HYPERBOLIC GEOMETRY AND COXETER GROUPS

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ABSTRACT

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This paper will examine first the three spaces of constant curvature: Euclidean, spherical, and hyperbolic. Next, we consider the definitions and properties associated with Coxeter groups, reflection groups, and geometric reflection groups. This leads us to an interesting theorem about polytopes with angles of the form $\frac{\pi}{m_{ij}}$ for $m_{ij} \in \mathbb{N} \cup \{\infty\}$ and how they tessellate these model spaces. These m_{ij} give a Coxeter matrix and corresponding Coxeter group. We list each of these possible polytopes in two dimensional Euclidean space and two dimensional spherical space. In hyperbolic space, there are infinitely many possibilities (based on the Gauss Bonnet theorem) and we specifically investigate the right angled case. From this theorem, we can also investigate regular polygons in $\mathbb{H}^2$ with angles of the form $\frac{2\pi}{k}$ and their tessellations. Lastly, we use the upper half plane model to construct a right angled hexagon centered at i.

This work is dedicated to the educators who taught me many things about writing and mathematics. More importantly, they taught me about life, faith, and how to inspire students to be their absolute best. It is because of Sr. Carolyn Ratkowski, Mrs. Laurel Marmul, Mrs. Ellen Purrenhage Taber, Mrs. Gale Schwalm, Mr. JC Mellor, Dr. Margret Höft, and my aunt, Aileen Lockhart, that I have been able to give my best toward this and all of my endeavors.

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CHAPTER 1

INTRODUCTION TO SPACES OF CONSTANT CURVATURE

There are three spaces of constant curvature in each dimension $n \geq 2$: $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$. We consider the spherical case, $\mathbb{S}^n$, to have curvature 1 (positive); the Euclidean case, $\mathbb{E}^n$, to have curvature 0 (neutral); and the hyperbolic case $\mathbb{H}^n$ to have curvature -1 (negative). Let $\mathbb{X}^n$ represent any of the three spaces of constant curvature listed above. We will consider each of these spaces as a metric space. A **metric space** is a set M together with a distance function $d : M \times M \to [0, \infty)$ so that d satisfies the following conditions for all $x, y, z \in M$:

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$, and
- $d(x, z) \leq d(x, y) + d(y, z)$.

The metric for each space will be defined as the model space is described. We can also define a **topological space** X together with a family of open sets O. This family of open sets must satisfy the following properties:

- O contains both the empty set and X,

- O is closed under taking arbitrary unions, and
- O is closed under taking finite intersections.

A metric space (X, d) can be given a topology by declaring that $B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$ is open for all $x \in X$ and all $\epsilon > 0$ and allowing O to be as large as necessary so that it contains all of X.

1.1 Standard Model of $\mathbb{E}^n$

The standard model for Euclidean space is the vector space $\mathbb{R}^n$ where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is a vector. We define the Euclidean inner product between any two vectors $x, y \in \mathbb{E}^n$:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

With the norm of a single vector x:

$$||x|| = \sqrt{\langle x, x\rangle}$$

The metric for this space is the standard Euclidean norm between any two vectors x and y defined by:

$$d(x, y) = ||x - y||$$

We consider Euclidean n-space to have the same structure as $\mathbb{R}^n$ except there is no analogous origin point in $\mathbb{E}^n$. The angle θ between two vectors in $\mathbb{R}^n$ can be found using the metric and inner product defined above:

$$\theta = \arccos\left(\frac{\langle x,y\rangle}{||x||||y||}\right)$$

We consider a **hyperplane** in $\mathbb{E}^n$ to be any translated linear subspace of dimension

$n-1$. Consider the two portions of $\mathbb{E}^n$ formed on either side of a hyperplane, these are called **half-spaces**. When we consider half-spaces, we include the bounding hyperplane. In Euclidean space, we consider a **convex polytope** to be the compact intersection of a finite number of such half-spaces. We also define a convex polyhedral cone C as the intersection of a finite number of linear half spaces in $\mathbb{R}^n$. A convex polyhedral cone in $\mathbb{R}^n$ is **essential** if the intersection with a sphere centered at the origin does not contain any pair of antipodal points. Colloquially, for a convex polyhedral cone to be essential, we want to avoid "orange slices" when intersecting with a sphere.

1.2 Standard Model of $\mathbb{S}^n$

Let $\mathbb{S}^n$ be the hypersurface in $\mathbb{R}^{n+1}$ of vectors x such that $||x|| = 1$. For example, we can consider the unit sphere in $\mathbb{R}^4$. Another example is the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$. We apply the standard inner product and metric from Euclidean space to the vectors in $\mathbb{S}^n$. A convex polytope in $\mathbb{S}^n$ is the intersection of $\mathbb{S}^n$ with an essential convex polyhedral cone in $\mathbb{R}^{n+1}$, defined above.

1.3 Models of $\mathbb{H}^n$

There are three main models of hyperbolic space. The first is the hyperboloid model in $\mathbb{R}^{n,1}$, an $(n+1)$-dimensional vector space. Consider the symmetric bilinear form defined by:

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1} \tag{1.1}$$

for $x = (x_1, \cdots, x_{n+1})$ and $y = (y_1, \cdots, y_{n+1})$. The space defined by $\langle x, x\rangle = -1$ is a hyperboloid of two sheets; in this model we consider only one sheet with $x_{n+1} > 0$. To define a hyperplane in $\mathbb{H}^n$, we must consider the hyperplanes in $\mathbb{R}^{n,1}$ that are positive definite with respect to the bilinear form (1.1); such vectors $v \in \mathbb{R}^{n,1}$ have $\langle v, v\rangle > 0$. We define a hyperplane in $\mathbb{H}^n$ to be the intersection of $\mathbb{H}^n$ with a positive definite hyperplane in $\mathbb{R}^{n,1}$.

We also define a **negative light cone** as the set $\{x \in \mathbb{R}^{n,1} \mid \langle x, x\rangle \leq 0, x_{n+1} \geq 0\}$. A convex polytope in $\mathbb{H}^n$ is its intersection with a convex polyhedral cone in $\mathbb{H}^n$ so that $C - \{0\}$ is contained in the interior of a negative light cone.

The next two models of hyperbolic space are specific to two dimensions, but they can be generalized to higher dimensions. First, the Poincaré disk which is the unit disk: $\{x \in \mathbb{R}^2 \mid \langle x, x\rangle < 1\}$. Lines in the Poincaré unit disk are chords or circular arcs meeting the boundary at a perpendicular. The boundary itself is not included in the Poincaré disk model. Bridson and Haefliger [3] define the distance between any two points x, y on the Poincaré disk by:

$$d(x,y) = \operatorname{arccosh}\left(1 + 2\frac{||x-y||^2}{(1-||x||^2)\,(1-||y||^2)}\right)$$

We can also describe the maps from the hyperboloid model to the Poincaré disk model. Consider the points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, 0)$ with $x, y \in \mathbb{R}^3$ so that x is on the hyperboloid with $\langle x, x\rangle = -1$ and y is on the Poincaré disk. Then we have the following conversion from Poincaré disk to the hyperboloid [1]:

$$y_i = \frac{x_i}{1 + x_3}$$

We also have the following conversions from the hyperboloid to the Poincaré disk:

$$x_i = \begin{cases} \frac{2y_i}{1-y_1-y_2} & \text{for} \quad i = 1,2 \\ \frac{1+y_1+y_2}{1-y_1-y_2} & \text{for } i = 3 \end{cases}$$

This particular map projects the hyperboloid in 3-space onto the Poincaré disk by taking the line through $(0, 0, -1)$ and projecting the point from the hyperboloid onto the corresponding point on the Poincarè disk.

The model of hyperbolic 2-space focused on in this paper is the upper half plane model. In two dimensions this model consists of all $x+iy \in \mathbb{C}$ with $y > 0$, that is $\{x+iy \in \mathbb{C} \mid y > 0\}$. In this model, we consider lines to be semi-circles centered on the real axis of the form $\{x+iy \in \mathbb{C} | (x-c)^2 + y^2 = r^2\}$ for $c, r \in \mathbb{R}$. Additionally, we consider vertical lines of the form $x = k$. To find the distance between any two points $A = x_1 + iy_1$ and $B = x_2 + iy_2$, we must consider the line that contains these two points. Let E_1 and E_2 be the endpoints on the real axis of this line so that $E_1 \leq x_1 \leq x_2 \leq E_2$. Then we define the cross ratio between A and B as:

$$(E_1, A; B, E_2) = \frac{(B - E_1)(A - E_2)}{(B - E_2)(A - E_1)} \tag{1.2}$$

Then the hyperbolic distance between A and B is $d_{hyp}(A, B) = \log(E_1, A; B, E_2)$. In this model, angles are still considered in the Euclidean sense, but clearly distance is distorted and increases quickly near the real axis.

1.4 Angle Sum

Within these model spaces, there are strict implications for the angle sum of any convex polygon. First, let E_1 and E_2 be half spaces in $\mathbb{X}^n$ so that $E_1 \cap E_2 \neq \emptyset$ and these half spaces have bounding hyperplanes H_1 and H_2, respectively. Let $x \in H_1 \cap H_2$. Let $u_1 \in E_1$ be the unit normal vector at x with respect to H_1 and let $u_2 \in E_2$ be the unit normal vector at x with respect to H_2. We define $\theta = \cos^{-1} \langle u_1, u_2 \rangle$ to be the **exterior dihedral angle** along $H_1 \cap H_2$. We call $\pi - \theta$ the **dihedral angle**. In two dimensions, the dihedral is also referred to as an **interior angle**. We must also define the area of a polygon. In $\mathbb{E}^2$ and $\mathbb{S}^2$ we consider the area of a polygon as defined by the integral over the polygon with the standard Euclidean measure. In the upper half plane model $\mathbb{H}^2$, Anderson [2] defines the hyperbolic area as the integral:

$$area_{\mathbb{H}}(P^2) = \int_{P^2} \frac{1}{\text{Im}(z)^2} dxdy = \int_{P^2} \frac{1}{y^2} dxdy$$

Then we can consider the following version of the Gauss-Bonnet theorem.

Theorem 1. The Gauss-Bonnet Theorem *[5] Let $P^2 \subseteq \mathbb{X}^2$ be a polygon with interior angles $\alpha_1, \alpha_2, \cdots, \alpha_m$, then*

$$\mathcal{E}\,Area(P^2) + \sum(\pi - \alpha_i) = 2\pi \tag{1.3}$$

Where $\mathcal{E} \in \{-1, 0, 1\}$ and is equal to the curvature of the space.

This implies, in particular, that the angle sum for any m-gon will be strictly greater than $(m-2)\pi$ in $\mathbb{S}^2$, strictly less than $(m-2)\pi$ in $\mathbb{H}^2$, and exactly $(m-2)\pi$ in $\mathbb{E}^2$.

1.5 Isometries of $\mathbb{X}^n$

Ultimately, we want to consider not only the polytopes within these spaces, but also the isometries that can act on polytopes each space. In general, an **isometry** ϕ from a metric space (X, d_1) to another metric space (Y, d_2) is a distance preserving map. That is,

$$d_1(x_1, x_2) = d_2(\phi(x_1), \phi(x_2))$$

for all $x_1, x_2 \in X$. The set of isometries for a metric space is a group under composition. Translations, reflections, and rotations are examples of isometries in Euclidean space. We can consider the group of isometries for each of the three model spaces, described in Bridson and Haefliger.

Theorem 2. *[3, Theorem 2.24] The following groups are isomorphic.*

1. *$Isom(\mathbb{E}^n) \cong \mathbb{R}^n \rtimes O(n)$.*

2. *$Isom(\mathbb{S}^n) \cong O(n+1)$.*

3. *$Isom(\mathbb{H}^n) \cong O(n,1)_+$.*

Where $O(n)$ is the orthogonal group of $n \times n$ matrices with the property $AA^T = I$. The subgroup $O(n,1) \subset \mathrm{GL}(n+1,\mathbb{R})$ are those $(n+1) \times (n+1)$ matrices that leave the metric (1.1) on the hyperboloid invariant. That is, $A \in O(n,1)$ if and only if

$$AA^T = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$$

Because we only consider the sheet with $x_{n+1} > 0$, we want to avoid those matrices that interchange the two sheets, thus we take $O(n,1)_+ \subset O(n,1)$ to be the set of isometries that preserve the upper half sheet. The matrices in $O(n,1)_+$ are those in $O(n,1)$ that have a positive entry in the lower right corner.

In general, let $\mathbb{X}^n$ represent $\mathbb{S}^n$, $\mathbb{E}^n$, or $\mathbb{H}^n$. We want to consider the group of isometries on $\mathbb{X}^n$, denoted $\mathrm{Isom}(\mathbb{X}^n)$. Let H be a linear hyperplane in $\mathbb{E}^n$ defined by a unit normal vector u. Then the following map is a **reflection** in H: $r_H(x) = x - 2\langle x, u\rangle u$ for $x \in \mathbb{E}^n$. The reflection r_H will fix all $x \in H$ and $r_H(u) = -u$. This will determine r_H for all other vectors in $\mathbb{E}^n$. If H is not a linear hyperplane, then it is of the form $H + v$ so that the vectors in this hyperplane all look like $h + v$ for some $h \in H$. Then translate everything by $-v$, apply $r_H(x)$ as before and then translate everything back by $+v$. This will give a reflection in the hyperplane $H + v$. This definition of reflection will preserve distances and therefore preserve $\mathbb{S}^{n-1}$. A reflection can be defined in the hyperboloid model in a similar way, except using the symmetric bilinear form (1.1) instead of the standard inner product $\langle \cdot, \cdot \rangle$. One can verify the following lemma.

Lemma 3. *Every element in* $\mathrm{Isom}(\mathbb{X}^n)$ *can be written as a product of reflections.*

Thus, the isometry group can be generated by a set of reflections each of which clearly has order two. This does not imply that every isometry is a reflection or that every isometry has order two. It simply says that any isometry can be written as the product of reflections.

CHAPTER 2

THE DEFINITION OF COXETER GROUPS

2.1 The Dihedral Groups

A **dihedral group** is a group generated by two involutions: elements that have order two. Consider the finite dihedral groups, denoted D_n, defined in the following way. Given a line L in $\mathbb{R}^2$ and a line L' so that the angle between L and L' is $\frac{\pi}{n}$ for $n \geq 2$. Let ρ_1 be the reflection across L and ρ_2 be the reflection across L'. Then $\tau = \rho_1 \circ \rho_2$ is a rotation through $\frac{2\pi}{n}$. Thus, τ has order n. We now define D_n to be the group generated by ρ_1 and ρ_2. Alternatively, but equivalently, we can define the dihedral group D_n to be the group of symmetries of a regular n-gon so that $|D_n| = 2n$. The finite dihedral groups can be presented in terms of a reflection and rotation, namely, $D_n = \langle \tau, \rho \mid \tau^n = \rho^2 = 1, \rho\tau^{-1} = \tau\rho \rangle$ but we wish to consider groups generated by elements of order two so we will use the presentation generated by two reflections.

2.2 Coxeter Groups

The dihedral groups defined above are groups generated by two elements of order two. We want to consider a larger class of groups that are also generated by elements of order two, but these groups may have more than two generators. First, consider the **Coxeter Matrix** $M = (m_{st})$ on a set S. The Coxeter matrix is an $S \times S$ symmetric matrix with entries from $\mathbb{N} \cup \{\infty\}$ so that

$$m_{st} = \begin{cases} 1 & \text{if } s = t \\ \geq 2 & \text{otherwise} \end{cases} \tag{2.1}$$

From a Coxeter Matrix, we can encode the information into a **Coxeter Diagram** in the following way. Each element in the set S will be represented by one vertex or node.

- If $m_{st} = 2$ then the two vertices are not connected in the Coxeter Diagram.
- If $m_{st} = 3$ then the nodes are connected by an unlabeled edge.
- If $m_{st} \geq 4$, then the vertices are connected by an edge and that edge is labeled with the entry m_{st}.

Coxeter [4] had a third notation used for classifying symmetry groups. It is a bracket notation that is used to concisely describe the information encoded in the edges of a Coxeter diagram.

Consider the dihedral group presented above with $S = \{\rho_1, \rho_2\} \subset D_n$ for $n \geq 3$. Then the Coxeter Matrix corresponding to the set S is:

$$M = \begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$$

for any $n \in \mathbb{N}$ so that $n \geq 2$. The corresponding Coxeter diagram would contain two vertices, with an edge connecting them labeled if $n \geq 4$ and corresponding bracket notation $[n]$.

If W is a group and $S \subset W$ is a set of elements of order 2 which generate W, then we call (W, S) a **pre-Coxeter system**. We can construct a Coxeter matrix and Coxeter diagram

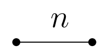

Figure 2.1: Coxeter diagram for D_n

for these pre-Coxeter systems in the following way: for each pair of generators $s, t \in S$, let $m_{st} = |st|$, that is m_{st} is the smallest integer such that $(st)^{m_{st}} = 1$. This gives an $S \times S$ symmetric matrix with entries corresponding to the relations of S. When the Coxeter matrix gives the only relations necessary to define the group, we call W a **Coxeter Group**. If (W, S) is a pre-Coxeter system then W is the homomorphic image of a Coxeter Group G. When these two groups are isomorphic, then we say W is a Coxeter group and S is the fundamental set of generators. A Coxeter group is generated by elements of order two according to the relations prescribed in the corresponding Coxeter matrix (or diagram). If $m_{st} = 2$, then the two elements $s, t \in W$ actually commute. The dihedral groups described above are Coxeter groups with two generators.

CHAPTER 3

REFLECTION GROUPS AND MIRROR STRUCTURE

3.1 Reflection Systems

First, a definition. A **graph** is a pair $\Gamma = (V, E)$ where V is a set of vertices and E is a set of edges, equipped with an incidence relation $* \subseteq V \times E$ such that any edge is incident to two (not necessarily distinct) vertices. We then say that the vertex lies on the edge and the edge has the vertex as an endpoint. A path of length n from x to y is a sequence of edges and vertices $x = x_0, e_1, x_1, \ldots, x_{n-1}, e_n, y = x_n$ such that the edge e_i has endpoints x_{i-1} and x_i for $i = 1, \ldots, n-1$. We can put a metric d on the graph Γ by declaring each edge to be isometric to the real unit interval. In this way a path of length n in the definition above also has length n with respect to metric d. We call the graph Γ equipped with such a metric a **simplicial graph**.

Let W be a group with subset $R \subset W$ so that each element in R has order two and $\langle R \rangle = W$. Consider an action of W by left multiplication on the vertices of a connected simplicial graph Ω with base point $v_0 \in Vert(\Omega)$. This means that W is realized as a group of isometries of Ω that preserve the sets of vertices and edges. We consider cases in which R

is closed under conjugation, and for each edge in Ω there is a unique element in R which flips that edge; that is, if $r \in R$ is applied to the edge $\{w_1, w_2\}$ then $r \cdot w_1 = w_2$ and $r \cdot w_2 = w_1$. Conversely, we also want each $r \in R$ to flip at least one edge in Ω. Let S be the set of reflections that flip the base point v_0. A system (W, R) which satisfies all of these conditions is called a **pre-reflection system**. We consider Ω^r to be the set of midpoints of the edges flipped by r. If for every $r \in R$, $\Omega - \Omega^r$ has two distinct connected components, then we call (W, R) a **reflection system**. This implies that for each $r \in R$, the action of multiplication on the left by r interchanges the connected components of $\Omega - \Omega^r$.

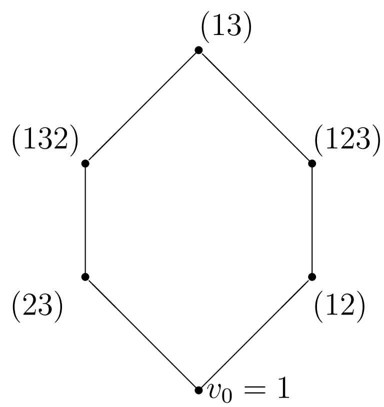

Figure 3.1: Reflection system Ω_1 for S_3 with $\mathcal{S}_1 = \{(12), (23)\}$

For example, consider the group of permutations S_3 with $\mathcal{S}_1 = \{(12), (23)\}$ in Figure 3.1 and $\mathcal{S}_2 = \{(12), (13), (23)\}$ in Figure 3.2. Conjugation of the elements in S_3 by both $\mathcal{S}_1$ and $\mathcal{S}_2$ will result in the same set of reflections $R = \{(12), (23), (13)\}$. For Ω_1, the element (23) flips the edge $\{v_0, (23)\}$ and the edge $\{(13), (123)\}$; the element (12) flips the edge $\{v_0, (12)\}$ and the edge $\{(13), (132)\}$; and the element (13) flips the edge $\{(23), (132)\}$ and the edge $\{(12), (123)\}$. So each element in the set R flips at least one edge and every edge is flipped by an element in R. For each of these reflections, $\Omega - \Omega^r$ is made up of two distinct components, thus $(S_3, \mathcal{S}_1)$ is a reflection system.

Ω_2 gives a pre-reflection system but does not satisfy the conditions for a reflection system. For example, the element $(1, 3)$ interchanges both $\{v_0, (13)\}$ and $\{(12), (132)\}$. Figure 3.2 shows that this reflection will not give two distinct components for $\Omega - \Omega^r$.

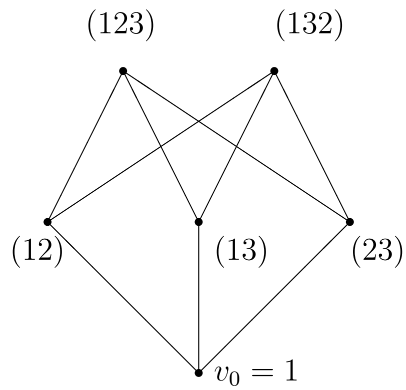

Figure 3.2: Pre-reflection system Ω_2 for S_3 with $\mathcal{S}_2 = \{(12), (13), (23)\}$

3.2 Deletion and Exchange Conditions

Consider a pre-reflection system (W, R) with graph Ω having v_0 as a base point. As before, let S be the set of reflections that flip the edges on the base point v_0. Define a **word** in S to be $\mathbf{s} = (s_1, \ldots, s_k)$ where $s_i \in S$. We will show that the word $\mathbf{s}$ defines an edge path in Ω that starts at the base point v_0. We consider $w(\mathbf{s}) = s_1 \cdots s_k$ to be the value of the word in W. We call $\mathbf{s}$ a **reduced expression** if it is a word of minimum length for $w(\mathbf{s})$, that is, if $l(w(\mathbf{s})) = k$.

Proposition 4. *There is a one to one correspondence between the set of words in S and the set of edge paths starting at v_0*

Proof: First, note that the set R of reflections is the set of elements in W which are conjugate to an element in S. Suppose $\mathbf{s} = (s_1, \ldots, s_k)$ is a word in S. Then for $0 \leq i \leq k$, define $w_i \in W$ so that $w_0 = 1$ and $w_i = s_1 \cdots s_i$ for $i \geq 1$. Then we can define $r_i = w_{i-1} s_i w_{i-1}^{-1} \in R$. It follows that $r_i \cdots r_1 = w_i$. As defined above, $\mathbf{s} = (s_1, \cdots, s_k)$ defines an edge path in Ω starting at v_0. One verifies easily that the action of $w_{i-1} \in W$ sends the edge $\{v_0, s_i v_0\}$ to $\{v_{i-1}, v_i\}$ so that v_{i-1} is, in fact, adjacent to v_i. As defined, r_i is the reflection that flips the edge $\{v_{i-1}, v_i\}$ in the following way:

$$\begin{aligned} r_i v_i &= w_{i-1} s_i w_{i-1}^{-1} v_i \\ &= w_{i-1} s_i w_{i-1}^{-1} w_i v_0 \\ &= w_{i-1} s_i w_{i-1}^{-1} w_{i-1} s_i v_0 \\ &= v_{i-1} \end{aligned}$$

So given a word in S we can construct an edge path in Ω starting at the base point v_0. Conversely, consider the edge path $(v_0, \ldots, v_k)$ which begins at v_0. Then let r_i be the reflection that flips $\{v_{i-1}, v_i\}$ and let $w_i = r_i \cdots r_1$. Then define $s_i = w_{i-1}^{-1} r_i w_{i-1}$. To show $s_i \in S$, we must show that s_i interchanges v_0 and a vertex adjacent to v_0

$$\begin{aligned} s_i v_0 &= w_{i-1}^{-1} r_i w_{i-1} v_0 \\ &= r_1 \ldots r_{i-1} r_i r_{i-1} \ldots r_1 v_0 \\ &= r_1 \ldots r_{i-1} r_i r_{i-1} \ldots r_2 v_1 \\ & \quad \ldots \\ &= r_1 v_2 \\ &= v_1 \end{aligned}$$

Thus, s_i interchanges v_0 and v_1 so that $s_i \in S$. This allows us to determine the word $\mathbf{s}$=$(s_1, \ldots, s_k)$. Thus, there is a one to one correspondence between the set of words in S and the set of edge paths starting at the base point in $v_0 \in \Omega$ $\square$

We now discuss two equivalent conditions pertaining to length of a word $\mathbf{s} \in S$ when (W, S) is a pre-Coxeter system.

Theorem 5. Deletion & Exchange Conditions *[5, Theorem 3.2.16]. If (W, S) is a pre-Coxeter system, then the following are equivalent:*

(D) *Deletion: If $\mathbf{s} = (s_1, \ldots, s_k)$ is a word in S such that $k > l(w(\mathbf{s})))$, then there are* *indices $i < j$ so that the sub word*

$$\mathbf{s}' = (s_1, \ldots, \widehat{s_i}, \ldots, \widehat{s_j}, \ldots, s_k)$$

is also an expression for $w(\mathbf{s})$.

(E) *Exchange: Given a reduced expression for $\mathbf{s} = (s_1, \ldots, s_k)$ for $w \in W$ and an element $s \in S$, then either $l(sw) = l(w) + 1 = k + 1$ or there is is an index i such that*

$$w = ss_1 \cdots \widehat{s_i} \cdots s_k$$

The exchange conditions implies that there are two possibilities for sw. First, $l(sw) = l(w) + 1$ so that a reduced expression for sw can be found by multiplying w on the left by s. Second, if $l(sw) = l(w) - 1$, then the reduced expression for w started with s and thus has been reduced in length.

Consider the graph Ω in figure 3.1 with $\mathcal{S} = \{s_1, s_2\}$ so that $s_1 = (12)$ and $s_2 = (23)$. Then the element $(132) = s_1s_2s_1s_2$ is an expression for the element (132), but $l(s_1s_2s_1s_2) = 4$; we can delete the first and last elements in this expression so that $(132) = s_2s_1$ and $l(s_2s_1) = 2 = l(132)$

For an example of the exchange condition, consider the Cayley graph of the dihedral group D_4. Then Ω is an octagon so that $S = \{\rho_1, \rho_2\}$ as in section 2.1. Consider the element $w = \rho_1\rho_2$, then $l(\rho_1\rho_2)$=2. If we multiply w by $\rho_2 \in S$, then:

$$\rho_2 w = \rho_2\rho_1\rho_2$$

and this is a reduced expression and $l(\rho_2 w) = 3 = l(w) + 1$. We can also multiply w by $\rho_1 \in S$, then:

$$\rho_1 w = \rho_1\rho_1\rho_2 = \rho_2.$$

Clearly, $l(\rho_1 w) \neq l(w) + 1$. but $w = \rho_1 \widehat{\rho_1} \rho_2$ so that the exchange condition is satisfied.

3.3 Mirror Structure

A **mirror structure** on a topological space X consists of an index set S and a set of closed subspaces $(X_s)_{s \in S}$. We call the subspaces X_s the **mirrors** of X. The space X together with a mirror structure is a mirrored space over S. Consider the following example: let each vertex of an equilateral triangle represent and element of the set $S = \{1, 2, 3\}$. We can consider the actions of S_3 acting on the space X in figure 3.3. Then each permutation of S_3 takes the space X and reflects it to another portion of the triangle, resulting in the following reflections of figure 3.4. The mirrored space X in figure 3.3 is a strict fundamental domain for the action of S_3 resulting in the equilateral triangle, Y.

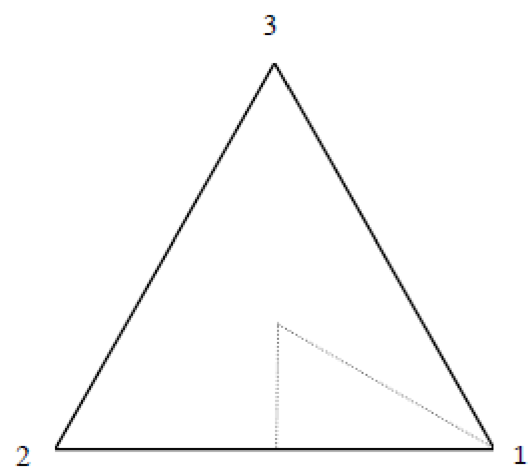

Figure 3.3: X with a mirror structure over {1,2,3}

We say that a closed subspace X of a space Y is a **strict fundamental domain** for a G-action if for each point $y \in Y$, there exists an element in $\sigma \in G$ so that σ maps a point $x \in X$ to y, that is $\sigma \cdot x = y$. If the point x is in the interior of X, then this σ is unique.

We can also consider an example of the group $\mathbb{Z}^2$ acting on $\mathbb{E}^2$ in the following way.

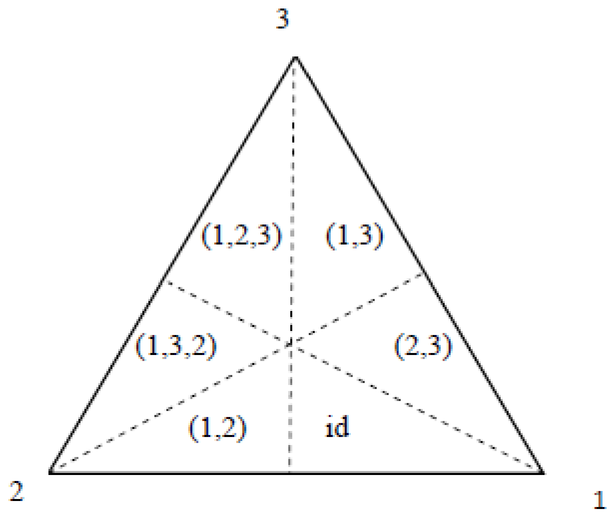

Figure 3.4: The actions of S_3 on X

Define a rule $\mathbb{Z}^2 \times \mathbb{E}^2 \to \mathbb{E}^2$ as follows: let $(z_1, z_2) \cdot (x_1, x_2) = (x_1 + z_1, x_2 + z_2)$. The action is a translation horizontally by z_1 units and a translation vertically by z_2 units. Clearly, this is an isometry of $\mathbb{E}^2$ for each $(z_1, z_2) \in \mathbb{Z}^2$ and one can verify easily that this defines an action of $\mathbb{Z}^2$ on $\mathbb{E}^2$. The one by one square in figure 3.5 is a strict fundamental domain for this action on $\mathbb{E}^2$.

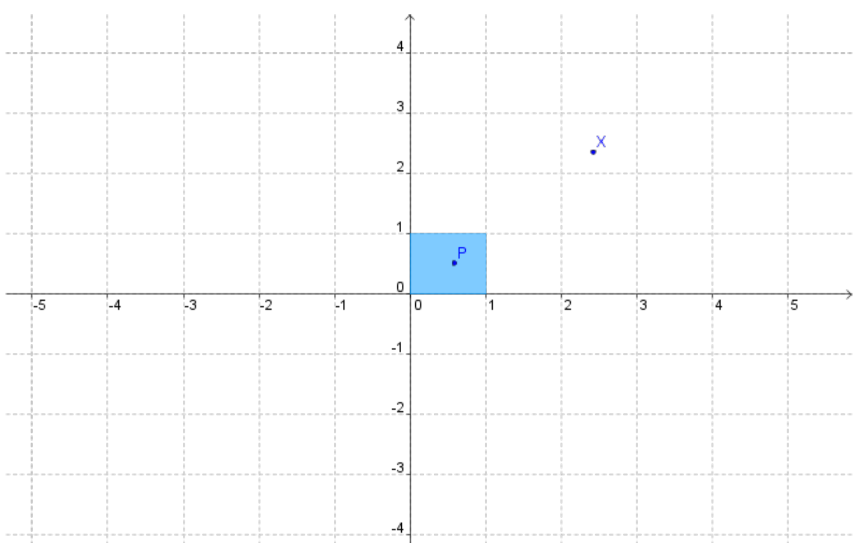

Figure 3.5: A strict fundamental domain for the action of $\mathbb{Z}^2$ on $\mathbb{E}^2$.

CHAPTER 4

GEOMETRIC REFLECTION GROUPS

We want to again consider the spaces of constant curvature denoted by $\mathbb{X}^n$. In this section, we will draw connections between the geometry of these spaces and the Coxeter groups and reflection systems defined previously.

4.1 Convex Polytopes in $\mathbb{X}^n$

To discuss the geometry in $\mathbb{X}^n$, we must first discuss convex polytopes with dihedral angles, defined in section 1.4. Furthermore, let $\{E_1, \ldots, E_k\}$ be a set of half spaces in $\mathbb{X}^n$ so that their common intersection is non-empty. As before, let $H_1, \ldots, H_k$ be the bounding hyperplanes for these half spaces. We say that $\{E_1, \ldots, E_k\}$ has **non obtuse dihedral angles** if for every pair of indices $(i \neq j)$, either the intersection of the bounding hyperplanes is empty, that is $H_i \cap H_j = \emptyset$; or if the intersection is nonempty, $\theta \leq \frac{\pi}{2}$ where θ is the dihedral angle along the intersection.

We can now extend the definition of non obtuse dihedral angles to a convex polytope $P^n \subset \mathbb{X}^n$. Consider the codimension-one faces of the polytope P^n. If these codimension-one faces satisfy the conditions outlined above–that is each pair does not meet or meets at an angle $\theta \leq \frac{\pi}{2}$–then we say that the polytope P^n has non obtuse dihedral angles. If, in fact, each pair of hyperplanes meets at a $\frac{\pi}{2}$ angle, then we say that P^n is right angled. Such a polytope is **simple** if exactly n codemension-one faces meet at each vertex. One can show that any convex polytope in $\mathbb{X}^n$ with non obtuse dihedral angles is simple.

4.2 Reflection Groups of Convex Polytopes

The following theorem describes under what conditions a polytope in a space of constant curvature will tile the space. We consider $\{F_1, F_2, \ldots, F_k\}$ to be the codimension one faces of the polytope P^n.

Theorem 6. *[5, Theorem 6.4.3] Suppose P^n is a simple convex polytope in $\mathbb{X}^n$, $n \geq 2$, with dihedral angles of the form $\frac{\pi}{m_{ij}}$ so that $m_{ij} \in \mathbb{N}$ whenever $F_i \cap F_j \neq \emptyset$. If the intersection is empty, then let $m_{ij} = \infty$ Let W be the Coxeter group defined by the Coxeter matrix $M = (m_{ij})$, then P^n is a strict fundamental domain for the W-action on $\mathbb{X}^n$.*

The conclusion of this theorem implies that $\mathbb{X}^n$ will be tiled by congruent copies of the polytope P^n. Because we assume that the polytopes have non obtuse dihedral angles, we actually do not need the simplicity condition in the statement of the theorem. As stated above, any convex polytope with non obtuse dihedral angles is simple.

The polytopes in these spaces satisfy the Gauss-Bonnet Theorem (1.3). Because of this, we can actually list all such polytopes in both $\mathbb{S}^2$ and $\mathbb{E}^2$. To see this, let P be an m-gon in $\mathbb{X}^2$ with non-obtuse angles $\alpha_i = \frac{\pi}{m_i} \leq \frac{\pi}{2}$ for $m_i \in \mathbb{N}$ and $i = 1, 2, \ldots, m$.

Consider first $P \subset \mathbb{S}^2$. Then, $\Sigma\alpha_i > (m-2)\pi$ implies that P must be a triangle, any other m-gon with non obtuse angles will not satisfy the Gauss-Bonett Theorem. This implies

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} > 1.$$

This quickly gives only four such possibilities for (m_1, m_2, m_3): (2,3,3), (2,3,4), (2,3,5) and

(2,2,n) for $n \geq 2$. The first three symmetry triples correspond to the symmetry groups of the platonic solids:

- (2,3,3) corresponds to the symmetry group of a tetrahedron S_4 of type A_3.
- (2,3,4) corresponds to the symmetry group of the cube or, its dual, the octahedron $2^3 \rtimes S_3$ of type C_3.
- (2,3,5) corresponds to the symmetry group of the dodecahedron or, its dual, the icosahedron $\mathrm{Alt}(5) \times C_2$ of type H_3.

While $(2, 2, n)$ corresponds to the direct product of $C_2 \times D_n$.

Next, consider $P \subset \mathbb{E}^2$ Again, each angle $\alpha_i \leq \frac{\pi}{2}$ and $\Sigma\alpha_i = (m-2)\pi$ implies that $m \leq 4$. There is only one possibility for $m = 4$, namely a rectangle with $m_1 = m_2 = m_3 = m_4 = 2$. This corresponds to the dihedral group $D_\infty \times D_\infty$. Moreover, If $m = 3$, we have:

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 1$$

Which gives the following possibilities for (m_1, m_2, m_3): (2,3,6), (2,4,4), and (3,3,3). These reflections yield the Euclidean triangle groups:

- The (2,3,6) case corresponds to the Coxeter group with diagram $\widetilde{G}_2$.
- The (2,4,4) case corresponds to the Coxeter group with diagram $\widetilde{C}_2$.
- The (3,3,3) case corresponds to the Coxeter group with diagram $\widetilde{A}_2$.

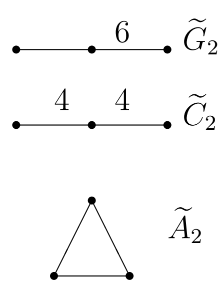

Figure 4.1: Coxeter diagrams for the Euclidean Triangle Groups

In the spherical and Euclidean cases, the Gauss Bonnet theorem restricts the the number of polytopes we can construct to be a strict fundamental domain for the reflection groups $\text{Isom}(\mathbb{X}^2)$. The hyperbolic case gives an infinite number of possibilities for the strict fundamental domain of any such tessellation. For example, in $\mathbb{H}^2$ for any $m \geq 5$, there exists an m-gon with right angles only, any such right angled polygon will tessellate hyperbolic two space.

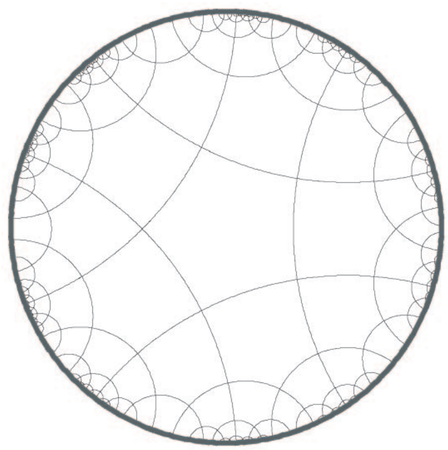

Figure 4.2: Tessellation of the Poincaré disk with right angled pentagons

4.3 Regular Polygons in $\mathbb{H}^2$

The conditions in Theorem 6 can be extended to regular polygons in $\mathbb{H}^2$ in the following corollary.

Corollary 7. *Let Q be a regular m-gon in hyperbolic 2-space with angles $\alpha_i = \frac{2\pi}{k}$. Any such polygon will tessellate $\mathbb{H}^2$ with k copies of Q meeting at each vertex.*

Proof: Essentially, we will decompose this regular m-gon into $2m$ congruent triangles that each have one right angle. These triangles will satisfy the conditions of Theorem 6. If k is even, the conditions of Theorem 6 are immediately met, but we can still complete the construction to reduce the strict fundamental domain to a triangle. Divide the m-gon into m triangles meeting at the center with angles $\frac{\pi}{k}$, $\frac{\pi}{k}$, and $\frac{2\pi}{m}$. The sides of these triangles will bisect the angles of Q at each vertex. Further divide one of these triangles into two right triangles by bisecting the $\frac{2\pi}{m}$ angle at the center and one edge of the polygon. Call one of these triangles P. Now the triangle P has angles $\frac{\pi}{k}$, $\frac{\pi}{2}$, and $\frac{\pi}{m}$. As long as the original m-gon satisfies the conditions of the Gauss Bonnet Theorem, P will have angle sum strictly less than π, which is necessary. Now, P satisfies the conditions of Theorem 6 with corresponding Coxeter diagram 4.3.

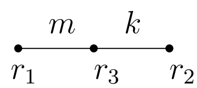

Figure 4.3: Coxeter diagram for the triangle P

The triangle P is a strict fundamental domain for the action of the reflection group $W = \langle r_1, r_2, r_3 \rangle$ in the notation of Theorem 6. We still must show that this tessellation with strict fundamental domain P is also a tessellation by our original m-gon Q. Consider the subgroup $W' = \langle r_1, r_3 \rangle$, then W' is the dihedral group of order $2m$ defined in section 2.1. This implies that the original m-gon $Q = \bigcup_{w \in W'} wP$.

Let the set $U \subset W$ be a complete set of coset representatives for W'. Consider the set $\mathcal{T} = \{uQ \mid u \in U\}$ which is a set of images of Q after the action of the coset representatives from U. We want to show that the images of this action give a tessellation of $\mathbb{H}^2$. First, suppose that there are two coset representatives $u, u' \in U$ so that $uQ \cap uQ'$ is nonempty. Then this implies that $uwP = u'w'P$ for some $w, w' \in W'$. Since P satisfies the conditions of theorem 6, it is a strict fundamental domain for the action of W on $\mathbb{H}^2$. Therefore, either $uQ \cap uQ' = \emptyset$ or u and u' represent the same coset of W'. This implies that if we select $u \in U$ then the images of Q in $\mathcal{T}$ will not overlap.

Next, we must show that the action $\mathcal{T}$ covers all of $\mathbb{H}^2$. To show this, consider the set $\widetilde{W} = \{uw \mid u \in U, w \in W'\}$. Clearly, $\widetilde{W} \subseteq W$. Since U is the set of coset representatives of W', $W \subseteq \widetilde{W}$; thus, $\widetilde{W} = W$. Then, because P is a strict fundamental domain for the action by W, every point of $\mathbb{H}^2$ belongs to some tile of $\mathcal{T}$. So this action covers all of $\mathbb{H}^2$ with no gaps or overlaps, thus it is a tessellation by the m-gon Q. $\square$

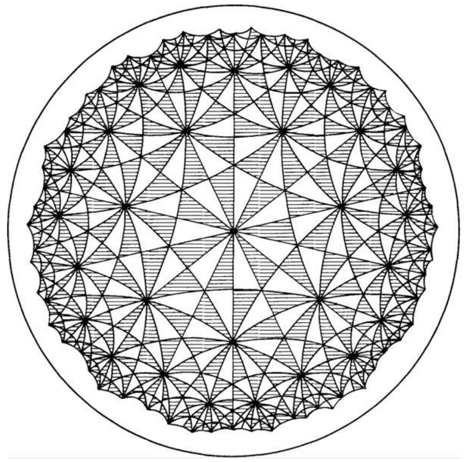

Figure 4.4: Hyperbolic 2-space tiled with a regular heptagon subdivided into right triangles

For example, consider figure 4.4. This is a tessellation of the Poincaré disk by triangles, but it is also a tessellation by a regular heptagon with angles $\alpha_i = \frac{\pi}{3}$. The shading of the figure shows the construction of subdividing the heptagon into right triangles.

CHAPTER 5

CREATING A RIGHT ANGLED HEXAGON CENTERED AT i

As stated previously, there exists a right angled m-gon in hyperbolic 2-space for all $m \geq 5$. We will construct a right angled hexagon in the upper half plane model centered at i. This hexagon can be decomposed into six triangles with the $\frac{\pi}{2}$ angles being bisected at each vertex. These triangles will have angles $\frac{\pi}{4}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ at the center. As in the proof of Corollary 7, we will again consider the triangle obtained by bisecting the $\frac{\pi}{3}$ angle at the center and one edge of the hexagon. Let β be this line through i that makes a $\frac{\pi}{6}$ angle with the imaginary axis, then one of these triangles is formed by the bisected side of the hexagon, β, and the imaginary axis. Now, β is of the form $(x-\gamma)^2 + y^2 = \rho^2$. To find γ, consider the derivative of β at $(0, i)$:

$$y' = \frac{-x+\gamma}{y}$$
$$y'(0,1) = \gamma$$

Because we want β to meet the imaginary axis at an $\frac{\pi}{6}$ angle, we have:

$$\gamma = \frac{\sin(-\frac{\pi}{3})}{\cos(\frac{-\pi}{3})} = -\sqrt{3}$$

This implies β is the line $2^2 = (x+\sqrt{3})^2 + y^2$.

Let l_1 be one of the sides of the hexagon that is bisected by the imaginary axis. Then l_1 is of the form $x^2 + y^2 = r_1^2$ with $r_1 < 1$. This will be the first side–or hyperplane–of our right angled hexagon and the point of intersection between l_1 and β will be a vertex of our hexagon.

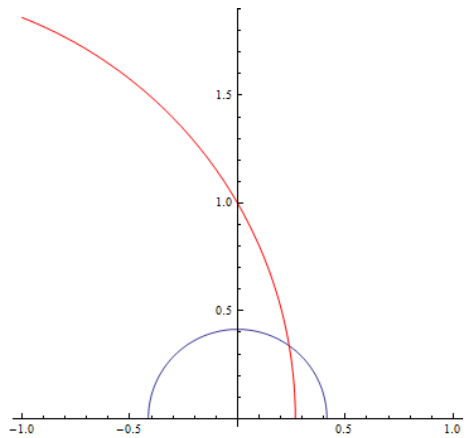

Figure 5.1: The line β in red and the line l_1 in blue in $\mathbb{H}^2$

We want l_1 to intersect β at a $\frac{\pi}{4}$ angle. Let $v_1 = (x_1, y_1)$ be this point of intersection. Consider the vectors $u_1 = \langle 1, \frac{-x_1}{y_1} \rangle$ and $u_2 = \langle 1, \frac{-x_1+\sqrt{3}}{y_1} \rangle$ given by the tangent vectors of β and l_1 at v_1. We can then solve for v_1 with the following system of equations:

$$\begin{aligned} \cos\frac{\pi}{4} &= \frac{u_1 \cdot u_2}{\|u_1\| \|u_2\|} \\ 2^2 &= (x_1 + \sqrt{3})^2 + y_1^2 \end{aligned}$$

The solution to this system gives us the first vertex of the hexagon. Using this vertex we can find the center c_2 of l_2 by making the slope of the tangent line at v_1 perpendicular to the tangent line of l_1. The value of the radius follows.

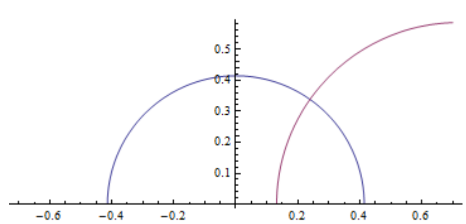

Figure 5.2: The lines l_1 and l_2

To find the next vertex, consider the line through $(0, i)$ that makes a $\frac{\pi}{2} = \frac{\pi}{6} + \frac{\pi}{3}$ with the imaginary axis. This line must be $x^2 + y^2 = 1$. It will intersect the line l_2 at a $\frac{\pi}{4}$ angle to gives us $v_2 = (x_2, y_2)$. Again, find the center c_3 of l_3 by making the slope of the tangent line at v_2 perpendicular to the tangent of l_2. l_4 will have to be centered at the origin so that its midpoint is on the imaginary axis. The radius will be $\frac{1}{r_1}$, it will intersect l_3 at $v_3 = (x_3, y_3)$.

This construction yields v_1, v_2, v_3, and the other 3 vertices can be found by taking $v_4 = (-x_1, y_1)$, $v_5 = (-x_2, y_2)$, and $v_6 = (-x_3, y_3)$.

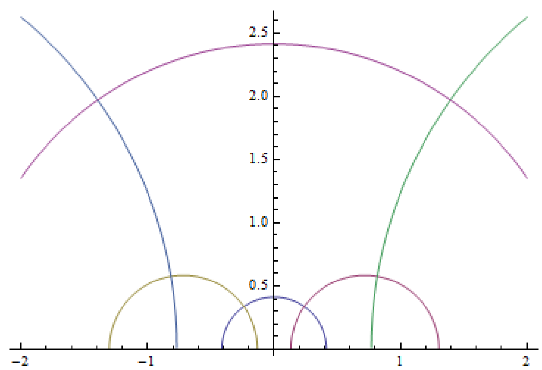

Figure 5.3: Right angled hexagon centered at i in the upper half plane model

5.1 Generalizing The Construction

We can construct a right angled m-gon for $m \geq 5$ in a similar way. Let β_1 be the line through $(0, i)$ so that β meets the imaginary axis at a $\frac{\pi}{m}$ angle. Clearly, β is of the form $(x - \gamma)^2 + y^2 = \rho^2$. We find γ by considering the derivative of β at $(0, i)$.

$$y' = \frac{-x+\gamma}{y}$$

$$y'(0,1) = \gamma$$

Because we want β to meet the imaginary axis at a $\frac{\pi}{m}$ angle,

$$\gamma = \frac{\sin\left(\frac{3\pi}{2} + \frac{\pi}{m}\right)}{\cos\left(\frac{3\pi}{2} + \frac{\pi}{m}\right)} = \frac{\sin\left(\frac{\pi(3m+2)}{2m}\right)}{\cos\left(\frac{\pi(3m+2)}{2m}\right)}$$

As before, let l_1 be the line $x^2 + y^2 = r_1^2$ for $r_1 < 1$. We want β_1 to intersect l_1 at a $\frac{\pi}{4}$ angle. Let $u_1 = \langle 1, \frac{-x_1}{y_1} \rangle$ and $u_2 = \langle 1, \frac{-x_1-\gamma}{y_1} \rangle$. The first vertex of the m-gon will be the solution to the following system:

$$\begin{aligned} \cos\tfrac{\pi}{4} &= \tfrac{u_1 \cdot u_2}{\|u_1\|\|u_2\|} \\ 1 &= (x_1 - \gamma)^2 + y_1^2 \end{aligned}$$

We then use the slope of the tangent line of l_1 at v_1 to find the center and radius of l_2: $(x - c_2)^2 + y^2 = r_2^2$. To determine the next vertex, again consider a line β_2 through $(0, i)$, this time with a $\frac{\pi}{m} + \frac{2\pi}{m} = \frac{3\pi}{m}$. This line should intersect l_2 at a $\frac{\pi}{4}$ angle so that it bisects the angle at the next vertex. Again, consider the tangent vectors at $v_2 = (x_2, y_2)$ on β_2 and l_2 so that they satisfy the system of equations above for v_2. We can repeat this process, finding the find the slope of the tangent line at v_2 and using the perpendicular slope to determine the center and radius of l_3: $(x - c_3)^2 + y^2 = r_3^2$. Continue this process for $\lfloor \frac{m}{2} \rfloor$ steps to find the vertices with $x_i > 0$. To find the vertices with $x_i < 0$, simply take the reflection of $v_i = (x_i, y_i)$ across the y-axis. If m is even, this will give all of the vertices. If m is odd, the final vertex will be on the imaginary axis.

The mathematics of this paper was investigated and developed by H.S.M Coxeter, a mathematician at the University of Toronto. There is another side to this work that lies in art. The Dutch artist M.C. Escher lived from 1898-1972. He had no formal training in mathematics but was very curious about tiling the plane with an infinite number of congruent shapes. Escher and Coxeter established a correspondence to discuss the mathematics of Escher's work. While Escher was grateful for Coxeter's input, he often lamented that he did not understand much of Coxeter's writing. After receiving a copy of Escher's *Circle Limit III*, Coxeter responded with three pages of mathematics. Schattschneider [8] recounts Escher's response: "Three pages of explanation of what I actually did...It is a pity that I understood nothing, absolutely nothing of it."

Escher's *Circle Limit* works correspond to tessellations of the Poincaré disk. The associated symmetry groups can be written in Coxeter's bracket notation. For example, *Circle*

Limit II corresponds to the symmetry group represented $[3^+, 8]$ and *Circle Limit IV* corresponds to the symmetry group represented by $[4^+, 6]$.

As Coxeter continued to develop this mathematical theory, he admired Escher's work. Even more than that, Coxeter investigated the mathematics behind Escher's work. While Escher did not understand Coxeter's mathematics, Coxeter vindicated the mathematics behind Escher's work. Strauss [10] notes: "The result, Prof. Coxeter said with the awe dripping from every word, is astounding. 'He got it absolutely right to the millimetre, absolutely to the millimetre... Unfortunately, he didn't live long enough to see my mathematical vindication.'" It is quite incredible that Escher achieved this kind of mathematical accuracy with no mathematical training. It was purely intuition and curiosity that led him to investigate the isometries and tessellations that produced beautiful works of art.

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