i

APPLICATIONS OF EXTREME POINT METHODS TO PROBABILITY THEORY

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A Dissertation

Submitted to the Graduate School of Bowling Green
State University in partial fulfillment of
the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 1975

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ABSTRACT

In 1966, S. Johansen made use of Choquets' Theorem, a result in Functional Analysis, to obtain the Levy-Khinchine canonical form of infinitely divisible characteristic functions. This representation was first discovered in the 1930's and has numerous applications in Probability Theory. Johansen's work was very gratifying in the sense that it displayed much interplay between the disciplines of Functional Analysis and Probability. The purpose of this thesis is to continue exploration of extreme point methods and their applications to Probability Theory.

The major part of this dissertation concerns itself with extending Johansen's work to include n-dimensional infinitely divisible characteristic functions. If one is to follow the ingenious methods of Johansen, one has to know solutions of various functional equations in this more general setting. These projects occupy the first portion of the thesis.

In the final section of this paper, we derive Khinchine's representation of unimodal distributions through the use of extreme points.

ACKNOWLEDGEMENT

Many people have shown an interest and provided much encouragement and motivation during my graduate career at Bowling Green State University. Among those, I would like to give special thanks to Hassoon Al-Amiri, W. Charles Holland, R. G. Laha, Eugene Lukacs, Francis and Ann O'Connor, Vijay Rohatgi, M. Satyanarayana and Raymond Snipes.

I am extremely indebted to my advisor, Professor Laha, for introducing me to an area which combines several branches of mathematics and for the many stimulating conversations which proved to be so helpful in preparing this thesis.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARY MATERIAL

This paper concerns applications of the Krein-Milman Theorem and its corollaries to Probability Theory. Investigations of this nature were initiated by S. Johansen, in 1966, where he used these ideas to obtain the Levy-Khinchine canonical representation of infinitely divisible characteristic functions. We shall further explore these methods to extend Johansen's result to R^n and also obtain an integral representation of unimodal distributions.

Concerning the organization of this thesis, it is convenient to separate the preliminary material into two categories. The remainder of this chapter contains the background material from Probability Theory and Functional Analysis which are of interest to this study. On the other hand, there appears an appendix to Chapter II which contains statements and proofs of technical results, used in Chapters II and III, which would otherwise interupt the main discussion. The principal results are included in Chapters II, III and IV. Let us begin with some basic concepts from Functional Analysis.

DEFINITION Let V be a vector space over K, the real or complex field. Let $C \subseteq V$ be a non-empty convex set. A point $x_0 \in C$ is called an <u>extreme point</u> of C if whenever $x,y \in C$ and $0 < \lambda < 1$ and

$$x_0 = \lambda x + (1 - \lambda)y \qquad \text{then} \qquad x_0 = x = y.$$

These are several examples in R^n and more general vector spaces where extreme points are either absent or have no significance. Thus it is of interest to know when a convex set necessarily has extreme points and what role the extreme points play. The following basic theorem was discovered by Krein and Milman.

THEOREM (Krein-Milman)

Let V be a locally convex topological vector space, and let $C \subseteq V$ be a non-empty compact, convex set. Then $\mathrm{Ext}(C)$, the set of extreme points of C, is non-empty and C is the closed convex hull of $\mathrm{Ext}(C)$.

In order for the Krein-Milman Theorem to be applicable, we must be guaranteed of the compactness of the convex set in question. We now mention the result which produces most of the compact, convex sets.

Let E be a normed linear space over K, and denote by E^* the collection of all continuous linear functionals on E. The weak* topology on E^* is the topology generated by the following collection of basic neighborhods:

$$\{U(f,\epsilon,x_1,\ldots,x_N): \ f \in E^*, \ \epsilon > 0, \ N \in \mathbb{N}, \ x_1,\ldots,x_N \in E\}$$

where $U(f;\epsilon,x_1,\ldots,x_N) = \{g \in E^*: \ |g(x_j) - f(x_j)| < \epsilon$

$\text{for all } j, \ 1 \leq j \leq N\}$

Thus, a net $\{f_\alpha\} \subseteq E^*$ converges to $f \in E^*$ in the weak* topology, denoted $f_\alpha \xrightarrow{w^*} f$, if and only if $f_\alpha(x) \to f(x)$ for all $x \in E$. In other words, the weak* topology is equivalent to the topology of pointwise convergence. It is easy to show that E^* equipped with the weak* topology forms a Hausdorff locally convex topology vector space, and a linear functional $\ell: E^* \to K$ is weak* continuous if and only if there exists an $x_0 \in E$ such that $\ell(f) = f(x_0)$ for all $f \in E^*$. For example, if $E = L'(R^n)$, then $E^* = L^\infty(R^n)$. And for bounded measurable functions f_n, $f \in E^*$ we have $f_n \overset{w^*}{\to} f$ if and only if

$$\lim_{m \to \infty} \int_{R^n} f_m(t)\, h(t)dt = \int_{R^n} f(t)\, h(t)dt$$

for all $h \in L'(R^n)$. And a linear mapping $\ell: L^\infty(R^n) \to K$ is weak* continuous if and only if there exists an $h \in L'(R^n)$ such that

$$\ell(f) = \int_{R^n} f(t)\, h(t)dt \quad \text{for all} \quad f \in L^\infty(R^n).$$

The weak* topology has a very important compactness property to which we now turn our attention.

THEOREM (Banach-Alaoglu)

Let E be a separable normed linear space over K, and let $B^* = \{f \in E^*: ||f|| \leq 1\}$. Then B^* is a metrizable, compact, convex set in the weak* topology of E^*.

Consequently any weak* closed subset of B^* is weak* compact and hence our compact, convex sets are in abundance. In particular, $B_\infty = \{f \in L^\infty(R^n): \text{ess sup}\{|f(x)|: x \in R^n\} \leq 1\}$ is a weak* compact set. The question remains as to how all of this fits into Probability and Statistics. A tool that has been shown to be quite useful in these disciplines is that of an integral representation. A result which can easily be deduced from the Krein-Milman Theorem is the following version of Choquet's Theorem.

THEOREM (Choquet)

Let C be a metrizable, compact, convex set in a locally convex space, E. Then for all $x_0 \in C$, there exists a finite, regular measure P on the Borel subsets of C with

$$P(C \setminus \mathrm{Ext}(C)) = 0$$

and yielding the following integral representation:

$$x_0 = \int_{\mathrm{Ext}(C)} x \, dP(x).$$

This is to be interpreted as a weak integral in the following sense: For all continuous linear functional f on E we have

$$f(x_0) = \int_{\mathrm{Ext}(C)} f(x) \, dP(x).$$

And we note that the right hand side is an ordinary Lebesgue integral.

We now turn to some of the concepts from Probability Theory that can be investigated by appealing to the above notions from Functional Analysis. We shall call a function $F: R \to R$ a distribution function, (generalized distribution function) if F is right continuous, monotone non-decreasing, $F(-\infty) = 0$ and $F(+\infty) = 1$ $(F(+\infty) = C \geq 0)$. Associated to each distribution, F, there exists a unique continuous function, φ: $R \to C$, called the characteristic function of F, and given by the Lebesgue-Stieltges Integral

$$\varphi(t) = \int_R e^{ixt}\, dF(x).$$

A characteristic function φ is said to be infinitely divisible if for each integer $N \geq 1$, there is a characteristic function f_N such that $\varphi(t) = (f_N(t))^N$ for all $t \in R$. For example, if φ is any characteristic function and $a \geq 0$, then $\exp\{a(\varphi(t) - 1)\}$ is an infinitely divisible characteristic function. An important property of infinitely divisible characteristic functions is that they have no zeros, and consequently, a branch of the logarithm exists so that $\log \varphi(t)$ is a continuous, finite valued function on R. This leads us to the following theorem of Levy and Khinchine.

THEOREM (Levy-Khinchine Representation)

A characteristic function φ is infinitely divisible if and only if $\log \varphi$ admits the representation

$$\log \varphi(t) = i\gamma t + \int_R \left(e^{ixt} - 1 - \frac{ixt}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG(x)$$

where $\gamma \in R$, G is a generalized distribution function, and the integrand is defined by continuity at zero to be $\frac{-t^2}{2}$. Moreover this representation is unique.

These discussions raise two related questions. When is a continuous function $\varphi: R \to C$ a characteristic function? When is a continuous function $f: R \to C$ such that $f(t) = \log \varphi(t)$ where φ is an infinitely divisible characteristic function.

Positive Definiteness is involved in the solutions to both problems. A function $h: R \to C$ is said to be positive definite if for each $n \in N$, $\alpha_1,\ldots,\alpha_n \in C$, and $x_1,\ldots,x_n \in R$ we have

$$\sum_{j=1}^{n} \sum_{k=1}^{n} h(x_j - x_k)\, \alpha_j\, \bar{\alpha}_k \geq 0.$$

THEOREM (Bochner)

A continuous function $\varphi: R \to C$ is a characteristic function if and only if φ is positive definite.

One consequence of the Bochner Theorem is that it provides the necessary tools to answer the second question.

THEOREM (Johansen)

A continuous function $f: R \to C$ satisfying $f(t) = \overline{f(-t)}$ is the logarithmn of an infinitely divisible characteristic function φ if and only if f satisfies the following condition.

(*) For any $n \in N$, $x_1,\ldots,x_n \in R$, and $\alpha_1,\ldots,\alpha_n \in C$ with $\sum_{1}^{n} \alpha_j = 0$, we have $\sum_{j=1}^{n} \sum_{k=1}^{n} f(x_j - x_k)\, \alpha_j\, \bar{\alpha}_k \geq 0.$

We note that the collection of positive definite functions or those functions satisfying condition (*) is convex. With little effort, one can establish the Bochner Theorem by first establishing the degenerate characteristic functions i.e. $\varphi(t) = \exp\{i\alpha t\}$, $\alpha \in R$, as extreme points and then appealing to Choquet's Theorem. S. Johansen attempted the more difficult problem of proving the Levy-Khinchine Theorem using these techniques. He investigated the following set:

$$K = \{f: f \text{ satisfies condition } (*),\ f(0) \leq 0, \text{ and } \int_0^1 f(u)du = -1\}$$

He first defined a suitable topology so that K became a compact, convex set. Then he determined the extreme points of K to be:

(i) $f(t) \equiv -1$

(ii) $f_0(t) = -3t^2$

(iii) $f_\beta(t) = (e^{i\beta t} - 1 - i\beta t \dfrac{2(1 - \cos\beta)}{\beta^2})\ (1 - \dfrac{\sin\beta}{\beta})^{-1}$

And by utilizing Choquet's Theorem, Johansen had derived the very useful Levy-Khinchine representation.

By reviewing the above probabilistic concepts, it is easy to see that their extension to R^n is immediate. The defining equation for a characteristic function on R^n is given by:

$$\varphi(t) = \int_{R^n} e^{i<x,t>} dF(x)$$

where F is a n-dimensional distribution function, and for vectors $x = (x_1,\ldots,x_n)$ and $t = (t_1,\ldots,t_n)$ belonging to R^n, $<x,t>$ is defined to be $<x,t> = \sum_1^n x_j t_j$. We also set $||x|| = <x,x>^{1/2}$.

The notions of infinite divisibility and positive definiteness are exactly the same as the one-dimensional case. The purpose of the next chapters is to extend Johansen's result to R^n.

Before so doing, we first present solutions to two functional equations. The first is well-known as it deals with the normal distribution and a proof can be found in Parthasarathy. The second generalizes the cosine function on R^n and appears to be original. We also remark that Proposition 2 can be extended to a locally compact abelian group.

PROPOSITION 1. Let $\theta\colon\ R^n \to R$ be continuous, non-negative and satisfy

$$2\,\theta(x) + 2\,\theta(y) = \theta(x+y) + \theta(x-y) \quad \text{for all} \quad x,y \in R^n.$$

Then there exists an $n \times n$ non-negative semidefinite matrix P such that

$$\theta(x) = \langle Px,x \rangle \quad \text{for all} \quad x \in R^n.$$

PROPOSITION 2. Let $A\colon\ R^n \to R$ be a continuous function satisfying:

(i) $A(0) = 1$

(ii) A is bounded

(iii) $2\,A(x)\,A(y) = A(x+y) + A(x-y)$

then there exists a vector $\beta \in R^n$ such that

$$A(x) = \cos \langle \beta,x \rangle \quad \text{for all} \quad x \in R^n.$$

PROOF: We first iterate (iii) for an arbitrary vector $x_0 \in R^n$.

$$1 + A(2\,x_0) = 2\,A^2(x_0)$$

$$1 + A(4\,x_0) = 2\,A^2(2\,x_0) = 2(2\,A^2(x_0) - 1)^2$$

$$1 + A(2^{k+1}\,x_0) = 2\,A^2(2^k\,x_0) = 2(\underbrace{2(\cdots 2(2\,A^2(x_0) - 1)^2 - 1)^2 \cdots}_{k} - 1)^2$$

So we see at once if $|A(x_0)| > 1$, then A would be unbounded. Appealing to the same argument, if $A \not\equiv 1$ and using the connectedness of R^n, we conclude that A is onto $[-1,1]$. Assume the latter.

Let $y_0 \in R^n$ be such that $A(y_0) = 0$. From (iii) we see that $A(2\,y_0) = -1$ and for any, $x \in R^n$,

$$A(x + 2\,y_0) = -A(x).$$

Define $f\colon\ R^n \to R$ by

$$f(x) = A(y_0 - x) = A(x - y_0).$$

then $2\,A(x)\,A(y) + 2\,f(x)\,f(y)$

$$\begin{aligned}
&= 2\,A(x)\,A(y) + 2\,A(y_0 - x)\,A(y_0 - y) \\
&= 2\,A(x)\,A(y) + A(2\,y_0 - x - y) + A(-x + y) \\
&= A(x + y) + A(x - y) - A(x + y) + A(x - y) \\
&= 2\,A(x - y).
\end{aligned}$$

We now show A is positive definite. Let $N \in \mathbb{N}$, $x_1,\ldots,x_N \in R^n$, $\alpha_1,\ldots,\alpha_N \in C$, then

$$\sum_{j=1}^{N} \sum_{k=1}^{N} A(x_j - x_k)\, \alpha_j\, \bar{\alpha}_k = \Sigma\, \Sigma\; A(x_j)\, A(x_k)\, \alpha_j\, \bar{\alpha}_k$$

$$+ \Sigma\, \Sigma\; f(x_j)\, f(x_k)\, \alpha_j\, \bar{\alpha}_k = |\Sigma\, A(x_j)\, \alpha_j|^2 + |\Sigma\, f(x_j)\, \alpha_j|^2 \geq 0.$$

Thus by the n-dimensional Bochner Theorem, there exists a symmetric (since A is real) probability measure P on R^n such that

$$A(t) = \int_{R^n} \cos < t,x > dP(x) \quad \text{for all} \quad t \in R^n.$$

Let $t \in R^n$ be arbitrary. Then

$$0 = A(2t) + 1 - 2\, A^2(t) = \int_{R^n} \cos < 2t,x > + 1\, dP(x) - 2(\int_{R^n} \cos < t,x > dP(x)^2$$

$$= 2[\int_{R^n} \frac{\cos < 2t,x > + 1}{2}\, dP(x) - (\int_{R^n} \cos < t,x > dP(x)^2]$$

$$= 2[\int_{R^n} \cos^2 < t,x > dP(x) - (\int_{R^n} \cos < t,x > dP(x))^2]$$

$= 2\,\mathrm{Var}(\cos < t,X >)$ where X is a random vector with induced measure P.

Hence for all $t \in R^n$, $\cos < t,x >$ is almost surely constant [P].

Let $E_t = \{x \in R^n: \cos < t,x > \text{ is constant}\}$ be such that $P(E_t) = 1$. Now set $E = \bigcap_{t \in Q^n} E_t$, where Q^n denote the countable dense subset of R^n consisting of all vectors in R^n with rational components. Using the continuity of P, we have $P(E) = 1$. Thus E is non-empty and so let $x_0 \in E$. Then $-x_0 \in E$ and suppose $y \in E$. Then

$$\cos < t,y > = \cos < t,x_0 > \quad \text{for all} \quad t \in Q^n.$$

and consequently $\cos < t,y > = \cos < t,x_0 >$ for all $t \in R^n$. Thus $y = x_0$ or $-x_0$. Hence $E = \{x_0,-x_0\}$. Since P is symmetric $P\{x_0\} = 1/2 = P\{-x_0\}$ and finally

$$A(t) = \int_{R^n} \cos < t,x > dP(x) = \frac{1}{2} (\cos < t,x_0 > + \cos < t,-x_0 >)$$

$$= \cos < t,x_0 > \quad \text{as asserted.}$$

CHAPTER II

THE n-DIMENSIONAL LEVY-KHINCHINE REPRESENTATION

In this chapter we present a proof of the Levy-Khinchine for real n-dimensional infinitely divisible characteristic functions. The approach here is similiar to the one-dimensional case established by Johansen, [4], in 1966. In his paper, Johansen gave a necessary and sufficient condition for a continuous function to be the logarithmn of an infinitely divisible characteristic function. His result was easily extended to R^n and was written out in detail by Prakasa Rao [8]. It reads as follows:

PROPOSITION 1. Let φ: $R^n \to C$ be continuous and hermitian ($\varphi(t) = \overline{\varphi(-t)}$ for all $t \in R^n$). Then φ is the logarithmn of an infinitely divisible characteristic function if and only if φ satisfies the following condition (*):

(*) For any choice of $N \in N$, $x_1,\ldots,x_N \in R^n$ and $\alpha_1,\ldots,\alpha_N \in C$ with $\sum_1^N \alpha_j = 0$, we must have

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \varphi(x_j - x_k)\, \alpha_j \bar{\alpha}_k \geq 0.$$

Let Q be the collection of all real continuous functions, φ, on R^n, satisfying $\varphi(t) = \varphi(-t)$ and condition (*). And set $Q_0 = \{\varphi \in Q\colon \varphi(0) = 0\}$. Various properties of functions belonging to Q are given in the appendix. However, we state the frequently used ones below.

PROPOSITION 2. The following assertions hold:

(i) if $\varphi \in Q_0$, $-\frac{1}{2} \leq \varepsilon \leq \frac{1}{2}$ and $\alpha \in R^n$, then the function $\psi(u) = \varphi(u) + \varepsilon\,(\varphi(u + \alpha) + \varphi(u - \alpha) - 2\varphi(\alpha))$ also belongs to Q_0.

(ii) if $\varphi \in Q_0$ and $n \in N$, then for all $u \in R^n$, we have $n^2\,\varphi(u) \leq \varphi(nu) \leq 0$.

(iii) $\varphi \in Q$ if and only if for all $\alpha \in R^n$, the function $f(u) = \varphi(u) - \frac{1}{2}\,(\varphi(u + \alpha) + \varphi(u - \alpha)$ is positive definite.

We now make a detailed investigation of the following subset of Q. Let

$$K = \{\varphi \in Q:\ \varphi(0) \leq 0 \text{ and } \int_E \varphi(u)du = -2^{n-1}\}$$

where $E = \{x = (x_1,\ldots,x_n) \in R^n:\ -1 < x_j \leq 1,\ 1 \leq j \leq n - 1,$

$$\text{and} \qquad 0 \leq x_n \leq 1\}$$

Following Johansen, a function $f \in Q$ is called degenerate if $f(u) \equiv f(0)$, and two functions $f_1, f_2 \in Q$ are said to be of the same type if there exists an $a \geq 0$, $b \in R$ such that $f_2(u) = a\,f_1(u) + b$ for all $u \in R^n$. The sets Q and K are connected in the following manner: If $f \in Q$, there exists a function $g \in K$ of the same type as f. In the sequel, we shall establish the compactness and extreme points of K, and with the aid of Choquet's Theorem, conclude our main theorem.

THEOREM 1 Let $f\colon R^n \to R$ be continuous and satisfy condition (*). Then there exists a real number $a \in R$, a negative semi-definite matrix A, and a symmetric measure μ defined on the Borel subsets of R^n with $\mu\{0\} = 0$ such that

$$f(t) = a + \langle At,t \rangle + \int_{R^n} (\cos \langle \beta,t \rangle - 1) \frac{1+||\beta||^2}{||\beta||^2} d\mu(\beta)$$

Moreover this representation is unique, and takes the usual form of the Levy-Khinchine Representation by assuming $f(0) = 0$.

We will only be concerned with the existence of such a representation, as the uniqueness is given in Takano [10]. As to be expected, the proof requires a number of steps. Our initial task is determination of Ext(K), the set of extreme points of K. For the remainder of the discussion, we will let $\mathfrak{N}$ be those functions in K corresponding to normal distributions. i.e.

$$\mathfrak{N} = \{\varphi \in K: \ 2\varphi(u) + 2\varphi(\alpha) = \varphi(u + \alpha) + \varphi(u - \alpha) \text{ for all } u, \alpha \in R^n\}.$$

for all $u, \alpha \in R^n\}$.

PROPOSITION 3. Ext(K) consists of the following functions:

(i) $f(u) \equiv -1$.

(ii) $f(u) = c_\beta (\cos \langle \beta,u \rangle - 1)$, $\beta \in R^n \setminus \{0\}$, and c_β is the appropriate constant.

(iii) Ext($\mathfrak{N}$).

PROOF: Let $f \in K$. By integrating the inequality $f(u) \leq f(0)$ over E we obtain $-1 \leq f(0) \leq 0$. Suppose $f \in \mathrm{Ext}(K)$ and let $\lambda = -f(0)$. Then $\lambda = 0$ or 1, for otherwise, we could express f as

$$f(u) = (1 - \lambda) \left(\frac{f(u) - f(0)}{1-\lambda}\right) + \lambda(-1).$$

If $f(0) = -1$, we must have $f(u) = -1$ for all $u \in E$. But the evenness of f forces f to be constant on a neighborhood of the origin, and hence f is necessarily constant. Thus $f(u) \equiv -1$.

Now assume $f(0) = 0$. Let $\alpha \in R^n$ and $0 \leq \epsilon \leq \frac{1}{2}$. Then

$$\psi^{\pm}_{\alpha,\epsilon}(u) = f(u) \pm \epsilon\,(f(u + \alpha) + f(u - \alpha) - 2f(\alpha))$$

belongs to Q_0 and

$$\lim_{\epsilon \to 0} \int_E \psi^{\pm}_{\alpha,\epsilon}(t)d(t) = \int_E f(t)dt = -2^{n-1}$$

Fix on $\epsilon > 0$ such that $\psi^{\pm}_{\alpha,\epsilon} \not\equiv 0$, and choose positive numbers $a_1(\alpha), a_2(\alpha)$ and functions $\varphi_1, \varphi_2 \in K$ such that

$$\psi^{+}_{\alpha,\epsilon}(u) = a_1(\alpha)\,\varphi_1(u)$$

$$\psi^{-}_{\alpha,\epsilon}(u) = a_2(\alpha)\,\varphi_2(u)$$

Now
$$f = \frac{1}{2}\,(\psi^{+}_{\alpha,\epsilon} + \psi^{-}_{\alpha,\epsilon})$$

$$= \frac{1}{2} (a_1(\alpha)\, \varphi_1(u) + a_2(\alpha)\, \varphi_2(u))$$

and by integrating over E, we conclude that $a_1(\alpha) + a_2(\alpha) = 1$. So by assuming f to be an extreme point, it follows that f satisfies the following identity:

$$f(u) = \frac{1}{a_1(\alpha)} f(u) + \varepsilon(f(u + \alpha) + f(u - \alpha) - 2\, f(\alpha)$$

Thus for an appropriately defined $A(\alpha)$ we have for all $u, \alpha \in R^n$,

$$A(\alpha)\, f(u) = f(u + \alpha) + f(u - \alpha) - 2\, f(\alpha) \tag{1}$$

By interchanging u and α and subtracting we arrive at

$$A(u)\, f(\alpha) - 2\, f(\alpha) = A(\alpha)\, f(u) - 2\, f(u) \tag{2}$$

Let us first determine the function A.

Case 1: $A(u) \equiv 2$, then f satisfies the functional equation $2\, f(u) + 2\, f(\alpha) = f(u + \alpha) + f(u - \alpha)$ so that $f \in \mathfrak{N}$. But if $f \in \mathrm{Ext}(K)$, it must be the case that $f \in \mathrm{Ext}(\mathfrak{N})$.

Case 2: $A(\alpha_0) \neq 2$. Then $f(u) = c(A(u) - 2)$ (3)

where $c = f(\alpha_0)/_{A(\alpha_0) - 2}$. Now $c \neq 0$ and by substituting (3) into (1) we get

$$c\,A(\alpha)\,(A(u) - 2) = c(A(u + \alpha) - 2) + c(A(u - \alpha) - 2)$$

$$- 2c(A(\alpha) - 2)$$

or

$$A(\alpha)\,A(u) = A(u + \alpha) + A(u - \alpha) \tag{4}$$

and $A(0) = 2$.

For any $f \in Q_0$ we have $|f(u + \alpha) + f(u - \alpha) - 2f(\alpha)| \leq - 2f(u)$ and consequently $|A(u)| \leq 2$. Thus the only solution to (4) is given by

$$A(u) = 2 \cos < \beta, u > \qquad \text{for some} \qquad \beta \neq 0.$$

In terms of f, we have

$$f(u) = 2c(\cos < \beta, u > - 1)$$

and so $c_\beta = 2c = (\int_E 1 - \cos < \beta, u > du)^{-1}\, 2^{n-1}$, thus

$$f(u) = c_\beta\,(\cos < \beta, u > - 1).$$

It remains to prove that the above functions are in fact extreme points. We first show that the function $f(u) \equiv -1$ is an extreme point of K.

Let $f_1, f_2 \in K$ and $0 \leq \lambda \leq 1$ be such that

$-1 = \lambda f_1(u) = (1 - \lambda)\, f_2(u)$. It then follows that $f_i(0) = -1$ and so by arguing as above $f_i \equiv -1$ and thus $f(u) \equiv -1$ is an extreme point of K.

We next observe that $\mathfrak{N}$ is an extremal subset of K and hence if $f \in \mathrm{Ext}(\mathfrak{N})$ then $f \in \mathrm{Ext}(K)$.

We finally establish the extremal nature of $f_\beta(u) = c_\beta(\cos < \beta, u > - 1)$ where $\beta \in R^n \setminus \{0\}$. Let $f_1, f_2 \in K$ and $0 < \lambda < 1$ and suppose that

$$f_\beta = \lambda f_1 + (1 - \lambda)\, f_2.$$

Let $\alpha \in R^n$, for $i = 1,2$ we define $f_{i,\alpha}$ as

$$f_{i,\alpha}(u) = f_i(u) - \frac{1}{2}\,(f_i(u + \alpha) + f_i(u - \alpha)).$$

Then $f_{i,\alpha}$ is positive definite and

$$\lambda f_{1,\alpha}(u) + (1 - \lambda)\, f_{2,\alpha}(u) = c_{\beta,\alpha} \cos < \beta, u >.$$

Thus each $f_{i,\alpha}$ must be of the form

$$f_{i,\alpha}(u) = A_i(\alpha) \cos < \beta, u >. \qquad \text{(See Appendix page }\quad)$$

Thus each f_i satisfies for all $u, \alpha \in R^n$

$$f_i(u) - \frac{1}{2}\,(f_i(u + \alpha) + f_i(u - \alpha)) = A_i(\alpha) \cos < \beta, u >$$

which implies, upon letting $u = 0$, that

$$f_i(\alpha) = -A_i(\alpha)$$

and so

$$f_i(u) - \frac{1}{2}\,(f_i(u+\alpha) + f_i(u-\alpha) - 2f_i(\alpha)\cos < \beta,u >) = 0 \tag{5}$$

Or by interchanging u and α

$$f_i(\alpha) - \frac{1}{2}\,(f_i(u+\alpha) + f_i(u-\alpha) - 2f_i(u)\cos < \beta,\alpha >) = 0 \tag{6}$$

And substracting yields

$$f_i(u) - f_i(\alpha) + f_i(\alpha)\cos < \beta,u > - f_i(u)\cos < \beta,\alpha > = 0$$

or $f_i(u)\,(1 - \cos < \beta,\alpha >) = f_i(\alpha)\,(1 - \cos < \beta,u >)$ and hence $f_i(u) = c_i\,(1 - \cos < \beta,u >)$. But since $f_i \in K$, we must have $c_i = -c_\beta$ and f_β is an extreme point as asserted. This completes the proof of Proposition 3.

An enumeration of functions belonging to $\mathrm{Ext}(\mathfrak{N})$ is not necessarily for this development, however this list of functions is given in the appendix.

We still have to prove that K is compact in a suitably chosen topology. We first show that K a uniformly bounded and then appeal to the compactness of the unit ball in $L^\infty(R^n)$.

LEMMA 1. The set K is uniformly bounded by a constant multiple of the function $G(t) = 1 + ||t||^2$.

PROOF: Let $f \in K$ and suppose $f(0) = 0$. Let $\alpha = (\alpha_1,\ldots,\alpha_n) \in R^n$ and define $\phi \in Q_0$ as follows:

$$\phi(u) = f(u) + \frac{1}{2}(f(u+\alpha) + f(u-\alpha) - 2f(\alpha)).$$

Since $\phi(u) \le \phi(0) = 0$, we obtain

$$f(\alpha) \ge f(u) + \frac{1}{2}(f(u+\alpha) + f(u-\alpha)).$$

Now integrate this inequality over I^n, the n-dimensional unit square, to conclude

$$f(\alpha) \ge \int_{I^n} f(u)du + \frac{1}{2}\int_{I^n} f(u+\alpha) + f(u-\alpha)du$$

$$\ge \int_E f(u)du + \frac{1}{2}\int_{-\alpha+I_n} f(u)du + \frac{1}{2}\int_{\alpha+I_n} f(u)du$$

Letting $2t = u$ in the latter integrals, we have

$$f(\alpha) \ge -2^{n-1} + 2^{n-1}\int_{\frac{-\alpha+I_n}{2}} f(2t)dt + 2^{n-1}\int_{\frac{\alpha+I_n}{2}} f(2t)dt.$$

Since $f(2t) \ge 2^2 f(t)$ we have

$$f(\alpha) \geq -2^{n-1} + 2^{n+1} \int_{\frac{-\alpha+I_n}{2}} f(t)dt + 2^{n+1} \int_{\frac{\alpha+I_n}{2}} f(t)dt.$$

Now assume $||\alpha|| \leq 1$, then $-1 \leq \alpha_j \leq 1$ for $1 \leq j \leq n$, where $\alpha = (\alpha_1,\ldots,\alpha_n)$. Then $\frac{\pm \alpha + I_n}{2} \subseteq \{x = (x_1,\ldots,x_n)\colon -1 \leq x_j \leq 1$ for $1 \leq j \leq n\} = E \cup (-E)$. Hence

$$f(\alpha) \geq -2^{n-1} + 2^{n+1} \int_{E \cup (-E)} f(t)dt + 2^{n+1} \int_{E \cup (-E)} f(t)dt$$

$$= -2^{n-1} + 2^{n+3} (-2^{n-1}) = -M_1.$$

Thus for all $f \in K$ with $f(0) = 0$ and for all α, $||\alpha|| \leq 1$, we have $f(\alpha) \geq -M_1$. Now let $\alpha \in R^n \setminus \{0\}$ be arbitrary. Then letting [ ] denote the largest integer less than or equal to, we have

$$0 \geq f(\alpha) = f\left(\frac{(1 + [||\alpha||])\alpha}{1 + [||\alpha||]}\right) \geq (1 + [||\alpha||]^2 \quad f\left(\frac{\alpha}{1 + [||\alpha||]}\right)$$

$$\geq - M(1 + ||\alpha||^2).$$

So if $f \in K$ with $f(0) = 0$ and $\alpha \in R^n$ we have

$$|f(\alpha)| \leq M(1 + ||\alpha||^2).$$

For arbitrary $f \in K$ we have $f(0) \geq -1$. So by repeating the above argument we get $|f(\alpha) - f(0)| \leq M(1 + ||\alpha||^2)$ and the desired conclusion is an immediate consequence of this last inequality.

Using the usual limiting techniques, a continuous even function $\varphi \in Q$ is and only if φ satisfy the condition (**).

(**) if $h \in C_c(R^n)$ is any continuous function with compact support and $\int_{R^n} h(u)du = 0$ then $\int_{R^n}\int_{R^n} \varphi(u - v)\, h(u)\, \overline{h(v)}du\, dv \geq 0$.

We now define $\widetilde{Q}$ to be the set of all measurable functions, φ, essentially bounded on every bounded neighborhood of the origin and satisfying $\varphi(t) = \varphi(-t)$ a.e. and condition (**). We want to identify Q with $\widetilde{Q}$ and in order to so do we must show that each equivalence class of $\widetilde{Q}$ contains one and only one function from Q.

PROPOSITION 4. For any $\varphi \in \widetilde{Q}$, there exists a function $\varphi_1 \in Q$ such that $\varphi = \varphi_1$ almost everywhere with respect to Lebesgue measure on R^n.

PROOF: See Appendix.

Utilizing this fact, we can identify K with the corresponding set of equivalence classes

$$\widetilde{K} = \{\varphi \in \widetilde{Q}: \int_E \varphi(u)du = -2^{n-1}$$

$$\text{and } \varphi(u) \leq 0 \text{ a.e.}\}.$$

We also know that the functions in K are bounded by $G(t) = M(1 + ||t||^2)$. Thus the mapping, T, defined on K by

$T(\varphi) = \varphi/G$ is an injective mapping into $B_\infty = \{f \in L^\infty(R^n): \ ||f||_\infty \leq 1\}$.

By appealing to the weak* compactness of B_∞, $T(K)$ will compact and K will be compact in the topology induced by T if and only if $T(K)$ or equivalently $T(\widetilde{K})$ is weak* closed.

PROPOSITION 5. $T(\widetilde{K})$ is a weak* closed subset of B_∞.

PROOF: Let $f_j \in \widetilde{Q}$ for $j = 1,2,\ldots$ and suppose $\frac{f_j}{G} \xrightarrow{w^*} f = \frac{fG}{G}$. We must show $fG \in \widetilde{K}$. Since $f_j \leq 0$ a.e., it follows that $fG \leq 0$ a.e. Now $\frac{f_j}{G} \xrightarrow{w^*} \frac{fG}{G}$ entails for all $h \in L'(R)$. $\lim_{j \to \infty} \int_{R^n} \frac{f_j}{G} h = \int_R \frac{fG}{G} h$.

In particular, with $h = G\chi_E$ we have

$$\lim_{j \to \infty} \int_{R^n} \frac{f_j}{G} h = \lim_{j \to \infty} \int_E f_j = -2^{n-1}$$

$$= \int_{R^n} \frac{fG}{G} h = \int_E fG$$

$$\text{therefore } \int_E fG = -2^{n-1}.$$

Also for any $g \in C_c(R^n)$ we have

$\lim_{j \to \infty} \int_{R^n} f_j(t)\, g(t)dt = \int_{R^n} f(t)\, G(t)\, g(t)dt$. And so let $g \in C_c(R^n)$ with $\int_{R^n} g(u)du = 0$. Let $h = g * g'$ where $g'(x) = \overline{g(-x)}$, then $h \in C_c(R^n)$ and

$$0 \leq \int_{R^n} \int_{R^n} f_j(x - y)\, g(x)\, \overline{g(y)} dy$$

$$= \int_{R^n} f_j(x)\, h(x) dx \rightarrow \int_{R^n} f(x)\, G(x)\, h(x) dx$$

$$= \int_{R^n} \int_{R^n} (fG)\,(x - y)\, g(x)\, \overline{g(y)} dx\, dy$$

thus $fG \in \widetilde{K}$ and hence the desired conclusion.

We now turn to the proof of Theorem 1.

Let $f \in K$. By Choquet's Theorem there exists a non-negative regular Borel measure P on $T(K)$ such that P is concentrated on the extreme points of $T(K)$ and $f/G = \int_{T(K)} \psi\, dP(\psi) = \int_{Ext(T(K))} \psi\, dP(\psi)$.

Letting $B = Ext(T(K))$, this integral is to be interpreted in the following sense: *For every weak* continuous linear functional* ℓ we have

$$\ell(f/G) = \int_B \ell(\psi)\, dP(\psi)$$

And consequently for each $h \in L'(R^n)$

$$\int_{R^n} h(t)\, \frac{f(t)}{G(t)}\, dt = \int_B \left(\int_R h(t)\, \psi(t) dt\right) dP(\psi).$$

Before utilizing the Fubini Theorem, we first verify that the function L_h: $R^n \times B \to R$ defined by $L_h(t,\psi) = h(t)\,\psi(t)$ is integrable for $dt \times dP$.

The measurability of L_h is given in the appendix and since $h \in L'(R^n)$ and P is a finite measure and $|L_h(t,\psi)| \leq |h(t)|$ we conclude that L_h is summable for the product measure and hence the Fubini Theorem applies.

Let $h \in L'(R^n)$ and express $\psi \in B$ as $\psi = \varphi/G$ where $\varphi \in \mathrm{Ext}(K)$. Then

$$\int_{R^n} h(t)\,\frac{f(t)}{G(t)}\,dt = \int_B \left(\int_{R^n} h(t)\,\frac{\varphi(t)}{G(t)}\,dt\right) dP(\varphi)$$

$$= \int_{R^n} \frac{h(t)}{G(t)} \left(\int_B \varphi(t)\,dP(\varphi)\right)dt.$$

And this in turn implies

$$f(t) = \int_B \varphi(t)\,dP(\varphi) \qquad \text{a.e.} \tag{8}$$

We now verify the right hand side of (8) is continuous in t.

Suppose $t_m \to t_0$, then for every $\varphi \in \mathrm{Ext}(K)$, $\varphi(t_m) \to \varphi(t_0)$. Let $n \in \{0,1,2,\ldots\}$ and define U_n: $B \to R^n$ by

$$U_n(\varphi/G) = \varphi(t_n)$$

$$U(\varphi/G) = \varphi(t_0)$$

Now Ext(K) are uniformly bounded in a neighborhood of t_0 and hence

$$|U_m(\varphi/G)) \leq M \qquad \text{for } m = 0,1,2,\ldots$$

and $U_m \to U_0$ as $m \to \infty$. So by the Dominated Convergence Theorem,

$$\lim_{m \to \infty} \int_B \varphi(t_m)\, dP(\varphi) = \lim_{m \to \infty} \int_B U_m(\varphi/G)\, dP(\varphi)$$

$$= \int_B U_0(\varphi/G)\, dP(\varphi) = \int_B \varphi(t_0)\, dP(\varphi).$$

Since the left hand side of (8) is surely continuous in t, we have

$$f(t) = \int_B \varphi(t)\, dP(\varphi) \qquad \text{everywhere.}$$

We now analyze $\int_B \varphi(t)\, dP(\varphi)$.

We will use the following notation.

$$S = \{(x_1,\ldots,x_n) \in R^n: \ x_1 > 0, \ \ x_j \in R \quad 2 \leq j \leq n\}$$

$$\cup\{(x_1,\ldots,x_n) \in R^n: \ x_1 = 0, \ \ x_2 > 0, \ \ x_j \in R, \quad 3 \leq j \leq n\}$$

$$\cup\{(x_1,\ldots,x_n) \in R^n: \ x_j = 0 \quad 1 \leq j \leq n-1, \ \ x_n > 0\}$$

$$B_1 = \{f/G \in B: \ f(t) = C_\beta(\cos <\beta,t> - 1), \ \ \beta \in R^n\backslash\{0\}$$

$$= \{f/G \in B: \ f(t) = C_\beta(\cos < \beta,t > - 1), \ \ \beta \in S\}$$

$$B_2 = \{f/G \in B: \ f \in \mathrm{Ext}(\mathfrak{N})\}.$$

Then B_2 is a measurable subset of B and when identifying $x \in R^n$ with $-x$, one concludes that B_1 is homeomorphic with S, and thus

$$\int_B \varphi(t)\, dP(\varphi) = \int_{B_1} \varphi(t)\, dP(\varphi) + \int_{B_2} \varphi(t)\, dP(\varphi) + \int_{\{-\frac{1}{G}\}} \varphi(t)\, dP(\varphi)$$

$$= \int_S C_\beta(\cos < \beta,t > - 1)\, dP(\varphi) - P\{-\tfrac{1}{G}\} + \int_{B_2} \varphi(t)\, dP(\varphi).$$

Now the function $g(t) = \int_{B_2} \varphi(t)\, dP(\varphi)$ belongs to $\mathfrak{N}$ and consequently may be expressed as $g(t) = < At,t >$ where A is a negative semi-definite matrix. Combining all this, we now rewrite (8) as

$$f(t) = -P\{-\tfrac{1}{G}\} + < At,t > + \int_S C_\beta(\cos < \beta,t > - 1)\, dP(\beta).$$

Finally if $f \in Q$, then there exists unique numbers $a_1 \geq 0$, $b \in R$ such that $a_1 f + b \in K$, and consequently there exists a real number a, a negative semi-definite matrix A, and a symmetric finite measure P on R^n with $P\{0\} = 0$ such that

$$f(t) = a + < At,t > + \int_R (\cos < \beta,t > - 1)\ \frac{1 + ||\beta||^2}{||\beta||^2}\ dP(\beta), \text{ and}$$

this completes the proof of our main theorem.

APPENDIX TO CHAPTER II

Chapters II and III are concerned with the Levy-Khinchine representation of the logarithmn of n-dimensional infinitely divisible characteristic functions. In this section, we present the technical properties of such functions. Here we shall let $Q = \{\varphi\colon\ R^n \to C\colon\ \varphi = f + ig,\ \varphi(t) = \overline{\varphi(-t)},$ and φ satisfies condition $(*)\}$ where as before

(*) For any choice $N \in N$, $x_1,\ldots,x_N \in R^n$, and $\alpha_1,\ldots,\alpha_N \in C$ with $\sum_1^N \alpha_j = 0$, we have $\sum_{j=1}^N \sum_{k=1}^N \varphi(x_j - x_k)\ \alpha_j\ \overline{\alpha_k} \geq 0$.

We shall let $Q_0 = \{\varphi \in K\colon\ \varphi(0) = 0\}$ and retain $K = \{\varphi \in Q\colon\ \varphi$ real-valued, $\varphi(0) \leq 0$, and $\int_E \varphi(t)dt = -2^{n-1}\}$.

LEMMA 0. Let $\varphi\colon\ R^n \to C$ be any complex function with $\varphi(0) = 0$ and $\varphi(u) = \overline{\varphi(-u)}$. Let $S \subseteq R^n$ be a finite subset and h be any complex function defined on S. The following three conditions are equivalent:

(1) $\sum_{u \in S} \sum_{v \in S} \varphi(u - v)\ h(u)\ \overline{h(v)} \geq 0$ for all (S,h) with $\sum_{u \in S} h(u) = 0$

(2) $\sum_{u \in S} \sum_{v \in S} \exp\{\lambda(\varphi(u - v))\}\ h(u)\ \overline{h(v)} \geq 0$ for all (S,h) and for all $\lambda \geq 0$.

(3) $\sum_{u \in S} \sum_{v \in S} (\varphi(u - v) - \varphi(u) - \varphi(-v))\ h(u)\ \overline{h(v)} \geq 0$ for all (S,h).

PROOF: This lemma is given in Prakasa Rao [8]. See also Johansen [4].

LEMMA 1. Let $\varphi \in Q$, $x_0 \in R^n$, $a \geq 0$, $b \in R$. Then the function $\psi: R^n \to C$ defined by $\psi(t) = a\,\varphi(t) + i < x_0, t > + b$ also belongs to Q.

PROOF: We must show ψ satisfies condition (*). Let $N \in N$, $x_1, \ldots, x_N \in R^n$ and $\alpha_1, \ldots, \alpha_N$ be complex numbers with $\sum_1^N \alpha_j = 0$.

Then $\Sigma\,\Sigma\, \psi(x_j - x_k)\, \alpha_j\, \overline{\alpha_k} = a\, \Sigma\,\Sigma\, \varphi(x_j - x_k)\, \alpha_j\, \overline{\alpha_k}$

$+ i\, \Sigma\,\Sigma\, (< x_0, x_j > - < x_0, x_k >)\, \alpha_j\, \overline{\alpha_k} + b\, \Sigma\,\Sigma\, \alpha_j\, \overline{\alpha_k} = a\, \Sigma\,\Sigma\, \varphi(x_j - x_k) \alpha_j\, \overline{\alpha_k}$

since $\Sigma\, \alpha_j = 0$. Thus $\varphi \in Q$ entails $\psi \in Q$.

LEMMA 2. Let $\varphi \in Q$, $g_1, \ldots, g_M \in R^n$, $\gamma_1, \ldots, \gamma_M \in C$, then $\psi(u) = \Sigma\,\Sigma\, \varphi(u - g_j + g_k)\, \gamma_j\, \overline{\gamma_k}$ is an element of Q.

PROOF: Let $N \in N$ and $x_1, \ldots, x_N \in R^n$, $\alpha_1, \ldots, \alpha_N \in C$ be such that $\sum_1^N \alpha_j = 0$. We set

$$y_{i,k} = x_k - g_i \qquad 1 \leq k \leq N, \qquad 1 \leq i \leq M$$

$$\beta_{i,k} = \gamma_i\, \alpha_k \qquad 1 \leq k \leq N, \qquad 1 \leq i \leq M$$

Then $\sum_{i,k} \beta_{i,k} = 0$ and

$$\sum_k \sum_\ell \psi(x_k - x_\ell)\, \alpha_k\, \overline{\alpha_\ell} = \sum_i \sum_j \sum_k \sum_\ell \varphi(x_k - g_i - x_\ell + g_j)\, \alpha_i\, \overline{\alpha_j}\, \gamma_k\, \overline{\gamma_\ell}$$

$$= \sum_{i,k} \sum_{j,\ell} \varphi(y_{i,k} - y_{\ell,j})\, \beta_{i,k}\, \overline{\beta}_{j,\ell} \geq 0$$

COROLLARY: If $\varphi \in Q$, $-\frac{1}{2} \leq \varepsilon \leq \frac{1}{2}$, $\theta \in R$, and $\alpha \in R^n$. Then the function

$$\psi(u) = \varphi(u) + \varepsilon\,(e^{-i\theta}\, \varphi(u + \alpha) + e^{i\theta}\, \varphi(u - \alpha))$$

belongs to Q.

PROOF: In the above lemma let $g_1 = 0$, $g_2 = \alpha$ $\gamma_1 = 1$, $\gamma_2 = \lambda\, e^{i\theta}$ where $\lambda \in R$. Then we get $\psi(u) = \varphi(u) + \lambda\, e^{i\theta}\, \varphi(u - \alpha) + \lambda\, e^{-i\theta}$ $\varphi(u + \alpha) + \lambda^2\, \varphi(u) \in Q$ and so $\frac{\psi(u)}{1+\lambda^2} \in Q$. And the result follows from $-\frac{1}{2} \leq \frac{\lambda}{1+\lambda^2} \leq \frac{1}{2}$ for all $\lambda \in R$.

Henceforth, we shall express $\varphi = f + ig$.

LEMMA 3. If $\varphi \in Q_0$, then $f(u) \leq 0$ for all $u \in R^n$.

PROOF: In condition (*), let $x_1 = 0$, $x_2 = u$, $\alpha_1 = 1$, $\alpha_2 = -1$ and we get that $-\varphi(u) - \overline{\varphi(u)} \leq 0$ and thus $f(u) \leq 0$.

LEMMA 4. If $\varphi \in Q_0$, and $N \geq 2$, then

(i) $N^2\, f(u) \leq f(Nu) \leq 0$

(ii) $|g(Nu) - N\, g(u)| \leq -N(N - 1)\, f(u)$

Proof (i) It remains to show for all $n \geq 2$, all $u \in R^n$, $n^2\, f(u) \leq f(nu)$. Let $\alpha \in R^n$. Then $\varphi_1(u) = \varphi_1(u) - \frac{1}{2}\,(\varphi(u + \alpha)$

$+ \varphi(u - \alpha) - \varphi(\alpha) - \varphi(-\alpha)) \in Q_0$ which implies that $\mathrm{Re}\ \varphi_1(u) \leq 0$. Thus $2 f(u) - f(u + \alpha) - f(u - \alpha) + 2 f(\alpha) \leq 0$ or

$$(1) \qquad f(u) - f(u - \alpha) + 2 f(\alpha) \leq f(u + \alpha) - f(u)$$

Let $N \geq n \geq 2$, $u = j\alpha$ and sum from $j = 1$ to $n - 1$. Then

$$\sum_{j=1}^{n-1} f(j\alpha) - f((j - 1)\alpha) + 2(n - 1) f(\alpha) \leq \sum_{j=1}^{n-1} f(j + 1)\alpha) - f(j\alpha),$$

and so $f((n - 1)\alpha) + 2(n - 1) \ f(\alpha) \leq f(n\alpha) - f(\alpha)$. We now sum $n = 1$ to N. Thus

$$\sum_{n=1}^{N} f((n - 1)\alpha) + 2(n - 1) \ f(\alpha) \leq \sum_{n=1}^{N} f(n\alpha) - f(\alpha)$$

or
$$f(\alpha) \sum_{n=1}^{N} (2 n - 1) \leq \sum_{n=1}^{N} f(n\alpha) - f((n - 1)\alpha)$$

which implies that $N^2 f(\alpha) \leq f(N\alpha)$.

(ii) Here we use the fact that if $\alpha \in R^n$, the function $\varphi_2(u) = \varphi(u) - \frac{i}{2} (\varphi(u + \alpha) - \varphi(u - \alpha) - \varphi(\alpha) + \varphi(-\alpha))$ belongs to Q_0. Consequently

$$(2) \quad \mathrm{Re}\ \varphi_2(u) = f(u) + \frac{1}{2} (g(u + \alpha) - g(u - \alpha) - 2 g(\alpha)) \leq 0.$$

or $g(u + \alpha) - g(\alpha) \leq g(\alpha) - g(\alpha - u) - 2 f(u)$. Again suppose $n \geq 2$, $\alpha = ju$, and sum from $j = 1$ to $n - 1$. $\sum_{j=1}^{n-1} g((j + 1)u) - g(ju)$ $\leq \sum_{j=1}^{n-1} g(ju) - g((j - 1)u) - 2 f(u)$ which implies that

$g(nu) - g(u) \leq g((n - 1)u) - 2(n - 1) f(u)$. Now summing $n = 1$ to N, we have that

$$(3) \qquad g(Nu) - N\, g(u) \leq -N(N - 1)\, f(u).$$

By repeating the above argument applied to the inequality

$$(4) \qquad 2\, f(u) - (g(u + \alpha) - g(u - \alpha) - 2\, g(\alpha)) \leq 0$$

we see that $N(N - 1)\, f(u) \leq -N\, g(u) + g(Nu)$ and so $|g(Nu) - N\, g(u)| \leq -N(N - 1)\, f(u)$.

COROLLARY. If $\varphi \in Q_0$ and $f(u) \equiv 0$ on a neighborhood of the origin, then $f(u) \equiv 0$.

LEMMA 5. For all $\varphi \in Q_0$, we have the inequality $|\varphi(u - v) - \varphi(u) - \varphi(-v)|^2 \leq 4\, f(u)\, f(v)$ holding for all $u, v \in R^n$.

PROOF: In condition (3) of Lemma 0, let $S = \{u,v\}$ and α and β be arbitrary complex numbers. Then $(-\varphi(u) - \varphi(-u))|\alpha|^2 + (-\varphi(v) - \varphi(-v))|\beta|^2$

$+ (\varphi(u - v) - \varphi(u) - \varphi(-v))\, \alpha\bar{\beta} + (\varphi(v - u) - \varphi(v) - \varphi(-u))\, \beta\bar{\alpha} \geq 0$.

This implies the matrix

$$\begin{pmatrix} -2\, f(u) & \varphi(u - v) - \varphi(u) - \varphi(-v) \\ \varphi(v - u) - \varphi(v) - \varphi(-u) & -2\, f(v) \end{pmatrix}$$

is hermitian and positive definite. Hence the determinant is ≥ 0 and the result follows.

LEMMA 6. Suppose $\varphi \in Q_0$, and for all $u \in R^n$ and $r \in R$, $f(ru) = r^2 f(u)$. Then $g(ru) = rg(u)$.

PROOF: Let $n_1, n_2 \in N$ and $u \in R^n$. From Lemma 5, we see that $(f(n_1 + n_2)u) - f(n_1u) - f(n_2u))^2 + (g((n_1 + n_2)u) - g(n_1u) - g(n_2u))^2$ $\leq 4\ f(n_1u)\ f(n_2u) = 4\ n_1^2\ n_2^2\ f(u)$ and thus $g((n_1+n_2)u) + g(n_1u)\_g(n_2u)$. So $g(2u) = 2\ g(u)$ and the result follows by induction, and the usual method of considering the rationals then the reals.

THEOREM 1. Normal Characterization.

Suppose $\varphi \in Q_0$ and $f(u + v) + f(u - v) = 2\ f(u) + 2\ f(v)$ for all $u, v \in R^n$. Then $g(u + v) = g(u) + g(v)$ for all $u, vv \in R^n$.

PROOF: Since $\varphi \in Q_0$, the function $\psi(u) = \varphi(u) - \frac{1}{2}(\varphi(u + \alpha) - \varphi(u - \alpha)$ $- 2\ f(\alpha)) \in Q_0$. But $\mathrm{Re}\ \psi(u) \equiv 0$ and thus by Lemma 5, $\mathrm{Im}\ \psi$ is linear. Let $q = \mathrm{Im}\ \psi$. Then $q(u) = g(u) - \frac{1}{2}(g(u + \alpha) + g(u - \alpha))$. Appealing to Lemma 6, we have

$$(1) \qquad n\ q(u) = q(nu) = g(nu) - \frac{1}{2}(g(nu + \alpha) + g(nu - \alpha))$$

$$= n\ g(u) - \frac{1}{2}(g(nu + \alpha) + g(nu - \alpha))$$

and consequently $n\ g(u + \alpha) + n\ g(u - \alpha) = g(nu + \alpha) + g(nu - \alpha)$. Now the homogeneity of g yields $g(u + \alpha) + g(u - \alpha) = g(u + \frac{\alpha}{n}) + g(u - \frac{\alpha}{n})$ for all $n \in N$. Thus by letting n approach infinity, we have for all

$\alpha \in R^n$, $2\,g(u) = g(u + \alpha) + g(u - \alpha)$. Interchanging u and α and using the oddness of g we have that $g(u + \alpha) = g(u) + g(\alpha)$ which completes the proof.

We shall let $C_c(R^n) = \{h\colon\ R^n \to C\colon\ h$ is a continuous complex function with compact support$\}$.

LEMMA 7. Suppose φ is a continuous complex function on R^n and $\varphi(t) = \overline{\varphi(-t)}$ for all $t \in R^n$. Then $\varphi \in Q \Leftrightarrow \varphi$ satisfies condition (**) where

(**) for all $h \in C_c(R^n)$ with $\int_{R^n} h(u)du = 0$ we have

$$\int_{R^n} \int_{R^n} \varphi(u - v)\, h(u)\, h(v) du\, dv \geq 0.$$

PROOF: The sufficiency of the condition follows from the usual limiting techniques.

Let $\varphi \in Q$ and $h \in C_c(R^n)$ with $\int_{R^n} h(u)du = 0$. Then for all finite choices $x_1,\dots,x_N \in R^n$ we have $\Sigma\,\Sigma\,(\varphi(x_j - x_k) - \varphi(x_j) - \varphi(-x_k) - \varphi(0))\, h(x_j)\, h(x_k) \geq 0$. We now integrate this inequality over the N-fold product of K_h, the support of h, with respect to $dx_1,\dots,dx_N$. Thus

$$(1)\quad \sum_{j=1}^{N} \sum_{k=1}^{N} \overbrace{\int_{K_h} \cdots \int_{K_h}}^{N} (\varphi(x_j - x_k) - \varphi(x_j) - \varphi(-x_k) - \varphi(0))\, h(x_j)\, \overline{h(x_k)}\; dx_1,\dots,dx_N \geq 0$$

$$(2) \quad = N\, m(K_h)^{N-1} \int_{K_h} -2 \operatorname{Re} \varphi(x)\, |h(x)|^2\, dx$$

$$+ N(N-1)\, m(K_h)^{N-2} \int_{K_h} \int_{K_h} (\varphi(x-y) - \varphi(x) - \varphi(-y) - \varphi(0))$$

$$h(x)\, \overline{h(y)}\, dx\, dy \geq 0.$$

Here m denotes n-dimensional Lebesgue measure. In (2), divide by $N(N-1)\, m(K_h)^{N-2}$ and let N approach ∞, to obtain

$$(3) \quad \int_{K_h} \int_{K_h} (\varphi(x-y) - \varphi(x) - \varphi(-y) - \varphi(0))\, h(x)\, \overline{h(y)}\, dx\, dy \geq 0$$

But recalling that $\int_{K_h} h(u)du = \int_{R^n} h(u)du = 0$, (3) reduces to

$\int_{R^n} \int_{R^n} \varphi(x-y)\, h(x)\, \overline{h(y)}\, dx\, dy \geq 0$ as asserted.

We now define $\tilde{Q}$ to be the collection of all measurable functions, $\tilde{\varphi}$, essentially bounded on every bounded neighborhood of the origin for which $\tilde{\varphi}$ satisfies condition (**) and $\varphi(t) = \varphi(-t)$ a.e. We now mention the theorem relating Q and $\tilde{Q}$.

THEORM 2. If $\varphi_1 \in \tilde{Q}$ there is a continuous function $\varphi \in Q$ such that $\varphi_1 = \varphi$ almost everywhere with respect to n-dimensional Lebesgue measure.

PROOF: Let $k \in C_c(R^n)$ and $\alpha \in R^n$. Define $h(u) = k(u) - k(u + \alpha)$. Then $h \in C_c(R^n)$ and $\int_{R^n} h(u)du = 0$. Thus

$$0 \leq \int \int \varphi_1(u - v)\, h(u)\, \overline{h(v)}\, du\, dv$$

$$= \int \int \varphi_1(u - v)\, (k(u) - k(u + \alpha))\, \overline{(k(v) - k(v + \alpha))}\, du\, dv$$

$$= \int \int (2\, \varphi_1(u - v) - \varphi_1(u - v + \alpha) - \varphi_1(u - v - \alpha))\, k(u)\, \overline{k(v)} du\, dv$$

Consequently the function $\psi_\alpha(u) = \varphi_1(u) - \frac{1}{2}(\varphi_1(u + \alpha) + \varphi_1(u - \alpha))$ is positive definite. In view of the Fubini Theorem, the function $\eta(u) = \int \psi_\alpha(u) d\alpha$ is also positive definite. By the Cartan-Godement Theorem, see Edwards [2], there is a continuous positive definite function η_1 with $\eta = \eta_1$ a.e. But

$$\eta(u) = \varphi_1(u) - \frac{1}{2} \int_{I^n + u} \varphi_1(t) dt - \frac{1}{2} \int_{I^n - u} \varphi_1(t) dt$$

and the integrals are continuous functions of u. Thus φ_1 equals almost everywhere a continuous function and the result now follows.

From the proof of the above theorem, we see that φ is a logarithmn of an infinitely divisible characteristic function if and only if for all $\alpha \in R^n$, the function $\psi_\alpha(u) = \varphi(u) - \frac{1}{2}(\varphi(u + \alpha) + \varphi(u - \alpha))$ is positive definite.

LEMMA 8. Suppose f_1 and f_2 are real continuous positive definite functions on R^n and for some $\beta \in R^n$ and $0 < \lambda < 1$ we have $\cos < \beta, u > = \lambda\, f_1(u) = (1 - \lambda)\, f_2(u)$ for all $u \in R^n$. Then there exists constants c_1 and c_2 such that $f_i(u) = c_i \cos < \beta, u >$ for $i = 1, 2,$.

PROOF: Let F, F_1, and F_2 be the corresponding generalized distributions of $\cos < \beta,u >$, $f_1(u)$ and $f_2(u)$. By the n-dimensional Uniqueness Theorem, F is a symmetric two point distribution and $F = \lambda F_1 + (1-\lambda)F_2$. Thus each F_i must be a symmetric two point generalized distribution with the same jump points as F, hence the result.

This concludes over study of the general properties of logarithmns of infinitely divisible characteristic functions. We now mention the technical results used in Chapter II. We shall adhere to the notation in that chapter by letting

$$Q = \{f\colon\ R^n \to R\colon\ f(u) = f(-u) \text{ and } f \text{ satisfies condition } (*)\}$$

$$K = \{f \in Q\colon\ f(0) \leq 0,\ \int_E f(u)du = -2^{n-1}\}$$

$$\mathfrak{N} = \{f \in K\colon\ f(u+\alpha) + f(u-\alpha) = 2\,f(u) + 2\,f(\alpha) \text{ for all } u,\ \alpha \in R^n\}.$$

$G(t) = M(1 + ||t||^2)$ is the function which uniformly bounds K. Then by setting $K^* = \{f/G\colon\ f \in K\}$ we know that K^* is a compact metric space. So K^* together with the collection of its Borel subsets forms a measurable space.

LEMMA 9. Define $\ell\colon\ R^n \times K^* \to R$ by $\ell(x,f/G) = f(x)/G(x)$. Then ℓ is a measurable mapping.

PROOF: For each $x \in R^n$, let $C(x)$ be a cube with center x and a fixed side length. We first define $L\colon\ R^n \times K^* \to R$ by

$L(x,f/G) = \int_{C(x)} \frac{f(u)}{G(u)}\,du$. We now show L is continuous. Let $x \in R^n$, $f \in K$. Let $x_j \to x$ in R^n and $\frac{f_j}{G} \overset{w^*}{\to} \frac{f}{G}$.

Let M be a bounded measurable subset of R^n for which $C(x) \subseteq M$ and for all j sufficiently large, $C(x_j) \subseteq M$. Since f_j/G is uniformly bounded for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \subseteq M$ is measurable and $m(A) < \delta$ then $\int_A \frac{|f_j(u)|}{G(u)}\,du < \epsilon$ for $j = 1,2,\ldots.$

Since $C(x_j) \to C(x)$, for all j large $m(C(x) \setminus C(x_j)) < \delta$ and $m(C(x_j) \setminus C(x)) < \delta$. and $\left|\int_{C(x)} \frac{f_j(t) - f(t)}{G(t)}\,dt\right| < \epsilon$. Hence

$$\left|\int_{C(x_j)} \frac{f_j(t)}{G(t)}\,dt - \int_{C(x)} \frac{f(t)}{G(t)}\,dt\right| = \left|\int_{C(x)} \frac{f_j}{G} - \int_{C(x)-C(x_j)} \frac{f_j}{G}\right.$$

$$\left. + \int_{C(x_j)-C(x)} \frac{f_j}{G} - \int_{C(x)} \frac{f}{G}\right| < 3\epsilon.$$

Hence L is continuous. Now let $j \in N$, and $C_j(x)$ be the cube with center x and side length $1/j$. Then $L_j(x,f/G) = \frac{1}{m(C_j(x))} \int_{C_j(x)} \frac{f(t)}{G(t)}\,dt$ is continuous and $\lim_{j \to \infty} L_j(x,f/G) = \ell(x,f/G)$ and the measurablity of ℓ now follows.

LEMMA 10. Let $h \in L'(R^n)$, $x \in R^n$. Then $h_x(t) = \frac{1 + ||t||^2}{1+||t-x||^2}\, h(t - x)$ also belongs to $L'(R^n)$.

PROOF: For any $x \in R^n$, there exists a T such that $\frac{1 + ||t||^2}{1+||t-x||^2} \leq T$.

and if $h \in L'(R^n)$ then $h_1(t) = h(t - x) \in L'(R^n)$ and the lemma follows from Holder's Inequality

LEMMA 11. Suppose $f_j, f \in K$, $j = 1,2,\ldots$ and $\frac{f_j}{G} \xrightarrow{w^*} \frac{f}{G}$. Then for all $x \in R^n$ and all $h \in L'(R^n)$, we have $\lim_{j\to\infty} \int_{R^n} h(t-x) \frac{f_j(t)}{G(t-x)}\, dt$ $= \int_{R^n} \frac{h(t-x)\, f(t)}{G(t-x)}\, dt.$

PROOF: Let $x \in R^n$, $h \in L'(R^n)$, by lemma 10,

$\lim_{j\to\infty} \int_{R^n} h_x(t) \frac{f_j(t)}{G(t)}\, dt = \int_{R^n} h_x(t) \frac{f(t)}{G(t)}\, dt.$ But

$\int_{R^n} h_x(t) \frac{f_j(t)}{G(t)}\, dt = \int_R h(t - x) \frac{f_j(t)}{G(t-x)}\, dt$ and the result is now immediate.

We shall now make further investigations of $\mathfrak{N} = \{f \in K:$ $f(u + \alpha) + f(u - \alpha) = 2\, f(u) + 2\, f(\alpha)$ for all $u, \alpha \in R^n$.

LEMMA 12. Suppose $f \in \mathfrak{N}$ and $f = \varphi$ a.e. Then for almost all $(x,t) \in R^{2n}$ we have $\varphi(x + t) + \varphi(x - t) = 2\,\varphi(x) + 2\,\varphi(t)$.

PROOF: Define $F_1, F_2\colon R^{2n} \to R$ by

$$F_1(x,t) = f(x) + f(t)$$

$$F_2(x,t) = \varphi(x) + \varphi(t)$$

Then $\{(x,t) \in R^{2n}\colon F_1(x,t) \neq F_2(x,t)\}$

$$\subseteq \{(x,t):\ t \in R^n,\ f(x) \neq \varphi(x)\}$$

$$\cup \{(x,t):\ x \in R^n,\ f(t) \neq \varphi(t)\}$$

Let $\mu^{(n)}$ denote Lebesgue measure on R^n. Then $\mu^{(2n)}\ \{(x,t) \in R^n:$ $F_1(x,t) \neq F_2(x,t)\} = 0$. Also $\{(x,t) \in R^n:\ F_1(x+t,x-t) \neq F_2(x+t,x-t)\}$ has 2n-dimensional Lebesgue measure 0.

Consequently

$$F_2(x,t) = F_1(x,t) = \frac{1}{2} F_1(x+t,x-t) = \frac{1}{2} F_2(x+t,x-t)$$

and thus

$$\mu^{(2n)}\ \{(x,t) \in R^{2n}:\ \varphi(t+x) + \varphi(t-x) \neq 2\,\varphi(t) + 2\,\varphi(x)\} = 0$$

and hence the result.

LEMMA 13. Suppose $f_j \in \mathfrak{N}$ $j = 1,2,\ldots$ and $f \in K$ and $\frac{f_j}{G} \overset{w^*}{\to} \frac{f}{G}$, then $f \in \mathfrak{N}$.

PROOF: We shall in fact show that for all $x \in R^n$, $\lim\limits_{j\to\infty} f_j(x)$ exists.

Let $x \in R^n$ and $h \in L'(R^n)$. In view of Lemma 11,

$$\lim_{j\to\infty} \int_{R^n} \frac{h(t)}{G(t)} (f_j(t-x) + f_j(t+x) - 2\,f_j(t))dt$$

exists and

$$0 = \int_{R^n} \frac{h(t)}{G(t)} (f_j(t - x) + f_j(t + x) - 2 f_j(t) - 2 f_j(x))dt$$

for $j = 1,2,\ldots$. Consequently $\lim_{j \to \infty} f_j(x) \int_{R^n} \frac{h(t)}{G(t)} dt$ exists and thus $\lim_{j \to \infty} f_j(x)$ exists. And so $F(x) = \lim_{j \to \infty} f(x)$ satisfies the defining functional equation and $F = f$ a.e. So by Lemma 12, $f \in \mathfrak{N}$. It is now convenient to identify $\mathfrak{N}$ with a collection of quadratic forms. If $f \in \mathfrak{N}$, then there exists $a_1,\ldots,a_n \in R$ such that

$$f(x) = \sum_{i,j} a_{ij}\, x_i x_j \qquad \text{where} \qquad x = (x_1,\ldots,x_n).$$

Since $f \leq 0$, we must have $a_i \leq 0$ $1 \leq i \leq n$. and hence the matrix $A = (a_{ij})$ is a symmetric $n \times n$ negative semi-definite matrix. Recall $E = \{(x_1,\ldots,x_n)$: $-1 \leq x_i \leq 1$ $1 \leq i \leq n - 1$, $0 \leq x_n \leq 1$.

If we integrate f over E we obtain

$$\int_{x_1=-1}^{1} \int_{x_{n-1}=-1}^{1} \int_{x_n=0}^{1} \sum_{i,j} a_{ij}\, x_i\, x_j\, dx_1,dx_2,\ldots,dx_n = \frac{1}{3} \sum_{1}^{n} a_j$$

$$= \frac{1}{3} \operatorname{tr} A.$$

In order to determine the extreme points of $\mathfrak{N}$, it is necessary and sufficient to enumerate the extreme points of the following set.

$P = \{A$: A a non-negative semi-definite $n \times n$ matrix with $\operatorname{tr} A = 1\}$.

PROPOSITION. The extreme points of P are precisely those matrices A of the form $A = u\,u^t$ when $u = \begin{bmatrix} u_1 \\ u_n \end{bmatrix}$, u^t is the transpose of u, and $||u||^2 = \sum_1^n u_j^2 = 1$.

PROOF: Let $\mathfrak{G}$ be the collection of all $n \times n$ matrices $U = [u^1_j, \ldots, u^n]$ where $\{u', \ldots, u^n\}$ forms an othonormal set for R^n. And let Δ^{n-1} be the $(n-1)$ simplex

$$\Delta^{n-1} = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in R^n \mid \lambda_j \geq 0 \quad 1 \leq j \leq n \text{ and } \sum_1^n \lambda_j = 1\}.$$

Then we have a mapping (in fact, continuous) of $\Delta^{n-1} \times \mathfrak{G}$ onto P given by $(\lambda, u) \rightarrow A$ where $A = \sum_1^n \lambda_j\, u^j\, (u^j)^t$. Thus P is compact and connected.

Suppose $A = \sum_1^n \lambda_j\, (u^j)\, (u^j)^t$ with $0 < \lambda_1 < 1$. Then we can write

$$A = \lambda_1\, B + (1 - \lambda_1)\, C \quad \text{where} \quad B = (u')\,(u')^t$$

$$C = \sum_2^n \frac{\lambda_j\, u^j (u^j)^t}{(1 - \lambda_1)}$$

both belong to P. And since A is unequal to B and C, A is not an extreme point of P. This shows the only possible extreme points of P are the rank 1 projections $u\,u^t$ where $||u|| = 1$.

It remains to show that the rank 1 projections are extreme points of P Let $u \in R^n$ with $||u|| = 1$. And suppose $0 < \lambda < 1$, $A, B \in P$ and $u\,u^t = \lambda A + (1 - \lambda)\,B$.

Let $v \in R^n$ be perpendicular to u. Then $0 = < u\,u^t\,v, v >$ $= \lambda < Av, v > + (1 - \lambda) < Bv, v >$.

But A, B are both non-negative semi-definite and so $< Av, v > = 0$ and $< Bv, v > = 0$. But this implies that $Av = 0 = Bv$. This implies that the null space of $u\,u^t$, A, and B are identical, and since $\operatorname{tr} A = \operatorname{tr} B = 1$, we necessarily have $A = B = u\,u^t$.

Let us now translate this result to $\mathfrak{N}$. Let $f \in \mathfrak{N}$, and let A be the matrix representing f. Then we know that A is a negative semi-definite matrix and $-2^{n-1} = \int_E f(u)du = \frac{1}{3} \operatorname{tr} A$. Thus

$f \in \operatorname{Ext}(\mathfrak{N}) \Leftrightarrow A_1 \in \operatorname{Ext}(-P)$ where $A_1 = \frac{1}{3\cdot 2^{n-1}} A$. And consequently

$f \in \operatorname{Ext}(\mathfrak{N}) \Leftrightarrow$ there exists a vector $u \in R^n$ with $||u|| = 1$ and $A = -3\cdot 2^{n-1}\, u\,u^t$. And thus an explicit determination of $\operatorname{Ext}(\mathfrak{N})$ is completed.

As was indicated earlier, a listing of the extreme points is unnecessary, however the above results can be combined to formulate the following theorem.

THEOREM 3. Let $K = \{f \in Q:\ f$ real valued, $f(0) \leq 0$, and $\int_E f(u)du = -2^{n-1}\}$. Let $\mathfrak{N} = \{f \in K:\ f(u+\alpha) + f(u-\alpha) = 2\,f(u) + 2\,f(\alpha)$ for all $u, \alpha \in R^n\}$.

Then $\mathfrak{N}$ is a weak* closed subset of K and hence weak* compact. The Krein-Milman Theorem guarantees that $\operatorname{Ext}(\mathfrak{N})$ is non-empty, and $f \in \operatorname{Ext}(\mathfrak{N})$ if and only if the matrix A corresponding to f is of the form $A = -3\cdot 2^{n-1}\, u\,u^t$ where u is a unit vector in R^n.

CHAPTER III

THE GENERAL LEVY-KHINCHINE REPRESENTATION

In this chapter, we make further investigation of general continuous functions φ: $R^n \to C$ which are logarithmns of infinitely divisible characteristic functions. Again, we shall let Q stand for this collection of functions, and write $\varphi = f + ig$. We start with redefining the set K and show that K is uniformly bounded and the extreme points of K are precisely those functions which appear in the Levy-Khinchine integrand.

Let $K = \{\varphi \in Q\colon\ \varphi = f + ig,\ \varphi(0) \leq 0,\ \int_E f(u)du = -2^{n-1},$ and for all i $1 \leq i \leq n,\ \int_0^1 g(0,\ldots,x_i,\ldots 0)dx_i = 0\}$.

We may sometimes use f and g indiscriminately, but they will always stand for the real and imaginary parts of the φ in question.

LEMMA 1. Suppose $\varphi \in K$ and g is a continuous linear functional. Then $g \equiv 0$.

PROOF: If g is a continuous linear functional, there exists a vector $\beta \in R^n$ such that $g(u) = \langle \beta,u \rangle$ for all $u \in R^n$. Write

$\beta = (\beta_1,\ldots,\beta_n)$, and let $i \in \{1,2,\ldots,n\}$. Then $0 = \int_0^1 g(0,\ldots,x_i,\ldots,0)dx_i = \int_0^1 \beta_i\, x_i\, dx_i = \frac{\beta_i^2}{2}$ and so $\beta = 0$, and hence $g(u) \equiv 0$.

DEFINITION. $\varphi \in Q$ is called degenerate if $f(u) \equiv f(0)$.

$\varphi_1, \varphi_2 \in Q$ are said to be of the same type if there exists an $a > 0$, $b \in R^n$, $c \in R$ such that $a\,\varphi_1(u) + i \langle b,u \rangle + c = \varphi_2(u)$ for all $u \in R^n$.

LEMMA 2. For every non-degenerate $\varphi \in Q$ there is a $\varphi_1 \in K$ of the same type as φ.

PROOF: Let $\varphi \in Q$ be non-degenerate. Define $\varphi_0(u) = \varphi(u) - \varphi(0)$, then $\varphi_0 \in Q_0$. Now φ_0 is non-degenerate and so $f_0 \leq 0$ and $f_0 \not\equiv 0$. Thus if $\int_E \varphi_0(t)dt = A + iB$, we must have $A < 0$. For each $i = 1,2,\ldots,n$, set $t_i = \int_0^1 g_0(0,\ldots,x_i,\ldots,0)dx_i$ and $t = (t_1,\ldots,t_n)$. By defining $\varphi_1(u) = \frac{2^{n-1}}{-A}\ (\varphi_0(u) - 2i < u,t >)$ we see that $\varphi_1 \in K$ and φ_1 is of the same type as φ.

LEMMA 3. There exists a real number c such that for all $\varphi \in K$, we have $|f(u)| \leq c(1 + ||u||^2)$.

PROOF: This argument was presented in Chapter II.

LEMMA 4. Let $\varphi \in K$ with $\varphi(0) = 0$. Then there exists a c such that $\int_0^1 |g(0,\ldots,x_i,\ldots,0)|\, dx_i \leq c$ for all $i = 1,2,\ldots,n$.

PROOF: Let $i \in \{1,2,\ldots,n\}$ be fixed. We shall use the inequality $2\ f(u) - (g(u + \alpha) - g(u - \alpha) - 2\ g(\alpha)) \leq 0$ for all $u, \alpha \in R^n$ (See Appendix, page 30). Let $\alpha_0 = (0,\ldots,\alpha_i^0,\ldots,0)$ and $\beta_0 = (0,\ldots,\beta_i^0,\ldots,0)$ be such that $g(\alpha_0) = \max\limits_{0 \leq \alpha_i \leq 1} g(0,\ldots,\alpha_i,\ldots,0)$ and $g(\beta_0) = \min\limits_{0 \leq \alpha_i \leq 1} g(0,\ldots,\alpha_i,\ldots,0)$. First assume that $\alpha_i^0 < \beta_i^0$. In the above inequality, let $\alpha = \alpha_0$, and $u = \beta_0 - \alpha_0$. We then have $-2\ f(\beta_0 - \alpha_0) \geq 2\ g(\alpha_0) - g(\beta_0) + g(\beta_0 - 2\ \alpha_0) \geq g(\alpha_0)$ since $g(\beta_0 - 2\ \varphi_0) \geq g(\beta_0) - g(\alpha_0)$.

Since $\varphi \in K$ entails $\bar{\varphi} \in K$, if $\alpha_i^0 > \beta_i^0$ we then would replace g by $-g$, and obtain $-2\ f(\alpha_0 - \beta_0) \geq -g(\beta_0)$. Since $\int_0^1 g(0,\ldots,x_i,\ldots,0)dx_i = 0$, we have $\int_0^1 |g(0,\ldots,x_i,\ldots,0)|\,dx_i =$ $= 2\int_0^1 g^+(0,\ldots,x_i,\ldots,0)dx_i = 2\int_0^1 g^-(0,\ldots,x_i,\ldots,0)dx_i$.

Recalling that f's are uniformly bounded on every bounded neighborhood of 0, if $\alpha_i^0 < \beta_i^0$, we have $\int_0^1 |g(0,\ldots,x_i,\ldots,0)|\,dx_i \leq 2\ g(\alpha_0) \leq -2\ f(\alpha_0 - \beta_0) \leq c$. Or if $\alpha_i^0 > \beta_i^0$, $\int_0^1 |g(0,\ldots,x_i,\ldots,0)|\,dx_i \leq -2\ g(\beta_0) \leq -2\ f(\alpha_0 - \beta_0) \leq c$. And if $\alpha_0 = \beta_0$, then $g(0,\ldots,x_i,\ldots,0) \equiv 0$ on $0 \leq x_i \leq 1$.

So if $\varphi \in K$ with $\varphi(0) = 0$, and $i \in \{1,2,\ldots,n\}$, there exists a real number c such that

$$\int_0^1 |g(0,\ldots,x_i,\ldots,0)|\,dx_i \leq c.$$

LEMMA 5. Let $\varphi \in K$ with $\varphi(0) = 0$. Then there is a real number c such that $|g(\alpha)| \leq c$ for all $\alpha \in R^n$ of the form $\alpha = (0,\ldots,\alpha_i,\ldots,0)$ where i is arbitrary and $-1 \leq \alpha_i \leq 1$.

PROOF: Let u and α be of the form specified in the lemma. By integrating the inequality $2\ f(0,\ldots,u_i,\ldots,0) \pm (g(0,\ldots,u_i - \alpha_i,\ldots,0)$ $- g(0,\ldots,u_i - \alpha_i,\ldots,0) - 2\ g(0,\ldots,\alpha_i,\ldots,0)) \leq 0$ over I^n we have $|g(0,\ldots,\alpha_i,\ldots,0)| \leq \int_0^1 |f(0,\ldots,u_i,\ldots,0)|\,du_i + \frac{1}{2}\int_0^1 |g(0,\ldots,u_i+\alpha_i,\ldots,0)|\,du_i$ $+ \frac{1}{2}\int_0^1 |g(0,\ldots,u_i - \alpha_i,\ldots,0)|\,du_i$.

By the oddness of g we have

$$|g(0,\ldots,\alpha_i,\ldots,0)| \le \int_0^1 |f(0,\ldots,u_i,\ldots,0)|\,du_i$$

$$+ \frac{1}{2}\int_{\alpha_i-1}^{\alpha_i+1} |g(0,\ldots,u_i,\ldots,0)|\,du_i$$

$$\le 2^{n-1} + \int_{\frac{\alpha_i-1}{2}}^{\frac{\alpha_i+1}{2}} |g(0,\ldots,2u_i,\ldots,0)|\,du_i$$

Since $-1 \le \alpha_i \le 1$ and $|g(2u)| \le 2|g(u)| + 2|f(u)$

$$|g(0,\ldots,\alpha_i,\ldots,0)| \le 2^{n-1} + 2\int_{-1}^{1} |g(0,\ldots,u_i,\ldots,0)| + |f(0,\ldots,u_i,\ldots,0)|\,du_i$$

$$\le c \text{ by lemmas 3 and 4.}$$

LEMMA 6. If $\varphi \in K$ with $\varphi(0) = 0$, then there is a real number c such that $|g(u)| \le c(1 + ||u||^2)$ for all $u \in R^n$.

PROOF: By lemma 5, there exists a c_0 such that $|g(e_i)| \le c_0$ for all g where e_i is the standard basis vector $e_i = (0,\ldots,1,\ldots,0)$. To complete the proof we make use of the inequalities
$|g(u + v) - g(u) - g(v)|^2 \le 4\ f(u)\ f(v)$ and
$|g(nu) - n\ g(u)| \le -n(n - 1)\ f(u)$ for all $u, v \in R^n$, $n \in N$. (See Appendix page , page). Let $U = \{x = (x_1,\ldots,x_n)\colon\ -1 \le x_j \le 1$ for all $j = 1,\ldots,n\}$. Let $x \in U$ with $x = (x_1,\ldots,x_n)$. First let $u = (x_1,0,\ldots,0)$ and $v = (0,x_2,0,\ldots,0)$. Then
$|g(x_1,x_2,0,\ldots,0)|^2 \le 4\ f(u)\ f(v) + |g(u)|^2 + |g(v)|^2 \le c_1$ by

lemmas 2 and 4. Now let $u = (x_1, x_2, 0, \ldots, 0)$ and $v = (0, 0, x_3, 0, \ldots, 0)$ and again we get $|g(x_1, x_2, x_3, 0, \ldots, 0)| \leq c_2$. Continuing in this manner, we easily see that there exists c such that if $\varphi \in K$ with $\varphi(0) = 0$ and $x \in u$, then $|g(x)| \leq c$.

Now let $t \in R^n \backslash \{0\}$ be arbitrary. Then

$$|g(t)| = \left|g\left(\frac{(1 + [||t||]\ t)}{1 + [||t||]}\right)\right| \leq (1 + [||t||])\ \left|\ g\left(\frac{t}{1 + [||t||]}\right)\right.$$

$$+ (1 + [||t||])\ ([||t||])\ \left|f\left(\frac{t}{1 + [||t||]}\right)\right| \leq c(1 + ||t||)$$

$$+ c(1 + ||t||)\ ||t|| \leq c'(1 + ||t||^2)$$

and the lemma now follows.

Realizing if $\varphi \in K$, then $-1 \leq \varphi(0) \leq 0$, we have the following theorem.

THEOREM 1. There exists a real number c such that for all $\varphi \in K$, $|\varphi(t)| \leq c(1 + ||t||^2)$ for all $t \in R^n$.

We now determine the function belonging to $\mathrm{Ext}(K)$. By the Normal Characterization Theorem, (See Appendix, page 33), if $\varphi \in K$ and f satisfies $f(u + \alpha) + f(u - \alpha) = 2\ f(u) + 2\ f(\alpha)$, then g must be linear. But if g is linear, and $\varphi \in K$, then $g \equiv 0$. So we shall again let $\mathfrak{N} = \{\varphi \in K\colon\ f(u + \alpha) + f(u - \alpha) = 2\ f(u) + 2\ f(\alpha)$ for all $u, \alpha \in R^n\}$.

PROPOSITION 1. If $\varphi \in \mathrm{Ext}(K)$, then

(i) $\varphi(u) \equiv -1$ or

(ii) $\varphi(u) = c_\beta(e^{i<\beta,u>} - 1 - 2i < \gamma_\beta, u >)$ for appropriate $c_\beta \in R$ and $\gamma_\beta \in R^n$ or

(iii) $\varphi(u) \in \mathrm{Ext}(\mathfrak{M})$.

PROOF: This proof somewhat parallels the argument given in Chapter II.

Let $\varphi \in \mathrm{Ext}(K)$, then we know $\varphi(0) = -1$ or 0. If $\varphi(0) = -1$, then $f(u) \equiv -1$ and so g is linear. Which in turn implies that $g \equiv 0$. and $\varphi(u) \equiv -1$.

Now assume $\varphi(0) = 0$. Let $\alpha \in R^n$ and $0 \leq \varepsilon \leq \frac{1}{2}$. Then the functions $\psi^{\pm}_{\alpha,\varepsilon}(u) = \varphi(u) \pm \varepsilon\,(\varphi(u+\alpha) + \varphi(u-\alpha) - \varphi(\alpha) - \varphi(-\alpha))$ belong to Q_0. As before, fix in ε, $0 < \varepsilon \leq \frac{1}{2}$ such that $\psi^{\pm}_{\alpha,\varepsilon}$ are non-degenerate, and pick $\varphi_1, \varphi_2 \in Q_0 \cap K$ to be of the same type as $\psi^{+}_{\alpha,\varepsilon}$ and $\psi^{-}_{\alpha,\varepsilon}$ respectively. Thus for suitably chosen positive numbers $a_1(\alpha)$, $a_2(\alpha)$ and vectors $b_1(\alpha)$, $b_2(\alpha)$ we have

$$(1)\qquad \begin{aligned} \psi^{+}_{\alpha,\varepsilon}(u) &= a_1(\alpha)\,\varphi_1(u) + i < b_1(\alpha), u > \\ \psi^{-}_{\alpha,\varepsilon}(u) &= a_2(\alpha)\,\varphi_2(u) + i < b_2(\alpha), u > \end{aligned}$$

However, $\varphi = \frac{1}{2}\psi^{+}_{\alpha,\varepsilon} + \frac{1}{2}\psi^{-}_{\alpha,\varepsilon}$ implies

$$(2)\qquad \varphi(u) = \frac{1}{2}a_1(\alpha)\,\varphi_1(u) + \frac{1}{2}a_2(\alpha)\,\varphi_2(u) + \frac{1}{2} < b_1(\alpha) + b_2(\alpha), u >$$

By taking the real part of (2), and integrating over E, we see that $a_1(\alpha) + a_2(\alpha) = 2$. Now $\varphi \in K$ and

$\frac{a_1(\alpha)}{2}\ \varphi_1(u) + \frac{a_2(\alpha)}{2}\ \varphi_2(u) \in K$, so $b_1(\alpha) + b_2(\alpha) = 0$. Thus $\varphi(u) = \lambda\, \varphi_1(u) + (1 - \lambda)\, \varphi_2(u)$ for some λ, $0 \leq \lambda \leq 1$. But if φ is extreme, $\varphi = \varphi_1 = \varphi_2$. Collecting this, if $\varphi \in \mathrm{Ext}(K)$, for suitably chosen $A(\alpha) \in R$, and $B(\alpha) \in R^n$

$$(3)\quad A(\alpha)\, \varphi(u) = \varphi(u + \alpha) + \varphi(u - \alpha) - \varphi(\alpha) - \varphi(-\alpha) - 2i < u, B(\alpha) >$$

for all $u, \alpha \in R^n$.

Let us first consider the real part of (3). Then we get

$$(4)\quad A(\alpha)\, f(u) = f(u + \alpha) + f(u - \alpha) - 2\, f(\alpha).$$

The analysis of Chapter II, shows that either $A(\alpha) \equiv 2$, and $f \in \mathrm{Ext}(\mathfrak{N})$ or $A(\alpha) = 2 \cos < \beta, \alpha >$ for some $\beta \in R^n \setminus \{0\}$ and $f(u) = c_\beta(\cos < \beta, u > - 1)$ where $c_\beta = 2^{n-1}(\int_E 1 - \cos < \beta, t > dt)$. In the former, we also know that if $\varphi = f + ig$ and $f \in \mathfrak{N}$, then $g \equiv 0$. Assuming the latter, we turn to the solutions of the imaginary part of (3), namely

$$(5)\qquad A(\alpha)\, g(u) = g(u + \alpha) - g(u - \alpha) - 2 < u, B(\alpha) >.$$

Let $\alpha_1 \in R^n$ be such that $A(\alpha_1) = \cos < \beta, \alpha_1 > = 0$ and $\sin < \beta, \alpha_1 > = 1$. In (5), let $\alpha = \alpha_1$, $u = t + \alpha_1$ to obtain $0 = g(t + 2\,\alpha_1) + g(t) - 2 < t + \alpha_1, B(\alpha_1) >$ and so

$$(6) \qquad g(t + 2\,\alpha_1) + g(t) = 2 < t + \alpha_1, B(\alpha_1) >$$

In particular if $t = 0$, $g(2\,\alpha_1) = < 2\,\alpha_1, B(\alpha_1)>$. Now let α be arbitrary, and $u = 2\,\alpha_1$. Then $2 \cos < \beta,\alpha > g(2\,\alpha_1) =$
$= g(2\,\alpha_1 + \alpha) + g(2\,\alpha_1 - \alpha) - 2 < 2\,\alpha_1, B(\alpha) >$ or

$$(7) \quad 2 \cos < \beta,\alpha > 2 < \alpha_1, B(\alpha_1) > = g(2\,\alpha_1 + \alpha) + g(2\,\alpha_1 - 2) - 4 < \alpha_1, B(\alpha) >$$

$$= g(2\,\alpha_1 + \alpha) + g(\alpha) + g(2\,\alpha_1 - \alpha) + g(-\alpha) - 4 < \alpha_1, B(\alpha) >$$

$$= 2 < \alpha + \alpha_1, B(\alpha_1) > + 2 < -\alpha + \alpha_1, B(\alpha_1) > - 4 < \alpha_1, B(\alpha) >$$

$$= 4(< \alpha_1, B(\alpha_1) > - < \alpha_1, B(\alpha) >).$$

Thus $4 < \alpha_1, B(\alpha) > = 4(< \alpha_1, B(\alpha_1) > - \cos < \beta,\alpha > < \alpha_1, B(\alpha_1) >)$ which implies that

$$(8) \qquad < \alpha_1, B(\alpha) > = < \alpha_1, B(\alpha_1) > (1 - \cos < \beta,\alpha >)$$

for all $\alpha \in R^n$. In (5), let $\alpha = t + \alpha_1$, $u = \alpha_1$
$A(t + \alpha_1)\, g(\alpha_1) = g(t + 2\,\alpha_1) + g(-t) - 2 < \alpha_1, B(t + \alpha_1)$. But
$A(t + \alpha_1) = -2 \sin < t,\beta >$, $g(t + 2\,\alpha_1) = -g(t) + 2 < t + \alpha_1, B(\alpha_1) >$ and
$2 < \alpha_1, B(t + \alpha_1) > = 2 < \alpha_1, B(\alpha_1) > (1 - \cos < \beta, t + \alpha_1 >) =$
$2 < \alpha_1, \beta(\alpha_1) > (1 + \sin < \beta, t >)$. Thus

(9) $-2 \sin < t,\beta > g(\alpha_1) = -2\, g(t) - 2 < \alpha_1,\beta(\alpha_1) > \sin < \beta,t >$

$+ 2 < t,B(\alpha_1) >$ and finally

(10) $g(t) = (g(\alpha_1) - < \alpha_1,B(\alpha_1) >) \sin < t,\beta > + < t,B(\alpha_1) >$

$= c(c_\beta(\sin < \beta,t > + < t,B(\alpha_1) >)).$

We now determine c and $B(\alpha_1)$. Since $\int_0^1 g(0,\ldots,x_i,\ldots,0)dx_i = 0$ for all i, we see that $B(\alpha_1) = -2(\gamma_1,\ldots,\gamma_n)$ where $\gamma_j = \dfrac{1 - \cos \beta_j}{\beta_j}$ if $\beta_j \neq 0$ and $\gamma_j = 0$ if $\beta_j = 0$. Let $\gamma_\beta = (\gamma_1,\ldots,\gamma_n)$. Then we have $g(t) = c(c_\beta(\sin < t,\beta > - 2 < t,\gamma_\beta >))$. Substituting $f(u) = c_\beta(\cos < \beta,u > - 1)$ in the inequality $|g(2u) - 2\, g(u)| \leq -2\, f(u)$ we see that

$$|g(2u) - 2\, g(u)| = |c(c_\beta(\sin < 2u,\beta > - 2 < 2u,\gamma_\beta >))$$

$$-2c(c_\beta(\sin < u,\beta > -2 < u,\gamma_\beta >))|$$

$$= |c|\ |c_\beta|\ (2 \sin < u,\beta > \cos < u,\beta > - 2 \sin < u,\beta >|$$

$$\leq -2\, f(u) = 2|c_\beta|\ |\cos < u,\beta > - 1|.$$

Thus $|c|\ |\sin < u,\beta >| \leq 1$ for all $u \in R^n$. So $|c| \leq 1$. But if φ is extreme $c = \pm 1$. However the φ corresponding to $c_1 = -1$ and β

is the same as if $c_1 = 1$ and $-\beta$. So the solution in the second case is $\varphi_\rho(t) = c_\beta(e^{i<\beta,t>} - 1 - 2i < u,\gamma_\beta >)$ where

$$c_\beta = 2^{n-1}(\int_E 1 - \cos < \beta,u > du)^{-1} \text{ and } \gamma_\beta = (\frac{1 - \cos \beta_1}{\beta_1},\ldots,\frac{1 - \cos \beta_n}{\beta_n})$$

Appealing to the arguments given in Chapter II, we will have completed the task of determining $\mathrm{Ext}(K)$ provided we prove

PROPOSITION 2: For any $\beta \in R^n\backslash\{0\}$, the function $\varphi_\beta(t) = c_\beta(e^{i<\beta,t>} - 1 - 2 < t,\gamma_\beta >)$ defined above, is an extreme point of K.

PROOF: Suppose $\varphi_1, \varphi_2 \in K$ and $0 \leq \lambda \leq 1$ are such that $\varphi_\beta = \lambda \varphi_1 + (1 - \lambda) \varphi_2$. Fix $\alpha \in R^n$ and define L_α: $K \to Q_0$ by the rule

$$(1)\quad (L_\alpha \varphi)(u) = 2 \varphi(u) - [e^{-i<\alpha,\beta>}(\varphi(u + \alpha) - \varphi(\alpha))$$

$$+ e^{i<\alpha,\beta>}(\varphi(u - \alpha) - \varphi(-\alpha))].$$

Then $L_\alpha \varphi_\beta \in Q_0$ and so $\mathrm{Re}\, L_\alpha \varphi_\beta \leq 0$. We also note that if $\varphi', \varphi'' \in K$ and $0 \leq \gamma \leq 1$, then $L_\alpha(\gamma \varphi' + (1-\gamma)\varphi'') = \gamma L_\alpha \varphi' + (1-\gamma)L_\alpha \varphi''$. Now

$$(2)\quad (L_\alpha \varphi_\beta)(u) = L_\alpha(c_\beta(e^{i<\beta,u>} - 1 - 2i < u,\gamma_\beta >)$$

$$= 2(c_\beta(e^{i<\beta,u>} - 1 - 2i < u,\gamma_\beta >)) -$$

$$[e^{-i<\alpha,\beta>}c_\beta(e^{i<\beta,u+\alpha>} - 1 - 2i < u + \alpha,\gamma_\beta > -$$

$$e^{i<\beta,\alpha>} + 1 + 2i < \alpha,\gamma_\beta >) + e^{i<\alpha,\beta>}c_\beta(e^{i<u-\alpha,\beta>}$$

$$- 1 - 2i < u - \alpha,\gamma_\beta > - e^{-i<\alpha,\beta>} + 1 - 2i < \alpha,\gamma_\beta >)]$$

$$= c_\beta[2(e^{i<\beta,u>} - 1 - 2i<u,\gamma_\beta>) - (e^{i<\beta,u>} - 1 - e^{-i<\alpha,\beta>}2i<u,\gamma_\beta)$$

$$+ e^{i<\beta,u>} - 1 - e^{i<\alpha,\beta>}2i < u,\gamma_\beta >)]$$

$$= c_\beta \; 2(e^{i<\beta,u>} - 1 - 2i < u,\gamma_\beta > - e^{i<\beta,u>} + 1 + ie^{-i<\alpha,\beta>}<u,\gamma_\beta>$$

$$+ ie^{i<\alpha,\beta>}< u,\gamma_\beta >).$$ Thus

$$(3) \quad (L_\alpha \varphi_\beta)(u) = c_\beta \; 2(2i < u,\gamma_\beta >) \left(\frac{e^{i<\alpha,\beta>} + e^{-1<\alpha,\beta>}}{2} - 1\right)$$

$$= 2\, c_\beta(2i < u,\gamma_\beta >) (\cos < \beta,\alpha > - 1).$$

Thus $\operatorname{Re} L_\alpha \varphi_\beta \equiv 0$, and since $L_\alpha \varphi_\beta = \lambda L_\alpha \varphi + (1-\lambda) L_\alpha \varphi_2$ we have $\operatorname{Re} L_\alpha \varphi_i \equiv 0$ for $i = 1,2$. Henceforth, we shall let φ stand for either φ_1 or φ_2. Since $\operatorname{Re} L_\alpha \varphi \equiv 0$, there exists a vector $A(\alpha) \in R^n$ such that $L_\alpha \varphi(u) = 2i < A(\alpha),u >$ for all $u \in R^n$. Thus φ satisfies the following identity

(4) $\varphi(u) - \frac{1}{2}(e^{-i<\alpha,\beta>}(\varphi(u+\alpha)-\varphi(\alpha))+e^{i<\alpha,\beta>}(\varphi(u-\alpha)-\varphi(-\alpha)))$

$= i < A(\alpha),u >$ for all $u \in R^n$. By taking the imaginary parts of both sides of (4), we obtain

$$< A(\alpha),u > = g(u) - \frac{1}{2}[\cos < \alpha,\beta > (g(u + \alpha) - g(\alpha))$$

$$- \sin < \alpha,\beta > (f(u + \alpha) - f(\alpha) + \cos < \alpha,\beta > (g(u - \alpha)$$

$$+ g(\alpha)) + \sin < \alpha,\beta > (f(u - \alpha) - f(\alpha))]$$

(5) $$= g(u) + \frac{1}{2}\sin < \alpha,\beta > (f(u + \alpha) - f(u - \alpha)$$

$$- \frac{1}{2}\cos < \alpha,\beta > (g(u + \alpha) + g(u - \alpha)).$$

We now determine $A(\alpha)$. We shall use $[\beta]$ to denote the vector space spanned by β. Let $\alpha_1 \in [\beta]$ be such that $\cos < \beta,\alpha_1 > = 1$ and $\cos < \beta,\alpha_1/2 > = -1$. Let $\alpha \in R^n$. We shall now show $< A(\alpha),u > = \frac{1}{2}(1 - \cos < \beta,\alpha >) < A(\alpha_1/2),u >$ for all $u \in R^n$.

If $u \perp \beta$, then $f_\beta(u) = 0$ and so $f(u) = 0$. Then $f(u+\alpha) = f(\alpha)$ and $g(u + \alpha) = g(u) + g(\alpha)$ for all $\alpha \in R^n$. Therefore $< A(\alpha),u > = g(u) - \frac{1}{2}\cos < \alpha,\beta > 2\, g(u) = g(u)\,(1 - \cos < \alpha,\beta >)$. On the other hand, $< A(\alpha_1/2),u > = 2\, g(u)$ and consequently $< A(\alpha),u > = \frac{1}{2}(1 - \cos < \beta,\alpha >) < A(\alpha_1/2),u >$ for all $u \perp \beta$. Now $< A(\alpha),\alpha_1 > = g(\alpha_1)\,(1 - \cos < \alpha,\beta >)$ from (5) and $f(\alpha_1) = 0$. Thus

$< A(\alpha_1/2),\alpha_1 > = 2\, g(\alpha_1)$ and $\frac{1}{2}(1 - \cos < \beta,\alpha_1 >)\ < A(\alpha_1/2),\alpha_1 > =$ $g(\alpha_1)\,(1 - \cos < \alpha,\beta >) = < A(\alpha)^2,\alpha_1 >$. Now let $u \in [\beta] = [\alpha_1]$. Then $u = k\,\alpha_1$ for some $k \in R$. Then $< A(\alpha),u > = k < A(\alpha_1),\alpha_1 >$ $= k(\frac{1}{2}(1 - \cos < \beta,\alpha >)) < A(\alpha_{1/2}),\alpha_1 > = \frac{1}{2}(1 - \cos < \beta,\alpha >) < A(\alpha_1/2),u>$.

Finally let $u \in R^n$ be arbitrary. Then we may express $u = v_1 + v_2$ when $v_1 \perp \beta$ and $v_2 \in [\beta]$. Using the above computations we have

$$< A(\alpha),u > = < A(\alpha),v_1 > + < A(\alpha),v_2 >$$

$$= \frac{1}{2}(1 - \cos < \beta,\alpha >)\,(< A(\alpha_1/2),v_1 > +$$

$$< A(\alpha_1/2),v_2 >) = \frac{1}{2}(1 - \cos < \beta,\alpha >) < A(\alpha_1/2),u > .$$

Since $< A(\alpha),u > = \frac{1}{2}(1 - \cos < \beta,\alpha >) < A(\alpha_1/2),u >$ for all u, we must have

$$(6) \qquad A(\alpha) \equiv \frac{1}{2}(1 - \cos < \beta,\alpha >)\, A(\alpha_1/2).$$

Let $c = \dfrac{A(\alpha_1/2)}{2}$. By setting $\psi(u) = e^{-i<u,\beta>}(\varphi(u) - i < c,u >)$ we see that

$$(7) \quad \psi(u) - \frac{1}{2}[\psi(u + \alpha) + \psi(u - \alpha) - e^{-i<\beta,u>}(\psi(\alpha) + \psi(-\alpha))]$$

$$= e^{-i<u,\beta>}[\varphi(u) - i < c,u > - \frac{1}{2}(e^{-i<\alpha,\beta>}(\varphi(u + \alpha)$$

$$-i < c,u + \alpha > - \varphi(\alpha) + i < c,\alpha >) + e^{i<\alpha,\beta>}(\varphi(u - \alpha)$$

$$-i < c,u - \alpha) - \varphi(-\alpha) - i < c,\alpha >))].$$

Applying (4) and (6) to (7) we have (7)

$$= e^{-i<u,\beta>}[\varphi(u) - \frac{1}{2}(e^{-i<\alpha,\beta>}(\varphi(u + \alpha) - \varphi(\alpha)) + e^{i<\alpha,\beta>}(\varphi(u-\alpha)-\varphi(-\alpha)))$$

$$- i < c,u > - \frac{1}{2}(e^{-i<\alpha,\beta>}(-i < c,u > + e^{i<\alpha,\beta>}(-i < c,u >))]$$

$$= e^{-i<u,\beta>}[i < c,u > (1 - \cos < \beta,\alpha >) - i < c,u >$$

$$- i < c,u > (\frac{e^{i<\alpha,\beta>} + e^{-i<\alpha,\beta>}}{2})] = e^{-i<u,\beta>}[i<c,u>(1-\cos<\beta,\alpha>)$$

$$- i < c,u > (1 - \cos < \beta,\alpha >)] = 0$$

Thus $\psi(u) = e^{-i<u,\psi>}(\varphi(u) - i < c,u >)$ satisfies $\psi(u) = \overline{\psi(-u)}$ and

$$(8) \quad \psi(u) - \frac{1}{2}[\psi(u + \alpha) + \psi(u - \alpha) - e^{-i<\beta,u>}(\psi(\alpha) + \psi(-\alpha))] = 0.$$

Express $\psi = p + iq$, take the real part of 8, and interchange u and α to get

$$(9) \quad p(u) - \frac{1}{2}(p(u + \alpha) + p(u - \alpha) - 2\cos < \beta,\alpha > p(\alpha)) = 0$$

$$p(\alpha) - \frac{1}{2}(p(u - \alpha) + p(\alpha - u) - 2\cos < \beta,u > p(u)) = 0$$

And by subtracting, we have

$$p(u)\ (1 - \cos < \beta,\alpha >) = p(\alpha)\ (1 - \cos < \beta,u >)$$

or

(10) $$p(u) = c_1(1 - \cos < \beta,u >)$$

for some $c_1 \in R$. Next take the imaginary parts of (8).

$q(u) - \frac{1}{2}(q(u + \alpha) + q(u - \alpha) + \sin < \beta,u > 2\ p(\alpha)) = 0$. This implies

(11) $$q(u) - \frac{1}{2}(q(u + \alpha) + q(u - \alpha)) = c_1(1 - \cos < \beta,\alpha >)^2 \sin < \beta,u >.$$

Set $F(u) = q(u) - c_1 \sin <\beta,u >$. By using the usual trigonometric rules and the oddness of q, we see that F is linear, and hence there is an $a \in R^n$ such that $F(u) = < a,u >$ for all $u \in R^n$. This in turn yields $q(u) = c_1 \sin < \beta,u > + < a,u >$. Collecting this we have

(12) $$e^{-i<\beta,u>}(\varphi(u) - i < c,u >) = \psi(u)$$

$$= p(u) + iq(u)$$

$$= c_1(1 - \cos < \beta,u >)$$

$$+ i\ c_1 \sin < \beta,u > + i < a,u >$$

$$= c_1(1 - e^{i<\beta,u>}) + i < a,u >.$$

which yields

$$\varphi(u) - i < c,u > = c_1(e^{i<\beta,u>} - 1) + i\, e^{i<\beta,u>} < a,u >. \tag{13}$$

Taking the real parts of (13), we have (14) $f(u) = c_1(\cos < \beta,u > - 1) - \sin < \beta,u > < a,u >$. If $a \neq 0$, it would be easy to produce a $u \in R^n$ such that $f(u) > 0$. Consequently $a = 0$. and $f(u) = c_1(\cos < \beta,u > - 1)$. But since $f \in K$ we must have $c_1 = c_\beta$. Returning to (13), we see that $g(u) = c_\beta(\sin < \beta,u >) + < c,u >$. But the restrictions of K on g force $c = -2\, c_\beta\, \gamma_\beta$, where γ_β is as above. Thus $\varphi(u) = c_\beta(e^{i<\beta,u>} - 1 - 2\, i < u,\gamma_\beta >) = \varphi_\beta(u)$ which completes the proof of Proposition 2.

It would be very gratifying to conclude the Levy-Khinchine Representation by appealing to Choquet's Theorem. However at present, the fact that K is compact can not be established.

CHAPTER IV

AN APPLICATION TO UNIMODAL DISTRIBUTIONS

In this chapter, Khinchine's representation of unimodal distributions is obtained through the use of extreme points.

DEFINITION Let $F: R \to R$ be monotone non-decreasing, right continuous, and $0 \leq F(-\infty) \leq F(+\infty) \leq 1$. If F is convex on $(-\infty, 0)$ and F is concave on $(0, +\infty)$, F is called a (generalized) unimodal distribution.

The concept of unimodality is closely related to absolute continuity as the following can be established. See Lukacs [6].

LEMMA 1. F is a unimodal distribution if and only if there exist two non-negative, integrable functions f, g, $f: (-\infty, 0) \to R$ is non-decreasing $g: (0, +\infty) \to R$ is non-increasing and

$$F(x) = \begin{cases} F(-\infty) + \int_{-\infty}^{x} f(u)du & \text{if} \quad x < 0 \\ \\ F(+\infty) - \int_{x}^{\infty} g(u)du & \text{if} \quad x > 0 \end{cases}$$

With the aid of Lemma 1, Khinchine established the following theorem.

THEOREM 1. A distribution F is unimodal if and only if there exists a corresponding distribution G yielding the following representation:

$$F(x) = \begin{cases} \lambda_1 + \lambda_2[G(x) - x \int_{-\infty}^{x} \frac{dG(\alpha)}{\alpha}] & \text{if} \quad x < 0 \\ \lambda_1 + \lambda_2\, G(0) & \text{if} \quad x = 0 \\ \lambda_1 + \lambda_2[G(x) + x \int_{x}^{\infty} \frac{dG(\alpha)}{\alpha}] & \text{if} \quad x > 0. \end{cases}$$

here $\lambda_1 = F(-\infty)$, $\lambda_2 = F(+\infty) - F(-\infty)$. $G(-\infty) = 0$, $G(+\infty) = 1$.

Using integration by parts and Theorem 1, assuming $\lambda_1 = 0$, $\lambda_2 = 1$, we obtain the following corollary concerning the corresponding characteristic functions.

COROLLARY 1 A characteristic function φ is of a unimodal distribution if and only if there is a characteristic function g such that

$$\varphi(t) = \frac{1}{t} \int_0^t g(u)du.$$

Our goal is to obtain Theorem 1 using the method of extreme points. For this purpose, we make a detailed investigation of the following set K: $K = \{F\colon R \to R\colon F \text{ is a unimodal distribution}\}$.

We first want to establish K is a compact, convex set, but it is necessary to consider equivalence classes of K as follows:

$$\tilde{K} = \{f\colon R \to R\colon \text{ there is an } F \in K,\ f = F \text{ a.e.}\}$$

We may identify K with $\tilde{K}$. In order to establish the compactness of K, it suffices to do the same for $\tilde{K}$.

LEMMA 2. $\tilde{K}$ is a weak* closed subset of B, where $B = \{f \in L^{\infty}(R): \operatorname*{ess\,sup}_{x \in R} |f(x)| \leq 1\}$.

PROOF: Let $f_n \in \tilde{K}$ $n = 1,2,\ldots,$ and $f \in B$ be such that $f_n \xrightarrow{w^*} f$. Choose $F_n \in K$ such that $F_n = f_n$ a.e. By the Weak Compactness Theorem (Loeve, [5]) there exists a subsequence $\{n_k\}$ and a distribution function G such that $F_{n_k} \xrightarrow{w} G$. This implies G is unimodal, (Lukacs, [6]) and $F_{n_k}(t) \to G(t)$ a.e. (t). Hence $f_{n_k} \to G$ a.e. So by the Lebesgue Dominated Convergence Theorem, for all $g \in L'(R)$ we have

$$\lim_{k \to \infty} \int_R f_{n_k}(t)\, g(t)dt = \int_R G(t)\, g(t)dt.$$

And, hence for all $g \in L'(R)$ it is the case that

$$\int_R G(t)\, g(t)dt = \int_R f(t)\, g(t)dt.$$

This implies $f(t) = G(t)$ a.e. so $f \in \tilde{K}$.

Thus we have established that K is a compact set and clearly K is convex, so that the Krein-Milman Theorem applies. To establish the extreme points we introduce the following concept.

DEFINITION Let F be a unimodal distribution. F is called an

<u>elementary</u> <u>unimodal</u> <u>distribution</u> if, in addition, F satisfies the following two conditions:

(i) $F(-\infty) = 0$, $F(+\infty) = 1$

(ii) If G and $F - G$ are both unimodal distributions, then there is a real number, λ, $0 \leq \lambda \leq 1$, such that $G = \lambda F$.

THEOREM 2. The extreme points of K are the elementary unimodal distributions together with the functions $F_{-\infty} \equiv 1$ and $F_{+\infty} \equiv 0$.

PROOF: Clearly $F_{-\infty}$, $F_{+\infty}$ are extreme points of K. Let $F \in K$ be an elementary unimodal distribution. We now show F is an extremal point of K. Let $F_1, F_2 \in K$ and $0 \leq \lambda \leq 1$ be such that $F = \lambda F_1 + (1 - \lambda) F_2$. Since it is true of F, we necessarily have

$$F_1(-\infty) = 0 = F_2(-\infty)$$

and
$$F_1(+\infty) = 1 = F_2(+\infty).$$

If $0 < \lambda < 1$ then $F, \lambda F_1, F - \lambda F_1 \in K$ so since F is elementary, there exists an $\alpha \geq 0$ such that $F = \alpha F_1$. But $F(+\infty) = F_1(+\infty)$, therefore $\alpha = 1$. So $F = F_1 = F_2$ and as a result F is an extreme point of K.

Conversely, let F be an extreme point of K. If F is constant, then $F = F_{-\infty}$ or $F_{+\infty}$. So assume $F(+\infty) - F(-\infty) > 0$.

Suppose $0 < F(+\infty) < 1$. Then

$$F = F(+\infty) \left(\frac{F}{F(+\infty)}\right) + (1 - F(+\infty)) F_{+\infty}.$$

This forces $F(+\infty) = 1$. Now suppose $0 < F(-\infty) < 1$. Then $\frac{F(x) - F(-\infty)}{1 - F(-\infty)}$ and $\frac{F(x) + F(-\infty)}{1 + F(-\infty)}$ belong to K, and

$$F(x) = \frac{1 + F(-\infty)}{2} \left[\frac{F(x) + F(-\infty)}{1 + F(-\infty)}\right]$$

$$+ \frac{1 - F(-\infty)}{2} \left[\frac{F(x) - F(-\infty)}{1 - F(-\infty)}\right]$$

but if F is to be extreme, it must be that $F(-\infty) = 0$. Finally, let $G \in K$ be such that $G \neq 0$, F and $F - G \in K$.

Define $F_1(x) = \frac{G(x)}{G(+\infty)}$ and $F_2(x) = \frac{F(x) - G(x)}{1 - G(+\infty)}$ then $F(x) = G(+\infty)\ F_1(x) + (1 - G(+\infty))\ F_2(x)$ but F is extreme, hence $F = F_1$ so $\lambda F = G$ for some $\lambda > 0$. Thus F is an elementary unimodal distribution.

LEMMA 3. Let F be an extreme point of K. Then $F(0) = 0$ or 1.

PROOF: Suppose to the contrary $0 < F(0) < 1$. Define $G(x)$ to be $F(x)$ or $F(0)$ accordingly as $x < 0$ or $x \geq 0$.

Then G and $F - G$ are unimodal but G is not a multiple of F since $F(+\infty) = 1$.

THEOREM 3. The elementary unimodal distributions are the following:

$$\alpha < 0, \quad F_\alpha(x) = \begin{cases} 0 & x \leq \alpha \\ 1 - x/\alpha & \alpha \leq x \leq 0 \\ 1 & x \geq 0 \end{cases}$$

$\alpha = 0, \quad F_0(x) = \epsilon(x)$

$$\alpha > 0, \quad F_\alpha(x) = \begin{cases} 0 & x \leq 0 \\ x/\alpha & 0 \leq x \leq \alpha \\ 1 & x \geq \alpha. \end{cases}$$

PROOF: We first show each $F_\alpha(x)$, $\alpha \in R$, is an elementary unimodal distribution. Clearly F_0 is elementary. Let $\alpha < 0$ be fixed. Let $G \in K$ be such that $F_\alpha - G$ also belongs to K. Thus $1 - G(x)$ and $G(x)$ are both non-decreasing on $(0, +\infty)$, hence $G(x) = G(0)$ if $x \geq 0$. Let $\gamma = G(0)$. If $\gamma = 0$, $G \equiv 0$. So assume $\gamma > 0$. Claim $G = \gamma F_\alpha$. Since G and F_α are convex on $(-\infty, 0)$, there exists non-decreasing non-negative functions g and f_α such that

$$G(x) = \int_{-\infty}^{x} g(u)du \qquad x < 0$$

and
$$F_\alpha(x) = \int_{-\infty}^{x} f_\alpha(u)du \qquad x < 0.$$

Now $f_\alpha(x) = -\frac{1}{\alpha} I_{[\alpha,0]}$. Thus $(F_\alpha - G)(x) = \int_{-\infty}^{x} f_\alpha(u) - g(u)du$ and $f_\alpha - g$ is non-decreasing and non-negative. Hence $g(u) = c\, I_{[\alpha,0]}$. Thus

$$\gamma = G(0) = \int_{-\infty}^{0} g(u)du = c|\alpha| \; .$$

Thus
$$g(u) = \gamma f_\alpha(u).$$

So $G = \gamma F_\alpha$. Similarily if $\alpha > 0$, F_α is an elementary unimodal distribution.

Thus it remains to show if F is elementary, then $F(x) = \epsilon(x)$ or a suitable uniform distribution.

Let $a = F(0-)$. If $F(0) = 0$, then $a = 0$ and F is continuous at a. If $F(0) = 1$, then $a = 0$ or 1. For if $0 < a < 1$, pick $\epsilon > 0$ such that $0 < (1 - \epsilon)\, a < (1 + \epsilon)\, a < 1$. Now define F_1 and F_2 as

$$F_1(x) = \begin{cases} (1 + \epsilon)\, F(x) & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

$$F_2(x) = \begin{cases} (1 - \epsilon)\, F(x) & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0. \end{cases}$$

then $F = \frac{1}{2} F_1 + \frac{1}{2} F_2$ which contradicts F as being extreme. So $a = 0$ or 1. If $a = 0$, $F(x) = \epsilon(x)$, otherwise F is continuous at 0. Suppose $F(0) = 1$ and $F(x) \neq \epsilon(x)$. Then there is an $f \in L'(-\infty, 0)$, $f \geq 0$ a.e., $\int_{-\infty}^{0} f(u)du = 1$ and f is non-decreasing.

Let $K_f = \text{supp } f = \overline{\cup[\alpha, 0]}$ where the union is taken over all intervals $[\alpha, 0]$ such that $f(x) > 0$ a.e. on $[\alpha, 0]$. Let $\gamma \in K_f$ be such that

$$0 < \int_{-\infty}^{\gamma} f(u)du < 1.$$

Define

$$f_1(x) = \begin{cases} f(x) & \text{if} \quad x < \gamma \\ f(\gamma) & \text{if} \quad x \geq \gamma \end{cases}$$

and

$$F_1(x) = \begin{cases} \int_{-\infty}^{x} f_1(t)dt & \text{if} \quad x \leq 0 \\ F_1(0) & \text{if} \quad x > 0 \end{cases}$$

Then F_1 is unimodal and $F - F_1$ is unimodal. Now since F is elementary, there is an $\alpha = \alpha(\gamma)$ such that

$$F_1 = \alpha F.$$

But this implies

$$f_1(x) = \alpha f(x) \quad \text{a.e.} \quad x \leq 0.$$

But if $x \leq \gamma$, $f_1(x) = f(x)$, therefore $\alpha(\gamma) = 1$ which implies if $x > \gamma$, $f(x) = f(\gamma)$. But $\gamma \in K_f$ was arbitrary, thus f is a.e. constant on K_f. Thus $K_f = [\alpha,0]$ for some α, and $f = -1/\alpha$ since $\int_{-\infty}^{0} f(u)du = 1$.

Hence F is the uniform distribution prescribed above. Proceeding in exactly a similar manner, if $F(0) = 0$, then if F is to be extreme, $F = F_\alpha$ for some $\alpha > 0$.

Collecting the above, we now formulate the following theorem.

THEOREM 4. K is a compact, convex set in the weak* topology of $L^{\infty}(R)$. Moreover the extreme points are precisely the following functions:

$$F_{-\infty}(x) \equiv 1$$

$$F_{\alpha}(x) = (1 - \frac{x}{\alpha}) I_{[\alpha,0]} + I_{(0,\infty)} \quad \alpha < 0,$$

$$F_0(x) = \epsilon(x) = I_{[0,\infty)}$$

$$F_{\alpha}(x) = \frac{x}{\alpha} I_{[0,\alpha]} + I_{(\alpha,+\infty)} \quad \alpha > 0$$

$$F_{+\infty}(x) \equiv 0.$$

Now let $[-\infty,+\infty]$ denote the compactified real line. Define Φ: $[-\infty,+\infty] \to K$ by $\Phi(\alpha) = F_{\alpha}$, then clearly Φ is a homeomorphism, so that Ext(K), the set of extreme points of K, is compact and homeomorphic to $[-\infty,+\infty]$.

So, in view of Choquet's Theorem, for each $F \in K$, then is a probability measure P on $[-\infty,+\infty]$ such that

$$F = \int_{[-\infty,+\infty]} F_{\alpha} \, dP(\alpha).$$

This integral is to be interpreted in the following sense: For every weak* continuous linear functional ℓ we have

$$\ell(F) = \int_{[-\infty,+\infty]} \ell(F_\alpha)\, dP(\alpha)$$

Consequently, for each $h \in L'(R)$, we have

$$\int_R h(t)\, F(t)\, dt = \int_{[-\infty,+\infty]} \left(\int_R h(t)\, F_\alpha(t) dt\right) dP(\alpha)$$

$$= \int_R h(t) \left(\int_{[-\infty,+\infty]} F_\alpha(t)\, dP(\alpha)\right) dt.$$

The last interchange of integrals is permitted by the Fubini Theorem. But since this holds for all $h \in L'(R)$ we have

$$F(t) = \int_{[-\infty,+\infty]} F_\alpha(t)\, dP(\alpha)$$

$$= P\{-\infty\} + \int_R F_\alpha(t)\, dP(\alpha)$$

Let us consider $\int_R F_\alpha(t)\, dP(\alpha)$. Since for fixed t, the integral is continuous in α, we can consider

$$\int_R F_\alpha(t)\, dG(\alpha) \qquad \text{where } G \text{ is given by}$$

$$G(x) = P(-\infty,x].$$

If $t = 0$,

$$F_\alpha(0) = \begin{cases} 1 & \text{if } \alpha \leq 0 \\ 0 & \text{if } \alpha > 0 \end{cases}$$

thus $F(0) = \int_R F_\alpha(0)\, dG(\alpha) = G(0)$. If $t < 0$, and $\alpha < 0$,

$$F_\alpha(t) = \begin{cases} 1 - t/\alpha & \text{if } \alpha \leq t \leq 0 \\ 0 & \text{if } t < \alpha \end{cases}, \quad F_0(t) = 0, \text{ and } F_\alpha(t) = 0 \text{ for } \alpha > 0$$

Consequently $\int_R F_\alpha(t)\, dG(\alpha) = \int_{-\infty}^{t} (1 - \frac{t}{\alpha})\, dG(\alpha)$.

$$= G(t) - t \int_{-\infty}^{t} \frac{dG(\alpha)}{\alpha} .$$

If $t > 0$, and $\alpha \leq t$, $F_\alpha(t) = 1$. And if $\alpha > t$, $F_\alpha(t) = t/\alpha$.

Thus $\int_R F_\alpha(t)\, dG(\alpha) = G(t) + t \int_{t}^{\infty} \frac{dG(\alpha)}{\alpha}$.

Collecting all this, we now state the following Theorem.

THOEREM 5. For each $F \in K$, then exists a probability distribution G on R, and real numbers λ_1, $\lambda_2 \geq 0$, $\lambda_1 + \lambda_2 \leq 1$ such that

$$F(x) = \begin{cases} \lambda_1 + \lambda_2 [G(x) - x \int_{-\infty}^{x} \frac{dG(\alpha)}{\alpha} & \text{if } x < 0 \\ \lambda_1 + \lambda_2 G(0) & \text{if } x = 0 \\ \lambda_1 + \lambda_2 [G(x) + x \int_{x}^{\infty} \frac{dG(\alpha)}{\alpha} & \text{if } x > 0 \end{cases}$$

here $\lambda_1 = F(-\infty)$, $\lambda_2 = F(+\infty) - F(-\infty)$. By ignoring these constants and assuming F to be a probability distribution, we see F has the following representation:

$$F(x) = \int_{-\infty}^{x} [-\int_{-\infty}^{y} \frac{dG(\alpha)}{\alpha}]dy \qquad x < 0$$

$$= \int_{(0,x)} [\int_{y}^{\infty} \frac{dG(\alpha)}{\alpha}]dy + G(0) \quad x > 0.$$

By using this last form and integration by parts, it is easy to establish that if φ is the characteristic function of F and ξ is the characteristic function of G, then for all $t \in R$,

$$\varphi(t) = \frac{1}{t} \int_{0}^{t} \xi(u)du.$$

So by the Uniqueness Theorem (Lukacs [6]), we immediately conclude the probability distribution G occurring in Theorem 5 is also unique.

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