ISOLATED POINT THEOREMS FOR UNIFORM ALGEBRAS ON MANIFOLDS

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ABSTRACT

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Suppose A is a uniform algebra on a compact Hausdorff space X. In 1957, Andrew Gleason conjectured that if (i) the maximal ideal space of A is X, and (ii) each point of X is a one-point Gleason part for A, then A must be $C(X)$, the collection of all complex-valued continuous functions on X. Subsequently, a weaker conjecture, known as Peak Point Conjecture, was considered in which condition (ii) was replaced by the stronger condition that "each point of X is a peak point for A". In fact, one can consider a stronger conjecture, referred as Isolated Point Conjecture, by considering a weaker condition "each point of X is isolated in the Gleason metric for A" in place of condition (ii). However, all of these three conjectures fail by a counterexample produced by Brian Cole in 1968. In 2001, John Anderson and Alexander Izzo proved that the Peak Point Conjecture is true for uniform algebras generated by collections of C^1 functions on a compact two-dimensional real manifold-with-boundary of class C^1. In the same year, Anderson, Izzo and John Wermer together proved that the same conjecture is true for uniform algebras generated by polynomials on compact subsets of real-analytic three-dimensional submanifolds of complex Euclidean spaces. In this dissertation, we will prove Gleason's conjecture, and the Isolated Point Conjecture for the earlier mentioned classes of uniform algebras considered by Anderson, Izzo and Wermer. In view of the relations of isolated point (in the Gleason metric) with Gleason part and peak point, it is sufficient to consider the Isolated Point Conjecture, the strongest of all the three conjectures. More explicitly, we will prove that the Isolated Point Conjecture is true for uniform algebras generated by collections of C^1 functions on a compact two-dimensional real manifold-with-boundary of class C^1, as well as for uniform algebras generated by polynomials on compact subsets of real-analytic three-dimensional submanifolds of complex Euclidean spaces. Hence, in particular, these results will generalize the corresponding results proved by Anderson, Izzo and Wermer.

To my grandparents and all family members

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CHAPTER 1: INTRODUCTION

A *uniform algebra* on a compact Hausdorff space X is a uniformly closed subalgebra of $C(X)$, the algebra of all complex-valued continuous functions on X, that contains all the constant functions on X and separates the points of X. In this dissertation, we consider some questions related to certain conjectures regarding the structure of uniform algebras. The conjectures that we will consider here involve five important notions of the theory of uniform algebras, namely, *maximal ideal spaces, peak points, point derivations, Gleason parts* and *isolated points* (in the *Gleason metric*). These notions will be defined in the next chapter. First, we will discuss the relations between these notions.

Throughout this dissertation, if not specified otherwise, A will denote an arbitrary uniform algebra on a compact Hausdorff space X. We assume, for the rest of this chapter, that the maximal ideal space of A is X. This condition is a necessary condition for A to be $C(X)$. For an arbitrary point p in X, consider the following four statements:

(a) p is a peak point for A;

(b) there is no non-zero point derivation on A at p;

(c) p is a one-point Gleason part for A;

(d) p is an isolated point in the Gleason metric for A.

It can be shown that (a) $\Rightarrow$ (b) $\Rightarrow$ (d) (see [8, Corollary 1.6.7] and [8, Theorem 1.6.2] respectively), and (a) $\Rightarrow$ (c) $\Rightarrow$ (d) (the first implication is easy and the second one is obvious). Moreover, it easily follows that if $A = C(X)$, then each of the statements (b), (c) and (d) holds for all points

p in X. In addition, if X is metrizable, then $A = C(X)$ implies that the statement (a) holds for all points p in X. In 1957, Andrew Gleason [13] conjectured that if the statement (c) holds for all points p in X, then A must be $C(X)$. Explicitly, the following conjecture was made.

Conjecture 1.1 (Gleason's Conjecture)**.** *If the maximal ideal space of A is X and every point of X is a one-point Gleason part for A, then $A = C(X)$.*

Subsequently, the following two conjectures were considered.

Conjecture 1.2 (Peak Point Conjecture)**.** *If the maximal ideal space of A is X and every point of X is a peak point for A, then $A = C(X)$.*

Conjecture 1.3 (Point Derivation Conjecture)**.** *If the maximal ideal space of A is X and there is no non-zero point derivation for A, then $A = C(X)$.*

In fact, one can also consider the following stronger conjecture that will be referred to as the Isolated Point Conjecture.

Conjecture 1.4 (Isolated Point Conjecture)**.** *If the maximal ideal space of A is X and every point of X is isolated in the Gleason metric for A, then $A = C(X)$.*

In 1959, Errett Bishop [6] showed that if X is a compact subset of the complex plane $\mathbb{C}$, the Peak Point Conjecture is true for the uniform algebra $R(X)$, the uniform closure of the collection of all rational functions with no poles on X. More generally, an equivalent statement of his result is the following:

Theorem 1.5 ([8], Theorem 3.3.3)**.** *Suppose X is a compact subset of $\mathbb{C}$. Then $R(X) = C(X)$ if and only if almost all (with respect to the two-dimensional Lebesgue measure on $\mathbb{C}$) points of X are peak points for $R(X)$.*

In 1968, the first three conjectures, namely, Gleason's Conjecture, the Peak Point Conjecture and the Point Derivation Conjecture were disproved by Brian Cole in his Ph.D. dissertation

[9]. Although Cole did not consider the Isolated Point Conjecture, it still fails by the same counterexample he used for disproving the other three conjectures. However, the failure of all these conjectures gave rise to an interesting question: Are there classes of uniform algebras for which these conjectures are still true? In the case of the Peak Point Conjecture, affirmative answers to the above question have been given by John Anderson, Alexander Izzo and John Wermer. In 2001, Anderson and Izzo first proved that the Peak Point Conjecture is true for uniform algebras generated by collections of C^1 functions on a compact two-dimensional real manifold-with-boundary of class C^1 [2, Theorem 4.1]. In the same year, Anderson, Izzo and Wermer together proved that the same conjecture is true for uniform algebras generated by polynomials on compact subsets of real-analytic three-dimensional submanifolds of $\mathbb{C}^n$ [4, Theorem 1.1].

In this dissertation, our goal is to prove each of the other three conjectures, namely, Gleason's Conjecture, the Point Derivation Conjecture and the Isolated Point Conjecture for the earlier mentioned classes of uniform algebras considered by Anderson, Izzo and Wermer. In view of the relations of isolated point (in the Gleason metric) with Gleason part, point derivation and peak point, it is sufficient to consider the Isolated Point Conjecture, the strongest of all the four conjectures.

In Chapter 2, we will provide, for reader's convenience, various important notions which will be used in the later chapters.

In Chapter 3, we will prove that the Isolated Point Conjecture is true for uniform algebras generated by collections of C^1 functions on a compact two-dimensional real manifold-with-boundary of class C^1. Our proof uses Bishop's peak point theorem for rational approximation, that is, Theorem 1.5 mentioned earlier.

In Chapter 4, we will prove that the Isolated Point Conjecture is true for uniform algebras generated by polynomials on compact subsets of real-analytic three-dimensional submanifolds of $\mathbb{C}^n$. Our proof uses the two-dimensional result from Chapter 3.

In Chapter 5, we state, again for reader's convenience, some important results that will be used throughout this dissertation.

CHAPTER 2: PRELIMINARIES

For the reader's convenience, in this chapter, we provide a list of examples, define various important notions and state few results. These will be used throughout this dissertation.

EXAMPLES

We discuss some important examples of uniform algebras on compact subsets of complex Euclidean spaces. Consider the following four subalgebras of $C(X)$, where X is a compact subset of $\mathbb{C}^n$.

(a) The uniform closure of the collection of all polynomials on X, denoted by $P(X)$.

(b) The uniform closure of the collection of all rational functions with no poles on X, denoted by $R(X)$.

(c) The collection of all continuous functions that are holomorphic in the interior of X, denoted by $A(X)$.

(d) The uniform closure of the collection of all continuous functions that are holomorphic in a neighborhood (dependent on the function) of X, denoted by $O(X)$.

It is easy to see that each of these subalgebras is a uniform algebra on X, and that $P(X) \subseteq R(X) \subseteq O(X) \subseteq A(X) \subseteq C(X)$. Each of these inclusions may be proper. In the case of $X = \overline{\mathbb{D}}$, the closed unit disc in $\mathbb{C}$, it can be shown that $P(\overline{\mathbb{D}}) = R(\overline{\mathbb{D}}) = O(\overline{\mathbb{D}}) = A(\overline{\mathbb{D}})$. This uniform algebra is known as the *disc algebra on disc*. In the case of $X = \Gamma$, the unit circle in $\mathbb{C}$, the uniform algebra $P(\Gamma)$ is known as the *disc algebra on circle*.

MAXIMAL IDEAL SPACE

A *multiplicative linear functional* on A is a non-zero linear functional ϕ on A that is multiplicative, that is, $\phi(fg) = \phi(f)\phi(g)$ for f, g in A. Each x in X gives rise to a multiplicative linear functional ϕ_x defined by $\phi_x(f) = f(x)$ (for all f in A), known as the *point evaluation functional* on A at x. The *maximal ideal space* $\mathfrak{M}_A$ of A consists of all maximal ideals of A. By the well-known one-to-one correspondence between maximal ideals and multiplicative linear functionals, $\mathfrak{M}_A$ can be thought of as the collection of all multiplicative linear functionals on A. In view of this, $\mathfrak{M}_A$ can be topologized with the relative weak*-topology that it inherits as a subset of the dual space A^* of A. By identifying each point x of X with the corresponding point evaluation functional ϕ_x in $\mathfrak{M}_A$, we can regard X as a closed subset of $\mathfrak{M}_A$. A necessary condition for A to be $C(X)$ is that the maximal ideal space of A is X, that is, $\mathfrak{M}_A = X$.

GELFAND TRANSFORM

The *Gelfand transform* of a function f in A is the function $\hat{f}\colon \mathfrak{M}_A \longrightarrow \mathbb{C}$ defined by $\hat{f}(\phi) = \phi(f)$. In particular, when $\phi = \phi_x$, the point evaluation functional at a point x in X, we obtain $\hat{f}(\phi_x) = \phi_x(f) = f(x)$. So, regarding X as a subset of $\mathfrak{M}_A$, for each f in A, we can view $\hat{f}$ as a continuous extension of f from X to $\mathfrak{M}_A$.

PEAK POINT

A point x in X is called a *peak point* for A if there exists f in A such that $f(x) = 1$ and $|f(y)| < 1$ for all y in $X \setminus \{x\}$.

POINT DERIVATION

A *point derivation* at a point x in X is a linear functional $\psi\colon A \longrightarrow \mathbb{C}$ that satisfies the Leibniz rule: for f, g in A,

$$\psi(fg) = \psi(f)g(x) + f(x)\psi(g).$$

GLEASON PART

Let ϕ, ψ be in $\mathfrak{M}_A$. Then the formula

$$\|\phi - \psi\|_A = \sup\{|\phi(f) - \psi(f)| : f \in A, \|f\|_\infty \leq 1\}$$

defines a metric on $\mathfrak{M}_A$, called the *Gleason metric* on $\mathfrak{M}_A$. This metric is nothing but the restriction of the dual metric on A^* to $\mathfrak{M}_A$. Using this metric, Gleason [13] introduced a non-trivial equivalence relation $\sim$ on $\mathfrak{M}_A$ defined by $\phi \sim \psi$ if and only if $\|\phi - \psi\|_A < 2$. For a proof of the fact that $\sim$ is an equivalence relation on $\mathfrak{M}_A$, see [8, Theorem 2.6.3]. The equivalence classes of $\mathfrak{M}_A$ under this equivalence relation are called the *Gleason parts* (or, simply *parts*) for A.

BOUNDARY

A subset Y of X is called a *boundary* for A if each function in A attains its maximum modulus on Y, that is, for each f in A, there exists y in Y such that $\|f\|_\infty = |f(y)|$. An obvious trivial example of a boundary for A is X itself. It can be established that the intersection of all boundaries for A that are closed is again a boundary for A (see [17, Theorem 7.4]). This unique minimal closed boundary is known as the *Shilov boundary*.

ESSENTIAL SET

The *essential set* for A, a notion first introduced by Herbert Bear, is the unique minimal closed subset E of X with the property that A contains every continuous function on X which vanishes on E ([5, Corollary 1] or see [8, Theorem 2.8.1]). Bear proved that the restriction of a uniform algebra to its essential set is uniformly closed [5, Theorem 2] (and hence forms a uniform algebra). Moreover, if the original uniform algebra is defined on its maximal ideal space, then the restricted uniform algebra is also defined on its maximal ideal space. More precisely, the statement of Bear's result is the following:

Theorem 2.1 ([5], Theorem 4)**.** *Suppose E is the essential set for a uniform algebra A on X. Then the maximal ideal space of $A|_E$ is E if and only if the maximal ideal space of A is X.*

REAL-ANALYTIC SUBVARIETY

Let U be an open subset of $\mathbb{C}^n$. A closed subset V of U is said to be a *real-analytic subvariety* of U if for each point p in V, there exists a neighborhood $W \subseteq U$ of p in $\mathbb{C}^n$ and real-valued functions f_j $(j = 1, \ldots, m)$ which are real-analytic in W, so that

$$V \cap W = \{q \in W : f_j(q) = 0, j = 1, \ldots, m\}.$$

A point p in V is called a *regular point* of V if there is a neighborhood O of p in $\mathbb{C}^n$ such that $V \cap O$ is a real-analytic submanifold of O. A point of V that is not a regular point is called a *singular point* of V. The set of all regular points of V will be denoted as V_{reg}, whereas the set of all singular points of V will be denoted as V_{sing}. As a manifold, the dimension of V in a neighborhood of a regular point is constant on connected components of V_{reg}. The maximum of these dimensions over all connected components of V_{reg} is defined to be the dimension of V. The following result is regarding the Hausdorff measure of the singular set of a real-analytic subvariety of the real-analytic manifold $\mathbb{C}^n$.

Lemma 2.2 ([10], Section 3.4.10)**.** *Suppose V is an m-dimensional real-analytic subvariety of an open subset U of $\mathbb{C}^n$. Then $\mathcal{H}^{m-1}(V_{\text{sing}} \cap C)$ is finite for each compact subset C of U, where $\mathcal{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure.*

ANALYTIC DISC

Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$. An *analytic disc* is a non-constant one-to-one continuous map $A\colon \overline{\mathbb{D}} \longrightarrow \mathbb{C}^n$ which is holomorphic in $\mathbb{D}$. By the boundary of the analytic disc A, we will mean the restricted map $A|_{\partial\mathbb{D}}$, that is, the restriction of A to the unit circle $\partial\mathbb{D}$. Often in the literature, the analytic disc and its boundary are identified with their images in $\mathbb{C}^n$.

CHAPTER 3: TWO DIMENSIONAL ISOLATED POINT THEOREM

In this chapter, we establish an isolated point theorem for uniform algebras generated by smooth functions on a compact two-dimensional real manifold-with-boundary. More precisely, we prove the following result that will be referred to as the Two-dimensional Isolated Point Theorem.

Theorem 3.1 (Two-dimensional Isolated Point Theorem)**.** *Suppose M is a compact two-dimensional real manifold-with-boundary of class C^1. Let A be a uniform algebra on M generated by a collection of C^1 functions. If*

(i) *the maximal ideal space of A is M, and*

(ii) *every point of M is isolated in the Gleason metric for A,*

then $A = C(M)$.

In 2001, Anderson and Izzo proved a similar result [2, Theorem 4.1]. In their result, the condition (ii) is replaced by the stronger condition that "every point of M is a peak point for A". The proof of Theorem 3.1 that is presented here is very similar to their proof of [2, Theorem 4.1] and only differs in those places where peak point has been used by Anderson and Izzo. However, first we will prove some useful lemmas.

Lemma 3.2. *Suppose A is a uniform algebra on X and Y is a closed subset of X. Let $B = \overline{A|_Y}$, the uniform closure in $C(Y)$ of the algebra $A|_Y = \{f|_Y \in C(Y) : f \in A\}$. If a point in Y is isolated in the Gleason metric for A, then it is also isolated in the Gleason metric for B.*

Proof. First, we define a map $T\colon A \longrightarrow B$ by $T(f) = f|_Y$. Clearly, T is a multiplicative linear operator between Banach spaces. In fact, T is bounded with $\|T\| = 1$ since $\|T(f)\|_\infty = \|f|_Y\|_\infty \leq$

$\|f\|_\infty$ for all f in A and $T(1_X) = 1_Y$. So, $\|T^*\| = \|T\| = 1$, where $T^*\colon B^* \longrightarrow A^*$, the adjoint of T, is given by $T^*(\phi) = \phi \circ T$. Next, note that T is a bounded linear operator with range $A|_Y$ that is dense in B. So, by applying duality, we see that T^* is injective. Hence, for ϕ, ψ in B^* with $\phi \neq \psi$, we obtain

$$0 < \|\phi \circ T - \psi \circ T\| = \|T^*(\phi - \psi)\| \leq \|\phi - \psi\|.$$

Fix a point p in Y. Note that, at p, if ϕ_p is the point evaluation functional on B, then $\phi_p \circ T$ is the point evaluation functional on A. Also, if ϕ is in $\mathfrak{M}_B$, then $\phi \circ T$ is in $\mathfrak{M}_A$. Hence, by taking ϕ in $\mathfrak{M}_B$ and $\psi = \phi_p$, we see from the preceding inequality that if p is isolated in the Gleason metric for A, then p is also isolated in the Gleason metric for B.

□

Lemma 3.3. *Suppose A is a uniform algebra on X, and Y is a closed subset of X. Let $\tilde{B}$ be a uniform algebra on Y containing $A|_Y$ and with maximal ideal space Y, that is, $\mathfrak{M}_{\tilde{B}} = Y$. If a point in Y is isolated in the Gleason metric for A, then it is also isolated in the Gleason metric for $\tilde{B}$.*

Proof. Suppose that $p \in Y$ is isolated in the Gleason metric for A. Now if $B = \overline{A|_Y}$, then clearly $B \subseteq \tilde{B}$. Hence, p is isolated in the Gleason metric for B, by Lemma 3.2. So, there exists $\delta > 0$ such that $\|\phi - \phi_p\|_B \geq \delta$ for all ϕ in $\mathfrak{M}_B$ with $\phi \neq \phi_p$, where ϕ_p denotes the point evaluation functional on B at p.

Fix q in $Y \setminus \{p\}$. Then, clearly $\phi_p \neq \phi_q$ as A separates the points of X. Consequently, we obtain

$$\begin{aligned} \|p - q\|_{\tilde{B}} &= \sup\{|g(p) - g(q)| : g \in \tilde{B}, \|g\|_\infty \leq 1\} \\ &\geq \sup\{|f(p) - f(q)| : f \in B, \|f\|_\infty \leq 1\} \\ &= \|\phi_p - \phi_q\|_B \geq \delta. \end{aligned}$$

Since q in Y is arbitrary and the maximal ideal space of $\tilde{B}$ is Y, the above inequality shows that p is isolated in Gleason metric for $\tilde{B}$.

□

Now we provide a partial converse of the Lemma 3.2.

Lemma 3.4. *Suppose Y is compact Hausdorff spaces, and X is a closed subset of Y. Let A be a uniform algebra on X with maximal ideal space X. Then, $B = \{f \in C(Y) : f|_X \in A\}$ is a uniform algebra on Y with maximal ideal space Y. If a point in X is isolated in the Gleason metric for A, then it is also isolated in the Gleason metric for B. Moreover, each point in $Y \setminus X$ is a one-point Gleason part for B.*

Proof. It easily follows that B is a uniform algebra on Y from the fact that A is a uniform algebra on X. To see that the maximal ideal space of B is Y, first note that both A and B have same essential set, say E, and also $B|_E = A|_E$. Then, from the hypothesis that the maximal ideal space of A is X, we obtain that the maximal ideal space of B is Y, by applying Theorem 2.1.

To verify the second assertion, let $p \in X$ be isolated in the Gleason metric for A. Then, there exists $\delta > 0$ such that $\|p - q\|_A \geq \delta$ for all q in $X \setminus \{p\}$. Note that $\|r - s\|_A = \|r - s\|_B$, for r, s in X. So, in particular, $\|p - q\|_B \geq \delta$ for all q in $X \setminus \{p\}$. Next, for q in $Y \setminus X$, by Urysohn's lemma (Theorem 5.1), there exists h in $C(Y)$ with $0 \leq h \leq 1$ such that $h(X) = \{0\}$ and $h(q) = 1$. Then, h is in B and $\|p - q\|_B \geq |h(p) - h(q)| = 1$, for q in $Y \setminus X$. Hence, $\|p - q\|_B \geq \delta_0 = \min(\delta, 1) > 0$ for all q in $Y \setminus \{p\}$. Therefore, p is isolated in the Gleason metric for B.

To prove the last assertion, let a be a point in $Y \setminus X$. Consider a point b in $Y \setminus \{a\}$. Then, by Urysohn's lemma (Theorem 5.1), there exists k in $C(Y)$ with $-1 \leq k \leq 1$ such that $k(X \cup \{b\}) = \{-1\}$ and $k(a) = 1$. Clearly k is in B, and $2 = |k(a) - k(b)| \leq \|a - b\|_B (\leq 2)$. Therefore, $\|a - b\|_B = 2$, and consequently, a is a one-point Gleason part.

□

Lemma 3.5. *Suppose X and Y are compact Hausdorff spaces and $f\colon X \longrightarrow Y$ is a homeomorphism. If A is a uniform algebra on X with maximal ideal space $\mathfrak{M}_A$, then the following hold:*

(a) $B = \{g \circ f^{-1} : g \in A\}$ *is a uniform algebra on Y,*

(b) $F\colon A \longrightarrow B$ *defined by $F(g) = g \circ f^{-1}$ is an isomorphism, and*

(c) *if φ is in $\mathfrak{M}_B$, the maximal ideal space of B, then $\tilde{\varphi}\colon A \longrightarrow \mathbb{C}$ defined by $\tilde{\varphi} = \varphi \circ F$ is in $\mathfrak{M}_A$. Also, for ϕ, ψ in $\mathfrak{M}_B$, $\|\tilde{\phi} - \tilde{\psi}\|_A = \|\phi - \psi\|_B$.*

Proof.

(a) The proof of the fact that B is a uniform algebra follows from the fact that A is a uniform algebra and f is a bijection.

(b) Since f is a bijection, it follows that $\tilde{f}$ is an isometric isomorphism between uniform algebras.

(c) The first assertion can be easily verified. For the proof of the second assertion, first note that $F^*(\varphi) = \tilde{\varphi}$ for all φ in $\mathfrak{M}_B$, where $F^*\colon B^* \longrightarrow A^*$ denotes the adjoint of F. Since the operator F is a surjective isometry, it follows that F^* is also an isometry. Moreover, the restriction of F^* to $\mathfrak{M}_B$ is again an isometry. So, for ϕ, ψ in $\mathfrak{M}_B$, we get

$$\|\tilde{\phi} - \tilde{\psi}\|_A = \|F^*(\phi) - F^*(\psi)\|_A = \|F^*(\phi - \psi)\|_A = \|\phi - \psi\|_B.$$

□

An *annihilating measure* for A is a regular complex Borel measure μ on X so that $\int f\, d\mu = 0$ for each f in A. The collection of all annihilating measures for A is denoted by $A^\perp$.

Given a regular complex Borel measure μ on X and f in $C(X)$, we define a new measure $f_*(\mu)$ on $\mathbb{C}$ by $f_*(\mu)(K) = \mu(f^{-1}(K))$ for each Borel subset K of $\mathbb{C}$. The measure $f_*(\mu)$ is often called the *push-forward measure* of μ. If g is an $f_*(\mu)$-integrable function, then it follows that

$$\int_{\mathbb{C}} g\, d(f_*(\mu)) = \int_X g \circ f\, d\mu.$$

The following sufficient condition for a uniform algebra to be the collection of all continuous functions is due to Anderson and Izzo.

Lemma 3.6 ([2], Lemma 2.1)**.** *Let A_0 be a dense subset of A. If $f_*(\mu) = 0$ for each f in A_0 and each measure μ in $A^\perp$, then $A = C(X)$.*

Let μ be a complex Borel measure on $\mathbb{C}$ with compact support. Then, the *Cauchy transform* $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(z) = \int \frac{d\mu(w)}{w - z}$$

for all z in $\mathbb{C}$ such that the integral converges absolutely.

Let M be a compact n-dimensional real manifold-with-boundary of class C^1, and $\mathcal{F}$ be a collection of complex-valued C^1 functions on M. Then, the set $E = \{p \in M : df_1 \wedge \ldots \wedge df_n(p) = 0$ for each n-tuple $f_1, \ldots, f_n$ in $\mathcal{F}\}$ is called the *exceptional set* of $\mathcal{F}$. Here we state a result of Michael Freeman that plays a crucial role in proving the main result in this chapter.

Theorem 3.7 ([11], Theorem 3.2)**.** *Let M be a compact two-dimensional real manifold-with-boundary of class C^1, and A be a uniform algebra on M generated by a collection $\mathcal{F}$ of C^1 functions with exceptional set E. Suppose that the maximal ideal space of A is M. If f is in $A \cap C^1(M)$, and μ is in $A^\perp$, then $\widehat{f_*(\mu)} = 0$ almost everywhere on $\mathbb{C} \setminus f(E)$ with respect to the Lebesgue measure.*

Finally, we present a proof of the Two-dimensional Isolated Point Theorem, that is, Theorem 3.1.

Proof of Theorem 3.1. Let A_0 denote the collection of all C^1 functions in A, and E be the exceptional set of A_0. Note that A_0 is dense in A. Then, by Lemma 3.6, it is sufficient to show that $f_*(\mu) = 0$ for each f in A_0 and each measure μ in $A^\perp$. So, fix a function f in A_0, and a measure μ in $A^\perp$. Then, by Theorem 3.7, $\widehat{f_*(\mu)} = 0$ almost everywhere on $\mathbb{C} \setminus f(E)$. Consequently, $f_*(\mu)$ is supported on $f(E)$ and $f_*(\mu) \perp R(f(E))$. It, therefore, suffices to show that $R(f(E)) = C(f(E))$.

Next, let S be the set of all critical values of f. Note that S is compact. By Sard's theorem, $m(S) = 0$, where m denotes the two-dimensional Lebesgue measure on $\mathbb{C}$. Also, $m(f(\partial M)) = 0$ and, hence, $m(S \cup f(\partial M)) = 0$ where ∂M denotes the boundary of M. In particular, $m(f(E) \cap$

$(S \cup f(\partial M))) = 0$. So, by the Hartogs-Rosenthal theorem (Theorem 5.2), we obtain

$$R(f(E) \cap (S \cup f(\partial M))) = C(f(E) \cap (S \cup f(\partial M))).$$

Now, we claim the following:

For each z_0 in $f(E) \setminus (S \cup f(\partial M))$, there exists a closed disc D centered at z_0 such that $R(f(E) \cap D) = C(f(E) \cap D)$.

If the preceding claim holds, we easily obtain a countable collection $\{D_n\}_{n=1}^{\infty}$ of closed discs with

$$R(f(E) \cap D_n) = C(f(E) \cap D_n)$$

and

$$f(E) \setminus (S \cup f(\partial M)) = \bigcup_{n=1}^{\infty} D_n.$$

Then, $\{f(E) \cap D_n : n \in \mathbb{N}\} \cup \{f(E) \cap (S \cup f(\partial M))\}$ is a countable collection of compact sets with union $f(E)$ which is compact. Hence, by a theorem of Herbert Alexander (Theorem 5.3), we conclude that $R(f(E)) = C(f(E))$.

To prove the claim, fix z_0 in $f(E) \setminus (S \cup f(\partial M))$. Then, by a corollary of the Inverse Function Theorem, there exists a closed disc D, centered at z_0, such that $f^{-1}(D)$ is a disjoint union of finitely many compact subsets $U_1, U_2, \ldots, U_t$ of M, such that f maps each U_j $(j = 1, 2, \ldots, t)$ diffeomorphically onto D.

Denote $\overline{A|_{U_j}}$, the uniform closure of the subalgebra $A|_{U_j}$ of $C(U_j)$, by A_j for $j = 1, 2, \ldots, t$. Since $f|_{U_j}$ is a diffeomorphism of U_j onto D, by part (a) of Lemma 3.5, $B_j = \{h \circ (f|_{U_j})^{-1} : h \in A_j\}$ is a uniform algebra on D. Moreover, by part (b) of Lemma 3.5, $f|_{U_j}$ induces an isomorphism from A_j onto B_j, given by $h \longmapsto h \circ (f|_{U_j})^{-1}$.

Let $E_j = E \cap U_j$ for $j = 1, 2, \ldots, t$. Note that for each $j = 1, 2, \ldots, t$, the uniform algebra B_j is generated by the collection $\mathcal{F}_j = \{h \circ (f|_{U_j})^{-1} : h \in A_0\}$ of C^1 functions. Since $f \circ (f|_{U_j})^{-1} = z$ on D, it follows that $f(E_j) = \{w \in D : \frac{\partial g}{\partial \bar{z}}(w) = 0 \text{ for all } g \in \mathcal{F}_j\}$. Therefore, by Theorem 5.6, $k|_{f(E_j)}$ is in $R(f(E_j))$ for k in B_j $(j = 1, 2, \ldots, t)$. Since every point of M is isolated in the

Gleason metric for A, by Lemma 3.2, we obtain that every point of U_j is also isolated in the Gleason metric for A_j $(j = 1, 2, \ldots, t)$. So, by part (c) of Lemma 3.5, every point of D is isolated in the Gleason metric for B_j $(j = 1, 2, \ldots, t)$. Since $f(E_j)$ is the maximal ideal space of $R(f(E_j))$, applying Lemma 3.3, we obtain that every point of $f(E_j)$ is isolated in the Gleason metric for $R(f(E_j))$ $(j = 1, 2, \ldots, t)$. Consequently, for all $j = 1, 2, \ldots, t$, each point of $f(E_j)$ is a peak point for $R(f(E_j))$ because the peak points for $R(f(E_j))$ are precisely the isolated points (in the Gleason metric) for $R(f(E_j))$ by Theorem 5.7. Therefore, $R(f(E_j)) = C(f(E_j))$ for all $j = 1, 2, \ldots, t$, by Bishop's peak point theorem for rational approximation (Theorem 1.5). Also, note that $f(E_j)$ $(j = 1, 2, \ldots, t)$ and $f(E) \cap D$ are compact subsets of $\mathbb{C}$. Finally, since $\bigcup_{j=1}^{t} f(E_j) = f(E) \cap D$, applying Alexander's theorem (Theorem 5.3) again, we obtain $R(f(E) \cap D) = C(f(E) \cap D)$. This proves the claim and, hence, the theorem.

□

The following theorem generalizes Theorem 3.1 to uniform algebras on compact subsets of smooth two-dimensional manifolds.

Theorem 3.8. *Suppose X is a compact subset of M, a two-dimensional real manifold-with-boundary of class C^1, and A is a uniform algebra on X generated by continuous functions that extend to be C^1 on a neighborhood of X. If*

(i) *the maximal ideal space of A is X, and*

(ii) *each point of X is isolated in the Gleason metric for A,*

then $A = C(X)$.

Proof. First, we choose a compact submanifold-with-boundary N of M containing X. Next, define $B = \{f \in C(N) : f|_X \in A\}$. Then, using condition (i) and by applying the Lemma 3.4, we see that B is a uniform algebra on N with maximal ideal space N. Also, it easily follows that B is generated by C^1 functions from the fact that A is generated by continuous functions that extend to be C^1 on a neighborhood of X. Moreover, from condition (ii), each point of X is isolated in the

Gleason metric for B by Lemma 3.4. Since a one-point Gleason part is, in particular, an isolated point in the Gleason metric, again from Lemma 3.4, we see that each point in $N \setminus X$ is also isolated in the Gleason metric for B. Thus, by Theorem 3.1, $B = C(N)$ and consequently, $A = C(X)$.

□

The hypothesis that "each point is isolated in the Gleason metric" in Theorem 3.1 can be weakened by assuming "almost every point is isolated in the Gleason metric" (note that the notion of a set of measure zero is well defined on a manifold of class C^1). In fact, the same proof remains valid in this case too because $R(f(E_j)) = C(f(E_j))$ even when almost every point of $f(E_j)$ is isolated in the Gleason metric for $R(f(E_j))$. Consequently, the hypothesis of Theorem 3.8 can be weakened by assuming that only almost every point in the space has the corresponding property.

CHAPTER 4: EMBEDDED THREE DIMENSIONAL ISOLATED POINT THEOREM

In this chapter, we establish an isolated point theorem for uniform algebras generated by polynomials on a compact subset of a three-dimensional real-analytic manifold-with-boundary embedded in $\mathbb{C}^n$.

Let X be a compact subset of $\mathbb{C}^n$. The *polynomial convex hull* $\hat{X}$ (or $X\widehat{\ }$) of X is defined as the set

$$\hat{X} = \{z \in \mathbb{C}^n : |p(z)| \leq \sup_{x \in X} |p(x)| \text{ for all polynomial } p\}.$$

Moreover, X is *polynomially convex* if $\hat{X} = X$. In fact, the polynomial convex hull $\hat{X}$ of X can be viewed as the maximal ideal space of the uniform algebra $P(X)$ [12, Chapter III, Theorem 1.2]. In the complex plane, that is, for $n = 1$, polynomial convexity is equivalent to a simple geometric condition: $X \subseteq \mathbb{C}$ is polynomially convex if and only if the complement $\mathbb{C} \setminus X$ is connected [12, Chapter III, Theorem 1.3]. However, in higher dimensional complex Euclidean spaces, this notion is much more complicated.

Now, we state the main result of this chapter that will be referred to as the Embedded Three-dimensional Isolated Point Theorem.

Theorem 4.1 (Embedded Three-dimensional Isolated Point Theorem)**.** *Suppose M is a real-analytic three-dimensional submanifold of $\mathbb{C}^n$. Assume that X is a compact subset of M such that the boundary ∂X of X relative to M is a two-dimensional submanifold of class C^1. If*

(i) *X is polynomially convex, and*

(ii) *every point of X is an isolated point in the Gleason metric for $P(X)$,* *then $P(X) = C(X)$. [Here ∂X denotes the union of the topological boundary of X relative to M and the set $X \cap \partial M$.]*

The proof of this theorem depends heavily on work of Anderson, Izzo and Wermer.

The following lemma [3, Lemma 2.3] plays a crucial role in proving the three-dimensional peak point theorem ([3, Theorem 1.1]). In fact, this lemma has been repeatedly used in that proof.

Lemma 4.2 ([3], Lemma 2.3)**.** *Suppose A is a uniform algebra on a compact Hausdorff space X. Also, assume that the maximal ideal space of A is X, and every point of X is a peak point for A. If Y is a closed subset X, then the maximal ideal space of $\overline{A|_Y}$ is Y, and every point of Y is a peak point for $\overline{A|_Y}$.*

The preceding lemma is not true if we replace "peak point" by "isolated point (in the Gleason metric)". To verify this claim, we will give an example. However, for that we first need the notion of *universal root algebra* of a uniform algebra, the existence of which was first shown by Cole [9].

Theorem 4.3 ([9], Theorem 2.4)**.** *Suppose A is a uniform algebra on X. There exist a compact Hausdorff space $\tilde{X}$ and a uniform algebra $\tilde{A}$ on $\tilde{X}$ with an associated continuous map $\tilde{\pi}\colon \tilde{X} \longrightarrow X$ such that*

(i) $\tilde{\pi}^*(h) = h \circ \tilde{\pi}$ *defines an embedding of A into $\tilde{A}$;*

(ii) *every function in $\tilde{A}$ has a square root in $\tilde{A}$;*

(iii) $\tilde{A} \cap \tilde{\pi}^*(C(X)) = \tilde{\pi}^*(A)$*;*

(iv) $\partial_{\tilde{A}} = \tilde{\pi}^{-1}(\partial_A)$*, where ∂_A and $\partial_{\tilde{A}}$ are the respective Shilov boundaries of A and $\tilde{A}$;*

(v) $\tilde{A}|_{\tilde{\pi}^{-1}(x)}$ *is dense in $C(\tilde{\pi}^{-1}(x)$ for each $x \in X$;*

(vi) *if the maximal ideal space of A is X, then the maximal ideal space of $\tilde{A}$ is $\tilde{X}$.*

The uniform algebra $\tilde{A}$ on $\tilde{X}$ is called the universal root algebra of A.

In particular, $\tilde{A}$ is nontrivial whenever A is nontrivial, and $\partial_{\tilde{A}} \neq \tilde{X}$ whenever $\partial_A \neq X$. Moreover, from [9, Lemma 1.1] and condition (ii) of the preceding theorem, it follows that every point of the maximal ideal space of $\tilde{A}$ is one-point Gleason part for $\tilde{A}$, and there is no nonzero point derivation for $\tilde{A}$.

Example 4.4. *Let $\overline{\mathbb{D}}$ denote the closed unit disc in the complex plane. Then, consider the disc algebra $A(\overline{\mathbb{D}})$ on the disc $\overline{\mathbb{D}}$. Let A be the universal root algebra of $A(\overline{\mathbb{D}})$ on $X = \tilde{\overline{\mathbb{D}}}$. First, note that $A \neq C(X)$ as $A(\overline{\mathbb{D}}) \neq C(\overline{\mathbb{D}})$. Since $\overline{\mathbb{D}}$ is the maximal ideal space of $A(\overline{\mathbb{D}})$, the maximal ideal space of A is X. Also, every point of X is a one-point Gleason part for A, and there is no nonzero point derivation for A. So, in particular, every point of X is isolated in the Gleason metric for A. Note that the Shilov boundary of the disc algebra $A(\overline{\mathbb{D}})$ is the unit circle in the complex plane. Hence, the Shilov boundary, say Y, of A is a proper closed subset of X. Since the Shilov boundary Y of A is a closed boundary for A, it can be easily shown that the $A|_Y$, the restriction of the uniform algebra A to Y, is also a uniform algebra on Y and is isometrically isomorphic to A on X. However, isomorphic uniform algebras have homeomorphic maximal ideal spaces. Hence, the maximal ideal space of $A|_Y$ is X, not Y.*

Next, we prove the following result that strengthens Bear's result (Theorem 2.1).

Lemma 4.5. *Let L be a closed subset of X containing the essential set E for A. Then, $A|_L$ is uniformly closed in $C(L)$. Moreover, the maximal ideal space of $A|_L$ is L if and only if the maximal ideal space of A is X.*

Proof. To show that $A|_L$ is uniformly closed in $C(L)$, let $f \in \overline{A|_L}$, the uniform closure of $A|_L$ in $C(L)$. Then, there exists a sequence $(f_n)_{n=1}^{\infty} \subseteq A$ such that $f = \lim_{n\to\infty} f_n|_L$. So, in particular, $f|_E = \lim_{n\to\infty} f_n|_E$ as $E \subseteq L$. Then, $f|_E \in A|_E$ as $f_n|_E \in A|_E$ for all $n \in \mathbb{N}$ and $A|_E$ is uniformly closed in $C(E)$. Next, note that, by the Tietze Extension Theorem, f can be extended continuously to $\tilde{f}$ on X, that is, there exists $\tilde{f}$ in $C(X)$ such that $\tilde{f}|_L = f$. Then, clearly $\tilde{f}|_E = f|_E$ is in $A|_E$. However, that implies $\tilde{f} \in A$ as E is the essential set for A. Hence, $f = \tilde{f}|_L \in A|_L$. This shows that $\overline{A|_L} \subseteq A|_L$ and consequently, $A|_L$ is uniformly closed in $C(L)$.

To prove the second assertion, note that E is also the essential set for $A|_L$. Then, by Theorem 2.1, the maximal ideal space of $A|_L$ is L if and only if the maximal ideal space of $A|_E$ is E if and only if the maximal ideal space of A is X.

□

Now we define a notion that generalizes the notion of polynomial convexity. Let K be a closed subset of X. The *A-convex hull* $\hat{K}$ of K is defined as the set

$$\hat{K} = \{\phi \in \mathfrak{M}_A : |\phi(f)| \leq \sup_{x \in K} |f(x)| \text{ for all } f \text{ in } A\}.$$

Moreover, K is *A-convex* if $\hat{K} = K$. In fact, the A-convex hull $\hat{K}$ of K can be viewed as the maximal ideal space of the uniform algebra $\overline{A|_K}$ [12, Chapter II, Theorem 6.1].

In the case of a uniform algebra generated by Lipschitz functions on a compact metric space, the following lemma gives a partial converse to the Lemma 4.5. The proof of this lemma is very similar to that of [3, Lemma 2.1] by Anderson and Izzo. For the definition and the properties of Hausdorff measure in a metric space, see [10].

Lemma 4.6. *Suppose K is a compact metric space, and A is a uniform algebra on K generated by a collection of Lipschitz functions. Also, suppose that maximal ideal space of A is K. If $K = Y \cup S$ where Y is a compact, A-convex subset of K and $S \subset K$ is a set with two-dimensional Hausdorff measure zero, then Y contains the essential set E for A.*

A subset F of X is called a *set of antisymmetry* for A if every function in A which is real-valued on F must be constant on F. For the proof of the above result, we use the antisymmetric decomposition of a uniform algebra due to Bishop (Theorem 5.4).

Proof of Lemma 4.6. We claim that it is sufficient to show that for every x in $S \setminus Y$ and every y in K with $x \neq y$, there is a real-valued function f in A such that $f(x) \neq f(y)$. To see the claim, first note each set of antisymmetry for A then either is a singleton or else contained in Y. Now, let $g \in C(K)$ with $g|_Y = 0$. To show that g is in A, let F be a set of antisymmetry. If F is a singleton, then clearly $g|_F \in A|_F$. If F is not a singleton, then $F \subseteq Y$ and hence $g|_F = 0 \in A|_F$. Then, by Bishop's antisymmetric decomposition (Theorem 5.4), $g \in A$ and, consequently, $E \subseteq Y$.

Now, fix x in $S \setminus Y$ and y in K with $x \neq y$. Since Y is A-convex, so is $Y \cup \{y\}$ by [3, Lemma 3.1]. Consequently, there exists a function p in A with $|p| \leq \frac{1}{2}$ on $Y \cup \{y\}$ and $p(x) = 1$. In fact, p can be taken to be Lipschitz as A is generated by Lipschitz functions. Next, consider the subset $M = p(K) \cap \{z \in \mathbb{C} : |z| \leq \frac{7}{8}\}$ of $p(K)$ and $z = 1 \notin M$. By Urysohn's lemma (Theorem 5.1), there is a real-valued function $h \in C(p(K))$ with $h(M) = \{0\}$ and $h(1) = 1$. We claim that h belongs to $R(p(K))$ locally. Since the two-dimensional Hausdorff measure of S is zero, it follows that the two-dimensional Hausdorff measure and hence the two-dimensional Lebesgue measure of $p(S)$ is zero. Then, by the Hartogs-Rosenthal theorem (Theorem 5.2), $R(L) = C(L)$ for each compact subset $L \subset p(S)$. So, in particular, $R(N) = C(N)$, where $N = p(K) \cap \{z \in \mathbb{C} : |z| \geq \frac{3}{4}\}$ is a compact subset of $p(S)$. Therefore, h belongs to $R(N)$. Also, h belongs to $R(M)$ as $h|_M = 0$. Hence, $h \in R(p(K))$ by the localization theorem (Theorem 5.5). Since $p \in A$ and K is the maximal ideal space of A, by the functional calculus, it follows that $h \circ p \in A$. Thus, $h \circ p \in A$ is a real-valued function with $(h \circ p)(x) \neq (h \circ p)(y)$.

□

In fact, as a corollary we obtain the following result of Anderson and Izzo.

Corollary 4.7 ([3], Lemma 2.1)**.** *Suppose K is a compact metric space, and A is a uniform algebra on K generated by a collection of Lipschitz functions. Also, suppose that maximal ideal space of A is K. If $K = Y \cup S$ where Y is a compact subset of K with $\overline{A|_Y} = C(Y)$ and $S \subset K$ is a set with two-dimensional Hausdorff measure zero, then $A = C(K)$.*

Proof. First, note that Y is A-convex as $\overline{A|_Y} = C(Y)$. Then, by Lemma 4.6, Y contains the essential set E for A. To show $A = C(K)$, let $f \in C(K)$. Then, $f|_Y \in C(Y) = \overline{A|_Y}$. Since E is a closed subset of Y, $f|_E \in \overline{A|_E}$. However, $\overline{A|_E} = A|_E$ as E is the essential set for A. So, $f|_E \in A|_E$, and, consequently, $f \in A$. This shows that $A = C(K)$.

□

Suppose M is a real submanifold of $\mathbb{C}^n$, of class C^1, and p is a point in M. In general, the real tangent space $T_p(M)$ of M at p is not a complex vector subspace of $T_p(\mathbb{C}^n) \simeq T_p(\mathbb{R}^{2n})$. The largest complex vector subspace of $T_p(M)$, denoted by $H_p(M)$, is called the *holomorphic tangent space* of M at p and the complex dimension of it is called the *Cauchy-Riemann rank* (in short, *CR rank*) of M at p. If $H_p(M)$ is non-trivial, then M is said to have a *complex tangent* at p. On the contrary, M is called *totally real* if it has no complex tangent at any point. The following lemma characterizes the points where a real submanifold of $\mathbb{C}^n$ has a complex tangent. A proof of this lemma can be found in [4, Lemma 2.5].

Lemma 4.8 ([4], Lemma 2.5)**.** *Suppose M is a real m-dimensional submanifold of $\mathbb{C}^n$. Then M has a complex tangent at p if and only if $dz_I(p) = 0$ as a form on M, for all m-tuples I.*

The following result is due to Anthony O'Farrell, Kenneth Preskenis and David Walsh [15].

Proposition 4.9 ([15], Theorem 2)**.** *Suppose K is a holomorphically convex compact set, and K_0 is a closed subset of K such that $K \setminus K_0$ is a totally real submanifold of $\mathbb{C}^n$, of class C^1. Then, a continuous function f is in $O(K)$ if and only if there exists g in $O(K)$ with $f = g$ on K_0.*

Next, we state two corollaries of Proposition 4.9. The first corollary and its proof is in [4, Corollary 2.4]. The proof of the second one follows from the well known fact that a polynomially convex set is also holomorphically convex and from the definition of essential set.

Corollary 4.10 ([4], Corollary 2.4)**.** *Suppose K is a polynomially convex compact set, and K_0 is a closed subset of K such that $K \setminus K_0$ is a totally real submanifold of $\mathbb{C}^n$, of class C^1. If $P(K_0) = C(K_0)$, then $P(K) = C(K)$.*

Corollary 4.11. *Suppose K is a polynomially convex compact set, and K_0 is a closed subset of K such that $K \setminus K_0$ is a totally real submanifold of $\mathbb{C}^n$, of class C^1. Then, K_0 contains the essential set E for $P(K)$.*

Next, let E be the set of points at which M, a real smooth submanifold of $\mathbb{C}^n$, has a complex tangent. If K is a compact subset of M, we will prove that under the isolated point hypothesis, the interior of $E \cap K$ in M is empty. In fact, we will prove the following lemma:

Lemma 4.12. *Suppose M is a real m-dimensional submanifold of $\mathbb{C}^n$, of class C^2. Also, assume that*

(i) *K is polynomially convex, and*

(ii) *every point of K is an isolated point in the Gleason metric for $P(K)$.*

Then $E \cap K$ has empty interior in M, where E is the set of all points at which M has a complex tangent.

In fact, in case of class C^2 manifold, the preceding lemma generalizes a similar result [4, Lemma 3.2] proved by Anderson, Izzo and Wermer. They proved that the same conclusion is true when the condition (ii) is replaced by the stronger condition that "every point of K is a peak point for $P(K)$". To establish [4, Lemma 3.2], Anderson, Izzo and Wermer used Lemma 4.2 repeatedly. However, Lemma 4.2 fails if we replace "peak point" by "isolated point", as discussed earlier. Hence, for proving Lemma 4.12, we take a different approach. We first show that the interior of E must contain the boundary of an analytic disc if M is of class C^2. The proof of this fact is due to Wermer (obtained via a personal communication).

Lemma 4.13. *Suppose M is a real m-dimensional submanifold of $\mathbb{C}^n$, of class C^2. Let E be the set of points at which M has a complex tangent. Assume that U is an open subset of $\mathbb{C}^n$ so that $M \cap U$ is a non-empty subset of E. Then $M \cap U$ contains the boundary of an analytic disc.*

Proof. Denote the CR rank of M at q by $r(q)$, and put $r_0 = \min\{r(q) : q \in M \cap U\}$. Note that $\{q \in M \cap U : r(q) = r_0\}$ is a relatively open subset of $M \cap U$. Then, there is a non-empty open subset $V \subseteq \mathbb{C}^n$ with $V \subseteq U$ such that $M \cap V$ is non-empty and $r(q) = r_0$ for all $q \in M \cap V$.

Next, note that $2r_0 \leq m$ by definition of CR rank. Also, $r_0 \geq 1$ as $M \cap U \subseteq E$. Then, (by a result in [7, § 12.5]) without loss of generality we can find a generic m-dimensional CR submanifold M_0 of $\mathbb{C}^{m-r_0}$, of class C^2 and a CR map $g\colon M_0 \longrightarrow \mathbb{C}^{n-m+r_0}$ so that $M \cap V = \{(\zeta, g(\zeta)) : \zeta \in M_0\}$. Note that g, being a CR map, can be written as $g = (g_1, \ldots, g_{n-m+r_0})$, where each $g_i\colon M_0 \longrightarrow \mathbb{C}$ is a CR function for $i = 1, \ldots, n-m+r_0$. Note that $m - r_0 < m \leq 2(m - r_0)$.

So, by an approximation theorem of M. Salah Baouendi and Francois Treves (Theorem 5.9), there exists an open subset N_0 of M_0 such that for each $i = 1, \ldots, n - m + r_0$, there is a sequence of polynomials $({p^i}_n)_{n=1}^{\infty}$ on $\mathbb{C}^{m-r_0}$ that converges uniformly to g_i on N_0. Since M_0 is a generic CR manifold with CR rank r_0, by a result of Bishop [1, Theorem 18.7], there exists an analytic disc Δ in $\mathbb{C}^{m-r_0}$ with boundary $\partial\Delta$ contained in N_0. So, in particular, for each $i \in \{1, \ldots, n - m + r_0\}$, the sequence $({p^i}_n)_{n=1}^{\infty}$ converges uniformly to g_i on the boundary $\partial\Delta$ of Δ. Also, for each $i = 1, \ldots, n - m + r_0$, the Maximum Modulus Principle on Δ implies that the sequence $({p^i}_n)_{n=1}^{\infty}$ converges uniformly on Δ to a function, say, G_i which is analytic in the interior of Δ and $G_i = g_i$ on $\partial\Delta$. Now, define a map $G\colon \Delta \longrightarrow \mathbb{C}^{n-m+r_0}$ by $G = (G_1, \ldots, G_{n-m+r_0})$. Note that G is a continuous map that is analytic in the interior of Δ and agrees with g on $\partial\Delta$. Then, $A\colon \Delta \longrightarrow \mathbb{C}^n$ given by $A(\zeta) = (\zeta, G(\zeta))$ is an analytic map with $A(\partial\Delta) = \{(\zeta, g(\zeta)) : \zeta \in \partial\Delta\} \subseteq M \cap V$. Thus, the image $A(\Delta)$ of A is an analytic disc in $\mathbb{C}^n$ whose boundary lies in $M \cap V \subseteq M \cap U$.

□

Finally, we prove Lemma 4.12.

Proof of Lemma 4.12. We claim that K contains no analytic disc. Suppose that the claim is not true, that is, there is an analytic disc $\Phi\colon \overline{\mathbb{D}} \longrightarrow \mathbb{C}^n$ with $\Phi(\overline{\mathbb{D}}) \subseteq K$. Then, for z in $\mathbb{D} \setminus \{0\}$, we obtain

$$\begin{aligned} \|z - 0\|_{A(\overline{\mathbb{D}})} &= \sup\{|g(z) - g(0)| : g \in A(\overline{\mathbb{D}}), \|g\|_\infty \leq 1\} \\ &\geq \sup\{|f(\Phi(z)) - f(\Phi(0))| : f \in P(K), \|f\|_\infty \leq 1\} \\ &= \|\Phi(z) - \Phi(0)\|_{P(K)}. \end{aligned}$$

Since Φ is one-to-one, by condition (ii), there exists $\delta > 0$ such that $\|\Phi(z) - \Phi(0)\|_{P(K)} \geq \delta$ for all z in $\mathbb{D} \setminus \{0\}$. Therefore, $\|z - 0\|_{A(\overline{\mathbb{D}})} \geq \delta$, for all z in $\mathbb{D} \setminus \{0\}$. Hence, 0 is an isolated point of $\overline{\mathbb{D}}$ in the Gleason metric for $A(\overline{\mathbb{D}})$, but this is a contradiction. So, the claim is true, that is, K does not contain any analytic disc.

Next suppose, on the contrary to our assertion, that $E \cap K$ has non-empty interior in M. Then, there is a non-empty open subset U of $\mathbb{C}^n$ with $M \cap U \subseteq E \cap K$. Therefore, by Lemma 4.13, there exists an analytic disc $\Psi\colon \overline{\mathbb{D}} \longrightarrow \mathbb{C}^n$ whose boundary lies in $M \cap U (\subseteq E \cap K) \subseteq K$. Moreover, by condition (i), the analytic disc $\Psi(\overline{\mathbb{D}})$ is contained in K. However, this contradicts the fact that K does not contain any analytic disc. Consequently, $E \cap K$ has empty interior.

□

We now state a crucial result that is due independently to Alexander and Nessim Sibony. We thank Edgar Stout for pointing out this result.

Theorem 4.14 ([1], Corollary 21.10)**.** *Suppose K is a compact subset of $\mathbb{C}^n$ with polynomial convex hull $\hat{K}$. If $L = \hat{K} \setminus K$, then for every z in L and $r > 0$, the set $L \cap \mathbb{B}_n(z; r)$ has positive two-dimensional Hausdorff measure. (Here, $\mathbb{B}_n(z; r)$ denotes the open ball with center $z \in \mathbb{C}^n$ and radius r.)*

Finally, we prove the Embedded Three-dimensional Isolated Point Theorem, that is, Theorem 4.1.

Proof of Theorem 4.1. Let E be the set of all points at which M has a complex tangent. Also, let X_0 be the interior of X relative to M, and Ω_0 be an open subset of $\mathbb{C}^n$ with $X_0 = X \cap \Omega_0$. Define $\tilde{E} = E \cap X_0$ and $K_0 = \partial X \cup \tilde{E}$. Note that K_0 is compact because each limit point of $\tilde{E}$, that is not in $\tilde{E}$, belongs to ∂X. Also, note X is polynomially convex by assumption, and $X \setminus K_0$ is a totally real submanifold of Ω_0. Hence, by Corollary 4.10, to show $P(X) = C(X)$ it suffices to prove that $P(K_0) = C(K_0)$. Moreover, by Corollary 4.11, K_0 contains the essential set for $P(X)$ and, hence, by Lemma 4.5, K_0 is polynomially convex.

It easily follows from Lemma 4.8 that $\tilde{E}$ is a real-analytic subvariety of Ω_0. Let $\tilde{E}_{\text{c}}$ be the set of all points at which $\tilde{E}_{\text{reg}}$ itself has complex tangent, and set $Z = \partial X \cup \tilde{E}_{\text{sing}} \cup \tilde{E}_{\text{c}}$. It follows that Z is compact and that $K_0 \setminus Z (= \tilde{E}_{\text{reg}} \setminus \tilde{E}_{\text{c}})$ is a totally real, real-analytic submanifold of Ω_0. So, again by Corollary 4.10, to show $P(K_0) = C(K_0)$ it suffices to prove that $P(Z) = C(Z)$. Moreover, by Corollary 4.11, Z contains the essential set for $P(K_0)$ and, hence, by Lemma 4.5, Z is polynomially convex.

Finally, to show $P(Z) = C(Z)$, we apply Corollary 4.7 with $Y = \partial X$ and $S = \tilde{E}_{\text{sing}} \cup \tilde{E}_{\text{c}}$. First, we verify that $\mathcal{H}^2(S) = 0$, that is, $\mathcal{H}^2(\tilde{E}_{\text{sing}} \cup \tilde{E}_{\text{c}}) = 0$. By Lemma 4.12, $E \cap X$ and, hence, $\tilde{E}$

has no interior in M. Therefore, the dimension $\tilde{E}$ is at most two. So, $\mathcal{H}^1(\tilde{E}_{\text{sing}} \cap C) < \infty$ for every compact subset C of Ω_0, by Lemma 2.2. Now, covering Ω_0 by countably many compact sets, we obtain $\mathcal{H}^2(\tilde{E}_{\text{sing}}) = 0$. Note that $\tilde{E}_c$ is a real-analytic subvariety of Ω_0 follows from Lemma 4.8. To prove $\mathcal{H}^2(\tilde{E}_{\text{c}}) = 0$, fix a point p in $\tilde{E}_{\text{reg}}$. Since $\tilde{E}_{\text{reg}}$ is open in $\tilde{E}$, and $\tilde{E}$ is open in K_0, clearly $\tilde{E}_{\text{reg}}$ is open in K_0. Therefore, there is $r > 0$ such that $\overline{B(p;r)} \cap K_0 \subseteq \tilde{E}_{\text{reg}}$. Denote $\overline{B(p;r)} \cap K_0$ by K_p. Then, K_p is a compact subset of $\tilde{E}_{\text{reg}}$. Note K_p, being an intersection of two polynomially convex sets, is polynomially convex. Also, by applying Lemma 3.2 using condition (ii), note that every point of K_0 is an isolated point in the Gleason metric for $P(K_0)$. Therefore, Lemma 4.12 implies that $\tilde{E}_{\text{c}} \cap K_p$ has empty interior in $\tilde{E}_{\text{reg}}$. Since the point $p \in \tilde{E}_{\text{reg}}$ is arbitrary, it follows that $\tilde{E}_c$ is a real-analytic subvariety of Ω_0 of dimension at most one. Consequently, $\mathcal{H}^2(\tilde{E}_{\text{c}}) = 0$.

Next, we verify that $P(Y) = C(Y)$, that is, $P(\partial X) = C(\partial X)$. To do this we will first show that ∂X is polynomially convex. Because ∂X is a subset of the polynomially convex set Z, the polynomial convex hull $\widehat{\partial X}$ of ∂X is contained in Z. So, $\widehat{\partial X} \setminus \partial X \subseteq Z \setminus \partial X \subseteq \tilde{E}_{\text{sing}} \cup \tilde{E}_{\text{c}}$. However, the two-dimensional Hausdorff measure of $\tilde{E}_{\text{sing}} \cup \tilde{E}_{\text{c}}$ is zero. Hence, by Theorem 4.14, $\widehat{\partial X} \setminus \partial X$ is empty, that is, ∂X is polynomially convex. Next, by applying Lemma 3.2 using condition (ii), we see that every point of ∂X is an isolated point in the Gleason metric for $P(\partial X)$. Finally, by applying the Two-dimensional Isolated Point Theorem (Theorem 3.1), we conclude that $P(\partial X) = C(\partial X)$.

$\square$

CHAPTER 5: APPENDIX

For the reader's convenience, in this chapter, we state some important results. These results have been used throughout this dissertation.

Theorem 5.1 ([14], Theorem 33.1)**.** *Let X be a normal space; let A and B be disjoint closed dubsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map $f\colon A \longrightarrow [a, b]$ such that $f(x) = a$ for every x in A, and $f(x) = b$ for every x in B.*

Theorem 5.2 (**Hartogs-Rosenthal Theorem**, [8], Theorem 3.2.4)**.** *Suppose X is a compact subset of $\mathbb{C}$. If the two-dimensional Lebesgue measure of X is zero, then $R(X) = C(X)$.*

Theorem 5.3 ([17], Theorem 26.4)**.** *Let $\{X_n\}_{n=1}^{\infty}$ be sequence of compact subsets of $\mathbb{C}$ with compact union X. If $R(X_n) = C(X_n)$ for all n, then $R(X) = C(X)$.*

Theorem 5.4 ([8], Theorem 2.7.5)**.** *Let A be a uniform algebra on X. Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be the collection of distinct maximal sets of antisymmetry for A. Then*

(i) *$X = \cup_{\alpha\in\Lambda} X_\alpha$, and $X_\alpha \cap X_\beta$ is empty for $\alpha, \beta \in \Lambda$;*

(ii) *$A|_{X_\alpha}$ is uniformly closed, and X_α is a set antisymmetry for A $(\alpha \in \Lambda)$;*

(iii) *$A = \{f \in C(X) : f|_{X_\alpha} \in A|_{X_\alpha}$ for every $\alpha \in \Lambda\}$.*

Theorem 5.5 ([12], Section II, Theorem 10.3)**.** *Let K be a compact subset $\mathbb{C}$, and $f \in C(K)$. If every point $z \in K$ has a neighborhood $U(z)$ such that $f \in R(K \cap \overline{U(z)})$, then $f \in R(K)$.*

Theorem 5.6 ([8], Corollary 3.2.2)**.** *Let K be a compact subset of $\mathbb{C}$. Let $f \in C^1(U)$ for some neighborhood U of K. If $\frac{\partial f}{\partial \bar{z}} = 0$ on K, then $f|_K \in R(K)$.*

Theorem 5.7 ([8], Corollary 3.3.10)**.** *Let K be a compact subset of $\mathbb{C}$. The isolated points for $R(K)$, in the Gleason metric, are precisely the peak points for $R(K)$.*

Theorem 5.8 ([16], Theorem 1.6.2)**.** *A compact subset K of $\mathbb{C}^n$ with $\mathcal{H}^1(K) = 0$ is polynomially convex and satisfies $P(K) = C(K)$.*

Theorem 5.9 ([7], § 13, Theorem 1)**.** *Suppose p is a point in a generic CR submanifold M of $\mathbb{C}^n$, of class C^2 with real dimension m, $n \leq m \leq 2n$. Given an open neighborhood U_1 of p in M, there exists another open neighborhood U_2 of p in M containing U_1 so that each CR function of class C^1 on U_1 can be uniformly approximated on U_2 by a sequence of entire functions in $\mathbb{C}^n$.*

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