HYPERCYCLIC ALGEBRAS AND AFFINE DYNAMICS

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ABSTRACT

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An operator T on a Fréchet space X is said to be *hypercyclic* if it has a dense orbit. In that case, the set $HC(T)$ of hypercyclic vectors for T is a dense G_δ subset of X. In most cases the set $HC(T) \cup \{0\}$ is not a vector space. However, Herrero and Bourdon showed that if T is hypercyclic then $HC(T)$ contains a hypercyclic manifold, that is a dense linear subspace of X except for the origin. In a different direction, a great amount of research has been carried out in the search of hypercyclic subspaces, that is infinite dimensional closed subspaces contained (excluding the origin) in $HC(T)$. It is not always the case that a hypercyclic operator has a hypercyclic subspace. For instance, Rolewicz's operator on ℓ^2 does not have a hypercyclic subspace, but on the other hand all hypercyclic convolution operators on the space $H(\mathbb{C})$ of entire functions have hypercyclic subspaces.

If the space X is a Fréchet algebra, continuing the search for structure in $HC(T)$ one may ask whether $HC(T) \cup \{0\}$ contains an algebra. In that direction, Aron, Conejero, Peris and Seoane-Sepúlveda showed that the translation operators on $H(\mathbb{C})$ do not support a hypercyclic algebra. On the other hand, Shkarin and independently Bayart and Matheron showed that the complex differentiation operator D on $H(\mathbb{C})$ has a hypercyclic algebra.

In the present dissertation we first continue the search for hypercyclic algebras in the setting of convolution operators on $H(\mathbb{C})$. Following Bayart and Matheron's techniques, we extend their above mentioned result with Shkarin, by establishing that $P(D)$ supports a hypercyclic algebra whenever P is a non-constant polynomial vanishing at 0.

With a different approach we provide a geometric condition on the set $\{z : |\Phi(z)| \leq 1\}$ which ensures the existence of hypercyclic algebras for $\Phi(D)$ with $\Phi \in H(\mathbb{C})$ of exponential type. This new approach not only recovers the result of Shkarin-Bayart and Matheron but also gives hypercyclic algebras for convolution operators $\Phi(D)$ which do not satisfy the conditions $\Phi(0) = 0$ or that Φ be a polynomial, such as $I + D, De^D, e^D - 1$, or $cos(D)$.

Answering a question of Seoanne-Sepúlveda, we show that the operator D supports hypercyclic algebras that are not singly generated. We next consider hypercyclic algebras beyond the setting of convolution operators. For instance, we provide abstract criteria for the existence of hypercyclic algebras, which in a sense generalize familiar results from Linear Dynamics. We also show that every hypercyclic weighted backward shift operator on ℓ^2 supports a hypercyclic algebra.

Finally, on a completely different direction we study the dynamic behavior of affine maps, that is, maps of the form $A = T + a$ where T is a linear map and a is a vector of the underlying space. We prove that in many cases the dynamic behavior of A is identical to that of its linear part T. We also show that if A is hypercyclic then T has to be hypercyclic as well. The converse is not true however by an example due to Shkarin, who provided a hypercyclic operator T on ℓ^2 and a specific $a \in \ell^2$ such that $A = T + a$ is not hypercyclic. Furthermore, we generalize several results from linear dynamics to the affine setting, as well as discuss some open questions and provide partial answers to those.

To my beloved wife Eleni.

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PREFACE

Chapter 1 lists basic definitions, standard notation and some fundamental results from topological dynamics and linear dynamics that are used in this dissertation. The reader familiar with linear dynamics is prompted to skip Chapter 1 and refer to it on his/her own will.

Chapter 2 is completely devoted to the search of hypercyclic algebras for convolution operators. In the first part we prove that convolution operators induced by polynomials vanishing at the origin support hypercyclic algebras. In the second part we provide a sufficient condition for a convolution operator (not necessarily induced by a polynomial) to support a hypercyclic algebra, and then we apply it to several examples of both polynomial and transcedental convolution operators.

Chapter 3 deals with more examples of hypercyclic algebras, like two generated hypercyclic algebras for the differentiation operator, and hypercyclic algebras for weighted shifts. We also provide general criteria for the existence of hypercyclic algebras which resemble classical criteria from the main theory of Linear Dynamics.

Finally, in Chapter 4 we consider the dynamics of affine maps. We relate some results from the theory of Linear Dynamics to this new setting, seeking to understand the interplay of the dynamics of an affine map and that of its linear part.

CHAPTER 1 PRELIMINARIES

1.1 **Topological Dynamics.**

If X is a metric space, and $T : X \to X$ is a continuous map, then we can consider the sequence of iterates $\{I, T, T^2, T^3, \ldots)$ of T acting on X. Our aim is to study the dynamical properties of this sequence, which is essentially to understand its asymptotic behavior. We will sometimes refer to the pair (X, T) as a "dynamical system".

In many cases, the less interesting for us, we have a regular behavior of the sequence of iterates. We are mostly interested in cases were the sequence behaves "wildly", incorporating a notion of chaos.

In the following definition we let the sequence of iterates act on a specific point of the space.

Definition. 1.1. *Let X be a metric space, and $T : X \to X$ a continuous map. For an element $x \in X$ we define its orbit under T to be the set*

$$Orb(x, T) = \{x, Tx, T^2x, \ldots\}.$$

It is obvious from the definition, that $Orb(x, T)$ is the smallest T-invariant subset of X containing x.

The next definition will turn out to be of great importance for what follows. It provides a way to compare dynamical systems, and therefore, to transfer results from a known dynamical system to an unknown one.

Definition. 1.2. *Let X and Y be metric spaces, and $T : X \to X$, $S : Y \to Y$ be continuous maps. Then we say that T is quasiconjugate to S, if there exists a continuous map $\phi : Y \to X$ with dense range, such that*

$$T \circ \phi = \phi \circ S$$

In addition, if ϕ happens to be a homeomorphism, then we say that T and S are conjugate.

As we will see, conjugate maps, have identical dynamical behavior.

Definition. 1.3. *Let X be a metric space and $T : X \to X$ continuous. Then T is said to be topologically transitive, if for any pair U, V of nonempty open sets, there exists $n \in \mathbb{Z}_+$, such that $T^n(U) \cap V \neq \emptyset$.*

The following result relates topological transitivity of a dynamical system, to the property of having a dense orbit.

Theorem 1.4 (Birkhoff Transitivity Theorem [15])**.** *Let T be a continuous map on the separable, complete metric space X without isolated points. Then the following are equivalent:*

1. *T is topologically transitive,*
2. *T has a dense orbit.*

If one of the above holds, then the set of all points $x \in X$ for which $Orb(x, T)$ is dense, is a dense G_δ-set.

The final part of Birkhoff's theorem is indicative of a "0-1" behavior. Either there will be no point $x \in X$ with dense orbit, or there are going to be plenty of them.

On the other extreme of a point with dense orbit is a periodic point. The orbit of a periodic point behaves very smoothly and except for trivial cases it is nowhere dense in the space.

Definition. 1.5. *Let X be a metric space, and $T : X \to X$ a continuous map. A point $x \in X$ is called a periodic point of T, if there exists $n \in \mathbb{Z}_+$ such that $T^n x = x$.*

There are several nonequivalent definitions of chaos. One of the most widely used is due to Devaney and it is stated as follows.

Definition. 1.6. *A continuous self-map T on a metric space X is called chaotic, if it is topologically transitive and it has a dense set of periodic points.*

In fact, Devaney's original definition of chaos included a third condition, the so called "sensitive dependence on initial conditions", which provides the dynamical system with a "butterfly effect" behavior. However, it was proved by Banks-Brooks-Cairns-Davis-Stacey [6], that sensitive dependence on initial conditions was implied by topological transitivity together with the existence of a dense set of periodic points. Thus, the definition above is equivalent to Devaney's.

We also notice, that if X is complete and has no isolated points, the map T is chaotic if it has a dense set of points having dense orbit, and a dense set of periodic points.

The following property is a strong form of topological transitivity. It demands that the orbit of a nonempty open set U under T, to eventually intersect every open set V.

Definition. 1.7. *If X is a metric space and $T : X \to X$ is continuous, then T is said to be mixing, if for any pair U, V of nonempty open sets, there is an $N \in \mathbb{Z}_+$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$.*

Definition. 1.8. *Let X, Y be metric spaces and $T : X \to X$, $S : Y \to Y$ be continuous maps. We define the map $T \times S$ on $X \times Y$ endowed with the product topology, by*

$$(T \times S)(x, y) = (Tx, Sy).$$

Proposition. 1.9. *If T is a continuous map on a metric space X, then if $T \times T$ is topologically transitive, T is also topologically transitive. T is mixing if and only if $T \times T$ is mixing.*

Definition. 1.10. *If T is a continuous map on the metric space X, it is called weakly mixing if $T \times T$ is topologically transitive.*

By Proposition 10, the property of weak mixing is stronger than topological transitivity, but weaker than the mixing property. Weak mixing is equivalent to the condition that for any 4-tuple U_1, U_2, V_1, V_2 of nonempty open sets, there exists an $n \in \mathbb{N}$ such that $T^n(U_1) \cap V_1 \neq \emptyset$, and $T^n(U_2) \cap V_2 \neq \emptyset$.

We close this first section with the observation that conjugate dynamical systems share the same dynamical properties.

Proposition. 1.11. *The properties of topological transitivity, weak mixing, mixing and chaoticity are preserved under quasi conjugacy. This means that if the map S has any of the above properties, and T is quasi conjugate to S, then T satisfies also the same property.*

1.2 **Linear Dynamics.**

1.2.1 **Introduction.**

From now on, we will restrict our interest to the case were X is a vector space with some suitable metric, and $T : X \to X$ is a continuous linear operator. Let us begin with some preliminary definitions.

Definition. 1.12. *Let X be a vector space over $\mathbb{K}$. A functional $p : X \to \mathbb{R}_+$ is called a seminorm if*

1. $p(x+y) \leq p(x) + p(y)$, and

2. $p(\lambda x) = |\lambda| p(x)$,

for all $x, y \in X$ and $\lambda \in \mathbb{K}$.

On a vector space X if a sequence of seminorms $\{p_n\}_{n\in\mathbb{N}}$ is defined, then we can consider the family of sets $\{y \in X : p_n(x-y) < \epsilon\}$, for $x \in X, n \in \mathbb{N}$ and $\epsilon > 0$, as a subbase for a topology on X. This is called the topology induced by the sequence of seminorms $\{p_n\}_{n=1}^{\infty}$. If moreover, $\{p_n\}_{n=1}^{\infty}$ is separating, which means $\cap_{n=1}^{\infty} p_n^{-1}\{0\} = \{0\}$, then the formula

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, p_n(x-y)) \tag{1.2.1}$$

defines a metric on X. This metric is also translation invariant, which means that

$$d(x+z, y+z) = d(x, y),$$

for all $x, y, z \in X$.

Definition. 1.13. *A Fréchet space is a vector space X, endowed with a separating sequence of seminorms $\{p_n\}_{n=1}^{\infty}$, which is complete in the metric given by (1).*

Definition. 1.14. *Let X and Y be Fréchet spaces. Then a continuous linear map $T : X \to Y$ is called an operator. The set of all operators from X to Y is denoted by $L(X, Y)$, and the set of all operators from X to X by $L(X)$.*

If X and Y are Fréchet spaces whose topologies are induced by the sequences of seminorms $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$, then we define $X \oplus Y$ to be the vector space $X \times Y$ endowed with the sequence of seminorms $r_n(x, y) = p_n(x) + q_n(y)$, for $n \in \mathbb{N}, x \in X, y \in Y$. The sequence $\{r_n\}_{n=1}^{\infty}$ induces the product topology on $X \times Y$.

Definition. 1.15. *Let X and Y be Fréchet spaces and $T \in L(X), S \in L(Y)$. Then we define the operator $T \oplus S \in L(X \oplus Y)$ by,*

$$T \oplus S(x, y) = (T \times S)(x, y) = (Tx, Sy),$$

for $(x, y) \in X \oplus Y$.

1.2.2 **Basic Definitions and Results.**

Now we are ready to transfer the theory developed in the first section, to this new linear setting.

Definition. 1.16. *Let X be a Fréchet space, and $T \in L(X)$. Then T is called hypercyclic if there exists $x \in X$ such that $orb(x, T)$ is dense in X. In that case x is called a hypercyclic vector for T, and the set of all hypercyclic vectors of T is denoted by $HC(T)$.*

Birkhoff's Transitivity Theorem 1.4 can be restated using the new terminology.

Theorem 1.17 (Birkhoff Transitivity Theorem)**.** *Let T be an operator on the Fréchet space X. Then T is hypercyclic if and only if it is topologically transitive. In this case, the set $HC(T)$, is a dense G_δ-set.*

Therefore, for a hypercyclic operator T, the set $HC(T)$ of hypercyclic vectors of T is a dense G_δ-set. Concerning its algebraic structure, we have the following simple application of Baire's Category Theorem.

Theorem 1.18. *If T is a hypercyclic operator on the Fréchet space X, then*

$$X = HC(T) + HC(T)$$

which means that every vector $x \in X$ can be written as the sum of two hypercyclic vectors.

Kitai [27] noticed that when T is hypercyclic on a complex Banach space its adjoint T^* cannot have eigenvalues. More generally, we have the following.

Theorem 1.19 (Herrero[26], Bourdon[16], Bès[10], Wengenroth[38])**.** *If T is a hypercyclic operator on a real or complex Fréchet space X and $x \in X$ is a hypercyclic vector for T, then*

$$\{p(T)x : p \text{ is a polynomial } \} \setminus \{0\}$$

is a dense set of hypercyclic vectors. In particular, every hypercyclic operator admits a dense linear subspace consisting, except for 0, of hypercyclic vectors.

As an immediate corollary of Theorem 1.19 we get that $HC(T)$ is connected. However, as Bayart and Matherón noticed [7, Theorem 1.33] one can say much more about the topological structure of $HC(T)$.

Theorem 1.20. *Let X be a separable Fréchet space and $T \in L(X)$ be hypercyclic. Then $HC(T)$ is homeomorphic to X.*

Therefore, we can conclude that the set $HC(T)$, for a hypercyclic operator T, is a connected, dense G_δ-set, containing a dense linear subspace consisting except from 0, from hypercyclic vectors, and satisfying that every $x \in X$ can be written as the sum of two hypercyclic vectors.

The following theorem says that hypercyclicity is a purely infinite dimensional property.

Theorem 1.21 (Rolewicz [33])**.** *There are no hypercyclic operators on a finite dimensional Fréchet space.*

In the remaining part of this subsection, we present some necessary conditions for an operator to be hypercyclic.

We recall that for a Fréchet space X its dual space X^* is defined to be the vector space of all continuous linear functionals on X. If $T : X \to X$ is an operator, then its dual operator $T^* : X^* \to X^*$, is defined by

$$T^*(f)(x) = f(Tx),\ f \in X^*,\ x \in X.$$

Theorem 1.22. *If T is a hypercyclic operator on the Fréchet space X, then its adjoint $T^* : X^* \to X^*$ has no eigenvalues.*

We recall the following definition.

Definition. 1.23. *Let T be an operator on the Banach space X. Then T is called power bounded if*

$$\sup_{n\geq 0} ||T^n|| < \infty.$$

Since every orbit of a power bounded operator is bounded we get the following.

Theorem 1.24. *If T is a power bounded operator on the Banach space X, then T fails to be hypercyclic.*

Compact operators play a central role in Banach space theory. Their dynamic properties though, are rather poor.

Definition. 1.25. *If X is a Banach space, the operator T on X is compact, if the image of the closed unit ball of X, has compact closure.*

Theorem 1.26. *No hypercyclic compact operator exists on a Banach space.*

1.2.3 **Criteria for Hypercyclicity.**

Besides the definition, the only tool we mentioned to prove hypercyclicity until now is Birkhoff's Transitivity Theorem. In this subsection we provide several sufficient conditions that ensure hypercyclicity. Most of them, are refinements of the following criterion due to Kitai.

Theorem 1.27 (Kitai's Criterion [27])**.** *Let T be an operator on the Fréchet space X. If there exist dense subsets X_0 and Y_0 of X and a map $S : Y_0 \to Y_0$ such that*

1. $T^n x \to 0, \forall x \in X_0$,
2. $S^n y \to 0, \forall y \in Y_0$, *and*
3. $TSy = y, \forall y \in Y_0$.

Then T is mixing.

It turns out that we can substitute the full sequence (n) for the exponents of iterates of T and S in Kitai's criterion by some subsequence (n_k). In this case though, we no longer have the mixing property as a result.

Theorem 1.28 (Gethner-Shapiro [21])**.** *Let X be a Fréchet space and T an operator on X. Suppose there exist dense subsets X_0, Y_0 of X a subsequence (n_k) of (n), and a map $S : Y_0 \to Y_0$ such that,*

1. $T^{n_k} x \to 0, \forall x \in X_0$,
2. $S^{n_k} y \to 0, \forall y \in Y_0$, *and*
3. $TSy = y, \forall y \in Y_0$.

Then T is weakly mixing.

A slight modification of the Gethner-Shapiro criterion due to J. Bès [9] is the following.

Theorem 1.29 (Hypercyclicity Criterion)**.** *Let T be an operator on the Fréchet space X. If there exist dense subsets X_0, Y_0 of X, a subsequence (n_k) of (n), and a sequence of maps $S_{n_k} : Y_0 \to X$ satisfying the following,* *1.* $T^{n_k}x \to 0, \forall x \in X_0$*,*

2. $S_{n_k}y \to 0, \forall y \in Y_0$*, and*

3. $TS_{n_k}y \to y, \forall y \in Y_0$*.*

Then T is weakly mixing.

The next result says that the Hypercyclicity Criterion in fact characterizes weak mixing.

Theorem 1.30 (Bès-Peris [13])**.** *Let T be an operator on the Fréchet space X. Then T satisfies the Hypercyclicity Criterion if and only if T is weakly mixing.*

1.2.4 **Examples.**

After having presented all the fundamental notions and results, we give several examples of hypercyclic operators. It turns out that hypercyclicity occurs quite often, and that many familiar operators are hypercyclic.

If we consider the space $H(\mathbb{C})$ of all entire functions, endowed with the increasing sequence of seminorms

$$p_n(f) = sup_{z \in B(0,n)} |f(z)|,$$

then $H(\mathbb{C})$ is turned into a Fréchet space. The topology induced by the above sequence of seminorms is the topology of uniform convergence on compact subsets. The following example is due to Birkhoff and it is an application of Birkhoff's Transitivity Theorem.

Example. 1.31 (Birkhoff [14])**.** *On the space $H(\mathbb{C})$, we consider the translation operator,*

$$\tau_a f(z) = f(z+a), a \neq 0.$$

Then τ_a is hypercyclic.

Example. 1.32 (MacLane [29])**.** *The differentiation operator*

$$D : f \to f'$$

on $H(\mathbb{C})$, is hypercyclic.

Example. 1.33 (Rolewicz [33])**.** *On the space $X = l^p, p \in [1, \infty)$ or $X = c_0$, we consider the operator*

$$T = \lambda B,\ (x_1, x_2, x_3, \ldots) \mapsto \lambda(x_2, x_3, x_4, \ldots)$$

where B is the backward shift. Then T is hypercyclic precisely when $|\lambda| > 1$.

It turns out that Birkhoff's, MacLane's and Rolewicz's operators are both mixing and chaotic. The following class of operators extends Example 1.33.

Example. 1.34 (Unilateral Weighted Backward Shifts, Salas [34])**.**

On the sequence space $X = l^p, p \in [1, \infty)$, or $X = c_0$, we consider the weighted (unilateral) backward shift,

$$B_w(x_1, x_2, \ldots) = (w_2 x_2, w_3 x_3, \ldots)$$

where the sequence $w = (w_n)$ is a bounded sequence of non zero scalars. Then the following hold,

1. B_w is hypercyclic if and only if $\sup_{n\geq 1} \prod_{k=1}^{n} |w_k| = \infty$,

2. B_w is mixing if and only if $\lim_{n\to\infty} \prod_{k=1}^{n} |w_k| = \infty$.

3. B_w is chaotic if and only if $\sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} |w_k|^p} < \infty$.

Example. 1.35 (Perturbation of the identity by a weighted backward shift, Salas [34])**.** *Let X be the space $l^p, 1 \leq p < \infty$ or c_0, and B_w be the weighted backward shift with weight sequence $w = (w_n)_{n\in\mathbb{N}}$, such that $\sup_{n\geq 0} |w_n| < \infty$. Then the operator $I + B_w$ is mixing.*

We note that Example 1.35 provides compact perturbations of the identity operator that are mixing, what may seem counterintuitive given that the identity operator is far from being hypercyclic!

We recall that an entire function $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$ is of exponential type provided that there exist constants $A, C \in \mathbb{C}$ such that $|\Phi(z)| \leq C\, e^{A|z|}$, for all $z \in \mathbb{C}$. In that case the operator

$$\Phi(D) : H(\mathbb{C}) \to H(\mathbb{C}), f \mapsto \sum_{n=0}^{\infty} a_n D^n f$$

is well defined and continuous.The following result extends Example 1.31 and Example 1.32.

Theorem 1.36 (Godefroy and Shapiro [22]). *The following conditions are equivalent for an operator $L : H(\mathbb{C}) \to H(\mathbb{C})$.*

1. *$LD = DL$, where D is the differentiation operator,*
2. *$L\tau_a = \tau_a L$ for each $a \in \mathbb{C}$, where τ_a is the translation by a,*
3. *There is a complex Borel measure μ on $\mathbb{C}$ with compact support such that*

$$Lf(z) = \int f(z+w)d\mu(w), z \in \mathbb{C}.$$

4. *$L = \Phi(D)$, for some $\Phi \in H(\mathbb{C})$ of exponential type.*

Furthermore, if $\Phi \in H(\mathbb{C})$ is non-constant and of exponential type, then the operator $\Phi(D)$ is mixing and chaotic. In particular $\Phi(D)$ is hypercyclic.

The above theorem justifies the name "Convolution Operators" for the operators on $H(\mathbb{C})$ which commute with the differentiation operator D.

We already mentioned in Theorem 1.19 that every hypercyclic operator has a hypercyclic manifold, that is, a dense linear manifold consisting entirely (except the origin) of hypercyclic vectors. A different situation happens with the existence of so-called *hypercyclic subspaces*, that is, of *closed* and infinite dimensional subspaces consisting entirely (but the origin) of hypercyclic vectors. For instance, each closed subspace (except the origin) of the hypercyclic vectors for the Rolewicz operator $2B$ on l^2 is finite dimensional (see Montes [31]). Moreover, a weakly mixing operator on a complex, separable, infinite dimensional Banach space has a hypercyclic subspace if and only if its essential spectrum intersects the closed unit disc (see León Gonzales and Montes [23], see also Chan[18], Chan and Taylor [19]). On the other hand, all hypercyclic convolution operators on $H(\mathbb{C})$ support a hypercyclic subspace, thanks to the collective works of Bernal and Montes [8], Petersson [32], Shkarin [36], and Menet [30].

1.2.5 **Three major results.**

In this final section we state three central results in the field of Linear Dynamics.

For an operator T, if some power T^p is hypercyclic, we automatically get that T is also hypercyclic. The following result due to Ansari, says that also the converse is true.

Theorem 1.37 (Ansari [1])**.** *Let T be an operator on the Fréchet space X. Then for every $p \in \mathbb{N}$, $HC(T) = HC(T^p)$. In particular, T^p is hypercyclic for every $p \in \mathbb{N}$, if T is hypercyclic.*

We recall that a subspace of a metric space is somewhere dense, if the interior of its closure is nonempty. In other words, if its closure contains a nonempty open set.

Theorem 1.38 (Bourdon-Feldman [17])**.** *Let T be an operator on the Fréchet space X. If there exists $x \in X$ such that $orb(x, T)$ is somewhere dense in X, then it is dense in X.*

Let us notice, that the Bourdon-Feldman theorem is another example of a "0-1" behavior. An orbit of an operator T is going to be either nowhere dense, or (everywhere) dense.

Theorem 1.39 (León-Saavedra and Muller [28])**.** *Let T be an operator on the Fréchet space X. Then for any $\lambda \in \mathbb{K}$ such that $|\lambda| = 1$, the operators T and λT have the same hypercyclic vectors (possibly none).*

CHAPTER 2 CONVOLUTION OPERATORS SUPPORTING HYPERCYCLIC ALGEBRAS

A special task in linear dynamics is to understand the algebraic and topological properties of the set

$$HC(T) = \{f \in X : \ \{f, Tf, T^2 f, \dots\} \text{ is dense in } X\}$$

of hypercyclic vectors for a given operator T on a topological vector space X. It is well known that in general $HC(T)$ is always connected and is either empty or contains a dense infinite-dimensional linear subspace (but the origin), see [38]. Moreover, when $HC(T)$ is non-empty it sometimes contains (but zero) a closed and infinite dimensional linear subspace, but not always [8, 23]; see also [7, Ch. 8] and [25, Ch. 10].

When X is a topological algebra it is natural to ask whether $HC(T)$ can contain, or must always contain, a subalgebra (but the origin) whenever it is non-empty. Both questions have been answered by considering convolution operators on the space $X = H(\mathbb{C})$ of entire functions on the complex plane $\mathbb{C}$, endowed with the compact-open topology; that convolution operators (other than scalar multiples of the identity) are hypercyclic was established by Godefroy and Shapiro [22], see also [14, 29, 5], together with the fact that convolution operators on $H(\mathbb{C})$ are precisely those of the form

$$f \overset{\Phi(D)}{\mapsto} \sum_{n=0}^{\infty} a_n D^n f \quad (f \in H(\mathbb{C}))$$

where $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{C})$ is of (growth order one and finite) exponential type (i.e., $|a_n| \leq M \frac{R^n}{n!} (n = 0, 1, \dots)$, for some $M, R > 0$) and where D is the operator of complex differentiation. Aron et al [3, 4] showed that no translation operator τ_a

$$\tau_a(f)(z) = f(z+a) \ \ f \in H(\mathbb{C}), z \in \mathbb{C}$$

can support a hypercyclic algebra, in a very strong way:

Theorem 2.1. (Aron, Conejero, Peris, Seoane) *For any positive integer p and any $f \in H(\mathbb{C})$, the non-constant elements of the orbit of f^p under τ_a are those entire functions for which the multiplicities of their zeros are integer multiples of p.*

In stark contrast with the translations operators, Aron et al showed that the collection of entire functions f for which every power f^n $(n = 1, 2, \dots)$ is hypercyclic for D is residual in $H(\mathbb{C})$. Later, Shkarin [36, Th. 4.1] and with a different approach Bayart and Matheron [7, Th. 8.26] showed that D supports a hypercyclic algebra:

Theorem 2.2 (Shkarin, Bayart and Matheron)**.** *The set of entire functions that generate an algebra consisting entirely (but the origin) of hypercyclic vectors for D is residual in $H(\mathbb{C})$.*

Motivated by the work of Aron et al, Bayart and Matheron, and Shkarin, we consider the following problem:

Problem. 2.3. *Which convolution operators*

$$\Phi(D) : H(\mathbb{C}) \to H(\mathbb{C})$$

support a hypercyclic algebra?

In Section 2.1 we use the techniques of Bayart and Matheron and extend Theorem 2.2 to operators of the form $P(D)$, with P polynomial vanishing at the origin. In section 2.2 we use a new approach, and establish a geometric condition for a convolution operator to support a hypercyclic algebra. As applications, we get hypercyclic algebras for convolution operators induced by polynomials not vanishing at the origin, as well as of convolution operators induced by transcedental functions.

2.1 **The Bayart-Matheron approach**

The main result of this section is the following:

Theorem 2.4. *Let Ω be a simply connected planar domain and $H(\Omega)$ the space of holomorphic functions on Ω endowed with the compact open topology. Let Φ be a non-constant polynomial with $\Phi(0) = 0$. Then the set of functions $f \in H(\Omega)$ that generate a hypercyclic algebra for $\Phi(D)$ is residual in $H(\Omega)$.*

The proof of Theorem 2.4 follows that of [7, Th. 8.26]. We postpone the proof of Proposition 2.5 for later.

Proposition. 2.5. *Let Φ be a polynomial with $\Phi(0) = 0$. Then for each pair (U, V) of non-empty open subsets of $H(\Omega)$ and each $m \in \mathbb{N}$ there exists $P \in U$ and $q \in \mathbb{N}$ so that*

$$\begin{cases} \Phi(D)^q(P^j) = 0 \quad \textit{for } 0 \leq j < m, \\ \Phi(D)^q(P^m) \in V. \end{cases} \tag{2.1.1}$$

Proof of Theorem 2.4. For any $g \in H(\Omega)$ and $\alpha \in \mathbb{C}^m$, we let $g_\alpha := \alpha_1 g + \cdots + \alpha_m g^m$. Let $(V_k)_k$ be a countable local basis of open sets of $H(\Omega)$. For each $(k, s, m) \in \mathbb{N}^3$ we let $\mathcal{A}(k, s, m)$ denote the set of $f \in H(\Omega)$ that satisfy the following property

$$\forall \alpha \in \mathbb{C}^m \text{with } \alpha_m = 1 \text{ and } \|\alpha\|_\infty \leq s,\ \exists q \in \mathbb{N} :\ \Phi(D)^q(f_\alpha) \in V_k. \tag{2.1.2}$$

Each such $\mathcal{A}(k, s, m)$ is open and dense in $H(\Omega)$, thanks to Proposition 2.14. By Baire's Theorem,

$$\mathcal{A} = \cap_{k,s,m \in \mathbb{N}} \mathcal{A}(k, s, m)$$

is residual in $H(\Omega)$. Let $f \in \mathcal{A}$, and let g be in the algebra generated by f. Since a vector is hypercyclic if and only if any non-zero scalar multiple of it is hypercyclic, we may assume $g = f_\alpha = \alpha_1 f + \alpha_2 f^2 + \cdots + \alpha_{m-1} f^{m-1} + f^m$. Then g is clearly hypercyclic for $\Phi(D)$. Indeed, given any non-empty open set U of $H(\Omega)$, let $k \in \mathbb{N}$ so that $V_k \subset U$. Pick $s > \|\alpha\|_\infty$. Then since $f \in \mathcal{A}(k,s,m)$ we know that there exists q satisfying (2.1.2). Hence

$$\Phi(D)^q g = \Phi(D)^q(\alpha_1 f + \alpha_2 f^2 + \cdots + \alpha_{m-1} f^{m-1} + f^m) \in V_k \subset U.$$

□

Proof of Proposition 2.5. Let $\Phi(z) = z^r \sum_{j=0}^{k} a_j z^j$, with $a_0 \neq 0$ and $r \in \mathbb{N}$, and let $(A, B) \in U \times V$ be polynomials. Enlarging the degree of B if necessary, we may assume that degree$(B) = p \in r\mathbb{N}$ and $p > m$. It suffices to show the following.

Claim. 2.6. *For any large $n \in r\mathbb{N}$ there exist $(c_0, \ldots, c_p) = (c_0(n), \ldots, c_p(n)) \in \mathbb{C}^{p+1}$ so that*

$$\begin{cases} R := R_n = z^n \sum_{j=0}^{p} c_j z^j \quad \textit{and} \\ q := q_n = \frac{m}{r} n + (m-1)\frac{p}{r} \end{cases}$$

satisfy the following:

(i) $\Phi(D)^q((A+R)^j) = 0$ *for* $0 \le j < m$,

(ii) $\Phi(D)^q((A+R)^m) = \Phi(D)^q(R^m) = B$,

(iii) $R_n \underset{n\to\infty}{\to} 0$ *in* $H(\Omega)$.

To show the claim, notice that for each $0 \le \ell \le m-1$ and each $s \in \mathbb{N}$ we have the inequality

$$\text{degree}(A^s R^\ell) \le \text{constant} + (m-1)n < (m-1)p + mn = rq$$

for all large n. Hence, since $\Phi(D)^q = (\sum_{\ell=0}^{k} a_\ell D^\ell)^q \, D^{rq}$, it follows that for large n we have

$$\begin{aligned} \Phi(D)^q \, ((A+R)^j) \; &= 0 \;\text{ for } 0 \le j < m, \text{ and} \\ \Phi(D)^q((A+R)^m) \; &= \Phi(D)^q(R^m). \end{aligned}$$

So (i) holds, as well as the first equality in (ii), regardless of the selection $c = (c_0, \dots, c_p)$. Next, for each $s \in \mathbb{N}$ and multi-index $\gamma = (\gamma_0, \dots, \gamma_s) \in \mathbb{N}_0^{s+1}$ we let

$$|\gamma| = \sum_{j=0}^{s} \gamma_j \text{ and } \widehat{\gamma} = \sum_{j=0}^{s} j\gamma_j = \sum_{j=1}^{s} j\gamma_j.$$

Also, for $c = (c_0, \dots, c_s) \in (\mathbb{C} \setminus \{0\})^{s+1}$ and $|\gamma| = m$, we let

$$c^\gamma = \prod_{j=0}^{s} c_j^{\gamma_j} \text{ and } \binom{m}{\gamma} = \frac{m!}{\gamma_0!\gamma_1! \dots \gamma_s!}.$$

With this notation we have

$$\begin{aligned}
\Phi(D)^q(R^m) &= \left(\sum_{\beta \in \mathbb{N}_0^{k+1} : |\beta| = q} \binom{q}{\beta} a^\beta \; D^{rq+\widehat{\beta}} \right) \left(\sum_{\alpha \in \mathbb{N}^{p+1} :\ |\alpha| = m} \binom{m}{\alpha} c^\alpha \; z^{nm+\widehat{\alpha}} \right) \\
&= \sum_{(\alpha,\beta)\in A} \binom{m}{\alpha} c^\alpha \binom{q}{\beta} a^\beta \; D^{rq+\widehat{\beta}} z^{nm+\widehat{\alpha}} \\
&= \sum_{(\alpha,\beta)\in A} \binom{m}{\alpha} c^\alpha \binom{q}{\beta} a^\beta \; \frac{(nm+\widehat{\alpha})!}{(\widehat{\alpha} - \widehat{\beta} - (m-1)p)!} z^{\widehat{\alpha} - \widehat{\beta} - (m-1)p}
\end{aligned}$$

where

$$\begin{aligned}
A &= \left\{ (\alpha, \beta) \in \mathbb{N}_0^{p+1} \times N_0^{k+1} :\ |\alpha| = m,\ |\beta| = q, \text{ and } rq + \widehat{\beta} \le nm + \widehat{\alpha} \right\} \\
&= \left\{ (\alpha, \beta) \in \mathbb{N}_0^{p+1} \times N_0^{k+1} :\ |\alpha| = m,\ |\beta| = q, \text{ and } mp - p \le \widehat{\alpha} - \widehat{\beta} \right\}.
\end{aligned}$$

Thus

$$\Phi(D)^q(R^m) = \sum_{i=0}^{p} \sum_{(\alpha,\beta)\in A_i} \binom{m}{\alpha} c^\alpha \binom{q}{\beta} a^\beta \; \frac{(nm+\widehat{\alpha})!}{i!} \; z^i$$

where for each $i = 0, \dots p$

$$A_i = \{(\alpha, \beta) \in A :\ \widehat{\alpha} - \widehat{\beta} = i + (m-1)p\}.$$

In particular, $B = \sum_{i=0}^{p} b_i z^i = \Phi(D)^q(R^m)$ if and only if $c = (c_0, \ldots, c_p)$ is a solution of the system

$$b_i = \sum_{(\alpha,\beta)\in A_i} \binom{m}{\alpha} c^{\alpha} \binom{q}{\beta} a^{\beta} \, \frac{(nm+\widehat{\alpha})!}{i!} \qquad (0 \le i \le p). \tag{2.1.3}$$

We finish the proof of the claim using the following remark.

Remark. 2.7. *For each fixed $0 \le \ell \le p$, the following occurs:*

(a) Each $(\alpha, \beta) \in A_\ell$ must satisfy $\alpha_0 = \cdots = \alpha_{\ell-1} = 0$. Otherwise, if $\alpha_s > 0$ with $0 \le s \le \ell - 1$ we'd have since $|\alpha| = m$ that $pm-(p-\ell) = \ell+(m-1)p \le \widehat{\alpha} - \widehat{\beta} \le \widehat{\alpha} \le s\alpha_s + p(m-\alpha_s) = pm - (p-s)\alpha_s \le pm - (p-s)$, a contradiction.

(b) If $(\alpha, \beta) \in A_\ell$ satisfies that $\alpha_\ell > 0$, then $\beta = (q, 0, \ldots, 0)$ and $\alpha_\ell = 1$, $\alpha_p = m - 1$, and $\alpha_j = 0$ for $j \neq \ell, p$. This is forced from (a) and the inequalities $pm - (p - \ell) = \widehat{\alpha} - \widehat{\beta} \le \widehat{\alpha} = \sum_{j=\ell}^{p} j\alpha_j \le \ell\alpha_\ell + p(m - \alpha_\ell) = pm - (p - \ell)\alpha_\ell$.

(c) Let $A'_\ell := A_\ell \setminus \{((0, \ldots, 0, \underbrace{1}_{\ell^{th}}, 0, \ldots, 0, m-1), (q, 0, \ldots, 0))\}$. Then from (a) and (b) each $(\alpha, \beta) \in A'_\ell$ satisfies that $\alpha_0 = \cdots = \alpha_\ell = 0$.

(d) If $\beta \in \mathbb{N}_0^{k+1}$ satisfies $|\beta| = q$ and $\widehat{\beta} \in \{0, \ldots, \ell\}$, then $\binom{q}{\beta} \le q^\ell$ and $|a^\beta| \le (max\{|a_0|, \ldots, |a_k|\})^\ell$.

Now, thanks to Remark 2.7 the system (2.1.3) is upper triangular and thus solvable, and any solution to (2.1.3) satisfies (ii) for sufficiently large n. To see (iii), it suffices to show that there exists $w > 1$ so that for each $\ell = 0, 1, \ldots, p$ we have

$$c_{p-\ell} = \mathrm{O}\left(\frac{w^n}{[(mn+mp)!]^{\frac{1}{m}}}\right) \quad \text{as } n \to \infty. \tag{2.1.4}$$

Condition (2.1.4) ensures that $R_n \underset{n\to\infty}{\to} 0$ in $H(\Omega)$ as for each $M > 0$ we have $M^{n+i}|c_i| \underset{n\to\infty}{\to} 0$.

Indeed, by (2.1.4) and Stirling's formula

$$\begin{aligned}\left(M^{n+i}|c_i|\right)^m &\leq \frac{M^{(n+i)m}w^{mn}}{(mn+mp)!}\\ &= o\left(\frac{(Mw)^{mn+mp}}{(\frac{mn+mp}{e})^{mn+mp}}\right)\\ &= O\left(\frac{eMw}{mn+mp}\right)^{mn+mp} \underset{n\to\infty}{\to} 0.\end{aligned}$$

So we finish by proving (2.1.4) by induction on ℓ. Taking $i = p$ in (2.1.3) we get -since in this case $(\alpha,\beta) \in A_p \Leftrightarrow \alpha = (0,\dots,0,m)$ and $\beta = (q,0,\dots,0)$- that

$$\begin{aligned}p!b_p &= \sum_{(\alpha,\beta)\in A_p} \binom{m}{\alpha}\binom{q}{\beta} a^\alpha c^\beta (nm+\widehat{\alpha})!\\ &= \binom{m}{(0,\dots,0,m)}\binom{q}{(q,0,\dots,0)} a_0^q\, c_p^m (nm+mp)!\end{aligned}$$

Thus

$$c_p^m = \frac{p!b_p}{a_0^q\,(nm+mp)!} \tag{2.1.5}$$

and (2.1.4) holds for $\ell = 0$. Inductively, suppose there exists $w_{\ell-1} > 1$ so that

$$c_{p-j} = \mathrm{O}\left(\frac{w_{\ell-1}^n}{[(mn+mp)!]^{\frac{1}{m}}}\right)(n\to\infty)$$

for each $j = 0,\dots,\ell-1$. We want to show that for some $w > 1$

$$c_{p-\ell} = \mathrm{O}\left(\frac{w^n}{[(mn+mp)!]^{\frac{1}{m}}}\right)(n\to\infty). \tag{2.1.6}$$

Now, taking $i = p-\ell$ in (2.1.3) we have by Remark 2.7(b) and (c) that

$$\begin{aligned}(p-\ell)!\,b_{p-\ell} &= \sum_{(\alpha,\beta)\in A_{p-\ell}} \binom{m}{\alpha}\binom{q}{\beta} a^\beta c^\alpha (nm+\widehat{\alpha})!\\ &= mc_{p-\ell}c_p^{m-1}\,(nm+mp-\ell)!\,a_0^q + K_n,\end{aligned} \tag{2.1.7}$$

where $K_n = \sum_{(\alpha,\beta)\in A'_{p-\ell}} \binom{m}{\alpha}\binom{q}{\beta} a^\beta c^\alpha (nm+\widehat{\alpha})!$. Also, thanks to (2.1.5) we have that

$$\begin{aligned} c_p^{m-1}\,(nm+mp-\ell)! &= \frac{(p!b_p)^{1-\frac{1}{m}}\,[(nm+mp)!]^{\frac{1}{m}}}{(a_0^{1-\frac{1}{m}})^q\,\prod_{j=0}^{\ell-1}(nm+mp-j)} \\ &\sim \frac{[(nm+mp)!]^{\frac{1}{m}}}{(a_0^{1-\frac{1}{m}})^q\,n^\ell}\ \ (n\to\infty). \end{aligned} \tag{2.1.8}$$

Now, let $(\alpha,\beta)\in A'_{p-\ell}$ be fixed. Notice that $(\widehat{\alpha},\widehat{\beta})\in\{(mp-j,j) : j=0,\ldots,\ell\}$, and thus by Remark 2.7(d) that $\binom{q}{\beta}\le q^\ell$ and $|a^\beta|\le\|a\|_\infty^\ell$. Moreover, thanks to Remark 2.7(c) and our inductive hypothesis we also have

$$\begin{aligned} |c^\alpha| = |c_{p-\ell+1}^{\alpha_{p-\ell+1}}\cdots c_p^{\alpha_p}| &= \mathrm{O}\left(\left(\frac{w_{\ell-1}^n}{[(nm+mp)!]^{\frac{1}{m}}}\right)^{\alpha_{p-\ell+1}+\cdots+\alpha_p}\right) \\ &= \mathrm{O}\left(\frac{w_{\ell-1}^{nm}}{(nm+mp)!}\right). \end{aligned}$$

Hence

$$\left|\sum_{(\alpha,\beta)\in A'_{p-\ell}}\binom{m}{\alpha}\binom{q}{\beta}a^\beta c^\alpha(nm+\widehat{\alpha})!\right| = \mathrm{O}\left(n^\ell w_{\ell-1}^{mn}\right)\ \ (n\to\infty). \tag{2.1.9}$$

So by (2.1.7), (2.1.8) and (2.1.9) we have

$$c_{p-\ell} = \mathrm{O}\left(\frac{n^{2\ell}w_{\ell-1}^{mn}}{[(nm+mp)!]^{\frac{1}{m}}\,a_0^{\frac{q}{m}}}\right)\ \ (n\to\infty)$$

Thus any $w > w_{\ell-1}^m a_0^{-\frac{1}{r}}$ satisfies (2.1.6), and Claim 2.6 holds. The proof of Proposition 2.5 is now complete.

$\square$

2.2 **A new approach**

Theorem 2.1,Theorem 2.2, and Theorem2.4 motivate the following question.

Question. 2.8. *Let $\Phi \in H(\mathbb{C})$ be of exponential type so that the convolution operator $\Phi(D)$ supports a hypercyclic algebra. Must Φ be a polynomial? Must $\Phi(0) = 0$?*

We answer the above questions in the negative, by establishing for example that $\Phi(D)$ supports a hypercyclic algebra when $\Phi(z) = \cos(z)$ and when $\Phi(z) = ze^z$ (Example 2.16 and Example 2.17), as well as when $\Phi(z) = (a_0 + a_1 z^n)^k$ with $|a_0| \leq 1$ and $0 \neq a_1$ and when $\Phi(z) = e^z - a$ with $0 < a \leq 1$ (Corollary 2.15 and Example 2.18). All such examples satisfy a geometric condition on the level set $\{z : |\Phi(z)| = 1\}$ that ensures $\Phi(D)$ to have a hypercyclic algebra:

Theorem 2.9. *Let $\Phi \in H(\mathbb{C})$ be of finite exponential type so that the level set $\{z : |\Phi(z)| = 1\}$ contains a non-trivial, strictly convex compact arc Γ_1 satisfying*

$$\mathit{conv}(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\}) \subseteq \Phi^{-1}(\mathbb{D}). \tag{2.2.1}$$

Then the set of entire functions that generate a hypercyclic algebra for the convolution operator $\Phi(D)$ is residual in $H(\mathbb{C})$.

Here for any $A \subset \mathbb{C}$ the symbol conv(A) denotes its convex hull. We also recall that for a planar smooth curve $\mathcal{C}$ with parametrization $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = x(t) + iy(t)$, its signed curvature at a point $P = \gamma(t_0) \in \mathcal{C}$ is given by

$$\kappa(P) := \frac{x'(t_0)y''(t_0) - y'(t_0)x''(t_0)}{|\gamma'(t_0)|^3}.$$

and its unsigned curvature at P is given by $|\kappa(P)|$. It is well-known that $|\kappa(P)|$ does not depend on the parametrization chosen, and that the signed curvature $\kappa(P)$ depends only on the choice of orientation seleted for $\mathcal{C}$. It is simple to see that any straight line segment has zero curvature. We say that $\mathcal{C}$ is *strictly convex* provided it contains no straight segment. In particular, any planar smooth curve with zero-free curvature is stricty convex. Notice also that for the particular case when $\mathcal{C}$ is given by the graph of a function $y = f(x)$, $a \le x \le b$, (and oriented from left to right), its signed curvature at a point $P = (x_0, f(x_0))$ is given by

$$\kappa(P) = \frac{y''(x_0)}{(1 + (y'(x_0))^2)^{\frac{3}{2}}}.$$

In particular, $\kappa < 0$ on $\mathcal{C}$ if and only if $y = f(x)$ is concave down (i.e., $(1-s)f(a) + sf(b) < f((1-s)a + sb)$ for any $s \in (0,1)$).

We first note the following invariant for composition operators with homothecy symbol

Lemma. 2.10. *Let $\Phi \in H(\mathbb{C})$ be of exponential type, and let $\varphi : \mathbb{C} \to \mathbb{C}$, $\varphi(z) = az$ be a homothecy on the plane, where $0 \neq a \in \mathbb{C}$. Then $\Phi_a := C_\varphi(\Phi)$ is of exponential type and*

$$C_\varphi(HC(\Phi_a(D))) = HC(\Phi(D)).$$

In particular, the algebra isomorphism $C_\varphi : H(\mathbb{C}) \to H(\mathbb{C})$ maps hypercyclic algebras of $\Phi_a(D)$ onto hypercyclic algebras of $\Phi(D)$.

Proof. For each $f \in H(\mathbb{C})$ we have $C_\varphi(f)(z) = f(az)$ $(z \in \mathbb{C})$, and thus

$$D^k C_\varphi(f) = a^k C_\varphi D^k(f) \ \ (k = 0, 1, 2, \dots).$$

Hence given $\Phi(z) = \sum_{k=0}^\infty c_k z^k$ of exponential type $\Phi_a := C_\varphi(\Phi)$ is clearly of exponential type and

$$\begin{aligned}\Phi(D)C_\varphi(f) &= \sum_{k=0}^\infty c_k D^k C_\varphi(f) = \sum_{k=0}^\infty c_k a^k C_\varphi D^k(f)\\ &= C_\varphi\left(\sum_{k=0}^\infty c_k a^k D^k\right)(f)\\ &= C_\varphi \Phi_a(D)(f). \quad (f \in H(\mathbb{C})).\end{aligned}$$

So $\Phi_a(D)$ is conjugate to $\Phi(D)$ via the algebra isomorphism C_φ. $\square$

Remark. 2.11.

1. *Lemma 2.10 is a particular case of the following Comparison Principle for Hypercyclic Algebras.* Any operator $T : X \to X$ on a Fréchet Algebra X that is quasi-conjugate via a multiplicative operator $Q : Y \to X$ to an operator $S : Y \to Y$ supporting a hypercyclic algebra must support a hypercyclic algebra. Indeed, if A is a hypercyclic algebra for S then $Q(A) = \{Qy : y \in A\}$ is a hypecyclic algebra for T.

2. *If $\Phi \in H(\mathbb{C})$ satisfies the assumptions of Theorem 2.9, then so will $\Phi_a := C_\varphi(\Phi)$ for any non-trivial homothecy $\varphi(z) = az$. Indeed, notice that for any $r > 0$ we have*

$$a\Phi_a^{-1}(r\partial\mathbb{D}) = \Phi^{-1}(r\partial\mathbb{D}).$$

   Hence if $\Gamma \subset \Phi^{-1}(r\partial\mathbb{D})$ is a smooth arc satisfying

$$\textit{conv}(\Gamma \cup \{0\}) \setminus (\Gamma \cup \{0\}) \subset \Phi^{-1}(r\mathbb{D}),$$

   then $\Gamma_a := \frac{1}{a}\Gamma \subset \Phi_a^{-1}(r\partial\mathbb{D})$ is a smooth arc satisfying

$$\textit{conv}(\Gamma_a \cup \{0\}) \setminus (\Gamma_a \cup \{0\}) \subset \Phi_a^{-1}(r\mathbb{D}).$$

   Moreover, if Γ is a strictly convex, compact, simple and non-closed arc whose convex hull does not contain the origin, say, then Γ_a will share each corresponding property as these are invariant under homothecies. In particular, the angle difference between the endpoints of Γ is the same as the corresponding quantity in Γ_a.

We make use of the following three results. The first one ellaborates on the geometric assumption of Theorem 2.9. By $\mathbb{D}$ and $\partial\mathbb{D}$ we respectively denote the open unit disc centered at 0 and its boundary, and by $\text{Arg}(z)$ the Principal argument of a non-zero scalar z.

Proposition. 2.12. *Let $\Phi \in H(\mathbb{C})$ and let $\Gamma \subset \Phi^{-1}(r\partial\mathbb{D})$ be a simple, strictly convex arc with endpoints z_1, z_2 satisfying $0 < Arg(z_1) < Arg(z_2) < \pi$ and $Re(z_1) \neq Re(z_2)$, where $r > 0$. Suppose that $0 \notin conv(\Gamma)$ and that*

$$\Omega := conv(\Gamma \cup \{0\}) \setminus (\Gamma \cup \{0\}) \subset \Phi^{-1}(r\mathbb{D}). \tag{2.2.2}$$

Then $S(0, z_1, z_2) \setminus \Gamma$ consists of two connected components of which Ω is the bounded one, where

$$S(0, z_1, z_2) = \{0 \neq w \in \mathbb{C} : \ Arg(z_1) \leq Arg(w) \leq Arg(z_2)\}.$$

Moreover,

$$\Omega = \{tz : \ (t, z) \in (0, 1) \times \Gamma\} = \{tz : \ (t, z) \in (0, 1) \times conv(\Gamma)\},$$

and $\partial\Omega = [0, z_1) \cup (0, z_2) \cup \Gamma$. In addition,

$$\Gamma \cap ([a, b] \times (0, \infty)) = Graph(f) \cup \{z_1, z_2\}$$

for some smooth function $f : I \to \mathbb{R}$, where I is the closed interval with endpoints $Re(z_1)$ and $Re(z_2)\}$ and where f is concave up if $Re(z_1) < Re(z_2)$ and concave down if $Re(z_2) < Re(z_1)$.

Proof. Since $|\Phi| \leq r$ on $\mathrm{conv}(\Gamma \cup \{0\})$, by (2.2.2) and the Maximum Modulus Principle we have

$$\Gamma \cap \mathrm{int}(\mathrm{conv}(\Gamma \cup \{0\})) = \emptyset. \tag{2.2.3}$$

We claim that

$$\Gamma \subset \{0 \neq w \in \mathbb{C} : \ \mathrm{Arg}(w) \in [\mathrm{Arg}(z_1), \mathrm{Arg}(z_2)]\}. \tag{2.2.4}$$

To see this, notice that since $0 \notin \mathrm{conv}(\Gamma)$ the arc Γ cannot intersect the ray $\{te^{i(\mathrm{Arg}(z_2)+\pi)} : t \geq 0\}$, and by (2.2.3) it cannot intersect the interior of the triangle $\mathrm{conv}\{0, z_1, z_2\}$, either. Also, notice that if H denotes the open half-plane not containing 0 and with boundary

$$\partial H = \{z_1 + t(z_2 - z_1) : \ t \in \mathbb{R}\},$$

then

$$\emptyset = \Gamma \cap H \cap \{0 \neq w \in \mathbb{C} : \ \text{Arg}(w) < \text{Arg}(z_1)\}, \tag{2.2.5}$$

as any $z \in \Gamma \cap H$ with $\text{Arg}(z) < \text{Arg}(z_1)$ would make $z_1 \in \text{int}(\text{conv}(\{z, z_2, 0\})$, contradicting (2.2.3). Finally, since Γ is simple it now follows from (2.2.5) that

$$\emptyset = \Gamma \cap \{0 \neq w \in \mathbb{C} : \ \text{Arg}(w) \in [\pi + \text{Arg}(z_2), 2\pi) \cup [0, \text{Arg}(z_1))\},$$

and thus any $w \in \Gamma$ satisfies $\text{Arg}(z_1) \leq \text{Arg}(w)$. By a similar argument, each $w \in \Gamma$ satisfies $\text{Arg}(w) \leq \text{Arg}(z_2)$, and (2.2.4) holds. Next, using (2.2.3) and the continuity of the argument on $S(0, z_1, z_2)$ it is simple now to see that for each $\theta \in [\text{Arg}(z_1), \text{Arg}(z_2)]$ the ray

$$\{te^{i\theta} : \ t \geq 0\}$$

intersects Γ at exactly one point, giving the desired description for Γ. For the final statement, assume $\text{Re}(z_2) < \text{Re}(z_1)$ (the case $\text{Re}(z_1) < \text{Re}(z_2)$ follows with a similar argument).

Notice that for each $x = t\text{Re}(z_2) + (1-t)\text{Re}(z_1)$ with $0 < t < 1$ there exists a unique $y \in \mathbb{R}$ so that

$$(x, y) \in \Gamma \ \text{ with } \ y \in [t\text{Im}(z_2) + (1-t)\text{Im}(z_1), \infty). \tag{2.2.6}$$

Indeed, the continuous path Γ from z_1 to z_2 lies in $S(0, z_1, z_2)$ and only meets the closed triangle $\text{conv}(\{0, z_1, z_2\})$ at z_1 and z_2, so the existence of a y veryfing (2.2.6) follows (it also follows for the cases $t = 0, 1$, in which case there may exist up to two values per endpoint, by (2.2.4)). To see the uniqueness, if $y_2 > y_1 > t\text{Im}(z_2) + (1-t)\text{Im}(z_1)$ with $(x, y_1), (x, y_2) \in \Gamma$, then

$$(x, y_1) \in \text{int}(\text{conv}(\{z_1, z_2, x + iy_2\}) \cap \Gamma \subset \Omega \cap \Gamma = \emptyset,$$

a contradiction. Hence (2.2.6) defines a smooth function $f : [\text{Re}(z_1), \text{Re}(z_2)] \to (0, \infty)$ whose graph Γ_0 is a subarc of Γ, provided that if at either endpoint $x \in \{\text{Re}(z_1), \text{Re}(z_2)\}$ there are two values y satisfying $x + iy \in \Gamma$ we let $f(x)$ be the largest of such two values. $\square$

The next lemma is used to apply the forthcoming Proposition 2.14.

Lemma. 2.13. *Let $\Phi \in H(\mathbb{C})$ be of exponential type supporting a a non-trivial, strictly convex compact arc Γ_1 contained in $\Phi^{-1}(\partial\mathbb{D})$ so that*

$$\textit{conv}(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\}) \subseteq \Phi^{-1}(\mathbb{D}).$$

Then for each $m \in \mathbb{N}$ there exist $r > 1$, a strictly convex smooth arc $\Gamma \subset \Phi^{-1}(r\partial\mathbb{D})$ and $\epsilon > 0$ so that

$$\textit{conv}(\Gamma \cup \{0\}) \setminus \Gamma) \subseteq \Phi^{-1}(r\mathbb{D}). \tag{2.2.7}$$

and

$$\Lambda + \sum_{k=1}^{j} \frac{1}{m}\Gamma \subset \Omega \quad \textit{and} \quad \sum_{k=1}^{j} \frac{1}{m}\Gamma \subset \Omega \ \textit{ for each } 1 \le j < m, \tag{2.2.8}$$

where

$$\Omega := \textit{conv}(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\})$$

$$\Lambda := \Omega \cap D(0, \epsilon) \cap \textit{conv}(\Gamma \cup \{0\}).$$

Proof. Since Γ_1 is strictly convex, replacing it by a subarc if necessary we may further assume by Remark 2.11(2) that Γ_1 is simple and with endpoints z_1, z_2 satisfying $0 < \text{Arg}(z_1) < \text{Arg}(z_2) < \pi$ and $\text{Re}(z_2) < \text{Re}(z_1)$ and so that $0 \notin \text{conv}(\Gamma_1)$. By Proposition 2.12,

$$\Omega = \{tz : \ (t, z) \in (0, 1) \times \text{conv}(\Gamma_1)\} \subset S(0, z_1, z_2), \tag{2.2.9}$$

with $\partial\Omega = [0, z_1) \cup \Gamma_1 \cup (0, z_2)$ and we may assume Γ_1 is the graph of a concave down function $f : [\text{Re}(z_2), \text{Re}(z_1)] \to (0, \infty)$ (i.e., replacing z_j by $z_j' = \text{Re}(z_j) + if(\text{Re}(z_j))$ $(j = 1, 2)$ if necessary). Now, pick $z_0 \in \Gamma_1 \setminus \{z_1, z_2\}$ with $\Phi'(z_0) \neq 0$, and let $w_0 := \Phi(z_0) = e^{i\theta_0}$, where $\theta_0 \in [0, 2\pi)$. Choose $\rho > 0$ small enough so that the only solution to

$$\Phi(z) = w_0$$

in $D(z_0, \rho)$ is at $z = z_0$, and so that $D(z_0, \rho) \cap ([0, z_1] \cup [0, z_1']) = \emptyset$. Next, pick

$$0 < s < \min\{|\Phi(z) - w_0| : \ |z - z_0| = \rho\}$$

and let $0 < \delta < \min\{1, s\}$ so that the polar rectangle

$$R_\delta := \{z = re^{i\theta} : \ (r, \theta) \in [1 - \delta, 1 + \delta] \times [\theta_0 - \delta, \theta_0 + \delta]\}$$

is contained in $D(w_0, s)$. Then

$$g : R_\delta \to D(z_0, \rho), \ g(w) = \frac{1}{2\pi i} \int\limits_{|z - z_0| = \rho} \frac{z\Phi'(z)}{\Phi(z) - w} dz$$

defines a univalent holomorphic function satisfying that

$$\Phi \circ g = \text{identity on } R_\delta, \tag{2.2.10}$$

see e. g. [20, p. 283]. So $W := g(R_\delta)$ is a connected compact neighborhood of z_0, and Φ maps W biholomorphically onto R_δ. Hence for each $1 - \delta \leq r \leq 1 + \delta$

$$\eta_r := g(R_\delta \cap r\partial\mathbb{D})$$

is a smooth arc contained in $W \cap \Phi^{-1}(r\partial\mathbb{D})$. In particular, $\eta_1 = W \cap \Gamma_1$ is a strictly convex subarc of Γ_1. Next, notice that since

$$W \cap \Omega \quad \text{and} \quad W \cap \text{Ext}(\Omega)$$

are the two connected components of $g(R_\delta \setminus \partial\mathbb{D}) = W \setminus \eta_1$ and $\Omega \subseteq \Phi^{-1}(\mathbb{D})$, by (2.2.10) the homeomorphism $g : R_\delta \setminus \partial\mathbb{D} \to W \setminus \eta_1$ must satisfy

$$g(R_\delta \cap \text{Ext}(\mathbb{D})) = W \cap \text{Ext}(\Omega)$$
$$g(R_\delta \cap \mathbb{D}) = W \cap \Omega.$$

Hence

$$W \cap \overline{\text{Ext}(\Omega)} = \bigcup_{1 \leq r \leq 1+\delta} \eta_r$$

and g induces a smooth homotopy among the curves $\{\eta_r\}_{1 \leq r \leq 1+\delta}$. Namely, each η_r $(1 \leq r \leq 1-\delta)$ has the Cartesian parametrization

$$\eta_r : \begin{cases} X(r,t) \\ Y(r,t) \end{cases} \qquad \theta_0 - \delta \leq t \leq \theta_0 + \delta,$$

where $X, Y : [1-\delta, 1+\delta] \times [\theta_0 - \delta, \theta_0 + \delta] \to \mathbb{R}$ are given by

$$X(r,t) := \text{Re}(g)(re^{it})$$
$$Y(r,t) := \text{Im}(g)(re^{it}).$$

Recall that given a point $P = g(re^{i\theta})$ in W, the (signed) curvature $\kappa^{\eta_r}(P)$ of η_r at P is given by

$$\kappa^{\eta_r}(P) = \frac{\frac{\partial X}{\partial t}(r,\theta)\frac{\partial^2 Y}{\partial^2 t}(r,\theta) - \frac{\partial Y}{\partial t}(r,\theta)\frac{\partial^2 X}{\partial^2 t}(r,\theta)}{\left((\frac{\partial X}{\partial t}(r,\theta))^2 + (\frac{\partial Y}{\partial t}(r,\theta))^2\right)^{\frac{3}{2}}},$$

Hence the map $K : W \to \mathbb{R}$, $K(g(re^{it})) := \kappa^{\eta_r}(P)$, is continuous. Now, since η_1 is strictly convex there exists some $P = g(e^{i\theta_1})$ in η_1 for which each of $\kappa^{\eta_1}(P)$, $\frac{\partial X}{\partial t}(1, \theta_1)$ is non-zero. Hence by the continuity of K and of $\frac{\partial X}{\partial t}$ we may find some $0 < \delta' < \delta$ so that the polar rectangle

$$R_{\delta'} := \{z = re^{i\theta} :\ (r, \theta) \in [1 - \delta', 1 + \delta'] \times [\theta_1 - \delta', \theta_1 + \delta']\}$$

is contained in the interior of R_δ and so that K and $\frac{\partial X}{\partial t}$ are bounded away from zero on $g(R_{\delta'})$ and on $R_{\delta'}$, respectively.

In particular, either $\frac{\partial X}{\partial t} > 0$ or $\frac{\partial X}{\partial t} < 0$ on $R_{\delta'}$, and either $K > 0$ or $K < 0$ on $g(R_{\delta'})$. So each $\eta_r \cap g(R_{\delta'})$ $(1 \leq r < 1 + \delta')$ is the graph of a smooth function

$$f_r : (a_r, b_r) \to (0, \infty),$$

with

$$(a_r, b_r) = \begin{cases} (X(r, \theta_1 - \delta'), X(r, \theta_1 + \delta')) & \text{if } \frac{\partial X}{\partial t} > 0 \text{ on } R_{\delta'} \\ (X(r, \theta_1 + \delta'), X(r, \theta_1 - \delta')) & \text{if } \frac{\partial X}{\partial t} < 0 \text{ on } R_{\delta'}. \end{cases}$$

Since $g(re^{it}) \underset{r \to 1}{\to} g(e^{it})$ uniformly on $t \in [\theta_1 - \delta, \theta_1 + \delta]$, there exists $0 < \delta'' < \delta'$ so that

$$\sup_{1 \leq r \leq 1+\delta''} a_r = a < b = \inf_{1 \leq r \leq 1+\delta''} b_r$$

and thus each

$$\eta'_r = \{(x, f_r(x));\ x \in [a, b]\}$$

is a subarc of η_r. In particular, $f_1 = f$ on $[a, b]$ must be a concave down function, and so must be each f_r with $1 \leq r \leq 1 + \delta''$. Thus choosing $r > 1$ close enough to 1 the arc $\Gamma := \eta'_r$ satisfies

$$\text{conv}(\Gamma \cup \{0\}) \setminus (\Gamma \cup \{0\}) \subset \Phi^{-1}(r\mathbb{D})$$

and

$$\sum_{k=1}^{j} \frac{1}{m} \Gamma \subset \Omega \ \text{ for } j = 1, \ldots, m - 1.$$

By the compactness of Γ we may now get $\epsilon > 0$ small enough so that the subsector

$$\Lambda := \Omega \cap D(0,\epsilon) \cap \text{conv}(\Gamma \cup \{0\})$$

satisfies that

$$\Lambda + \sum_{k=1}^{j} \frac{1}{m}\Gamma \subset \Omega \text{ for } j = 1, \ldots, m-1,$$

and Lemma 2.13 holds. $\square$

Finally, we recall the following key ingredient used to establish Theorem 2.9.

Proposition. 2.14. (Bayart-Matheron [7, Th. 8.26]) *Let T be an operator on a (real or complex) topological vector algebra X that is a Baire space and so that each triple (U, V, W) of non-empty open subsets of X with $0 \in W$ and each $m \in \mathbb{N}$ there exists $P \in U$ and $q \in \mathbb{N}$ so that*

$$\begin{cases} T^q(P^j) \in W & \textit{for } 0 \leq j < m, \\ T^q(P^m) \in V. \end{cases} \tag{2.2.11}$$

Then T supports a hypercyclic algebra. Indeed, the set of f in X that generate an algebra contained in $HC(T) \cup \{0\}$ is residual in X.

We are ready now to prove the main result of this section.

Proof of Theorem 2.9. Let U, V and W be non-empty open subsets of $H(\mathbb{C})$, with $0 \in W$, and let $1 < m \in \mathbb{N}$ be fixed. By Proposition 2.14, it suffices to find some $f \in U$ and $q \in \mathbb{N}$ so that

$$\begin{aligned} \Phi(D)^q(f^j) &\in W \quad \text{for } j = 1, \ldots, m-1, \\ \Phi(D)^q(f^m) &\in V. \end{aligned} \tag{2.2.12}$$

Now, let $r > 1$, let $\Gamma \subset \Phi^{-1}(r\partial\mathbb{D})$ and let Ω and the subsector Λ be given by Lemma 2.13. Since Γ and Λ have accumulation points in $\mathbb{C}$ there exist $(a_k, b_k, \lambda_k, \gamma_k) \in \mathbb{C} \times \mathbb{C} \times \Lambda \times \Gamma$ $(k =$

$1, \ldots, p)$ so that

$$(A, B) := (\sum_{k=1}^{p} a_k e^{\frac{\lambda_k z}{m}}, \sum_{k=1}^{p} b_k e^{\gamma_k z}) \in U \times V.$$

Next, set $R = R_q = \sum_{k=1}^{p} c_k e^{\frac{\gamma_k z}{m}}$, where for each $1 \leq k \leq p$ the scalar $c_k = c_k(q)$ is some solution of

$$z^m - \frac{b_k}{(\Phi(\gamma_k))^q} = 0.$$

Notice that for any $k = 1, \ldots, p$ we have $|\Phi(\gamma_k)| = r > 1$ and thus $|c_k|^m = \frac{|b_k|}{|\Phi(\gamma_k)|^q} \underset{q \to \infty}{\to} 0$. So

$$R = R_q \underset{q \to \infty}{\to} 0. \tag{2.2.13}$$

For $1 \leq j \leq m$ we have

$$(A + R)^j = \sum_{\ell = (u,v) \in \mathcal{L}_j} \binom{j}{\ell} a^u \, c^v \, e^{(\frac{u \cdot \lambda + v \cdot \gamma}{m})z}$$

where $\mathcal{L}_j$ conists of those multiindexes $\ell = (u, v) \in \mathbb{N}_0^p \times \mathbb{N}_0^p$ satisfying $|\ell| := |u| + |v| = \sum_{k=1}^{p} u_k + \sum_{k=1}^{p} v_k = j$ and where for each $\ell = (u, v) \in \mathcal{L}_j$

$$a^u := a_1^{u_1} \, a_2^{u_2} \cdots a_p^{u_p},$$
$$c^v := c_1^{v_1} \, c_2^{v_2} \cdots c_p^{v_p}, \text{ and}$$
$$\binom{j}{\ell} = \frac{j!}{u_1! \cdots u_p! v_1! \cdots v_p!}.$$

So for $1 \leq j \leq m$ we have

$$\Phi(D)^q((A + R)^j) = \sum_{\ell \in \mathcal{L}_j} U_{j,\ell},$$

where

$$\begin{aligned} U_{j,\ell} &= \binom{j}{\ell} a^u \, c^v \, \left(\Phi(\frac{u \cdot \lambda + v \cdot \gamma}{m})\right)^q e^{(\frac{u \cdot \lambda + v \cdot \gamma}{m})z} \\ &= \binom{j}{\ell} a^u \, b^{\frac{v}{m}} \, \left(\frac{\Phi(\frac{u \cdot \lambda + v \cdot \gamma}{m})}{\prod_{k=1}^{p} \Phi(\gamma_k)^{\frac{v_k}{m}}}\right)^q e^{(\frac{u \cdot \lambda + v \cdot \gamma}{m})z}. \end{aligned}$$

Now, notice that if $\{e_1,\dots,e_p\}$ denotes the standard basis of $\mathbb{C}^p$, our selections of $(c_1,\dots,c_p)$ ensure that

$$\Phi^q(D)((A+R)^m)-B=\sum_{\ell\in\mathcal{L}_m^*}U_{m,\ell}, \tag{2.2.14}$$

where

$$\mathcal{L}_m^*=\{\ell=(u,v)\in\mathcal{L}_m:\ |u|=0 \text{ and } \ v\notin\{me_1,\dots,me_p\}\}.$$

Also, for each $1\le j\le m$ and $\ell=(u,v)\in\mathcal{L}_j$ with $|v|<m$ we have

$$U_{j,\ell}\underset{q\to\infty}{\to}0,$$

as our selections of Λ and Γ give by (2.2.8) that $\frac{u\cdot\lambda+v\cdot\gamma}{m}\in\Omega$ and thus

$$|\Phi(\frac{u\cdot\lambda+v\cdot\gamma}{m})|<1<r=|\Phi(\gamma_1)|=\cdots=|\Phi(\gamma_p)|.$$

Hence since each $\mathcal{L}_j$ is finite we have

$$\Phi(D)^q((A+R_q)^j)\underset{q\to\infty}{\to}0\quad(1\le j<m). \tag{2.2.15}$$

Finally, recall that by Lemma 2.13 we have

$$\text{conv}(\Gamma_r)\setminus\Gamma_r\subseteq\Phi^{-1}(r\mathbb{D}).$$

Hence if $\ell=(u,v)\in\mathcal{L}_m^*$ with $|v|=m$ (so $\|v\|_\infty<m$ and $u=0$) we also have that $U_{m,\ell}\underset{q\to\infty}{\to}0$, as

$$|\Phi(\frac{u\cdot\lambda+v\cdot\gamma}{m})|=|\Phi(\frac{v\cdot\gamma}{m})|<r=|\Phi(\gamma_1)|^{\frac{v_1}{m}}\dots|\Phi(\gamma_p)|^{\frac{v_p}{m}}.$$

Thus

$$\Phi^q(D)((A+R_q)^m)\underset{q\to\infty}{\to}B,$$

and (2.2.12) follows by (2.2.13) and (2.2.15). $\square$

Theorem 2.9 gives another extension of Theorem 2.2 and complements Theorem 2.4

Corollary. 2.15. *Let* $P(z) = (a_0 + a_1 z^k)^n$ *with* $|a_0| \leq 1$*,* $a_1 \neq 0$*, and* $k, n \in \mathbb{N}$*. Then* $P(D)$ *supports a hypercyclic algebra on* $H(\mathbb{C})$*.*

Proof. Notice first that $Q_1(z) = a_0 + z^k$ satisfies the assumptions of Theorem 2.9, and hence so does $Q_2(z) = a_0 + a_1 z^k$, by Remark 2.11. The conclusion now follows by a result due to Ansari [1] that the set of hypercyclic vectors for an operator T coincides with the corresponding set of hypercyclic vectors for any given iterate T^n $(n \in \mathbb{N})$. □

We may also apply Theorem 2.9 to non-polynomials.

Example. 2.16. *The operator* $\cos(D)$ *supports a hypercyclic algebra on* $H(\mathbb{C})$*. Indeed,* $\Phi(z) = \cos(z)$ *is of exponential type, and*

$$1 \geq |\Phi(z)|^2 = |\cos(z)|^2 = \cos^2(x) + \sinh^2(y) \ \ (z = x + iy, x, y \in \mathbb{R}).$$

So $\Gamma_1 = \{(x, f(x)) : \ 0 \leq x \leq \pi\} \subset \Phi^{-1}(\partial\mathbb{D})$ *for the smooth function* $f : [0, \pi] \to [0, \infty)$*,* $f(x) = \sinh^{-1}(\sin(x))$*, which is concave down since it's second derivative* $f''(x) = \frac{-2\sin(x)}{(1+\sin^2(x))^{\frac{3}{2}}}$ *is negative on* $(0, \pi)$*. Now*

$$\textit{conv}(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\})$$

is the region bounded by the graph of f *and the* $x - axis$*, on which* $|\Phi| < 1$*, and the conclusion follows by Theorem 2.9.*

The next two examples should be contrasted with [3, Corollary 2.4].

Example. 2.17. *The operator* $T = D\tau_1 = De^D$ *on* $H(\mathbb{C})$*, where* τ_1 *is the translation operator* $g(z) \mapsto g(z+1), g \in H(\mathbb{C})$ *supports a hypercyclic algebra.*

Let $\Phi(z) = ze^z$*. Clearly* Φ *is of exponential type, so we may check the conditions of Theorem 2.9. Writing* $z = x + iy$ *we get*

$$|f(z)| = 1 \Leftrightarrow y^2 = e^{-2x} - x^2 \tag{2.2.16}$$

The above equation has solutions provided the function $\phi(x) = e^{-2x} - x^2$ *satisfies that* $\phi(x) \geq 0$*. By doing some elementary calculus, one shows that* ϕ *is strictly decreasing on* $\mathbb{R}$ *and has a unique solution say* $r \in (0, 1)$*. Thus the graph of the function*

$$h(x) = \sqrt{e^{-2x} - x^2}, x \in (-\infty, r]$$

lies in $f^{-1}(\partial\mathbb{D})$*. Taking derivatives, we get that* $h' < 0$ *and* $h'' < 0$ *on* $(0, r)$*, thus* h *is strictly decreasing and concave down on* $[0, r]$*. Furthermore, it is evident that the sector*

$$S = \{z = x + iy \in \mathbb{C} : 0 \leq x < r, 0 \leq y < h(x)\}$$

lies in $f^{-1}(\mathbb{D})$*. Thus, the strictly convex arc*

$$\Gamma_1 = \{z = x + iy \in \mathbb{C} : 0 \leq x \leq r, y = h(x)\}$$

satisfies the conditions of Theorem 2.9, which guarantees the existence of a hypercyclic algebra for the operator $f(D)$*.*

Example. 2.18. *For each* $0 < a \leq 1$*, the operator* $T = \tau_1 - aI = e^D - aI$ *supports a hypercyclic algebra. To see this, we will show that the exponential type function* $\Phi(z) = e^z - a, z \in \mathbb{C}$ *satisfies the assumptions of Theorem 2.9. If* $z = x + iy$ *then an easy calculation shows that*

$$|\Phi(z)| \leq 1 \Leftrightarrow e^{2x} - 2a \cos y e^x + a^2 - 1 \leq 0. \tag{2.2.17}$$

If we restrict $y \in [0, \frac{\pi}{4}]$ *inequality 2.2.17 has solution* $x \leq ln(a \cos y + \sqrt{1 - a^2 sin^2 y})$*. Hence, setting*

$$\Gamma_1 = \{z = x + iy \in \mathbb{C} : 0 \leq y \leq \frac{\pi}{4}, x = ln(a \cos y + \sqrt{1 - a^2 sin^2 y})\}$$

we get that $\Gamma_1 \subset \Phi^{-1}(\partial\mathbb{D})$, *and that*

$$\{z = x + iy \in \mathbb{C} : 0 \leq y \leq \frac{\pi}{4}, x < ln(a\cos y + \sqrt{1 - a^2 sin^2 y})\} \subset \Phi^{-1}(\mathbb{D}).$$

Moreover, since $0 < a \leq 1$ *and* $0 \leq y \leq \frac{\pi}{4}$, *it follows that*

$$x = ln(a\cos y + \sqrt{1 - a^2 sin^2 y}) > 0$$

and that

$$\frac{dx}{dy} = -\frac{a\sin y}{\sqrt{1 - a^2 sin^2 y}} < 0,$$
$$\frac{d^2x}{dy^2} = -\frac{a\cos y}{(1 - a^2 sin^2 y)^{3/2}} < 0.$$

Hence, the function $x = ln(a\cos y + \sqrt{1 - a^2 sin^2 y}), y \in [0, \frac{\pi}{4}]$ *is positive, decreasing and concave down. It follows that* $conv(\Gamma_1) \setminus \Gamma_1 \subset \Phi^{-1}(\mathbb{D})$ *and hence, by Theorem 2.9 that* $\Phi(D)$ *has a hypercyclic algebra as claimed.*

Since any quasi-conjugacy induced by a linear and multiplicative map preserves algebras as well as hypercyclic vectors, we conclude in Corollary 2.20 below the existence of hypercyclic algebras on spaces of smooth functions on the real line $C^\infty(\mathbb{R}, \mathbb{C})$. We first need the following remark.

Remark. 2.19. *(Godefroy and Shapiro) The restriction operator*

$$\begin{aligned} \mathcal{R} : & H(\mathbb{C}^N) \to C^\infty(\mathbb{R}^N, \mathbb{C}) \\ f = & f(z_1, \ldots, z_N) \mapsto f(x_1, \ldots, x_N) \end{aligned}$$

is continuous, of dense range, and multiplicative, and for any complex polynomial $P = P(z_1, \ldots, z_N)$ *we have*

$$\mathcal{R}P(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_N}) = P(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N})\mathcal{R}.$$

Theorem 2.9, together with Remark 2.19 and Theorem 2.4 give now the following corollary.

Corollary. 2.20. *Let $P \in H(\mathbb{C})$ be either a non-constant polynomial vanishing at zero, or so that the level set $\{z : \ |\Phi(z)| = 1\}$ contains a non-trivial, strictly convex compact arc Γ_1 satisfying*

$$\textit{conv}(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\}) \subseteq P^{-1}(\mathbb{D}).$$

Then the operator $P(\frac{d}{dx})$ supports a hypercyclic algebra on $C^\infty(\mathbb{R}, \mathbb{C})$. In particular, $T = aI + b\frac{d}{dx}$ support a hypercyclic algebra on $C^\infty(\mathbb{R}, \mathbb{C})$ whenever $|a| \leq 1$ and $0 \neq b$.

We don't know whether 2.4 is fully implied by Theorem 2.9:

Question. 2.21. *Let $P \in H(\mathbb{C})$ be a non-constant polynomial vanishing at the origin. Must it support a non-trivial strictly convex compact arc $\Gamma_1 \subset \{z : \ |P(z)| = 1\}$ so that conv$(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\}) \subseteq P^{-1}(\mathbb{D})$?*

We conclude the section with the following question for future research.

Question. 2.22. *Let $\Phi(D) : H(\mathbb{C}) \to H(\mathbb{C})$ be a convolution operator* not *supporting a hypercyclic algebra.*

(i) (Aron) Can Φ be a (non-constant) polynomial?

(ii) Must $\Phi \in H(\mathbb{C})$ be either constant or of the form $\Phi(z) = e^{az}$?

CHAPTER 3 OTHER TOPICS ON HYPERCYCLIC ALGEBRAS

3.1 Two generated hypercyclic hypercyclic algebras

As we saw earlier the operator D on $H(\mathbb{C})$ supports a hypercyclic algebra. However, all the proofs of this result provide singly generated hypercyclic algebras for D. Therefore, it becomes natural to ask whether D supports multigenerated hypercyclic algebras as well. This question has a trivial positive answer since if A is a hypercyclic algebra for D, the algebra generated by any finite subset of $A \setminus \{0\}$ will also be hypercyclic. However in the case of two generated hypercyclic algebras Seoane-Sepúlveda posed the following question.

Question. 3.1. (Seoane-Sepúlveda) *Does there exist a pair of algebraically independent functions which altogether generate a hypercyclic algebra for the differentiation operator?*

Our first aim is to provide an affirmative answer to the above mentioned question. In what follows we denote the algebra generated by the entire functions $f_1, \ldots, f_n$ by $A(f_1, \ldots, f_n)$. Also, for a polynomial $P \in \mathbb{C}[z_1, \ldots z_k], k \geq 1$, by $\|P\|_\infty$ we denote the maximum of the absolute values of its coefficients.

Theorem 3.2. *The set of algebraically dependent pairs of functions is a set of first category in the space $H(\mathbb{C}) \times H(\mathbb{C})$. More precisely, the set*

$$\{(f, g) \in H(\mathbb{C}) \times H(\mathbb{C}) : \exists h \in H(\mathbb{C}), \text{ such that } A(f, g) \subset A(h)\}$$

is of first category in $H(\mathbb{C}) \times H(\mathbb{C})$.

Proof. We first introduce some notation. If P is a nonzero polynomial in $H(\mathbb{C})$, let

$$\begin{aligned} &supp(P) = \{j \in \mathbb{Z}_+ : P^{(j)}(0) \neq 0\}, \\ &M(P) = \max\{\frac{|P^{(j)}(0)|}{j!} : 0 \leq j \leq \deg(P)\} \text{ and} \\ &m(P) = \min\{\frac{|P^{(j)}(0)|}{j!} : j \in supp(P)\}. \end{aligned}$$

Set also

$$N = (H(\mathbb{C}) \times \{0\}) \cup (\{0\} \times H(\mathbb{C}))$$

which is a union of two nowhere dense subsets of $H(\mathbb{C})^2$. Now for any $m \in \mathbb{N}$ we let

$$F_m = \{P \in \mathbb{C}[z] : P(0) = 0, 1 \leq \deg(P) \leq m \text{ and } \frac{1}{m(P)}, M(P) \leq m\}$$

and we observe that F_m is a compact subset of $H(\mathbb{C})$. Finally, for $m \in \mathbb{N}$ we set

$$A_m = \{(P(h), Q(h)) \in H(\mathbb{C})^2 : h \in H(\mathbb{C}), P, Q \in F_m\}.$$

Our aim is to show that each set A_m is nowhere dense in $H(\mathbb{C})^2$. To do this we break the argument in two steps.

Step 1: Here we show that for each $m \in \mathbb{N}$, the set A_m is closed.

Let $\{(f_n, g_n)\}_{n=1}^{\infty}$ in A_m such that $(f_n, g_n) \to (f, g) \in H(\mathbb{C})^2$. This means that for each $n \in \mathbb{N}$ there exists tuple $(P_n, Q_n, h_n) \in F_m \times F_m \times H(\mathbb{C})$ such that $f_n = P_n(h_n)$ and $g_n = Q_n(h_n)$. By the compactness of F_m passing to a subsequence if necessary, we may assume that $P_n \to P$ and $Q_n \to Q$ for some $P, Q \in F_m$. Passing to a further subsequence we may also assume that $P_n^{(j)} \neq 0$ for each $j \in supp(P)$ and $n \in \mathbb{N}$.

We claim that the sequence $\{h_n\}$ is locally uniformly bounded. If not, there is $K \subset \mathbb{C}$ compact and sequence $\{z_n\}_{n=1}^{\infty} \subset K$ such that $|h_n(z_n)| \geq n$ for each $n \in \mathbb{N}$. Then, if we set $m^* =$

$\max(supp(P))$ we have

$$|f_n(z_n)| = |P_n(h_n(z_n))| = |\sum_{j\in supp(P)} \frac{P_n^{(j)}(0)}{j!} h_n^j| \geq$$
$$n^{m^*}\left(\frac{1}{m} - m \sum_{j\in supp(P)\setminus\{m^*\}} \frac{1}{n^{m^*-j}}\right) \to \infty \text{ as } n \to \infty$$

contradicting that $f_n \to f$.

By Montel's theorem passing to a further subsequence we may assume that there is $h \in H(\mathbb{C})$ such that $h_n \to h$. It is an easy exercise now to see that $P_n(h_n) \to P(h)$ and $Q_n(h_n) \to Q(h)$ hence, $f = P(h)$ and $g = Q(h)$ and thus A_m is closed.

Step 2: Here we show that for each $m \in \mathbb{N}$, $int(A_m) = \emptyset$.

If we assume that there exist nonempty open subsets of $H(\mathbb{C})$ say U, V such that $U \times V \subset A_m$, then we may pick $(P, Q) \in U \times V$ such that $\deg(P), \deg(Q) > m$ and so that $\deg(P), \deg(Q)$ are relatively prime. Then $P = P_1(h)$ and $Q = Q_1(h)$ for some $P_1, Q_1 \in F_m$, $h \in H(\mathbb{C})$. Now, h is not constant since P is not, and h does not have essential singularity at ∞ since in that case $P_1(h) = P$ would also have an essential singularity at ∞, which is not true. Hence, h is a nonconstant polynomial. Also, we have that

$$\deg P = \deg P_1 \deg h$$
$$\deg Q = \deg Q_1 \deg h.$$

By the above display we get that $\deg h = 1$ and hence that $m < \deg P = \deg P_1 \leq m$ which is a contradiction.

Having established that the sets A_m are nowhere dense for each $m \in \mathbb{N}$ we conclude the proof of the theorem as follows,

$$\{(f,g) \in H(\mathbb{C})^2 : \exists h \in H(\mathbb{C}) \text{ such that } A(f,g) \subset A(h)\} =$$
$$\{(P(h),Q(h)) \in H(\mathbb{C})^2 : h \in H(\mathbb{C}), P,Q \in \mathbb{C}[z] \text{ with } P(0) = Q(0) = 0\} =$$
$$N \cup \bigcup_{m=1}^{\infty} A_m,$$

and the set $N \cup \bigcup_{m=1}^{\infty} A_m$ is of first category in $H(\mathbb{C})^2$. $\square$

Proposition. 3.3. *Let X be a topological F-Algebra and let T be a continuos operator on X. Suppose that for each non-empty open subsets U_1, U_2, V of X, and each nonzero polynomial $P : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ which vanishes at the origin, there exist $(f_1, f_2) \in U_1 \times U_2$ and $q \in \mathbb{N}$ so that*

$$T^q P(f_1, f_2) \in V. \tag{3.1.1}$$

Then the set of elements (f,g) of $X \times X$ for which the algebra $A(f,g)$ generated by f and g is a hypecyclic algebra for T is residual in $X \times X$.

Proof. Let $(V_k)_{k\geq 1}$ be a basis for the topology of X. For any polynomial $P \in \mathbb{C}[z_1, z_2]$ of degree m we use the standard notation $P = (P_1, \dots, P_m)$ to denote its decomposition in j- homogeneous polynomials ($j = 1, \dots, m$). This means that $P = \sum_{j=1}^{m} P_j$, where $P_j(z_1, z_2) = \sum_k^j a_{k,j-k} z_1^k z_2^{j-k}$. Now, for each $(k, s, m) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ let $A(k, s, m)$ consist of those (f_1, f_2) in $X \times X$ satisfying that for each $P = (P_1, \dots, P_m) : \mathbb{C}^2 \to \mathbb{C}$ polynomial with $\|P\|_\infty \leq s$ and $\|P_m\|_\infty = 1$, there exists some $q \in \mathbb{N}$ so that

$$T^q P(f_1, f_2) \in V_k.$$

Each $A(k, s, m)$ is open and thanks to the assumption (3.1.1) it is dense in $X \times X$. Since $Y = \{(f,g) \in X \times X : \ f = 0 \text{ or } g = 0\}$ is nowhere dense in $X \times X$, by Baire the set

$$A := \bigcap_{(k,s,m)\in\mathbb{N}^3} A(k,s,m) \setminus Y$$

is residual in $X \times X$. Pick $(f_1, f_2) \in A$. If $0 \neq g \in X$ belongs to the algebra generated by f_1 and f_2, then

$$g = P(f_1, f_2)$$

for some $P = (P_1, \dots, P_m)$ with each $P_j : \mathbb{C}^2 \to \mathbb{C}$ a j-homogeneous polynomial and $\|P_m\|_\infty \neq 0$. Re-scaling g if necessary, we may assume that $\|P_m\|_\infty = 1$. Given any $k \in \mathbb{N}$, taking an integer $s > \|P\|_\infty$ we know that since $(f_1, f_2) \in A(k, s, m)$ there exists some $q \in \mathbb{N}$ for which

$$T^q g = T^q P(f_1, f_2) \in V_k.$$

Thus, $A(f_1, f_2)$ is a hypercyclic algebra for T which completes the proof. $\square$

Theorem 3.4. *The set of pairs $(f_1, f_2) \in H(\mathbb{C})^2$ such that $A(f, g)$ is a hypercyclic algebra for the operator D is residual in $H(\mathbb{C})^2$.*

Proof. Let U_1, U_2, V be non empty open subsets of $H(\mathbb{C})$, and $P : \mathbb{C}^2 \to \mathbb{C}$ be a nonzero polynomial with $P(0, 0) = 0$. If $\deg P = m, m \geq 1$, then P has the form

$$P(z_1, z_2) = \sum_{i+j=1}^{m} a_{ij} z_1^i z_2^j.$$

Set

$$g = \max\{i, j : a_{i,j} \neq 0\}.$$

If there is an i_0 such that $g = i_0$, set

$$s = \min\{j : a_{gj} \neq 0\},$$

otherwise, there will be a j_0 such that $g = j_0$ and then we set

$$s = \min\{i : a_{ig} \neq 0\}.$$

Define $K_P := a_{gs}$ if the first case holds or $K_P := a_{sg}$ if the second one does. Therefore, K_P is uniquely determined by P, $0 < K_P \leq m$, and K_P is the coefficient a_{ij} corresponding to the highest power that occurs in P accompanied by the smallest one. So if

$$P(z_1, z_2) = 2z_1^6 z_2^5 + 3z_1^7 z_2^2 + 4z_1^7 z_2^3 + 5z_2^7 + 6z_1^2 z_2^7,$$

say, then $K_P = 3$, the coefficient of $z_1^7 z_2^2$. Now, let $\Lambda = (\frac{1}{3m}, \frac{1}{2m}) \subset (0, 1)$, and notice that $\underbrace{\Lambda + \cdots + \Lambda}_{m-times} \subset \mathbb{D}$. Consider the point $r = 1 + \frac{1}{4m}$ and notice that $-r + \Lambda \subset \mathbb{D}$. Take D a disc around $-r$ such that

$$D + \underbrace{\Lambda + \cdots + \Lambda}_{l \leq m-times} \subset \mathbb{D}.$$

Define $\Gamma := \{|z| = r\} \cap D$. Since the sets Λ and Γ have accumulation points, we can find $A \in U_1$, $B \in U_2$ and $C \in V$ such that $A = \sum_{i=1}^{p} a_i e^{\lambda_i z}$, $B = \sum_{i=1}^{p} b_i e^{\lambda_i z}$, and $C = \sum_{i=1}^{p} c_i e^{\gamma_i z}$. For $1 \leq i \leq p$ and $n \in \mathbb{N}$ let $r_i = r_i(n)$ be a solution of the equation

$$r_i^g = \frac{n^s c_i}{K_P \gamma_i^n}.$$

Since $\Gamma \subset \overline{\mathbb{D}}^c$, $r_i \to 0$ as $n \to \infty$. Set

$$R_n(z) = \sum_{i=1}^{p} r_i e^{\frac{\gamma_i}{g} z}.$$

Clearly, $R_n \to 0$ as $n \to \infty$. In what follows we are going to use the following classical multinomial notation. For each $p \in \mathbb{N}$ and multi-index $\gamma = (\gamma_0, \ldots, \gamma_s) \in \mathbb{N}_0^{p+1}$ we let

$$|\gamma| = \sum_{j=0}^{p} \gamma_j.$$

Also, for $c = (c_0, \ldots, c_p) \in (\mathbb{C} \setminus \{0\})^{p+1}$ and $|\gamma| = m$, we let

$$c^\gamma = \prod_{j=0}^{p} c_j^{\gamma_j} \text{ and } \binom{m}{\gamma} = \frac{m!}{\gamma_0!\gamma_1! \ldots \gamma_p!}.$$

Finally, if $x, y \in \mathbb{C}^p$ we denote by $x \cdot y$ the usual inner product of x and y. For $n \in \mathbb{N}$ we have

$$P(A + R_n, B + \frac{1}{n}) = \sum_{i+j=1}^{m} a_{ij}(A + R_n)^i (B + \frac{1}{n})^j =$$
$$\sum_{i+j=1}^{m} \sum_{(u,v,k,\ell)\in N_{i,j}} a_{ij} \binom{i}{u, v} \binom{j}{k, \ell} \frac{a^u b^k c^{(v/g)}}{K_P^{(|v|/g)} n^{(\ell-(|v|s/g))} \gamma^{(nv/g)}} e^{((u+k)\cdot\lambda + \frac{v\cdot\gamma}{g})z},$$

where $N_{i,j} = \{(u, v, k, \ell) \in \mathbb{N}_0^p \times \mathbb{N}_0^p \times \mathbb{N}_0^p \times \mathbb{N}_0 : |u| + |v| = i, |k| + \ell = j\}$. Hence,

$$D^n P(A + R_n, B + \frac{1}{n}) = \sum_{i+j=1}^{m} a_{ij} \sum_{(u,v,k,\ell)\in N_{ij}} \binom{i}{u, v} \binom{j}{k, \ell} \frac{a^u b^k c^{(v/g)}}{K_P^{(|v|/g)} n^{(\ell-(|v|s/g))}}$$
$$\left(\frac{(u + k) \cdot \lambda + (v \cdot \gamma)/g}{\gamma^{v/g}}\right)^n e^{((u+k)\lambda+(v\cdot\gamma)/g)z}$$

We will show that $D^n P(A + R_n, B + \frac{1}{n}) \to C$ as $n \to \infty$. If $|u| \neq 0$ or $|k| \neq 0$, then by the definition of the sets Λ and Γ, $|(u + v) \cdot \lambda + \frac{v\cdot\gamma}{g}| < 1$ while $|\gamma^{(v/g)}| = r^{(|v|/g)} \geq 1$, so

$$\left(\frac{(u + k) \cdot \lambda + \frac{v\cdot\gamma}{g}}{\gamma^{(v/g)}}\right)^n \to 0 \text{ as } n \to \infty.$$

If $|u| = 0 = |k|$, and $|v| < g$ or if $|v| = g$ but $v \neq e_l, l = 1, \ldots, p$, where e_l is the standard basis vector for $\mathbb{C}^p$, then $|(u + k) \cdot \lambda + \frac{v\cdot\gamma}{g}| = |\frac{v\cdot\gamma}{g}| < r$, and $|\gamma^{(v/g)}| = r^{(|v|/g)} = r$, since now $|v| > 0$. Hence, we get again that

$$\left(\frac{(u + k) \cdot \lambda + \frac{v\cdot\gamma}{g}}{\gamma^{(v/g)}}\right)^n \to 0 \text{ as } n \to \infty.$$

Finally, if $|u| = 0 = |k|, v = e_l$ for some $l \in \{1, \ldots, p\}$, and $\ell > s$, then

$$\frac{(u+k)\cdot\lambda + \frac{v\cdot\gamma}{g}}{\gamma^{(v/g)}} = \frac{\gamma_i}{\gamma_i} = 1,$$

but $\ell - \frac{|v|s}{g} = \ell - s > 0$ hence

$$\frac{1}{n^{\ell - \frac{|v|s}{g}}} \to 0 \text{ as } n \to \infty.$$

Thus,

$$\lim_{n\to\infty} \frac{1}{n^{(\ell-(|v|s/g))}} \left(\frac{(u+k)\cdot\lambda + (v\cdot\gamma)/g}{\gamma^{v/g}}\right)^n = 0$$

whenever $i \neq g$, $j \neq s$, or $v \neq ge_l$ for any $l \in \{1, \ldots, p\}$. We conclude that

$$\lim_{n\to\infty} D^n P(A + R_n, B + \frac{1}{n}) = a_{gs} \sum_{i=1}^{p} \frac{c_i}{K_P} e^{\gamma_i z} = C$$

by the definition of K_P. Taking n large enough to ensure that $A + R_n \in U_1$, $B + \frac{1}{n} \in U_2$, and $D^n P(A+R_n, B+\frac{1}{n}) \in V$, and applying the previous proposition with $(f_1, f_2) = (A+R_n, B+\frac{1}{n})$ we get the result. $\square$

Now combining Theorem 3.2 and Theorem 3.4 we get the following corollary which provides the answer to Question 3.1

Corollary. 3.5. *The operator D on $H(\mathbb{C})$ supports a two generated hypercyclic algebra not contained in a singly generated one.*

Remark. 3.6. *Corollary 3.5 holds for operators $\Phi(D)$, where Φ is entire function of exponential type which satisfies that $0 \in \Phi^{-1}(\mathbb{D})$ and that $\{|\Phi(z)| = 1\}$ contains a nontrivial strictly convex arc Γ such that*

$$conv(\Gamma \cup \{0\}) \setminus (\Gamma \cup \{0\}) \subset \Phi^{-1}(\mathbb{D}).$$

As a result we get for example, that $D + \frac{1}{2}I$ and De^D contain a two-generated hypercyclic algebra.

We conclude this section with some questions for future research.

Questions.

1. Does D+I contain a two generated hypercyclic algebra?

2. Do Theorem 3.2 and Theorem 3.4 extend to any $n \in \mathbb{N}$? In other words, is it true that for each $2 \leq n \in \mathbb{N}$ the operator D supports an n-generated hypercyclic algebra, that is not contained in an $(n-1)$-generated hypercyclic algebra?

3. Does D support an infinitely generated hypercyclic algebra?

3.2 **Hypercyclic Algebras for weighted backward shifts**

Here we change the setting by considering $\ell^p, 1 \leq p < \infty$ the space of p-summable complex sequences endowed with the pointwise product which gives ℓ^p a Banach algebra structure. For a sequence of complex numbers $w = (w_n) \in \ell^\infty$ let B_w be the weighted backward shift operator acting on ℓ^p. Salas has proved that B_w is hypercyclic if and only if $\sup_{n\geq 1} \prod_{\nu=1}^n |w_\nu| = \infty$. In that case it is natural to ask whether B_w supports a hypercyclic algebra. We make use of the following proposition which is a more primitive version of 2.14.

Proposition. 3.7. *Let X be an F- algebra with the following property. For each U, V nonempty open subsets, and each p nonzero polynomial with $p(0) = 0$, there exist $x \in U$ and $q \in \mathbb{N}$ such that $T^q p(x) \in V$. Then there is a residual set of vectors which generate a hypercyclic algebra for T.*

Proof. Let $\{V_k\}_{k=1}^\infty$ be a basis for the topology of X. For each $(k, s, m) \in \mathbb{N}^3$ set $A(k, s, m)$ to be the set of $x \in X$ such that for each p nonzero monic polynomial with $p(0) = 0$, $\deg p \leq m$, and $\|p\|_\infty \leq s$, there exists $q \in \mathbb{N}$ such that $T^q p(x) \in V_k$. Each set $A(k, s, m)$ is open and, by the assumption dense. Hence by the Baire Category theorem

$$\bigcap_{(k,s,m)\in\mathbb{N}^3} A(k, s, m)$$

is a dense G_δ-set, and any vector $x \in \bigcap_{(k,s,m)\in\mathbb{N}^3} A(k, s, m)$ is a generator of a hypercyclic algebra for T. $\square$

Theorem 3.8. *If B_w is hypercyclic, then it supports a hypercyclic algebra.*

Proof. We prove the theorem by making use of the proposition. Let $U, V \subset \ell^p$ open and nonempty, and let $x = (x_1, \dots, x_k, 0, 0, \dots) \in U$ and $y = (y_1, \dots, y_k, 0, 0, \dots) \in V$, $(k \in \mathbb{N})$. Let $\epsilon > 0$ such that if $\|x - x_0\|_p < \epsilon$ then $x_0 \in U$. Consider $p \in \mathbb{C}[z]$ nonzero polynomial with $p(0) = 0$. Then there exists $\delta > 0$ such that whenever $|z| < \delta$, there exists z_0 with $|z_0| < \frac{\epsilon}{2k^{1/p}}$ and $p(z_0) = z$. Since $\sup_{n\geq 1} \prod_{\nu=1}^{n} |w_\nu| = \infty$, and $w \in \ell^\infty$, there exists $N \in \mathbb{N}$, $N > k$ such that for each $i = 1, \dots, k$

$$|\frac{y_i}{w_{i+1} \dots w_{N+i}}| < \delta.$$

For every $i \in \{1, \dots, k\}$ choose r_i such that

$$|r_i| < \frac{\epsilon}{2k^{1/p}}, \text{ and } p(r_i) = \frac{y_i}{w_{i+1} \dots w_{N+i}}.$$

Set $r = (0, 0, \dots, \underbrace{r_1}_{N}, r_2, \dots, r_k, 0, 0, \dots)$. Now since $\|r\|_p = \frac{\epsilon}{2}$, $x + r \in U$ and

$$B_w^{N-1} p(x + r) = B_w^{N-1} p(r) = y \in V.$$

By Proposition 3.7, the shift B_w supports a hypercyclic algebra. □

Remark. 3.9. *Several authors in the definition of the weighted shift operator, demand the weight sequence $w = (w_n)$ to consist of positive (real) numbers. In that case we may let B_w act on the space of real p-summable sequences $\ell^p, 1 \leq p < \infty$. Salas's condition on hypercyclicity of B_w remains the same but in the real case B_w can never support a hypercyclic algebra. Indeed, for any $x \in \ell^p$, its square x^2 has only nonnegative entries, and hence by the assumption on the weight sequence w, the same is true for $B_w^n x^2, n \in \mathbb{N}$. This means that x^2 fails to be a hypercyclic vector for B_w for any $x \in \ell^p$. However, the question remains open if we consider the real case but with weight sequence $w \in \mathbb{R}^{\mathbb{N}}$.*

3.3 **Criteria for the existence of hypercyclic algebras**

3.3.1 **An eigenvalue criterion for hypercyclic algebras**

Theorem 3.10. (Eigenvalue Criterion for Hypercyclic Algebras) *Let T be an operator acting on a separable commutative topological vector algebra X that is a Baire space. Suppose that for each $m \in \mathbb{N}$ there exist X_0, Y_0 subsets of X so that*

(i) Each of span(X_0) and span$\{y^m : \ y \in Y_0\}$ is dense in X,

(ii) For each $(u, v) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with $v < m$ and $1 \leq u + v \leq m$

$$(\prod_{j=1}^{u} X_0)(\prod_{j=1}^{v} Y_0) \subset \cup_{|\lambda|<1} \mathit{Ker}(T - \lambda I), \text{ and}$$

(iii) For each $y_1, \ldots, y_m \in Y_0$

$$y_1 \cdots y_m \in \begin{cases} \cup_{|\lambda|>1} \mathit{Ker}(T - \lambda I) & \text{if } y_1 = \cdots = y_m \\ \cup_{|\lambda| < \prod_{k=1}^{p} |\lambda_k|^{\frac{1}{m}}} \mathit{Ker}(T - \lambda I) & \text{otherwise,} \end{cases}$$

where λ_i, the eigenvalue of y_i^m, is in $(|\cdot| > 1)$.

Then T supports a hypercyclic algebra. Moreover, the set of vectors f in X that generate a hypercyclc algebra for T is residual in X.

Proof. Let U and V be non-empty open subsets of X. By (i), there exist $x_j \in X_0$ and $y_j \in Y_0$ and non-zero scalars a_j, b_j $(j = 1, \ldots, p)$ so that

$$x = a_1 x_1 + \cdots + a_p x_p \in U \ \text{ and } y = b_1 y_1 + \cdots + b_p y_p \in V.$$

For each $n \geq 1$ let

$$r_n := \sum_{k=1}^{p} \lambda_k^{-\frac{n}{m}} c_k y_k,$$

where for each $k = 1, \ldots, p$, the scalar c_k is a solution of $z^m = b_k$ and λ_k is the eigenvalue of y_k^m. Notice that by (iii) each λ_k has modulus larger than one, so

$$r_n \underset{n\to\infty}{\to} 0.$$

Also, for each $1 \leq j \leq m$ we have

$$(x + r_n)^j = \sum_{\ell=(u,v)\in\mathcal{L}_j} \binom{j}{\ell} a^u b^v (\prod_{k=1}^{p} \lambda_k^{-\frac{nv_k}{m}})\, x^u y^v,$$

where $\mathcal{L}_j = \{\ell = (u, v) \in \mathbb{N}_0^p \times \mathbb{N}_0^p : \ |u| + |v| = j\}$ and for each $\ell = (u, v) \in \mathcal{L}_j$

$$a^u = \prod_{k=1}^{p} a_k^{u_k},\ b^v = \prod_{k=1}^{p} b_k^{v_k},\ x^u = \prod_{k=1}^{p} x_k^{u_k},\ \text{ and } y^v = \prod_{k=1}^{p} y_k^{v_k}.$$

Now, when $j < m$ we know by (ii) that for each $\ell = (u, v) \in \mathcal{L}_j$ the vector $x^u y^v$ has eigenvalue of modulus strictly less than one, so (since $|\lambda_k| > 1$ for each k) we have

$$T^n((x + r_n)^j) \to 0 \quad \text{ for each } j < m.$$

On the other hand, we have

$$T^n((x + r_n)^m) - y = \sum_{\ell=(u,v)\in\mathcal{L}_m^*} \binom{m}{\ell} a^u b^v\, (\prod_{k=1}^{p} \lambda_k^{-\frac{nv_k}{m}})\, T^n(x^u y^v),$$

where $\mathcal{L}_m^*$ consists of those multiindexes $\ell = (u, v) \in \mathcal{L}_m$ for which $\frac{1}{m}v$ is not an element of the standard basis of $\mathbb{C}^p$. Thus the assumption (iii) ensures that

$$T^n((x + r_n)^m) \underset{n\to\infty}{\to} y.$$

The conclusion now follows by Proposition 2.14. $\square$

3.3.2 A Gethner-Shapiro type criterion for hypercyclic algebras

Proposition 2.14 by Bayart and Matheron or more generaly Proposition 3.7 are obviously a stronger version of Birkhoff's Transitivity Theorem. Similarly, the Eigenvalue Criterion presented above (Theorem 3.10) is the analogue of the Godefroy-Shapiro Criterion ([22] or see [25, Theorem 3.1]) in the setting of hypercyclic algebras. The next criterion may be viewed as an adaptation of the Hypercyclicity Criterion to ensure the existence of hypercyclic algebras.

Theorem 3.11. *Let X be a commutative, topological F- algebra and $T \in L(X)$. Suppose that for each $m \in \mathbb{N}$ there exist dense subsets $D_{m,1}$ and $D_{m,2}$ of X such that each $x \in D_{m,2}$ has an m^{th} root in X (i.e. there exists $y \in X$ such that $y^m = x$) and with the property that whenever $x_n \to 0$ with $x_n \in D_{m,2}$, then $x_n^{1/m} \to 0$ for any selection of m^{th} roots. (This condition holds for any subset of ℓ^p). Assume also that there exist functions $S_m = S : D_{m,2} \to D_{m,2}$ and subsequences $n_k(m) = n_k$ such that the following conditions hold:*

1. *$S^{n_k} y \to 0$, as $k \to \infty$, $\forall y \in D_{m,2}$,*
2. *$T^{n_k}(x^i(S^{n_k}y)^{j/m}) \to 0$ as $k \to \infty$, for any $0 < i \leq m, 0 \leq j < m$ such that $1 \leq i + j \leq m, x \in D_{m,1}, y \in D_{m,2}$,*
3. *$T^{n_k} S^{n_k} y \to y$, as $k \to \infty$, $\forall y \in D_{m,2}$.*

Then the set of generators for a hypercyclic algebra for the operator T is residual in X.

Proof. Let $m \in \mathbb{N}$ and U, V, W nonempty open subsets of X with $0 \in W$. Take $x \in U \cap D_{m,1}$ and $y \in V \cap D_{m,2}$. For each $k \in \mathbb{N}$ consider the vector

$$z_k := x + (S^{n_k} y)^{1/m}.$$

By condition 1we have that $z_k \to x$.On the other hand, for each $0 < j < m$ we have,

$$T^{n_k}(z_k^j) = \sum_{i=0}^{j} \binom{j}{i} T^{n_k} x^i (S^{n_k} y)^{j/m} \to 0,$$

by condition 2. Finally,

$$T^{n_k}(z_k^m) = \sum_{i=0}^{m} \binom{m}{i} T^{n_k} x^i (S^{n_k} y)^{\frac{m-1}{m}} \to 0$$

by conditions 2 and 3. Proposition 2.14 now yelds the existence of a residual set of generators for a hypercyclic algebra for the operator T. □

We observe that any operator satisfying the assumptions of the Eigenvalue Criterion (Theorem 3.10) also satisfy the assumptions of Theorem 3.11. We note the latter is more general by providing a second proof of Theorem 3.8.

Example. 3.12. *We consider as in the previous section the operator B_w acting on ℓ^p (complex case). For each $m \in \mathbb{N}$, set $D_{m,1} = D_{m,2} = c_{00}$, and $S_m = S = B_w^{-1}$ on c_{00},*

$$S(x_1, x_2, \dots) = (0, w_2^{-1} x_1, w_3^{-1} x_2, \dots), (x_n) \in c_{00}.$$

It is not hard to see that since $\sup_{n\geq 1} \prod_{\nu=1}^{n} |w_\nu| = \infty$ and $w \in \ell^\infty$, there is subsequence $(n_k) \subset \mathbb{N}$ such that

$$S^{n_k} y \to 0, \forall y \in c_{00}.$$

Therefore, Condition 1 of Theorem 7 holds, and Condition 3 is automatic by the choice of S. For any $x, y \in c_{00}$ and $n \in \mathbb{N}$ large enough we have $x^i (S^n y)^{j/m} = 0$, hence

$$B_w(x^i (S^{n_k} y)^{j/m}) = 0, \textit{ for any } n \textit{ large enough},$$

and condition 2 is also established.

According to the previous example any hypercyclic weighted backward shift satisfies Theorem 3.8 . The next example shows however, that there are hypercyclic weighted backward shifts which do not satisfy the Eigenvalue Criterion.

Example. 3.13. *Let* $B_w : \ell^1 \to \ell^1$ *be the weighted backward shift with weight sequence* $w = (w_n)_{n=1}^{\infty}$ *defined inductively as follows.*

Let $n_1 = 1$ *and* $n_1' = 3$*. Then for each* $k \geq 2$ *integer find* $n_k > n_{k-1}'$ *such that*

$$2^{n_1 - n_1' + n_2 - n_2' + \cdots + n_{k-1} - n_{k-1}' + n_k} \geq k,$$

and $n_k' > n_k$ *such that*

$$2^{n_1 - n_1' + n_2 - n_2' + \cdots + n_k - n_k'} \leq \frac{1}{(n_k + k)^{n_k + k}}.$$

Set $N_k = n_k + k, k \in \mathbb{N}$*. Now take*

$$w = (1, 2, 2^{-3}, \underbrace{2, \ldots, 2}_{n_2}, \underbrace{2^{-n_2'}}_{w_{N_2+1}}, \underbrace{2, \ldots, 2}_{n_3}, \underbrace{2^{-n_3'}}_{w_{N_3+1}}, \ldots).$$

Clearly, $\|w\|_\infty = 2$ *and* $\prod_{\nu=1}^{N_k} w_\nu \geq k \to \infty$ *as* $k \to \infty$*, hence* B_w *is hypercyclic.*

For a $\lambda \in \mathbb{C}$ *to be an eigenvalue for* B_w *it is necessary and sufficient to exist* $x = (x_n)_{n=1}^{\infty} \in \ell^1$ *nonzero such that*

$$B_w(x) = \lambda x \Leftrightarrow x_{k+1} = \lambda w_2^{-1} x_k = \lambda^k w_2^{-1} \ldots w_{k+1}^{-1} x_1, k \geq 1.$$

Therefore $x = x_1(1, \lambda w_2^{-1}, \lambda^2 w_2^{-1} w_3^{-1}, \ldots)$*, where* $x_1 \in \mathbb{C} \setminus \{0\}$ *is arbitrary. Now,* $x \in \ell^1$ *if and only if*

$$\sum_{n=2}^{\infty} |\lambda|^n w_2^{-1} \ldots w_{n+1}^{-1} < \infty.$$

But $|\lambda|^{N_k} w_2^{-1} \ldots w_{N_k+1}^{-1} \geq |\lambda|^{N_k} N_k^{N_k} \to \infty$ *as* $k \to \infty$*, whenever* $\lambda \neq 0$*. We conclude that* $\lambda = 0$ *is the only eigenvalue for* B_w *and the corresponding eigenspace is* $Ker(B_w)$*. The later one though is a closed proper subspace of* ℓ^1 *and thus* B_w *can not satisfy the Eigenvalue Criterion. However, as marked before,* B_w *being hypercyclic satisfies Theorem 3.8.*

CHAPTER 4 AFFINE DYNAMICS

4.1 Affine Dynamics

As we know from geometry an *affine map* on a vector space is the composition of a linear map with a translation. In this section we study the dynamical behavior of affine maps, and try to relate it with the dynamical behavior of their linear part. Essentially every theorem or question from linear dynamics, has a corresponding statement in affine dynamics simply by replacing linear operators by affine maps. Therefore, it becomes natural to see to what extend can one generalize the results from linear dynamics to the setting of affine maps.

Definition. 4.1. *Let X be a Fréchet space and $A : X \to X$ a map on X. We will say that A is an affine map if there exist $T \in L(X)$ and $a \in X$ such that*

$$A(x) = T(x) + a, \forall x \in X.$$

Sometimes we will use the notation A_a for the affine map $A = T + a$.

It is obvious from the above definition that the affine map A_a is not linear except from the trivial case when $a = 0$. For simplicity though, we are going to adopt the notation and terminology from linear dynamics. For the rest of the section unless otherwise specified, X will be a separable Fréchet space.

Definition. 4.2. *An affine map A on X is said to be hypercyclic provided that there exists a vector $x \in X$ such that its orbit under A*

$$Orb(x, A) = \{x, Ax, A^2x, \dots\},$$

is dense in X. In such case x is called a hypercyclic vector for A. Also we denote the set of all hypercyclic vectors for A by $HC(A)$.

Since the space X is separable and has no isolated points, Birkhoff's Transitivity Theorem says that the affine map A is hypercyclic if and only if it is topologically transitive. In that case the set $HC(A)$ will be a dense G_δ subset of X.

Now let as before $A = T + a$ be an affine map on X and take an $x \in X$. For $n \in \mathbb{Z}_+$ we have

$$A^n x = T^n x + \sum_{i=0}^{n-1} T^i a. \tag{4.1.1}$$

The above relationship shows that the dynamical behavior of A depends partly on the dynamical behavior of its linear part T, as well as on the behavior of the sequence of operators $\{\sum_{i=0}^{n} T^i\}_{n=0}^{\infty}$. Therefore, it seems that the presence of the summation in the right hand side of 4.1.1 could increase the chaotic behavior of the system, potentially lower it (in the case of cancellations), or play no role (if the family $\{\sum_{i=0}^{n} T^i\}_{n=0}^{\infty}$ is stable enough). However, the next result by Shkarin [35] shows that the dynamics of T and $\{\sum_{i=0}^{n} T^i\}_{n=0}^{\infty}$ are closely related.

Proposition. 4.3 (Shkarin)**.** *Let X be a Fréchet space and $T \in L(X)$. Then T is hypercyclic if and only if the sequence $\{\sum_{i=0}^{n} T^i\}_{n=0}^{\infty}$ is universal and moreover,*

$$(I - T)HC(T) \subset \mathcal{U}(\{\sum_{i=0}^{n} T^i\}_{n=0}^{\infty}) \subset HC(T).$$

The next proposition, shows that for many a's the dynamical behavior of A_a is equivalent to that of its linear part T.

Proposition. 4.4. *If $a \in (I - T)(X)$ then $A_a = T + a$ and T are conjugate via an affine map. Specifically, if $c \in (I - T)^{-1}(a)$ and $\phi : X \to X$ is given by $\phi(x) = x + c$, then $A_a \circ \phi = \phi \circ T$. In particular,*

$$HC(A_a) = HC(T) + c.$$

Proof. If $c \in X$ such that $c - Tc = a$ then for $x \in X$ we have, $A \circ \phi(x) = A(x + c) = T(x + c) + a = Tx + c = T \circ \phi(x)$. $\square$

Therefore, if I-T is onto then A_a is always hypercyclic, provided T is. But even if $I - T$ is not onto, assuming hypercyclicity for T, we get that $(I-T)(X)$ is dense in X, so there is a dense set of indices a, for which A_a is hypercyclic. Actually we can say much more as the next result indicates.

Proposition. 4.5. *If $T \in L(X)$ is hypercyclic, then the set*

$$\mathcal{A} = \{a \in X : A_a = T + a \textit{ is hypercyclic}\} \tag{4.1.2}$$

is a dense G_δ subset of X. Indeed, $\mathcal{A}$ is homeomorphic to X.

Proof. If $\{U_k\}_{k\in\mathbb{N}}$ is a basis for the topology of X, we define

$$\mathcal{U}_{i,j} = \{a \in X : \exists n \in \mathbb{N}, A_a^n(U_i) \cap U_j \neq \emptyset\}.$$

$\mathcal{U}_{i,j}$ is open. Indeed, let $a \in \mathcal{U}_{i,j}$, then there exists $n \in \mathbb{N}$ such that $A_a^n(U_i) \cap U_j \neq \emptyset$, so, there is an $x \in U_i$ such that $A_a^n x \in U_j$. Since the map

$$a \mapsto A_a^n x = T^n x + \sum_{i=0}^{n-1} T^i a$$

is continuous, there is a neighbourhood W of a in X, such that for the above $n \in \mathbb{N}$ and $x \in X$, we have $A_z^n x \in U_j, \forall z \in W$, which means that $W \subset \mathcal{U}_{i,j}$.

$\mathcal{U}_{i,j}$ is also dense. Since for $a \in (I-T)(X)$, A_a is hypercyclic hence topological transitive, we get that $(I-T)(X) \subset \mathcal{U}_{i,j}$ hence $\mathcal{U}_{i,j}$ is dense.

Now Baire category theorem completes the proof since

$$\mathcal{A} = \bigcap_{i,j=1}^{\infty} \mathcal{U}_{i,j}$$

is a G_δ set containing the dense linear subspace $(I-T)(X)$ (see [7, p. 17]). $\square$

By Proposition 4.5, it becomes natural to ask whether the set $\mathcal{A}$ defined in 4.1.2 for a hypercyclic operator T is always all of X. Shkarin [37] constructed a hypercyclic operator T on a Hilbert space, such that A_a is not hypercyclic for some choice of a, proving that $\mathcal{A}$ could fail to be the full space.

Example. 4.6 (Shkarin)**.** *Let $B_w \in L(\ell^2)$ the weighted backward shift with weight sequence $w = (e^{-2n})$, and $a = (\frac{1}{n+1}) \in \ell^2$. Then the operator $T = I + B_w$ is hypercyclic, while the affine map $A_a = T + a$ is not.*

Remark. 4.7. *Although Shkarin's example shows that for a hypercyclic $T \in L(X)$ the set $\mathcal{A} = \{a \in X : A_a = T + a \text{ is hypercyclic}\}$ can fail to be X, Proposition 4.5 still ensures in those cases the existence of vectors $a \in \mathcal{A} \setminus (I - T)(X)$. Whenever $I - T$ is not surjective, $(I - T)(X)$ is a proper subspace of X and hence of first category in X while $\mathcal{A}$ is residual in X.*

The next proposition provides a necessary condition for an affine map to be hypercyclic. That is, the adjoint of its linear part must have empty point spectrum. This generalizes a classical result from linear dynamics.

Proposition. 4.8. *Let $T \in L(X)$ such that T^* has an eigenvalue. Then for any $a \in X$, the affine map $A_a = T + a$ is not hypercyclic.*

Proof. By the assumption, there is a $\lambda \in \mathbb{K}$, and a $y^* \in X^*, y^* \neq 0$, such that

$$T^* y^* = \lambda y^*.$$

Let $a \in X$, and suppose there is an $x \in X$ such that $Orb(A_a, x)$ is dense in X. If $\lambda \neq 1$, for each

$n \in \mathbb{N}$ we get

$$\begin{aligned}
&< A_a^n x, y^* >=< T^n x + \sum_{i=0}^{n-1} T^i a, y^* >= \\
&=< T^n x, y^* > + < \sum_{i=0}^{n-1} T^i a, y^* >= \\
&=< x, (T^*)^n y^* > + \sum_{i=0}^{n-1} < a, (T^*)^i y^* >= \\
&= \lambda^n < x, y^* > + < a, y^* > \sum_{i=0}^{n-1} \lambda^i = \\
&= \lambda^n < x, y^* > + < a, y^* > \frac{\lambda^n - 1}{\lambda - 1} = \\
&\lambda^n (< x, y^* > + \frac{< a, y^* >}{\lambda - 1}) - \frac{< a, y^* >}{\lambda - 1}.
\end{aligned}$$

Therefore, we get that the set $\{< A_a^n x, y^* >: n \in \mathbb{N}\}$ is not dense in $\mathbb{K}$, contradicting the assumption that $Orb(A_a, x)$ is dense in X.

The case $\lambda = 1$ is similar, as in this case

$$< A_a^n x, y^* >=< x, y^* > + n < a, y^* > \text{ for each } n \in \mathbb{N}.$$

Therefore, in any case, A_a is not hypercyclic. □

Proposition 4.8implies that an affine map cannot be hypercyclic if its linear part is not hypercyclic. We notice that this does not follow immediately from Proposition 4.3 , since two sets may fail to be dense even if their sum is dense in the space. An instance of this phenomenon is the case of two proper complemented subspaces.

Theorem 4.9. *If A_a is hypercyclic for some $a \in X$, then T is also hypercyclic.*

Proof. We observe first that the hypercyclicity of A_a implies that $I - T$ has dense range. If by means of contradiction we assume that the range of $I - T$ is not dense, then by the Hahn-Banach theorem one easily gets that T^* has a nontrivial fixed point, which is it has $\lambda = 1$ as an eigenvalue.

By the previous proposition, we get that A_a can not be hypercyclic, contradicting the assumption.

Leting now $x \in X$ be a hypercyclic vector for A_a and applying the operator $I - T$ to the equality

$$A_a^n x = T^n x + \sum_{i=0}^{n-1} T^i a$$

we get that

$$(I - T)A_a^n x = T^n(x - Tx - a) + a.$$

Since the operator $I - T$ has dense range, we conclude that the set

$$\{T^n(x - Tx - a) : n \in \mathbb{N}\}$$

is dense in X, thus T is hypercyclic. □

Corollary. 4.10. *If X is a Banach space and $T \in L(X)$ is a compact or a power bounded operator, then for any $a \in X$, the affine map A_a is not hypercyclic.*

Corollary. 4.11. *There exist no hypercyclic affine maps on a finite dimensional space.*

In the proof of Theorem 4.9 we established that if A_a is hypercyclic for some $a \in X$ then the range of $I - T$ is dense. The conclusion of the theorem though, provides an immediate generalization of this fact.

Corollary. 4.12. *If A_a is hypercyclic for some $a \in X$ then $p(T)$ has dense range for each nonzero polynomial p.*

It is also natural to study other notions close to hypercyclicity, like weak mixing, mixing or chaoticity on affine maps. Since all those properties are preserved under conjugacies, it is clear that if $a \in (I - T)(X)$, the affine map A_a has any given of these properties if and only if its linear part T has it. Moreover, as with Proposition 4.5 we have the following

Proposition. 4.13. *If T is weakly mixing, then the set of all $a \in X$ for which $A_a = T + a$ is also weakly mixing is a dense G_δ-set homeomorphic to X.*

Proof. Let $\mathcal{U} = \{a \in X : A_a \; is \; weakly \; mixing\}$, and $\{U_n\}_{n\in\mathbb{N}}$ a basis for the topology of X. We define

$$\mathcal{U}_{i,j,k,l} = \{a \in X : \exists n \in \mathbb{N}, A_a^n(U_i) \cap U_j \neq \emptyset \text{ and } A_a^n(U_k) \cap U_l \neq \emptyset\}.$$

Each $\mathcal{U}_{i,j,k,l}$ is open. Indeed, let $a \in \mathcal{U}_{i,j,k,l}$, then there exist $n \in \mathbb{N}$, $x \in U_i$, and $y \in U_k$, such that $A_a^n x \in U_j$ and $A_a^n y \in U_l$. Since the maps

$$a \mapsto A_a^n x \quad \text{and} \quad a \mapsto A_a^n y$$

are continuous, we can find a neighbourhood W of a, such that $A_z^n x \in U_j$, and $A_z^n y \in U_l$, for all $z \in W$. This means that $W \in \mathcal{U}_{i,j,k,l}$.

Each $\mathcal{U}_{i,j,k,l}$ is also dense since it contains the dense subspace $(I - T)(X)$. Hence by the Baire Category Theorem,

$$\mathcal{U} = \bigcap_{i,j,k,l} \mathcal{U}_{i,j,k,l}$$

is a dense G_δ subset of X, and it is homeomorphic to X since it contains the dense linear subspace $(I - T)(X)$. □

Proposition. 4.14. *If $A_a = T + a$, and $A_b = T + b$ satisfy that $A_a \oplus A_b$ is hypercyclic, then T is weakly mixing. In particular, if A_a is weakly mixing for some $a \in X$, then T is also weakly mixing.*

Proof. We have that $A_a \oplus A_b(x, y) = (Tx + a, Ty + b) = (Tx, Ty) + (a, b) = T \oplus T(x, y) + (a, b)$. Thus,

$$A_a \oplus A_b = T \oplus T + (a, b).$$

So if $A_a \oplus A_b$ is hypercyclc, then $T \oplus T$ is hypercyclic, so T is weakly mixing. □

The last assertion of Proposition 4.14 also follows from the quasi- conjugacy established in the following proposition.

Proposition. 4.15. *If an affine map $A_a = T + a$ is chaotic, weakly mixing, or mixing then its linear part T is also chaotic, weakly mixing, or mixing, respectively.*

Proof. We notice that since for $x \in X$, $A_a x = Tx + a$, then $(I - T)A_a x = T(x - Tx - a) + a$ which is

$$(I - T)A_a x - a = T(x - Tx - a). \tag{4.1.3}$$

Define $\tau_{-a} : X \to X$, the affine map $\tau_{-a}(x) = x - a$. Then consider the map $\psi_a = \tau_{-a} \circ (I - T)$. Notice that since A_a is in every case hypercyclic, T is hypercyclic, therefore $I - T$ and hence ψ_a have dense range. Now equation 4.1.3 becomes

$$\psi_a \circ A_a = T \circ \psi_a.$$

This means that T is a quasi factor of A_a, and hence inherits the dynamical properties from A_a. $\square$

The next example shows that the León-Müller theorem does not hold in full generality for affine maps.

Example. 4.16. *First we notice that if $|\lambda| = 1$, $a \in (\lambda^{-1}I - T)(X)$, and $\phi(x) = x + c$ for some $c \in (\lambda^{-1}I - T)^{-1}(a)$, then $\lambda A_a \circ \phi = \phi \circ \lambda T$ which means that λA_a and λT are conjugate and furthermore, $HC(\lambda A_a) = HC(\lambda T) + c = HC(T) + c$ by the León-Müller theorem. Now, consider the operator $T = I - B$ on $X = \ell^2$ where $B(x_n)_{n=1}^{\infty} = (x_{n+1})_{n=1}^{\infty}$ the unilateral backward shift. We know that T is hypercyclic, and let $h = (h_n)_{n=1}^{\infty} \in HC(T)$. For fixed $\lambda \neq 1$ with $|\lambda| = 1$ we set*

$$\begin{aligned}
c_1 &= (\frac{\lambda}{\lambda - 1}h_2, \frac{\lambda}{\lambda - 1}h_3, \frac{\lambda}{\lambda - 1}h_4, \dots),\\
c_2 &= (\frac{\lambda}{\lambda - 1}h_2 - h_1, \frac{\lambda}{\lambda - 1}h_3 - h_2, \frac{\lambda}{\lambda - 1}h_4 - h_3, \dots),\\
a &= (\frac{\lambda}{\lambda - 1}h_3 - h_2, \frac{\lambda}{\lambda - 1}h_4 - h_3, \frac{\lambda}{\lambda - 1}h_5 - h_4, \dots).
\end{aligned}$$

It is straightforward to check that $(\lambda^{-1}I - T)c_1 = a = (I - T)c_2$, and $c_1 - c_2 = h$. Thus, since

$0 \notin HC(T)$, we get that $HC(T) + (c_1 - c_2) \neq HC(T)$ which yields that $HC(\lambda A_a) \neq HC(A_a)$. Nevertheless, both λA_a and A_a are hypercyclic. In general, if $|\lambda| = 1$ and $a \in (\lambda^{-1}I - T)^{-1}(X) \cap (I - T)^{-1}(X)$, then since as noticed before λA_a is conjugate to λT, and A_a is conjugate to T, it holds that λA_a is hypercyclic if and only if A_a is hypercyclic.

Example 4.16 shows that even if we cannot hope to get the full León-Müller theorem for affine maps the following question remains still open.

Question. 4.17. *If $a \in X$ and $|\lambda| = 1$, is it the case that λA_a is hypercyclic if and only if A_a is hypercyclic?*

4.2 **Open Questions and Remarks**

The open questions in Affine Dynamics are numerous since as we noticed before, every theorem from Linear Dynamics has an analogue statement for affine maps. Because of the conjugacy between T and A_a whenever $a \in (I-T)(X)$, most of the properties of T pass to those specific A_a. However, since the conjugating map is affine and not linear some results become slightly different in the affine setting. For instance, $HC(T)$ contains a dense linear subspace except from the origin, whenever $HC(T) \neq \emptyset$, yields that $HC(A_a)$ contains a dense affine manifold, except from one point, whenever $a \in (I - T)(X)$ and $HC(A_a) \neq \emptyset$.

The fact that if $x \in HC(T)$ then $p(T)x \in HC(T)$ for each nonzero polynomial p is not true for affine maps. Indeed, if T is hypercyclic consider $d \in HC(T)$. Then $d - Td \in HC(T)$. Set $c = Td$, and $a = c - Tc$ so that T and A_a are conjugate via the affine map $\phi(x) = x + c$. Then $d = (d - Td) + c \in HC(A_a)$, but $Td \notin HC(A_a)$, since $Td = (Td - c) + c = 0 + c$ and $0 \notin HC(T)$. However, it is true that if $x \in HC(A_a)$ then $p(T)x - p(T)c + c \in HC(A_a)$ for any nonzero polynomial p.

An interesting question is whether $HC(A_a)$ is always connected. We notice that when $a \in (I - T)(X)$ the answer is clearly positive since then the set $HC(A_a)$ is just a translation of $HC(T)$. At Shkarin's example though (Example 4.6), $HC(A_a) = \emptyset$ while $HC(T) \neq \emptyset$, hence it is not always the case that $HC(A_a)$ is a translation of $HC(T)$. If one could establish that $HC(A_a)$ is connected for all $a \in X$ then appart from its own interest, using a result by Shkarin [35] it would follow that A_a is hypercyclic if and only if A_a^m is hypercyclic, and furthermore, $HC(A_a) = HC(A_a^m)$, $m \in \mathbb{Z}_+, a \in X$. Let's notice that this generalization of Ansari's theorem follows immediately in the case when $a = c - Tc, c \in X$, since in this case A_a^m is conjugate to $T^m, m \in \mathbb{Z}_+$ via the affine map $\phi(x) = x + c, x \in X$, and hence $HC(A_a^m) = HC(T^m) + c = HC(T) + c = HC(A_a)$.

The last theorem whose analogue to the affine setting we discuss here is the Bourdon-Feldman theorem. The question is, if $A_a = T + a$ has a somewhere dense orbit must this orbit be (everywhere) dense in X? If there exists λ such that $(\lambda I - T)(X)$ is dense in X which, by the Hahn-Banach theorem, is equivalent to the statement that T^* has an eiganvalue, we know that A_a does not have a dense orbit for any $a \in X$. In that case, if λ is an eigenvalue for T^* with corresponding eigenvector y^* as we showed in the proof of Proposition 8, for $x \in X$ and $n \in \mathbb{N}$, we have

$$< A_a^n x, y^* >= \begin{cases} \lambda^n(< x, y^* > + \frac{<a,y^*>}{\lambda-1}) - \frac{<a,y^*>}{\lambda-1}, \lambda \neq 1, \\ < x, y^* > + n < a, y^* >, \lambda = 1. \end{cases}$$

Therefore, since any nonzero continuous functional is an open map and hence, it maps somewhere dense sets to somewhere dense sets, we conclude that A_a does not have a somewhere dense orbit either. In the case when T^* does not have eigenvalues but $a \in (I - T)(X)$, since A_a is conjugate to T, we immediately get that a somewhere dense orbit for A_a must be dense in X. The question though remains open for those affine maps $A_a = T + a$ with $\sigma_p(T^*) = \emptyset$ and $a \notin (I - T)(X)$.

BIBLIOGRAPHY

[1] S. Ansari. *Hypercyclic and cyclic vectors.* J. Funct. Anal. 128 no. 2, 374–383, 1995.

[2] R. M. Aron, L. Bernal-González, D. Pellegrino, and J. B. Seoane-Sepúlveda. *Lineability: The search for linearity in mathematics*. Monographs and Research Notes in Mathematics. Chapman and Hall/CRC 2015.

[3] R. M. Aron, J. A. Conejero, A. Peris, and J. B. Seoane-Sepúlveda. *Powers of hypercyclic functions for some classical hypercyclic operators.* Integral Equations Operator Theory, 58(4):591–596, 2007.

[4] R. M. Aron, J. A. Conejero, A. Peris, and J. B. Seoane-Sepúlveda. *Sums and products of bad functions.* In Function spaces, volume 435 of *Contemp. Math.*, pages 47–52. Amer. Math. Soc., Providence, RI, 2007.

[5] R. Aron and D. Markose. *On universal functions.* J. Korean Math. Soc., 41(1):65–76, 2004. Satellite Conference on Infinite Dimensional Function Theory.

[6] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey. *On Devaney's Definition of Chaos.* The American Mathematical Monthly. 99:332-334, 1992.

[7] F. Bayart and É. Matheron. *Dynamics of linear operators*, volume 179 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2009.

[8] L. Bernal-González and A. Montes-Rodríguez. *Non-finite-dimensional closed vector spaces of universal functions for composition operators.* J. Approx. Theory, 82(3):375–391, 1995.

[9] J. P. Bès. *Three Problems on Hypercyclic Operators,* Kent State University (1998)

[10] J. P. Bès. *Invariant Manifolds of Hypercyclic Vectors for the Real Scalar Case.* Proceedings of the American Mathematical Society. Vol. 127, No. 6 (Jun., 1999), pp. 1801-1804

[11] J. Bès, J.A. Conejero, and D. Papathanasiou. *Convolution operators supporting hypercyclic algebras.* J. Math. Anal. Appl., 445 , 1232–1238, 2017.

[12] J. Bès, J.A. Conejero, and D. Papathanasiou. *Transcedental Convolution operators supporting hypercyclic algebras.* Preprint, 2017.

[13] J. Bès and A. Peris, *Hereditarily hypercyclic operators,* J. Funct. Anal. 167:1 (1999), 94–112.

[14] G. D. Birkhoff. *Démonstration d'un théorème élémentaire sur les fonctions entières.* C. R. Math. Acad. Sci. Paris, 189(2):473–475, 1929.

[15] G. D.Birkhoff. *On the periodic motions of dynamical systems.* Acta Math., 50 (1927), 359–379.

[16] P. S. Bourdon. *Invariant manifolds of hypercyclic vectors.* Proc. Amer. Math. Soc. 118 (1993), 845–847.

[17] P. S. Bourdon, N. S. Feldman. *Somewhere dense orbits are everywhere dense.* Indiana Univ. Math. J., 52 (3) (2003), pp. 811-819

[18] K. C. Chan. *Hypercyclicity of the operator algebra for a separable Hilbert space.* J. Operator Theory 42 (1999), 231–244.

[19] K.C. Chan and R.D. Taylor. *Hypercyclic subspaces of a Banach space.* Integr Equations Operator Theory (2001) 41: 381.

[20] T. W. Gamelin. *Complex Analysis.* Springer, New York, 2001.

[21] R. M. Gethner, J.H. Shapiro. *Universal vectors for operators on spaces of holomorphic functions.* Proc. Amer. Math. Soc., 100 (1987), pp. 281-288

[22] G. Godefroy and J. H. Shapiro. *Operators with dense, invariant, cyclic vector manifolds.* J. Funct. Anal., 98(2):229–269, 1991.

[23] M. González, F. León-Saavedra, and A. Montes-Rodríguez. *Semi-Fredholm theory: hypercyclic and supercyclic subspaces.* Proc. London Math. Soc. (3), 81(1):169–189, 2000.

[24] K. G. Grosse-Erdmann. *Universal families and hypercyclic operators.* Bull. Amer. Math. Soc. 36, 345-381, 1999.

[25] K.-G. Grosse-Erdmann and A. Peris. *Linear chaos*. Universitext. Springer, London, 2011.

[26] D. A. Herrero. *Limits of hypercyclic and supercyclic operators*. J. Funct. Anal. 99 (1991), 179190.

[27] C. Kitai. *Invariant Closed Sets for Linear Operators.* Thesis, Univ. of Toronto (1982).

[28] F. León-Saavedra and V. Müller. *Rotations of hypercyclic and supercyclic operators*. Integral Equations Operator Theory 50 (2004), 385–391.

[29] G. R. MacLane. *Sequences of derivatives and normal families.* J. Analyse Math., 2:72–87, 1952.

[30] Q. Menet. *Hypercyclic subspaces and weighted shifts*. Adv. Math. 255 (2014), 305–337.

[31] A. Montes-Rodríguez. *Banach spaces of hypercyclic operators*. Michigan Math. J. 43 (1996), 419–436.

[32] H. Petersson. *Hypercyclic subspaces for Fréchet space operators*. J. Math. Anal. Appl. 319 (2006), 764–782.

[33] S. Rolewicz. *On orbits of elements.*.Studia Mathematica 32.1 (1969), 17-22.

[34] H.N. Salas. *Hypercyclic weighted shifts.* Trans. Amer. Math. Soc. 347:993–1004, 1995.

[35] S. Shkarin. *Universal elements for non-linear operators and their applications.* J. Math. Anal. Appl. 348:193–210, 2008.

[36] S. Shkarin. *On the set of hypercyclic vectors for the differentiation operator.* Israel J. Math. 180:271–283, 2010.

[37] S. Shkarin. *On supercyclicity of operators from a supercyclic semigroup.* J. Math. Anal. Appl. 382:516–522, 2011.

[38] J. Wengenroth. *Hypercyclic operators on non-locally convex spaces.* Proc. Amer. Math. Soc., 131(6):1759–1761 (electronic), 2003.