^{-1}A_n^{-1}\ldots A_2^{-1}h_2, \ldots,$$

$$A_{n+k-1}^{-1}A_{n+k-2}^{-1}\ldots A_k^{-1}h_k, 0, 0, 0, \ldots)\|^2.$$

By the definition of the norm, the right hand side term of the above expression is equal to

$$\sum_{j=1}^{k} \|A_{n+j-1}^{-1} A_{n+j-2}^{-1} \dots A_j^{-1} h_j\|^2.$$

Thus, we can now apply (2.6.5) to get

$$\begin{aligned} \sum_{j=1}^{k} \|A_{n+j-1}^{-1} A_n^{-1} \dots A_j^{-1} h_j\|^2 &\leq \sum_{j=1}^{k} \left(\frac{1}{\alpha}\right)^{2n} \|h_j\|^2 \\ &\leq \left(\frac{1}{\alpha}\right)^{2n} \sum_{j=1}^{k} \|h_j\|^2, \end{aligned}$$

which converges to 0 as $n \to \infty$. Hence, by the hypercyclicity criterion , we conclude that T^* is hypercyclic. This completes the proof. □

Next, we address the question of when a pure quasinormal operator T has a hypercyclic adjoint T^*. For this, we need a few preliminary results.

Proposition 2.6.6. *Let μ be a Borel measure on a subset X of $\mathbb{C}$. Let $\Phi \in L^\infty(X)$ and $M_\Phi : L^2(X) \to L^2(X)$ be the multiplication operator defined by $M_\Phi f = \Phi f$ for all $f \in L^2(X)$. Let $\alpha > 1$. Then $\|M_\Phi f\| \geq \alpha \|f\|$ for all $f \in L^2(X)$ if and only if $|\Phi| \geq \alpha$ a.e.*

Proof. Suppose that $|\Phi| \geq \alpha$ a.e. Then for all $f \in L^2(X)$,

$$\|M_\Phi f\|^2 = \|\Phi f\|^2 = \int_\Omega |\Phi f|^2 d\mu \geq \alpha^2 \int_\Omega |f|^2 d\mu = \alpha^2 \|f\|^2.$$

Conversely, suppose there exists a measurable subset E of X with positive measure such that $|\Phi| < \alpha$. Let $f = \chi_E$.

We have

$$\|M_\Phi f\|^2 = \int_X |\Phi \chi_E|^2 d\mu < \alpha^2 \int_X |\chi_E|^2 d\mu \leq \alpha^2 \int_X \|f\|^2 d\mu = \alpha^2 \|f\|^2.$$

Hence $\|M_\Phi f\| < \alpha\|f\|$, as desired. □

The following Corollary asserts that if $A : L \to L$ is a positive operator which is bounded from below by a scalar $\alpha > 1$ then it is invertible.

Corollary 2.6.7. *Let $\alpha > 1$. If A is a positive operator on a Complex Hilbert space L such that $\|Af\| \geq \alpha\|f\|$ for all $f \in L$ then A is invertible.*

Proof. By the Spectral Theorem, A is unitarily equivalent to $M_\Phi : L^2(X,\mu) \to L^2(X,\mu)$ for some $\Phi \in L^\infty(X,\mu)$, where μ is a Borel measure on a subset X of $\mathbb{C}$. So, it suffices to prove the Corollary by taking $A = M_\Phi : L^2(X,\mu) \to L^2(X,\mu)$. Since then $\|M_\Phi f\| \geq \alpha\|f\|$ for all $f \in L^2(X,\mu)$, Proposition 2.6.6 implies $|\Phi| \geq \alpha$ a.e., and so $\frac{1}{|\Phi|} < \frac{1}{\alpha}$ a.e. Thus $M_{\frac{1}{\Phi}}$ is bounded on $L^2(X,\mu)$. This implies that the inverse

$$A^{-1} = M_\Phi^{-1} = M_{\Phi^{-1}} = M_{\frac{1}{\Phi}}$$

exists and completes the proof. □

The converse of Corollary 2.6.7 holds since an invertible operator must be bounded from below.

Recall if S is any pure quasinormal operator on a Hilbert space H then by Brown's Theorem S is unitarily equivalent to the infinite matrix

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ A & 0 & 0 & 0 & \dots \\ 0 & A & 0 & 0 & \dots \\ 0 & 0 & A & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \end{pmatrix}$$

on $\bigoplus_{n=1}^{\infty} L$ where A is a positive operator on the Hilbert space L with $\ker A = \{0\}$. Clearly, this positive operator $A : L \to L$ is self-adjoint by Lemma 2.1.7. Furthermore if there exists scalar

$\alpha > 1$ such that $\|Af\| \geq \alpha\|f\|$ for all $f \in L$, then A is invertible by Corollary 2.6.7. By Corollary 2.6.3, $\|Af\| \geq \alpha\|f\|$ for all $f \in L$ is equivalent to $\|T\tilde{h}\| \geq \alpha\|\tilde{h}\|$ for all $\tilde{h}$ in $H = \bigoplus_{n=1}^{\infty} L$. Now by taking all A_n in Theorem 2.6.4 to be the same positive operator A, we have proved the desired result.

Corollary 2.6.8. *Let H be an infinite-dimensional Hilbert space and let $T : H \to H$ be a pure quasinormal operator on H. If there exists $\alpha > 1$ such that T satisfies $\|T\tilde{f}\| \geq \alpha\|\tilde{f}\|$ for all $\tilde{f}$ in H, then the adjoint T^* of T is hypercyclic on H.*

We summarize Corollary 2.6.8 by saying that every pure quasinormal operator on a separable, infinite-dimensional, complex Hilbert space, bounded below by 1 has a hypercyclic adjoint. In the next Chapter, we focus on the supercyclicity aspect of the adjoint of these operators.

CHAPTER 3 SUPERCYCLICITY OF THE ADJOINT OF PURE QUASINORMAL OPERATORS

3.1 Introduction

The content of this Chapter is part of the paper by Chan and Phanzu [20]. It contains new results on the supercyclicity of the adjoint of a pure quasinormal operator. We prove that the adjoint T^* of every pure quasinormal operator $T : H \to H$ on a separable, infinite-dimensional, complex Hilbert space H is supercyclic, by using some of the results in the previous Chapter. It is our intention here to obtain such a supercyclic vector constructively and make no assumption on the boundedness from below of the operator.

As in Chapter 2, we will see that Brown's Theorem [14, p. 135] is put to use again since the adjoint T^* of a pure quasinormal operator $T : H \to H$ is unitarily equivalent to the adjoint of an operator that is in the form of the operator (2.6.2). In fact, in section 3.3, we show that the adjoint of an operator that is in the form of the operator (2.6.2) is supercyclic, and since unitary equivalence preserves supercyclicity, this yields our result. Note that this phenomenon may be explained by the result of Hilden and Wallen [34], who showed that every unilateral weighted backward shift is indeed supercyclic.

3.2 Preliminary New Results

In section 2.6, we proved that the adjoint of a pure quasinormal operator is hypercyclic precisely when the pure quasinormal operator is bounded below by a scalar $\alpha > 1$. Here we address the question about what happens if we let the pure quasinormal operator be bounded below by a scalar $\alpha \in (0, 1]$. We examine this question by first defining an operator that is a scalar multiple of an operator that is in the form of the operator (2.6.2). It turns out that the adjoint of such operator is hypercyclic. We then apply the same technique to obtain the supercyclicity of the adjoint of a pure quasinormal operator bounded below by a scalar $\alpha \in (0, 1]$. The details of this discussion are contained in the proofs of the following two preliminary results.

Corollary 3.2.1. *Let L be a separable, infinite dimensional, complex Hilbert space and $H = \bigoplus_{n=1}^{\infty} L$ with respect to which an operator $T : H \to H$ is given by*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ A_1 & 0 & 0 & 0 & \dots \\ 0 & A_2 & 0 & 0 & \dots \\ 0 & 0 & A_3 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \end{pmatrix},$$

where each operator $A_n : L \to L$ is an invertible self-adjoint bounded linear operator. Assume further that there exists a scalar α satisfying $1 \geq \alpha > 0$ such that $\|A_n g\| \geq \alpha \|g\|$ for all $g \in L$. Then the adjoint T^* of T is supercyclic.

Proof. Put $\beta = \alpha + 1$ and let $T_0 = \frac{\beta}{\alpha} T$. Clearly, for all $n \geq 1$, we have that each $\frac{\beta}{\alpha} A_n$ is an invertible self-adjoint bounded linear operator. Now let $\tilde{h}$ be a vector in H. By using Lemma 2.6.3, we get

$$\|T_0 \tilde{h}\| = \left\| \frac{\beta}{\alpha} T \tilde{h} \right\| = \frac{\beta}{\alpha} \|T \tilde{h}\| \geq \frac{\beta}{\alpha} \alpha \|\tilde{h}\| = \beta \|\tilde{h}\|.$$

Hence, $\|T_0 \tilde{h}\| \geq \beta \|\tilde{h}\|$ for all $\tilde{h} \in H$. Since $\beta > 1$, it follows from Theorem 2.6.4 that $T_0^* = \frac{\beta}{\alpha} T^*$ is hypercyclic, and hence T^* is supercyclic. □

Corollary 3.2.2. *Let H be a separable, infinite-dimensional, complex Hilbert space and let $T : H \to H$ be a pure quasinormal operator on H. If there exists scalar α satisfying $1 \geq \alpha > 0$ such that $\|T \tilde{f}\| \geq \alpha \|\tilde{f}\|$ for all $\tilde{f}$ in H, then the adjoint T^* of T is supercyclic on H.*

Proof. Let $T_0 = \frac{\beta}{\alpha} T$ where, as in the previous Corollary, we let $\beta = \alpha + 1$. Clearly, T_0 is pure quasinormal as a product of a pure quasinormal operator T with a scalar $1 + \frac{1}{\alpha}$. Let $\tilde{h} \in H$. By assumption,

$$\|T_0 \tilde{h}\| = \left\| \frac{\beta}{\alpha} T \tilde{h} \right\| = \frac{\beta}{\alpha} \|T \tilde{h}\| \geq \frac{\beta}{\alpha} \alpha \|\tilde{h}\| = \beta \|\tilde{h}\|.$$

So we have $\|T_0\tilde{h}\| \geq \beta\|\tilde{h}\|$ for all $\tilde{h} \in H$. Since $\beta > 1$ and T_0 is pure quasinormal, we apply Corollary 2.6.8 to get that the adjoint $T_0^* = \frac{\beta}{\alpha}T^*$ is hypercyclic. Therefore, T^* is supercyclic and the proof is complete. □

So far we have observed that the interval on which must lie a lower bound of the forward shifting operator T as in Theorem 2.6.4 and Corollary 3.2.1 plays a crucial role in determining the hypercyclicity or supercyclicity of the adjoint T^* of the operator T.

In the following section, we want to obtain supercyclicity results of the adjoint of a pure quasinormal operator by closely looking at the backward shifting operator with underlying conditions as made explicit below.

3.3 Construction of the Supercyclic Vector

In Corollary 2.6.8 and corollary 3.2.2, we proved that the adjoint of every pure quasinormal operator is supercyclic when the operator is bounded from below by some scalar α where $\alpha > 1$ or $1 \geq \alpha > 0$. However the next theorem shows that this condition about the boundedness from below of the operator is no longer needed. In fact, let us define a backward shifting operator T on the direct sum $\bigoplus_{i=0}^{\infty} L^2(\mu)$ of square integrable functions by the following formula

$$T(f_0, f_1, f_2, \ldots) = (A_1 f_1, A_2 f_2, A_3 f_3, \ldots), \tag{3.3.1}$$

where each A_i is a left multiplication operator $M_{\varphi_i} : L^2(\mu) \to L^2(\mu)$ induced by a function $\varphi_i \in L^\infty(\mu)$, and given by $M_{\varphi_i} f = \varphi_i f$ for all $f \in L^2(\mu)$. Note the Spectral Theorem implies that every positive operator $A : L \to L$ on a separable complex Hilbert space L is unitarily equivalent to a left multiplication operator. Hence if we take all A_i in (3.3.1) to be the same positive operator $A = M_\varphi$, with $\varphi \neq 0$ a.e, then the operator T reduces to the adjoint of the pure quasinormal operator in the Brown's Theorem [14, p. 135]. As we state the Theorem, we impose that $\varphi_n \neq 0$ a.e. so that the left multiplications $A_n : L^2(\mu) \to L^2(\mu)$ have null kernels. The measure μ is a σ-finite measure. This choice of measure μ is well explained for instance in the book of Conway

[21, Proposition 4.7, p. 273].

Theorem 3.3.2. *Let* $(\varphi_n)_{n\geq 1} \subset L^\infty(\mu)$ *be a uniformly bounded sequence such that* $\varphi_n \neq 0$ *a.e. Define* $A_n : L^2(\mu) \to L^2(\mu)$ *by* $A_n g = \varphi_n g$ *for all* $g \in L^2(\mu)$*. If* $T : \bigoplus_{i=0}^{\infty} L^2(\mu) \to \bigoplus_{i=0}^{\infty} L^2(\mu)$ *is defined by*

$$T(f_0, f_1, f_2, \ldots) = (A_1 f_1, A_2 f_2, A_3 f_3, \ldots), \tag{3.3.3}$$

then T *is supercyclic.*

Before proving the theorem by constructing a supercyclic vector for the operator T, we need a few Lemmas. The first lemma is to show us how to find a function h for which a vector of the form $A_1 A_2 \ldots A_n h$ can be used to approximate a given square integrable function f, under an extra hypothesis that $|\varphi_i| > 0$ a.e.

Lemma 3.3.4. *Let* $\{\varphi_i \in L^\infty(\mu) : i \geq 1\}$ *be a family of essentially bounded functions satisfying* $|\varphi_i| > 0$ *a.e. Let* $A_i = M_{\varphi_i} : L^2(\mu) \to L^2(\mu)$ *be defined by*

$$M_{\varphi_i} g = \varphi_i g \quad for\ all\ g \in L^2(\mu).$$

For any $f \in L^2(\mu)$ *and* $\epsilon > 0$ *and any integer* $n \geq 1$*, there exists* $h \in L^2(\mu)$ *such that*

$$\|A_1 A_2 \ldots A_n h - f\| < \epsilon.$$

Proof. Let $E_0 = \{|f| > 1\}$ and for $n \geq 1$, let $E_n = \left\{\frac{1}{n+1} < |f| \leq \frac{1}{n}\right\}$. Thus $X = \bigcup_{n=0}^{\infty} E_n$ and we have

$$\infty > \int_X |f|^2 d\mu = \sum_{n=0}^{\infty} \int_{E_n} |f|^2 d\mu \geq \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \mu(E_n).$$

Hence, for all $n \geq 0$, $\mu(E_n) < \infty$, and furthermore, there exists $N \geq 1$ such that

$$\sum_{n=N+1}^{\infty} \int_{E_n} |f|^2 d\mu < \frac{\varepsilon^2}{2} \tag{3.3.5}$$

Let $F = \bigcup\limits_{m=0}^{N} E_m$ and so $\mu(F) < \infty$.
Note that for all $i = 1, 2, 3, \ldots$,

$$\left\{|\varphi_i| < \frac{1}{n+1}\right\} \subset \left\{|\varphi_i| < \frac{1}{n}\right\}.$$

Hence, for all $i = 1, 2, 3, \ldots$,

$$\begin{aligned}
\lim_{n\to\infty} \mu\left(\left\{|\varphi_i| < \frac{1}{n}\right\} \cap F\right) &= \mu\left(\bigcap_{n=1}^{\infty} \left\{|\varphi_i| < \frac{1}{n}\right\} \cap F\right) \\
&= \mu\left(\{|\varphi_i| = 0\} \cap F\right) \\
&= 0,
\end{aligned}$$

by our hypothesis that $\varphi_i > 0$ a.e.

Hence, for all i, there exists $M_i \geq 1$ such that if $n \geq M_i$, then

$$\mu\left(\left\{|\varphi_i| < \frac{1}{n}\right\} \cap F\right) < \frac{\varepsilon^2}{2^i}. \tag{3.3.6}$$

In particular, for all i, we have

$$\mu\left(\left\{|\varphi_i| < \frac{1}{M_i}\right\} \cap F\right) < \frac{\varepsilon^2}{2^i}.$$

Let

$$G = F \cap \bigcap_{i=1}^{\infty} \left\{|\varphi_i| \geq \frac{1}{M_i}\right\}.$$

So,

$$F \setminus G = F \cap \bigcup_{i=1}^{\infty} \left\{|\varphi_i| < \frac{1}{M_i}\right\}.$$

Hence,

$$\mu(F \setminus G) = \mu\left(F \cap \left(\bigcup_{i=1}^{\infty} \left\{|\varphi_i| < \frac{1}{M_i}\right\}\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} \left(F \cap \left\{|\varphi_i| < \frac{1}{M_i}\right\}\right)\right),$$

and by the countable sub-additivity of the measure, we get

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty}\left(F\cap\left\{|\varphi_i|<\frac{1}{M_i}\right\}\right)\right) &\leq \sum_{i=1}^{\infty}\mu\left(F\cap\left\{|\varphi_i|<\frac{1}{M_i}\right\}\right)\\
&= \sum_{i=1}^{\infty}\mu\left(\left\{|\varphi_i|<\frac{1}{M_i}\right\}\cap F\right)\\
&< \sum_{i=1}^{\infty}\frac{\varepsilon^2}{2^i},\ \text{by (3.3.6)}\\
&= \varepsilon^2.
\end{aligned}$$

Thus $\mu(F\backslash G)<\varepsilon^2$. Since $f\in L^2(\mu)$, the absolute continuity of the integral (see [45], p. 267-268) implies that

$$\int_{F\backslash G}|f|^2d\mu<\frac{\varepsilon^2}{2}. \tag{3.3.7}$$

Since $|\varphi_i|\geq\frac{1}{M_i}$ on G, we have $|\varphi_i^{-1}|\leq M_i$ and hence for all $n\geq 1$,

$$\varphi_n^{-1}\varphi_{n-1}^{-1}\dots\varphi_2^{-1}\varphi_1^{-1}f\chi_G\in L^2(\mu).$$

So let $h=\varphi_n^{-1}\varphi_{n-1}^{-1}\dots\varphi_2^{-1}\varphi_1^{-1}f\chi_G$. Then $h\in L^2(\mu)$ and, clearly,

$$A_1A_2\dots A_{n-1}A_nh=f\chi_G.$$

Hence using equations $(3.3.5)$ and $(3.3.7)$, we get

$$\begin{aligned}
\|A_1A_2\dots A_nh-f\|^2 &= \int_X|f\chi_G-f|^2\\
&= \int_{F^c}|f\chi_G-f|^2+\int_{F\backslash G}|f\chi_G-f|^2+\int_G|f\chi_G-f|^2\\
&= \int_{F^c}|f|^2+\int_{F\backslash G}|f|^2\\
&< \frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}\\
&= \varepsilon^2.
\end{aligned}$$

Thus,

$$\|A_1A_2\ldots A_nh - f\| < \varepsilon$$

which completes the proof. □

Notice that for each given f_i with $1 \leq i \leq N$ and any positive integer n, Lemma 3.3.4 provides us with a function h_i that facilitates the approximation of f_i by the vectors $A_iA_{i+1}\ldots A_{n+i-1}h_i$. This approximation is important and will serve in proving relevant statements in the next Lemma.

Lemma 3.3.8. *Let* $H = \bigoplus_{n=0}^{\infty} L^2(\mu)$ *and* $\varphi_i \in L^\infty(\mu)$*. For* $i \geq 1$*, let* $A_i : L^2(\mu) \to L^2(\mu)$ *be defined by* $A_if = \varphi_if$*. Let* $T : H \to H$ *be defined by*

$$T(f_0, f_1, f_2, f_3, \ldots) = (A_1f_1, A_2f_2, A_3f_3 \ldots).$$

For any vector $\tilde{f} = (f_1, f_2, \ldots, f_N, 0, 0, 0, \ldots)$ *in* H *and any* $\varepsilon, \delta > 0$*, and any integer* $n \geq 1$*, there exists* $\tilde{h} \in H$ *and there exists* $a > 0$ *such that*

(i) $\|\tilde{h}\| < \delta$ and

(ii) $\|aT^n\tilde{h} - \tilde{f}\| < \varepsilon$ and

(iii) there exists k such that $T^k\tilde{h} = 0$.

Proof. For each given f_i with $1 \leq i \leq N$, and any positive integer n, by Lemma 3.3.4 there exists $h_i \in L^2(\mu)$ such that

$$\|A_iA_{i+1}\ldots A_{n+i-1}h_i - f_i\| < \frac{\varepsilon}{\sqrt{N}} \tag{3.3.9}$$

Let $a = \delta^{-1}\sqrt{N}\max_{1\leq j\leq N}\|h_j\|$, and for the given integer n, let

$$\tilde{h} = \frac{1}{a}\left(\overbrace{0, 0, \ldots, 0}^{n \text{ zeros}}, h_1, h_2, \ldots, h_N, 0, 0, 0, \ldots\right) \in H.$$

We have

$$\|\tilde{h}\|^2 = \frac{1}{a^2}\sum_{i=1}^{N}\|h_i\|^2 = \sum_{i=1}^{N}\frac{\delta^2}{N}\frac{\|h_i\|^2}{\max\limits_{1\le j\le N}\|h_j\|^2} \le \frac{\delta^2}{N}\sum_{i=1}^{N}1 = \delta^2$$

establishing statement (i). We proceed to show statement (ii).

$$\begin{aligned}
& \|aT^n\tilde{h} - \tilde{f}\|^2 \\
= \; & \|T^n(\overbrace{0,0,\ldots,0}^{n \text{ zeros}}, h_1, h_2, \ldots, h_N, 0,0,0,\ldots) - (f_1, f_2, \ldots, f_N, 0,0,0,\ldots)\|^2 \\
= \; & \|(A_1A_2\ldots A_nh_1, A_2A_3\ldots\ldots A_{n+1}h_2, \ldots, A_NA_{N+1}\ldots A_{n+N-1}h_N, \\
& 0,0,0,\ldots) - (f_1, f_2, \ldots, f_N, 0,0,0,\ldots)\|^2 \\
= \; & \|(A_1A_2\cdots A_nh_1 - f_1, A_2A_3\cdots A_{n+1}h_2 - f_2, \ldots, A_NA_{N+1}\ldots A_{n+N-1}h_N \\
& -f_N, 0,0,0,\ldots)\|^2.
\end{aligned}$$

Now

$$\begin{aligned}
& \|(A_1A_2\cdots A_nh_1 - f_1, A_2A_3\cdots A_{n+1}h_2 - f_2, \ldots, A_NA_{N+1}\ldots A_{n+N-1}h_N \\
& -f_N, 0,0,0,\ldots)\|^2 \\
= \; & \sum_{i=1}^{N}\|A_iA_{i+1}\ldots A_{n+i-1}h_i - f_i\|^2 \\
< \; & \sum_{i=1}^{N}\frac{\varepsilon^2}{N} = \varepsilon^2 \text{ by } (3.3.9).
\end{aligned}$$

Thus $\|aT^n\tilde{h} - \tilde{f}\|^2 < \varepsilon^2$, which establishes statement (ii).

To show statement (iii), note that

$$T^n\tilde{h} = \frac{1}{a}(A_1A_2\cdots A_nh_1, A_2A_3\cdots A_{n+1}h_2, \ldots, A_NA_{N+1}\cdots A_{n+N-1}h_N, 0,0,0,\ldots).$$

Hence by taking T^N on both sides, we have $T^{n+N}\tilde{h} = 0$. Therefore there exists $k \ge n+N$ such that $T^k\tilde{h} = 0$, establishing statement (iii). $\square$

In Lemma 3.3.8, we used an approximation argument to show that for any vector $\tilde{f} = (f_1, f_2, \ldots, f_N, 0, 0, 0, \ldots)$ in the Hilbert space H and any $\varepsilon, \delta > 0$, and any integer $n \geq 1$, there exists $\tilde{h} \in H$ and there exists $a > 0$ such that

$$\|aT^n\tilde{h} - \tilde{f}\| < \varepsilon.$$

This means precisely that the open ball $B(\tilde{f}, \varepsilon)$ centered at $\tilde{f}$ with radius ε in the norm-topology of the Hilbert space contains an element of the form $aT^n\tilde{h}$. In what follows, we infer this method to constructing a supercyclic vector for the backward shifting operator

$$T(f_0, f_1, f_2, \ldots) = (A_1 f_1, A_2 f_2, A_3 f_3, \ldots).$$

Proof of Theorem 3.3.2. Let $H = \bigoplus_{i=0}^{\infty} L^2(\mu)$, and $B(g, r)$ be the open ball given by $\{h \in H : \|h - g\| < r\}$. Let $\left\{B(\tilde{f}_i, \varepsilon_i) : i \geq 1\right\}$ be a countable base of the norm-topology of H, where each $\tilde{f}_i$ is of the form

$$\tilde{f}_i = (f_1, f_2, \ldots, f_N, 0, 0, 0, \ldots)$$

for some positive integer N.

We now use Lemma 3.3.8 to construct a supercyclic vector $\tilde{x}$ as follows.

For the ball $B(\tilde{f}_1, \varepsilon_1)$, we apply the previous Lemma to get $\tilde{h_1} \in H$, $a_1 > 0$, $n_1 \geq 1$, $k_1 \geq 1$ such that

i) $\|\tilde{h_1}\| < \frac{1}{2}$,

ii) $\|a_1 T^{n_1}\tilde{h_1} - \tilde{f_1}\| < \frac{\varepsilon_1}{2}$, and

iii) $T^{k_1}\tilde{h_1} = 0$.

Inductively, for all $j \geq 2$, we apply the previous Lemma to obtain

$$\tilde{h_j} \in H, \ \ a_j > 0, k_j > n_j > k_{j-1}, \tag{3.3.10}$$

and furthermore if we let $b = \max(1, \|T\|)$, then

1) $\|\tilde{h_j}\| < \min\left(\frac{1}{2^j}, \frac{\varepsilon_1}{2^j a_1 b^{n_1}}, \cdots, \frac{\varepsilon_{j-1}}{2^j a_{j-1} b^{n_{j-1}}}\right)$,

2) $\|a_j T^{n_j}\tilde{h_j} - \tilde{f_j}\| < \frac{\varepsilon_j}{2}$, and

3) $T^{k_j}\tilde{h_j} = 0$.

Since $\|\tilde{h_j}\| < \frac{1}{2^j}$ for each positive integer j, we can define a vector $\tilde{x}$ in H by the absolutely convergent series $\tilde{x} = \sum_{j=1}^{\infty} \tilde{h_j}$.

We now show that $\tilde{x}$ is a supercyclic vector as follows.

Note that $a_m T^{n_m}\tilde{x} = a_m \sum_{j=1}^{\infty} T^{n_m}\tilde{h_j}$, and $T^{n_m}\tilde{h_j} = 0$ if $j < m$ (because $n_j > k_{j-1} > n_{j-1} > k_{j-2} > \ldots$ by equation (3.3.10)), and hence

$$a_m T^{n_m}\tilde{x} = a_m \sum_{j=m}^{\infty} T^{n_m}\tilde{h_j} = a_m T^{n_m}\tilde{h_m} + \sum_{j=m+1}^{\infty} a_m T^{n_m}\tilde{h_j}. \tag{3.3.11}$$

Therefore, applying (3.3.11) along with statements (1) and (2) above, we get

$$\begin{aligned}
\|a_m T^{n_m}\tilde{x} - \tilde{f_m}\| &\leq \|a_m T^{n_m}\tilde{h_m} - \tilde{f_m}\| + \sum_{j=m+1}^{\infty} a_m \|T\|^{n_m}\|\tilde{h_j}\| \\
&< \frac{\varepsilon_m}{2} + \sum_{j=m+1}^{\infty} a_m b^{n_m}\frac{\varepsilon_m}{2^j a_m b^{n_m}} \\
&= \frac{\varepsilon_m}{2} + \sum_{j=m+1}^{\infty} \frac{\varepsilon_m}{2^j}.
\end{aligned}$$

After a change of variables on the sum of the above expression, we obtain

$$\|a_m T^{n_m}\tilde{x} - \tilde{f_m}\| < \frac{\varepsilon_m}{2} + \sum_{j=1}^{\infty} \frac{\varepsilon_m}{2^{m+j}} = \frac{\varepsilon_m}{2} + \frac{\varepsilon_m}{2^m}\sum_{j=1}^{\infty}\frac{1}{2^j}$$

$$= \frac{\varepsilon_m}{2} + \frac{\varepsilon_m}{2^m} \leq \varepsilon_m,$$

for all $m \geq 1$. Thus $a_m T^{n_m} \tilde{x} \in B(\tilde{f_m}, \varepsilon_m)$ for all $m \geq 1$.

It follows that $\tilde{x}$ is a supercyclic vector and the proof is complete. □

By taking all A_i in Theorem 3.3.2 to be the same positive operator $A = M_\varphi$, the operator T reduces to the adjoint of the pure quasinormal operator in Brown's Theorem [23, p. 135]. We have thus obtained the desired result.

Corollary 3.3.12. *Every pure quasinormal operator on a separable, infinite-dimensional, complex Hilbert space has a supercyclic adjoint.*

Corollary 3.3.12 improves Wogen's result in [52] that the adjoint S^* of a pure quasinormal operator S must be cyclic.

Quasinormal operators being a more general class of subnormal operators, we want to examine a necessary condition for the adjoint of a pure subnormal operator to have supercyclic vectors. We begin with the next proposition which says that an operator having an extension whose adjoint is supercyclic has a supercyclic adjoint.

Proposition 3.3.13. *Let S be a bounded linear operator on a Hilbert space H and assume that S has an extension T on a Hilbert space K. If T^* is supercyclic then S^* is supercyclic.*

Proof. Suppose T^* is supercyclic. There exists $f \in K$ such that

$$\{\alpha T^{*n} f : n \geq 0, \alpha \in \mathbb{C}\}$$

is dense in K.

Since T is an extension of S, $T|_H = S$. Hence, $SH \subset H$ implies $TH \subset H$.

<u>Claim 1:</u> $T^* H^\perp \subset H^\perp$.

<u>Proof of Claim 1.</u> Let $y \in T^* H^\perp$. We show that $y \in H^\perp$.

$y \in T^* H^\perp \Rightarrow \exists x \in H^\perp$ such that $y = T^* x$.

Let $l \in H$. We have:

$\langle y, l \rangle = \langle T^* x, l \rangle = \langle x, Tl \rangle = 0$. □

<u>Claim 2:</u> $S^* = PT^*$, where $P : K \to K$ is the orthogonal projection of K onto H.

<u>Proof of Claim 2.</u> Let $u, v \in H$.

We have

$$\langle u, PT^*v\rangle = \langle (PT^*)^*u, v\rangle = \langle TPu, v\rangle = \langle Tu, v\rangle = \langle Su, v\rangle.$$

Hence, $S^* = PT^*$. □

<u>Claim 3:</u> Write $f = h + h'$, where $h \in H$ and $h' \in H^{\perp}$. If f is supercyclic vector for T^* then h is supercyclic vector for S^*.

<u>Proof of Claim 3.</u> Let f be supercylic vector for T^* and let P be the orthogonal projection of K onto H. Then for any vector $g \in H$ and any $\lambda \in \mathbb{C}$, we can use claim 2 to get

$$\langle \lambda S^{*n}h, g\rangle = \langle \lambda PT^{*n}Pf, g\rangle = \lambda\langle f, (PT^{*n}P)^*g\rangle = \lambda\langle f, PT^nPg\rangle.$$

Also, we have

$$\lambda\langle f, PT^nPg\rangle = \lambda\langle f, PT^ng\rangle = \lambda\langle f, T^ng\rangle = \langle \lambda T^{*n}f, g\rangle.$$

Now the set $\{\langle \lambda T^{*n}f, g\rangle : n \geq 0, \lambda \in \mathbb{C}\}$ being dense in $\mathbb{C}$, implies that the set $\{\langle \lambda S^{*n}h, g\rangle : n \geq 0, \lambda \in \mathbb{C}\}$ is also dense in $\mathbb{C}$. Therefore, $Pf = h$ is supercyclic vector for S^*. This finishes the proof of the proposition. □

The following is an immediate consequence of Corollary 3.3.12 and proposition 3.3.13.

Corollary 3.3.14. *If $S : H \to H$ has a pure quasinormal extension $Q : K \to K$, then the adjoint S^* is supercyclic.*

Recall that an operator A is pure if it has no reducing subspaces on which it is normal. That is, the only reducing subspace on which A is normal is the zero subspace. We mentioned earlier that if an operator S is subnormal and N is a normal extension of S, then $N \oplus M$ is also a normal extension of S for any normal operator M. Hence every subnormal operator has a non pure extension.

However, if a pure subnormal operator has a pure quasinormal extension then Corollary 3.3.14 directly implies the following.

Corollary 3.3.15. *If $S : H \to H$ is a pure subnormal operator which has a pure quasinormal extension $Q : K \to K$, then the adjoint S^* is supercyclic.*

CHAPTER 4 SOME EXISTING RESULTS IN THE THEORY OF SUBNORMAL OPERATORS

In this Chapter, we study more existing results in the theory of subnormal operators. As mentioned in the general introduction of this dissertation, the main reason for treating these operators is the fact that the class of subnormal operators is larger than the class of quasinormal operators and that it is a natural generalization of normal operators. Hence for obvious reasons, we want to have the kind of some how in-depth information about the subnormal operators. If $A : H \to H$ is an operator on H and e_0 is a vector in H, e_0 is called a *star-cyclic vector* for A if H is the smallest reducing subspace for A that contains e_0. So one such information about subnormal operators could be for instance how the cyclic subnormal operators are characterized in comparison to the star-cyclic normal operators which are given as the multiplication operators N_μ on $L^2(\mu)$ where μ is a compactly supported measure on the complex numbers (See for instance [21, Theorem 3.4, p. 269]).

We should also mention here our result obtained in Chapter 3 that every pure subnormal operator having a pure quasinormal extension has a supercyclic adjoint is an improvement in a special case of a result by Feldman [26], who proved that every pure subnormal operator has a cyclic adjoint. In Chapter 5 of this dissertation, we mention and discuss open questions relating normal, quasinormal, hyponormal, and subnormal operators and the Invariant Subspace Problem.

4.1 General Subnormal Operators

We start with some classic examples.

Example 4.1.1. (Bergman Operators). *Let G be a bounded region in the Complex plane. Let*

$$L^2(G) = \left\{ f : G \to \mathbb{C} |\ f \textit{Lebesgue measurable and} \int_G |f|^2 dA < \infty \right\}$$

be the L^2-space of area measure restricted to G.

Let

$$L_a^2(G) = \left\{ f : G \to \mathbb{C} | \ f \text{ analytic on } G \text{ and } \int_G |f|^2 dA < \infty \right\}$$

be the subspace consisting of analytic functions belonging to $L^2(G)$. Clearly $L^2(G)$ is a Hilbert space. The subspace $L_a^2(G)$, which is also a Hilbert space (see for instance [23, p. 176]), is called the Bergman space.

Definition 4.1.2. *The Bergman operator for a region G is the operator $S : L_a^2(G) \to L_a^2(G)$ defined by $Sf = zf$ for all $f \in L_a^2(G)$.*

If $N : L^2(G) \to L^2(G)$ is an operator defined by $Nf = M_z f = zf$ for all $f \in L^2(G)$, it is easy to see that N is bounded and that $N^* f = M_{\bar{z}} f = \bar{z} f$ for all $f \in L^2(G)$. This implies that N is a normal extension of S and hence the Bergman operator is subnormal.

A special case is when the region G is equal to the open unit disk $\mathbb{D}$ with its normalized area measure $\frac{1}{\pi} dA$. The following Lemma is elementary and certainly well-known for a general operator.

Lemma 4.1.3. *Let K be a Hilbert space and H a closed subspace of K. Let $P : K \to K$ be the orthogonal projection of K onto H. If $A \in B(H)$ and $T \in B(K)$ such that $T|_H = A$ then $A^* = PT^*|_H$.*

Proof. Let $f, g \in H$. Then

$$\langle f, A^* g \rangle_H = \langle Af, g \rangle_H = \langle Tf, g \rangle_H = \langle Tf, g \rangle_K = \langle f, T^* g \rangle_K.$$

Since $T^* g = P(T^* g) + (I - P)(T^* g)$, we get that

$$\begin{aligned} \langle f, T^* g \rangle_K &= \langle f, P(T^* g) + (I - P)(T^* g) \rangle_K \\ &= \langle f, PT^* g \rangle_K + \langle f, (I - P)(T^* g) \rangle_K \\ &= \langle f, PT^* g \rangle_H \end{aligned}$$

which completes the proof. □

By applying Lemma 4.1.3 to the Bergman operator, we immediately obtain the following Corollary.

Corollary 4.1.4. *If $S : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ is the Bergman operator, its adjoint S^* is given by $S^*f = P(N^*f) = P(\bar{z}f)$ for all $f \in L_a^2(\mathbb{D})$, where $P : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ is the orthogonal projection onto $L_a^2(\mathbb{D})$.*

Since $L_a^2(\mathbb{D})$ is a Hilbert space, it has an orthonormal basis. To find it, Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n, g(z) = \sum_{n=0}^{\infty} b_n z^n$ are analytic functions on the unit disk $\mathbb{D}$. Then

$$\begin{aligned}
\langle f, g\rangle_{L_a^2} &= \int_{\mathbb{D}} \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{k=0}^{\infty} \overline{b_k}\bar{z}^k\right)\frac{dA}{\pi} \\
&= \int_{\mathbb{D}} \left(\sum_{n} a_n r^n e^{in\theta}\right)\left(\sum_{k} \overline{b_k} r^k e^{-ik\theta}\right)\frac{dA}{\pi} \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \sum_{n,k} a_n\overline{b_k} r^{n+k} e^{i(n-k)\theta} r\,dr\,d\theta.
\end{aligned}$$

A change of the order of integration gives

$$\begin{aligned}
\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \sum_{n,k} a_n\overline{b_k} r^{n+k} e^{i(n-k)\theta} r\,dr\,d\theta &= \frac{1}{\pi}\int_0^1\int_0^{2\pi} \sum_{n,k} a_n\overline{b_k} r^{n+k} e^{i(n-k)\theta} r\,d\theta\,dr \\
&= \frac{2\pi}{\pi}\int_0^1 \sum_{n} a_n\overline{b_n} r^{2n+1}\,dr \\
&= \sum_{n=0}^{\infty} \frac{a_n\overline{b_n}}{n+1}.
\end{aligned}$$

It follows that $\|f\|_{L_a^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$ for all $f \in L_a^2(\mathbb{D})$. Furthermore, if $n \neq k$ then $z^n \perp z^k$. Therefore $\|z^n\|^2 = \frac{1}{n+1}$ and so $\|z^n\| = \frac{1}{\sqrt{n+1}}$. Let $e_n = \frac{z^n}{\|z^n\|} = \sqrt{n+1}z^n, n \geq 0$. Then the set $\{e_n : n = 0, 1, 2, \ldots\}$ is an orthonormal basis of $L_a^2(\mathbb{D})$.

The following Lemma says more about the orthogonal complement of the Bergman space.

Lemma 4.1.5. *For all* $k \geq 1$, $\bar{z}^k \in L_a^2(\mathbb{D})^{\perp}$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in L_a^2(\mathbb{D})$. For $k \geq 1$, we have

$$\begin{aligned}\langle f, \bar{z}^k \rangle &= \frac{1}{\pi}\int_{\mathbb{D}} f(z) z^k dA \\ &= \frac{1}{\pi}\int_0^1 \int_0^{2\pi} \sum_{n=0}^{\infty} a_n z^n r^k e^{ik\theta} r d\theta dr \\ &= \frac{1}{\pi}\int_0^1 \int_0^{2\pi} \sum_{n=0}^{\infty} a_n r^{n+k+1} e^{i(n+k)\theta} d\theta dr \\ &= 0.\end{aligned}$$

This completes the proof. □

Let us recall that if M is a closed subspace of a Hilbert space H and $P : H \to H$ is the orthogonal projection onto M, let $\{e_n : n \geq 0\}$ be an orthonormal basis of M and $\{h_n : n \geq 0\}$ an orthonormal basis of $M^{\perp}$. If $f = \sum_{n=0}^{\infty} a_n e_n + \sum_{n=0}^{\infty} b_n h_n \in H$ then

$$Pf = \sum_{n=0}^{\infty} a_n e_n = \sum_{n=0}^{\infty} \langle f, e_n \rangle e_n. \tag{4.1.6}$$

In regards to the orthogonal projection onto $L_a^2(\mathbb{D})$, we note that the square integrable function $f(z) = |z|^2$ on the unit disk is not analytic and it turns out that its projection is a constant. More precisely, the following Corollary holds.

Corollary 4.1.7. *Let* $P : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ *be the orthogonal projection onto* $L_a^2(\mathbb{D})$. *Then*

$$P(|z|^2) = \frac{1}{2}, \tag{4.1.8}$$

a constant function.

Proof. Let $n \geq 0$. Let $\{e_n : n = 0, 1, 2, \ldots\}$ be the orthonormal basis of $L_a^2(\mathbb{D})$. We have

$$
\begin{aligned}
\langle |z|^2, e_n\rangle_{L_a^2(\mathbb{D})} &= \langle |z|^2, \sqrt{n+1}z^n\rangle_{L_a^2(\mathbb{D})} \\
&= \int_{\mathbb{D}} |z|^2\sqrt{n+1}\bar{z}^n \frac{dA}{\pi} \\
&= \frac{\sqrt{n+1}}{\pi}\int_0^1\int_0^{2\pi} r^2 r^n e^{-in\theta} r d\theta dr \\
&= \begin{cases} 0, \text{ if } n \geq 1 \\ \frac{1}{2}, \text{ if } n = 0. \end{cases}
\end{aligned}
$$

Hence using (4.1.6), we get that

$$P(|z|^2) = \langle |z|^2, e_0\rangle e_0 = \frac{1}{2}$$

which completes the proof. □

By Corollary 4.1.4 and Lemma 4.1.5, we obtain that

$$SS^*1 = S(PM_{\bar{z}}1) = S(P\bar{z}) = 0.$$

By Corollary 4.1.4 and Corollary 4.1.7, we get that

$$S^*S1 = S^*(S1) = S^*z = P(M_{\bar{z}}z) = P(\bar{z}z) = P(|z|^2) = \frac{1}{2}.$$

Thus, $SS^* \neq S^*S$, i.e. the Bergman operator S is not normal. Since $N : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ defined by $Nf = M_z f = zf$ for $f \in L^2(\mathbb{D})$ is normal, N is a normal extension of S. Thus $S : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ is subnormal.

The following examples of subnormal operators can be found in [24, p. 28].

Example 4.1.9. *Let μ be a compactly supported positive measure on $\mathbb{C}$ and let $P^2(\mu) = \overline{\{\text{polynomials}\}}^{L^2(\mu)}$ be the closure of the analytic polynomials in $L^2(\mu)$. Define*

$S_\mu : P^2(\mu) \to P^2(\mu)$ by $(S_\mu f)(z) = zf(z)$ for all $f \in P^2(\mu)$. As in the previous example, if $N_\mu : L^2(\mu) \to L^2(\mu)$ is defined by $(N_\mu f)(z) = zf(z)$ for all $f \in L^2(\mu)$, then N_μ is normal. If p is an analytic polynomial, then $N_\mu p = zp$ is also a polynomial. By passing to limits, we see that N_μ leaves $P^2(\mu)$ invariant and thus S_μ is a subnormal operator with N_μ as a normal extension.

A generalization of the preceding example is the following.

Example 4.1.10. *Let μ be a compactly supported positive measure on $\mathbb{C}$ and let K be a compact subset of $\mathbb{C}$ containing the support of μ. Let $\mathrm{Rat}(K)$ denote the set of rational functions with poles off K. Let $R^2(K,\mu) = \overline{\mathrm{Rat}(K)}^{L^2(\mu)}$ be the closure of $\mathrm{Rat}(K)$ in $L^2(\mu)$. Define S on $R^2(K,\mu)$ by $Sf = zf$. Then S is subnormal with N_μ as a normal extension.*

The spaces $P^2(\mu)$ and $R^2(K,\mu)$ are among the most important in the theory of subnormal operators. In the following section we see how they contribute to the understanding of the structure of these operators by providing a larger class of subnormal operators.

4.2 Cyclic Subnormal Operators

Let H be a complex Hilbert space and let $A \in B(H)$. let $K \subset \mathbb{C}$ be a compact subset containing $\sigma(A)$, the spectrum of A. We recall that

$$\begin{aligned}\mathrm{Rat}(K) &= \{\text{ rational functions with no poles at points in } K\} \\ &= \left\{\frac{p(z)}{q(z)} : p, q \text{ polynomials and } q(z) \neq 0 \text{ for all } z \in K\right\}.\end{aligned}$$

If

$$\frac{p}{q}(z) = \frac{\alpha(z-a_1)\dots(z-a_n)}{\beta(z-b_1)\dots(z-b_n)},$$

then

$$\frac{p}{q}(A) = \alpha(A-a_1)\dots(A-a_n)\beta^{-1}(A-b_1)^{-1}\dots(A-b_n)^{-1}.$$

If $\frac{p}{q} \in Rat(K)$, then $b_1, \dots, b_n \notin K$, and since $\sigma(A) \subseteq K$, we have $b_1, \dots, b_n \notin \sigma(A)$. Hence the terms

$$(A - b_1)^{-1}, \ldots, (A - b_n)^{-1}$$

exist.

Observe that if $S \in B(H)$ is subnormal, $N \in B(K)$ and $N =$ mne (S), then the fact that $\sigma(N) \subseteq \sigma(S)$ (See for instance [23, p. 131]) implies that $f(N)$ is defined whenever $f \in$ Rat(K) and $K \supseteq \sigma(S)$.

The next two definitions are taken from the book of Conway [23].

Definition 4.2.1. *Let $A \in B(H)$ and let K be a compact subset of $\mathbb{C}$ containing $\sigma(A)$. A vector $e_0 \in H$ is said to be a Rat(K)-cyclic vector for A if*

$$H = \overline{\{f(A)e_0 : f \in \mathrm{Rat}(K)\}}.$$

The operator A is Rat*(K)-cyclic if it has a* Rat*(K)-cyclic vector.*

Definition 4.2.2. *Let $A : H \to H$ be a bounded linear operator on a Hilbert space H. A vector $e_0 \in H$ is said to be a star-cyclic vector for A if H is the smallest reducing subspace for A that contains e_0. The operator A is star-cyclic if it has a star-cyclic vector.*

The following Theorem unitarily identifies any star-cyclic normal operator as the operator $N_\mu = M_z$, the multiplication operator by z on $L^2(\mu)$, like the one in Example 4.1.9. Its proof can be found in Conway [21, p. 269].

Theorem 4.2.3. *A normal operator $N : H \to H$ is star-cyclic if and only if N is unitarily equivalent to N_μ for some compactly supported measure μ on $\mathbb{C}$. If e_0 is a star-cyclic vector for N, then μ can be chosen such that there is an isomorphism $V : H \to L^2(\mu)$ with $Ve_0 = 1$ and $VNV^{-1} = N_\mu$. Under these conditions, V is unique.*

There are analogous results for subnormal operators. The following Theorem which characterizes Rat(K)-cyclic subnormal operators can be found in [24].

Theorem 4.2.4. *Let K be a compact subset of $\mathbb{C}$ and Suppose $S : H \to H$ is subnormal and has a* Rat*(K)-cyclic vector e_0, and let $N : K \to K$ be mne (S). Then there exists an isomorphism $U : K \to L^2(\mu)$ such that*

(i) $UH = R^2(K, \mu)$*;*

(ii) $Ue_0 = 1$*;*

(iii) $UNU^{-1} = N_\mu$*;*

(iv) If $V = U|_H$, then $V : H \to R^2(K, \mu)$ is an isomorphism and $VSV^{-1} = N_\mu|_{R^2(K,\mu)}$.

Proof. Note that if $\alpha \notin \sigma(S)$, then $(S-\alpha)^{-1}$ exists and $(S-\alpha)^{-1}H = H$, and so $H = (S-\alpha)H$. Since $N|_H = S$, $H = (N - \alpha)H$ and so

$$(N - \alpha)^{-1}H = H \tag{4.2.5}$$

Now on the one hand, since e_0 is a Rat(K)-cyclic vector for S, we have

$$\begin{aligned} H &= \overline{\{\phi(S)e_0 : \phi \in \text{Rat}(K)\}} \\ &= \overline{\{\phi(N)e_0 : \phi \in \text{Rat}(K)\}} \quad \text{by } (4.2.5). \end{aligned}$$

On the other hand, since N=mne (S), using proposition 2.2.5, we have that

$$\begin{aligned} K &= \overline{\text{span}\{N^{*n}f : n \geq 0, f \in H\}} \\ &= \overline{\text{span}\{N^{*n}\phi(N)e_0 : n \geq 0, \phi \in Rat(K)\}}. \end{aligned}$$

Let

$$L = \overline{\text{span}\{N^{*n}N^k e_0 : n, k \geq 0\}}.$$

Clearly $L \subset K$ and L is a reducing subspace of N. The Stone-Weierstrass Theorem implies that the set C(K) of continous functions on K is the uniform closure of the linear span of $\{\bar{z}^n z^k : n, k \geq 0\}$

, that is,

$$\mathrm{C}(K) = \overline{\mathrm{span}\{\bar{z}^n z^k : n, k \geq 0\}}$$

and, since $\mathrm{Rat}(K) \subseteq \mathrm{C}(K)$, we have $\phi(N)e_0 \in L$ for all $\phi \in \mathrm{Rat}(K)$. Thus $H \subseteq L$, and since $N = mne(S)$, $L = K$. Hence e_0 is a star-cyclic vector for N.

By Theorem 4.2.3, there is a measure μ with compact support in $\mathbb{C}$ and an isomorphism $U : K \to L^2(\mu)$ such that $Ue_0 = 1$ and $UNU^{-1} = N_\mu$. That is, (ii) and (iii) hold. It follows that $U\phi(N) = \phi(N_\mu)U$ for every bounded Borel function ϕ on $\sigma(N)$. Hence, $U\phi(N)e_0 = \phi(N_\mu)Ue_0 = \phi(N_\mu)1 = \phi(z)$. In particular, if $u \in \mathrm{Rat}(K)$,

$$Uu(S)e_0 = Uu(N)e_0 = u. \tag{4.2.6}$$

To show (i), note that $R^2(K, \mu) = \overline{\mathrm{Rat}(K)}^{L^2(\mu)}$. Let $g \in UH$. Write $g = Uh$ for some $h \in H = \overline{\{\phi(S)e_0 : \phi \in \mathrm{Rat}(K)\}}$. Then there exists a sequence $(\phi_n)_n \subset \mathrm{Rat}(K)$ such that

$$\phi_n(S)e_0 \to h.$$

Because U is an isometry, it is continuous, and so

$$\phi_n = U\phi_n(S)e_0 \to Uh = g$$

by (4.2.6). This implies that $g \in R^2(K, \mu)$, i.e., $UH \subseteq R^2(K, \mu)$.

Similarly, if $g \in R^2(K, \mu)$, then there exists a sequence $(\phi_n)_n \subset \mathrm{Rat}(K)$ such that

$$U\phi_n(S)e_0 = \phi_n \to g$$

by (4.2.6). This means that $g \in U\overline{\{\phi(S)e_0 : \phi \in \mathrm{Rat}(K)\}} = UH$, i.e., $R^2(K, \mu) \subseteq UH$.

To show (iv), observe that it $V = U|_H$, then $VH = R^2(K, \mu)$ by (1), and so $V : H \to R^2(K, \mu)$ is an isomorphism. Finally, since $U : K \to L^2(\mu)$ and $V : H \to R^2(K, \mu)$ are

isomorphisms, and $S = N|_H$, $V = U|_H$, it is clear that $VSV^{-1} = N_\mu|_{R^2(K,\mu)}$.
This completes the proof. $\square$

Since every cyclic operator is obviously $\mathrm{Rat}(K)-$ cyclic, the following Corollary by Bram and I.M. Singer (see Conway [24, Corollary 5.3, p. 52]) holds. Compared to the operator N_μ in Theorem 4.2.3, it provides an analogous characterization of the subnormal operators S_μ in Example 4.1.10.

Corollary 4.2.7. *An operator $S : H \to H$ is cyclic subnormal if and only if S is unitarily equivalent to $S_\mu : P^2(\mu) \to P^2(\mu)$ for some compactly supported measure μ on $\mathbb{C}$.*

Definition 4.2.8. *If $\mathcal{S} \subseteq B(H)$, the commutant of $\mathcal{S}$ is the set of all operators A in $B(H)$ such that $AS = SA$ for all S in $\mathcal{S}$. We denote by $\{\mathcal{S}\}'$, the commutant of $\mathcal{S} \subseteq B(H)$.*

The following Theorem by Yoshino [53] which can also be found in [23] uses measure theory, and it completely categorizes the commutant of a class of operators.

Theorem 4.2.9. (Yoshino [53]). *Let μ be a measure with support contained in the compact subset K of $\mathbb{C}$. If $S = N_\mu|_{R^2(K,\mu)}$, then*

$$\{S\}' = \{M_\phi : \phi \in R^2(K,\mu) \cap L^\infty(\mu)\},$$

where $M_\phi f = \phi f$ for all f in $R^2(K,\mu)$.

Proof. If $\phi \in R^2(K,\mu) \cap L^\infty(\mu)$ and $f \in \mathrm{Rat}(K)$ then $\phi f \in R^2(K,\mu)$. Hence $M_\phi : R^2(K,\mu) \to R^2(K,\mu)$ is a linear operator that commutes with $M_z = S$. Conversely, fix an operator A in $\{S\}'$ and put $\phi = A(1)$; so $\phi \in R^2(K,\mu)$. It is easy to see that if $f \in \mathrm{Rat}(K)$, then $AM_f = M_f A$. Hence for any f in $\mathrm{Rat}(K)$,

$$Af = AM_f 1 = M_f A1 = f\phi. \tag{4.2.10}$$

If $f \in R^2(K,\mu)$, there is a sequence $\{f_n\} \subseteq \mathrm{Rat}(K)$ such that $\int |f - f_n|^2 d\mu \to 0$. By passing to a subsequence, we may suppose that $f_n \to f$ μ-a.e.. Hence

$$0 = \lim \|Af - Af_n\| = \lim \|Af - \phi f_n\|$$

by (4.2.10). But $\phi f_n \to \phi f$ μ-a.e.. Thus, $Af = \phi f$ for all $f \in R^2(K, \mu)$.

It remains to show that $\phi \in L^\infty(\mu)$. To do this, we may suppose that $\|A\| = 1$. Let $\Delta = \{z : |\phi(z)| > 1\}$. We must show that $\mu(\Delta) = 0$ (and hence $\|\phi\|_\infty \leq 1$). For every positive integer n,

$$\|1\|^2 \geq \|A^n(1)\|^2 = \|\phi^n\|^2 = \int |\phi|^{2n} d\mu \geq \int_\Delta |\phi|^{2n} d\mu.$$

But $|\phi(z)|^{2n} \uparrow \infty$ on Δ. By the Monotone convergence Theorem, it must be that $\mu(\Delta) = 0$, and this completes the proof. □

The next result is an immediate consequence of Yoshino's Theorem.

Corollary 4.2.11. *If μ is a compactly supported measure on $\mathbb{C}$, then*

$$\{S_\mu\}' = \{M_\phi : \phi \in P^2(\mu) \cap L^\infty(\mu)\}.$$

Corollary 4.2.12. *If $S = N_\mu|_{R^2(K,\mu)}$ and T is a bounded operator such that $TS = ST$, then* $\ker T$ *is a reducing subspace for S.*

Proof. Since $TS = ST$, $T \in \{S\}'$ and by Theorem 4.2.9, there is a function ϕ in $R^2(K, \mu) \cap L^\infty(\mu)$ such that $Tf = \phi f$ for all $f \in R^2(K, \mu)$. Note that since $\phi \in L^\infty(\mu)$, the set

$$\Delta = \{z : \phi(z) = 0\}$$

has positive measure. Now if $f \in R^2(K, \mu)$, on the one hand, we have for all $z \in \Delta$, $Tf(z) = \phi(z)f(z) = 0$; on the other hand, if $z \in \Delta^c$, then $Tf(z) = \phi(z)f(z) = 0$ only if $f(z) = 0$. It follows that

$$\ker T = \{f \in R^2(K, \mu) : f = 0 \; \mu - a.e \; on \; \{z : \phi(z) = 0\}\}.$$

Finally, since $\ker(T)$ clearly reduces T, it follows that $\ker T$ is a reducing subspace for S. This completes the proof. □

An additional reference on rationally cyclic subnormal operators can be found in McCarthy [40]. He proved that if a subnormal operator S has the property that each of its invariant subspaces must be hyperinvariant, then S must have a star-cyclic vector.

CHAPTER 5 REMARKS AND OPEN QUESTIONS

It is a result in [23, p. 117] that every subnormal operator on a Hilbert space has a quasinormal extension. Note that this assertion does not allow us to conclude that such operator would have a supercyclic adjoint. However the results of Corollary 2.6.8 in Chapter 2 and of Corollary 3.2.2 and Corollary 3.3.12 in Chapter 3 on the existence of hypercyclic and supercyclic vectors respectively for the adjoint of quasinormal operators led to the conclusion in Corollary 3.3.15 that every pure subnormal operator having a pure quasinormal extension must have a supercyclic adjoint. Thus this result is an improvement in a special case of a result by Feldman [26], who proved that the adjoint of every pure subnormal operator is cyclic. Yet, it is unknown whether every pure subnormal operator must have a supercyclic adjoint.

One way to tackle this problem was to ask the following question in light of Theorem 3.1.9 in [23]. Must every pure subnormal operator have a pure quasinormal extension? We answered this question by a negative which gave us a different perspective of trying other ways around. So alternatively, we asked the following question. If S is a pure subnormal operator, must there exist a generalized backward shift B such that $S^*B = BS^*$ and $\ker S^* \supset \ker B$? Note that by Theorem 3.6 in [29], a positive answer to this question would seem to answer in the positive whether every pure subnormal operator has a supercyclic adjoint.

The open questions in operator theory are numerous and many still remain unsolved. We mention here a few more questions regarding the hypercyclicity or supercyclicity of the operators we have studied as well as some of their implications to the Invariant Subspace Problem we mentioned earlier in this dissertation.

First, we would like to know more about the absolutely convergent supercyclic vector for the adjoint of the pure quasinormal operator. In fact, the adjoint Q^* of a pure quasinormal operator Q is unitarily equivalent to an operator that is in the form (3.3.3) of the operator T given in Theorem

3.3.2. Recall that the supercyclic vector $\tilde{x}$ is of the form

$$\tilde{x} = \sum_{j=1}^{\infty} \tilde{h}_j$$

where each $\tilde{h}_j$ is a vector in the Hilbert space and satisfies $\|\tilde{h}_1\| < \frac{1}{2}$, and for all $j \geq 2$,

$$\|\tilde{h}_j\| < \min\left(\frac{1}{2^j}, \frac{\varepsilon_1}{2^j a_1 b^{n_1}}, \cdots, \frac{\varepsilon_{j-1}}{2^j a_{j-1} b^{n_{j-1}}}\right),$$

where $a_j > 0$, $\varepsilon_j > 0$, and $b = \max(1, \|T\|)$. Given that it is unknown [27] if the adjoint of every pure hyponormal is cyclic. The following question is relevant.

Question 5.0.1. *Can we construct a typical supercyclic vector for the adjoint of pure hyponormal operator?*

Next, in light of Theorem 4.2.3, we see that on a finite-dimensional Hilbert space, every star-cyclic quasinormal operator is unitarily equivalent to the multiplication operator N_μ on $L^2(\mu)$. We may then ask the following question.

Question 5.0.2. *Can we obtain an analogous representation on infinite-dimensional Hilbert spaces?*

Looking back at the Invariant Subspace Problem, though it still remains unsolved for a separable Hilbert space, most attempts have been made in the positive direction, that is, trying to prove that every operator has a nontrivial closed invariant subspace. A striking case was of Lomonosov's theorem [38] who proved the existence of hyperinvariant subspaces for non-scalar multiple of the identity operators commuting with a non-zero compact operator.

Interestingly, among the operators we have studied, namely, normal, quasinormal, subnormal, and hyponormal, all have nontrivial closed invariant subspaces except the hyponormal operators [37]. This intriguing fact may indeed raise a lot of questions. For instance, given that hyponormal operators cannot be hypercyclic or supercyclic [10] , this suggests that they may have nontrivial closed invariant subspaces. For instance, it is known that compact hyponormal operators are normal [1], [5], and hence such operators must have nontrivial closed invariant subspaces. Consequently, the following question can be of interests.

Question 5.0.3. *What are the non-compact hyponormal operators in infinite-dimensional separable Hilbert spaces with the property that they commute with a non-zero compact operator?*

By Lomonosov's Theorem such operators will prove to have nontrivial closed invariant subspaces.

Lastly one more problem close to our study arises from Fuglede-Putnam Theorem which says that if N and M are normal operators on H and K, and $B : K \to H$ is an operator such that $NB = BM$, then $N^*B = BM^*$. In the recent years an extensive amount of publications has been done in relationship to this theorem, including a generalization to unbounded operators (see for instance [41], [42]). However, Generalizations to subnormal operators or even to quasinormal operators have failed as shown for instance in [23, p. 199], [35]. Thus we might want to ask whether weakening conditions on these operators could perhaps lead to some improvements.

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