APPLICATIONS OF ENTIRE FUNCTION THEORY TO THE SPECTRAL SYNTHESIS OF DIAGONAL OPERATORS

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ABSTRACT

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A diagonal operator acting on the space $H(B(0,R))$ of functions analytic on the disk $B(0,R)$ where $0 < R \leq \infty$ is defined to be any continuous linear map on $H(B(0,R))$ having the monomials z^n as eigenvectors. In this dissertation, examples of diagonal operators D acting on the spaces $H(B(0,R))$ where $0 < R < \infty$, are constructed which fail to admit spectral synthesis; that is, which have invariant subspaces that are not spanned by collections of eigenvectors. Examples include diagonal operators whose eigenvalues are the points $\{n^a e^{2\pi ij/b} : 0 \leq j < b\}$ lying on finitely many rays for suitably chosen $a \in (0,1)$ and $b \in \mathbb{N}$, and more generally whose eigenvalues are the integer lattice points $\mathbb{Z} \times i\mathbb{Z}$. Conditions for removing or perturbing countably many of the eigenvalues of a non-synthetic operator which yield another non-synthetic operator are also given. In addition, sufficient conditions are given for a diagonal operator on the space $H(B(0,R))$ of entire functions (for which $R = \infty$) to admit spectral synthesis.

This dissertation is dedicated to my family who believed in me even when I did not believe in myself. Especially to my Grandma, who I know would be proud of me.

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Table of Contents

CHAPTER 1

Invariant Subspaces, Diagonal Operators and Spectral Synthesis

1.1 Introduction

The purpose of this document is to study the closed invariant subspaces of analogues of diagonal operators acting not on Hilbert spaces, but instead on spaces of functions analytic on regions in the complex plane. The general setting for our study concerns continuous linear operators $T : \mathcal{X} \to \mathcal{X}$, where $\mathcal{X}$ is a complete metrizable topological vector space. Recall that a closed subspace $\mathcal{M}$ of $\mathcal{X}$ is **invariant** for T if $Tx \in \mathcal{M}$ whenever $x \in \mathcal{M}$. Examples of invariant subspaces include the closed linear span of eigenvectors for T if any exist, and more generally the closed linear spans $\overline{\vee\{T^n x : n \geq 0\}}$ of orbits $\{T^n x : n \geq 0\}$ of vectors $x \in \mathcal{X}$. In fact, $\overline{\vee\{T^n x : n \geq 0\}}$ is the smallest closed invariant subspace for T containing x. However, it may be that such a subspace coincides with all of $\mathcal{X}$; in this case, we say that x is a **cyclic vector** for T and that T is a **cyclic operator**. If T has no non-trivial invariant subspaces then every non-zero vector is cyclic. Consequently, examples of non-trivial invariant subspaces for T are obtained from its non-cyclic vectors. A long-standing open problem in operator theory is to determine whether or not every operator acting on a separable Hilbert space has a non-trivial invariant subspace; it is the Invariant Subspace Problem.

The importance of cyclic vectors and invariant subspaces derives in part from Linear Algebra, the study of operators on finite-dimensional spaces. In particular, every linear map T on $\mathbb{C}^n$ is known to have an eigenvector x with associated eigenvalue λ which generates an invariant subspace $R(\lambda) \equiv \vee\{\cup_k Ker((T-\lambda)^k)\}$ for T, called the root space for T. The map T, when restricted to this root space, is given by the sum of Jordan cells. A Jordan cell is a matrix (with respect to an appropriately chosen basis for $R(x)$) having λ's on the main diagonal, ones on the super-diagonal, and zeros elsewhere. Each Jordan cell is a cyclic operator having as cyclic vectors any column vector whose last coordinate is non-zero. The operator T may be viewed as the assemblage of each of its "parts," namely, the restrictions of T to its root spaces. More precisely, the Jordan Decomposition Theorem states that every linear map on $\mathbb{C}^n$ is similar to the direct sum of Jordan cells. It follows that a linear map T on $\mathbb{C}^n$ is cyclic if and only if the diagonal entries of its Jordan cells are distinct. In this case, a vector is cyclic for T if and only if it is the sum of cyclic vectors for the Jordan cells.

The first infinite-dimensional generalization of $\mathbb{C}^n$ that one might study is a separable Hilbert space $\mathcal{H}$. In an effort to better understand an operator $T : \mathcal{H} \to \mathcal{H}$, it seems natural to try to decompose T into its "parts" or restrictions of T to its invariant subspaces. However, the Invariant Subspace Problem, which remains an open problem, is to determine whether or not every operator on a separable Hilbert space has a non-trivial invariant subspace. This problem has been solved in several special cases but not in general. For example, Per Enflo [14] has constructed a Banach space on which no operator has an invariant subspace, however, there are examples of Banach spaces on which it is known (see [2], for instance) that every operator has non-trivial invariant subspaces. In view of the Jordan Decomposition Theorem on $\mathbb{C}^n$, it seems natural to believe that every operator on a separable Hilbert space has non-trivial invariant subspaces, namely its root spaces. However, operators on a Hilbert space need not have any eigenvalues, such as the forward shift. Other operators may have uncountably many eigenvalues, such as the backward shift whose eigenvalues are precisely the points in the open unit disk.

Thus, in general, one can not hope to identify or characterize the set of cyclic vectors or lattice of invariant subspaces of an arbitrary operator on a separable Hilbert space. For this reason, research in operator theory is often of one of two types; either study as many different aspects of a certain class of operators as one can, or try to decide one specific property of all operators. This dissertation is of the former type; in particular, we study analogues of direct sums of Jordan cells (reminiscent of the Jordan Decomposition Theorem), called diagonal operators which act on spaces of functions analytic on a region in the complex plane, and attempt to decide whether or not their invariant subspaces are of a special type (reminiscent of root spaces).

We now describe the class of operators in question by revisiting the finite-dimensional setting of $\mathbb{C}^n$. Recall, a linear map on $\mathbb{C}^n$ is similar to the direct sum of Jordan cells, which have a constant on the main diagonal, ones on the super-diagonal, and zeros elsewhere. A simple generalization to a Hilbert space is to consider a single Jordan cell with eigenvalue zero and ones along the super-diagonal, which is precisely the shift operator. In a seminar paper of 1949, Beurling showed that a closed subspace is invariant for the shift operator S, when viewed as acting on the Hardy space H^2, if and only if it has the form BH^2 where B is a so-called inner function. It then follows, as a corollary, that a function $f \in H^2$ is cyclic for S if and only if f has no inner divisor; that is, if f is a so-called outer function.

Another generalization of an arbitrary operator, which can be thought of as the direct sum of Jordan cells, acting on $\mathbb{C}^n$ to Hilbert spaces is to take the direct sum of many one-by-one Jordan cells, instead of one large Jordan cell. This is a so-called diagonal operator. More precisely, if $\mathcal{H}$ is a Hilbert space with an orthonormal basis $\{e_n\}$ and $D : \mathcal{H} \to \mathcal{H}$ is a continuous linear operator, we say D is a **diagonal operator with eigenvalues** $\{\lambda_n\} \subset \mathbb{C}$ if $D(e_n) = \lambda_n e_n$ for all $n \geq 0$. Hence, D is a diagonal operator if every basis element e_n is an eigenvector for D. Thus, for diagonal operators some simple examples of closed invariant subspaces are the closed linear spans of arbitrary collections of eigenvectors $\overline{\vee\{e_n : n \in N\}}$ where $N \subseteq \mathbb{N}$. It may seem that these constitute the entire collection of closed invariant subspaces of D. However, in 1921, Wolff presented an example showing this need not be the case. As a result, the invariant subspaces and cyclic vectors of diagonal operators acting on Hilbert spaces became an active area of research carried out by Wolff [47], Wermer [45], Scroggs [40], Brown, Shields, and Zeller [8], Sarason [38], Nikol'skii [32] and [33], and Sibilev [44], amongst others. One of the central open problems is to determine conditions for a diagonal operator acting on a separable Hilbert space to have closed invariant subspaces consisting only of spaces spanned by the eigenvectors they contain. Such operators are said to be **synthetic** or to **admit spectral synthesis**, otherwise we say the operator is **non-synthetic**. Wolff's example demonstrated the existence of a non-synthetic diagonal operator acting on the Hilbert space ℓ^2. The concept of spectral synthesis has been extended to analogues of diagonal operators acting on spaces of functions analytic on regions in the complex plane in the work of Deters, Marin, Seubert, and Wade ([11]-[13], [31], [41]-[43]). However, it was not known whether or not there exist diagonal operators on such spaces which fail spectral synthesis.

In this dissertation, we show that there exist non-synthetic diagonal operators acting on spaces of functions analytic on the unit disk which fail spectral synthesis by constructing examples analogous to Wolff's 1921 example.

In the rest of this chapter, we discuss the known equivalent conditions for a diagonal operator to admit spectral synthesis on a Hilbert space, the space of entire functions $H(\mathbb{C})$, and the space of functions analytic on the unit disk $H(\mathbb{D})$. We also discus examples of non-synthetic diagonal operators acting on a ℓ^2, as well as several known results that can be used to test for synthesis.

In Chapter 2, we show the diagonal operator on $H(\mathbb{D})$ with eigenvalues $\mathbb{Z} \times i\mathbb{Z}$ fails to admit spectral synthesis.

In Chapter 3, we show the diagonal operator on $H(\mathbb{D})$ with eigenvalues $\{n^a e^{2\pi i j/b} : 0 \leq$

$j < b\}$ for suitably chosen $a \in (0,1)$ and $b \in \mathbb{N}$, fails synthesis.

In Chapter 4, we give conditions for modifying (that is, adding, rearranging, deleting, or perturbing) countably many eigenvalues of a non-synthetic operator acting on $H(\mathbb{D})$ to yield another non-synthetic operator. We illustrate these results using the examples of non-synthetic operators obtained in Chapters 2 and 3.

In Chapter 5, we give a sufficient condition for a diagonal operator acting on the space of entire functions to admit spectral synthesis. In particular, we strengthen a result of Leontev's [25] which asserts that if D is a diagonal operator acting on $H(\mathbb{C})$ with eigenvalues $\{\lambda_n\}$, which satisfy $\{|\lambda_n|\}$ is increasing and $0 < \liminf_{n\to\infty} |\lambda_n|/n \leq \limsup_{n\to\infty} |\lambda_n|/n < \infty$, then D admits spectral synthesis. We demonstrate that the condition $\liminf_{n\to\infty} |\lambda_n|/n > 0$ is not necessary.

For the convenience of the reader, we include an appendix containing an overview of the results from the theory of entire functions which are necessary in our results. All of the information given can be found in the books of Boas [5], Levin [29] and [30], Holland [18], and Rubel [36], amongst others.

1.2 The Hilbert Space Case

In this section, we discuss the relevant background information regarding the spectral synthesis of diagonal operators acting on a separable Hilbert space. In Section 1.5, we have the analogous discussion for the spectral synthesis of diagonal operators acting on the space of entire functions, and in Section 1.6, we have the analogous discussion for diagonal operators acting on the space of functions analytic on the unit disk.

Cyclic vectors and invariant subspaces of diagonal operators acting on a separable Hilbert space have been studied extensively since at least 1921 by Wolff [47], Wermer [45], Scroggs [40], Brown, Shields, and Zeller [8], Sarason [38] and [39], Nikol'skii [32] and [33], and Sibilev [44], amongst others. The following theorem, extracted from these references, gives several equivalent conditions for a diagonal operator acting on a separable Hilbert space to admit spectral synthesis.

Theorem 1.1. *Let $\mathcal{H}$ be a separable complex Hilbert space and let D be any bounded linear operator on $\mathcal{H}$ for which there exists an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ and a sequence $\{\lambda_n\}$ of complex numbers for which $De_n = \lambda_n e_n$ for all $n \geq 0$. Then $\{\lambda_n\}$ is bounded. Moreover, D is cyclic if and only if $\lambda_m \neq \lambda_n$ for all $m \neq n$, and in this case, the following are equivalent:*

(i) *D admits spectral synthesis,*

(ii) *a vector x is cyclic for D if and only if $\langle x, e_n \rangle \neq 0$ for all n,*

(iii) *there does not exist a sequence $\{\omega_n\}$ of complex numbers in ℓ^1, not all zero, for which $\sum_{n=0}^{\infty} \omega_n \lambda_n^k \equiv 0$ for all $k \geq 0$,*

(iv) *there does not exist a sequence $\{\omega_n\}$ of complex numbers in ℓ^1, not all zero, for which the Wolff-Denjoy series $\sum_{n=0}^{\infty} \frac{\omega_n}{z-\lambda_n} \equiv 0$ for all z with $|z| > \sup |\lambda_n|$,*

(v) *there does not exist a sequence $\{\omega_n\}$ of complex numbers in ℓ^1, not all zero, for which the complex measure $\mu \equiv \sum_{n=0}^{\infty} \omega_n \delta_{\{\lambda_n\}}$ consisting of point masses at the λ_n with weights ω_n annihilates the polynomials,*

(vi) *there does not exist a sequence $\{\omega_n\}$ of complex numbers in ℓ^1, not all zero, for which the exponential series $\sum_{n=0}^{\infty} \omega_n e^{\lambda_n z} \equiv 0$ on the complex plane,*

(vii) *every closed invariant subspace for D is also invariant for the adjoint D^* of D,*

(viii) *the weakly closed algebra generated by D and the identity is the commutant of D, and*

(ix) *there does not exist a bounded complex domain Ω such that $\sup\{|f(z)| : z \in \Omega\} = \sup\{|f(z)| : z \in \Omega \cap \{\lambda_n\}\}$ for all f bounded and analytic on Ω.*

If, in addition, the λ_n lie in the open unit disk and accumulate only on the unit circle, then conditions (i)-(ix) are equivalent to *(x)* *not almost every point of the unit circle is in the non-tangential cluster set of* $\{\lambda_n\}$, *and*

(xi) *the map* $T : H^\infty \to \ell^\infty(\mu)$ *from the space of functions bounded and analytic on the open unit disk to* $\ell^\infty(\mu)$, *where* $\mu = \sum_{n=0}^{\infty} \delta_{\{\lambda_n\}}$ *is the measure consisting of point masses at the eigenvalues defined by* $T : f \to \{f(\lambda_n)\}$, *is not an isometry.*

The equivalent conditions given in the preceding theorem demonstrate the diverse nature of spectral synthesis. Conditions *(i)*, *(ii)*, *(vii)*, and *(viii)* are all purely operator theoretic statements involving invariant subspaces, cyclic vectors, adjoints, commutants, and the algebra generated by D. Condition *(iii)* is a combinatoric statement about moments. Conditions *(iv)* and *(vi)* regard, in some sense, the linear independence of $\{1/(z - \lambda_n)\}$ and $\{e^{\lambda_n z}\}$, respectively. Conditions *(v)* and *(xi)* are functional analytic statements about measures and isometries. Conditions *(ix)* and *(x)* are purely geometrical statements about so-called dominating sequences.

The equivalence of several of the diverse conditions in the preceding theorem can be established easily. The combinatoric Condition *(iii)* is easily seen to be equivalent to the "linear independence" Condition *(vi)* by observing $\sum_{n=0}^{\infty} \omega_n \lambda_n^k = g^{(k)}(0)$ for all $k \geq 0$, where $g(z) \equiv \sum_{n=0}^{\infty} \omega_n e^{\lambda_n z} \in H(\mathbb{C})$ (and recalling, $g(z) \equiv 0$ if and only if $g^{(k)}(0) = 0$ for all $k \geq 0$). Similarly, Condition *(iii)* is easily seen to be equivalent to the measure theoretic Condition *(v)* by observing $\sum_{n=0}^{\infty} \omega_n \lambda_n^k = \int z^k d\mu$ for all $k \geq 0$, where $\mu \equiv \sum_{n=0}^{\infty} \omega_n \delta_{\{\lambda_n\}}$ is the measure consisting of weighted point masses. The equivalence of several of the operator theoretic conditions is discussed in further detail in Section 1.3.

The interpretations of several of the equivalent conditions for spectral synthesis given in Theorem 1.1 provide some insight into the behavior of the eigenvalues. For example, Condition *(ix)* states that the operator fails to admit spectral synthesis if and only if the sequence of eigenvalues $\{\lambda_n\}$ is a so-called dominating sequence; that is, D is non-synthetic whenever the eigenvalues are "thick enough" to recapture the supremum of $|f(z)|$ for any function f bounded and analytic on some domain $\Omega \subset \{z \in \mathbb{C} : |z| < 1\} \equiv \mathbb{D}$. In view of the

Maximum Modulus Principle, this condition requires that the points $\{\lambda_n\}$ be "thick enough" near the boundary of Ω for D to fail synthesis on $\mathcal{H}$. Condition *(iv)* regards representing the zero function as a Wolff-Denjoy series. Conditions for such representations to be unique, when they exist, have been studied extensively by Borel [6], Beurling [4], and Sibilev [44]. Condition *(vi)*, regards representing the zero function as an exponential series, which has been studied extensively by Leontev [25]-[28] and Korobeinik [21]-[24].

It might seem reasonable to believe that each condition in the preceding theorem holds for any diagonal operator. Surprisingly, this is not always the case. The first example of a diagonal operator acting on a separable Hilbert space which failed to admit spectral synthesis was given by Wolff in 1921. The details of the example can be found in [33] and [47], however, due to its simplicity and elegance we include it here.

Example 1.1. *Wolff's Example*

As usual, we let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk. Let $\{D_j : j \geq 1\}$ be any collection of disks $D_j = \{z \in \mathbb{C} : |z - \lambda_j| \leq r_j\}$, which covers almost all of the entire unit disk; that is, for which $m_2(\mathbb{D} \setminus \cup_{j=1}^{\infty} D_j) = 0$ where m_2 denotes planar Lebesgue measure. Then, for any z such that $|z| > 1$, we have

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{D}} \frac{dm_2(\lambda)}{\lambda - z} &= \frac{1}{\pi}\int_{\mathbb{D}} -\frac{1}{z}\left(\frac{1}{1-(\lambda/z)}\right) dm_2(\lambda) \\
&= \frac{1}{\pi}\int_{\mathbb{D}} -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{\lambda}{z}\right)^n dm_2(\lambda) \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{re^{i\theta}}{z}\right)^n rdrd\theta \\
&= -\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \sum_{n=0}^{\infty}\frac{r^n e^{i\theta n}}{z^{n+1}} rdrd\theta \\
&= -\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\int_0^{2\pi}\int_0^1 r^{n+1}e^{i\theta n}drd\theta \\
&= -\frac{1}{\pi z}\int_0^{2\pi}\int_0^1 rdrd\theta \\
&= -\frac{1}{z}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{D}}\frac{dm_2(\lambda)}{\lambda - z} &= \frac{1}{\pi}\int_{\cup D_j}\frac{dm_2(\lambda)}{\lambda - z}\\
&= \frac{1}{\pi}\sum_{j=1}^{\infty}\int_{B(\lambda_j,r_j)}\frac{dm_2(\lambda)}{\lambda - z}\\
&= \frac{1}{\pi}\sum_{j=1}^{\infty}\int_{B(\lambda_j,r_j)} -\frac{1}{z}\left(\frac{dm_2(\lambda)}{1-(\lambda/z)}\right)\\
&= \frac{1}{\pi}\sum_{j=1}^{\infty}\int_{B(\lambda_j,r_j)} -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{\lambda}{z}\right)^k dm_2(\lambda)\\
&= -\frac{1}{\pi}\sum_{j=1}^{\infty}\int_0^{2\pi}\int_0^{r_j}\sum_{k=0}^{\infty}\frac{(\lambda_j + re^{i\theta})^k}{z^{k+1}} r\,dr\,d\theta\\
&= -\frac{1}{\pi}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\frac{\lambda_j^k}{z^{k+1}}\int_0^{2\pi}\int_0^{r_j} r\,dr\,d\theta\\
&= -\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\frac{\lambda_j^k r_j^2}{z^{k+1}}\\
&= \sum_{k=0}^{\infty}\left(-\frac{1}{z^{k+1}}\sum_{j=1}^{\infty} r_j^2\lambda_j^k\right).
\end{aligned}$$

Hence,

$$-\frac{1}{z} = \sum_{k=0}^{\infty}\left(-\frac{1}{z^{k+1}}\sum_{j=1}^{\infty} r_j^2\lambda_j^k\right).$$

Equating the Laurent series coefficients, we have that $\sum_{j=1}^{\infty} r_j^2 = 1$ and $\sum_{j=1}^{\infty} r_j^2\lambda_j^k = 0$ for all $k \geq 1$. Hence, $\{r_j^2\} \in \ell^1$, and if we define $\omega_j = r_j^2\lambda_j$ for $j \geq 1$, then $\sum_{j=1}^{\infty}|\omega_j| = \sum_{j=1}^{\infty} r_j^2|\lambda_j| \leq \sum_{j=1}^{\infty} r_j^2 = 1$ and $\sum_{j=1}^{\infty}\omega_j\lambda_j^k = \sum_{j=1}^{\infty} r_j^2\lambda_j^{k+1} = 0$ for all $k \geq 0$. Hence, the diagonal operator D having eigenvalues $\{\lambda_j\}$ satisfies Condition *(iii)* of Theorem 1.1, and thus fails spectral synthesis on ℓ^2.

Wolff's example led the way for years of research in this area (that is, representing the zero function by series of the form $\sum_{j=0}^{\infty}\omega_j\lambda_j^k$) by many prominent mathematicians and has been extended to sequences $\{\lambda_n\}$ of distinct complex numbers which are not necessarily bounded. The following four such examples appear on page 128 of *Operators, Functions and*

Systems, I by Nikol'skii [32].

Example 1.2. *Extensions of Wolff's Example*

In 1936, Natason showed that for the sequence $\{\lambda_n\} = \{n\}$, there exists a sequence $\{\omega_n\}$ of complex numbers for which $0 < \sum_{n=0}^{\infty} |\omega_n||\lambda_n|^k < \infty$ and $\sum_{n=0}^{\infty} \omega_n \lambda_n^k \equiv 0$ for all $k \geq 0$. In 1959, Makarov generalized Natason's example to include any sequence $\{\lambda_n\}$ for which $|\lambda_n| \to \infty$. In 1968, Markus showed that, for all sequences $\{\lambda_k\}$ of distinct complex numbers there exists a sequence $\{\epsilon_k\}$ such that $\sum_{k=0}^{\infty} \epsilon_k |\lambda_k|^n < \infty$ for every $n \geq 0$, and for every sequence $\{\omega_k\}$ satisfying $|\omega_k| \leq C\epsilon_k$ for every $k \geq 0$, and $\sum_{k=0}^{\infty} \omega_k \lambda_k^n = 0$ for every $n \geq 0$, we have $\omega_k \equiv 0$. Hence, if the sequence $\{\omega_k\}$ decays quicker than $\{\epsilon_k\}$, then the moments $\sum_{k=0}^{\infty} \omega_k \lambda_k^n \equiv 0$ only when the coefficients are identically zero. In 1995, Sibilev showed that for any decreasing sequence $\{\epsilon_k\}$ of positive numbers, the following are equivalent:

1. for all bounded sequences $\{\lambda_k\}$ of distinct points and for all sequences $\{\omega_k\}$ such that $|\omega_k| \leq C\epsilon_k$ for all $k \geq 1$, $\sum_{k=1}^{\infty} \omega_k \lambda_k^n = 0$ for all $n \geq 0$ implies $\omega_k = 0$ for all $k \geq 0$, and

2. $\sum_{k=1}^{\infty} (1/k^2) \log (1/\epsilon_k) = \infty$.

The purpose of this dissertation is to provide examples of cyclic diagonal operators acting on the space of functions analytic on the unit disk $H(\mathbb{D})$ which fail to admit spectral synthesis, by providing analogues to Wolff's example on ℓ^2. By definition, diagonal operators acting on $H(\mathbb{D})$, as well as on the space of entire functions $H(\mathbb{C})$, have as eigenvectors the monomials z^n with associated eigenvalues $\{\lambda_n\}$. We will see in these two settings that the existence of a non-synthetic diagonal operator is equivalent to a moment condition $\sum_{n=0}^{\infty} \omega_n \lambda_n^k \equiv 0$ for $k \geq 0$, where $\{\omega_n\}$ satisfies a certain decay rate. However, the exact nature of the decay rate of $\{\omega_n\}$ versus the growth rate of $\{\lambda_n\}$ is the essential ingredient that defines the condition for non-synthesis on each space. In the following section, we preview the results that we state in Sections 1.5 and 1.6 regarding conditions for non-synthesis on $H(\mathbb{C})$ and $H(\mathbb{D})$, by examining the analogues of the moment Condition *(iii)* of Theorem 1.1. Using these results we deduce that Wolff's example, as well as those of Natason and Makarov, do not yield examples of non-synthetic diagonal operators acting on spaces of functions analytic on a region in the complex plane.

1.3 The Moment Condition

Let $\mathcal{H}$ denote a separable Hilbert space with an orthonormal basis $\{e_n\}$. A vector $x = \sum_{n=0}^{\infty} a_n e_n$ is in $\mathcal{H}$ if and only if $\{a_n\} \in \ell^2$. If D is a linear map on $\mathcal{H}$ having e_n as eigenvectors with associated eigenvalues λ_n, then D is given formally by $D : \sum_{n=0}^{\infty} a_n e_n \to \sum_{n=0}^{\infty} \lambda_n a_n e_n$. In this case, D is a continuous linear operator on $\mathcal{H}$ if and only if $\{\lambda_n\} \in \ell^\infty$, by the Principle of Uniform Boundedness [9, page 95]. We define a **diagonal operator acting on $\mathcal{H}$ having eigenvalues** $\{\lambda_n\}$ to be an operator D acting on $\mathcal{H}$ for which there exists a sequence $\{\lambda_n\} \subset \mathbb{C}$ such that $\{\lambda_n\} \in \ell^\infty$ and $D(e_n) = \lambda_n e_n$ for all $n \geq 0$.

A vector $x = \sum_{n=0}^{\infty} a_n e_n \in \mathcal{H}$ is non-cyclic for D if and only if there exists a non-zero linear functional L, in the dual $\mathcal{H}^*$ of $\mathcal{H}$, such that $L(D^k x) \equiv 0$ for all $k \geq 0$, by the Hahn-Banach Theorem [9, page 78]. If we define $l_n = L(e_n)$ and $\omega_n = a_n l_n$ for all $n \geq 0$, we have for non-cyclic x that $0 = L(D^k x) = L(D^k(\sum_{n=0}^{\infty} a_n e_n)) = L(\sum_{n=0}^{\infty} a_n \lambda_n^k e_n) = \sum_{n=0}^{\infty} \omega_n \lambda_n^k$ for all $k \geq 0$. Observe this is Condition *(iii)* of Theorem 1.1, which is equivalent to the diagonal operator D acting on $\mathcal{H}$ failing spectral synthesis. Since $\{a_n\} \in \ell^2$ and $\{l_n\} \in \ell^2$ (as $\{l_n\}$ corresponds to the linear functional $L \in \mathcal{H}^* \cong \ell^2$), we have $\{\omega_n\} \in \ell^1$. We note this process can be reversed; that is, given $\{\omega_n\} \in \ell^1$, we can factor $\omega_n = a_n l_n$ where $\{a_n\} \in \ell^2$ and $\{l_n\} \in \ell^2$.

Hence the decay rate of the sequence $\{\omega_n\}$ in Condition *(iii)* of Theorem 1.1 depends on the space $\mathcal{H}$ and its dual. As mentioned in Section 1.2, the moment condition $0 = \sum_{n=0}^{\infty} \omega_n \lambda_n^k$, will be satisfied if there exists a balance between the growth rate of $\{\lambda_n\}$ and the decay rate of $\{\omega_n\}$, where the growth of $\{\lambda_n\}$ reflects the continuity of D on $\mathcal{H}$ and the decay of $\{\omega_n\} = \{a_n l_n\}$ reflects the membership of $x \in \mathcal{H}$ and $L \in \mathcal{H}^*$. It is this balance that distinguishes examples of non-synthetic diagonal operators acting on Hilbert spaces from examples of non-synthetic diagonal operators acting on spaces of functions analytic on a region. In Sections 1.5 and 1.6, we observe that the non-synthesis of diagonal operators on $H(\mathbb{C})$ and $H(\mathbb{D})$ is equivalent to a moment condition $0 = \sum_{n=0}^{\infty} \omega_n \lambda_n^k$, analogous to Condition *(iii)* for a diagonal operator acting on a Hilbert space being non-synthetic. The difference in each setting will be the required decay rate of $\{\omega_n\}$. Moreover, on a Hilbert space the eigenvalues $\{\lambda_n\}$ of a diagonal operator are bounded, while on both $H(\mathbb{C})$ and $H(\mathbb{D})$, the eigenvalues can be unbounded.

Let $H(\mathbb{C})$ denote the vector space of functions analytic on the entire complex plane $\mathbb{C}$. A function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $H(\mathbb{C})$ if and only if $\limsup_{n\to\infty} |a_n|^{1/n} = 0$, by the Radius of Convergence Formula. When endowed with the topology of uniform convergence on compacta, $H(\mathbb{C})$ is a complete locally convex topological vector space. The topology of $H(\mathbb{C})$ is induced by the invariant metric $\rho(f,g) \equiv \sum_{n=0}^{\infty} ||f-g||_n/[2^n(1+||f-g||_n)]$, where $||h||_n \equiv \sup\{|h(z)| : |z| \leq n\}$. If D is a linear map on $H(\mathbb{C})$ having the monomials z^n as eigenvectors with associated eigenvalues λ_n, then D is given formally by $D : \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} \lambda_n a_n z^n$. In this case, D is a continuous linear operator on $H(\mathbb{C})$ if and only if $\limsup_{n\to\infty} |\lambda_n|^{1/n} < \infty$, by an application of the Closed Graph Theorem [31, Lemma 1]. We define a **diagonal operator acting on $H(\mathbb{C})$ having eigenvalues $\{\lambda_n\}$** to be any operator D acting on $H(\mathbb{C})$ for which there exists a sequence $\{\lambda_n\} \subset \mathbb{C}$ such that $\limsup_{n\to\infty} |\lambda_n|^{1/n} < \infty$ and $D(z^n) = \lambda_n z^n$ for all $n \geq 0$.

A vector $f(z) \in H(\mathbb{C})$ is non-cyclic for D if and only if there exists a non-zero linear functional L, in the dual $H^*(\mathbb{C})$ of $H(\mathbb{C})$, such that $L(D^k f) \equiv 0$ for all $k \geq 0$ (see [37, Rudin], [29, Levin], or [19, Iyer]). If we define $l_n = L(z^n)$ and $\omega_n = a_n l_n$ for all $n \geq 0$, we have for non-cyclic f that $0 = L(D^k f) = L(D^k(\sum_{n=0}^{\infty} a_n z^n)) = L(\sum_{n=0}^{\infty} a_n \lambda_n^k z^n) = \sum_{n=0}^{\infty} \omega_n \lambda_n^k$ for all $k \geq 0$. Since $\limsup_{n\to\infty} |a_n|^{1/n} = 0$ and $\limsup_{n\to\infty} |l_n|^{1/n} < \infty$ (as $\{l_n\}$ corresponds to the linear functional $L \in H^*(\mathbb{C})$), we have $\limsup_{n\to\infty} |\omega_n|^{1/n} = 0$. For the moment condition $0 = \sum_{n=0}^{\infty} \omega_n \lambda_n^k$ to be satisfied, there must exist a balance between the growth rate of $\{\lambda_n\}$ and the decay rate of $\{\omega_n\}$. The growth of $\{\lambda_n\}$, namely that $\limsup_{n\to\infty} |\lambda_n|^{1/n} < \infty$, reflects the continuity of the operator D on $H(\mathbb{C})$, and the decay rate of $\{\omega_n\}$, namely that $\limsup_{n\to\infty} |\omega_n|^{1/n} = 0$, allows for $\{\omega_n\}$ to be factored into $\{a_n l_n\}$ where $\{a_n\}$ is such that $\limsup_{n\to\infty} |a_n|^{1/n} = 0$ (which guarantees the corresponding vector is in $H(\mathbb{C})$), and $\{l_n\}$ is such that $\limsup_{n\to\infty} |l_n|^{1/n} < \infty$ (which guarantees the corresponding functional is in $H^*(\mathbb{C})$).

Let $H(\mathbb{D}) = H(B(0,1))$ denote the vector space of functions analytic on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $H(\mathbb{D})$ if and only if $\limsup_{n\to\infty} |a_n|^{1/n} \le 1$, by the Radius of Convergence Formula. When endowed with the topology of uniform convergence on compacta, $H(\mathbb{D})$ is a complete locally convex topological vector space where the topology of $H(\mathbb{D})$ is induced by the invariant metric $\rho(f,g) \equiv \sum_{n=0}^{\infty} ||f-g||_n/[2^n(1+||f-g||_n)]$, where $||h||_n \equiv \sup\{|h(z)| : |z| \le (1-1/n)\}$. If D is a linear map on $H(\mathbb{D})$ having the monomials z^n as eigenvectors with associated eigenvalues λ_n, then D is given formally by $D : \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} \lambda_n a_n z^n$. In this case, D is a continuous linear operator on $H(\mathbb{D})$ if and only if $\limsup_{n\to\infty} |\lambda_n|^{1/n} \le 1$, by an application of the Closed Graph Theorem [12, Proposition 1]. We define a **diagonal operator acting on $H(\mathbb{D})$ having eigenvalues** $\{\lambda_n\}$ to be any operator D acting on $H(\mathbb{D})$ for which there exists a sequence $\{\lambda_n\} \subset \mathbb{C}$ such that $\limsup_{n\to\infty} |\lambda_n|^{1/n} \le 1$ and $D(z^n) = \lambda_n z^n$ for all $n \ge 0$.

A vector $f(z) \in H(\mathbb{D})$ is non-cyclic for D if and only if there exists a non-zero linear functional L, in the dual $H^*(\mathbb{D})$ of $H(\mathbb{D})$, such that $L(D^k f) \equiv 0$ for all $k \ge 0$ [37, Rudin]. If we define $l_n = L(z^n)$ and $\omega_n = a_n l_n$ for all $n \ge 0$, we have for non-cyclic f that $0 = L(D^k f) = L(D^k(\sum_{n=0}^{\infty} a_n z^n)) = L(\sum_{n=0}^{\infty} a_n \lambda_n^k z^n) = \sum_{n=0}^{\infty} \omega_n \lambda_n^k$ for all $k \ge 0$. Since $\limsup_{n\to\infty} |a_n|^{1/n} \le 1$ and $\limsup_{n\to\infty} |l_n|^{1/n} < 1$ [9, page 116], we have $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$. For the moment condition $0 = \sum_{n=0}^{\infty} \omega_n \lambda_n^k$ to be satisfied, there must exist a balance between the growth rate of $\{\lambda_n\}$ and the decay rate of $\{\omega_n\}$. The growth of $\{\lambda_n\}$, namely that $\limsup_{n\to\infty} |\lambda_n|^{1/n} \le 1$, reflects the continuity of the operator D on $H(\mathbb{D})$, and the decay rate of $\{\omega_n\}$, namely that $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$, allows for $\{\omega_n\}$ to be factored into $\{a_n l_n\}$

where $\{a_n\}$ is such that $\limsup_{n\to\infty} |a_n|^{1/n} \le 1$ (which guarantees the corresponding vector is in $H(\mathbb{D})$), and $\limsup_{n\to\infty} |l_n|^{1/n} < 1$, (which guarantees the corresponding functional is in $H^*(\mathbb{D})$).

In Wolff's Example 1.1, we saw that for certain sequences $\{\lambda_n\}$, there exist sequences $\{\omega_n\} \in \ell^1$ such that $\sum_{n=0}^{\infty} \omega_n \lambda_n^k = 0$ for all $k \ge 0$. Thus, the diagonal operator acting on a separable Hilbert space with eigenvalues $\{\lambda_n\}$ fails to admit spectral synthesis. However, Wolff's sequence $\{\omega_n\}$ cannot be factored into the product of two sequences; one corresponding to a vector in $H(\mathbb{C})$ (or $H(\mathbb{D})$) and the other corresponding to a linear functional in $H^*(\mathbb{C})$ (or $H^*(\mathbb{D})$) [44, Sibilev]. Hence, Wolff's example does not yield a non-synthetic operator on either of these spaces. This is also the case with the examples of Natason and Makarov discussed in Example 1.2.

1.4 A Preview of the Main Results

The purpose of this document is to produce diagonal operators acting on the space of functions analytic on the unit disk which fail to admit spectral synthesis. From the preceding discussion, it is sufficient to find a sequence $\{\lambda_n\}$ for which $\limsup_{n\to\infty} |\lambda_n|^{1/n} \le 1$ and a non-zero sequence $\{\omega_n\}$ for which $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$ and such that $\sum_{n=0}^{\infty} \omega_n \lambda_n^k \equiv 0$ for all $k \ge 0$. We showed in Section 1.2, that the moment condition $0 = \sum_{n=0}^{\infty} \omega_n \lambda_n^k$ is equivalent to the condition $0 \equiv \sum_{n=0}^{\infty} \omega_n e^{\lambda_n z}$ for all z. An analogous equivalence holds on $H(\mathbb{D})$; in particular, whenever $\{\lambda_n/n : n \ge 1\}$ is bounded, the moment condition is equivalent to $\sum_{n=0}^{\infty} \omega_n e^{\lambda_n z} \equiv 0$ for all z in a disk centered about the origin. In Chapter 2 of this dissertation, we show that the diagonal operator having as eigenvalues $\mathbb{Z} \times i\mathbb{Z} \equiv \{m + in : m, n \in \mathbb{Z}\}$ fails synthesis on $H(\mathbb{D})$. In Chapter 3, we show that diagonal operators having as eigenvalues sequences of the form $\{n^a e^{2\pi i j/b} : 0 \le j < b\}$ for suitably chosen constants $a \in (0,1)$ and $b \in \mathbb{N}$, fail synthesis on $H(\mathbb{D})$.

In the remainder of this section, we outline briefly the technique, which is due to Ermenko, used to prove these results. Let D be a diagonal operator having as eigenvalues $\{\lambda_n\}$, either $\mathbb{Z} \times i\mathbb{Z}$ or a set of the form $\{n^a e^{2\pi i j/b} : 0 \leq j < b\}$ for suitably chosen constants $a \in (0,1)$ and $b \in \mathbb{N}$. In either case, $\inf \{\alpha : \sum_{n=1}^{\infty} 1/|\lambda_n|^{\alpha} < \infty\} > 1$, hence, any entire function $S(\lambda)$ having simple zeros at λ_n has order $\rho > 1$ [5, Boas]. If follows that there exist constants $\alpha, \beta > 0$, for which $|S(\lambda)| > \alpha e^{\beta|\lambda|^{\rho}}$ whenever λ avoids a disjoint collection of balls $B(\lambda_n, r_n)$. In view of which, by the Residue Theorem, $\sum_{n=0}^{\infty} e^{\lambda_n z}/S'(\lambda_n) = \lim_{r\to\infty} \int_{C_r} (e^{\lambda z}/S(\lambda))d\lambda = 0$ for appropriately chosen contours C_r which avoid the balls $B(\lambda_n, r_n)$. It follows from the Inverse Function Theorem [15, Gamelin] and Schwarz' Lemma that $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$ where $\omega_n \equiv 1/S'(\lambda_n)$. Hence, $0 \equiv \sum_{n=0}^{\infty} \omega_n e^{\lambda_n z}$ for all z near the origin, and by the discussion in Section 1.3, D fails to admit spectral synthesis on $H(\mathbb{D})$.

1.5 The Case $H(\mathbb{C})$-The Space of Entire Functions

In this section, we discuss the relevant background information regarding the spectral synthesis of diagonal operators acting on the space of entire functions $H(\mathbb{C})$. In particular, we state the analogue of Theorem 1.1, that is, we provide equivalent conditions for diagonal operators acting on $H(\mathbb{C})$ to admit spectral synthesis.

Cyclic vectors, invariant subspaces, and the spectral synthesis of diagonal operators acting on the space of entire functions $H(\mathbb{C})$ have been studied by Deters, Marin, and Seubert ([13], [31], and [41]). As mentioned before, a diagonal operator on a Hilbert space is cyclic if and only if the eigenvalues are distinct. The same results holds in $H(\mathbb{C})$ as is stated in the following proposition [31, Proposition 3]. We also state a result giving equivalent conditions for a vector in $H(\mathbb{C})$ to be cyclic for a diagonal operator D [31, Proposition 2].

Proposition 1.1. *Let D be a diagonal operator on $H(\mathbb{C})$ having eigenvalues $\{\lambda_n\}$. Then D is cyclic if and only if $\lambda_m \neq \lambda_n$ whenever $m \neq n$.*

Proposition 1.2. *Let D be a diagonal operator on $H(\mathbb{C})$ having eigenvalues $\{\lambda_n\}$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be any entire function. The following are equivalent:*

(i) *f fails to be cyclic for D,*

(ii) *the closed linear span of the orbit $\{\sum_{n=0}^{\infty} a_n \lambda_n^k z^n : k \geq 0\}$ of f under D is not all of $H(\mathbb{C})$, and*

(iii) *there exists a sequence $\{l_n\}$ of complex numbers, not all zero, for which $\sup |l_n|^{1/n} < \infty$ and $0 \equiv \sum_{n=0}^{\infty} l_n a_n \lambda_n^k$ for all $k \geq 0$.*

As in Theorem 1.1 for a Hilbert space, several equivalent conditions (most of which are analogues to the conditions on a Hilbert space) for a diagonal operator acting on $H(\mathbb{C})$ to admit spectral synthesis have been obtained in [31], as the following theorem demonstrates.

Theorem 1.2. *Let D be any cyclic diagonal operator on $H(\mathbb{C})$ having distinct eigenvalues $\{\lambda_n\}$. Then the following are equivalent:*

(i) *D admits spectral synthesis,*

(ii) *every closed invariant subspace of D is the closed linear span of $\{z^n : n \in N\}$ where N is an arbitrary set of nonnegative integers,*

(iii) *for every function $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ in $H(\mathbb{C})$, span$\{D^j f : j \geq 0\} =$ span$\{z^r : a_r \neq 0\}$,*

(iv) *every entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \neq 0$ for all $n \geq 0$ is cyclic for D,*

(v) *there do not exist sequences $\{a_n\}$ and $\{l_n\}$ of complex numbers with $a_n \neq 0$ for all $n \geq 0$, $\limsup_{n\to\infty} |a_n|^{1/n} = 0$, $\sup |l_n|^{1/n} < \infty$, and $\{l_n\}$ not identically zero, such that $0 \equiv \sum_{n=0}^{\infty} a_n l_n \lambda_n^k$ for all $k \geq 0$, and*

(vi) *there does not exist a sequence $\{\omega_n\}$ of complex numbers, not identically zero, for which $\limsup_{n\to\infty} |\omega_n|^{1/n} = 0$ and $0 \equiv \sum_{n=0}^{\infty} \omega_n \lambda_n^k$ for all $k \geq 0$.*

If, in addition, $\{\lambda_n/n : n \geq 1\}$ is bounded, then $\sum_{n=0}^{\infty} d_n e^{\lambda_n z}$ is entire whenever $\limsup_{n\to\infty} |d_n|^{1/n} = 0$ and conditions (i)-(vi) are equivalent to *(vii) there does not exist a sequence $\{\omega_n\}$ of complex numbers, not identically zero, for which $\limsup_{n\to\infty} |\omega_n|^{1/n} = 0$ and $0 \equiv \sum_{n=0}^{\infty} \omega_n e^{\lambda_n z}$ for all z in $\mathbb{C}$.*

The diverse equivalent conditions for spectral synthesis given in the preceding theorem provide interesting and enlightening information about the eigenvalues. Many of the conditions are almost identical to the conditions of Theorem 1.1 for a Hilbert space, which were discussed in detail following the theorem in Section 1.2. For example, Condition *(iv)* is analogous to Condition *(ii)* of Theorem 1.1, Condition *(vi)* to Condition *(iii)* of Theorem 1.1, and Condition *(vii)* is analogous to Condition *(vi)* of Theorem 1.1. However, we note that there are no analogous statements to Conditions *(vii)* and *(viii)*, for example, of Theorem 1.1 in Theorem 1.2, as we do not know how to precisely define the adjoint of an operator on $H(\mathbb{C})$. The discussion in Section 1.3 shows the equivalence of Conditions *(v)* and *(vi)*. We discuss the equivalences of several of the other conditions briefly here. Conditions *(i)* and *(ii)* are equivalent as Condition *(ii)* is the definition of spectral synthesis since a diagonal operator acting on $H(\mathbb{C})$ has the monomials z^n as eigenvectors. By this same reasoning it is clear that Condition *(ii)* is equivalent to Conditions *(iii)* and *(iv)*.

The first example of a diagonal operator acting on $H(\mathbb{C})$ which fails to admit spectral synthesis was produced by Henthorn in [16] where it is shown that if D is a cyclic diagonal operator on $H(\mathbb{C})$ having eigenvalues $\{\lambda_n\}$ such that $\limsup_{n\to\infty} |\lambda_n/\lambda_{n+1}| < 1$, that is, if $\{\lambda_n\}$ grows exponentially, then D fails synthesis on $H(\mathbb{C})$. Whether any such example exists on the space of functions analytic on the unit disk remained an open problem, which we address in this document.

1.6 The Case $H(\mathbb{D})$-The Space of Functions Analytic on the Disk

In this section, we discuss the relevant background information regarding the spectral synthesis of diagonal operators acting on the space of functions analytic on the unit disk $H(\mathbb{D})$.

In particular, we state the analogue of Theorems 1.1 and 1.2, which provide equivalent conditions for diagonal operators acting on $\mathcal{H}$ and $H(\mathbb{C})$ to admit spectral synthesis, to diagonal operators acting on $H(\mathbb{D})$.

In this document, we study operators acting on $H(\mathbb{D})$, however, we need not limit ourselves to the unit disk. If we let $H(B(0,R))$ denote the space of functions analytic on the disk $\{z \in \mathbb{C} : |z| < R\}$, then $\sum_{n=0}^{\infty} a_n z^n \in H(B(0,R))$ if and only if $\limsup_{n\to\infty} |a_n|^{1/n} \leq 1/R$. A linear map D with eigenvalues $\{\lambda_n\}$ is continuous on this space if and only if $\limsup_{n\to\infty} |\lambda_n|^{1/n} \leq 1$ [12, Proposition 1]. In view of which, for the purposes of our study, we can translate all definitions and results on $H(\mathbb{D})$ to $H(B(0,R))$. Hence, for simplicity, we will study operators acting on $H(\mathbb{D})$.

Cyclic vectors, invariant subspaces, and the spectral synthesis of diagonal operators acting on the space of functions analytic on the unit disk $H(\mathbb{D})$ have been studied be Deters, Seubert, and Wade ([11], [12], and [42]). We begin, as we did with $H(\mathbb{C})$, by stating results about the cyclicity of diagonal operators on $H(\mathbb{D})$ and conditions for vectors in $H(\mathbb{D})$ to be cyclic for a diagonal operator D [12, Theorem 1 and Lemma 1].

Proposition 1.3. *Let D be a diagonal operator on $H(\mathbb{D})$ having eigenvalues $\{\lambda_n\}$. Then D is cyclic if and only if $\lambda_m \neq \lambda_n$ whenever $m \neq n$.*

Proposition 1.4. *Let D be a diagonal operator on $H(\mathbb{D})$ having eigenvalues $\{\lambda_n\}$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be any function in $H(\mathbb{D})$. The following are equivalent:*

(i) f fails to be cyclic for D,

(ii) the closed linear span of the orbit $\{\sum_{n=0}^{\infty} a_n \lambda_n^k z^n : k \geq 0\}$ of f under D is not all of $H(\mathbb{D})$, and

(iii) there exists a sequence $\{l_n\}$ of complex numbers, not all zero, for which $\sup |l_n|^{1/n} < 1$ and $0 \equiv \sum_{n=0}^{\infty} l_n a_n \lambda_n^k$ for all $k \geq 0$.

In [11] and [12] Deters and Seubert present an analogous result to Theorems 1.1 and 1.2 giving equivalent conditions for a diagonal operator acting on $H(\mathbb{D})$ to admit spectral synthesis. Many of the conditions are similar to those on Hilbert spaces and $H(\mathbb{C})$, as the following theorem demonstrates.

Theorem 1.3. *Let D be any cyclic diagonal operator on $H(\mathbb{D})$ having distinct eigenvalues $\{\lambda_n\}$. Then the following are equivalent:*

(i) *D admits spectral synthesis,*

(ii) *every closed invariant subspace of D is the closed linear span of $\{z^n : n \in N\}$ where N is an arbitrary set of nonnegative integers,*

(iii) *every closed invariant subspace for D (other than the empty set and $\{0\}$) contains at least one monomial z^n for some $n \geq 0$,*

(iv) *every function $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ in $H(\mathbb{D})$ with $a_n \neq 0$ for all $n \geq 0$ is cyclic for D,*

(v) *there does not exist a sequence $\{\omega_n\}$ of complex numbers, not identically zero, for which $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$ and $0 \equiv \sum_{n=0}^{\infty} \omega_n \lambda_n^k$ for all $k \geq 0$,*

(vi) *the function $u(z) = \frac{1}{1-z}$ is cyclic for D,*

(vii) *for each $j \geq 0$ there is some sequence $\{p_n\}$ of polynomials such that $\lim_{n\to\infty} p_n(\lambda_k) = \delta_{j,k}$ and $\limsup_{n\to\infty} \sup_{k>j} |p_n(\lambda_k)|^{1/k} \leq 1$, and*

(viii) *if $\mathcal{A}$ is the algebra generated by D and the identity, that is, $\mathcal{A} \equiv \vee\{D^n : n \geq 0\}$, and we let $\mathcal{D}$ denote the set of diagonal operators on $H(\mathbb{D})$, then in the Strong Operator Topology, $\overline{\mathcal{A}} = \mathcal{D}$.*

If, in addition, $\{\lambda_n/n : n \geq 1\}$ is bounded, then $\sum_{n=0}^{\infty} \omega_n e^{\lambda_n z}$ is analytic on the open ball $B(0,\epsilon)$ containing the origin whenever $\{\omega_n\}$ is a sequence of complex numbers for which $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$ where $\epsilon \equiv [\ln(1/\limsup|\omega_n|^{1/n})]/[\sup\{|\lambda_n|/n\}]$. In this case, conditions (i)-(viii) are equivalent to

(ix) *there does not exist a sequence $\{\omega_n\}$ of complex numbers, not identically zero, for which $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$ and $0 \equiv \sum_{n=0}^{\infty} \omega_n e^{\lambda_n z}$ for all z in the open ball $B(0,\epsilon)$.*

Most of the conditions given in the preceding theorem are nearly identical to the conditions in Theorems 1.1 and 1.2 with the necessary changes on the decay rate of $\{\omega_n\}$ made, as discussed in Section 1.3. We remark on a few of the other conditions. Condition *(vi)* states that we need only check the cyclicity of one function, $u(z)$; if it is cyclic for D, then all of the closed invariant subspaces are known, but if not, then there is some closed invariant subspace that is not the closure of the span of some set of monomials. Condition *(vii)* gives a computational approach to checking synthesis by examining the growth of polynomials.

In this dissertation, as outlined in Section 1.4, we prove that non-synthetic diagonal operators acting on $H(\mathbb{D})$ do exist. More precisely, we use Condition *(ix)* of Theorem 1.3 to show the diagonal operator with the integer lattice points $\mathbb{Z} \times i\mathbb{Z}$ as eigenvalues fails to admit spectral synthesis on $H(\mathbb{D})$. In Chapter 3, we expand upon this example to generate an entire class of non-synthetic diagonal operators acting on $H(\mathbb{D})$.

1.7 Testable Conditions for Synthesis

In the preceding sections, lists of equivalent conditions for a diagonal operator acting on $H(\mathbb{C})$ or $H(\mathbb{D})$ to admit spectral synthesis were given in Theorems 1.2 and 1.3. However, most of the conditions, although sufficient for synthesis, are not convenient for determining if a given diagonal operator is synthetic. On both of these spaces a handful of results which are often more useful to determine if an operator is synthetic are known and are stated in this section.

Recall that on a Hilbert space a diagonal operator is cyclic if and only if its eigenvalues are bounded. The following result states that if the eigenvalues of a diagonal operator acting on $H(\mathbb{C})$ or $H(\mathbb{D})$ are bounded, then the operator is synthetic.

Theorem 1.4. *(see [31], [12]) Every cyclic diagonal operator D on $H(\mathbb{C})$ (or on $H(\mathbb{D})$) whose eigenvalues $\{\lambda_n\}$ are bounded admits spectral synthesis.*

It follows from this theorem that there exist cyclic diagonal operators acting on $H(\mathbb{C})$ and

$H(\mathbb{D})$ admitting spectral synthesis, the closure of whose eigenvalues $\{\lambda_n\}$ have non-empty interior. This is not the case for diagonal operators acting on a separable Hilbert space [40, Scroggs].

The following theorem asserts that diagonal operators with eigenvalues $\{\lambda_n\} = \{n^p\}$ for $p \leq 1$ are synthetic on $H(\mathbb{C})$ and $H(\mathbb{D})$. Moreover, unlike the Hilbert space case, it asserts that it is possible for a synthetic operator to have unbounded eigenvalues.

Theorem 1.5. *(see [31], [12]) Let D be a diagonal operator on $H(\mathbb{C})$ or $H(\mathbb{D})$ having eigenvalues $\{\lambda_n\}$. If $\{\lambda_n/n : n \geq 1\}$ is bounded and the real parts of the λ_n are strictly increasing, then D admits spectral synthesis.*

If we instead assert that the eigenvalues lie in a half-plane and satisfy a certain growth rate, namely that $\{\mu_n/n : n \geq 1\}$ is bounded, then the diagonal operator D with eigenvalues $\{\mu_n\}$ is synthetic on $H(\mathbb{C})$, thus strengthening Theorem 1.5. As an example the diagonal operator D having eigenvalues $\{\pm n\}$ admits synthesis on $H(\mathbb{C})$.

Theorem 1.6. *(see [41]) Let D be any cyclic diagonal operator having eigenvalues $\{\mu_n\}$ which lie in any half-plane and are such that $\{\mu_n/n : n \geq 1\}$ is bounded. Then D admits spectral synthesis on $H(\mathbb{C})$.*

The results in this dissertation suggest that, unlike the previous results, the synthesis of a diagonal operator not only depends on the growth rate of the eigenvalues, but also on their distribution in the complex plane. More specifically, we observe that a diagonal operator with eigenvalues $\{\sqrt{n}\}$ is synthetic, but a diagonal operator having as eigenvalues six copies of $\sqrt{n}$ placed symmetrically on six rays $e^{i\pi j/3}$, where $0 \leq j < 6$, fails to admit spectral synthesis on $H(\mathbb{D})$.

The next four theorems involve diagonal operators failing to admit spectral synthesis on $H(\mathbb{C})$ and $H(\mathbb{D})$. On a Hilbert space analogous results are given in the paper of Brown, Shields and Zeller [8], where the coefficients $\{\gamma_n\}$ are only required to be in ℓ^1.

Theorem 1.7. *(see [41]) Let D be a cyclic diagonal operator on $H(\mathbb{C})$ having eigenvalues $\{\lambda_n\}$ for which $\{\lambda_n/n : n \geq 1\}$ is bounded. Then D fails spectral synthesis if and only if for each complex number λ in $\overline{(\{\lambda_n\})}^C$, the complement of the closure of $\{\lambda_n\}$, there exists a sequence $\{\gamma_n\}$ of complex numbers, not identically zero, for which $\limsup_{n\to\infty} |\gamma_n|^{1/n} = 0$ and $e^{\lambda z} = \sum_{n=0}^{\infty} \gamma_n e^{\lambda_n z}$ for all complex numbers z.*

Theorem 1.8. *(see [42]) Let D be a cyclic diagonal operator on $H(\mathbb{D})$ with eigenvalues $\{\lambda_n\}$ such that $\{\lambda_n/n : n \geq 1\}$ is bounded. If D fails spectral synthesis and λ is in $\overline{(\{\lambda_n\})}^C$, the complement of the closure of the eigenvalues of D, then there exist coefficients $\{\gamma_n\}$ for which $\limsup_{n\to\infty} |\gamma_n|^{1/n} < 1$ and $e^{\lambda z} = \sum_{n=0}^{\infty} \gamma_n e^{\lambda_n z}$ on $B(0,\epsilon)$ where $\epsilon \equiv [\ln(1/\limsup |\gamma_n|^{1/n})]/[\sup\{|\lambda_n|/n\}]$. Conversely, if $e^{\lambda z} \equiv \sum_{n=0}^{\infty} \gamma_n e^{\lambda_n z}$ on some non-empty open disc $B(0,r)$ where $\lambda \neq \lambda_n$ for all $n \geq 0$ and $\limsup_{n\to\infty} |\gamma_n|^{1/n} < 1$, then D fails spectral synthesis.*

Theorem 1.7 is an extension of Theorem 1.2; that is, by Condition *(vii)* of Theorem 1.2 D fails to admit spectral synthesis if and only if the zero function can be represented as a Dirichlet series $\sum_{n=0}^{\infty} \omega_n e^{\lambda_n z}$, while Theorem 1.7 states that we can represent various exponential functions $e^{\lambda z}$ as Dirichlet series. Theorem 1.8 is the analogue on $H(\mathbb{D})$ of Theorem 1.7 on $H(\mathbb{C})$, however, it is somewhat less satisfying as it only concludes the representation holds on some neighborhood $B(0,\epsilon)$ of the origin as opposed to holding on all of $\mathbb{C}$ or on $\mathbb{D}$. Regardless, it is still an extension of Theorem 1.3 in the same way Theorem 1.7 extends Theorem 1.2.

The following results are applications of the work of Leontev and Korobeinik [21]-[28] concerning the possibility of representing analytic functions as generalized Dirichlet series $\sum_{n=0}^{\infty} a_n e^{\lambda_n z}$ on certain regions. Korobeinik has shown that under the condition $\ln n/\lambda_n \to 0$, a generalized Dirichlet series $\sum_{n=0}^{\infty} c_n e^{\lambda_n z}$ converges on a domain Ω_D if and only if $\limsup_{n\to\infty} (\ln |c_n|/|\lambda_n| + h(\arg \lambda_n)) \leq 0$, where $h(\theta)$ denotes the indicator function. Moreover, he has shown that every function $f(z)$ analytic on Ω_D can be expressed as $f(z) = \sum_{n=0}^{\infty} c_n e^{\lambda_n z}$ if and only if the zero function can be represented as such a series with the

$\{c_n\}$ not identically zero. Leontev and Korobeinik give numerous conditions for the zero function to be written as a generalized Dirichlet series. The following two theorems assert that if D is a cyclic diagonal operator acting on $H(\mathbb{C})$ or $H(\mathbb{D})$, respectively, then under certain conditions, every function analytic on a particular region Ω_D associated with D is representable as a generalized Dirichlet series on Ω_D.

Theorem 1.9. *(see [41]) Let D be a cyclic diagonal operator on $H(\mathbb{C})$ having eigenvalues $\{\lambda_n : n \geq 0\}$ which fails spectral synthesis. Suppose that there exists an entire function g of exponential type, not identically zero, for which $g(\lambda_n) = 0$ for all $n \geq 0$, and denote by Ω_D the interior of the convex compact set having supporting function $h_g(-\theta)$. If $\{\lambda_n/n : n \geq 1\}$ is bounded, then every function $f(z)$ analytic on Ω_D is representable as a generalized Dirichlet series $\sum_{n=0}^{\infty} b_n e^{\lambda_n z}$ in the sense that the series $\sum_{n=0}^{\infty} b_n e^{\lambda_n z}$ converges uniformly to $f(z)$ on every compact subset of Ω_D.*

Theorem 1.10. *(see [42]) Let D be a cyclic diagonal operator on $H(\mathbb{D})$ failing spectral synthesis whose eigenvalues $\{\lambda_n : n \geq 0\}$ are such that $\overline{\{\lambda_n\}} \neq \mathbb{C}$ and $\{\lambda_n/n : n \geq 1\}$ is bounded, and let $\{\omega_n\}$ be any nontrivial sequence for which $0 \equiv \sum_{n=0}^{\infty} \omega_n \lambda_n^k$ for all $k \geq 0$ and $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$. Define τ to be the supremum of the radii of all open balls contained in $\overline{(\{\lambda_n\})}^C$. Then for every entire function f of exponential type less than τ, there exists a sequence $\{b_n\}$ of complex numbers for which $f(z) = \sum_{n=0}^{\infty} b_n e^{\lambda_n z}$ on $B(0, \epsilon)$ where $\limsup_{n\to\infty} |b_n|^{1/n} = \limsup_{n\to\infty} |\omega_n|^{1/n} < 1$ and $\epsilon \equiv [\ln(1/\limsup |\gamma_n|^{1/n})]/[\sup\{|\lambda_n|/n\}]$.*

These results are analogous to a result [8, Theorem 3] of Brown, Shields and Zeller for diagonal operators acting on a Hilbert space $\mathcal{H}$, which asserts D fails spectral synthesis on $\mathcal{H}$ if and only if every entire function $f(z)$ can be represented as $f(z) = \sum_{n=0}^{\infty} b_n e^{\lambda_n z}$ where $\sum_{n=0}^{\infty} |b_n| < \infty$. On $H(\mathbb{C})$ and $H(\mathbb{D})$ the preceding results require that $\{\lambda_n/n : n \geq 1\}$ is bounded and $\overline{\{\lambda_n\}} \neq \mathbb{C}$; however, the latter condition is not required for a Hilbert space $\mathcal{H}$. On $H(\mathbb{C})$ and $H(\mathbb{D})$ we can represent any entire function $f(z)$ of order at most one and type less than the supremum of the radii of the largest ball contained in the complement of the closure of the λ_n, as $\sum_{n=0}^{\infty} b_n e^{\lambda_n z}$ for some $\{b_n\}$, while on $\mathcal{H}$ we can represent any entire function of order at most one regardless of its type in this way. On $\mathcal{H}$ and $H(\mathbb{C})$ the representation $f(z) = \sum_{n=0}^{\infty} b_n z^n$ holds on $\mathbb{C}$, while on $H(\mathbb{D})$ it only holds on some ball containing the origin.

The following two theorems give sufficient conditions for a diagonal operator acting on $H(\mathbb{C})$ to admit spectral synthesis in terms of the growth of the eigenvalues. These results, unlike the previous ones, do not require the condition $\{\lambda_n/n : n \geq 1\}$ is bounded.

Theorem 1.11. *(see [13]) Let D be a cyclic diagonal operator on $H(\mathbb{C})$ having eigenvalues $\{\lambda_n\}$. If for each $j \geq 0$, there exists a sequence $\{p_n(z)\}$ of polynomials for which $\lim_{n\to\infty} p_n(\lambda_k) = \delta_{j,k}$ and $\sup\{|p_n(\lambda_k)|^{1/k} : k \geq 0, n \geq 1\} < \infty$, then D admits spectral synthesis.*

This theorem yields several results, the following states that if there exists a non-trivial entire function of order ρ with zeros at the eigenvalues $\{\lambda_n\}$ and $\sup\{|\lambda_n|^\rho/n : n \geq 1\} < \infty$ then D admits spectral synthesis.

Theorem 1.12. *(see [13]) Let D be a cyclic diagonal operator on $H(\mathbb{C})$ having eigenvalues $\{\lambda_n\}$. If there exists a non-trivial entire function $E(z)$ of order ρ and finite type τ with $E(\lambda_n) \equiv 0$ for all $n \geq 0$ and $\sup\{|\lambda_k|^\rho/k : k \geq 1\} < \infty$, then D admits spectral synthesis.*

The preceding theorem follows from Theorem 1.11 by defining the sequence of polynomials as follows. Let m_j denote the order of the zero λ_j of $E(z)$ for all $j \geq 0$, then the function $E_j(z) \equiv E(z)/[(z-\lambda_j)^{m_j} E^{(m_j)}(\lambda_j)] \equiv \sum_{k=0}^{\infty} a_k z^k$ satisfies $E_j(\lambda_k) = \delta_{j,k}$ for all $j, k \geq 0$. Hence, defining $p_n(z) = \sum_{k=0}^{n} a_k z^k$ gives a sequence of polynomials satisfying the hypotheses of Theorem 1.11.

From Theorems 1.11 and 1.12, we observe that if the eigenvalues of a diagonal operator D can be expressed as $\lambda_n = p(n)$ where $p(z)$ is a polynomial, then D admits spectral synthesis on $H(\mathbb{C})$. In particular, we conclude that diagonal operators acting on $H(\mathbb{C})$ having as eigenvalues $\{n^q\}$ are synthetic for any positive integer q. The previous theorems and their consequences improve upon the other results mentioned in this section as they do not require the condition $\{\lambda_n/n : n \geq 1\}$ is bounded.

In Chapter 5, we prove another result which gives sufficient conditions for a diagonal operator on $H(\mathbb{C})$ to admit spectral synthesis. More precisely, we prove that if $\{\lambda_n\}$ is such that $\{\lambda_n/n : n \geq 1\}$ is bounded and $n(r) = \mathcal{O}(r)$, then the diagonal operator with eigenvalues $\{\lambda_n\}$ is synthetic on $H(\mathbb{C})$.

CHAPTER 2

A Non-synthetic Operator on the Space of Functions Analytic on the Unit Disk

Wolff's Example 1.1, gave the first example of a non-synthetic diagonal operator acting on a separable Hilbert space $\mathcal{H}$. In particular, Wolff showed that for certain bounded sequences $\{\lambda_n\}$ of distinct complex numbers there exist sequences $\{\omega_n\}$ in ℓ_1, not identically zero, such that $\sum_{n=0}^{\infty} \omega_n \lambda_n^k \equiv 0$ for all $k \geq 0$. By Condition *(iii)* of Theorem 1.1 such examples yield non-synthetic diagonal operators acting on ℓ^2. Henthorn, in her dissertation [16], proves that if D is a cyclic diagonal operator acting on $H(\mathbb{C})$ having eigenvalues $\{\lambda_n\}$ such that $\limsup_{n\to\infty} |\lambda_n/\lambda_{n+1}| < 1$, then D fails to admit spectral synthesis, hence giving examples of non-synthetic diagonal operators acting on $H(\mathbb{C})$. In this Chapter, we obtain an example of a diagonal operator acting on $H(\mathbb{D})$ which fails to admit spectral synthesis.

If D is a non-synthetic diagonal operator acting on $H(\mathbb{C})$ having as eigenvalues $\{\lambda_n\}$, then by Condition *(vi)* of Theorem 1.2, there exists a sequence $\{\omega_n\}$, not identically zero, such that $\limsup_{n\to\infty} |\omega_n|^{1/n} = 0$ and $\sum_{n=0}^{\infty} \omega_n \lambda_n^k \equiv 0$ for all $k \geq 0$. Moreover, if it is also the case that $\limsup_{n\to\infty} |\lambda_n|^{1/n} \leq 1$, then by Condition *(ix)* of Theorem 1.3, D is also non-synthetic when viewed as an operator acting on $H(\mathbb{D})$. Whether there exist diagonal operators which fail spectral synthesis when viewed as acting on $H(\mathbb{D})$, but admit spectral synthesis when viewed as acting on $H(\mathbb{C})$ remained an open question, which we answer affirmatively in this dissertation.

Throughout this chapter, we let D be the cyclic diagonal operator with eigenvalues at the integer lattice points $\mathbb{Z} \times i\mathbb{Z} \equiv \{m + in : m, n \in \mathbb{Z}\}$, and prove that D is non-synthetic on $H(\mathbb{D})$ but synthetic on $H(\mathbb{C})$. We let S_j denote the square with vertices $\pm(j + ij)$ and $\pm i(j + ij)$, for all $j \geq 0$, and define $\{\lambda_k\}$ to be the enumeration of $\mathbb{Z} \times i\mathbb{Z}$ defined by beginning on the positive real line and moving counterclockwise around larger and larger squares S_j; thus, $\lambda_0 = 0$; $\lambda_1 = 1$; $\lambda_2 = 1 + i$; $\lambda_3 = i$; $\lambda_4 = -1 + i$; $\lambda_5 = -1$; $\lambda_6 = -1 - i$; $\lambda_7 = -i$; $\lambda_8 = 1 - i$; $\lambda_9 = 2$;...$\lambda_{24} = 2 - i$; $\lambda_{25} = 3$... To show D, having eigenvalues $\{\lambda_k\}$, is non-synthetic on $H(\mathbb{D})$ we follow the outline given in Section 1.4. In fact, we show that the sequence $\{\omega_k\} \equiv \{1/S'(\lambda_k)\}$, where $S(z)$ is an entire function with zeros only at the points of $\{\lambda_k\}$ (all of which are simple), satisfies Condition *(ix)* of Theorem 1.3. In Section 2.1, we collect information on Weierstrass σ-functions which we use in Section 2.2, to determine the growth rate of the entire function S, and its derivative.

2.1 Weierstrass σ-functions

In this section, we define an entire function S having zeros at the integer lattice points $\mathbb{Z} \times i\mathbb{Z}$ by means of a canonical product and outline the necessary background information, details of which can be found in Whittaker and Watson [50, Chapter XX], to determine the growth rate of S and its derivative S', computations which appear in the following section. The function S is used to show that the diagonal operator D on $H(\mathbb{D})$ having as eigenvalues the integer lattice points $\mathbb{Z} \times i\mathbb{Z}$ is non-synthetic.

Throughout this section, we use the symbol $\sum_{m,n}$ to denote the summation over all integer values of m and n and the symbol $\sum'_{m,n}$ to denote the same sum except omitting the single term $m = 0 = n$. Similarly, we use the symbol $\Pi_{m,n}$ to denote the product over all integer values of m and n and the symbol $\Pi'_{m,n}$ to denote the same product except omitting the single term $m = 0 = n$. If ω_1 and ω_2 are two complex numbers for which ω_2/ω_1 has positive imaginary part, then we define the so-called Weierstrass $\wp$-function by $\wp(z) = \frac{1}{z^2} + \sum'_{m,n}\left(\frac{1}{(z-(2\omega_1 m+2\omega_2 n))^2} - \frac{1}{(2\omega_1 m+2\omega_2 n)^2}\right)$. The series for this elliptic function converges absolutely and uniformly on any compact set omitting its poles $\{2\omega_1 m + 2\omega_2 n : m, n \in \mathbb{Z}\}$. Rearranging the terms in the product for $\wp(z)$, we see that $\wp(z)$ is an even function, since

$$\begin{aligned}
\wp(-z) &= \frac{1}{(-z)^2} + \sum_{m,n}{}' \left(\frac{1}{(-z-2\omega_1 m-2\omega_2 n)^2} - \frac{1}{(2\omega_1 m+2\omega_2 n)^2}\right) \\
&= \frac{1}{z^2} + \sum_{m,n}{}' \left(\frac{1}{(z+2\omega_1 m+2\omega_2 n)^2} - \frac{1}{(2\omega_1 m+2\omega_2 n)^2}\right) \\
&= \wp(z).
\end{aligned}$$

By a similar argument, we see that the derivative of $\wp(z)$

$$\wp'(z) = -\frac{2}{z^3} + \sum_{m,n}{}' \frac{-2}{(z-2\omega_1 m-2\omega_2 n)^3} = -2\sum_{m,n} \frac{1}{(z-2\omega_1 m-2\omega_2 n)^3}$$

is an odd function, since

$$\begin{aligned}
\wp'(-z) &= -2\sum_{m,n} \frac{1}{(-z-2\omega_1 m-2\omega_2 n)^3} \\
&= 2\sum_{m,n} \frac{1}{(z+2\omega_1 m+2\omega_2 n)^3} \\
&= -\wp'(z).
\end{aligned}$$

Since the poles of $\wp'(z)$ form a lattice, one might suspect that $\wp(z)$ periodic. In fact, using the formula for $\wp'(z)$, we see that $\wp(z+2\omega_1) = \wp(z)$ (the constant of integration is seen to be zero upon letting $z = -\omega_1$ and recalling that $\wp(z)$ is even). The analogous argument shows that $\wp(z)$ also has period $2\omega_2$.

For the purposes of this dissertation, we consider the integer lattice $\mathbb{Z}\times i\mathbb{Z}$ obtained upon setting $\omega_1 = 1/2$ and $\omega_2 = i/2$. In this case, the associated $\wp$-function is defined by $\wp(z) = \frac{1}{z^2} + \sum'_{m,n}\left(\frac{1}{(z-(m+in))^2} - \frac{1}{(m+in)^2}\right)$. The unique function ζ for which $\frac{d}{dz}\zeta(z) = -\wp(z)$ and $\lim_{z\to 0}\{\zeta(z) - \frac{1}{z}\} = 0$ is given by $\zeta(z) = \frac{1}{z} + \sum'_{m,n}\left(\frac{1}{z-(m+in)} + \frac{1}{m+in} + \frac{z}{(m+in)^2}\right)$. It follows that ζ is an odd function. Since $\wp(z+1) = \wp(z)$ and $\wp(z+i) = \wp(z)$ and $\frac{d}{dz}\zeta(z) = -\wp(z)$, we see, upon integrating, that $\zeta(z+1) = \zeta(z) + 2\eta_1$ and $\zeta(z+i) = \zeta(z) + 2\eta_2$, where η_1, η_2 are constants of integration. In view of which, $\zeta(z+m) = \zeta(z)+2m\eta_1$ and $\zeta(z+in) = \zeta(z)+2n\eta_2$ for $m, n \in \mathbb{Z}$; the so-called quasi-periodicity of $\zeta(z)$. Letting $z = -\frac{1}{2}$ yields $\eta_1 = \zeta(\frac{1}{2})$, and letting $z = -\frac{i}{2}$ yields $\eta_2 = \zeta(\frac{i}{2})$.

In [46], relationships between the two constants of integration η_1 and η_2 are established for general lattices of periods $2\omega_1$ and $2\omega_2$. For example, integrating $\zeta(z)$ around a parallelogram P whose sides avoid the poles of $\zeta(z)$, and applying the Residue Theorem, yields $\eta_1\omega_2 - \eta_2\omega_1 = \pi i/2$ [46, 20.411, page 446]. For $\omega_1 = \frac{1}{2}$ and $\omega_2 = \frac{i}{2}$ we have

$$i\eta_1 - \eta_2 = \pi i.$$

Moreover,

$$\begin{aligned}
\eta_1 &= \zeta(iz) \\
&= \frac{1}{iz} + \sum_{m,n}{}'\left(\frac{1}{iz-(m+in)} + \frac{1}{m+in} + \frac{iz}{(m+in)^2}\right) \\
&= -\frac{i}{z} + \sum_{m,n}{}'\left(\frac{-i}{z-(n-im)} + \frac{-i}{n-im} + \frac{-iz}{(n-im)^2}\right) \\
&= -i\left(\frac{1}{z} + \sum_{m,n}{}'\left(\frac{1}{z-(m+in)} + \frac{1}{m+in} + \frac{z}{(m+in)^2}\right)\right) \\
&= -i\zeta(z) \\
&= -i\eta_2.
\end{aligned}$$

Combining these two relationships yields $\eta_1 = \frac{\pi}{2}$ and $\eta_2 = -\frac{i\pi}{2}$.

The unique entire function $S(z)$ for which $\frac{d}{dz}\log S(z) = \zeta(z)$ and $\lim_{z\to 0}\frac{S(z)}{z} = 1$ is given by

$$S(z) = z\prod_{m,n}{}' (1 - \frac{z}{m+in})e^{z/(m+in)+z^2/2(m+in)^2} = z\prod_{k=1}^{\infty}\left(1 - \frac{z}{\lambda_k}\right)e^{z/\lambda_k + z^2/2\lambda_k^2};$$

a so-called **Weierstrass σ-function**. Since $\sum'_{m,n}(1/|m+in|^2) < \infty$, S is a canonical product having zeros only at the points of $\mathbb{Z}\times i\mathbb{Z}$ (all of which are simple). In the following section, we determine the growth rate of S and its derivative S'.

2.2 The Growth Rate of $S(z)$ and $S'(z)$

In this section, we determine the growth rate of the function S defined in the previous section and its derivative S'. Although many of the computations detailed in this section can be found, for example in [34] and [29], we include them for the sake of completeness. Before we begin the computations to find the growth rate of $S(z) = z\prod'_{m,n}(1 - \frac{z}{m+in})e^{z/(m+in)+z^2/2(m+in)^2}$, we discuss two inequalities and a technical lemma. For $z \in \overline{B(0,1/2)}$, it can be shown that

$$\frac{1}{2}|z| \le |\log(1+z)| \le \frac{3}{2}|z|, \tag{2.1}$$

by examining the power series expansion of $\log(1+z)$ about $z = 0$ [10, Conway, page 165]. In our computations we invoke the following inequality.

Lemma 2.1. *For $z \in B(0,1)$, $|(1-z)e^z| \ge 1 - |z|^2$.*

Proof. Write $z = re^{i\theta}$, where $0 \le r < 1$ and $0 \le \theta < 2\pi$. Note for any $r \ge 0$, we have $e^r = \sum_{n=0}^{\infty} r^n/n! \ge 1 + r$. Hence,

$$\begin{aligned}
|(1-re^{i\theta})e^{re^{i\theta}}| &= |(1 - r(\cos\theta + i\sin\theta))||e^{r(\cos\theta + i\sin\theta)}| \\
&= |(1 - r\cos\theta) - ir\sin\theta|e^{r\cos\theta} \\
&= e^{r\cos\theta}\sqrt{(1-r\cos\theta)^2 + (-r\sin\theta)^2}
\end{aligned}$$

$$
\begin{aligned}
&= e^{r\cos\theta}\sqrt{1-2r\cos\theta+r^2} \\
&\geq \min_{\{\phi\in[0,2\pi]\}} e^{r\cos\phi}\sqrt{1-2r\cos\phi+r^2} \\
&= e^{r\cos 0}\sqrt{1-2r\cos 0+r^2} \\
&= e^{r}\sqrt{(1-r)^2} \\
&= (1-r)e^{r} \\
&\geq (1-r)(1+r) \\
&= 1-r^2 \\
&= 1-|z|^2,
\end{aligned}
$$

and so, $|(1-z)e^z| \geq 1-|z|^2$. □

Recall that $\{\lambda_k\}$ is the enumeration of the integer lattice points $\mathbb{Z}\times i\mathbb{Z}$ defined in the introduction to this chapter. The following technical lemma gives bounds on the index and the modulus of the eigenvalues lying on the squares S_j, where $j \in \{0,1,2,...\}$.

Lemma 2.2. *If $\lambda_m \in \{\lambda_k\}$ lies on the square S_j, then $j \leq |\lambda_m| \leq \sqrt{2}j$ and $(2j-1)^2 \leq m \leq 4(j^2+j)$.*

Proof. For any $\lambda_m \in S_j$, the smallest value of $|\lambda_m|$ occurs when $\lambda_m = j$, and the largest value of $|\lambda_m|$ occurs when $\lambda_m = j+ij$. Thus,

$$j = |j| \leq |\lambda_m| \leq |j+ij| = \sqrt{2}j.$$

There are $8j$ points from $\{\lambda_k\}$ on the square S_j. Hence, the smallest value for m such that $\lambda_m \in S_j$ is

$$m = 1+\sum_{i=0}^{j-1} 8i = 1+4j(j-1) = (2j-1)^2.$$

The largest value for m such that $\lambda_m \in S_j$ is

$$m = \sum_{i=1}^{j} 8i = 4j(j+1) = 4(j^2+j).$$

Hence, for $\lambda_m \in S_j$, we have

$$(2j-1)^2 \leq m \leq 4(j^2+j).$$

$\square$

In the following proposition, a type of periodicity of S is established using the quasi-periodicity of ζ which was discussed in Section 2.1.

Proposition 2.1. *For any $z \in \mathbb{C}$ and any $m+in \in \mathbb{Z}\times i\mathbb{Z}$, we have $|S(z)| = |S(z-(m+in))||e^{\pi[(z-(m+in))(m-in)+(m^2+n^2)/2]}|$.*

Proof. Using the quasi-periodicity of $\zeta(z)$ we can integrate the equation $\zeta(z+m+in) = \zeta(z+m) - in\pi = \zeta(z) + m\pi - in\pi$ to obtain $\log S(z+m+in) = \log S(z) + (m\pi - in\pi)z + c$, whence, $S(z+m+in) = e^{\log S(z) + (m\pi - in\pi)z + c} = S(z)e^{\pi(m-in)z+c}$. Letting $z = -\frac{1}{2}m - \frac{i}{2}n$, we have $S(-\frac{m}{2} - \frac{in}{2} + m + in) = S(-\frac{m}{2} - \frac{in}{2})e^{\pi(m-in)(-(m/2)-(in/2))+c}$. Since $S(z)$ is an odd function, we have $S(\frac{m}{2} + \frac{in}{2}) = -S(\frac{m}{2} + \frac{in}{2})e^{\pi(m-in)(-(m/2)-(in/2))+c}$. Hence $1 = -e^{\pi(m-in)(-(m/2)-(in/2))}e^c$, and therefore $e^c = -e^{\pi(m-in)((m/2)+(in/2))} = -e^{\pi((m^2/2)+(imn/2)-(imn/2)+(n^2/2))} = -e^{\pi((m^2+n^2)/2)}$. Thus $S(z+m+in) = S(z)e^{\pi(m-in)z+\pi((m^2+n^2)/2)}$, or, equivalently, $S(z) = -S(z-(m+in))e^{\pi((z-(m+in))(m-in)+(m^2+n^2)/2)}$. The result follows. $\square$

In the following proposition, we find a lower bound for $|S(z)|$ for z near the origin.

Proposition 2.2. *Whenever $|z| \leq \frac{1}{\sqrt{2}}$, $|S(z)| \geq |z|e^{-c|z|^4}$ where $c = 3\sum_{m,n}' \frac{1}{(m^2+n^2)^2}$.*

Proof. Observe

$$\left|\frac{S(z)}{z}\right| = \left|\prod_{m,n}{}' \left(1 - \frac{z}{m+in}\right) e^{z/(m+in)} e^{z^2/(2(m+in)^2)}\right|$$

$$\begin{aligned}
&= \left|\prod_{m,n\geq 0}{}' \left(1-\frac{z}{m+in}\right) e^{z/(m+in)} e^{z^2/(2(m+in)^2)} \left(1-\frac{z}{-m-in}\right) e^{z/(-m-in)} e^{z^2/(2(m+in)^2)}\right| \\
&\times \left|\prod_{m,n>0} \left(1-\frac{z}{m-in}\right) e^{z/(m-in)} e^{z^2/(2(m-in)^2)} \left(1-\frac{z}{-m+in}\right) e^{z/(-m+in)} e^{z^2/(2(-m+in)^2)}\right| \\
&= \prod_{m,n\geq 0}{}' \left|\left(1-\frac{z}{m+in}\right)\left(1+\frac{z}{m+in}\right) e^{z^2/((m+in)^2)}\right| \\
&\times \prod_{m,n>0} \left|\left(1-\frac{z}{m-in}\right)\left(1+\frac{z}{m-in}\right) e^{z^2/((m-in)^2)}\right| \\
&= \prod_{m,n\geq 0}{}' \left|\left(1-\left(\frac{z}{m+in}\right)^2\right) e^{(z/(m+in))^2}\right| \prod_{m,n>0}\left|\left(1-\left(\frac{z}{m-in}\right)^2\right) e^{(z/(m-in))^2}\right|.
\end{aligned}$$

Note $|(\frac{z}{m\pm in})^2| = \frac{|z|^2}{|m\pm in|^2} \leq \frac{(1/\sqrt{2})^2}{1} = \frac{1}{2}$, since it must be the case that at least one of m and n is nonzero. Applying the inequality from Lemma 2.1 gives

$$\begin{aligned}
\left|\frac{S(z)}{z}\right| &\geq \prod_{m,n\geq 0}{}' \left(1-\left|\left(\frac{z}{m+in}\right)^2\right|^2\right) \prod_{m,n>0}\left(1-\left|\left(\frac{z}{m-in}\right)^2\right|^2\right) \\
&= \prod_{m,n\geq 0}{}' \left(1-\left|\frac{z}{m+in}\right|^4\right) \prod_{m,n>0}\left(1-\left|\frac{z}{m-in}\right|^4\right).
\end{aligned}$$

Observe the largest $(1-|\frac{z}{m+in}|^4)$ can be is 1 which occurs when $z=0$. When z is such that $0<|z|<1/\sqrt{2}$, the quantity $(1-|\frac{z}{m+in}|^4)$ is at most $3/4$ which occurs when either $m=1$ and $n=0$, or $m=0$ and $n=1$; for all other possibilities of m and n, the quantity is smaller. Hence, $|S(z)/z| \leq 1$ for all $z \in \overline{B(0,1/\sqrt{2})}$. Therefore,

$$\begin{aligned}
\left|\frac{S(z)}{z}\right| &\geq e^{\log \prod'_{m,n\geq 0}(1-|\frac{z}{m+in}|^4)} e^{\log \prod_{m,n>0}(1-|\frac{z}{m-in}|^4)} \\
&= e^{\sum'_{m,n\geq 0}\log(1-|\frac{z}{m+in}|^4)} e^{\sum_{m,n>0}\log(1-|\frac{z}{m-in}|^4)}.
\end{aligned}$$

Moreover, since $\left|\frac{z}{m+in}\right|^4 \leq \frac{1}{2}$, by (2.1) we have that $\log(1-|\frac{z}{m\pm in}|^4) = -|\log(1-|\frac{z}{m\pm in}|^4)| \geq$

$-\frac{3}{2}|\frac{z}{m\pm in}|^4$, and so,

$$\begin{aligned}
\left|\frac{S(z)}{z}\right| &\geq e^{\sum'_{m,n\geq 0}-\frac{3}{2}|\frac{z}{m+in}|^4} e^{\sum_{m,n>0}-\frac{3}{2}|\frac{z}{m-in}|^4} \\
&= e^{-\frac{3}{2}|z|^4\sum'_{m,n\geq 0}\frac{1}{|m+in|^4}} e^{-\frac{3}{2}|z|^4\sum_{m,n>0}\frac{1}{|m-in|^4}} \\
&= e^{-\frac{3}{2}|z|^4\sum'_{m,n\geq 0}\frac{1}{(m^2+n^2)^2}} e^{-\frac{3}{2}|z|^4\sum_{m,n>0}\frac{1}{(m^2+n^2)^2}} \\
&\geq e^{-3|z|^4\sum'_{m,n\geq 0}\frac{1}{(m^2+n^2)^2}} \\
&= e^{-c|z|^4},
\end{aligned}$$

where $c \equiv 3\sum'_{m,n\geq 0}\frac{1}{(m^2+n^2)^2} \leq 3\frac{\pi^2}{3} = \pi^2$. Thus, we have a lower bound on $|\frac{S(z)}{z}|$ whenever $|z| \leq \frac{1}{\sqrt{2}}$. $\square$

We use Proposition 2.2 to find a bound on $|S(z)|$ for all $z \in \mathbb{C}$.

Proposition 2.3. *For $z \in \mathbb{C}$, we have $|S(z)| \geq |z-(m'+in')|e^{-(c/4)-(\pi/4)}e^{(\pi/2)|z|^2}$ where $m'+in' \in \mathbb{Z}\times i\mathbb{Z}$ is such that $|z-(m'+in')| = \inf\{|z-(m+in)| : m,n \in \mathbb{Z}\}$.*

Proof. Since $|z-(m'+in')| \leq \frac{1}{\sqrt{2}}$, by Propositions 2.1 and 2.2, we have that

$$\begin{aligned}
|S(z)| &= |S(z-(m'+in'))||e^{\pi[(z-(m'+in'))(m'-in')+((m'^2+n'^2)/2)]}| \\
&\geq |z-(m'+in')|e^{-c|z-(m'+in')|^4}|e^{\pi[(z-(m'+in'))(m'-in')+((m'^2+n'^2)/2)]}|.
\end{aligned}$$

Writing $z = x+iy$, we have that

$$\begin{aligned}
|e^{\pi((z-(m'+in'))(m'-in')+((m'^2+n'^2)/2))}| &= |e^{\pi(((x-m')+i(y-n'))(m'-in')+((m'^2+n'^2)/2))}| \\
&= e^{Re[\pi(xm'-m'^2-ixn'+im'n'+iym'+yn'-im'n'-n'^2+(m'^2/2)+(n'^2/2))]} \\
&= e^{Re[\pi(xm'+yn'-(m'^2/2)-(n'^2/2)+i(-xn'+ym'))]} \\
&= e^{\pi(xm'+yn'-((m'^2+n'^2)/2))} \\
&= e^{(\pi/2)(2xm'-m'^2+2yn'-n'^2)} \\
&= e^{(\pi/2)(x^2+y^2-(x^2-2xm'+m'^2+y^2-2yn'+n'^2))}
\end{aligned}$$

$$
\begin{aligned}
&= e^{(\pi/2)(x^2+y^2-((x-m')^2+(y-n')^2))} \\
&= e^{(\pi/2)(|z|^2-|z-(m'+in')|^2)} \\
&\geq e^{(\pi/2)(|z|^2-(1/\sqrt{2})^2)} \\
&= e^{(\pi/2)|z|^2-(\pi/4)}.
\end{aligned}
$$

Thus, $|S(z)| \geq |z-(m'+in')|e^{-(c/4)-(\pi/4)}e^{(\pi/2)|z|^2}$. $\square$

The preceding results demonstrate that S is of order 2 and type $\pi/2$. In the following proposition we obtain a lower bound for $|S'(\lambda_k)|$.

Proposition 2.4. $|S'(\lambda_k)| \geq \frac{e^{(\pi/2)(|\lambda_k|-(1/4))^2}}{e^{(c/4)+(\pi/4)}}$ *for all* $k \geq 0$.

Proof. Fix $\lambda_k = m+in$. Note $S(m+in) = 0$ but $S'(m+in) \neq 0$, since $m+in$ is a simple zero of S. Moreover, $S(z) \neq 0$ for all z with $0 < |z-(m+in)| \leq \frac{1}{4}$, since S only has zeros at the integer lattice points all of which are one unit apart. If $|z-(m+in)| = \frac{1}{4}$, then $|m+in| - \frac{1}{4} \leq |z| \leq |m+in| + \frac{1}{4}$, by the reverse triangle inequality. Furthermore, $\lambda_k = m+in$ is the closest lattice point to any such z. Thus, by Proposition 2.3, whenever $|z-(m+in)| = \frac{1}{4}$,

$$
\begin{aligned}
|S(z)| &\geq |z-(m+in)|e^{-(c/4)-(\pi/4)}e^{(\pi/2)|z|^2} \\
&= \frac{1}{4}e^{-(c/4)-(\pi/4)}e^{(\pi/2)|z|^2} \\
&\geq \frac{1}{4}e^{-(c/4)-(\pi/4)}e^{(\pi/2)(|m+in|-(1/4))^2}.
\end{aligned}
$$

If $\alpha_{mn} \equiv (1/4)e^{-(c/4)-(\pi/4)}e^{(\pi/2)(|m+in|-(1/4))^2}$, then, for each $w \in B(0,\alpha_{mn})$, there is a unique $z \in B(m+in, 1/4)$ with $S(z) = w$, by the Inverse Function Theorem [15, Gamelin, page 234]. The restriction $\hat{S}|_{S^{-1}(B(0,\alpha_{mn}))} : S^{-1}(B(0,\alpha_{mn})) \to B(0,\alpha_{mn})$ of S to the preimage $S^{-1}(B(0,\alpha_{mn}))$ of $B(0,\alpha_{mn})$ under S, is a bijection. Hence, the inverse $\hat{S}^{-1}$ of $\hat{S}$, exists and is analytic on $S^{-1}(B(0,\alpha_{mn}))$. Define $g(z) = 4(\hat{S}^{-1}(\alpha_{mn}z) - (m+in))$. Since $f_1(z) \equiv \alpha_{mn}z$ maps $B(0,1)$ onto $B(0,\alpha_{mn})$, $f_2(z) \equiv \hat{S}^{-1}(z)$ maps $B(0,\alpha_{mn})$ into $\hat{S}^{-1}(B(0,\alpha_{mn})) \subset B(m+$

$in, 1/4)$, and $f_3(z) \equiv 4(z - (m + in))$ maps $B(m + in, 1/4)$ onto $B(0.1)$, we have that $g : B(0,1) \to B(0,1)$ is analytic, and satisfies $g(0) = 4(\hat{S}^{-1}(0) - (m + in)) = 4((m + in) - (m+in)) = 0$, since $m+in$ is the only zero of S in $S^{-1}(B(0, \alpha_{mn}))$. Since $g'(z) = \frac{4\alpha_{mn}}{\hat{S}'(\hat{S}^{-1}(\alpha_{mn} z))}$ we have, by Schwarz' Lemma that

$$\begin{aligned} 1 &\geq |g'(0)| \\ &= \frac{4\alpha_{mn}}{\hat{S}'(\hat{S}^{-1}(0))} \\ &= \frac{4\alpha_{mn}}{S'(m+in)}, \end{aligned}$$

and so

$$\begin{aligned} |S'(\lambda_k)| &= |S'(m+in)| \\ &\geq 4\alpha_{mn} \\ &= e^{-(c/4)-(\pi/4)} e^{(\pi/2)(|m+in|-(1/4))^2} \\ &= \frac{e^{(\pi/2)(|m+in|-(1/4))^2}}{e^{(c/4)+(\pi/4)}} \\ &= \frac{e^{\pi/2(|\lambda_k|-1/4)}}{e^{(c/4)+(\pi/4)}}. \end{aligned}$$

□

In the preceding proof, the choice of $\frac{1}{4}$ as the radius of the disk centered at $m + in$ was not unique as we need only guarantee that for any z on the circle $|z - (m + in)| = \epsilon$ the closest lattice point is $m + in$. Therefore, we could use any value $\epsilon < 1/2$. We now use the bound obtained in the preceding proposition to establish that the sequence $\{\omega_k\} \equiv \{1/S'(\lambda_k)\}$ satisfies the decay rate necessary to show a diagonal operator D acting on $H(\mathbb{D})$ fails to admit spectral synthesis, as discussed in Section 1.3.

Proposition 2.5. $\limsup_{k\to\infty} \frac{1}{|S'(\lambda_k)|^{1/k}} < 1$.

Proof. By the preceding result, we have

$$\begin{aligned}
\frac{1}{|S'(\lambda_k)|^{1/k}} &\leq \left(\frac{e^{(c/4)+(\pi/4)}}{e^{(\pi/2)(|\lambda_k|-(1/4))^2}}\right)^{\frac{1}{k}} \\
&= \left(\frac{e^{(c/4)+(\pi/4)}}{e^{(\pi/2)(|\lambda_k|^2-(1/2)|\lambda_k|+(1/16))}}\right)^{\frac{1}{k}} \\
&= \left(\frac{e^{(c/4)+(7\pi/32)}e^{(\pi/4)|\lambda_k|}}{e^{(\pi/2)|\lambda_k|^2}}\right)^{\frac{1}{k}} \\
&= \frac{e^{(1/k)((c/4)+(7\pi/32))}e^{(\pi/4)(|\lambda_k|/k)}}{e^{(\pi/2)(|\lambda_k|^2/k)}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\limsup_{k\to\infty}\left|\frac{1}{S'(\lambda_k)}\right|^{1/k} &\leq \limsup_{k\to\infty}\frac{e^{(1/k)((c/4)+(7\pi/32))}e^{(\pi/4)(|\lambda_k|/k)}}{e^{(\pi/2)(|\lambda_k|^2/2)}} \\
&\leq \limsup_{k\to\infty} e^{(1/k)((c/4)+(7\pi/32))}\limsup_{k\to\infty}\frac{e^{(\pi/4)(|\lambda_k|/k)}}{e^{(\pi/2)(|\lambda_k|^2/k)}} \\
&= \limsup_{k\to\infty}\frac{e^{(\pi/4)(|\lambda_k|/k)}}{e^{(\pi/2)(|\lambda_k|^2/k)}}.
\end{aligned}$$

To estimate $\limsup_{k\to\infty}\frac{e^{(\pi/4)(|\lambda_k|/k)}}{e^{(\pi/2)(|\lambda_k|^2/k)}}$, we must establish bounds on $|\lambda_k|/k$ and $|\lambda_k|^2/k$. By Lemma 2.2, we have for $\lambda_k \in S_j$ that

$$\frac{|\lambda_k|}{k} \leq \frac{\sqrt{2}j}{(2j-1)^2}$$

and

$$\frac{|\lambda_k|^2}{k} \geq \frac{j^2}{4(j^2+j)}.$$

Hence,

$$\limsup_{k\to\infty}\frac{e^{(\pi/4)(|\lambda_k|/k)}}{e^{(\pi/2)(|\lambda_k|^2/k)}} \leq \limsup_{j\to\infty}\frac{e^{(\pi/4)(\sqrt{2}j/(2j-1)^2)}}{e^{(\pi/2)(j^2/4(j^2+j))}} = \frac{1}{e^{\pi/8}},$$

and so $\limsup_{k\to\infty}|1/S'(\lambda_k)|^{1/k} \leq e^{-\pi/8} < 1$. □

In the following proposition, we show that $\sum_{k=0}^{\infty} e^{\lambda_k z}/S'(\lambda_k)$ is analytic on a disk centered at the origin.

Proposition 2.6. $\sum_{k=0}^{\infty} \frac{e^{\lambda_k z}}{S'(\lambda_k)}$ *is analytic on* $B(0,\epsilon)$ *where* $\epsilon \equiv [\ln(1/\limsup_{k\to\infty}[|1/S'(\lambda_k)|^{1/k}])]/[\sup\{|\lambda_k|/k\}]$.

Proof. By Lemma 2.2, $\sup\{|\lambda_k|/k : k \geq 1\} \leq \sup\{\sqrt{2}j/(2j-1)^2 : j \geq 1\} = \sqrt{2}$. To prove this proposition it suffices to show the series $\sum_{k=0}^{\infty}(e^{\lambda_k z}/S'(\lambda_k))$ converges absolutely and uniformly on $B(0,\epsilon)$. Let C be any compact subset of $B(0,\epsilon)$. Then, for $z \in C$,

$$\begin{aligned}
\limsup_{k\to\infty}\left|\frac{e^{\lambda_k z}}{S'(\lambda_k)}\right|^{\frac{1}{k}} &\leq \limsup_{k\to\infty}\frac{e^{\frac{|\lambda_k|}{k}|z|}}{|S'(\lambda_k)|^{\frac{1}{k}}} \\
&< \limsup_{k\to\infty} e^{\frac{|\lambda_k|}{k}\epsilon}\limsup_{k\to\infty}\frac{1}{|S'(\lambda_k)|^{\frac{1}{k}}} \\
&\leq \limsup_{k\to\infty} e^{(|\lambda_k|/k)\ln(1/\limsup_{k\to\infty}|1/S'(\lambda_k)|^{1/k})/\sup\{|\lambda_k|/k:k\geq 1\}}\limsup_{k\to\infty}\frac{1}{|S'(\lambda_k)|^{\frac{1}{k}}} \\
&\leq \limsup_{k\to\infty}\frac{1}{\limsup_{k\to\infty}\left|\frac{1}{S'(\lambda_k)}\right|^{\frac{1}{k}}}\limsup_{k\to\infty}\frac{1}{|S'(\lambda_k)|^{\frac{1}{k}}} \\
&= 1.
\end{aligned}$$

Thus, by the Root Test, $\sum_{k=0}^{\infty}\frac{e^{\lambda_k z}}{S'(\lambda_k)}$ converges absolutely and uniformly on $B(0,\epsilon)$, proving the proposition. □

2.3 A Non-synthetic Operator on $H(\mathbb{D})$

Using the results of the preceding two sections, we prove the main result of this chapter, and in doing so, produce an example of a non-synthetic diagonal operator acting on $H(\mathbb{D})$.

Theorem 2.1. *The diagonal operator D on $H(\mathbb{D})$ with eigenvalues $\{\lambda_k\} = \mathbb{Z} \times i\mathbb{Z}$ fails to admit spectral synthesis.*

Proof. By Lemma 2.2, for all k with $\lambda_k \in S_j$, $|\lambda_k|^{1/k} \leq (\sqrt{2}j)^{1/(2j-1)^2}$, and so

$$\limsup_{k\to\infty}|\lambda_k|^{1/k} \leq \limsup_{j\to\infty}(\sqrt{2}j)^{1/(2j-1)^2} = 1.$$

Hence, D is a continuous linear operator acting on $H(\mathbb{D})$. The eigenvalues of D are distinct, hence D is cyclic by Proposition 1.3. The function $S(\lambda) = \lambda \prod' \left(1 - \frac{\lambda}{m+in}\right) e^{(\lambda/m+in)+(\lambda^2/2(m+in)^2)}$ has zeros only at the points of $\{\lambda_k\}$ (all of which are simple), and thus, the function $e^{\lambda z}/S(\lambda)$ has poles only at the points of $\{\lambda_k\}$ (all of which are simple) for any $z \in \mathbb{C}$. For each $r \in \mathbb{Z}^+$, let C_r denote the square with vertices $\pm(r + (1/2)) \pm i(r + (1/2))$, and apply the Residue Theorem to obtain,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_r} \frac{e^{\lambda z}}{S(\lambda)} d\lambda &= \sum_{\{k:|\lambda_k|<r\}} Res\left(\frac{e^{\lambda z}}{S(\lambda)}, \lambda_k\right) \\
&= \sum_{\{k:|\lambda_k|<r\}} \frac{e^{\lambda z}}{S'(\lambda)}\Big|_{\lambda=\lambda_k} \\
&= \sum_{\{k:|\lambda_k|<r\}} \frac{e^{\lambda_k z}}{S'(\lambda_k)}.
\end{aligned}$$

For a fixed $z \in \mathbb{C}$, by Proposition 2.3,

$$\begin{aligned}
\left|\int_{C_r} \frac{e^{\lambda z}}{S(\lambda)} d\lambda\right| &\leq \int_{C_r} \frac{|e^{\lambda z}|}{|S(\lambda)|}|d\lambda| \\
&\leq \int_{C_r} \frac{e^{|\lambda||z|}}{|\lambda - (m' + in')|e^{-(c/4)-(\pi/4)}e^{(\pi/2)|\lambda|^2}}|d\lambda| \\
&\leq \frac{(2r+1)^2 e^{(r+(1/2))|z|}}{(r + (1/2) - |m' + in'|)e^{-(c/4)-(\pi/4)}e^{(\pi/2)(r+(1/2))^2}} \\
&\to 0.
\end{aligned}$$

as $r \to \infty$. Thus, by Proposition 2.6,

$$\sum_{k=0}^{\infty} \frac{e^{\lambda_k z}}{S'(\lambda_k)} = \lim_{r\to\infty} \sum_{\{k:\lambda_k \in C_r^o\}} \frac{e^{\lambda_k z}}{S'(\lambda_k)} = \lim_{r\to\infty} \int_{C_r} \frac{e^{\lambda z}}{S(\lambda)} d\lambda = 0.$$

Hence, if we define $\omega_k = 1/S'(\lambda_k)$ for all $k \geq 0$, we have $\sum_{k=0}^{\infty} \omega_k e^{\lambda_k z} \equiv 0$ for all $z \in B(0, \epsilon)$ where ϵ is as defined in Proposition 2.6. Moreover, we have $\limsup_{k\to\infty} |\omega_k|^{1/k} < 1$ by Proposition 2.5, and $\sup\{|\lambda_k|/k : k \geq 1\} = \sqrt{2} < \infty$, by Lemma 2.2. Thus, by Condition *(xi)* of Theorem 1.3, D fails to admit spectral synthesis on $H(\mathbb{D})$. $\square$

The preceding theorem combined with the following theorem demonstrate that D is an example of a diagonal operator which is synthetic when viewed as acting on $H(\mathbb{C})$ but non-synthetic when viewed as acting on $H(\mathbb{D})$.

Theorem 2.2. *The diagonal operator D on $H(\mathbb{C})$ with eigenvalues $\{\lambda_k\}$ admits spectral synthesis.*

Proof. The entire function S of Theorem 2.1 is of order $\rho = 2$ and type $\tau = \pi/2$, by Proposition 2.3 (or see [29, page 128]). Furthermore, by Lemma 2.2, $\sup\{|\lambda_k|^2/k : k \geq 1\} \leq \sup\{2j^2/(2j-1)^2 : j \geq 1\} = 2$. Hence, by Theorem 1.12, D admits spectral synthesis on $H(\mathbb{C})$. $\square$

CHAPTER 3

A Class of Non-synthetic Operators on $H(\mathbb{D})$

Theorems 1.3 through 1.10 of Chapter 1 demonstrate that one major factor in determining whether or not a diagonal operator admits spectral synthesis is the rate of growth of its eigenvalues. For instance, the diagonal operator having eigenvalues $\lambda_n = \sqrt{n}$ admits spectral synthesis as an operator acting on $H(\mathbb{D})$ (see Theorem 1.5). However, in Chapter 2, we showed that the diagonal operator D acting on $H(\mathbb{D})$ having as eigenvalues the integer lattice points $\mathbb{Z} \times i\mathbb{Z}$ fails synthesis, even though $\{\lambda_k\}$ has growth on the order $\sqrt{n}$. In view of which, it appears as though the synthesis of an operator depends not only on the growth of its eigenvalues, but also on how they are distributed throughout the plane.

The purpose of this chapter is to investigate how distributing eigenvalues of a certain growth rate along various rays $\{z \in \mathbb{C} : \arg z = \theta\}$ in the complex plane affects the synthesis of the diagonal operator having the resulting points as its set of eigenvalues. In particular, we consider sequences of real values $\{n^{1/p}\}$, where $p \in \{2, 3, ...\}$, placed symmetrically on collections of rays resulting in sets of eigenvalues of the form $\{n^{1/p} e^{2\pi i j/s} : 0 \leq j < s\}$. For instance, when $p = 2$, we know that the diagonal operator on $H(\mathbb{D})$ having eigenvalues $\{\sqrt{n}\}$ admits spectral synthesis by Theorem 1.5. Similarly, the diagonal operator having eigenvalues $\{\pm\sqrt{n}\}$ also admits spectral synthesis by Theorem 1.6. However, we show in Section 3.1 that the diagonal operator on $H(\mathbb{D})$ having eigenvalues $\{\sqrt{n}e^{\pi ij/3} : 0 \leq j < 6\}$, consisting of six copies of the sequence $\{\sqrt{n}\}$ on six rays, fails spectral synthesis.

In Section 3.2, we show that any diagonal operator on $H(\mathbb{D})$ having sufficiently many copies of any real sequence growing on the order $n^{a/b}$ placed symmetrically along rays in the complex plane, where a/b is any rational number smaller than 1, fails spectral synthesis.

3.1 Diagonal Operators having Eigenvalues $\{n^{1/p}e^{2\pi ij/3p} : 0 \leq j < 3p\}$ for $p > 1$ are Non-synthetic

In this section, we show that the diagonal operator acting on $H(\mathbb{D})$ having eigenvalues $\{n^{1/p}e^{2\pi ij/3p} : 0 \leq j < 3p\}$ is non-synthetic, for any integer $p > 1$. We begin with two technical lemmas estimating infinite products which occur in our proofs.

Lemma 3.1. *For a fixed integer $n \geq 1$, $\prod_{j\neq n}^{\infty} |1 - \frac{n^2}{j^2}| = \frac{1}{2}$.*

Proof. Since $\sin \pi z = \pi z \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right)$, we have that

$$\begin{aligned}
\prod_{j\neq n} \left|1 - \frac{n^2}{j^2}\right| &= \lim_{z\to n} \left|\frac{\sin \pi z}{\pi z(1 - (z^2/n^2))}\right| \\
&= \lim_{z\to n} \left|\frac{\pi \cos \pi z}{\pi(z(-2z/n^2) + (1 - (z^2/n^2)))}\right| \\
&= \left|\frac{\cos \pi n}{-2}\right| \\
&= \frac{1}{2}.
\end{aligned}$$

□

Lemma 3.1 can be used to show the following estimate.

Lemma 3.2. *For fixed integers $n \geq 1$ and $q > 2$, $\prod_{j\neq n}^{\infty} |1 - \frac{n^q}{j^q}| \geq \frac{1}{2}(\frac{q}{2})^{\frac{n}{2}}$.*

Proof. Observe that

$$\prod_{j\neq n}^{\infty}\left|1-\frac{n^q}{j^q}\right| = \prod_{j\neq n}^{\infty}\frac{|j^q-n^q|}{j^q} = \lim_{N\to\infty}\prod_{j\neq n}^{N}\frac{|j^q-n^q|}{j^q} = \prod_{j=1}^{n-1}\frac{n^q-j^q}{j^q}\lim_{N\to\infty}\prod_{j=n+1}^{N}\frac{j^q-n^q}{j^q}.$$

We first estimate $\prod_{j=n+1}^{N}(j^q-n^q)/j^q$, where

$$\begin{aligned}\prod_{j=n+1}^{N}\frac{j^q-n^q}{j^q} &= \prod_{j=n+1}^{N}\frac{j^2-n^2}{j^2}\frac{j^{q-2}+\frac{n^2j^{q-2}-n^q}{j^2-n^2}}{j^{q-2}}\\ &= \prod_{j=n+1}^{N}\frac{j^2-n^2}{j^2}\prod_{j=n+1}^{N}\frac{j^q-n^q}{j^q-n^2j^{q-2}}.\end{aligned}$$

For $n+1\leq j\leq N$, we have $j^q-n^q\geq j^q-n^2j^{q-2}$, hence $\frac{j^q-n^q}{j^q-n^2j^{q-2}}\geq 1$. Thus, $\prod_{j=n+1}^{N}\frac{j^q-n^q}{j^q-n^2j^{q-2}}\geq 1$. We now estimate $\prod_{j=1}^{n-1}(n^q-j^q)/j^q$, where

$$\begin{aligned}\prod_{j=1}^{n-1}\frac{n^q-j^q}{j^q} &= \prod_{j=1}^{n-1}\frac{n^2-j^2}{j^2}\frac{j^{q-2}+\frac{n^q-n^2j^{q-2}}{n^2-j^2}}{j^{q-2}}\\ &= \prod_{j=1}^{n-1}\frac{n^2-j^2}{j^2}\prod_{j=1}^{n-1}\frac{j^{q-2}n^2-j^q+n^q-n^2j^{q-2}}{j^{q-2}(n^2-j^2)}\\ &= \prod_{j=1}^{n-1}\frac{n^2-j^2}{j^2}\prod_{j=1}^{n-1}\frac{n^q-j^q}{n^2j^{q-2}-j^q},\end{aligned}$$

by considering the terms $(n^q-j^q)/(n^2j^{q-2}-j^q)$ for various values of j. For $j=1$,

$$\frac{n^q-j^q}{n^2j^{q-2}-j^q}=\frac{n^q-1}{n^2-1}\geq\frac{q}{2}$$

since the function $f(n)=2n^q-qn^2+(q-2)$ is increasing for $n\geq 1$, and $f(1)=0$. For n even and $2\leq j\leq n/2$,

$$\frac{n^q-j^q}{n^2j^{q-2}-j^q}\geq\frac{q}{2}$$

since

$$\begin{aligned} 2n^q - qn^2j^{q-2} + (q-2)j^q &\geq 2n^q - qn^2(n/2)^{q-2} + (q-2)2^q \\ &= 2n^q - (q/2^{q-2})n^q + (q-2)2^q \\ &= (2-(q/2^{q-2}))n^q + (q-2)2^q \\ &\geq 0, \end{aligned}$$

as $2^{q-1} \geq q$ for any $q > 2$. The inequality $(n^q - j^q)/(n^2j^{q-2} - j^q) \geq (q/2)$ also holds for n odd by a similar argument. For $j \leq n-1$, we have that $j < n$, and so,

$$\frac{n^q - j^q}{n^2j^{q-2} - j^q} \geq 1.$$

Hence,

$$\prod_{j=1}^{n-1} \frac{n^q - j^q}{j^q} \geq \left(\frac{q}{2}\right)^{n/2} \prod_{j=1}^{n-1} \frac{n^2 - j^2}{j^2},$$

and so

$$\prod_{j=1, j\neq n}^{N} \frac{|n^q - j^q|}{j^q} \geq \prod_{j=1, j\neq n}^{N} \frac{|j^2 - n^2|}{j^2} \left(\frac{q}{2}\right)^{\frac{n}{2}}.$$

Thus, by Lemma 3.1,

$$\prod_{j=1, j\neq n}^{\infty} \left|1 - \frac{n^q}{j^q}\right| \geq \left(\frac{q}{2}\right)^{\frac{n}{2}} \prod_{j=1, j\neq n}^{\infty} \frac{|j^2 - n^2|}{j^2} = \frac{1}{2}\left(\frac{q}{2}\right)^{\frac{n}{2}}.$$

$\square$

Before proceeding to the main results of this chapter, we indicate a protocol for enumerating sets of eigenvalues as pertains to our study. We have already seen that a linear map D having each monomial z^n as an eigenvector with associated eigenvalue λ_n is continuous on $H(\mathbb{D})$ if and only if $\limsup_{n\to\infty} |\lambda_n|^{1/n} \leq 1$. Reordering the points in the set $\{\lambda_n\}$ does not necessarily result in a new sequence satisfying this condition. In view of which, it is not just the collection of eigenvalues, but their order, which affects the continuity of the linear map D. In fact, we see in the next chapter that there exists a synthetic diagonal operator on $H(\mathbb{D})$ a reordering of whose eigenvalues yields another diagonal operator on $H(\mathbb{D})$ which is non-synthetic. That is, a reordering of the eigenvalues of a diagonal operator need not preserve spectral synthesis.

Throughout the remainder of this chapter, we adopt the convention that any enumeration $\{\lambda_k\}$ of the set of points of the form $\{a_n e^{2\pi i j/p} : 0 \leq j < p\}$ where $\{a_n\}$ is an increasing sequence of positive numbers and p is a positive integer, be such that $\{|\lambda_k|\}$ is non-decreasing. Such an enumeration is always obtained by listing the points of the set by starting on the positive real axis and traversing circles of increasing radii a_n in the counterclockwise direction. In this case, $\lambda_0 = a_1$, $\lambda_1 = a_1 e^{2\pi i/p}$, $\lambda_2 = a_1 e^{4\pi i/p}$,..., etc.

We now show that the diagonal operator D on $H(\mathbb{D})$ having as eigenvalues $3p$ copies of the sequence $\{n^{1/p}\}$ placed on the $3p$ rays $\{z \in \mathbb{C} : \arg z = 2\pi i j\}$ for $0 \leq j < 3p$, fails spectral synthesis whenever p is an integer greater than 1. This theorem is generalized in the subsequent corollaries to include sequences $\{n^{a/b}\}$ for certain rational powers a/b. These results are valid whenever $\{\lambda_k\}$ is an enumeration of the eigenvalues for which $\{|\lambda_k|\}$ is non-decreasing. Throughout the proofs of this chapter, we invoke standard results from the theory of entire functions, and in particular those concerning canonical products, without individual citations. These standard results are collected in the appendix for the convenience of the reader.

Theorem 3.1. *The diagonal operator D on $H(\mathbb{D})$ having eigenvalues $\{n^{1/p} e^{2i\pi j/3p} : 0 \leq j < 3p\}$ fails spectral synthesis whenever p is an integer at least* 2.

Proof. Let $\{\lambda_k\}$ be any enumeration of the set $\{n^{1/p} e^{2i\pi j/3p} : 0 \leq j < 3p\}$ for which $\{|\lambda_k|\}$ is non-decreasing. The diagonal operator D is cyclic by Proposition 1.3 since the points $\{\lambda_k\}$ are distinct. The entire function $f(z) \equiv \prod_{n=1}^{\infty} (1 - \frac{z}{n^3})$ has order $1/3$ and zeros $\{n^3\}$ (all of which are simple) with density $\Delta = \lim_{n\to\infty} |a_n|^{1/3}/n = 1$. Hence, by Levin [29, pages 94 and 95],

$$\log|f(re^{i\theta})| \approx \frac{\pi\Delta r^{1/3}}{\sin(\pi/3)}\cos\frac{1}{3}(\theta-\pi)+o(r^{1/3})$$

outside the exceptional set $E \equiv \cup_{n=1}^{\infty}\overline{B(n^3,dn^2)}$ where d is any number in $(0,1]$. Since $(\pi\Delta/\sin(\pi/3))\cos((\theta-\pi)/3) \geq \pi/\sqrt{3}$, it follows that for every $\epsilon > 0$, there exists an R_ϵ such that

$$|f(re^{i\theta})| \geq e^{((\pi/\sqrt{3})-\epsilon)r^{1/3}}$$

whenever $r \geq R_\epsilon$ and $re^{i\theta}$ is not in E.

The entire function $S(z) \equiv f(z^{3p})$ has zeros only at the points $\{\lambda_k\}$ (all of which are simple) and is of order $1/p$. For each positive integer r we define $C_r \equiv \{z \in \mathbb{C} : |z| = \hat{r}\}$ to be the circle of radius $\hat{r}$ where $\hat{r} = ((r+1)^{1/p}+r^{1/p})/2$, and $(\hat{r}e^{i\theta})^{3p} \notin E$. Thus, no point λ_k lies on any C_r, and so $S(\lambda) \neq 0$ whenever $\lambda \in C_r$. Since $e^{\lambda z}/S(\lambda)$ has poles only at the points λ_k (all of which are simple), we have by the Residue Theorem that

$$\frac{1}{2\pi i}\int_{C_r}\frac{e^{\lambda z}}{S(\lambda)}d\lambda = \sum_{\{k:|\lambda_k|\leq\hat{r}\}}\frac{e^{\lambda_k z}}{S'(\lambda_k)}$$

for all $z \in \mathbb{C}$ for all $z \in \mathbb{C}$. Moreover,

$$\begin{aligned}
\left|\int_{C_r}\frac{e^{\lambda z}}{S(\lambda)}d\lambda\right| &\leq \int_0^{2\pi}\frac{|e^{\hat{r}e^{i\theta}z}|}{|S(\hat{r}e^{i\theta})|}\hat{r}|d\theta| \\
&\leq \int_0^{2\pi}\frac{\hat{r}e^{\hat{r}|z|}}{|f((\hat{r}e^{i\theta})^{3p})|}|d\theta| \\
&= \int_0^{2\pi}\frac{\hat{r}e^{\hat{r}|z|}}{|f(\hat{r}^{3p}e^{3ip\theta})|}|d\theta| \\
&\leq \frac{2\pi\hat{r}e^{\hat{r}|z|}}{e^{((\pi/\sqrt{3})-\epsilon)\hat{r}^p}} \\
&\to 0
\end{aligned}$$

as $r \to \infty$ since $p > 1$. Thus, $0 = \lim_{r\to\infty}\sum_{\{k:|\lambda_k|\leq\hat{r}\}}\frac{e^{\lambda_k z}}{S'(\lambda_k)}$. Furthermore, $S'(z) = 3pz^{3p-1}f'(z^{3p})$ where $f'(z) = \sum_{n=1}^{\infty}-\frac{1}{n^3}\prod_{j\neq n}(1-\frac{z}{j^3})$. For each $k \in \mathbb{N}$, $|\lambda_k| = m^{1/p}$ whenever

$3p(m-1) \le k < 3pm$. Hence,

$$\begin{aligned} |S'(\lambda_k)|^{\frac{1}{k}} &\geq \left[\left(3p(m^{1/p})^{3p-1}\right) \left| \frac{-1}{m^3} \prod_{j \neq m} \left(1 - \frac{m^3}{j^3}\right) \right|^{\frac{1}{3pm}} \right] \\ &\geq (3p)^{1/3pm} m^{-1/3pm} \left((1/2)(3/2)^{m/2}\right)^{\frac{1}{3pm}} \\ &\to \left(\frac{3}{2}\right)^{1/6p} \end{aligned}$$

as $m \to \infty$, by Lemma 3.2. Thus, $\limsup_{k\to\infty} 1/|S'(\lambda_k)|^{1/k} \le (2/3)^{1/6p} < 1$, and so $\sum_{k=0}^{\infty} \frac{e^{\lambda_k z}}{S'(\lambda_k)}$ is analytic in $B(0,\epsilon)$, where $\epsilon \equiv [\ln(1/\limsup 1/|S'(\lambda_k)|^{1/k})]/[\sup\{|\lambda_k|/k\}]$. Hence, D fails to admit spectral synthesis on $H(\mathbb{D})$ by Condition *(ix)* of Theorem 1.3. $\square$

The preceding theorem demonstrates that it is not only the rate of growth of the eigenvalues of a diagonal operator which affects the synthesis of the operator, but also their distribution throughout the plane. For instance, the diagonal operator D having eigenvalues $\{\sqrt{n}\}$ admits spectral synthesis on $H(\mathbb{D})$ (and on $H(\mathbb{C})$) by Theorem 1.5. However, the diagonal operator having eigenvalues $\{\sqrt{n}e^{\pi ij/3} : 0 \le j < 6\}$ consisting of six copies of $\{\sqrt{n}\}$ placed on the six rays $\{z \in \mathbb{C} : \arg z = j\}$ for $0 \le j < 6$ fails synthesis by Theorem 3.1.

The following two corollaries generalize Theorem 3.1 to include diagonal operators having as eigenvalues b copies of $\{n^{a/b}\}$ placed on the b rays $\{z \in \mathbb{C} : \arg z = j\}$ for $0 \le j < b$, for certain rational powers a/b smaller than 1.

Corollary 3.1. *The diagonal operator D on $H(\mathbb{D})$ having eigenvalues $\{n^{3/p}e^{2ij\pi/p} : 0 \le j < p\}$ fails synthesis whenever p is an integer greater than or equal to* 3.

Proof. Let $\{\lambda_k\}$ be any enumeration of the set $\{n^{3/p}e^{2ij\pi/p} : 0 \le j < p\}$ for which $\{|\lambda_k|\}$ is non-decreasing. The proof of this corollary is obtained from the proof of Theorem 3.1 applied to the entire functions $f(z) \equiv \prod_{n=1}^{\infty}(1 - \frac{z}{n^3})$ and $S(z) \equiv f(z^p)$ by replacing every occurrence of $3p$ with p. $\square$

Corollary 3.2. *The diagonal operator D on $H(\mathbb{D})$ having eigenvalues $\{n^{a/b}e^{2ij\pi/b} : 0 \le j < b\}$ fails synthesis whenever a and b are integers for which $b > a > 2$.*

Proof. Let $\{\lambda_k\}$ be any enumeration of the set $\{n^{a/b}e^{2ij\pi/b} : 0 \leq j < b\}$ for which $\{|\lambda_k|\}$ is non-decreasing. The proof of this corollary is similar to the proof of Theorem 3.1 applied to the entire functions $f(z) \equiv \prod_{n=1}^{\infty}(1 - \frac{z}{n^a})$ and $S(z) \equiv f(z^b)$. $\square$

3.2 A Generalization

In this section, we generalize Theorem 3.1 in several ways. In Theorem 3.1, the eigenvalues of the diagonal operator were precisely $3p$ copies of $\{n^{1/p}\}$ placed symmetrically on $3p$ rays $\{z \in \mathbb{C} : \arg z = j\}$ for $0 \leq j < 3p$. In the main result of this section, Theorem 3.2, the eigenvalues are only required to grow on the order of $n^{a/b}$ for rational powers a/b less than 1. For example, Theorem 3.2 shows that a diagonal operator on $H(\mathbb{D})$ whose eigenvalues are q copies of $\{n^p + n^{p-1}\}$ placed symmetrically on q rays $\{z \in \mathbb{C} : \arg z = j\}$ for $0 \leq j < q$, where p is a rational number smaller than 1 and q is an integer greater than $1/p$, is non-synthetic, an example which is not addressed by Theorem 3.1.

Although Theorem 3.1 is a consequence of the more general Theorem 3.2, we include both in this dissertation as the proof of Theorem 3.1 is more transparent than the proof of Theorem 3.2. The proofs of both results follow the techniques due to Ermenko outlined in Section 1.4, whereby we define $S(\lambda)$ to be an entire function with zeros only at the points of $\{\lambda_k\}$ (all of which are simple), where $\{\lambda_k\}$ is an enumeration of $\{a_n e^{2\pi ij/q} : 0 \leq j < q\}$. We then apply the Residue Theorem to obtain

$$\lim_{r\to\infty}\int_{C_r}\frac{e^{\lambda z}}{S(\lambda)}d\lambda = \lim_{r\to\infty}\sum_{\{k:\lambda_k\in C_r^o\}}\frac{e^{\lambda_k z}}{S'(\lambda_k)} = \sum_{k=0}^{\infty}\omega_k e^{\lambda_k z}$$

where C_r are appropriately chosen contours. Since the zeros of S consist of q copies of the sequence $\{a_n\}$ placed symmetrically on the q rays $\{\arg z = j\}$ for $0 \leq j < q$, it follows that $S(\lambda) = f(\lambda^q)$, where f is an entire function of order ρ at most $1/2$. Using a result of Levin we obtain the estimate

$$|f(re^{i\theta})| \geq e^{(H(\theta)-\epsilon)r^\rho}$$

for $re^{i\theta} \in \mathbb{C}$ with r sufficiently large and $re^{i\theta}$ not in some exceptional set E, where the indicator function $H(\theta)$ of f satisfies $\inf\{H(\theta) : 0 \le \theta < 2\pi\} \equiv 2\epsilon > 0$. The exceptional set E is obtained from the following two conditions guaranteeing that the points of $\{a_n\}$ are separated. That is, we say $\{a_n\}$ satisfies Condition (C) if there exists a $d > 0$ such that the closed balls $\{\overline{B(a_n, d|a_n|^{1-\rho/2})} : n \in \mathbb{N}\}$ are pairwise disjoint, and $\{a_n\}$ satisfies Condition (C') if $\{|a_n|\}$ is non-decreasing and there exists a $d > 0$ such that $|a_{n+1}| - |a_n| > d|a_n|^{1-\rho}$, where the closed balls $\{\overline{B(a_n, d|a_n|^{1-\rho})} : n \in \mathbb{N}\}$ are pairwise disjoint. For the sequences discussed in Theorem 3.1, Condition (C') is satisfied automatically since the zeros of f are $\{a_n\} = \{n^3\}$. However, in Theorem 3.2, we must include the hypothesis that one of the separation conditions hold. The exceptional set E is then $E \equiv \cup_{n=1}^{\infty} \overline{B(a_n, r_n)}$ where $r_n = d|a_n|^{1-\rho/2}$ (if Condition (C) holds) or $r_n = d|a_n|^{1-\rho}$ (if Condition (C') holds). We then have that

$$\lim_{r\to\infty} \int_{C_r} \frac{e^{\lambda z}}{S(\lambda)} d\lambda = 0$$

for contours C_r not intersecting E.

It is in the final step, that is, showing $\limsup_{k\to\infty} |\omega_k|^{1/k} = \limsup_{k\to\infty} \left|\frac{1}{S'(\lambda_k)}\right|^{1/k} < 1$, where the proofs of Theorems 3.1 and 3.2 differ. In Theorem 3.1, we obtain the estimates on $f'(a_n)$, and hence $S'(\lambda_k)$, by using the inequalities on the infinite products proven in Lemma 3.2. In this way, showing $\limsup_{k\to\infty} |\omega_k|^{1/k} < 1$ is a straightforward computation. In the proof of Theorem 3.2, we use the more abstract (yet effective) approach of invoking the Inverse Function Theorem [15, page 234], which relies on the separation of the zeros $\{a_n\}$ of f guaranteed by Condition (C) or (C') and the estimate $|f(re^{i\theta})| \ge e^{(H(\theta)-\epsilon)r^\rho}$ for $re^{i\theta} \notin E$ and r sufficiently large.

Before proceeding with the main result of this section, we establish the following technical lemma.

Lemma 3.3. *Suppose $\{a_n\}$ is a sequence of complex numbers whose convergence exponent ρ is less than 1. If $\Delta = \lim_{r\to\infty} \frac{n(r)}{r^\rho}$ exists and $\Delta \in (0, \infty)$, then $\lim_{n\to\infty} |a_n|^{1/n} = 1$.*

Proof. Since $\rho < 1$, we have that $\sum_{n=1}^{\infty} 1/|a_n| < \infty$. Hence, $|a_n| > 1$ eventually, and so $\liminf_{n\to\infty} |a_n|^{1/n} \geq 1$. Since $\Delta = \lim_{n\to\infty} (n/|a_n|^{\rho})$, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|(n/|a_n|^{\rho}) - \Delta| < \epsilon$ for all $n \geq N$. Hence $|a_n|^{\rho} < n/(\Delta - \epsilon)$ for $n \geq N$, and so $1 \leq \liminf_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} (n/(\Delta - \epsilon))^{1/(n\rho)} = 1$. The result follows. $\square$

Theorem 3.2. *Let $f(z)$ be an entire function of order $\rho \in (0, \frac{1}{2})$ whose zeros $\{a_n\}$ are all positive real numbers and are all simple. If*

(1) $\{a_n\}$ satisfies either Condition (C) or (C'),

(2) $\Delta = \lim_{r\to\infty} \frac{n(r)}{r^{\rho}}$ exists, where $\Delta \in (0, \infty)$,

and q is any integer greater than $1/\rho$, then the diagonal operator D on $H(\mathbb{D})$ with eigenvalues $\{a_n^{1/q} e^{2\pi ij/q} : 0 \leq j < q\}$ fails to admit spectral synthesis.

Proof. Let $\{\lambda_k\}$ be any enumeration of $\{a_n^{1/q} e^{2\pi ij/q} : 0 \leq j < q\}$ for which $\{|\lambda_k|\}$ is non-decreasing. By Theorem 5 of Levin [29, page 96], $\log|f(re^{i\theta})| \approx \frac{\pi r^{\rho}\Delta}{\sin\rho\pi} \cos\rho(\theta - \pi) + o(r^{\rho})$ outside of an exceptional set $E = \cup_{n=1}^{\infty} \overline{B(a_n, r_n)}$, where $r_n = d|a_n|^{1-(\rho/2)}$ (if $\{a_n\}$ satisfies Condition (C)) or $r_n = d|a_n|^{1-\rho}$ (if $\{a_n\}$ satisfies Condition (C')). Since $\rho < \frac{1}{2}$, $\cos\rho(\theta - 2\pi) > 0$ for $0 \leq \theta < 2\pi$, and so the indicator function $H(\theta) = \frac{\pi\Delta}{\sin\pi\rho} \cos\rho(\theta - 2\pi)$ is such that $\inf\{\frac{\pi\Delta}{\sin\pi\rho} \cos\rho(\theta - 2\pi) : 0 \leq \theta < 2\pi\} = 2\epsilon > 0$. Hence, there exists an $R_\epsilon > 0$ such that $|f(re^{i\theta})| \geq e^{\epsilon r^{\rho}}$ whenever $r \geq R_\epsilon$ and $re^{i\theta} \notin E$. The entire function $S(z) \equiv f(z^q)$ has zeros only at the points $\{\lambda_k\}$ (all of which are simple). For each positive integer r we define $C_r = \{z \in \mathbb{C} : |z| = \hat{r}\}$ to be the circle of radius $\hat{r} \equiv (a_r^{1/q} + a_{r+1}^{1/q})/2$, whenever $(\hat{r}e^{i\theta})^q \notin E$. Since $e^{\lambda z}/S(\lambda)$ has poles only at the points $\{\lambda_k\}$ (all of which are simple), we have by the Residue Theorem that

$$\frac{1}{2\pi i} \int_{C_r} \frac{e^{\lambda z}}{S(\lambda)} d\lambda = \sum_{\{k:|\lambda_k| \leq \hat{r}\}} \frac{e^{\lambda_k z}}{S'(\lambda_k)}.$$

Moreover, for $r \geq R_\epsilon$ and $z \in \mathbb{C}$ fixed,

$$\begin{aligned}
\left|\int_{C_r} \frac{e^{\lambda z}}{S(\lambda)} d\lambda\right| &\leq \int_{C_r} \frac{e^{|\lambda||z|}}{|S(\lambda)|}|d\lambda| \\
&= \int_0^{2\pi} \frac{e^{\hat{r}|z|}}{|f((\hat{r}e^{i\theta})^q)|}\hat{r}d\theta \\
&\leq \frac{2\pi\hat{r}e^{\hat{r}|z|}}{|f(\hat{r}^q e^{iq\theta})|}, 0 \leq q\theta \leq 2q\pi \\
&= \frac{2\pi\hat{r}e^{\hat{r}|z|}}{|f(\hat{r}^q e^{i\zeta})|}, 0 \leq \zeta \leq 2\pi \\
&\leq \frac{2\pi\hat{r}e^{\hat{r}|z|}}{e^{(H(\zeta)-\epsilon)\hat{r}^{q\rho}}} \\
&\leq \frac{2\pi\hat{r}e^{\hat{r}|z|}}{e^{\epsilon\hat{r}^{q\rho}}} \\
&\to 0
\end{aligned}$$

as $r \to \infty$ since $q\rho > 1$. Thus, $0 = \lim_{r\to\infty} \sum_{\{k:|\lambda_k|\leq\hat{r}\}} \frac{e^{\lambda_k z}}{S'(\lambda_k)}$.

It remains to show $\limsup_{k\to\infty} 1/|S'(\lambda_k)|^{1/k} < 1$. To this end, observe for $k \in \mathbb{N}$, $\lambda_k = a_n^{1/q}e^{2\pi ij/q}$ for some $n \in \mathbb{N}$ and $0 \leq j < q$. In fact, $|\lambda_k| = a_n^{1/q}$ whenever $(n-1)q \leq k < nq$. Since $S'(z) = qz^{q-1}f'(z^q)$, we have

$$S'(\lambda_k) = qa_n^{(q-1)/q}e^{2\pi ij(q-1)/q}f'(a_n).$$

Since the closed balls $\{\overline{B(a_n, r_n)} : n \in \mathbb{N}\}$ are pairwise disjoint, there exists radii $\hat{r}_n > r_n$ for which the open balls $\{B(a_n, \hat{r}_n)\}$ are pairwise disjoint. Thus, $f(a_n) = 0$, $f'(a_n) \neq 0$, and $f(z) \neq 0$ for any $z \in \mathbb{C}$ such that $0 < |z - a_n| \leq \hat{r}_n$. For $z = re^{i\theta} \in \partial B(a_n, \hat{r}_n)$ with $r \geq R_\epsilon$, we have that

$$|f(re^{i\theta})| \geq e^{(H(\theta)-\epsilon)r^\rho} \geq e^{\epsilon(|a_n|-\hat{r}_n)^\rho}.$$

If $\alpha_n \equiv e^{\epsilon(|a_n|-\hat{r}_n)^\rho}$, then, by the Inverse Function Theorem [15, page 234], we have that for each $\omega \in B(0, \alpha_n)$ there exists a unique $z \in B(a_n, \hat{r}_n)$ such that $f(z) = \omega$. Thus, the restriction $f|_{f^{-1}(B(0,\alpha_n))} : f^{-1}(B(0, \alpha_n)) \to B(0, \alpha_n)$ of f to the pullback $f^{-1}((B(0, \alpha_n))$ of

$B(0,\alpha_n)$ under f, is bijective, and so f^{-1} exists and is analytic on $f^{-1}(B(0,\alpha_n))$ The function $g(z) = (1/\hat{r}_n)(f^{-1}(\alpha_n z) - a_n) : B(0,1) \to B(0,1)$ is analytic with $g(0) = 0$. By Schwarz' Lemma,

$$1 \geq |g'(0)| = \left| \frac{\alpha_n}{\hat{r}_n f'(f^{-1}(0))} \right| = \frac{\alpha_n}{\hat{r}_n |f'(a_n)|},$$

and so

$$\frac{1}{|f'(a_n)|} \leq \frac{\hat{r}_n}{\alpha_n} = \frac{\hat{r}_n}{e^{\epsilon(|a_n|-\hat{r}_n)^\rho}}.$$

Hence,

$$\begin{aligned}
\limsup_{k\to\infty} \frac{1}{|S'(\lambda_k)|^{\frac{1}{k}}} &\leq \limsup_{n\to\infty} \frac{1}{|q a_n^{(q-1)/q} e^{2\pi i j(q-1)/q} f'(a_n)|^{\frac{1}{nq}}} \\
&\leq \limsup_{n\to\infty} \frac{1}{q^{1/nq} a_n^{(q-1)/nq^2}} \left(\frac{\hat{r}_n}{e^{\epsilon(|a_n|-\hat{r}_n)^\rho}} \right)^{\frac{1}{nq}} \\
&= \limsup_{n\to\infty} \frac{1}{e^{\frac{\epsilon}{nq}(|a_n|-\hat{r}_n)^\rho}},
\end{aligned}$$

where $\lim_{n\to\infty} (|a_n|^\rho/n) = (1/\Delta) > 0$, and $\lim_{n\to\infty} |\hat{r}_n|^{1/n} = 1$ (since $\lim_{n\to\infty} |a_n|^{1/n} = 1$ by Lemma 3.3). Thus, $\limsup_{k\to\infty} 1/|S'(\lambda_k)| < 1$, and so $\sum_{k=0}^{\infty} \frac{e^{\lambda_k z}}{S'(\lambda_k)}$ is analytic on $B(0,\epsilon)$ where $\epsilon \equiv [\ln(1/\limsup(1/|S'(\lambda_k)|^{1/k}))]/[\sup\{|\lambda_k|/k\}]$. Therefore, the diagonal operator D acting on $H(\mathbb{D})$ with eigenvalues $\{\lambda_k\}$ fails to admit spectral synthesis by Condition *(ix)* of Theorem 1.3. $\square$

In the proof of Theorem 3.2, we require that the indicator function $H(\theta)$ satisfies that $\inf\{H(\theta) : 0 \leq \theta < 2\pi\} > 0$, to conclude (as discussed before the proof of Theorem 3.2) $\lim_{r\to\infty} \int_{C_r} (e^{\lambda z}/S(\lambda))d\lambda = 0$ and $\limsup_{k\to\infty} 1/|S'(\lambda_k)|^{1/k} < 1$. This property of $H(\theta) = (\pi\Delta/\sin(\pi\rho))\cos\rho(\theta - \pi)$ is guaranteed since the zeros $\{a_n\}$ of f are positive, real numbers and the order ρ of f is strictly less than $1/2$. However, if the zeros of an entire function lie on a finite number of rays ψ_k, with densities Δ_k, then

$$H(\theta) \equiv \frac{\pi}{\sin \pi\rho} \sum_k \Delta_k \cos\rho(\theta - \psi_k - \pi)$$

[30, page 97]. In view of which, the hypothesis $\{a_n\} \subset \mathbb{R}^+$ of Theorem 3.2 can be weakened. For example, if the zeros $\{a_n\}$ of the entire function f lie on the negative real axis and f has order $\rho < 1/4$, then by an identical proof the diagonal operator D on $H(\mathbb{D})$ having as eigenvalues $\{a_n^{1/q} e^{2\pi i j/q} : 0 \leq j < q\}$ fails to admit spectral synthesis, where q is any integer greater than $1/\rho$. Moreover, if the $\{a_n\}$ lie on any finite number of rays and the condition on the order ρ of f is adjusted to guarantee $\inf\{H(\theta) : 0 \leq \theta < 2\pi\} > 0$, then the corresponding diagonal operator will be non-synthetic. As an example, if f is an entire function with zeros only at the points $\{a_n\} \equiv \{\pm n^5\}$ (all of which are simple), then f has order $\rho = 1/5$, and so, $H(\theta) = (\pi/\sin(\pi/5))(\cos(\theta - \pi)/5 + \cos(\theta - 2\pi)/5) > 0$ for all $0 \leq \theta < 2\pi$. Hence, the diagonal operator D on $H(\mathbb{D})$ having as eigenvalues $\{\pm n^{5/q} e^{2\pi i j/q} : 0 \leq j < q\}$ fails to admit spectral synthesis for any integer $q > 5$.

The technique used in Theorem 3.2 cannot be invoked to establish an analogous result on $H(\mathbb{C})$, as Condition *(2)* forces $\limsup_{k\to\infty} 1/|S'(\lambda_k)|^{1/k} > 0$ while to obtain non-synthesis on $H(\mathbb{C})$ we need $\limsup_{k\to\infty} 1/|S'(\lambda_k)|^{1/k} = 0$ (Condition *(vii)* of Theorem 1.2).

CHAPTER 4

Preserving Non-synthesis while Modifying the Eigenvalues

In this chapter we use the techniques of Chapters 2 and 3 to determine conditions under which adding, rearranging, deleting, or perturbing the eigenvalues of a non-synthetic diagonal operator produce another non-synthetic diagonal operator on $H(\mathbb{D})$.

Let D be a non-synthetic diagonal operator on $H(\mathbb{D})$ with eigenvalues $\{\lambda_n\}$ and suppose we modify the $\{\lambda_n\}$ to obtain a new sequence $\{\hat{\lambda}_n\}$. To determine if this modification produces a set of points which are the eigenvalues of another diagonal operator which is non-synthetic, we must first verify that the $\{\hat{\lambda}_n\}$ are the eigenvalues of some diagonal operator on $H(\mathbb{D})$, and if so, that the operator satisfies one of the conditions of Theorem 1.3.

Since D is continuous, we have that $\limsup_{n\to\infty} |\lambda_n|^{1/n} \leq 1$. Depending on the modification used to obtain the values $\{\hat{\lambda}_n\}$, it may or may not be the case that the set of points $\{\hat{\lambda}_n\}$ are the eigenvalues of a continuous linear map $\hat{D}$ sending z^n to $\hat{\lambda}_n z^n$. The following example demonstrates that rearranging the eigenvalues of a diagonal operator will not necessarily produce another diagonal operator.

Example 4.1. *Continuity Not Preserved*

Let $\{\lambda_n\} = \{n\}$ and let $\{\hat{\lambda}_n\}$ be the rearrangement of $\{\lambda_n\}$ such that $\{\hat{\lambda}_n\} = \{1, 10^2, 2, 10^4, 3, 10^6, ...\}$.

Then, $\limsup_{n\to\infty} |\hat{\lambda}_n|^{1/n} = \limsup_{n\to\infty} (10^{2n})^{1/n} = 100 > 1$, and thus, there does not exist a diagonal operator with eigenvalues $\{\hat{\lambda}_n\}$.

However, there are conditions for modifying the eigenvalues $\{\lambda_n\}$ of a diagonal operator which guarantees the existence of a diagonal operator with eigenvalues $\{\hat{\lambda}_n\}$. It is easy to see that if we delete a subsequence $\{\lambda_{n_k}\}$ from $\{\lambda_n\}$ such that $\{n_k/k : k \geq 1\}$ is bounded, then there exists a diagonal operator $\hat{D}$ having eigenvalues $\{\lambda_{n_k}\}$. In addition, if we add at most a finite number of eigenvalues in between each pair of elements from $\{\lambda_n\}$ or rearrange the eigenvalues within finite blocks only, we obtain modifications which yield the eigenvalues of a continuous operator.

Even if the modification $\{\hat{\lambda}_n\}$ of $\{\lambda_n\}$ yields a diagonal operator $\hat{D}$, it may or may not be the case that $\hat{D}$ is non-synthetic. By Condition *(v)* of Theorem 1.3, D is non-synthetic implies there exists a sequence $\{\omega_n\}$ of complex numbers, not identically zero, such that $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$ and $\sum_{n=0}^{\infty} \omega_n \lambda_n^k \equiv 0$ for all $k \geq 0$. However, it may not be the case that we can find a sequence $\{\hat{\omega}_n\}$ corresponding to $\{\hat{\lambda}_n\}$ which satisfies the necessary decay rate or the property $\sum_{n=0}^{\infty} \hat{\omega}_n \hat{\lambda}_n^k = 0$, both needed to conclude $\hat{D}$ is non-synthetic. The following example demonstrates that adding eigenvalues does not necessarily produce a non-synthetic diagonal operator, as no such $\{\hat{\omega}_n\}$ can exist.

Example 4.2. *Non-synthesis Not Preserved*
Let $\{\lambda_n\} = \{n^3\}$ and $\{\hat{\lambda}_n\} = \{n\}$, that is, to the set of eigenvalues $\{n^3\}$ we are adding in the remaining integers. The diagonal operator D having eigenvalues $\{\lambda_n\}$ is non-synthetic ([16]), however, although the diagonal operator $\hat{D}$ having eigenvalues $\{\hat{\lambda}_n\}$ exists, it is synthetic by Theorem 1.5.

In Section 4.1, we address the issue of adding countably many points to the eigenvalues $\{\lambda_n\}$ of a non-synthetic diagonal operator D on $H(\mathbb{D})$. Intuitively, it would seem this would always produce another non-synthetic diagonal operator, however, Example 4.2 proves this is not the case. In Section 4.2, we discuss rearrangements of the eigenvalues which would also seem to always preserve non-synthesis, but Example 4.1 demonstrates the existence of a diagonal operator is not even guaranteed. In both of these sections we discuss simple conditions on the way in which eigenvalues are added or rearranged which do preserve non-synthesis. In Sections 4.3 through 4.5, we address the issue of deleting eigenvalues. In particular, in Section 4.3, we show that any finite collection of eigenvalues can be deleted and the non-synthesis of the operator preserved. In Sections 4.4 and 4.5, we demonstrate that we can delete countable collections of eigenvalues from the sequences of eigenvalues $\{\lambda_n\}$ defined in Chapters 2 and 3 to obtain other non-synthetic operators. In Section 4.6, we discuss conditions on perturbations of eigenvalues which produce other non-synthetic operators. The discussions in Sections 4.1 through 4.3 are very simple arguments for general operators, while the discussions in Sections 4.4 through 4.6 are very technical in nature and follow similar techniques to those used in Chapters 2 and 3.

4.1 Adding Countably Many Eigenvalues

In this section, we discuss whether the addition of countably many elements to the eigenvalues of a non-synthetic diagonal operator will produce another non-synthetic diagonal operator. In this regard, suppose $\{\lambda_n\}$ are the eigenvalues of a non-synthetic diagonal operator D acting on $H(\mathbb{D})$. By Theorem 1.3, there exists a sequence $\{\omega_n\}$ of complex numbers, not identically zero, such that $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$ and $\sum_{n=0}^{\infty} \omega_n \lambda_n^m \equiv 0$ for all $m \geq 0$. Suppose $\{\hat{\lambda}_k\} = \{\lambda_n\} \cup \{\gamma_n\}$, and define

$$\hat{\omega}_k \equiv \begin{cases} \omega_n & \hat{\lambda}_k \in \{\lambda_n\} \\ 0 & \hat{\lambda}_k \in \{\gamma_n\}. \end{cases}$$

Then, $\sum_{k=0}^{\infty} \hat{\omega}_k \hat{\lambda}_k^m \equiv 0$ for all $m \geq 0$, and so, it appears we should obtain another non-synthetic operator. However, Example 4.2 shows that it need not be the case that $\limsup_{k\to\infty} |\hat{\omega}_k|^{1/k} < 1$.

The following theorem demonstrates that if finitely many eigenvalues are added in between each pair of eigenvalues of a non-synthetic diagonal operator, then the new diagonal operator also fails spectral synthesis.

Theorem 4.1. *Let D be a diagonal operator acting on $H(\mathbb{D})$ having distinct eigenvalues $\{\lambda_n\}$, and suppose $\hat{D}$ is a non-synthetic diagonal operator on $H(\mathbb{D})$ having as eigenvalues $\{\lambda_{n_k}\}$ where $\{n_k\}$ is a subsequence such that $\{n_k/k : k \geq 1\}$ is bounded. Then, D also fails synthesis.*

Proof. Since $\hat{D}$ is non-synthetic, there exists a sequence $\{\hat{\omega}_{n_k}\}$ such that $\limsup_{k\to\infty} |\hat{\omega}_{n_k}|^{1/k} < 1$ and $\sum_{k=0}^{\infty} \hat{\omega}_{n_k} \lambda_{n_k}^m \equiv 0$ for all $m \geq 0$, by Condition *(v)* of Theorem 1.3. Define for $n \geq 0$,

$$\omega_n \equiv \begin{cases} \hat{\omega}_{n_k} & n \in \{n_k\} \\ 0 & n \notin \{n_k\}. \end{cases}$$

Then, $\sum_{n=0}^{\infty} \omega_n \lambda_n^m = \sum_{k=0}^{\infty} \hat{\omega}_{n_k} \lambda_{n_k}^m \equiv 0$ for all $m \geq 0$. Moreover, $\limsup_{n\to\infty} |\omega_n|^{1/n} = \limsup_{k\to\infty} |\hat{\omega}_{n_k}|^{1/n_k} \leq \limsup_{k\to\infty} |\hat{\omega}_{n_k}|^{1/Mk} < 1$, where $M > 0$ is such that $n_k < Mk$ for all $k \geq 1$. Hence, D fails to admit spectral synthesis on $H(\mathbb{D})$. □

In [16] it is shown that if $\{\gamma_n\}$ is a sequence of distinct complex numbers with $|\gamma_n| \to \infty$, $\limsup_{n\to\infty} |\gamma_n|^{1/n} \leq 1$, and $|\gamma_n|/n^p$ increasing to infinity for some $p > 2$, then the diagonal operator with eigenvalues $\{\gamma_n\}$ fails spectral synthesis on $H(\mathbb{D})$. This result combined with the preceding theorem gives the following result.

Corollary 4.1. *Let $\{\lambda_n\}$ be a sequence of distinct complex numbers with $|\lambda_n| \to \infty$, $\limsup_{n\to\infty} |\lambda_n|^{1/n} \leq 1$, and having a subsequence $\{\lambda_{n_k}\}$ such that $|\lambda_{n_k}|/k^p$ increases to infinity for some $p > 2$ where $\{n_k/k : k \geq 1\}$ is bounded, then the diagonal operator with eigenvalues $\{\lambda_n\}$ is non-synthetic on $H(\mathbb{D})$.*

4.2 Rearranging Eigenvalues

In this section, we discuss rearrangements of the eigenvalues; that is, if D is a diagonal operator on $H(\mathbb{D})$ with eigenvalues $\{\lambda_n\}$, we consider $\{\lambda_{i(n)}\}$ where $\{i(n)\}$ is a rearrangement of $\{n\}$. If D is non-synthetic, then there exists a sequence $\{\omega_n\} \subset \mathbb{C}$, not identically zero, such that $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$ and $\sum_{n=0}^{\infty} \omega_n \lambda_n^k \equiv 0$ for all $k \geq 0$. Thus, $\sum_{n=0}^{\infty} \omega_{i(n)} \lambda_{i(n)}^k \equiv 0$ for all $k \geq 0$. However, as demonstrated in Example 4.1, it may not even be the case that a diagonal operator with eigenvalues $\{\lambda_{i(n)}\}$ exists.

Moreover, the following example demonstrates that the rearrangement of the eigenvalues of a synthetic diagonal operator can yield a non-synthetic diagonal operator.

Example 4.3. The diagonal operator D on $H(\mathbb{D})$ with eigenvalues $\{\lambda_n\} \equiv \{n\}$ admits spectral synthesis by Theorem 1.5. Consider the rearrangement of $\{n\}$, where $\{\lambda_{i(n)}\} = \{0, 1^3, 2, 2^3, 3, 3^3, 4, 4^3, 5, ...\}$; that is, each integer of the form m^3 is moved to the odd positions of $\{\lambda_{i(n)}\}$, and the even positions are the remaining integer values listed in increasing order. If we consider the subsequence $\{\lambda_{n_k}\} \equiv \{k^3\}$ where $\{n_k\} = \{2k-1\}$, by Corollary 4.1, we have that the diagonal operator $\hat{D}$ with eigenvalues $\{\lambda_{i(n)}\}$ fails spectral synthesis on $H(\mathbb{D})$.

However, if the eigenvalues are rearranged within finite blocks, then non-synthesis will be preserved, as the following theorem demonstrates. In addition, a nearly identical argument would show that synthesis is preserved under the same restrictions on the rearrangement.

Theorem 4.2. *Let D be a non-synthetic diagonal operator on $H(\mathbb{D})$ with eigenvalues $\{\lambda_n\}$. Let $i(n)$ be such that $n - c \leq i(n) \leq n + c$ for some constant $c > 0$ and all $n \in \mathbb{N}$. Then, the diagonal operator $\hat{D}$ with eigenvalues $\{\lambda_{i(n)}\}$ fails spectral synthesis.*

Proof. Since D is continuous, $\limsup_{n\to\infty} |\lambda_n|^{1/n} = M \leq 1$. Thus, for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $|\lambda_n|^{1/n} < M + \epsilon$. Moreover, for all n, $\lambda_{i(n)} = \lambda_{n\pm a}$ where $0 \leq a \leq c$. Thus, $\limsup_{n\to\infty} |\lambda_{i(n)}|^{1/n} < \limsup_{n\to\infty} (M+\epsilon)^{(n\pm a)/n} = M + \epsilon$, and therefore, $\hat{D}$ is continuous. An identical argument gives $\limsup_{n\to\infty} |\omega_{i(n)}|^{1/n} < 1$. Moreover, $\sum_{n=0}^{\infty} \omega_{i(n)} \lambda_{i(n)}^k \equiv 0$ for all $k \geq 0$, and so, $\hat{D}$ is non-synthetic. $\square$

In Chapter 2, it is shown that the diagonal operator D having as eigenvalues $\{\lambda_n\}$, an enumeration of the integer lattice points $\mathbb{Z} \times i\mathbb{Z}$, fails spectral synthesis. We chose the enumeration to begin on the positive real axis and traversed counterclockwise around larger and larger squares. However, as a consequence of the preceding theorem, we could rearrange the eigenvalues in any order along those squares and obtain the same result.

In Chapter 3, it is shown that the diagonal operator having eigenvalues $\{n^{1/p}e^{2\pi ij/3p} : 0 \leq j < 3p\}$ fails to admit spectral synthesis. In Theorem 3.1, we chose the enumeration of the eigenvalues to begin on the positive real axis and traversed counterclockwise around circles of increasing modulus. However, by Theorem 4.2, we could rearrange the eigenvalues on any circle, or even on every c circles, and preserve non-synthesis.

4.3 Deleting Finitely Many Eigenvalues

In this section, we show that finitely many of the eigenvalues of a non-synthetic diagonal operator acting on $H(\mathbb{D})$ can be deleted without affecting the non-synthesis of the operator.

Proposition 4.1. *Let D be a diagonal operator on $H(\mathbb{D})$ with distinct eigenvalues $\{\lambda_n\}$ that fails to admit spectral synthesis. Assume $\{\lambda_n/n : n \geq 1\}$ is bounded. If D' is the diagonal operator with eigenvalues $\{\{\lambda_n\} \setminus \{\lambda_0\}\}$, then D' fails to admit spectral synthesis on $H(\mathbb{D})$.*

Proof. Since D fails to admit synthesis, by Condition *(ix)* of Theorem 1.3, there exists a sequence $\{\omega_n\}$ of complex numbers, not identically zero, such that $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$ and $\sum_{n=0}^{\infty} \omega_n e^{\lambda_n z} = 0$ for all z near the origin. Thus, $-\omega_0 e^{\lambda_0 z} = \sum_{n=1}^{\infty} \omega_n e^{\lambda_n z}$, and so, $-\omega_0 = \sum_{n=1}^{\infty} \omega_n e^{(\lambda_n - \lambda_0)z}$. Differentiating both sides with respect to z gives $0 = \sum_{n=1}^{\infty} \omega_n(\lambda_n - \lambda_0)e^{(\lambda_n-\lambda_0)z}$. Moreover, $\limsup_{n\to\infty} |\lambda_n - \lambda_0|^{1/n} \leq 1$ since the operator $D - \lambda_0 I$ is continuous. Thus,

$$\limsup_{n\to\infty} |\omega_n(\lambda_n - \lambda_0)|^{1/n} \leq \limsup_{n\to\infty} |\omega_n|^{1/n} \limsup_{n\to\infty} |\lambda_n - \lambda_0|^{1/n} < 1.$$

Hence, the operator $D - \lambda_0 I$ having eigenvalues $\{\lambda_n - \lambda_0\}_{n=1}^{\infty}$ is a non-synthetic diagonal operator, and therefore, D' is also non-synthetic [11, Lemma 2]. $\square$

By repeating this process, finitely many eigenvalues can be removed and non-synthesis is preserved. A nearly identical proof would allow for deleting finitely many eigenvalues from a diagonal operator acting on $H(\mathbb{C})$. The preceding proposition only addresses the case when $\{\lambda_n/n : n \geq 1\}$ is bounded, which is sufficient for our results, however, in [11, Proposition 3] Deters proves the result for the general case using Condition *(v)* of Theorem 1.3. As a corollary of Proposition 4.1, we observe that removing finitely many eigenvalues from a synthetic diagonal operator produces another synthetic diagonal operator.

In the next two sections, we consider deleting countable collections of eigenvalues, and show that under certain conditions, but not always, non-synthesis is preserved.

4.4 Deleting Countably Many Eigenvalues Symmetrically

The following example demonstrates that, unlike finite collections, deleting countable collections of eigenvalues from a diagonal operator need not preserve non-synthesis.

Example 4.4. *Deleting Eigenvalues Does Not Preserve Non-synthesis*
The diagonal operator D acting on $H(\mathbb{D})$ with eigenvalues $\{n^{1/p}e^{2\pi ij/3p} : 0 \leq j < 3p\}$, where p is any integer at least 2, fails to admit spectral synthesis by Theorem 3.1. However, the diagonal operator $\hat{D}$ on $H(\mathbb{D})$ with eigenvalues $\{n^{1/p}\}$ admits spectral synthesis by Theorem 1.5.

The results of this section provide conditions under which deleting countably many eigenvalues from a non-synthetic diagonal operator produces another non-synthetic diagonal operator. As previously mentioned, we must first guarantee the existence of a diagonal operator having eigenvalues $\{\hat{\lambda}_n\}$. Recall from Section 1.3 that a linear map D on $H(\mathbb{D})$ such that $D(z^n) = \lambda_n z^n$ is continuous if and only if $\limsup_{n\to\infty} |\lambda_n|^{1/n} \leq 1$. The following proposition gives a sufficient (but not necessary) condition to preserve continuity when eigenvalues are deleted.

Proposition 4.2. *Let* $\{\lambda_n\}$ *be such that* $\limsup_{n\to\infty} |\lambda_n|^{1/n} \leq 1$*. Let* $\{\lambda_{n_k}\}$ *be a subsequence such that* $\{n_k/k : k \geq 1\}$ *is bounded. Then,* $\limsup_{k\to\infty} |\lambda_{n_k}|^{1/k} \leq 1$*.*

Proof. Let $M > 0$ be such that $n_k/k < M$ for all $k \geq 1$. Then, $\limsup_{k\to\infty} |\lambda_{n_k}|^{1/k} = \limsup_{k\to\infty} \left(|\lambda_{n_k}|^{1/n_k}\right)^{n_k/k} \leq \limsup_{k\to\infty} \max\{(|\lambda_{n_k}|^{1/n_k})^M, 1\} = 1$. □

The preceding result shows that a diagonal operator exists, however, Example 4.4 shows that this requirement on the deleted sequence is not enough to preserve the non-synthesis of the diagonal operator.

In Chapter 2, we proved that the diagonal operator D on $H(\mathbb{D})$ having as eigenvalues an enumeration $\{\lambda_n\}$ of the integer lattice points $\mathbb{Z} \times i\mathbb{Z}$ fails to admit spectral synthesis. In our first result regarding deleting countably many eigenvalues, we delete countably many lattice points to produce another non-synthetic diagonal operator $\hat{D}$ on $H(\mathbb{D})$. To do so, we consider a subsequence $\{\lambda_{n_k}\}$ such that

1. $0 \leq \inf\{\alpha : \sum_{k=1}^{\infty} 1/|\lambda_{n_k}|^{\alpha} < \infty\} < 2$ and

2. $\lambda_m \in \{\{\lambda_n\} \setminus \{\pm\lambda_{n_k}, \pm i\lambda_{n_k}\}\}$ is on the square S_j having vertices $\pm(j \pm ij)$, if and only if $aj^2 - o(j) \leq m \leq aj^2 + o(j)$ where $a > 0$ and $o(j)$ is a polynomial of degree at most one.

We then delete from $\{\lambda_n\}$ the collection $\{\pm\lambda_{n_k}, \pm i\lambda_{n_k}\}$ and obtain a non-synthetic diagonal operator $\hat{D}$ on $H(\mathbb{D})$ with eigenvalues $\{\{\lambda_n\} \setminus \{\pm\lambda_{n_k}, \pm i\lambda_{n_k}\}\}$.

To show D is non-synthetic, we obtained in Propositions 2.2 and 2.3, estimates on the entire function $S(z) = z\prod_{n=1}^{\infty}(1 - (z/\lambda_n))e^{z/\lambda_n + z^2/2\lambda_n^2}$ using that the exponential terms cancel as the eigenvalues appear in groups of four: the eigenvalue, its negative, its conjugate, and its conjugate's negative. In fact, we showed that S has order 2 and type $\pi/2$. To show $\hat{D}$ is non-synthetic, we obtain similar estimates on $\hat{S}(z) = \frac{S(z)}{f(z)}$, where $f(z) =$

$\prod_{k=1}^{\infty}\left(1-\frac{z}{\lambda_{n_k}}\right)e^{\frac{z}{\lambda_{k_n}}+\frac{z^2}{2\lambda_{k_n}^2}}\left(1+\frac{z}{\lambda_{n_k}}\right)e^{-\frac{z}{\lambda_{k_n}}+\frac{z^2}{2\lambda_{k_n}^2}}\left(1-\frac{z}{i\lambda_{n_k}}\right)e^{\frac{z}{i\lambda_{k_n}}-\frac{z^2}{2\lambda_{k_n}^2}}\left(1+\frac{z}{i\lambda_{n_k}}\right)e^{-\frac{z}{i\lambda_{k_n}}-\frac{z^2}{2\lambda_{k_n}^2}}$, since the exponential terms will cancel and Condition (1) forces $\hat{S}$ to also have order 2. To show D is non-synthetic, we defined $\omega_n = 1/S'(\lambda_n)$ for all $n \geq 0$ and proved $\{\omega_n\}$ satisfies Condition *(ix)* of Theorem 1.3. In Proposition 2.5, we proved that $\limsup_{n\to\infty}|\omega_n|^{1/n} < 1$, in part by using the fact that λ_n is on the square S_j if and only if $j \leq |\lambda_n| \leq \sqrt{2}j$ and if and only if $(2j-1)^2 \leq n \leq 4(j^2+j)$. Condition (2) allows for a similar argument to show $\hat{\omega}_n = 1/\hat{S}'(\lambda_n)$, for n such that $\lambda_n \notin \{\pm\lambda_{n_k}, \pm i\lambda_{n_k}\}$, satisfies Condition *(ix)* of Theorem 1.3, giving $\hat{D}$ is non-synthetic. We now present this result and give a more detailed outline of its proof.

Theorem 4.3. *Let D be the non-synthetic diagonal operator on $H(\mathbb{D})$ with eigenvalues $\{\lambda_n\}$, an enumeration of $\mathbb{Z}\times i\mathbb{Z}$. Let $\{n_k\}$ be a subsequence such that $0 \leq \inf\{\alpha : \sum_{k=1}^{\infty} 1/|\lambda_{n_k}|^{\alpha} < \infty\} = \rho < 2$. Define $\{\hat{\lambda}_p\}$ to be the enumeration of $\{\{\lambda_n\}\setminus\{\pm\lambda_{n_k}, \pm i\lambda_{n_k}\}\}$ defined by beginning on the positive real axis and traversing counterclockwise around larger and larger squares S_j. If $\hat{\lambda}_p$ is on S_j if and only if $aj^2 - o(j) \leq p < aj^2 + o(j)$ where $a > 0$ and $o(j)$ is a polynomial of degree at most one, then the diagonal operator $\hat{D}$ acting on $H(\mathbb{D})$ with eigenvalues $\{\hat{\lambda}_p\}$ fails to admit spectral synthesis.*

The proof of Theorem 4.3 follows the same technique as the proof that D is non-synthetic, given in Chapter 2, hence we provide an outline only. The operator $\hat{D}$ is continuous since $\hat{\lambda}_p$ is on the square S_j if and only if $j = |j| \leq |\hat{\lambda}_p| \leq |j+ij| = \sqrt{2}j$ and if and only if $aj^2 - o(j) \leq p < aj^2 + o(j)$, thus

$$\limsup_{p\to\infty}|\hat{\lambda}_p|^{1/p} \leq \limsup_{j\to\infty}(\sqrt{2}j)^{1/(aj^2-o(j))} = 1.$$

The entire function $S(z) = z\prod_{m,n}'(1-(z/(m+in)))e^{z/(m+in)}e^{z^2/(2(m+in)^2)}$ having simple zeros only at $\{\lambda_n\}$, is such that $|S(z)| \geq \alpha d(z)e^{\pi|z|^2/2}$ for all $z \in \mathbb{C}$, where $\alpha > 0$ and $d(z) = dist(z, \mathbb{Z}\times i\mathbb{Z}) = \inf\{|z-(m+ik)| : m, k \in \mathbb{Z}\}$, by Proposition 2.3. The entire function $f(z) = \prod_{k=1}^{\infty}(1-(z/\lambda_{n_k})^4)$ having simple zeros only at $\{\pm\lambda_{n_k}, \pm i\lambda_{n_k}\}$ is of order

$\rho < 2$ since $\inf\{\alpha : \sum_{k=1}^{\infty} 1/|\lambda_{n_k}|^{\alpha} < \infty\} < 2$. Hence, $|f(z)| \leq \beta e^{\tau|z|^{\rho+\epsilon}}$ for $0 < \epsilon < 2-\rho$, whenever $|z| > R$ for some $R > 0$, and constants β, τ. Thus, the entire function

$$\hat{S}(z) \equiv z \prod_{p=1}^{\infty} \left(1 - \frac{z}{\hat{\lambda}_p}\right) = \frac{S(z)}{f(z)}$$

has simple zeros only at $\{\hat{\lambda}_p\}$, and for $|z| > R$, satisfies

$$|\hat{S}(z)| = \frac{|S(z)|}{|f(z)|} \geq \frac{\alpha d(z) e^{(\pi/2)|z|^2}}{\beta e^{\tau|z|^{\rho+\epsilon}}}.$$

By the Residue Theorem and the previous estimate on $|\hat{S}(\lambda)|$, we have that

$$\sum_{p=1}^{\infty} \frac{e^{\hat{\lambda}_p z}}{\hat{S}'(\hat{\lambda}_p)} = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{C_r} \frac{e^{\lambda z}}{\hat{S}(\lambda)} d\lambda = 0,$$

where C_r are contours not passing through any of the lattice points. Using the Inverse Function Theorem and Schwarz' Lemma, as in Propositions 2.4 and 2.5, we have that

$$\frac{1}{|\hat{S}'(\hat{\lambda}_p)|} \leq \frac{\beta e^{\tau(|\hat{\lambda}_p|+(1/4))^{\rho+\epsilon}}}{\alpha e^{(\pi/2)(|\hat{\lambda}_p|-(1/4))^2}}.$$

Since $\hat{\lambda}_p$ lies on the square S_j if and only if $j \leq |\hat{\lambda}_p| \leq \sqrt{2}j$ and if and only if $aj^2 - o(j) \leq p < aj^2 + o(j)$, we have that

$$\begin{aligned}
\limsup_{p\to\infty} \frac{1}{|\hat{S}'(\hat{\lambda}_p)|^{\frac{1}{p}}} &\leq \limsup_{p\to\infty} \left(\frac{\beta e^{\tau(|\hat{\lambda}_p|+(1/4))^{\rho+\epsilon}}}{\alpha e^{(\pi/2)(|\hat{\lambda}_p|-(1/4))^2}}\right)^{\frac{1}{p}} \\
&\leq \limsup_{p\to\infty} \left(\frac{\beta}{\alpha e^{\pi/32}}\right)^{\frac{1}{p}} \left(\frac{e^{\pi|\hat{\lambda}_p|/4p} e^{\tau|\hat{\lambda}_p|^{\rho+\epsilon}/p}}{e^{\pi|\hat{\lambda}_p|^2/2p}}\right) \\
&\leq \limsup_{p\to\infty} \left(\frac{\beta}{\alpha e^{\pi/32}}\right)^{\frac{1}{p}} \limsup_{j\to\infty} \frac{e^{\sqrt{2}j\pi/4(aj^2-o(j))} e^{\tau(\sqrt{2}j)^{\rho+\epsilon}/(aj^2-o(j))}}{e^{\pi j^2/2(aj^2+o(j))}} \\
&= \frac{1}{e^{\pi/2a}} \\
&< 1,
\end{aligned}$$

and thus, $\hat{D}$ fails spectral synthesis by Condition *(ix)* of Theorem 1.3, where $\omega_p \equiv 1/\hat{S}'(\hat{\lambda}_p)$ for all $p \geq 0$.

By invoking the preceding theorem, from $\mathbb{Z} \times i\mathbb{Z}$ sets of eigenvalues such as $\{\pm p, \pm ip : p \in \mathbb{Z}^+\}$ can be deleted, and the corresponding diagonal operator $\hat{D}$ will fail spectral synthesis on $H(\mathbb{D})$. In the following section, we show it is not necessary to remove the eigenvalues symmetrically (that is, in groups of four) to preserve non-synthesis.

In Theorem 3.2, it is shown that a diagonal operator having eigenvalues $\{a_n^{1/q} e^{2\pi ij/q} : 0 \leq j < q\}$ fails to admit spectral synthesis whenever $\{a_n\}$ satisfies certain properties. As a specific example, if $\{a_n\} = \{n^3\}$ the diagonal operator with eigenvalues $\{n^{3/q} e^{2\pi ij/q} : 0 \leq j < q\}$ fails synthesis, where q is any integer greater than three. The following examples demonstrate that we can delete countably many of the $\{a_n\}$, thus countably many eigenvalues, and produce other non-synthetic diagonal operators. Note that in both examples we delete the eigenvalues in groups lying on circles.

Example 4.5. The diagonal operator $\hat{D}$ with eigenvalues $\{(2n)^{3/q} e^{2\pi ij/q} : 0 \leq j < q\}$ fails synthesis for any integer $q > 3$.

Proof. Observe that $\{a_n\} \equiv \{(2n)^3\}$ satisfies Condition (C) defined in Section 3.2, and $\Delta = \lim_{r\to\infty} n(r)/r^{1/3} = 1/2$, thus $\{a_n\}$ satisfies the hypotheses of Theorem 3.2. Hence, $\hat{D}$ fails spectral synthesis. $\square$

The previous example demonstrates that the roots of the cubes of all odd integers can be deleted and non-synthesis preserved. In this example, the convergence exponent of the deleted sequence is the same as the convergence exponent of the original sequence. The following example demonstrates this need not be the case, however, we must impose additional conditions on the deleted subsequence to guarantee the existence of a continuous operator and the existence of the angular density Δ.

Example 4.6. Let $\{a_n\}$ be the zeros of an entire function f of order ρ as in Theorem 3.2.

Let $\{a_{n_m}\}$ be a subsequence with convergence exponent $\rho_1 < \rho$, if any exist. The function

$$\hat{f}(z) \equiv \frac{\prod_{n=0}^{\infty} (1 - (z/a_n))}{\prod_{m=0}^{\infty} (1 - (z/a_{n_m}))}$$

is an entire function of order ρ with zeros $\{\{a_n\} \backslash \{a_{n_m}\}\}$. Define $\{\lambda_k\}$ to be an enumeration of $\{a_n^{1/q} e^{2\pi ij/q} : 0 \leq j < q\}$ and $\{\hat{\lambda}_p\}$ to be an enumeration of $\{a_{n_m}^{1/q} e^{2\pi ij/q} : 0 \leq j < q\}$, where $q > 1/\rho$ is an integer. Define $\{\gamma_t\}$ to be an enumeration of $\{\{\lambda_k\} \backslash \{\hat{\lambda}_p\}\}$. If $\limsup_{t\to\infty} |\gamma_t|^{1/t} = 1$ and $\lim_{t\to\infty} t/|\gamma_t|^{\rho} > 0$ exists, then the diagonal operator with eigenvalues $\{\gamma_t\}$ fails to admit spectral synthesis by Theorem 3.2.

As a specific example, let f be an entire function with zeros $\{\{n^3\} \cup \{n^4\}\}$. Then, if D is the diagonal operator defined in Theorem 3.2, we can delete all of the eigenvalues associated with the roots of n^4 and preserve non-synthesis. Furthermore, we could delete all of the eigenvalues associated with the roots of n^9 and preserve non-synthesis on $H(\mathbb{D})$.

All of the examples of this section involved deleting eigenvalues in a symmetric manner. In Theorem 4.3, groups of four eigenvalues were deleted (the lattice point, its negative, its conjugate, and its conjugate's negative), and in Examples 4.5 and 4.6, all eigenvalues lying on a circle were deleted. In the next section, we eliminate this restriction for deleting countably many points from $\mathbb{Z} \times i\mathbb{Z}$. An open problem is whether or not we could delete eigenvalues from $\{a_n^{1/q} e^{2\pi ij/q} : 0 \leq j < q\}$ in a non-symmetric manner; that is, not deleting all of the points lying on a given circle. This would aid in addressing the question of whether a minimum number of rays the eigenvalues need to lie on exists.

4.5 Deleting Countably Many Eigenvalues Without Symmetry

In the following theorem, eigenvalues are deleted from the integer lattice $\mathbb{Z} \times i\mathbb{Z}$, and unlike Theorem 4.3, they need not be deleted in groups of four. However, the additional condition

$\sum_{k=1}^{\infty} 1/|\lambda_{n_k}|^2 < \pi/2$ on the growth of the deleted eigenvalues $\{\lambda_{n_k}\}$ is imposed.

Theorem 4.4. *Let D be the non-synthetic diagonal operator on $H(\mathbb{D})$ with eigenvalues $\{\lambda_n\}$, an enumeration of $\mathbb{Z}\times i\mathbb{Z}$. Let $\{n_k\}$ be a subsequence such that $0 \le \inf\{\alpha : \sum_{k=1}^{\infty} 1/|\lambda_{n_k}|^{\alpha} < \infty\} = \rho < 2$ and $\sum_{k=1}^{\infty} 1/|\lambda_{n_k}|^2 < \pi/2$. Define $\{\hat{\lambda}_p\}$ to be the enumeration of $\{\{\lambda_n\} \setminus \{\lambda_{n_k}\}\}$ defined by beginning on the positive real axis and traversing counterclockwise around larger and larger squares S_j. If $\hat{\lambda}_p$ is on S_j if and only if $aj^2 - o(j) \le p < aj^2 + o(j)$ where $a > 0$ and $o(j)$ is a polynomial of degree at most one, then the diagonal operator $\hat{D}$ acting on $H(\mathbb{D})$ with eigenvalues $\{\hat{\lambda}_p\}$ fails to admit spectral synthesis.*

The proof of Theorem 4.4 follows almost identically to the proof of Theorem 4.3, except for the bound on $|f(z)| = |\prod_{k=1}^{\infty}(1-(z/\lambda_{n_k}))e^{(z/\lambda_{n_k})+(z^2/2\lambda_{n_k}^2)}|$. In this case, since the eigenvalues are not deleted in groups of four some of the exponential terms will not cancel, and so, we obtain

$$|f(z)| \le ce^{b|z|^{\rho+\epsilon}}e^{|z|^2\sum_{k=1}^{\infty}1/|\lambda_{n_k}|^2}$$

for some constants b and c, and $|z|$ sufficiently large. Thus, the requirement $\sum_{n=1}^{\infty} 1/|\lambda_{k_n}|^2 < \pi/2$, which was not necessary in Theorem 4.3, allows for the bound

$$|\hat{S}(z)| \ge \frac{\alpha d(z)e^{M|z|^2}}{ce^{b|z|^{\rho+\epsilon}}},$$

where $M = (\pi/2) - \sum_{n=1}^{\infty} 1/|\lambda_{k_n}|^2 > 0$ and $0 < \epsilon < 2-\rho$. The remainder of the argument for $\hat{D}$ to be non-synthetic follows exactly as the outline of the proof of Theorem 4.3 given in Section 4.2.

Theorem 4.4 has the following example as an immediate consequence.

Example 4.7. Let D be the diagonal operator with eigenvalues $\{\lambda_n\}$, where $\{\lambda_n\}$ is the enumeration of $\mathbb{Z} \times i\mathbb{Z}$ defined in Chapter 2. Define $\{\lambda_{n_k}\} \equiv \{k^2\}$. Then, $\rho = 1/2$ and $\sum_{k=1}^{\infty} 1/|\lambda_{n_k}|^2 = \sum_{k=1}^{\infty} 1/k^4 = \pi^4/90 < \pi/2$. Hence, a diagonal operator $\hat{D}$ acting on $H(\mathbb{D})$ with eigenvalues $\{\{m + in : m, n \in \mathbb{Z}\} \setminus \{m^2 : m \in \mathbb{Z}\}\}$ fails to admit spectral synthesis by Theorem 4.4.

4.6 Perturbing Eigenvalues

In this section, we discuss perturbing the eigenvalues of a non-synthetic diagonal operator to obtain another non-synthetic diagonal operator acting on $H(\mathbb{D})$. The first result of this section strengthens Theorem 3.2 as the eigenvalues are only required to lie in finitely many δ-sectors, instead of on finitely many rays.

Theorem 4.5. *Let p be an integer greater than two. Then, there exists a sequence $\{\theta_n\}$ of positive real numbers with $\theta_n < \delta$ for some $\delta > 0$, such that the diagonal operator with eigenvalues $\{n^{1/p}e^{i(\theta_n+2j\pi)/q} : 0 \leq j < q\}$ where $q > p$ is an integer, fails to admit spectral synthesis on $H(\mathbb{D})$.*

As the proof of Theorem 4.5 follows the same technique as the proof of Theorem 3.2 we include an outline only. The entire function $f(z) \equiv \prod_{n=1}^{\infty}(1-(z/n^3))$ has simple zeros $\{n^3\}$ and order $\rho = 1/3$, hence satisfies the hypotheses of Theorem 3.2. Thus, as in its proof, we have that

$$|f(re^{i\theta})| \geq e^{\epsilon r^{1/3}}$$

for r sufficiently large and $re^{i\theta}$ outside of some exceptional set E. The following result of Levin [29] can then be used to obtain a similar bound on another entire function.

Lemma 4.1. *([29, Lemma 1, page 98]) Let us assume that the set $\{a_n\}$ of the zeros of the canonical product*

$$\Pi(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\sum_{k=1}^{p} z^k/ka_n^k}$$

has a density with index $\rho(r)$, i.e., there exists the limit

$$\Delta = \lim_{r\to\infty}\frac{n(r)}{r^{\rho(r)}},$$

and suppose that $\rho = \lim_{r\to\infty}\rho(r)$ is not an integer. Let us denote by $\Pi^{\delta}(z)$ another canonical *product*

$$\Pi^\delta(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{a'_n}\right) e^{\sum_{k=1}^{p} z^k/k(a'_n)^k},$$

in which $|a'_n| = |a_n|$ *and* $|\arg a'_n - \arg a_n| < \delta$. *Then, for every* $\epsilon > 0$ *and* $\eta > 0$ *there exists a* $\delta > 0$ *such that*

$$|\log|\Pi(z)| - \log|\Pi^\delta(z)|| < \epsilon r^{\rho(r)}$$

for all z *that do not belong to some exceptional set of circles* C *with upper linear density less than* η.

Hence, there exists a $\delta > 0$ so that if $\{\lambda_n = r_n e^{i\theta_n}\}$ is a sequence such that $|\lambda_n| = r_n = n^3$, $\theta_n = |\arg\lambda_n - \arg n^3| < \delta$, and $f_\delta(z) = \prod_{n=1}^{\infty}(1-(z/\lambda_n))$, then $|\log|f_\delta(z)| - \log|f(z)|| < \epsilon_1|z|^{1/3}$ outside an exceptional set of circles C. Thus, for $z = re^{i\theta} \notin E \cup C$ and $|z| = r$ sufficiently large, we have the estimate

$$|f_\delta(re^{i\theta})| \geq e^{\log|f(re^{i\theta})| - \epsilon_1 r^{1/3}} \geq e^{\epsilon_2 r^{1/3}}$$

for some $\epsilon_2 > 0$. Applying the Residue Theorem and Inverse Function Theorem to $f_\delta(z)$, exactly as we did to $f(z)$ in the proof of Theorem 3.2, gives the diagonal operator D acting on $H(\mathbb{D})$ having as eigenvalues $\{n^{1/p}e^{i(\theta_n+2j\pi)/q} : 0 \leq j < q\}$ fails to admit spectral synthesis.

The second result we present regarding the perturbation of eigenvalues states that if D is a non-synthetic diagonal operator acting on $H(\mathbb{D})$ having eigenvalues $\{\lambda_n\}$, where the indicator function $H(\theta)$ of f is such that $\inf\{H(\theta) : 0 \leq \theta < 2\pi\} > 0$, then the diagonal operator having as eigenvalues $\{\mu_n\}$, where $\{\mu_n\}$ is such that $\lim_{n\to\infty}|\mu_n - \lambda_n|/|\lambda_n| = 0$, also fails synthesis. To establish this result we require two lemmas. The first, due to Korobeinik [23], asserts that the angular densities of $\{\lambda_n\}$ and $\{\mu_n\}$ are the same.

Lemma 4.2. *([23, page 124]) Suppose that a set* $\Lambda = \{\lambda_n\}$*, where* $|\lambda_n| \to \infty$*, has for all* θ

and θ_1 $(0 < \theta < \theta_1 \leq 2\pi)$*, except possibly a countable set* P*, the angular density*

$$\Delta_\Lambda(\theta, \theta_1) = \lim_{r\to\infty} n_\Lambda(r, \theta, \theta_1) r^{-\rho} = \Delta_\Lambda(\theta_1) - \Delta_\Lambda(\theta), \tag{4.1}$$

where, as usual, $n_\Lambda(r, \theta, \theta_1)$ *denotes the number of points of* Λ *lying in the sector* $\{\lambda : |\lambda| \leq r, \theta < \arg\lambda < \theta_1\}$ *and* $\Delta_\Lambda(\phi)$ *is a non-decreasing function defined by (4.1) up to an additive constant. Suppose further that the sequence* $M = \{\mu_n\}$ *is such that* $\lim_{n\to\infty}(|\mu_n - \lambda_n|/|\lambda_n|) = 0$*. If* $\theta, \theta_1 \notin P$ *and* $\Delta_\Lambda(\phi)$ *is continuous at* θ *and* θ_1*, then inside the angle* (θ, θ_1) *the angular density of* M*,* $\Delta_M(\theta, \theta_1) = \lim_{r\to\infty} n_M(r, \theta, \theta_1) r^{-\rho}$*, exists and is equal to* $\Delta_\Lambda(\theta, \theta_1)$*.*

The second lemma asserts that $\{\lambda_n\}$ and $\{\mu_n\}$ have the same convergence exponent.

Lemma 4.3. *Suppose that* $\lambda_n \to \infty$ *and* $\rho = \inf\{\alpha : \sum_{n=0}^{\infty} 1/|\lambda_n|^\alpha < \infty\} > 0$*. Let* $\{\mu_n\}$ *be such that* $\lim_{n\to\infty} |\mu_n - \lambda_n|/|\lambda_n| = 0$*. Then,* $\rho_1 = \inf\{\alpha : \sum_{n=0}^{\infty} 1/|\mu_n|^\alpha < \infty\} = \rho$*.*

Proof. Let $\epsilon > 0$ be given. Then, there exists a $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\frac{|\mu_n - \lambda_n|}{|\lambda_n|} < \epsilon.$$

Hence, for all $n \geq N$ and $\alpha > 0$,

$$\frac{1}{((1+\epsilon)|\lambda_n|)^\alpha} < \frac{1}{|\mu_n|^\alpha} < \frac{1}{((1-\epsilon)|\lambda_n|)^\alpha}.$$

Whence, $\rho_1 \equiv \rho$. □

We can then establish the following theorem.

Theorem 4.6. *Let* D *be a non-synthetic diagonal operator on* $H(\mathbb{D})$ *having eigenvalues* $\{\lambda_n\}$*. Let* $f(z)$ *be an entire function with simple zeros only at* $\{\lambda_n\}$*, and suppose the order* ρ *of* f *is strictly greater than one and* $M \equiv \inf\{H(\theta) : 0 \leq \theta < 2\pi\} > 0$ *where* $H(\theta)$ *is the indicator function for* f*. Let* $\{\mu_n\}$ *be a sequence of distinct complex numbers such that*

$\lim_{n\to\infty} |\mu_n - \lambda_n|/|\lambda_n| = 0$. *Then, the diagonal operator $\hat{D}$ with eigenvalues $\{\mu_n\}$ fails to admit spectral synthesis.*

Proof. Since $\lim_{n\to\infty} |\mu_n - \lambda_n|/|\lambda_n| = 0$, we have that

$$\limsup_{n\to\infty} |\mu_n|^{1/n} < \limsup_{n\to\infty} ((1+\epsilon)|\lambda_n|)^{1/n} \le 1,$$

where $\epsilon > 0$, and thus, $\hat{D}$ is continuous. Let $\hat{f}(z)$ be an entire function with simple zeros only at $\{\mu_n\}$. By applying Lemmas 4.2 and 4.3 as well as a result of Levin [29, Theorem 2, page 94], we have that $|\hat{f}(re^{i\theta})| > e^{Mr^\rho}$ for all r sufficiently large and $re^{i\theta}$ not belonging to some exceptional set E. Then applying identical arguments as in Theorem 3.2, we have that $\hat{D}$ fails to admit spectral synthesis. $\square$

As an immediate corollary of the preceding theorem, we observe that we can perturb the points of the integer lattice $\mathbb{Z} \times i\mathbb{Z} = \{m + ik : m, k \in \mathbb{Z}\}$ and obtain another non-synthetic diagonal operator.

Example 4.8. *Perturbing the Integer Lattice Points*

Let $S(z) = z\prod_{n=1}^{\infty}(1-(z/\lambda_n))e^{z/\lambda_n + z^2/2\lambda_n^2}$, the Weierstrass σ-function, where $\{\lambda_n\}$ is the enumeration of $\mathbb{Z} \times i\mathbb{Z}$ defined in Chapter 2. Define $\hat{S}(z) = \prod_{n=0}^{\infty}(1-(z/\mu_n))e^{z/\mu_n + z^2/2\mu_n^2}$, where $\{\mu_n\}$ is such that $\lim_{n\to\infty} |\mu_n - \lambda_n|/|\lambda_n| = 0$. We have shown that S has order $\rho = 2$ (Proposition 2.3). Hence, by a result of Levin [29, Theorem 2, page 91], $\log |S(re^{i\theta})| \approx H_S(\theta)r^2$, where $H_S(\theta) = \int_{\theta-2\pi}^{\theta}(\psi - \theta)\sin 2(\psi - \theta)d\Delta_S(\psi)$. Moreover, it is shown that $H_S(\theta) = \pi/2$ [29, page 128]. By Lemmas 4.2 and 4.3, we have that $H_{\hat{S}}(\theta) = \pi/2$. Thus, for z not in some exceptional set and $|z| = r$ large enough we have $|\hat{S}(re^{i\theta})| > e^{\pi r^2/2}$. Then applying nearly identical arguments to $\hat{S}$, as we did to S in Chapter 2, gives that the diagonal operator with eigenvalues $\{\mu_n\}$ fails to admit spectral synthesis on $H(\mathbb{D})$.

Note that as n gets large, that is, $|\lambda_n|$ gets large, μ_n can lie in a "significantly large" disk $B(\lambda_n, \epsilon|\lambda_n|)$, centered at λ_n allowing for "significant" perturbations of the integer lattice points.

The results and examples of this chapter demonstrate that the synthesis or non-synthesis of diagonal operators is not necessarily preserved when eigenvalues are added, rearranged, deleted, or perturbed. In some cases the modification of the eigenvalues $\{\lambda_n\}$ of a non-synthetic operator does not result in a continuous operator, and even when it does, it is not always the case that the operator is also non-synthetic. However, results and examples are given when the non-synthesis of an operator is preserved under modifications of the eigenvalues. In Sections 4.1 through 4.3, these results were proved for general non-synthetic diagonal operators, while in Sections 4.4 through 4.6 the examples involved using similar techniques to those used in Chapters 2 and 3. It would be interesting to determine universal conditions which would allow for modifications of the eigenvalues of any non-synthetic diagonal operator to preserve non-synthesis.

CHAPTER 5

A Sufficient Condition for Admitting Spectral Synthesis on $H(\mathbb{C})$

A consequence of Leontev's work [25] is that a diagonal operator D acting on $H(\mathbb{C})$ with distinct eigenvalues $\{\lambda_n\} \subset \mathbb{C}$, for which $0 < |\lambda_1| \leq |\lambda_2| \leq ...$ and $0 < \liminf_{n\to\infty} |\lambda_n|/n \leq \limsup_{n\to\infty} |\lambda_n|/n < \infty$, admits spectral synthesis on $H(\mathbb{C})$. In this chapter, we present a result which slightly improves Leontev's result. In particular, we replace the requirement $\liminf_{n\to\infty} \frac{|\lambda_n|}{n} > 0$ with the condition $n(r)/r$ is bounded, where $n(r) = \sum_{\{n:|\lambda_n|\leq r\}} 1$ counts the number of λ_n in $B(0,r)$, and prove the diagonal operator with such eigenvalues $\{\lambda_n\}$ admits spectral synthesis on $H(\mathbb{C})$. We also generate examples of synthetic operators on $H(\mathbb{C})$ which were not known to be synthetic by either Leontev's result of 1976, or the theorems stated in Section 1.7 from the work of Deters, Marin, and Seubert ([13], [31], [41]).

5.1 A Sufficient Condition for Synthesis on $H(\mathbb{C})$

In this section, we show a diagonal operator D acting on $H(\mathbb{C})$ with eigenvalues $\{\lambda_n\}$ is synthetic whenever $\{\lambda_n/n : n \geq 1\}$ is bounded and $n(r)/r$ is bounded. The condition $\{\lambda_n/n : n \geq 1\}$ bounded implies the eigenvalues cannot grow too fast as $|\lambda_n| \leq Mn$ for some $M > 0$. The condition $n(r)/r$ bounded implies there cannot be too many λ_n in disks $B(0,r)$.

To prove the result we consider a canonical product $L(\omega)$ having zeros only at $\{\lambda_n\}\cup\{-\lambda_n\}$. The condition $\{\lambda_n/n : n \geq 1\}$ bounded implies L has order at most one, while the condition $n(r)/r$ bounded implies L has order at least one, hence L has order one. We proceed by contradiction, that is, we invoke Condition *(vii)* of Theorem 1.2, which states that D fails to admit spectral synthesis if and only if there exists a sequence $\{\omega_n\}$, not identically zero, such that $\limsup_{n\to\infty} |\omega_n|^{1/n} = 0$ and $F(\omega) \equiv \sum_{n=0}^{\infty} \omega_n e^{\lambda_n \omega} \equiv 0$ for all $\omega \in \mathbb{C}$. In view of which, the Borel transform $B(\omega) \equiv \sum_{j=0}^{\infty} (a_j/\omega^{j+1})$ of $L_n(\omega) \equiv L(\omega)/(\omega - \lambda_n) \equiv \sum_{j=0}^{\infty} (a_j z^j/j!)$ (if the order of the zero at λ_n is one) or $L(\omega)/(\omega-\lambda_n)^2 \equiv \sum_{j=0}^{\infty} (a_j z^j/j!)$ (if the order of the zero at λ_n is two) satisfies

$$0 \equiv \frac{1}{2\pi i}\int_{\partial B(z,\epsilon)} F(\omega)B(\omega - z)d\omega = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\omega_m \sum_{j=0}^{\infty} a_j \int_{\partial B(z,\epsilon)} \frac{e^{\lambda_m \omega}}{(\omega - z)^{j+1}} d\omega = \omega_n L_n(\lambda_n) e^{\lambda_n z},$$

and so, $\omega_n = 0$ for all $n \geq 0$, a contradiction.

Before proceeding with the theorem, we prove two technical lemmas; the first shows that the conditions $\{\lambda_n/n : n \geq 1\}$ bounded and $n(r)/r$ bounded imply L has order one.

Lemma 5.1. *Suppose $\{\lambda_n\}$ is a sequence of distinct complex numbers such that $\{\lambda_n/n : n \geq 1\}$ is bounded and $n(r)/r$ is bounded. The canonical product $L(\omega) = \prod_{n=0}^{\infty}(1 - (\omega^2/\lambda_n^2))$ having zeros at $Z \equiv \{\lambda_n\}\cup\{-\lambda_n\}$, is of order one and finite type.*

Proof. Since $\{\lambda_n/n : n \geq 1\}$ is bounded, there exists an $M > 0$ such that $|\lambda_n| < Mn$ for all $n \geq 1$. Hence, $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^\alpha} \geq \sum_{n=1}^{\infty} \frac{1}{(Mn)^\alpha} = \infty$ for all $0 < \alpha \leq 1$. Whence, $\rho_1 \equiv \inf\{\alpha : \sum_{n=1}^{\infty} 1/|\lambda_n|^\alpha < \infty\} \geq 1$. By Theorem 2.5.8 of [5], $\rho_1 = \limsup_{n\to\infty} \log n(r)/\log r$, and since $n(r)/r$ is bounded there exists a $K > 0$ such that $n(r) \leq Kr$ for all r sufficiently large. Therefore, $\log n(r) \leq \log Kr = \log K + \log r$ for r large enough, and hence,

$$\rho_1 = \limsup_{r\to\infty} \frac{\log n(r)}{\log r} \leq \limsup_{r\to\infty} \frac{\log K + \log r}{\log r} = 1.$$

Thus, L has order one [5, Theorem 2.6.5]. Since $n(r)/r$ is bounded and $S(r) = \sum_{\{z\in Z:|z|\leq r\}} 1/z =$

0, by Lindelof's Theorem [5, Theorem 2.10.1], L has finite type. $\square$

The second technical lemma shows that the domain of convergence of the Borel transform of an entire function f of order one and finite type τ contains the complement of $B(0,\tau)$.

Lemma 5.2. *Let $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ be an entire function of order one and finite type τ. Then, $B(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}$ converges in the domain $\{z \in \mathbb{C} : |z| > \tau\}$.*

Proof. Since f is of order 1 and finite type τ, we have $\tau = \frac{1}{e} \limsup_{n\to\infty} n|\frac{a_n}{n!}|^{1/n}$ [36, Proposition 11.5]. By Stirling's Formula [5, page 6], $n! = n^n e^{-n}\sqrt{2\pi n} e^{\delta_n}$ where $1/(12n+1) < \delta_n < 1/12n$, and so, $\lim_{n\to\infty}(n!/n^n e^n)^{1/n} = \lim_{n\to\infty}(2\pi n)^{1/2n} e^{\delta_n/n} = 1$. Hence

$$
\begin{aligned}
\tau &= \frac{1}{e}\limsup_{n\to\infty} n\left|\frac{a_n}{n!}\right|^{\frac{1}{n}} \\
&= \limsup_{n\to\infty} \frac{n}{e}\left|\frac{a_n}{n!}\right|^{\frac{1}{n}} \\
&= \limsup_{n\to\infty} (n^n e^{-n})^{\frac{1}{n}}\left|\frac{a_n}{n!}\right|^{\frac{1}{n}} \\
&= \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\left(\frac{n^n e^{-n}}{n!}\right)^{\frac{1}{n}} \\
&= \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \lim_{n\to\infty}\left(\frac{n^n e^{-n}}{n!}\right)^{\frac{1}{n}} \\
&= \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.
\end{aligned}
$$

Thus, $\sum_{n=0}^{\infty} a_n z^n$ converges whenever $|z| < \frac{1}{\tau}$, by the Radius of Convergence Formula. If $|\omega| < \frac{1}{\tau}$, then $\sum_{n=0}^{\infty} |a_n\omega^{n+1}| = |\omega|\sum_{n=0}^{\infty}|a_n\omega^n| < \frac{1}{\tau}\sum_{n=0}^{\infty}|a_n\omega^n| < \infty$. Hence, $B(\omega) = \sum_{n=0}^{\infty} a_n\omega^{n+1}$ converges in the domain $\{\omega \in \mathbb{C} : |\omega| < 1/\tau\}$. Therefore, $B(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}$ converges outside the ball $B(0,\tau)$. $\square$

The main result of this chapter, and the only result of this dissertation regarding the synthesis of diagonal operators acting on the space of entire functions, is as follows.

Theorem 5.1. *Let D be a diagonal operator acting on $H(\mathbb{C})$ with distinct eigenvalues $\{\lambda_n\}$ for which*

(1) $\{\frac{\lambda_n}{n} : n \geq 1\}$ *is bounded, and*

(2) $n(r)/r$ *is bounded.*

Then, D *admits spectral synthesis.*

Proof. By means of contradiction, suppose D fails to admit spectral synthesis on $H(\mathbb{C})$. Thus, by Condition *(vii)* of Theorem 1.2, there exists a sequence $\{\omega_n\}$ of complex numbers, not identically zero, such that $\limsup_{n\to\infty} |\omega_n|^{1/n} = 0$ and $F(\omega) \equiv \sum_{m=0}^{\infty} \omega_m e^{\lambda_m \omega} \equiv 0$ for all $\omega \in \mathbb{C}$. Moreover, F is entire by Condition *(1)*. The canonical product $L(\omega) = \prod_{n=0}^{\infty}(1-(\omega^2/\lambda_n^2))$ has zeros $Z \equiv \{\lambda_n\} \cup \{-\lambda_n\}$ all having order one or two (if λ_n and $-\lambda_n$ are both eigenvalues for D). By Lemma 5.1, L has order one. For $n \geq 1$, define $L_n(\omega) = L(\omega)/(\omega - \lambda_n)$ if λ_n is a zero of order one, or $L_n(\omega) = L(\omega)/(\omega-\lambda_n)^2$ if λ_n is a zero of order two. Then, $L_n(\omega) = \sum_{j=0}^{\infty}(a_j\omega^j/j!)$ is an entire function with zeros $Z \setminus \lambda_n$. By Lemma 5.1, L_n is of type $\tau = \limsup_{j\to\infty}|a_j|^{1/j} < \infty$. Hence, by Lemma 5.2, $B(\omega) = \sum_{j=0}^{\infty} a_j/\omega^{j+1}$ converges in the domain $\{\omega \in \mathbb{C} : |\omega| > \tau\}$. Fix $z \in \mathbb{C}$ and $\epsilon > \tau$. For any $\omega \in \partial B(z,\epsilon)$, we have $|\omega - z| = \epsilon > \tau$. So any $\omega \in \partial B(z,\epsilon)$ is in the domain of convergence of $B(\omega - z)$. Since,

$$\sum_{j=0}^{\infty}\left|\frac{a_j}{(\omega-z)^{j+1}}\right| = \sum_{j=0}^{\infty}\frac{|a_j|}{\epsilon^{j+1}} = \frac{1}{\epsilon}\sum_{j=0}^{\infty}\frac{|a_j|}{\epsilon^j},$$

and

$$\limsup_{j\to\infty}\left(\frac{|a_j|}{\epsilon^j}\right)^{1/j} = \frac{1}{\epsilon}\limsup_{j\to\infty}|a_j|^{1/j} = \frac{\tau}{\epsilon} < 1,$$

$\sum_{j=0}^{\infty} a_j/(\omega-z)^{j+1}$ converges absolutely and uniformly on $\partial B(z,\epsilon)$, by the Root Test. Moreover,

$$\sum_{m=0}^{\infty}|\omega_m e^{\lambda_m\omega}| \leq \sum_{m=0}^{\infty}|\omega_m| e^{|\lambda_m||\omega|} = \sum_{m=0}^{\infty}|\omega_m| e^{|\lambda_m|(|z|+\epsilon)},$$

where

$$\limsup_{m\to\infty}(|\omega_m| e^{|\lambda_m|(|z|+\epsilon)})^{1/m} \leq \limsup_{m\to\infty}|\omega_m|^{1/m}\limsup_{m\to\infty} e^{|\lambda_m|(|z|+\epsilon)/m} = 0 < 1,$$

by Condition *(1)*. Thus, $\sum_{m=0}^{\infty} \omega_m e^{\lambda_m \omega}$ converges absolutely and uniformly on $\partial B(z,\epsilon)$, by the Root Test. Since $F(\omega) \equiv 0$, we have

$$\begin{aligned} 0 &\equiv \frac{1}{2\pi i} \int_{\partial B(z,\epsilon)} F(\omega) B(\omega - z) d\omega \\ &= \frac{1}{2\pi i} \int_{\partial B(z,\epsilon)} \sum_{m=0}^{\infty} \omega_m e^{\lambda_m \omega} \sum_{j=0}^{\infty} \frac{a_j}{(\omega - z)^{j+1}} d\omega \\ &= \frac{1}{2\pi i} \sum_{m=0}^{\infty} \omega_m \sum_{j=0}^{\infty} a_j \int_{\partial B(z,\epsilon)} \frac{e^{\lambda_m \omega}}{(\omega - z)^{j+1}} d\omega \\ &= \sum_{m=0}^{\infty} \omega_m \sum_{j=0}^{\infty} a_j \frac{\lambda_m^j}{j!} e^{\lambda_m z} \end{aligned}$$

by Cauchy's Integral Formula [10, Theorem 5.4]. Thus,

$$0 = \sum_{m=0}^{\infty} \omega_m L_n(\lambda_m) e^{\lambda_m z} = \omega_n L_n(\lambda_n) e^{\lambda_n z}$$

since $L_n(\lambda_m) = 0$ for all $m \neq n$. However, since λ_n is not a zero of L_n, $L_n(\lambda_n) \neq 0$ and $e^{\lambda_n z} \neq 0$, so $\omega_n = 0$ for all $n \geq 0$, a contradiction. The result holds. □

The hypotheses of the preceding theorem give some insight into the possible behavior of the eigenvalues $\{\lambda_n\}$ of a synthetic diagonal operator D acting on $H(\mathbb{C})$. Condition *(1)* asserts that $|\lambda_n|$ cannot grow very fast, in particular, $\{|\lambda_n|\}$ is bounded by Mn for some constant $M > 0$. On the other hand, by Condition *(2)*, $|\lambda_n|$ cannot grow too slow since there cannot be too many λ_n in disks $B(0, r)$. As an example, we quickly observe a diagonal operator with eigenvalues $\{\lambda_n\} = \{n\}$ admits spectral synthesis on $H(\mathbb{C})$; however, this also follows directly from Theorem 1.5. Moreover, by the preceding theorem, a diagonal operator acting on $H(\mathbb{C})$ with eigenvalues $\{\pm n, \pm in\}$ admits spectral synthesis, a conclusion which cannot be determined from any of the previously known results regarding synthesis on $H(\mathbb{C})$ stated in Section 1.7.

Joint work with Henthorn [16] suggests that a diagonal operator acting on $H(\mathbb{D})$ having as eigenvalues six copies of n placed on six rays $e^{i\pi j/3}$ for $0 \leq j < 5$, fails spectral synthesis. Since the eigenvalues satisfy Conditions *(1)* and *(2)* of Theorem 5.1, it appears as though the analogue of Theorem 5.1 for diagonal operators acting on $H(\mathbb{D})$ will not hold. In any event, the proof of Theorem 5.1 does not shed any light on the synthesis of diagonal operators acting on $H(\mathbb{D})$. Although the growth condition $\limsup_{n\to\infty} |\lambda_n|^{1/n} \leq 1$ for continuity on $H(\mathbb{D})$ is more restrictive than the growth condition $\limsup_{n\to\infty} |\lambda_n|^{1/n} < \infty$ for continuity on $H(\mathbb{C})$, the less restrictive decay rate on $\{\omega_n\}$, $\limsup_{n\to\infty} |\omega_n|^{1/n} < 1$ required for membership in $H^*(\mathbb{D})$ compared to $\limsup_{n\to\infty} |\omega_n|^{1/n} = 0$ required for membership in $H^*(\mathbb{C})$, discussed in Section 1.3, only guarantees $F(\omega) = \sum_{n=0}^{\infty} \omega_n e^{\lambda_n \omega}$ is analytic near the origin rather than entire as needed in the proof of Theorem 5.1.

5.2 Leontev's Result and Examples

Using Theorem 5.1, we establish the following corollary.

Corollary 5.1. *A diagonal operator D acting on $H(\mathbb{C})$ with distinct eigenvalues $\{\lambda_n\}$ such that $0 < \inf\{\frac{\lambda_n}{n} : n \geq 1\} \leq \sup\{\frac{\lambda_n}{n} : n \geq 1\} < \infty$ admits spectral synthesis.*

Proof. Clearly $\{\lambda_n\}$ satisfies Condition *(1)* of Theorem 5.1. Moreover, there exists $0 \neq a < b < \infty$ such that $a \leq \frac{|\lambda_n|}{n} \leq b$, for all $n \geq 1$. That is, $an \leq |\lambda_n| \leq bn$ and $(an)^\alpha \leq |\lambda_n|^\alpha \leq (bn)^\alpha$ for all $\alpha > 0$, hence $1/(bn)^\alpha \leq 1/|\lambda_n|^\alpha \leq 1/(an)^\alpha$. By the Comparison Test, $\sum_{n=0}^{\infty} 1/|\lambda_n|^\alpha \leq \sum_{n=0}^{\infty} 1/(an)^\alpha < \infty$ for $\alpha > 1$. Moreover, $\sum_{n=0}^{\infty} 1/|\lambda_n|^\alpha \geq \sum_{n=0}^{\infty} 1/(bn)^\alpha = \infty$ for $\alpha \leq 1$. Whence, $\inf\{\alpha : \sum_{n=0}^{\infty} 1/|\lambda_n|^\alpha < \infty\} = 1$. Since $an \leq |\lambda_n| \leq bn$, we have $n(m) \leq \frac{1}{b}m$ for any $m \in \mathbb{Z}$. Furthermore, for any $r \in \mathbb{R}^+$, $n(r) \leq (1/b)(r+1) \leq (2r/b)$. Hence, $n(r)/r$ is bounded, and D satisfies Condition *(2)* of Theorem 5.1. Therefore, D admits spectral synthesis on $H(\mathbb{C})$. $\square$

Note that Corollary 5.1 does not require that $\{|\lambda_n|\}$ is increasing as Leontev's results does, and thus, Leontev's result is a consequence of Corollary 5.1. The following example demonstrates that the hypothesis $\inf\{\frac{\lambda_n}{n} : n \geq 1\} > 0$ in Leontev's result is not a necessary condition for spectral synthesis on $H(\mathbb{C})$.

Example 5.1. Define $\lambda_n = \begin{cases} n & n \neq 10^k, k \geq 1 \\ 10^{k/2} & n = 10^k, k \geq 1 \end{cases}$.

First, note $\{\lambda_n\}$ are distinct and $\limsup |\lambda_n|^{1/n} = 1 < \infty$. Thus, if D is the diagonal operator with eigenvalues $\{\lambda_n\}$, it is continuous and cyclic.

Claim 1. $\{\lambda_n\}$ satisfies the hypotheses of Theorem 5.1.

Proof. For $n \neq 10^k$, where $k \geq 1$, we have $|\lambda_n/n| = |n/n| = 1$. For $n = 10^k$, where $k \geq 1$, we have, $|\lambda_n/n| = 10^{\frac{k}{2}}/n = 10^{\frac{k}{2}}/10^k = 1/10^{\frac{k}{2}} \leq 1$. Thus, $\{\lambda_n/n : n \geq 1\}$ is bounded. We show $\inf\{\alpha : \sum_{n=0}^{\infty} 1/|\lambda_n|^\alpha < \infty\} = 1$. To this end, consider

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^\alpha} &= \sum_{n \neq 10^k} \frac{1}{n^\alpha} + \sum_{n=10^k} \frac{1}{10^{\alpha k/2}} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^\alpha} + \sum_{k=1}^{\infty} \left(\frac{1}{10^{\alpha/2}}\right)^k \\ &= \sum_{n=1}^{\infty} \frac{1}{n^\alpha} + \frac{1}{(1-(1/10^{\alpha/2}))}, \end{aligned}$$

which is finite only when $\sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty$, thus for $\alpha > 1$. If $\{\lambda_n\} = \{n\}$, then $n(r) = \lfloor r \rfloor \leq r$ where $\lfloor r \rfloor$ is the greatest integer less than r. When we add in the powers of 10, we really only add in $10^{\frac{1}{2}}, 10^{\frac{3}{2}}, 10^{\frac{5}{2}}, \ldots$ Thus, for example, when $r = 100$, $n(r)$ will increase by 1; when $r = 10,000$, $n(r)$ increases by 2; when $r = 10^6$, $n(r)$ increases by 3, and so on. Clearly, $n(r) \leq 2\lfloor r \rfloor \leq 2r$. Hence, $n(r)/r$ is bounded. Therefore, by Theorem 5.1, D admits spectral synthesis on $H(\mathbb{C})$. $\square$

Claim 2. $\{\lambda_n\}$ does not satisfy the hypothesis of Corollary 5.1 (hence Leontev's result).

Proof. Consider $\inf\{\left|\frac{\lambda_n}{n}\right| : n \geq 1\} = \inf\{\frac{10^{k/2}}{10^k} : k \geq 1\} = \inf\{10^{-k/2} : k \geq 1\} = 0$. $\square$

In general, let $\{\hat{\lambda}_n\}$ be any sequence which satisfies the hypotheses of Leontev's result. Then, define

$$\lambda_n = \begin{cases} \hat{\lambda}_n & n \neq a^k, k \geq 1 \\ a^{k/2} & n = a^k, k \geq 1 \end{cases}$$

where $a \in (1, \infty)$. As long as we remove repeated values to guarantee the cyclicity of the operator, the diagonal operator acting on $H(\mathbb{C})$ with eigenvalues $\{\lambda_n\}$ will admit spectral synthesis by Theorem 5.1 but not Corollary 5.1 (or Leontev's result).

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Appendix A

Entire Function Theory

A.1 Introduction

The results of the preceding dissertation rely heavily on the theory of entire functions. This area is well-studied and well-documented, for example, see Boas [5], Conway [10], Holland [18], Levin [29] and [30], and Rubel [36], amongst others. For the convenience of the reader, many of the basic definitions and theorems that were necessary in the results of this document are presented in this chapter. Of central importance for our study is the growth of entire functions, both as a function of $|z|$ as well as the more refined measure of the growth along rays $\{z : \arg z = \theta\}$ for $0 \leq \theta < 2\pi$.

An entire function is a function $f(z)$ which is analytic in the whole complex plane. Entire functions are represented by their power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $\limsup_{n\to\infty} |a_n|^{1/n} = 0$. From this representation, we observe that all polynomials $p(z)$ are contained in the class of entire functions. Furthermore, polynomials are classified by their degree which is determined by the number of roots; the more roots a polynomial has, the higher its degree, and the faster it grows. This suggests the growth of an entire function is intimately related to its zeros. However, the relationship is much more complex than with polynomials, as there are many results that state if f grows "slowly" and its roots "pile up" in a domain, then $f(z) \equiv 0$. Moreover, entire functions can potentially have infinitely many zeros or no zeros, so to study their growth we must examine not only the number of zeros, but also the distribution of the zeros in the complex plane. One method for studying the growth of an entire function is to define the function $M_f(r) = \sup_{|z|=r} \{|f(z)|\}$, which as an application of the Maximum Modulus Principle increases monotonically. The function $M_f(r)$ measures the growth of f in disks centered at the origin.

Other questions that arise regard the growth of functions along different directions, and the relationship between this growth and the global growth determined by $M_f(r)$. Polynomials grow uniformly in all directions; that is, their growth as $z \to \infty$ depends only on $|z|$ not on $\arg z$. This is not necessarily the case for an entire function f with zeros $\{a_n\}$. By considering Liouville's and Picard's Theorems, it seems that a function with "small" global growth cannot decrease "too fast" in some direction, but also must grow on a "large enough" part of the complex plane. To discuss this issue further, we develop a method for measuring the growth of an entire function in different directions.

A.2 Growth as a Function of $|z|$

We first discuss the growth of entire functions in terms of their global growth; that is, we measure the growth of an entire function f by examining its growth on disks centered at the origin as characterized by the function $M_f(r)$, which is independent of direction. By an application of Cauchy's Estimate, if $\liminf_{r\to\infty} M_f(r)/r^\lambda = 0$ for $\lambda > 0$, then $f(z)$ is a polynomial of degree at most λ [30, Theorem 1, page 3]. Hence, to characterize entire functions according to their growth we need to compare them to monotonic functions that grow faster than any polynomial; an obvious choice is $e^{\alpha z^\beta}$, where α, $\beta > 0$ are constants.

In view of which, we say f is of **finite order** if there exists a constant $\lambda > 0$ such that $|f(z)| < e^{|z|^\lambda}$ for all $z \in \mathbb{C}$ with $|z|$ large enough. In this case, whenever $\lambda_1 > \lambda$, $|f(z)| < e^{|z|^{\lambda_1}}$, thus the inequality is satisfied for infinitely many λ's if it holds for one. Thus, we define the **order ρ of f** by $\rho \equiv \inf\{\lambda : |f(z)| < e^{|z|^\lambda}$ whenever $z \in \mathbb{C}$ with $|z| > R$ for some $R > 0\}$.

Hence, if f has order ρ, then for every $\epsilon > 0$ there exists an r_ϵ such that $|f(z)| > e^{r^{\rho+\epsilon}}$ whenever $z \in \mathbb{C}$ with $r = |z| > r_\epsilon$. Furthermore, there exists a sequence $\{r_n\}$ approaching infinity such that $|f(z_n)| < e^{r_n^{\rho-\epsilon}}$ where $z_n \in \mathbb{C}$ is such that $|z_n| = r_n$. Therefore, $M_f(r) < e^{r+\epsilon}$ for r large enough, and $M_f(r_n) > e^{r_n-\epsilon}$, and so clearly, $\rho = \limsup_{r\to\infty} (\log\log M_f(r)/\log r)$.

Using only the order to characterize the growth of entire functions is not always sufficient as it is possible to find two entire functions with the same order that behave very differently. For example, e^z and $\sin z$ are both functions of order one, yet have entirely different zero sets. We refine this measure of the growth by introducing the type of an entire function to further characterize its growth. We say an entire function f of order ρ is of **finite type** if there exists a $k > 0$ such that $M_f(r) < e^{kr^\rho}$ for r large enough. More precisely, we define the **type** τ **of** f by $\tau \equiv \inf\{k : M_f(r) < e^{kr^\rho}\}$. Then, if f is an entire function of order ρ and type τ, we have that for all $\epsilon > 0$ there exists an r_ϵ such that $|f(z)| < e^{(\tau+\epsilon)|z|^\rho}$ whenever $z \in \mathbb{C}$ with $|z| > r_\epsilon$, and there exists a sequence $\{r_n\}$ approaching infinity such that $|f(z_n)| > e^{(\tau-\epsilon)r_n^\rho}$ where $z_n \in \mathbb{C}$ is such that $|z_n| = r_n$. It then follows that $\tau = \limsup_{r\to\infty} (\log M_f(r)/r^\rho)$. An entire function is said to be of **exponential type** if either its order is less than one, or its order equals one and it has finite type.

Thus far we have defined the order and type of an entire function in two ways; by comparing its modulus to exponential functions, and in terms of the function $M_f(r)$. We can also define order and type in terms of the coefficients of the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of f. By an application of Cauchy's Estimate, if $M_f(r) < e^{Ar^k}$ for r large enough, then $|a_n| < (eAk/n)^{n/k}$ for n large enough. Moreover, if $|a_n| < (eAk/n)^{n/k}$ holds for n large enough, then $M_f(r) < e^{(A+\epsilon)r^k}$ for r large enough and $\epsilon > 0$, as an application of Stirling's formula. In this way, it can be shown $\rho = \limsup_{n\to\infty} (n\log n/\log(1/|a_n|))$ and $\tau = (1/\rho\epsilon)\limsup_{n\to\infty} (n\sqrt[n]{|a_n|^\rho})$ ([30, page 6] or [5, Theorem 2.2.10]). Using these formulas for ρ and τ we can easily create functions of any given order and type, as shown in the following example [30, page 7].

Example A.1. *Functions of Given Order and Type*

Let $0 < \rho < \infty$ and $0 < \tau < \infty$, we can then show:

1. $f(z) = \sum_{n=1}^{\infty} (e\tau\rho/n)^{1/\rho} z^n$ is of order ρ and type τ,

2. $f(z) = \sum_{n=2}^{\infty} (e\tau\rho/n \log n)^{n/\rho} z^n$ is of order ρ and type zero,

3. $f(z) = \sum_{n=2}^{\infty} (e\rho \log n)^{n/\rho} z^n$ is of order ρ and infinite type,

4. $f(z) = \sum_{n=2}^{\infty} (1/\log n)^n z^n$ is of finite order, and

5. $f(z) = \sum_{n=0}^{\infty} e^{-n^2} z^n$ is of order zero.

Using these tools we establish a relationship between the growth of an entire function, in terms of its order, and the distribution of its zeros $\{a_n\}$. To this end, we define the **convergence exponent** ρ_1 **of a sequence** $\{a_n\}$ by $\rho_1 \equiv \inf\{\alpha : \sum_{n=1}^{\infty} 1/|a_n|^{\alpha} < \infty\}$. If the number of points of $\{a_n\}$ is finite then $\rho_1 = 0$, and if the number of points of $\{a_n\}$ is countable, then the faster $|a_n| \to \infty$ the smaller the convergence exponent will be. This concept can be easily thought of, for example, in terms of sequences $\{a_n\} \equiv \{n^p\}$ where $\rho_1 = 1/p$. If we define the function $n(r)$ to be the counting function of $\{a_n\}$, $n(r) \equiv \sum_{|a_n| \leq r} 1$, then it can be shown that $n(r)$ is a nondecreasing function which is constant in intervals of the form $(|a_n|, |a_{n+1}|)$ whenever $\{|a_n|\}$ is increasing [18, Theorem 4.5.1]. Additionally, we can compute the convergence exponent by $\rho_1 = \limsup_{r\to\infty} (\log n(r)/\log r) = \limsup_{n\to\infty} (\log n/\log |a_n|)$. Moreover, as an application of Jensen's formula, it can be shown that $\rho_1 = \limsup_{r\to\infty} (\log n(r)/\log r) \leq \limsup_{r\to\infty} (\log\log M_f(r)/\log r) = \rho$. That is, the convergence exponent of the zeros of an entire function does not exceed the order of the function.

In fact, for certain entire functions the convergence exponent is equal to the order of the function. To define such functions, we consider a sequence of complex numbers $\{a_n\}$ such that $a_n \neq 0$ for any $n \geq 0$. Let $p \geq 0$ be an integer such that $\sum_{n=0}^{\infty} 1/|a_n|^{p+1} < \infty$, and define the infinite product $\Pi(z) = \prod_{n=0}^{\infty} G(z/a_n, p)$, where $G(u, 0) = (1 - u)$ and $G(u, p) = (1 - u)e^{u+u^2/2+\cdots+u^p/p}$ for $p > 0$, called the **Weierstrass primary factors**. Using the inequality $|\log G(u,p)| \leq \sum_{k=p+1}^{\infty} |u|^k/k \leq 2|u|^{p+1}$ for $|u| \leq 1/2$, we have that the **Weierstrass canonical product of genus** p $\Pi(z)$, converges absolutely and uniformly in every disk $\{z \in \mathbb{C} : |z| \leq R < \infty\}$. In this case, $\Pi(z)$ has simple zeros only at $\{a_n\}$ [10, Theorem 5.12], the order ρ of $\Pi(z)$ is equal to the convergence exponent of $\{a_n\}$ [36, Theorem 11.5], and the derivate $\Pi'(z) = \sum_{j=0}^{\infty} G'(z/a_j, p) \prod_{n \neq j} G(z/a_n, p)$ [15, page 355]. Moreover, by the Weierstrass Factorization Theorem, every entire function $f(z)$ can be written as $f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} G(z/a_n, p) = e^{g(z)} z^m \Pi(z)$, where $g(z)$ is an entire function, m is the order of the zero of f at $z = 0$ (possibly $m = 0$), and $\{a_n\}$ are the non-zero zeros of f. Hence, the order of f is the larger of the order of the non-zero entire function $e^{g(z)}$ and the canonical product $\Pi(z) = \prod_{n=1}^{\infty} G(z/a_n, p)$. Thus, the order of $f(z)$ is at least the convergence exponent of $\{a_n\}$. Furthermore, if f is an entire function of non-integer order ρ, then ρ is equal to the convergence exponent of the zero set of f, since the order of $g(z)$ does not exceed the genus of $\{a_n\}$ [30, page 31].

A.3 Growth Along Rays $\{z : \arg z = \theta\}$

A more refined measure of the growth of an entire function $f(z)$ is the growth along rays $\{\arg z = \theta\}$ for $0 \leq \theta < 2\pi$. If $f(\omega) = 0$, then $|f(z)|$ is small for z near ω. Thus, f may grow differently on a ray where countably many zeros lie than on a ray with finitely many zeros. In this section, we address this issue and find both lower and upper bounds for $|f(z)|$ which hold except on small regions containing the zeros, by examining the growth of f along rays.

For a sequence of complex numbers $\{a_n\}$ with convergence exponent ρ_1, we define the **density of** $\{a_n\}$ by $\Delta = \lim_{r\to\infty} (n(r)/r^{\rho_1})$, provided the limit exists. If it does not, we define the **upper density** and **lower density** by $\overline{\Delta} = \limsup_{r\to\infty} (n(r)/r^{\rho_1})$ and $\underline{\Delta} = \liminf_{r\to\infty} (n(r)/r^{\rho_1})$, respectively. It can be shown that $\overline{\Delta} = \limsup_{n\to\infty} (n/|a_n|^{\rho_1})$ and $\underline{\Delta} = \liminf_{n\to\infty} (n/|a_n|^{\rho_1})$ [30, page 17]. If we denote the number of zeros of f in the sector $\{z \in \mathbb{C} : |z| \leq r, \psi_1 \leq \arg z \leq \psi_2\}$ by $n_f(r, \psi_1, \psi_2)$, then we define the **angular density of**

the zeros of f by $\Delta_f(\psi_1, \psi_2) = \lim_{r\to\infty} (n_f(r, \psi_1, \psi_2)/r^\rho)$, provided the limit exists.

In order to describe the growth of an entire function $f(z)$ of order ρ along a ray $\{z : \arg z = \theta\}$, we define the **indicator function of** f by $h_f(\theta) = \limsup_{r\to\infty} \log|f(re^{i\theta})|/r^\rho$. If we consider the canonical product $\Pi(z) = \prod_{n=1}^{\infty} G(z/r_n, p)$, where $p < \rho < p + 1$ and $\{r_n\} \subset \mathbb{R}^+$ with $\lim_{r\to\infty} n(r)/r^\rho = \Delta$, then the asymptotic formula

$$\log|\Pi(re^{i\theta})| = \frac{\pi \Delta r^\rho}{\sin \pi\rho} \cos\rho(\theta - \pi) + \frac{o(r^\rho)}{\sin(\theta/2)},$$

for $0 < \theta < 2\pi$, can be established, where $o(r^\rho)$ denotes a function of order less than ρ [30, Lecture 12]. However, to make the given expression valid for $\theta = 0$ as well, we must exclude some exceptional set containing the zeros of $\Pi(z)$. To this end, a set of disks $\{C_j \equiv B(z_j, r_j) \subset \mathbb{C}\}$ will be called a C^0**-set** if $\lim_{R\to\infty} (1/R) \sum_{|z_j|<R} r_j = 0$. Then, outside of a C^0-set of disks the asymptotic relation

$$\log|\Pi(re^{i\theta})| = \frac{\pi\Delta}{\sin\pi\rho} r^\rho \cos\rho(\theta - \pi) + o(r^\rho)$$

holds uniformly with respect to θ, $0 \le \theta < 2\pi$ [30, Section 12.3].

The relations established in the preceding paragraph hold for a canonical product with real, positive zeros. However, similar asymptotic formulas can be established for less restrictive conditions on the zero set. In particular, if $\Pi(z)$ is a canonical product with zeros $\{a_n\}$ lying on a finite number of rays $\arg z = \psi_k$, having densities Δ_k with respect to r^ρ, where ρ is non-integer, then

$$\log|\Pi(z)| = \frac{\pi r^\rho}{\sin\pi\rho} \sum_k \Delta_k \cos\rho(\theta - \psi_k - \pi) + o(r^\rho),$$

for $\theta - 2\pi < \psi_k \le \theta$, outside an exceptional C^0-set. Moreover, if f is a function of non-integer order ρ, then $h(\theta) = (\pi/\sin\pi\rho) \int_{[0,2\pi]} \cos\rho(\theta - \psi - \pi) d\Delta(\psi)$ where Δ denotes the angular density of the zeros $\{a_n\}$ of f. Then, for the canonical product $\Pi(z)$ with zeros $\{a_n\}$, we have

$$\log |\Pi(z)| = r^\rho h(\theta) + o(r^\rho)$$

outside of an exceptional C^0-set [30, Section 13.2]. If $\Pi(z)$ is of integer order ρ, then we will have the same asymptotic formula except the indicator function will be given by $h(\theta) = \int_{[0,2\pi]} (\theta - \psi) \sin \rho(\theta - \psi) d\Delta(\psi) + \tau \cos \rho(\theta - \theta_0)$, where $\tau e^{i\theta_0} = \lim_{R\to\infty} (b_\rho + (1/\rho) \sum_{|a_n|\le R} (1/a_n^\rho))$ and b_ρ is the coefficient of z^ρ in the function $g(z)$ when f is written in the form given in the Weierstrass Factorization Theorem.

When the set $\{a_n\}$ has certain properties we may define the exceptional set more explicitly. As in Levin [29, Chapter II, Section 1], we say that $\{a_n\}$ satisfies Condition (C) if there exists a $d > 0$ such that $\{\overline{B(a_n, d|a_n|^{1-(\rho/2)})}\}_{n=0}^\infty$ is pairwise disjoint, and we say $\{a_n\}$ satisfies Condition (C') if $\{|a_n|\}$ is nondecreasing and there exists a $d > 0$ such that $|a_{n+1}| - |a_n| > d|a_n|^{1-\rho}$. These conditions guarantee that the points of $\{a_n\}$ cannot come arbitrarily close together. If (C) or (C') is satisfied then $\{a_n\}$ is called an **R-set**, while the disks $\{z : |z - a_n| \le d|a_n|^{1-(\rho/2)}\}$ (if (C) holds) and $\{z : |z - a_n| \le d|a_n|^{1-\rho}\}$ (if (C') holds), are called the **exceptional circles of the R-set**. Hence, in either case, the exceptional set is the union of all such disks and the asymptotic relation $\log |\Pi(z)| = r^\rho h(\theta) + o(r^\rho)$ holds outside of this exceptional set.

The asymptotic formulas discussed in this section give both a lower and upper bound for $|\Pi(z)|$ in terms of the indicator function. That is, they provide information regarding the growth of an entire function except on disks centered at the zeros by examining the growth of f along rays. The study of the theory of entire functions is extensive, while the information given in this chapter is a brief overview of the basic concepts. Further information can be found in Boas [5], Conway [10], Holland [18], Levin [29] and [30], and Rubel [36], amongst others.