ON THE HOMOTOPY PERTURBATION METHOD FOR NONLINEAR OSCILLATORS.

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ABSTRACT

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We study a family of nonlinear oscillators governed by the equation

$$\frac{d^2x}{dt^2} = -\beta x^p, \beta > 0 \tag{1}$$

Here β is a constant and p=1,3,5,... is necessarily odd to be an oscillator (the phase portrait needs to be closed and bounded). The initial condition is $x(0) = A, x'(0) = 0$. Our goal is to find the frequency of the oscillator and its trajectory for each p. We apply the Lindstedt-Poincaré method to a convex homotopy with the parameter λ:

$$x''(t) + x(t) = \lambda[x(t) - \beta x(t)^p], 0 \leq \lambda \leq 1. \tag{2}$$

Substituting the expansion

$$x(t) = \sum_{i=0}^{\infty} \lambda^i x_i(t)$$

and the expression

$$1 = \omega^2 - \lambda\alpha_1 - \lambda^2\alpha_2 - ... \tag{3}$$

into 1, the coefficient of $x(t)$ on the left side of (2), we obtain the first few ODE's for $x_i(t)$ such as Eq's (4.3.4)-(4.3.6) below after equating the coefficients of the powers of λ. The final target solution $x(t)$ and the frequency ω are obtained through a sequence of $x_i(t), \alpha_i, i = 1, 2, ...$ and setting λ to 1. It is necessary to kill the secular terms to have oscillatory behavior in the model.
To achieve this goal, we study about the homotopy perturbation method for nonlinear oscillators which have been demonstrated and discussed in chapter 2. Then we apply He's homotopy perturbation method for conservative truly nonlinear oscillators and find their improved approximate solutions [9]. By this approach, we can find a truly periodic solution and the period of the motion

as a function of the amplitude of oscillation. We take $p = 3$ in (1), so-called cubic oscillator and find that this approach works very well for the whole range of parameters and obtain excellent result match of frequencies between approximate and an exact one [9].

I would like to dedicate this work to my family, friends, and all my respected teachers.

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CHAPTER 1 INTRODUCTION

1.1 Perturbation Theory

Perturbation theory comprises mathematical methods to obtain an approximate solution to a problem, beginning with the exact solution of a related, simpler problem [44] . A main feature of this method is a middle step which breaks the problem into ”solvable” and ” perturbation” parts. Perturbation theory is beneficial if the problem we have cannot be solved exactly, but can be formulated with the addition of a ”small” term in the mathematical description of the exactly solvable problem. This theory leads to an expression for our desired solution in form of a ”formal power series” in some ”small” parameter, which is known as a perturbation series. The leading term of this power series indicates the solution of the exactly solvable problem and further terms of the series describe the deviation of the solution, due to the deviation from the initial problem. An example of power series with the small parameter ϵ can be expressed as

$$A = A_0 + \epsilon^1 A_1 + \epsilon^2 A_2 + ... \tag{1.1.1}$$

where A_0 is the known solution of the exactly solvable initial problem and A_1, A_2, ... indicate the higher- order terms which may be found iteratively by some systematic procedure. These higher-order term in the series become successively smaller due to the small parameter ϵ in the series. We obtain an approximate ”perturbation solution” by truncating the series, usually keeping the only the first two terms, the initial solution and the ”first-order” perturbation correction

$$A \approx A_0 + \epsilon A_1. \tag{1.1.2}$$

1.2 Regular Perturbation Theory

Most often, some mathematical problem cannot be solved exactly or, if the exact solution is available, it exposes such a complicated dependency in the parameter that it is difficult to use as such. However, it may be the case that a parameter can be identified, say ϵ, such that the solution is available and reasonably simple for $\epsilon = 0$. Then, we may wonder how this solution is altered for non-zero but small ϵ. Perturbation theory gives a systematic answer to this question.

1.2.1 Example

Consider the quadratic equation

$$x^2 - \epsilon x - 1 = 0 \tag{1.2.1}$$

It has two roots:

$$x_1 = \frac{\epsilon}{2} + \sqrt{1 + \frac{\epsilon^2}{4}}, \quad x_2 = \frac{\epsilon}{2} - \sqrt{1 + \frac{\epsilon^2}{4}}. \tag{1.2.2}$$

For small ϵ, by the Taylor series expansion, these roots are well approximated taking their first few terms (Fig. 1)

$$x_1 = 1 + \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + \mathcal{O}(\epsilon^3), \quad x_2 = -1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \mathcal{O}(\epsilon^3). \tag{1.2.3}$$

In Fig. (1.1), x_1 is the root plotted as a function of ϵ (solid line), which is compared with the approximations by truncation of the Taylor series at $\mathcal{O}(\epsilon^2)$, $x_1 = 1 + \frac{\epsilon}{2}$(dotted line), and $\mathcal{O}(\epsilon^3)$, $x_1 = 1 + \frac{\epsilon}{2} + \frac{\epsilon^2}{8}$(dashed line). We see that even though the approximations are a priori valid in the range $\epsilon \ll 1$ only, the approximation $x_1 = 1 + \frac{\epsilon}{2} + \frac{x^2}{8}$ is fairly good even up to $\epsilon = 2$.

By using regular perturbation theory, we can obtain (1.2.3) without prior knowledge of the exact solution of (1.2.1). It involves four steps.

STEP 1. Suppose that the solutions of (1.2.1) can be expanded by the Taylor series expansion in ϵ. Then

$$x = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \mathcal{O}(\epsilon^3), \tag{1.2.4}$$

where X_0, X_1, X_2 are to be determined.

STEP 2. Substitute (1.2.4) into (1.2.1) written as $x^2 - 1 - \epsilon x = 0$, and expand the left hand side

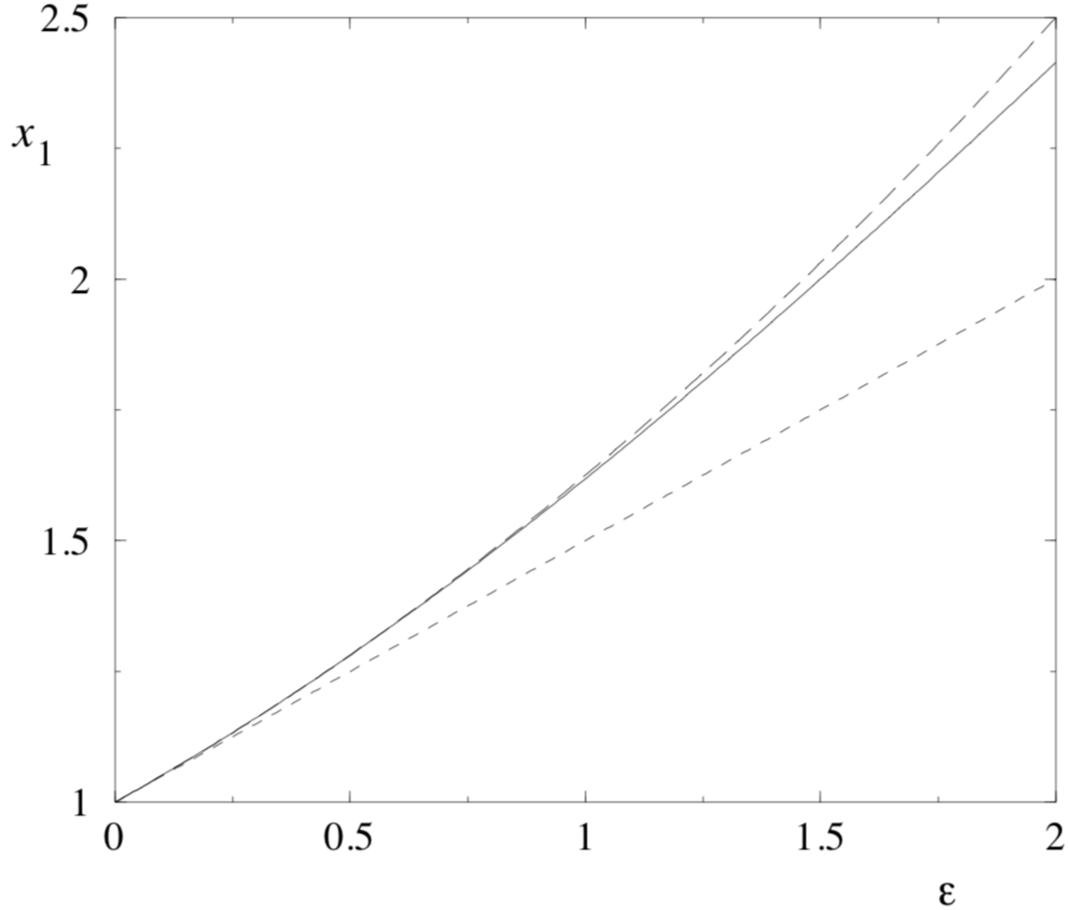

Figure 1.1 Regular Perturbation Theory (a)

of the resulting equation in power series of ϵ. Using

$$x^2 = X_0^2 + 2\epsilon X_0 X_1 + \epsilon^2(X_1^2 + 2X_0X_2) + \mathcal{O}(\epsilon^3), \epsilon x = \epsilon X_0 + \epsilon^2 X_1 + \mathcal{O}(\epsilon^3), \tag{1.2.5}$$

we get

$$0 = X_0^2 - 1 + \epsilon(2X_0X_1 - X_0) + \epsilon^2(X_1^2 + 2X_0X_2) + \mathcal{O}(\epsilon^3) \tag{1.2.6}$$

STEP 3. Equating to zero to the successive terms of the series in the left hand side, we get

$$\begin{aligned}
\mathcal{O}(\epsilon^0): \quad & X_0^2 - 1 &= 0, \\
\mathcal{O}(\epsilon^1): \quad & 2X_0X_1 - X_0 &= 0 \\
\mathcal{O}(\epsilon^2): \quad & X_1^2 + 2X_0X_2 - X_1 &= 0 \\
\mathcal{O}(\epsilon^3): \quad & &\ldots
\end{aligned} \tag{1.2.7}$$

STEP 4. Solving successively the sequence in (1.2.7), we have

$$\begin{aligned}
X_0 = 1, \qquad X_1 &= \frac{1}{2}, \qquad X_2 = \frac{1}{8}, \\
X_0 = -1, \qquad X_1 &= \frac{1}{2}, \qquad X_2 = \frac{-1}{8}.
\end{aligned} \tag{1.2.8}$$

We can check that we can recover (1.2.3) by substituting (1.2.8) into (1.2.4).

It may not be clear what is the advantage of regular perturbation theory from the previous example, because one can obtain (1.2.3) more directly from the Taylor expansion of the roots in (1.2.2).

We now consider the following equation to see the strength of regular perturbation theory

$$x^2 - 1 = \epsilon e^x. \tag{1.2.9}$$

In Fig. (1.2), dotted line and the dashed line are the plot of approximations by truncation of the Taylor series at $\mathcal{O}(\epsilon^2)$, $x_1 = 1 + \frac{\epsilon l}{2}$, and at $\mathcal{O}(\epsilon^3)$, $x_1 = 1 + \frac{\epsilon l}{2} + \frac{\epsilon^2 l^2}{8}$ respectively. Also the solid line is

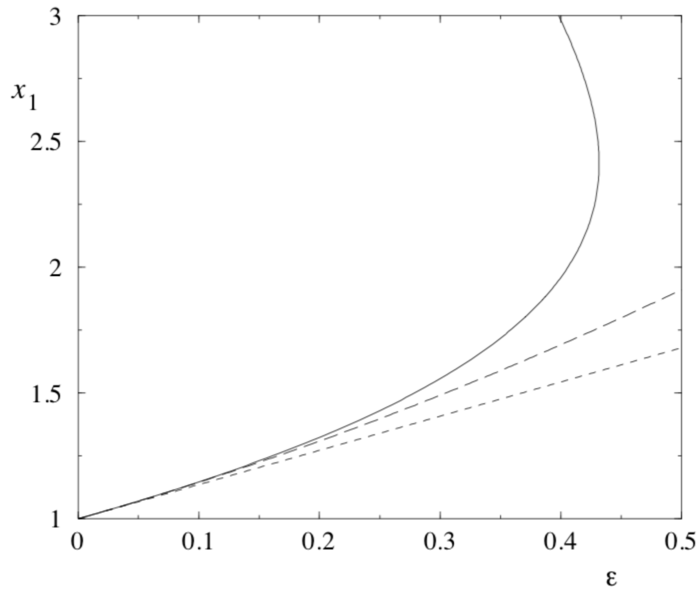

Figure 1.2 Regular Perturbation Theory (b)

the graph of two of the three solutions of (1.2.9) obtained numerically and plotted as a function of ϵ. Here, the direct method is not applicable, because, the solutions of this equation are not available. However, by perturbation theory, we can obtain the Taylor series expansion of these solutions. We introduce the expansion (1.2.4) as in Step 1. In Step 2, we recall $e^z = 1 + z + \frac{z^2}{2} + \mathcal{O}(z^3)$

$$\epsilon e^x = \epsilon e^{X_0+\epsilon X_1+\epsilon^2 X_2+\mathcal{O}(\epsilon^3)} = \epsilon e_0^X e^{\epsilon X_1+\epsilon^2 X_2+\mathcal{O}(\epsilon^3)} = \epsilon e_0^X + \epsilon^2 X_1 e_0^X + \mathcal{O}(\epsilon^3). \tag{1.2.10}$$

Substituting this expression into (1.2.9) expressed in the form of $x^2 - 1 - \epsilon e^x = 0$ and using (1.2.5), we get

$$X_0^2 - 1 + \epsilon(2X_0X_1 - e_0^X) + \epsilon^2(X_1^2 + 2X_0X_1 - X_1e_0^X) + \mathcal{O}(\epsilon^3) = 0. \tag{1.2.11}$$

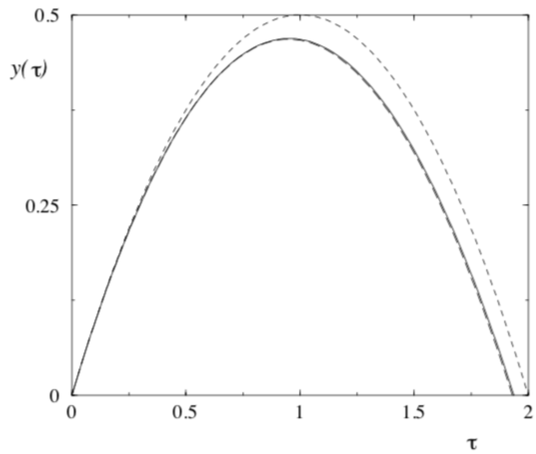

Figure 1.3 Regular Perturbation Theory (c)

Thus, the sequence of equations in Step 3 becomes

$$
\begin{aligned}
\mathcal{O}(\epsilon^0): &\quad X_0^2 - 1 &= 0, \\
\mathcal{O}(\epsilon^1): &\quad 2X_0X_1 - e_0^X &= 0, \\
\mathcal{O}(\epsilon^2): &\quad X_1^2 + 2X_0X_2 - X_1e_0^X = 0 \\
\mathcal{O}(\epsilon^3): &\quad ...
\end{aligned} \tag{1.2.12}
$$

from which we get Step 4 as

$$
\begin{aligned}
X_0 = 1, &\quad X_1 = \frac{e}{2}, &\quad X_2 = \frac{e^2}{8}, \\
X_0 = -1, &\quad X_1 = \frac{-1}{(2e)}, &\quad X_2 = \frac{-1}{(8e^2)}.
\end{aligned} \tag{1.2.13}
$$

In Fig. (1.3), the solid line is the exact solution line, which is compared with the approximations by truncation of the Taylor series at $\mathcal{O}(\epsilon)$ (dotted line), $\mathcal{O}(\epsilon^2)$ (dashed line), and $\mathcal{O}(\epsilon^3)$ (indistinguishable from solid line).

Equivalently,

$$x_1 = 1 + \frac{\epsilon l}{2} + \frac{\epsilon^2 e^2}{8} + \mathcal{O}(\epsilon^3),$$
$$x_2 = -1 - \frac{\epsilon}{(2e)} - \frac{\epsilon^2}{(8e^2)} + \mathcal{O}(\epsilon^3). \tag{1.2.14}$$

The expression for x_1 is compared to the numerical solution of (1.2.9) on Fig. (1.2).

Remark: In fact (1.2.9) has three solutions for $0 < \epsilon < \epsilon_1$, with $\epsilon_1 \approx 0.43$, and only one for $\epsilon_1 < \epsilon$. In (1.2.14) , the expansion of x_2 is the solution that exists for all $\epsilon > 0$; the solution of the expansion of x_1 in (1.2.14) disappears for $\epsilon_1 < \epsilon$; and the third solution (In Fig. (1.2): the solid line is the graph of a two-valued function) cannot be determined by regular perturbation.

1.2.2 Singular Perturbation Theory

For regular perturbation problems, the solution of the general problem converge to the solution of the limit problem when the parameter approaches the limit value. While, singular perturbation theory deals with the study of the problems featuring a parameter for which the solution of the problem at a limiting value of the parameter are different in character from the limit of the solution of the general problem.

Example Let us consider,

$$\epsilon x^2 + x + 1 = 0 \tag{1.2.15}$$

We see that the equation (1.2.15) is quadratic, therefore, it consists of two roots. As $\epsilon \longrightarrow 0$, the equation becomes

$$x + 1 = 0 \tag{1.2.16}$$

This equation is of first order. Therefore, x is discontinuous at $\epsilon = 0$. Perturbation of this kind are called singular perturbation problem. Let

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ... \tag{1.2.17}$$

Substituting this value into (1.2.15) we get

$$\epsilon(x_0^2 + 2\epsilon x_0 x_1 + ...) + x_0 + \epsilon x_1 + ... + 1 = 0$$
$$\Rightarrow \epsilon x_0^2 + 2\epsilon^2 x_0 x_1 + ... + x_0 + \epsilon x_1 + ... + 1 = 0$$
$$\Rightarrow \epsilon(x_0^2 + x_1) + x_0 + 1 = 0$$

Now, equate the co-efficient of like powers of ϵ, then,

$$x_0 + 1 = 0$$
$$x_1 + x_0^2 = 0$$

As $x_0 = -1, x_1 = -1$ therefore, one of the root is

$$x = -1 - \epsilon + ... \tag{1.2.18}$$

Thus as expected the above procedure we got only one root. Now, we investigate the exact solution i.e ,

$$x = \frac{1}{2\epsilon}(-1 \pm \sqrt{1 - 4\epsilon}) \tag{1.2.19}$$

By the binomial theorem we have

$$\begin{aligned} \sqrt{1-4\epsilon} &= 1 - 2\epsilon + \frac{\frac{1}{2}\frac{-1}{2}}{2!} \times (-4\epsilon)^2 + \cdots \\ &= 1 - 2\epsilon - 2\epsilon^2 + \cdots \end{aligned} \tag{1.2.20}$$

Putting (1.2.20) in (1.2.19), we have

$$\begin{aligned} x &= \frac{-1 + 1 - 2\epsilon - 2\epsilon^2 + \cdots}{2\epsilon} && = -1 - \epsilon + \cdots \\ x &= \frac{-1 - 1 + 2\epsilon + 2\epsilon^2 + \cdots}{2\epsilon} && = \frac{-1}{\epsilon} + 1 + \epsilon + \cdots \end{aligned} \tag{1.2.21}$$

Thus, both of the roots become as powers of ϵ but one starts with ϵ^{-1}. Therefore, it is no wondering that the assumed expansion in (1.2.17) is not able to produce the root (1.2.21). Consequently, we are not able to determine the second root by a perturbation method unless we know its form. In such cases, one recognizes that, if the order of the equation is not to be reduced, another tends to ∞ as $\epsilon \longrightarrow 0$ and thus , we assume that the leading term has the form as

$$x = \frac{y}{\epsilon^v} \tag{1.2.22}$$

where v must be greater than zero and required to be determined in the time of analysis. Substituting (1.2.22) in (1.2.15), we get

$$\epsilon^{1-2v}y^2 + \epsilon^{-v}y + 1 = 0$$

Since, $v > 0$, the second term is bigger than 1. Therefore the dominant part of the above equation is

$$\epsilon^{1-2v}y^2 + \epsilon^{-v}y = 0, \tag{1.2.23}$$

which demands that the index of ϵ to be the same.

$$1 - 2v = -v \qquad \Rightarrow v = 1 \tag{1.2.24}$$

For $v = 1$ we have $y = 0$ or -1. The case of $y = 0$ corresponds to the first root $x = -1 - \epsilon$. For $y = -1$, it corresponds to the second root. Therefore, from (1.2.22) , it follows

$$x = \frac{-1}{\epsilon} + ...$$

In order to determine more terms in the expansion of the second root, we try

$$x = \frac{-1}{\epsilon} + x_0 + ... \tag{1.2.25}$$

Plugging it into the equation (1.2.15), we see that

$$\begin{aligned}
&\epsilon(\frac{-1}{\epsilon}+x_0+...)^2-\frac{-1}{\epsilon}+x_0+...+1 &= 0,\\
\Rightarrow\quad &\epsilon(\frac{-1^2}{\epsilon}+\frac{2x_0}{\epsilon}+x_0^2+...)-\frac{-1}{\epsilon}+x_0+1+... &= 0,\\
\Rightarrow\quad &-2x_0+x_0+1+\mathcal{O}(\epsilon) &= 0
\end{aligned}$$

which implies $x_0=1$ and equation (1.2.25) becomes

$$x=-\frac{1}{\epsilon}+1+...$$

Alternatively, once if v has been found. We see (1.2.22) as a transformation of x to y. Now, substituting $x=\frac{y}{\epsilon}$ in (1.2.15) gives

$$y^2+y+\epsilon=0, \tag{1.2.26}$$

which may be solved to find both roots as ϵ does not multiply the highest order.

1.2.3 Perturbation Theory for Differential Equation

Ecample. Let us consider

$$\frac{d^2y}{dt^2}+\epsilon\frac{dy}{dt}+1=0, \qquad y(0)=0, \qquad \frac{dy}{dt}(0)=1. \tag{1.2.27}$$

Assume the expansion

$$y(t)=y_0(t)+\epsilon y_1(t)+\epsilon^2 Y_2(t)+\mathcal{O}(\epsilon^3) \tag{1.2.28}$$

Substituting the equation (1.2.28) into (1.2.27), we see that

$$\frac{d^2y}{dt^2}+\epsilon\frac{dy}{dt}+1=0$$

$$\begin{aligned}
\Rightarrow \frac{d^2}{dt^2}(y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \mathcal{O}(\epsilon^3)) + \epsilon \frac{d}{dt}(y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \mathcal{O}(\epsilon^3)) + 1 &= 0 \\
\Rightarrow \frac{d^2 y_0}{dt^2} + 1 + \epsilon(\frac{d^2 y_1}{dt^2} + \frac{dy_0}{dt}) + \epsilon^2(\frac{d^2 y_2}{dt^2} + \frac{dy_1}{dt} + \mathcal{O}(\epsilon^3) &= 0.
\end{aligned}$$

Now, equating the coefficient of ϵ, we get

$$\begin{aligned}
\frac{d^2 y_0}{dt^2} + 1 &= 0, \qquad y_0(0) = 0, \qquad \frac{dy_0}{dt}(0) &= 1, \\
\Rightarrow \frac{d^2 y_1}{dt^2} + \frac{dy_0}{dt} &= 0, \qquad Y_1(0) = 0, \qquad \frac{dy_1}{dt}(0) &= 0, \\
\Rightarrow \frac{d^2 y_2}{dt^2} + \frac{dy_1}{dt} &= 0, \qquad y_2(0) = 0, \qquad \frac{dy_1}{dt}(0) &= 0.
\end{aligned} \tag{1.2.29}$$

On solving the above equations, we obtain

$$y_0(t) = t - \frac{t^2}{2}, \tag{1.2.30}$$

$$y_1(t) = \frac{-t^2}{2} + \frac{t^3}{6}, \tag{1.2.31}$$

$$y_2(t) = \frac{t^3}{6} - \frac{t^4}{24}. \tag{1.2.32}$$

Substituting these values in equation ((1.2.28)) , we get the solution

$$y(t) = t - \frac{t^2}{2} + \epsilon(\frac{-t^2}{2} + \frac{t^3}{6}) + \epsilon^2(\frac{t^3}{6} - \frac{t^4}{24}) + \mathcal{O}(\epsilon^3)$$

CHAPTER 2 HOMOTOPY PERTURBATION METHOD

In recent years, with the rapid development of nonlinear science, mathematicians, scientists, and engineers have a great interest in solving the nonlinear problems by the analytical techniques. Traditional perturbation methods had been widely used, but they had their own limitations. They are based on the assumption that the equation must contain a small parameter [45]. This small parameter assumption greatly restricts the applications of perturbation techniques. To improve the situation, various other methods were proposed. One of the recently proposed methods is the homotopy perturbation method, which claims to be able to eliminate the small parameter assumption. It is the combination of classical perturbation technique and homotopy concept as used in topology. The homotopy perturbation method requires neither a small parameter nor a linear term in a differential equation. By the simple property of homotopy, the given problem is reduced into a special perturbation problem with the small embedding parameter, which is considered as a small parameter. Therefore, the method is termed homotopy perturbation method.

2.1 General Concept of HPM

To illustrate the general concept of homotopy perturbation method, consider nonlinear differential equations of the form

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{2.1.1}$$

with boundary condition

$$B(u, \frac{\partial u}{\partial n}), \qquad r \in \Gamma, \tag{2.1.2}$$

where A is a general differential operator, B is a boundary operator, Γ is the boundary of domain Ω , $f(r)$ is a known analytic function. The operator A has two parts L and N, where L and N represent linear and nonlinear parts of the equation. Equation (2.1.1) can be expressed as

$$L(u) + N(u) - f(r) = 0. \tag{2.1.3}$$

In the homotopy technique, a homotopy construction is

$$v(r,p) : \Omega \times [0,1] \longrightarrow \mathbb{R}, \tag{2.1.4}$$

which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \qquad p \in [0,1], \qquad r \in \Omega, \tag{2.1.5}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0. \tag{2.1.6}$$

Here, u_0 is the initial approximation of the equation (2.1.1) that satisfies the boundary condition. Now, from equation (2.1.5), we have

$$\begin{aligned} H(v,0) = L(v) - L(u_0) &= 0, \\ H(v,1) = A(v) - f(r) \quad &= 0. \end{aligned} \tag{2.1.7}$$

As the parameter p changes with the process from zero to unity, $v(r,p)$ changes from $u_0(r)$ to $u(r)$. This is called deformation in Topology, and $L(v)-L(u_0)$ and $A(v)-f(r)$ are said to be homotopic.

In this paper, we will use the embedding parameter p as a small parameter and then we assume that the solution of equation((2.1.5)) can be written as a power series of p.

$$v = v_0 + pv_1 + p^2 v_2 + ... \tag{2.1.8}$$

Setting $p = 1$, yields the approximate solution of equation (2.1.1)

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \tag{2.1.9}$$

The series((2.1.9)) is convergent for most cases, however the convergent rate depends upon the non-linear operator $A(v)$ [32] .

2.2 Application 1

We have the linear Schrödinger equation

$$u_t + iu_{xx} = 0, \qquad u(x,0) = 1 + \cosh(2x), [22] \tag{2.2.1}$$

where $u(x,t)$ is a complex function and $i^2 = -1$. Now construct a homotopy $(x,t,p) : \Omega \times [0,1] \to \mathbb{C}$ which satisfies

$$(1-p)(V_t - u_{0,t}) + p(V_t + iV_{xx}) = 0, \qquad p \in [0,1], \qquad (x,t) \in \Omega, \tag{2.2.2}$$

where $u_0(x,t) = V_0(x,0) = u(x,0)$ and $u_{0,t} = \frac{\partial u_0}{\partial t}$.

Assume a solution in the form

$$V(x,t) = V_0(x,t) + pV_1(x,t) + p^2 V_2(x,t) + \cdots . \tag{2.2.3}$$

and substitute to get $V_{0,t} + pV_{1,t} + p^2 V_{2,t} + \cdots - V_{0,t} - PV_{0,t} - p^2 V_{1,t} + p^3 V_{2,t} - \cdots + pV_{0,t} + pV_{0,t} + p^2 V_{1,t} + p^2 V_{2,t} + ip(V_{0,xx} + pV_{1,xx} + p^2 V_{2,xx} + \cdots = 0.$

Equating the terms with the identical powers of p yields

$$\begin{aligned}
p^0 &: V_{0,t} && = 0, \\
p^1 &: V_{1,t} + iV_{0,xx} && = 0, \\
p^2 &: V_{2,t} + iV_{1,xx} && = 0, \\
&\vdots \\
p^n &: V_{n,t} + iV_{n-1,xx} && = 0, n = 3,4,5,\cdots
\end{aligned} \tag{2.2.4}$$

with the initial conditions

$$V_i(x,0) = \begin{cases} 1+\cosh(2x), & i=0, \\ 0, & i=1,2,3,... \end{cases} \tag{2.2.5}$$

With the initial conditions (2.2.5) , the solution of (2.2.4) can be obtained as follows:

$$\begin{aligned} V_0(x,t) &= 1+\cosh(2x), \\ V_1(x,t) &= -4it\cosh(2x), \\ V_2(x,t) &= -8t^2\cosh(2x), \\ V_3(x,t) &= \frac{32}{3}it^2\cosh(2x), \\ V_4(x,t) &= \frac{32}{3}t^4\cosh(2x), \\ V_5(x,t) &= \frac{-128}{15}it^5\cosh(2x) \end{aligned} \quad . \tag{2.2.6}$$

In a similar manner the other components can also be obtained. Substituting (2.2.6) into (2.1.9), we get

$$u(x,t) = (1+\cosh(2x))\left(1 - 4it - 8t^2 + \frac{32}{3}it^3 + \frac{32}{3}t^4 - \frac{128}{15}it^5 - \cdots\right). \tag{2.2.7}$$

Consequently, using the Taylor series expansion of e^{-4it}, we can obtain the exact solution of (2.2.1) as

$$u(x,t) = 1+\cosh(2x)e^{-4it} \tag{2.2.8}$$

2.3 Application 2

Consider the nonlinear Schrödinger equation

$$iu_t + u_{xx} + 2|u|^2 = 0, \qquad u(x,0) = 2\text{sech}(2x). \tag{2.3.1}$$

We construct the homotopy $V(x,t,p) : \Omega \times [0,1] \to \mathbb{C}$ that satisfies

$$(1-p)(iV_t - iu_{0,t}) + p(iV_t + V_{xx} + 2|V|^2) = 0, \qquad p \in [0,1], \qquad (x,t) \in \Omega \tag{2.3.2}$$

or

$$(1-p)(iV_t - iu_{0,t}) + p(iV_t + V_{xx} + 2V\overline{V}) = 0 \tag{2.3.3}$$

where $\overline{V}$ is the conjugate of V.

Suppose that the series solution of (2.3.3) with its conjugate have the following forms:

$$V = V_0(x,t) + pV_1(x,t) + p^2V_2(x,t) + \cdots, \tag{2.3.4}$$

$$\overline{V} = \overline{V}_0(x,t) + p\overline{V}_1(x,t) + p^2\overline{V}_2(x,t) + \cdots. \tag{2.3.5}$$

Substituting (2.3.4) and (2.3.5) into (2.3.3) we get

$iV_t - iu_{0,t} - ipV_t + ipu_{0,t} + ipV_t + pV_{xx} + 2pV\overline{V} = 0$

or,

$iV_{0,t} + ipV_{1,t} + ip^2V_{2,t} + ip^3V_{3,t} + \cdots - ipV_{0,t} - ip^2V_{1,t} - ip^3V_{2,t} - \cdots + ipV_{0,t} + ip^2V_{1,t} + ip^3V_{2,t} + ip^4V_{3,t} + \cdots + pV_{0,xx} + p^2V_{1,xx} + p^3V_{2,xx} + p^4V_{3,xx} + \cdots + 2pV_0\overline{V}_0 + 2p^2V_0\overline{V}_1 + 2p^3V_0\overline{V}_2 + 2p^4V_0\overline{V}_3 + \cdots + 2p^2V_1\overline{V}_0 + 2p^3V_1\overline{V}_1 + 2p^4V_1\overline{V}_2 + 2p^5V_1\overline{V}_3 + \cdots + 2p^3V_2\overline{V}_0 + 2p^4V_2\overline{V}_1 + 2p^5V_2\overline{V}_2 + 2p^6V_2\overline{V}_3 + \cdots + 2p^4V_3\overline{V}_0 + 2p^5V_3\overline{V}_1 + 2p^6V_3\overline{V}_2 + 2p^7V_3\overline{V}_3 + \cdots = 0.$

Now, equating the terms with the identical powers of p yields:

$$\begin{aligned}
p^0 &: iV_{0,t} && = 0, \\
p^1 &: iV_{1,t} + V_{0,xx} + 2|V_0|^2 && = 0, \\
p^2 &: iV_{2,t} + V_{1,xx} + 2V_0\overline{V}_1 + 2V_1\overline{V}_0 && = 0, \\
p^3 &: iV_{3,t} + V_{2,xx} + 2V_0\overline{V}_2 + 2|V_1|^2 + 2V_2\overline{V}_0 && = 0, \\
&\vdots
\end{aligned} \tag{2.3.6}$$

with the following initial condition:

$$V_i(x,0) = \begin{cases} 2\text{sech}(2x), & i = 0, \\ 0, & i = 1,2,3,\cdots. \end{cases} \tag{2.3.7}$$

The solution of (2.3.6) with the initial condition (2.3.7) can be easily obtained as follows:

$$\begin{aligned}
V_0(x,t) &= 2\text{sech}(2x), \\
V_1(x,t) &= 8it\text{sech}(2x), \\
V_2(x,t) &= -16t^2\text{sech}(2x), \\
V_3(x,t) &= \frac{-63}{3}it^3\text{sech}(2x), \\
V_4(x,t) &= \frac{64}{3}t^4\text{sech}(2x), \\
V_5(x,t) &= \frac{256}{15}it^5\text{sech}(2x).
\end{aligned} \tag{2.3.8}$$

The further components can also be simply obtained in a similar manner. Substituting (2.3.8) into (2.1.9) yields

$$u(x,t) = 2\text{sech}(2x)\left(1 + 4it - 8t^2 - \frac{32}{3}it^3 + \frac{32}{3}t^4 + \frac{128}{15}it^5 - \cdots\right). \tag{2.3.9}$$

The exact solution of (2.3.1)

$$u(x,t) = 2\text{sech}(2x)e^{4it} \tag{2.3.10}$$

follows immediately upon using the Taylor series expansion of e^{4it} .

CHAPTER 3 ON THE CONVERGENCE OF HPM

3.1 Introduction

The seeking of a better and easy way to find the solution of nonlinear equations that illuminate the nonlinear phenomena of real life problems of science and engineering has recently received a continuing interest.

Various methods have been proposed to find the approximate solutions of nonlinear problems. One of the best techniques is the homotopy perturbation technique which is described in Chapter 2, and normally an infinite series is seen to converge rapidly to the exaact solution. There are numerous nonlinear problems[33, 34, 39, 38, 37, 17, 16, 50, 52, 30, 1, 29, 48, 2, 27, 28, 48] that have been treated with this method. Amongst them, the nonlinear equations arising in heat transfer [29] and [48], the nonlinear Schrödinger equations [17], the Painleve equation [27] , the Burger's equation [28] and the quadratic Riccati differential equation [2].

In term of biological fields, one may refer to the recent publications [24] and [23] for the application of the homotopy perturbation method. In this manner, we can see that the homotopy perturbation method is a powerful tool to solve the nonlinear problems in the field of science and engineering. However, the question behind the convergence of the homotopy perturbation method still remains unanswered. A routine convergence theorem for the homotopy perturbation method was outlined in [18] recently. However, the prior knowledge of the exact solution is required for this theorem of convergence. For examples of this theorem, see [18]. In this chapter, we investigate the convergence of the homotopy perturbation method from a mathematical point of view. The main goal of this chapter is to analyze the homotopy perturbation method theoretically and show eventually that under certain circumstances the method converges to the exact desired solution, without the prior knowledge of the exact solution. Another objective of this chapter is to point out the error estimate of the approximate solution. Some theorems are justified through basic examples that are well studied in the literature. The presented theory in this section provides the information not only about the convergence but also the interval of convergence for the homotopy series.

Rewriting (2.1.6) as

$$L(v) - L(u_0) = p[f(r) - L(u_0) - N(v)] \tag{3.1.1}$$

and substituting (2.1.8) into (3.1.1), we get

$$L\left(\sum_{i=0}^{\infty} v_i p^i\right) - L(u_0) = p\left[f(r) - L(u_0) - N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right]. \tag{3.1.2}$$

Thus

$$\sum_{i=0}^{\infty} L(v_i)p^i - L(u_0) = p\left[f(r) - L(u_0) - N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right]. \tag{3.1.3}$$

Applying Maclaurin expansion of $N(\sum_{i=0}^{\infty} v_i p^i)$ with respect to p, we get

$$N\left(\sum_{i=0}^{\infty} v_i p^i\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right)_{p=0} p^i. \tag{3.1.4}$$

Also from [31], we have

$$\left(\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right)_{p=0} = \left(\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} v_i p^i\right)\right)_{p=0}. \tag{3.1.5}$$

Thus, we have

$$N\left(\sum_{i=0}^{\infty} v_i p^i\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} v_i p^i\right)\right)_{p=0} p^i. \tag{3.1.6}$$

Or

$$N\left(\sum_{i=0}^{\infty} v_i p^i\right) = \sum_{n=0}^{\infty} H_i p^i, \tag{3.1.7}$$

where $H_n(v_0, v_1, \cdots, v_n) = \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} v_i p^i\right)\right)_{p=0}$, $n = 0, 1, 2, 3, \cdots$, are called He's polynomials [31]

Substituting (3.1.7) into (3.1.3), we have

$$\sum_{i=0}^{\infty} L(v_i)p^i - L(u_0) = p\left[f(r) - L(u_0) - \sum_{n=0}^{\infty} H_i p^i\right]. \tag{3.1.8}$$

Equating the like power terms of p, we get

$$\begin{aligned}
p^0 &: L(v_0) - L(u_0) = 0, \\
p^1 &: L(v_1) = f(r) - L(u_0) - H_0, \\
p^2 &: L(v_2) = -H_1, \\
&\vdots \\
p^n &: L(v_{n+1}) = -H_n, \\
&\vdots
\end{aligned} \tag{3.1.9}$$

and inverting L leads to

$$\begin{aligned}
v_0 &= u_0, \\
v_1 &= L^{-1}[f(r)] - u_0 - L^{-1}(H_0), \\
v_2 &= -L^{-1}(H_1), \\
&\vdots \\
v_{n+1} &= -L^{-1}(H_n), \\
&\vdots
\end{aligned} \tag{3.1.10}$$

We now prove the following theorem that justifies that homotopy perturbation method is convergence to exact solution of the problem.

Theorem 3.1.11. *[6]*

Homotopy perturbation method used the solution of (2.1.1) *is equivalent to determining the* *following sequence;*

$$s_n = v_1 + \cdots + v_n, s_0 = 0, \tag{3.1.12}$$

From the iterative scheme, we can write

$$s_{n+1} = -L^{-1}N_n(s_n + v_0) - u_0 + L^{-1}(f(r)), \tag{3.1.13}$$

where

$$N_n\left(\sum_{i=0}^{n} v_i\right) = \sum_{n=0}^{n} H_i, \qquad n = 0, 1, 2, \cdots \tag{3.1.14}$$

Proof. From (3.1.13),for $n = 0$ we have

$$\begin{aligned} s_1 &= -L^{-1}N_0(s_0 + v_0) - u_0 + L^{-1}(f(r)), \\ &= -L^{-1}(H_0) - u_0 + L^{-1}(f(r)). \end{aligned} \tag{3.1.15}$$

Then

$$v_1 = -L^{-1}(H_0) - u_0 + L^{-1}(f(r)). \tag{3.1.16}$$

For $n = 1$:

$$\begin{aligned} s_2 &= -L^{-1}N_1(s_1 + v_0) - u_0 + L^{-1}(f(r)) \\ &= -L^{-1}(H_0 + H_1) - u_0 + L^{-1}(f(r)), \\ &= -L^{-1}(H_1) + v_1. \end{aligned} \tag{3.1.17}$$

Since $s_2 = v_1 + v_2$, we have

$$v_2 = -L^{-1}(H_1). \tag{3.1.18}$$

We now argue by induction. Assume that $v_{k+1} = -L^{-1}(H_k), k = 1, 2, \cdots, n-1$. Thus,

$$\begin{aligned} s_{n+1} &= -L^{-1}N_n(s_n + v_0) - u_0 + L^{-1}(f(r)) \\ &= -L^{-1}\left(\sum_{n=0}^{n} H_i\right) - u_0 + L^{-1}(f(r)), \\ &= -\sum_{n=0}^{n} L^{-1}(H_i) - u_0 + L^{-1}(f(r)) \\ &= -L^{-1}(H_n) + v_1 + v_2 + \cdots + v_n \end{aligned} . \tag{3.1.19}$$

Then, by (3.1.13), we have

$$v_{n+1} = -L^{-1}(H_n), \tag{3.1.20}$$

which is (3.1.10). This completes the proof. ■ □

Further related theorems for the convergence of HPM in details are presented in [6].

3.2 Example

Consider the equation of Lane-Emden in the following form

$$u'' + \frac{2}{x}u' + u = x^5 + 30x^3, u(0) = 0, u'(0) = 0 \tag{3.2.1}$$

whose exact solution is $u(x) = x^5$. [6]

An application of homotopy perturbation method leads to the following equations

$$\begin{aligned} &(1-p)(v'' - v_0'') + p(v'' + \frac{2}{x}v' - x^5 - 30x^3) = 0 \\ &or, \qquad v'' - V_0'' = p(x^5 + 30x^3 - \frac{2}{x}v' - v - v_0'') \end{aligned} \tag{3.2.2}$$

The solution of (3.2.1) is given by

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{3.2.3}$$

$$= \sum_{i=0}^{\infty} v_i \tag{3.2.4}$$

Substituting (3.2.3) into (3.2.2), we get

$$\sum_{i=0}^{\infty} v_i'' - V_0'' = p(x^5 + 30x^3 - \frac{2}{x}\sum_{i=0}^{\infty} v_i' - \sum_{i=0}^{\infty} v_i - v_0'') \tag{3.2.5}$$

Equating the terms with like powers of p we get

$$\begin{aligned}
p^0 &: v_0 &&= 0,\\
p^1 &: v_1'' - x^5 - 30x^3 &&= 0,\\
p^2 &: v_2'' + \frac{2}{x}v_1' + v_1 &&= 0,\\
p^3 &: v_3'' + \frac{2}{x}v_2' + v_2 &&= 0,\\
&\vdots\\
p^n &: v_n'' + \frac{2}{x}v_{n-1}' + v_{n-1} &&= 0,\\
&\vdots
\end{aligned} \tag{3.2.6}$$

Now, we derive the solution of (3.2.6) with the initial conditions given in (3.2.1) as follows:

$$\begin{aligned}
v_0 &= 0\\
v_1 &= \frac{1}{42}x^7 + \frac{3}{2}x^5;\\
v_2 &= \frac{-1}{3024}x^9 - \frac{11}{252}x^7 - \frac{3}{4}x^5\\
v_3 &= \frac{25}{36288}x^9 + \frac{7}{216}x^7 + \frac{3}{8}x^5 + \frac{1}{332640}x^{11};\\
v_4 &= \frac{-137}{19958400}x^{13} + \frac{-271}{435456}x^9 - \frac{179}{9072}x^7 - \frac{3}{16}x^5 - \frac{1}{51891840}x^{13}\\
&\vdots
\end{aligned} \tag{3.2.7}$$

3.3 Definition

We define

$$\lambda_i = \begin{cases} \frac{||v_{i+1}||}{||v_i||}, & ||v_i|| \neq 0, \\ 0 & ||v_i|| = 0. \end{cases} \tag{3.3.1}$$

for every $i \in \mathbb{N}$.

From Theorem 3.2 of [6], $\sum_{n=0}^{\infty} v_i$ converges to exact solution for $0 \leq \lambda_i < 1$. If v_i and v_i' are derived from two different homotopies of the problem and $\lambda_i < \lambda_i'$ for each $i \in N$, the rate of convergence of $\sum_{n=0}^{\infty} v_i$ is higher than $\sum_{n=0}^{\infty} v_i'$.

Considering $||f(x)|| = \max_{0\leq x\leq 1} |f(x)|$, we get

$$\begin{aligned} \lambda_1 &= 0.5210503471, & \lambda_2 &= 0.5139910140, \\ \lambda_3 &= 0.5093374003, & \lambda_4 &= 0.5062439696, \\ \lambda_5 &= 0.5041785188, & \lambda_6 &= 0.5027965117. \end{aligned} \tag{3.3.2}$$

On the other hand, taking the linear part of the equation (3.2.1) as follows;

$$Lu = u'' + \frac{2}{x}u' \tag{3.3.3}$$

we construct the following homotopy

$$\left(w'' + \frac{2}{x}w'\right) - \left(u_0'' + \frac{2}{x}u_0'\right) = p(x^5 + 30x^3 - w - u_0'' - \frac{2}{x}u_0'). \tag{3.3.4}$$

Suppose that $u = w_0 + pw_1 + p^2w_2 + \cdots$, be the solution of (3.3.4) with $w_0 = u_0 = 0$.

Then the terms with the identical powers of p can be obtained as follows.

$$
\begin{aligned}
p^0 &: w_0 && = 0, \\
p^1 &: w_1'' + \frac{2}{x}w_1' - x^5 - 30x^3 && = 0, \\
p^2 &: w_2'' + \frac{2}{x}w_2' - w_1 && = 0, \\
p^3 &: w_3'' + \frac{2}{x}w_3' + w_2 && = 0, \\
&\vdots \\
p^n &: w_n'' + \frac{2}{x}w_n' - w_{n-1} && = 0, \\
&\vdots
\end{aligned}
\tag{3.3.5}
$$

The solution of (3.3.5) can be easily obtained with the help of MATLAB as follows:

$$
\begin{aligned}
w_0 &= 0 \\
w_1 &= \frac{1}{56}x^7 + x^5; \\
w_2 &= \frac{-1}{5040}x^9 - \frac{1}{56}x^7; \\
w_3 &= \frac{1}{665280}x^{11} + \frac{1}{5040}x^9; \\
w_4 &= \frac{-1}{121080960}x^{13} - \frac{-1}{665280}x^{11}; \\
&\vdots
\end{aligned}
\tag{3.3.6}
$$

and,

$$
\begin{aligned}
\lambda_1 &= 0.0177387914, & \lambda_2 &= 0.0110722610, \\
\lambda_3 &= 0:00756010906, & \lambda_4 &= 0.00548724954, \\
\lambda_5 &= 0:00416293765, & \lambda_6 &= 0.00326590091.
\end{aligned}
\tag{3.3.7}
$$

Thus, with the above result, it can be clearly concluded that the rate of convergence of homotopy (3.3.4) is more accurate than the (3.2.3).

CHAPTER 4 HPM FOR NONLINEAR OSCILLATORS

4.1 Introduction

In all areas of physics ,engineering and generally in every field of real world applications, we can see some simple systems for which the equations governing their behavior are easy to formulate but whose mathematical resolution is complicated [40, 46]. The reason behind this is that in most cases the systems are governed by nonlinear phenomena with nonlinear systems. The history for nonlinear oscillations in conservative systems goes far back to the origin of mathematics. A paradigm for such system is usually considered dealing with the simple pendulum [46, 7, 14] problems. It may be possible to replace nonlinear differential equations with the corresponding linear differential equation in many cases that approximates the original equation, but such linearization is not always feasible. In such cases, we should deal directly with actual nonlinear differential equation.

Solving nonlinear problems is a very difficult task and, in general, to get an approximation solution of a given nonlinear problem is often more difficult than a numerical one [46] . There are several methods that are in use to get an approximation solution of nonlinear problems, such as perturbation methods [46, 4, 35, 36, 36, 3, 26] , variational approaches [5, 49, 42, 42, 51, 5, 13, 8, 25, 20], decomposition [20, 47], parameter expansion [54], exp-function [21, 53, 15] or harmonic balance based methods [46, 12, 43]. Above mentioned technique in majority have been used in order to get the analytical approximations for the nonlinear pendulum [3, 12, 41, 10, 11]. In general, these approximate methods may be applied only for certain classes of problems and are within certain ranges of parameters.

In this section, we apply He's perturbation technique to get improved approximate solutions to conservative truly nonlinear oscillators. By this approach, we can get a truly periodic solution and the period of the motion as a function of the amplitude of oscillation. For this purpose, we take the case of the cubic oscillator in which we can see that this method works very well for the whole range of parameters. The agreement of the approximate frequencies with the exact one has been demonstrated and discussed in [9] .

Further, we can also see that this approach yields a very rapid convergence of the solution series and does not require a linear term and a perturbation parameter [9]. If this parameter exists, then this technique need not require that the parameter to be small. This perturbation approach has been very useful not only to nonlinear oscillators but also to other nonlinear problems [19].

4.2 Homotopy Perturbation Method Applications

4.2.1 The Cubic Oscillator

The non-dimensional differential equation describing the truly nonlinear oscillator [9] is

$$\frac{d^2x}{dt^2} + \beta x^3 = 0, \qquad \beta > 0 \tag{4.2.1}$$

with initial conditions $x(0) = A$ and $x'(0) = 0$.

By the homotopy perturbation method, we construct the homotopy of (4.2.1) as follows:

$$\frac{d^2x}{dt^2} + x = \lambda(x - \beta x^3), \qquad \beta > 0 \tag{4.2.2}$$

where λ is the homotopy parameter. When $\lambda = 0$, the equation (4.2.2) becomes linearized and for $\lambda = 1$ it takes the original form of the equation. Now, the expansion of the solution $x(t)$ with the use of the homotopy parameter p and the square of the unknown angular frequency ω can be written as follows:

$$x(t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t) + \cdots \tag{4.2.3}$$

$$1 = \omega^2 - \lambda\alpha_1 - \lambda^2\alpha_2 - \cdots , \tag{4.2.4}$$

where $\alpha_i,\ i = 1, 2, 3, \cdots$ are to be determined.

Now, substituting (4.2.3) and (4.2.4) into (4.2.2) we get;

$$\begin{aligned}&(x_0'' + \lambda x_1'' + \lambda^2 x_2'' + \cdots) + (\omega^2 - \lambda\alpha_1 - \lambda^2\alpha_2 - \cdots) \times (x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots) \\ =&\lambda[(x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots) - \beta(x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots)^3]\end{aligned} \tag{4.2.5}$$

Now, equating the terms with the identical powers of λ , we get a series of linear differential equations, amongst them we write only the first three

$$x_0'' + \omega^2 x_0 = 0, \qquad x_0(0) = A, \qquad x_0'(0) = 0, \tag{4.2.6}$$

$$x_1'' + \omega^2 x_1 = (1 + \alpha_1)x_0 - \beta x_0^3 \qquad x_1(0) = x_1'(0) = 0, \tag{4.2.7}$$

$$x_2'' + \omega^2 x_2 = \alpha_2 x_0 + (1 + \alpha_1)x_1 - 3\beta x_1 x_0^2 \qquad x_2(0) = x_2'(0) = 0. \tag{4.2.8}$$

In the above equations (4.2.6), (4.2.7) and (4.2.8), we have taken into account the following expression:

$$f(x) = f(x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots) \approx f(x_0) + \lambda x_1 f'(x_0) + \lambda^2[x_2 f'(x_0) + \frac{1}{2}x_1^2 f''(x_0)] + \mathcal{O}(\lambda^3) \tag{4.2.9}$$

where $f'(x) = df(x)/dx$.

Now, the solution of (4.2.6) is

$$x_0(t) = A\cos\omega t. \tag{4.2.10}$$

Substitution of this result into (4.2.7), we obtain the following differential equation for x_1:

$$x_1'' + \omega^2 x_1 = (1 + \alpha_1 - \frac{3}{4}\beta A^2)A\cos\omega t - \frac{1}{4}\beta A^3\cos^3\omega t \tag{4.2.11}$$

No secular terms in $x_1(t)$ require eliminating contributions proportional to $cos\omega t$ on the right side of the equation (4.2.11) .

Therefore,

$$1 + \alpha_1 - \frac{3}{4}\beta A^2 = 0$$

Or,

$$\alpha_1 = -1 + \frac{3}{4}\beta A^2 \tag{4.2.12}$$

and the solution of equation (4.2.11) can be obtained as follows:

$$x_1'' + \omega^2 x_1 = -\frac{1}{4}\beta A^3\cos^3\omega t$$

$$x_1(t) = -\frac{1}{32\omega^2}\beta A^3(\cos\omega t - \cos 3\omega t) \tag{4.2.13}$$

Following the exactly similar procedure at the second order, it is possible to obtain

$$\alpha_2 = -\frac{3}{128\omega^2}\beta^2 A^4, \qquad x_2(t) = -\frac{1}{1024\omega^4}\beta^2 A^5(\cos\omega t - \cos 5\omega t) \tag{4.2.14}$$

Thus, it is possible to obtain the approximate frequency, the period and the solution for each approximation [9].

4.3 Homotopy Perturbation Method for Nonlinear Oscillators of the Form $x'' = -\beta x^p$

We study a family of nonlinear oscillators governed by the equation

$$\frac{d^2x}{dt^2} = -\beta x^p, \beta > 0.$$

Here β is a constant and p=1,3,5,... is necessarily odd to be an oscillator (the phase portrait needs to be closed and bounded). The initial condition is $x(0) = A, x'(0) = 0$. Our goal is to find the frequency of the oscillator and its trajectory for each p. We apply the Lindstedt-Poincaré method to a convex homotopy with the parameter λ:

$$x''(t) + x(t) = \lambda[x(t) - \beta x(t)^p], 0 \leq \lambda \leq 1. \tag{4.3.1}$$

Substituting the expansion

$$x(t) = \sum_{i=0}^{\infty} \lambda^i x_i(t)$$

and the expression

$$1 = \omega^2 - \lambda\alpha_1 - \lambda^2\alpha_2 + ... \tag{4.3.2}$$

into 1, the coefficient of $x(t)$ on the left side of (4.3.1), we obtain the first few ODE's for $x_i(t)$ such as Eq's (4.3.4)-(4.3.6) below after equating the coefficients of the powers of λ. The final target solution $x(t)$ and the frequency ω are obtained through a sequence of $x_i(t), \alpha_i, i = 1, 2, ...$ and setting λ to 1. It is necessary to kill the secular terms to have oscillatory behavior in the model.

Theorem 4.3.3. *Let the sequence of approximation ODEs with the initial conditions $x_0(0) = A, x_0'(0) = 0$ and $x_i(0) = 0, x_i'(0) = 0, i \neq 0$, be defined as*

$$x_0'' + \omega^2 x_0 = 0, \tag{4.3.4}$$

$$x_1'' + \omega^2 x_1 = (1 + \alpha_1)x_0 - \beta x_0^p, \tag{4.3.5}$$

$$x_2'' + \omega^2 x_2 = \alpha_2 x_0 + (1 + \alpha_1)x_1 - p\beta x_1 x_0^{p-1}. \tag{4.3.6}$$

Then

$$x_0 = A\cos\omega t,$$

$$x_1(t) = \tilde{A}_0 \cos \omega t + \sum_{k=0}^{p'} A_k \cos \omega(p-2k)t, p' = (p-3)/2, \tag{4.3.7}$$

where $\tilde{A}_0$ *is from* (4.3.12) *and* A_k *is from* (4.3.11).

$$x_2'' + \omega^2 x_2 = \frac{\beta A^{p-1}}{2^{p-1}} \binom{p}{\frac{p-1}{2}} \sum_{k=0}^{p'} A_k \cos(p-2k)\omega t$$

$$- \beta p \frac{\tilde{A}_0 A^{p-1}}{2^{p-1}} \sum_{k=0}^{p'} \binom{p}{k} \cos(p-2k)\omega t$$

$$- \beta p \frac{A^{p-1}}{2^{p-1}} \sum_{l=0}^{p'} \sum_{k=l+(1-p)/2}^{l-(p+1)/4} A_{k-l+(p-1)/2} \cos(p-2k)\omega t$$

$$- \frac{\beta p}{2^{p-1}} \sum_{k=0}^{p'} A_k \binom{p-1}{\frac{p-1}{2}} \cos(p-2k)\omega t$$

$$- \frac{\beta p}{2^{p-1}} \sum_{l=1}^{p'} \sum_{k=\frac{p-1}{2}-l}^{p'} A_{l+k-(p-1)/2} \binom{p-1}{l} \cos(p-2k)\omega t.$$

We will derive this theorem in the next few pages. Since $x_0 = A \cos \omega t$ is obviously the solution of $x_0'' + \omega^2 x_0 = 0, x_0(0) = A, x'(0) = 0$. Let us start with x_1, the solution of

$$x_1'' + \omega^2 x_1 = (1 + \alpha_1)x_0 - \beta x_0^p, x_1(0) = x_1'(0) = 0. \tag{4.3.8}$$

Let $p'' = (p-1)/2$. Since

$$\cos^p(\omega t) = \frac{1}{2^{p-1}} \sum_{k=0}^{p''} \binom{p}{k} \cos(\omega(p-2k)t), p \text{ is odd,} \tag{4.3.9}$$

$$\begin{aligned} x_1'' + \omega^2 x_1 &= (1+\alpha_1)x_0 - \beta A^p \cos^p \omega t \\ &= (1+\alpha_1)x_0 - \frac{\beta A^p}{2^{p-1}} \sum_{k=0}^{p''} \binom{p}{k} \cos(\omega(p-2k)t). \end{aligned}$$

To eliminate the secular term, the coefficient of $\cos \omega t$ contained in the right side of the above equation must be zero, Thus, since $1 \leq p - 2k \leq p$ when $0 \leq k \leq p''$ we see that $k = (p-1)/2$ makes $p = 2k - 1$ and $\cos \omega t$ results. Setting

$$(1+\alpha_1)A - \frac{\beta A^p}{2^{p-1}} \binom{p}{\frac{p-1}{2}} = 0$$

we see that

$$\alpha_1 = -1 + \frac{\beta A^{p-1}}{2^{p-1}} \binom{p}{\frac{p-1}{2}}. \tag{4.3.10}$$

Thus the ODE now reads

$$x_1'' + \omega^2 x_1 = -\frac{\beta A^{p-1}}{2^{p-1}} \sum_{k=0}^{p'} \binom{p}{k} \cos \omega(p-2k)t, \quad p' = \frac{p-3}{2}$$

as $k = (p-1)/2$ is excluded. Now assume the particular solution takes the form of $x_s = \sum_{k=0}^{p'} A_k \cos \omega(p-2k)t$, which implies

$$x_s'' + \omega^2 x_s = \sum_{k=0}^{p'} A_k(\omega^2 - \omega^2(p-2k)^2) \cos \omega(p-2k)t.$$

Hence

$$\frac{\beta A^{p-1}}{2^{p-1}} \binom{p}{k} = A_k \omega^2[(p-2k)^2 - 1]$$

and we conclude that

$$A_k = \frac{\beta A^{p-1}}{2^{p-1}} \binom{p}{k} / \{\omega^2[(p-2k)^2 - 1]\}. \quad 0 \le k \le p' = (p-3)/2. \tag{4.3.11}$$

Let the general solution be $x = \tilde{A}_0 \cos \omega t + \sum_{k=0}^{p'} A_k \cos \omega(p-2k)t$ and $\tilde{A}_0$ can be determined by $x(0) = 0$:

$$\tilde{A}_0 = -\sum_{k=0}^{p'} A_k. \tag{4.3.12}$$

We conclude that

$$x_1(t) = \tilde{A}_0 \cos \omega t + \sum_{k=0}^{p'} A_k \cos \omega(p-2k)t, \tag{4.3.13}$$

where $\tilde{A}_0$ is from (4.3.12) and A_k is from (4.3.11).

Example. $p = 3$.

In this case $p' = 0$ and

$$\begin{aligned} x_1(t) &= \tilde{A}_0 \cos \omega t + A_0 \cos 3\omega t, \tilde{A}_0 = -A_0 = -\frac{\beta A^3}{32\omega^2}, \\ &= -\frac{\beta A^3}{32\omega^2}[\cos \omega t - \cos 3\omega t], \end{aligned}$$

which is Eq. (4.2.13) or Eq (38) in [9]. From (4.3.2) and (4.3.10), we get $\omega_1^2 = \frac{3\beta A^2}{4}$, which is the same as Eq. (4.2.12) or Eq. (41) of [9].

4.4 The Second Order Approximation x_2 and The Frequency Approximation α_2

$$x_2'' + \omega^2 x_2 = \alpha_2 x_0 + (1+\alpha_1)x_1 - \beta p x_1 x_0^{p-1}, \tag{4.4.1}$$

where

$$
\begin{aligned}
x_1 x_0^{p-1} &= \{\tilde{A}_0 \cos\omega t + \sum_{k=0}^{p'} A_k \cos(p-2k)\omega t\} A^{p-1}\cos^{p-1}\omega t \\
&= A^{p-1}\tilde{A}_0 \cos^p \omega t + \sum_{k=0}^{p'} A_k A^{p-1}\cos(p-2k)\omega t \cos^{p-1}\omega t \\
&:= S_1 + S_2. \qquad (4.4.2)
\end{aligned}
$$

From (4.3.9),

$$
S_1 = \frac{\tilde{A}_0 A^{p-1}}{2^{p-1}} \sum_{k=0}^{p''} \binom{p}{k} \cos(p-2k)\omega t, \quad p'' = (p-1)/2. \qquad (4.4.3)
$$

Noting $p-1$ being even, we have

$$
\cos^{p-1}(\omega t) = \frac{1}{2^{p-2}} \sum_{l=0}^{p'} \binom{p-1}{l} \cos(p-1-2l)\omega t + \frac{1}{2^{p-1}} \binom{p-1}{\frac{p-1}{2}}. \qquad (4.4.4)
$$

4.4.1 Computation of α_2

We need to find the coefficient of $\cos\omega t$ and set it equal to zero. Let us look at S_1: Since $1 \leq p-2k \leq p$ the presence of $\cos\omega t$ term corresponds to $p-2k=1$ or $k=(p-1)/2$. Contribution to $\cos\omega t$ from S_1 in (4.4.3) through (4.4.1) is

$$
-\beta p \frac{\tilde{A}_0 A^{p-1}}{2^{p-1}} \binom{p-1}{\frac{p-1}{2}}. \qquad (4.4.5)
$$

Now we look at

$$
S_2 = \sum_{k=0}^{p'} A_k A^{p-1} \cos(p-2k)\omega t \cos^{p-1}\omega t.
$$

From (4.4.4),

$$\sum_{k=0}^{p'} A_k \cos(p-2k)\omega t \cos^{p-1}\omega t = \sum_{k=0}^{p'}\sum_{l=0}^{p'} \binom{p-1}{l} \frac{A_k}{2^{p-2}} \cos(p-2k)\omega t \cos(p-1-2l)\omega t$$

$$+ \sum_{k=0}^{p'} \frac{A_k}{2^{p-1}} \binom{p-1}{\frac{p-1}{2}} \cos(p-2k)\omega t$$

$$:= J_1 + J_2,$$

where using product to sum formula we have

$$J_1 = \sum_{k=0}^{p'}\sum_{l=0}^{p'} \binom{p-1}{l} \frac{A_k}{2^{p-1}} \{(\cos(2p-2l-2k-1)\omega t + \cos(2l-2k+1)\omega t\}$$

$$:= J_{11} + J_{12}, \tag{4.4.6}$$

$$J_2 = \sum_{k=0}^{p'} \frac{A_k}{2^{p-1}} \binom{p-1}{\frac{p-1}{2}} \cos(p-2k)\omega t \quad (S_2 = A^{p-1}(J_1+J_2)). \tag{4.4.7}$$

Contribution of $\cos\omega t$

From J_2: Since $0 \le k \le p' \Rightarrow 3 \le p-2k \le p$, we have $p-2k \ne 1$ and there is no contribution from J_2,

From J_1: Since $0 \le l \le p'$ and $0 \le k \le p'$ imply $3 \le 2p-2l-2k-1 \le 2p-1$ and $2p-2l-2k+1$ is never one. No contribution from $\cos(2p-2l-2k-1)\omega t$ terms (the J_{11} term). Since $0 \le l \le p'$ and $0 \le k \le p'$ imply

$$4-p \le 2l-2k+1 \le p-2$$

we have for $p \ge 3$ the possibility of $2l-2k+1 = \pm 1$. So we need to handle

$$J_{12} = \frac{1}{2^{p-1}} \sum_{k=0}^{p'}\sum_{l=0}^{p'} \binom{p-1}{l} A_k \cos(2l-2k+1)\omega t$$

and more carefully $\frac{1}{2^{p-1}}\tilde{J}$,

$$\tilde{J} = \sum_{l=0}^{p'} \{ \sum_{k \le l}^{p'} \binom{p-1}{l} A_k \cos(2l-2k+1)\omega t + \sum_{k \ge l+1}^{p'} \binom{p-1}{l} A_k \cos(2l-2k+1)\omega t \}$$
$$:= \tilde{J}_1 + \tilde{J}_2,$$

where the splitting is based on whether $2l - 2k + 1$ is positive or negative. Thus $\tilde{J}_1$ contains $\cos \omega t$ terms from which we can extract

$$\sum_{l=0}^{p'} A_l \binom{p-1}{l} \cos \omega t \qquad (2l - 2k + 1 = 1 \Rightarrow k = l)$$

and from $\tilde{J}_2$ we can extract

$$\sum_{l=0}^{p'-1} A_{l+1} \binom{p-1}{l} \cos \omega t \quad (2l - 2k + 1 = -1 \Rightarrow k = l + 1).$$

Note the upper limit is $p' - 1$ due to the fact that

$$0 \le k \le p' \Rightarrow 0 \le l + 1 \le p' \Rightarrow 0 \le l \le p' - 1.$$

We conclude that the contribution of $\cos \omega t$ from S_2 in (4.4.2) is

$$\frac{A^p}{2^{p-1}} \{ \sum_{l=0}^{p'} A_l \binom{p-1}{l} \cos \omega t + \sum_{l=0}^{p'-1} A_{l+1} \binom{p-1}{l} \cos \omega t \} \tag{4.4.8}$$

4.4.2 Summary

The coefficient of $\cos\omega t$ in view of (4.4.1) is from the following contributions:

$$\begin{aligned} \alpha_2 x_0: &\quad \alpha_2 A \\ (1+\alpha_1)x_1: &\quad (1+\alpha_1)\tilde{A}_0 \quad (cf.(4.3.12)) \\ -\beta p x_1 x_0^p: &\quad -\beta p(\gamma_1+\gamma_2), \end{aligned}$$

where

$$\gamma_1 := \frac{A^p}{2^{p-1}}\{\sum_{l=0}^{p'} A_l \binom{p-1}{l} + \sum_{l=0}^{p'-1} A_{l+1}\binom{p-1}{l}\} \quad (cf.(4.4.8)) \tag{4.4.9}$$

and

$$\gamma_2 := -\beta p \frac{\tilde{A}_0 A^{p-1}}{2^{p-1}}\binom{p-1}{\frac{p-1}{2}}. \quad (cf.(4.4.5)) \tag{4.4.10}$$

Enforcing

$$\alpha_2 A + (1+\alpha_1)\tilde{A}_0 - \beta p(\gamma_1+\gamma_2) = 0,$$

we conclude that

$$\alpha_2 = \frac{1}{A}\left(-(1+\alpha_1)\tilde{A}_0 + \beta p\gamma_1 + \beta p\gamma_2\right). \tag{4.4.11}$$

Example. $p = 3$

The data are: $p' = 0$, $\tilde{A}_0 = -A_0$ (cf. (4.3.12)), $\gamma_2 = -\frac{3}{4}A_0A^2, \gamma_1 = \frac{A^2A_0}{4}$, $A_0 = \frac{\beta A^3}{32\omega^2}$ (cf. (4.3.11)), Thus,

$$\alpha_2 = -\frac{3}{128}\frac{\beta^2 A^4}{\omega^2}$$

which is the same as Eq. (4.2.14)Eq. (39) of [9].

4.4.3 The Right-Hand Side of the ODE for x_2

The ODE for x_2 , after eliminating the $\cos\omega t$ term is

$$x_2'' + \omega^2 x_2 = \frac{\beta A^{p-1}}{2^{p-1}}\binom{p}{\frac{p-1}{2}}\sum_{k=0}^{p'} A_k \cos(p-2k)\omega t + R, \tag{4.4.12}$$

where $R = -\beta p(S_1^F + S_2^F)$. Here S_i^F are the terms from S_i free of the $\cos\omega t$ terms. More specifically, from S_1, we have

$$S_1^F = \frac{\tilde{A}_0 A^{p-1}}{2^{p-1}} \sum_{k=0}^{p'} \binom{p}{k} \cos(p-2k)\omega t, \quad (cf.\text{(4.4.3), the last term deleted}) \tag{4.4.13}$$

and

$$\begin{aligned} S_2^F &= A^{p-1}(J_1^F + J_2^F) \\ &= A^{p-1}(J_1^F + J_2) \quad \text{(by(4.4.7), } J_2 = J_2^F) \\ &= A^{p-1}(J_{11}^F + J_{12}^F + J_2) \\ &= A^{p-1}(J_{11} + J_2 + J_{12}^F), \quad \text{(by (4.4.7), } J_{11} = J_{11}^F) \end{aligned} \tag{4.4.14}$$

where

$$J_{12}^F = \frac{1}{2^{p-1}} \sum_{l=0}^{p'} \sum_{k=0, k\neq l, l+1}^{p'} \binom{p-1}{l} A_k \cos(2l-2k+1)\omega t.$$

Example. $p = 3$.

$$\begin{aligned} \frac{3}{4}\beta A_0 A^2 \cos 3\omega t &= \frac{\beta A^{p-1}}{2^{p-1}} \binom{p}{\frac{p-1}{2}} \sum_{k=0}^{p'} A_k \cos(p-2k)\omega t, \\ S_1^F &= \frac{\tilde{A}_0 A^2}{4} \cos 3\omega t = \frac{-A_0 A^2}{4} \cos 3\omega t, \\ S_2^F &= A^2(J_{11} + J_2 + J_{12}^F), \\ J_{11} &= \sum_0^{p'} \sum_0^{p'} \frac{A_k}{4} \cos(2p - 2l - k - 1)\omega t \binom{2}{l} = \frac{A_0}{4} \cos 5\omega t, \\ J_2 &= \frac{A_0}{2} \cos 3\omega t, \quad (cf.\text{ (4.4.7)}), \\ J_{12}^F &= 0. \end{aligned}$$

Thus

$$x_2'' + \omega^2 x_2 = -\frac{3\beta A_0 A^2}{4}\cos 5\omega t, A_0 = \frac{\beta A^3}{32\omega^2}.$$

It is easily checked that a particular solution is

$$x_p = \frac{1}{1024}\frac{\beta^2 A^5}{\omega^4}\cos 5\omega t := \alpha \cos 5\omega t.$$

Let $x_2 = E\cos\omega t + \alpha\cos 5\omega t$ and use $x_2(0) = 0$ to conclude $E = -\alpha$. Thus

$$x_2(t) = -\frac{1}{1024}\frac{\beta^2 A^5}{\omega^4}(\cos\omega t - \cos 5\omega t)$$

which is the same as Eq. (4.2.14) or Eq. (39) of [9].

Next, we convert

$$\sum_{l=0}^{p'}\sum_{k=0}^{p'}\binom{p-1}{l} A_k \cos(2l-2k+1)\omega t = H_1 + H_2$$

in J_{11} of (4.4.6) into the sum of $\cos(p-2m)\omega t$ terms. Here

$$H_1 = \sum_{l=0}^{p'}\sum_{k\le l}^{p'}\binom{p-1}{l} A_k \cos(2l-2k+1)\omega t,$$

$$H_2 = \sum_{l=0}^{p'}\sum_{k\ge l+1}^{p'}\binom{p-1}{l} A_k \cos(2l-2k+1)\omega t$$

Conversion of H_1 minus $\cos\omega t$ terms: Set $2l-2k+1 = p-2m \ge 0$ and $p-2m \ne 1$ together with $0 \le k = (l+m) - \frac{p-1}{2} \le l \Rightarrow m \le \frac{p-1}{2}$ imply $m < \frac{p-1}{2}$ or $m \le p'$. So $H_1 - \cos\omega t$ terms is

$$\sum_{l=0}^{p'}\sum_{m\ge \frac{p-1}{2}-l}^{p'}\binom{p-1}{l} A_{l+m-\frac{p-1}{2}}\cos(p-2m)\omega t.$$

Conversion of H_2 minus $\cos\omega t$ terms: Set $2l-2k+1 = p-2m \le 0$ and $p-2m \ne -1$ imply $m \ge (p+3)/2$. On the other hand, $0 \le k = m+l-\frac{1-p}{2} \ge l+1 \Rightarrow m \ge \frac{p+1}{2}$. Together we need

$m \geq (p+3)/2$. So $H_2 - \cos\omega t$ terms is

$$\sum_{l=0}^{p'} \sum_{m=\frac{p+3}{2}-l}^{p+1-l} \binom{p-1}{l} A_{m-l-\frac{1-p}{2}} \cos(p-2m)\omega t,$$

which is empty since $p + \frac{1}{2} \leq l$ is impossible.

Thus we see that the non-$\cos\omega t$ terms in

$$\sum_{l=0}^{p'} \sum_{k=0}^{p'} \binom{p-1}{l} A_k \cos(2l-2k+1)\omega t$$

is

$$\sum_{l=0}^{p'} \sum_{m \geq \frac{p-1}{2}-l}^{p'} \binom{p-1}{l} A_{l+m-\frac{p-1}{2}} \cos(p-2m)\omega t.$$

Conclusion: the J_{12}^F in (4.4.6) takes the form:

$$J_{12}^F = \frac{1}{2^{p-1}} \sum_{l=0}^{p'} \sum_{m \geq \frac{p-1}{2}-l}^{p'} \binom{p-1}{l} A_{l+m-\frac{p-1}{2}} \cos(p-2m)\omega t. \tag{4.4.15}$$

One should note that m can take on negative values.

Finally, we convert J_{11} to $\cos(p-2m)$ form. Recall

$$J_{11} = \sum_{l=0}^{p'} \sum_{k=0}^{p'} \binom{p-1}{l} \frac{A_k}{2^{p-1}} \cos(2p-2l-2k-1)\omega t.$$

To this end,

$$\begin{aligned} 2p-2l-2k-1 = p-2m &\Rightarrow k = (2p-2l-1-p+2m)/2 \\ &\Rightarrow 0 \leq 2p-4l-2+4m \leq p+3 \\ &\Rightarrow l + \frac{1-p}{2} \leq m \leq l - \frac{p+1}{4}. \end{aligned}$$

Hence

$$J_{11} = \frac{1}{2^{p-1}} \sum_{l=0}^{p'} \sum_{m \geq l + \frac{1-p}{2}}^{l - \frac{p+1}{4}} \binom{p-1}{l} A_{m-l+\frac{p-1}{2}} \cos(p-2m)\omega t. \tag{4.4.16}$$

Example. $p = 3$. In this case $p' = 0$ and so $l = 0$ only. Note that the lower limit $l + \frac{1-p}{2} = -1$ and the upper limit $l - \frac{p+1}{4} = -1$, which imply that $m = 0$ only. Then

$$J_{11} = \frac{A_0}{4} \binom{2}{0} \cos 5\omega t = \frac{A_0}{4} \cos 5\omega t.$$

CHAPTER 5 CONCLUSION

We have studied the concept of homotopy perturbation method and the convergence of homotopy perturbation method for nonlinear systems. In general homotopy method exhibits good approximation properties, which motivated us to apply it to a family of nonlinear oscillator governed by the equation

$$\frac{d^2x}{dt^2} = -\beta x^p, \beta > 0,$$

, where β is constant and p is odd in order for the phase portrait of the oscillators to be closed and bounded. We conclude that homotopy perturbation method for this family of nonlinear oscillators can produce accurate frequency-amplitude relations and trajectories. What is not addressed in this thesis is the theoretic convergence issue related to the oscillators, which we plan to do in the future.

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