An Introduction to The Generalized Riemann Integral and Its Role in Undergraduate Mathematics Education

by

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Abstract

The Riemann integral is often introduced to undergraduate calculus students, as its definition and related theorems are relatively straightforward to understand. However, the Riemann integral is limited in its power to integrate a wide variety of functions. This paper introduces an alternate definition of the integral, known as the generalized Riemann integral. This version of the integral was introduced around 1960 by Ralph Henstock and Jaroslav Kurzweil, and its definition and theorems are almost as simple as the traditional Riemann integral, yet its power to integrate functions far surpasses Riemann's integral. This paper includes an overview of the most important theorems and definitions related to the generalized Riemann integral and explains how it can be used to supplement, or even replace, the Riemann integral in undergraduate calculus and analysis courses.

Preface

The goal of this discussion of the generalized Riemann integral is to give a quite recent and powerful definition of the integral which is almost as user-friendly as the Riemann integral. The hope is that students and instructors of undergraduate calculus and analysis courses can incorporate this particular definition of the integral into their coursework. Many expositions of the generalized Riemann integral have been given in the past 60 years (see [2] and [11]), but none of them have been written at a level which suggests usage in elementary calculus courses. This introduction is meant to clarify and bring to light the importance of this new definition of the integral. The only prerequisite is a mild acquaintance with the definition and major results of the Riemann integral. A knowledge of the derivative and differentiability of functions will also prove useful. The definition of the Riemann integral will be presented first, and then the exposition of the generalized Riemann integral will begin. For our purposes, we will only be focusing on functions in the plane ($\mathbb{R}^2$), defined on closed and non-infinite intervals. Formally, these are called compact intervals, but I will instead use the term closed interval throughout this paper, as this is a more familiar term from calculus. Thus, the intervals which we will consider will have the form $[a, b]$, where $a, b \in \mathbb{R}$ and where $a < b$. Functions defined on closed intervals are the ones most commonly dealt with in elementary calculus. Readers interested in integration on infinite intervals are referred to the concluding remarks in Chapter 9 at the end of this paper or see [2]. Others interested in integration of functions in $\mathbb{R}^n$ are referred to *The Generalized Riemann Integral* by McLeod [11]. In fact, few changes are needed when working with functions in $\mathbb{R}^n$ and McLeod presents the whole theory of the generalized Riemann integral from a viewpoint of functions in $\mathbb{R}^n$.

This paper is presented mainly in a theorem-proof format, but many examples are given of the generalized Riemann integral at work. The hope is that these examples illustrate the power of this definition of the integral and aid in understanding of the theorems and the proofs.

A section of this paper is devoted specifically to instructors who wish to incorporate this new definition of the integral into their courses. This goal is that the generalized Riemann integral can supplement, or even possibly replace, the traditional Riemann integral. However, the Riemann integral is still a solid starting point for any calculus student, and can be used as a launching pad into different theories of integration, including the one presented here.

Many of the ideas in this paper come from a book entitled *A Modern Theory of Integration* by Robert G. Bartle [2]. He presents a fairly comprehensive treatment of the generalized Riemann integral, and his intended audience is graduate or advanced undergraduate students of mathematics. This paper will hopefully give an even more user-friendly treatment of the integral than Bartle, but his book is still highly recommended to anyone who wishes to study the generalized Riemann integral in more depth. This paper also varies from Bartle's work in the sense that the aim of this paper is to give an introduction to students and especially instructors who would like to see a fresh and powerful theory of integration, but without some of the more advanced results which are not typically encountered until later mathematics coursework.

As a final note, the theory of the integral presented here is known by a variety of different names, such as the gauge integral, Henstock-Kurzweil integral, or simply the integral. I use the terminology generalized Riemann integral in order to establish its close connections with the Riemann integral. This name also seems to lend itself well to incorporating this integral into existing mathematics curricula.

Contents

Chapter 1

Historical Background of the Integral

Ever since ancient times, people have been fascinated with finding the area inside certain figures. This tradition is largely credited to the Greeks, and especially Archimedes (c. 287 BC-c. 212 BC), who used his mathematical ingenuity to approximate the area of a circle and to find the area between a line and a parabola. He relied on a geometric approach to solve his problem, now known as the method of exhaustion, where he used polygons of known area to make approximations of the unknown areas. For example, he first used hexagons and octagons to approximate the area of a circle. He then used figures with successively more sides, such that the area of the n-gon would approximate the area of circle more closely. Archimedes used this same procedure (using triangles) to determine the area between a line and a parabola [5]. This problem is still seen in calculus classes today, but is typically solved by making use of more modern theories of the integral. Thus, Archimedes' work with areas formed the foundation for the next two thousand years of calculus history.

Calculus, in the modern sense of the term, formally has its beginnings with Isaac Newton (1642-1727), who developed his idea of fluxions in the mid-1660s. These fluxions are analogous to the modern notion of the derivative. In the realm of integration, Newton was the first to formulate the "power rule" which is used widely in elementary calculus courses. He also developed the rudiments of the algebra of integrals, ideas which he set forth in his *De analysi* of 1669. From the title of his book, we get the modern term "analysis", which refers to the rigorous study of calculus concepts. Rigor in mathematics is the idea that any ideas given are backed up with formal proofs. In spite of Newton's groundbreaking work however, his theorems typically lacked these formal proofs, and some of the results he claimed to be true were later found to be false [4]. But even with the lack of rigor and formality in Newton's work, his ideas began a mathematical revolution which still continues today.

Gottfried Wilhelm Leibniz (1646-1716) was another father of modern calculus, and he did his work in France at the same time as Newton. In fact, there was, and still are, questions about whether Leibniz or Newton should be credited with founding calculus. Today, it is accepted by most mathematicians that the two individuals did their work independently, and did not plagiarize ideas from each other. Leibniz gave mathematics the modern integral symbol $\int$ and the differential symbol dx. He called this differential an "infinitesimal," which can be defined as an arbitrarily short distance or as a subinterval which has arbitrarily short length. Although this idea of the infinitesimal was the foundation for Leibniz's work, he seemed unsure of exactly how these infinitesimals behaved [4]. Some modern mathematicians still refer to the differential as an infinitesimal, but since calculus has now become rigorous, the infinitesimal now has a precise meaning and usage.

Leibniz followed the tradition of Archimedes by using polygons to approximate the area under curves. Leibniz used infinitesimal rectangles to complete these approximations, a concept employed by later mathematicians, including Riemann and Cauchy. An infinitesimal rectangle is a geometric figure which has an "infinitely small" base length and height equal to the function value at a point in the infinitesimal [4]. But similar to Newton, Leibniz's contributions to calculus were not rigorous, and he left some of his work unfinished. It would be the job of the future mathematicians to fill in the holes that both Newton and Leibniz left.

Although Leibniz developed the integral symbol, we actually get the term "integral" from Jakob Bernoulli (1654-1705), who began using the term in the 1680s.

In his time, the integral was used mainly to find the area under curves, in the way of Newton and Leibniz. Rigor was finally introduced to calculus and the integral beginning in the mid-18th century, when Jean-le-Rond d'Alembert (1717-1783) developed the notion of a "limit." This idea of a limit would eventually give a precise definition of Leibniz's infinitesimals. However, this early idea of limit was not the rigorous modern definition which is seen today, but was thought of as the value that a continuous function seemed to approach as we chose values close to a particular element of the function's domain [4]. In a sense, this is the intuitive notion of a limit, but it still needed a precise definition before it could be used to formalize calculus.

Augustin-Louis Cauchy (1789-1857) made numerous contributions to analysis, mostly based upon continuous functions and the idea of limit as introduced by d'Alembert. He even developed one of the first formulations of the Fundamental Theorem of Calculus based off of the limit idea. Although some of his work still lacked the rigor seen in modern analysis today, many of his results do have formal proofs and can be found in most any analysis or calculus textbook today [4].

Much of the rigor that is seen in modern analysis today came about as a result of new functions being introduced. These functions are sometimes referred to as pathological functions, and they exposed some of the shortcomings of the definitions and propositions that mathematicians were using up through the early 19th century. Even the mathematician Henri Poincare (1854-1912) complained, "Before when one would invent a new function it was to some practical end; today they are invented to demonstrate the errors in the reasoning of our fathers..." [15].

Peter Gustav Lejeune Dirichlet (1805-1859) was one of the first to create a pathological function that would set the stage for future advances in analysis. His function, also known as the characteristic function of the rationals, is defined on $[0, 1]$ as:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ represents the rational numbers. This function is discontinuous everywhere, and Dirichlet mainly created it to see how the analysis definitions of the day could handle such a bizarre function. Georg Friedrich Bernhard Riemann (1826-1866) was a student of Dirichlet, and he took it upon himself to determine the integrability of such a function. What resulted from Riemann's work was the modern and rigorous definition of the integral which is still introduced in the vast majority of elementary calculus and real analysis textbooks today. His definition of the integral will be given later in this paper, and the Dirichlet function will be studied more in depth. However, there was still one problem: Riemann's integral could not evaluate the area under Dirichlet's function. Riemann concluded that trying to find the integral of such a function was "nonsense" [4].

Since the time of Riemann, there have been many advances in the theory of integration, and this paper will explore the basics of some of the those advances. Other historical remarks will be given throughout this text. We will even find, contrary to Riemann's opinion, that the integral of Dirichlet's function is far from nonsense. We will now turn our attention to Riemann and his famous integral.

Chapter 2

The Riemann Integral and The Fundamental Theorem of Calculus

Riemann's definition of the integral, as noted above, is regarded as the first one which was truly rigorous, and it will be presented here since the definition of the generalized Riemann integral follows directly from it. Riemann's integral relies on the ε-δ definition of the limit and was first introduced in 1854 [4]. Any instructor or student who is familiar with the definition and results of Riemann's integral, both those given here and those commonly introduced in elementary calculus courses, should be able to understand the new material in later chapters. We will begin with a few prerequisites and then proceed with the definition.

Suppose that we have a function f, and we wish to integrate this function on a closed interval $I = [a, b]$. We must first **partition** this interval $I = [a, b]$ into a collection of closed subintervals I_i, where $I = I_1 \cup I_2 \cup ... \cup I_n$ and where the subintervals do not overlap (except at the endpoints). We will denote the partition as

$$P = \{I_1, I_2, I_3, ..., I_n\}$$

which is the set of all of the subintervals of I. Note that each $I_i \subseteq I$ and can also be written as $I_i = [x_{i-1}, x_i]$, for $i = 1, 2, 3, ..., n$ where

$$a = x_0 \leq x_1 \leq ... \leq x_{n-1} \leq x_n = b.$$

For each subinterval I_i in P, we can choose a point $t_i \in I_i$ which is known as a **tag**, or an **association point**, which we simply call t_i. A tag t_i is a point in the subinterval where $x_{i-1} \leq t_i \leq x_i$. Tags may be chosen arbitrarily in each of the subintervals. The set of ordered pairs

$$\dot{P} = \{(I_1, t_1), (I_2, t_2), ..., (I_n, t_n)\} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., n\}$$

is called a **tagged partition** of the interval I. These tags may be chosen arbitrarily and can be interior points of the subinterval or endpoints of the subinterval.

Furthermore, we can define the **norm** or **mesh** of the partition P to be

$$||P|| = \max\{x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1}\}.$$

In other words, the norm of the partition is the length of the longest subinterval in the partition of I. For our purposes, we define the **length** of a particular subinterval $[x_{i-1}, x_i]$ as $x_i - x_{i-1}$. Before proceeding any further, we will now present an example of partitioning an interval to aid in understanding.

Example. Suppose we have a function defined on the closed interval $[0, 1]$ and we want to partition this interval into 5 subintervals. Let these subintervals be $[0, \frac{1}{3}], [\frac{1}{3}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], [\frac{3}{4}, \frac{7}{8}]$, and $[\frac{7}{8}, 1]$. So we write the partition as

$$P = \left\{\left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, \frac{7}{8}\right], \left[\frac{7}{8}, 1\right]\right\}.$$

Now we can choose tags for each subinterval. Let the tags be $\frac{1}{6}$, $\frac{4}{9}$, $\frac{3}{5}$, $\frac{13}{16}$, and 1, respectively for each subinterval. We can then write the tagged partition as

$$\dot{P} = \left\{\left(\left[0, \frac{1}{3}\right], \frac{1}{6}\right), \left(\left[\frac{1}{3}, \frac{1}{2}\right], \frac{4}{9}\right), \left(\left[\frac{1}{2}, \frac{3}{4}\right], \frac{3}{5}\right), \left(\left[\frac{3}{4}, \frac{7}{8}\right], \frac{13}{16}\right), \left(\left[\frac{7}{8}, 1\right], 1\right)\right\}.$$

It is seen that each tag is either an interior point or an endpoint of its respective subinterval. Lastly, the norm of this partition is $\frac{1}{3}$ since the longest subinterval is $[0, \frac{1}{3}]$, which has length $\frac{1}{3} - 0 = \frac{1}{3}$.

We have one more definition and a brief lemma to present and we can then define the Riemann integral. The following definition will be of great importance throughout the remainder of the paper and the lemma will be of use to us later on when we verify certain results.

Definition. Given a function f defined on $[a, b]$, the **Riemann sum** is the term

$$\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

where $[x_{i-1}, x_i] \subseteq [a, b]$, where t_i is a tag in the subinterval $[x_{i-1}, x_i]$ and where $x_i - x_{i-1}$ is the length of the subinterval $[x_{i-1}, x_i]$.

Right-Left Lemma. *Suppose that $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., n\}$ is a tagged partition and let the tag t_k be an interior point of the subinterval $[x_{k-1}, x_k] \in \dot{P}$. If we create the tagged partition $\dot{P}'$ from $\dot{P}$ by adding a new partition point $t_k = \xi$ such that*

$$a = x_0 \leq \cdots \leq x_{k-1} < \xi < x_k \leq \cdots \leq x_n = b$$

where the two new subintervals $[x_{k-1}, \xi]$ and $[\xi, x_k]$ both have tag ξ, then the Riemann sum over the partition $\dot{P}$ will be equal to the Riemann sum over the partition $\dot{P}'$.

Proof. Suppose that the two new subintervals created are $[x_{k-1}, \xi]$ and $[\xi, x_k]$. Hence, ξ is the right endpoint of the subinterval $[x_{k-1}, \xi]$ and the left endpoint of the subinterval $[\xi, x_k]$. Now we have that

$$f(t_k)(x_k - x_{k-1}) = f(t_k)(\xi - x_{k-1}) + f(t_k)(x_k - \xi).$$

Since the equality holds, then these terms in the Riemann sums will be equal and consequently the Riemann sum over the tagged partition $\dot{P}$ will be equal to the Riemann sum over the tagged partition $\dot{P}'$. Since the Riemann sums are preserved in creating the new tagged partition, we conclude that this procedure is valid. Q.E.D.

The previous lemma allows us to split an existing subinterval into two new subintervals such that the tag $t_k = \xi$ is a right endpoint of the one subinterval and a left endpoint of the other subinterval. Furthermore, $\xi = t_k$ is the tag for both of the new abutting subintervals.

All of the definitions and results given so far in this chapter have been leading up to the definition of the Riemann integral and can be used to better understand its definition.

Definition of the Riemann Integral. Let f be a function on an interval $I = [a, b]$. Suppose that there is a number R such that for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., n\}$ is any tagged partition of $[a, b]$ where $x_i - x_{i-1} < \delta$ for $i = 1, 2, 3, ..., n$, then

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - R \right| \leq \varepsilon.$$

Then we write

$$R = \int_a^b f(x)dx$$

and say that R is the Riemann integral of f over the interval $[a, b]$.

This definition might be unfamiliar to many undergraduate students of mathematics, even though they might be adept at evaluating integrals using this definition. What is occurring in this definition is that first we are given an ε which is a small number greater than zero. We must find a constant δ such that when the lengths of each subinterval in a partition of $[a, b]$ are less than δ, then the absolute value of the difference between the Riemann sum and the value of the integral is less than the given ε. In other words, the ε forces us to choose a δ which will make the Riemann sum and the integral arbitrarily close. This integral is commonly thought of as the area under the curve, which is approximated with rectangles of height $f(t_i)$ and with length $x_i - x_{i-1}$. Although looking at the integral as simply the area under a curve is a narrow viewpoint due to its many other uses, it is a sufficient starting point for this paper. Nevertheless, we see the importance of defining appropriate subintervals and tags when working with the Riemann integral to ensure that the definition holds.

This definition of the integral is the one typically encountered in elementary calculus courses and the one most thoroughly studied by undergraduate students in mathematics. It is quite useful for simple functions that the textbooks often present. However, the limited power of this definition is seen quickly, specifically for those functions in which an antiderivative cannot be found or where the function is unbounded at a particular point in an interval. Even in elementary calculus, the textbooks will sometimes present functions which are not Riemann integrable (see [17], p. 227). For a simple example of this, consider the function

$$f(x) = \begin{cases} \frac{1}{x} & 0 < x \leq 1 \\ 0 & x = 0. \end{cases}$$

This function f is unbounded on $[0, 1]$. Thus, the limiting process used in the definition of the Riemann integral is not applicable in this case if we try to compute $\int_0^1 f(x)dx$. Hence, f is not Riemann integrable using the definition provided earlier in this chapter. The typical way of handling this situation in calculus classes is to extend the definition of the Riemann integral and make use of limits. However, it would be convenient if the integral definition could handle such issues without an extension.

Hence, we must ask ourselves whether there is an alternative way to integrate such functions in which Riemann's definition is not applicable. The answer to this question is a resounding "yes" and is explained somewhat in the historical note below.

Historical Note. Around the turn of the twentieth century, Henri Lebesgue (1875-1941) made note of the defects encountered in the definition of the Riemann integral and he formulated a new definition, now known as the Lebesgue integral. Today, this remains the most popular theory of integration among graduate students of mathematics [4]. Lebesgue's integral, however, relies on measure theory, which many students struggle to understand and which is rather complex. Arnaud Denjoy (1884-1974) and Oskar Perron (1880-1975) also created integrals around this time period which solved the issues of the Riemann integral, but which were also significantly more complex than the Riemann integral [9]. Beginning in the mid-twentieth century, two mathematicians named Ralph Henstock (1923-2007) and Jaroslav Kurzweil (b. 1926) independently sought to improve the technical complexities of the integrals introduced by Lebesgue, Denjoy, and Perron [2]. What resulted was a method of integration which was defined similarly to the Riemann integral, but which could integrate a much wider class of functions. The remainder of this paper is devoted to a discussion of the main idea related to the integral that Henstock and Kurzweil defined, which we will call the **generalized Riemann integral**.

We now conclude this chapter with the main motivation for defining the generalized Riemann integral: the Fundamental Theorem of Calculus. The Fundamental Theorem related to the Riemann integral will be given below, without proof. The Fundamental Theorem of Calculus for the generalized Riemann integral will be presented in a later chapter, and a proof will be given there.

Fundamental Theorem of Calculus. *Suppose we have a function F which is differentiable on $I = [a, b]$. If F' is Riemann integrable on $I = [a, b]$, then*

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Notice that one of the hypotheses of the theorem is that the function F' must be Riemann integrable. Above, it was discussed that there are many functions which are not Riemann integrable, so this theorem does not always hold. While this defect in the Riemann integral is often overlooked in calculus courses, it is not long before more advanced functions are introduced where their integrals cannot be evaluated using Riemann's method. The reason why we introduce the generalized Riemann integral is because the Fundamental Theorem of Calculus will hold for all differentiable functions, a very powerful result indeed. Furthermore, the new definition will only be slightly more difficult than Riemann's definition [2]. Thus, not only do we get a powerful definition, but also one which is relatively easy as compared with other theories of integration, such as Lebesgue, Denjoy, or Perron. Furthermore, all Riemann integrable functions are generalized Riemann integrable. This result will be explained in more detail in the next chapter. We will now move from Riemann's definition of the integral to the definition of the generalized Riemann integral given by Henstock and Kurzweil. This movement will only require one change to the definition, and this change opens up a world of integrable functions which were not previously accessible by using Riemann's definition.

Chapter 3

The Generalized Riemann Integral and Gauges

First, let us take a look back at the definition of Riemann's integral given in the previous chapter. Notice that the conclusion of the definition holds only when the length of each subinterval is less than a constant, which we called δ. In the definition of the generalized Riemann integral, instead of requiring that each subinterval have length less than the constant δ, we will require that each subinterval have length less than a particular function value, which we will call $\delta(t)$ for our purposes. This function is directly tied to the tags that we choose in the subintervals of the partition, hence the usage of t in the function $\delta(t)$. Under the Riemann integral, we would typically define the partition of the interval first, and then choose the tags in correspondence with the subintervals. However, under the generalized Riemann integral, we will choose the tags first in relation to the function $\delta(t)$, which in turn allows us to create a suitable partition in order to evaluate the integral [6]. We will call this function a **gauge**. The usage of a gauge is the only difference between definitions of the Riemann integral and the generalized Riemann integral, and it opens up a wider class of function which are integrable since we are no longer restricted to constant gauges.

Definition. If $I = [a, b] \subset \mathbb{R}$, then a function $\delta : I \longrightarrow \mathbb{R}$ is a **gauge** on I if $\delta(t) > 0$

for all $t \in I$. The interval $[t - \delta(t), t + \delta(t)]$ is known as the interval **controlled by the gauge**. This interval controlled by the gauge is important in understanding a concept known as δ-fineness, which will be explored below and which is key to being able to use the definition of the generalized Riemann integral.

In determining the function $\delta(t)$, the only condition imposed on $\delta(t)$ is that it is positive for all $t \in I$. However, not all of the gauges which we define will be useful, but the gauge will still allow us to find what are known as δ-fine partitions of an interval. Hence, given a gauge $\delta(t)$ on I, this gauge gives us a partition of I that is δ-fine. This fact will be stated and proven later in this chapter as Cousin's Lemma. First, we define a δ-fine partition below.

Definition. A tagged partition is said to be **δ-fine** if $[x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$ for all i, where $t_i \in [x_{i-1}, x_i]$. The figure below attempts to clarify the idea of what δ-fine means in a geometric sense. Notice that in the figure, the subinterval $[x_{i-1}, x_i]$ is a subset of the interval controlled by the gauge.

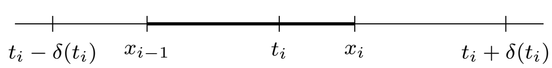

Figure 1: A geometric representation of δ-fineness.

Note. Only a tagged partition can be δ-fine, so instead of saying δ-fine tagged partition, we will merely refer to it as a δ-fine partition.

The purpose of defining this gauge on I and ensuring that there is a δ-fine partition of I is to allow smaller subintervals in places where the original function is changing rapidly and larger subintervals where the original function is changing slowly [8]. This method is useful since when the function is changing slowly or is nearly constant, the value of the Riemann sums will not change much even over large subintervals. On the other hand, when the function is changing rapidly, creating smaller subintervals will allow us to compute the Riemann sums with greater accuracy. For example, consider again the function

$$f(x) = \begin{cases} \frac{1}{x} & 0 < x \leq 1 \\ 0 & x = 0. \end{cases}$$

Near $x = 0$, f approaches a vertical asymptote, and f is changing rapidly near this value. Alternatively, as x becomes larger, values of f change more slowly. Thus, it would be appropriate to define a gauge which would allow us to take small subintervals near $x = 0$ and larger subintervals as x becomes greater.

We now have all of the necessary prerequisites for the definition of the generalized Riemann integral. Its definition is given below.

Definition. The **generalized Riemann integral** is defined as follows: Let f be a function on an interval $I = [a, b]$. Suppose that there is a number M such that for each $\varepsilon > 0$, there is a gauge $\delta(t) > 0$ such that if $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., n\}$ is any tagged partition of $[a, b]$ where $[x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$ for $i = 1, 2, 3, ..., n$ (i.e., $\dot{P}$ is δ-fine), then

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - M \right| \leq \varepsilon.$$

Then we write

$$M = \int_a^b f(x)dx$$

and say that M is the generalized Riemann integral of f over the interval $[a, b]$.

As referenced above, this new approach renders the Riemann integral a specific case of the generalized Riemann integral, specifically those instances where the gauge is a constant (i.e., $\delta(t) = \delta$). In fact, every Riemann integrable function is also generalized Riemann integrable [3]. This is precisely why we call this new theory of integration the generalized Riemann integral. By using these non-constant gauges in the new definition, functions which are not well-behaved can still be integrated by choosing the proper gauge.

We must now pause for a moment to consider the gauge which we define on a compact interval. How do we know that there is a δ-fine partition of the interval for a gauge that we construct? Could it possibly be that a particular gauge does not have a δ-fine partition of an interval? Clearly a result that guarantees the existence of a δ-fine partition would be useful since this is a crucial part of the definition of the new integral. In fact, Pierre Cousin, in the late nineteenth century, developed and proved a theorem which guarantees the existence of a δ-fine partition of an interval given a gauge $\delta(t)$ on that interval. This is popularly known as Cousin's Lemma, and is given below in keeping with the terminology used in this paper. (The original lemma was phrased slightly differently, but is still proved in a similar fashion. For a more traditional phrasing, see [16], pp. 166-167.)

Cousin's Lemma. *If $I = [a, b], a \neq b$, is a compact interval in $\mathbb{R}$ and $\delta(t)$ is a gauge on I, then there exists a δ-fine partition of I.*

Proof. We will do a proof by contradiction and use repeated bisection of the interval. First, we will suppose that I does not have a δ-fine partition. Now let $c = \frac{1}{2}(a + b)$ and bisect the interval I into $[a, c]$ and $[c, b]$. Next, we claim that one of these subintervals does not have a δ-fine partition. Since the union of two δ-fine partitions is also δ-fine, one of the intervals above must not have a δ-fine partition, or else $[a, b]$ would have a δ-fine partition. Let $I_1 = [a, c]$ if this subinterval does not have a δ-fine partition; otherwise, let $I_1 = [b, c]$. Relabel I_1 as $[a_1, b_1]$, and let $c_1 = \frac{1}{2}(a_1 + b_1)$ and bisect I_1 into $[a_1, c_1]$ and $[c_1, b_1]$. As before, one of these subintervals does not have a δ-fine partition. Let $I_2 = [a_1, c_1]$ if it does not have a δ-fine partition; otherwise, let $I_2 = [c_1, b_1]$. Relabel I_2 as $[a_2, b_2]$ and bisect this subinterval in the same manner as above. After repeated bisection, we will obtain an infinite sequence I_n of compact subintervals of $[a, b]$ which are nested, in the sense that

$$[a, b] = I \supset I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots .$$

According to our construction, none of these nested subintervals has a δ-fine partition. The Nested Intervals Property implies that there is a unique number z which lies in all of the subintervals I_n. From our definition of a gauge, we know that $\delta(z) > 0$. Now the length of the pth subinterval is $(b - a)/2^p$. By the Archimedean Property, we know that there is a $p \in \mathbb{N}$ such that the length of the pth subinterval is less than $\delta(z)$. Essentially, by taking p large enough, we can force the length of the subinterval to be less than $\delta(z)$. As a result, we know that $I_p \subset [z - \delta(z), z + \delta(z)]$. But this is precisely the definition of a δ-fine partition on the subinterval I_p. We have created a (trivial) δ-fine partition of I_p and therefore, we have reached a contradiction since we assumed that we could not create a δ-fine partition of I_p. So we conclude that for every gauge $\delta(t)$ in I, there exists a δ-fine partition of I. Q.E.D.

(This proof is based on the proof that Bartle gives in *A Modern Theory of Integration*, see [2], pp. 11-12.)

Historical Note. Pierre Cousin formulated this lemma around 1895, so the existence of δ-fine partitions was well known by the time of Henstock and Kurzweil [1]. Thus, by the turn of the twentieth century, all of the necessary prerequisites were in place for the development of the generalized Riemann integral. So it is interesting to note that it took almost another 65 years before the generalized Riemann integral was developed. In the meantime, Lebesgue, Perron, and Denjoy formulated their integrals, which were much more complex than the generalized Riemann integral. It is still a wonder as to why the simple idea of the integral presented here was developed relatively late in mathematical history [18].

We will begin our exploration of the generalized Riemann integral in the next chapter by looking at several functions for which we can construct gauges and evaluate using the definition of the generalized Riemann integral. The first two of these functions should be familiar to any sophomore level mathematics major, and are also Riemann integrable. For the third example, we return to Dirichlet's Function and show that while it is not Riemann integrable, it is in fact generalized Riemann integrable.

Chapter 4

Examples

Example 1. First, we will look at an example of a function which is Riemann integrable and use the definition of the generalized Riemann integral to construct a gauge and show that the generalized Riemann integral of the function gives the same value as the Riemann integral. Suppose $f(x) = 2x$ on $I = [a, b]$ with $a < b$.

From calculus, we can easily find the antiderivative of f, so we will introduce the function $F(x) = x^2$. Upon doing this, we can then write the Riemann sums of f over the partition of $[a, b]$ as telescoping sums. A telescoping sum is a series which has a finite number of terms after some or most of the terms have been canceled. The idea of telescoping sums will also be used in the proof of the Fundamental Theorem of Calculus in Chapter 6.

We will also need to make use of the Mean Value Theorem for derivatives which states that if F is continuous on $[x_{i-1}, x_i]$ and differentiable on (x_{i-1}, x_i), then there exists a $c_i \in (x_{i-1}, x_i)$ such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

Geometrically speaking, the Mean Value Theorem asserts that given a secant line with two endpoints on a curve, there exists at least one point on the curve between those endpoints where the slope of the tangent line to the curve at that point is equal to the slope of the secant line. We know that $F'(x) = f(x) = 2x$. Thus, F

is continuous everywhere and also differentiable everywhere, so the conclusion of the Mean Value Theorem holds. Multiplying both sides of the conclusion of the theorem by $x_i - x_{i-1}$ gives that

$$f(c_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1}).$$

Now $f(c_i) = 2c_i$, so

$$F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1}) = 2c_i(x_i - x_{i-1}), i = 1, 2, ..., n.$$

If we take the sum of the above expressions, we will obtain the telescoping sum

$$F(b) - F(a) = F(x_n) - F(x_0) = \sum_{i=1}^{n}[F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} 2c_i(x_i - x_{i-1}).$$

If $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., n\}$ is a tagged partition of $I = [a, b]$, then

$$\begin{aligned} F(b) - F(a) - \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) &= \sum_{i=1}^{n} 2c_i(x_i - x_{i-1}) - \sum_{i=1}^{n} 2t_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} 2(c_i - t_i)(x_i - x_{i-1}). \end{aligned}$$

If δ is a constant gauge on $I = [a, b]$ and if $\dot{P}$ is δ-fine, then since $c_i, t_i \in [x_{i-1}, x_i]$, we have that $|c_i - t_i| \leq 2\delta$. Hence it follows that

$$\begin{aligned} \left| F(b) - F(a) - \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \right| &\leq \sum_{i=1}^{n} 2|c_i - t_i|(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} 4\delta(x_i - x_{i-1}) = 4\delta(b - a). \end{aligned}$$

If $\varepsilon > 0$ is given, then we should choose a constant gauge $\delta(t) = \varepsilon/4(b - a)$. Since the choice of ε is arbitrary, we can see that f is generalized Riemann integrable and

$$\int_a^b 2x dx = b^2 - a^2.$$

Notice that from calculus and using the Riemann integral and the Fundamental Theorem of Calculus, we have that

$$\int_a^b 2x dx = x^2 \Big|_a^b = b^2 - a^2$$

which agrees with the answer we obtained by using the generalized Riemann integral.

Example 2. For this example, we are going to take a look at a simple step function which has one point of discontinuity. This step function is Riemann integrable, and we will now show that it is also generalized Riemann integrable. Let $I = [a, b]$, let $c \in (a, b)$, and let $\alpha, \beta \in \mathbb{R}$ where $\alpha \neq \beta$. Define the following function:

$$g(x) = \begin{cases} \alpha & a \leq x < c \\ \beta & c \leq x \leq b. \end{cases}$$

Notice that the function g is continuous at all points in $[a, b]$ except at $x = c$. Thus, we must focus our attention on this point of discontinuity. As a way of handling this situation, we will first force c to be the tag of two abutting subintervals, which each have length less than or equal to δ. As such, we can define the following gauge which accomplishes this task:

$$\delta(t) = \begin{cases} \frac{1}{2}|t - c| & t \neq c \\ \delta & t = c. \end{cases}$$

We can choose δ as needed, and below we will show the precise value for δ that allows the function to be handled using the generalized Riemann integral. Let $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., n\}$ be a δ-fine partition of $I = [a, b]$ which we will assume to be ordered in the sense that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Our choice of the gauge above forces c to be the tag of any subinterval within the partition that contains c. Why is this the case? First, if t is the tag of the subinterval which contains c and if $t = c$, then c is already the tag of that subinterval and we can force the subinterval containing c to have arbitrarily small length by choosing the proper value for δ in the gauge. This is advantageous since this subinterval is the one which contains the discontinuity. Now suppose that t is the tag of a subinterval which contains c, but where $t \neq c$. Now we find $\frac{1}{2}|t-c|$ and we know that the length of this subinterval containing both t and c must be less than $|t-c|$ per the gauge. Since this subinterval contains t, then the subinterval would have to have length at least $|t-c|$ if it also contained c, but since the length is less than $|t-c|$, we know that the subinterval cannot contain c. Hence, we have a contradiction and conclude that $t = c$. In other words, c must be the tag of subinterval which contains c. Now if we use the right-left lemma from Chapter 2, we can assume that c is the tag for two abutting subintervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$, where $x_k = c$. Now $g(t_i) = \alpha$ for $i = 1, 2, ..., k-1$, and thus the Riemann sum for the first $k-1$ terms is

$$\sum_{i=1}^{k-1} g(t_i)(x_i - x_{i-1}) = \alpha(x_{k-1} - a).$$

Furthermore, since $g(t_i) = \beta$ for $i = k, ..., n$, then the Riemann sum for the remaining terms is

$$\sum_{i=k}^{n} g(t_i)(x_i - x_{i-1}) = \beta(b - x_{k-1}).$$

From this, we know that

$$\sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) = \alpha(x_{k-1} - a) + \beta(b - x_{k-1}).$$

Since $x_{k-1} - a = (c-a) - (c - x_{k-1})$ and $b - x_{k-1} = (b-c) + (c - x_{k-1})$, we have that

$$\sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) = \alpha(c-a) + \beta(b-c) + (\beta - \alpha)(c - x_{k-1}).$$

Since the partition of $I = [a, b]$ is δ-fine, then $c - \delta \le x_{k-1} < c$ and thus $0 < c - x_{k-1} \le \delta$ and

$$\begin{aligned} \left| \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) - [\alpha(c-a) + \beta(b-c)] \right| &\le |\beta - \alpha|(c - x_{k-1}) \\ &\le |\beta - \alpha|\delta. \end{aligned}$$

Therefore, in our gauge, it will work to take $\delta(c) = \varepsilon/|\beta - \alpha|$. By doing so, we conclude that g is generalized Riemann integrable on $I = [a, b]$ and

$$\int_a^b g(x)dx = \alpha(c-a) + \beta(b-c).$$

(This example is taken from *A Modern Theory of Integration*, pp. 26-27, see [2].)

To handle this function in undergraduate calculus classes, the integrand would first need to be split since the function has a discontinuity and because there are two function values over the interval $[a, b]$. This is easily accomplished and the evaluation of the integral proceeds as follows:

$$\int_a^b g(x)dx = \int_a^c \alpha dx + \int_c^b \beta dx = \alpha x\Big|_a^c + \beta x\Big|_c^b = \alpha(c-a) + \beta(b-c).$$

This agrees with the answer we obtained using the generalized Riemann integral.

Example 3. Now, we will finally return to Dirichlet's function, which was first defined in Chapter 1. Recall that this function is not Riemann integrable but we are about to show that it is generalized Riemann integrable. We here give a more thorough explanation as to why the function is not Riemann integrable. Again, we define Dirichlet's function on $[0, 1]$ as

$$h(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

We will show that Dirichlet's function is not Riemann integrable by directly using the definition of the Riemann integral. Let $\varepsilon = \frac{1}{2}$ and let $\delta > 0$ be given. Now let $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., n\}$ be any tagged partition of $[0, 1]$ such that $x_i - x_{i-1} < \delta$ for $i = 1, 2, 3, ..., n$. Suppose the tags $t_i \notin \mathbb{Q}$. Then $h(t_i) = 0$ for all $t_i, i = 1, 2, 3, ..., n$, where n is the number of subintervals in the interval $[0, 1]$. According to the definition of the Riemann integral,

$$\left| \sum_{i=1}^{n} h(t_i)(x_i - x_{i-1}) - R \right| = |0 - R| = |-R| = R < \varepsilon = \frac{1}{2}$$

since $h(t_i) = 0$. (Note that R is the value of the Riemann integral). Now suppose the tags $s_i \in \mathbb{Q}$. Then $h(s_i) = 1$ for all $s_i, i = 1, 2, 3, ..., n$, where n is the number of subintervals in the interval $[0, 1]$. According to the definition of the Riemann integral,

$$\left| \sum_{i=1}^{n} h(s_i)(x_i - x_{i-1}) - R \right| = \left| \sum_{i=1}^{n} (x_i - x_{i-1}) - R \right| = |1 - R| < \varepsilon = \frac{1}{2}$$

since $h(s_i) = 1$ and the sum of the collection of subintervals over the interval $[0, 1]$ is 1. Notice that in the first case when $t_i \notin \mathbb{Q}$, we have that $R < \frac{1}{2}$. However, in the second case where $s_i \in \mathbb{Q}$, we have that $|1 - R| < \frac{1}{2}$ or that $\frac{1}{2} < R < \frac{3}{2}$. Clearly this is a contradiction, and since we are able to choose our tags arbitrarily, if the function is Riemann integrable, the choice of tags should not render this contradiction. Therefore, we conclude that $h(x)$ is not Riemann integrable. We will now show that Dirichlet's function is integrable using the generalized Riemann integral and has value 0.

We know that the rationals are a countably infinite set, and hence we will begin by enumerating the rationals in $[0, 1]$ as $\{r_1, r_2, ...\}$. Now, we must construct an appropriate gauge so that the definition of the generalized Riemann integral is applicable. Using the Riemann sums, we know that if we choose our tags t_i to be irrational, then $h(t_i) = 0$ and this contributes 0 to the Riemann sum. Thus, for irrational tags, we can choose any arbitrary value for the gauge without it affecting the Riemann sum. In our case, we will choose 1 for ease of computation. However, we must be cautious about choosing the gauge when our tags are rational, since the rationals do contribute to the Riemann sums. We must define a small interval around each t_i and determine the Riemann sum. Let $\varepsilon > 0$ be given and let the gauge be defined as

$$\delta(t_i) = \begin{cases} \frac{\varepsilon}{2^{j+1}} & t_i = r_j \\ 1 & t_i \notin \mathbb{Q}. \end{cases}$$

Now let $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., n\}$ be any tagged partition such that $[x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$ for all i (i.e., $\dot{P}$ is δ-fine). Then we need to show that

$$\left| \sum_{i=1}^{n} h(t_i)(x_i - x_{i-1}) - 0 \right| = \left| \sum_{i=1}^{n} h(t_i)(x_i - x_{i-1}) \right| \leq \varepsilon.$$

Now we can split this sum into the sum where the tags t_i are rational ($t_i = r_j$) and the sum where the tags t_i are irrational. Thus,

$$\left| \sum_{i=1}^{n} h(t_i)(x_i - x_{i-1}) \right| = \left| \sum_{t_i = r_j} h(t_i)(x_i - x_{i-1}) + \sum_{t_i \neq r_j} h(t_i)(x_i - x_{i-1}) \right|$$

where $\sum_{t_i = r_j} h(t_i)(x_i - x_{i-1})$ is the Riemann sum over the subintervals which have rational tags and where $\sum_{t_i \neq r_j} h(t_i)(x_i - x_{i-1})$ is the Riemann sum over the subintervals which have irrational tags. Now when t_i is rational, $h(t_i) = 1$ by the definition of h and when t_i is irrational, $h(t_i) = 0$ by the definition of h. Hence, the summation becomes

$$\left| \sum_{t_i = r_j} (x_i - x_{i-1}) \right|.$$

This summation represents the sum of the lengths of those subintervals which have rational tags. Now when t_i is rational, then $\delta(t_i)$ gives the value of $\frac{\varepsilon}{2^{j+1}}$. Hence the length of the subinterval around this tag t_i is less than or equal to $2\frac{\varepsilon}{2^{j+1}} = \frac{\varepsilon}{2^j}$ since t_i is the midpoint of the interval. Now suppose that the tag t_i is the tag for two consecutive subintervals. If this is the case, then t_i must the the right endpoint of one of the subintervals and the left endpoint of the other subinterval since the subintervals cannot overlap (except at the endpoints) by the definition of a partition. Since t_i is an endpoint for both of these subintervals, then each subinterval has length at most $\frac{\varepsilon}{2^{j+1}}$. Hence, the sum of the lengths of these two subintervals is less than or equal to $2\frac{\varepsilon}{2^{j+1}} = \frac{\varepsilon}{2^j}$. Therefore, in either case, any rational tag $t_i = r_j$ contributes at most $\frac{\varepsilon}{2^j}$ to the Riemann sum. Consequently,

$$\sum_{t_i=r_j}(x_i - x_{i-1}) \leq \sum_{t_i=r_j}\frac{\varepsilon}{2^j}.$$

Furthermore,

$$\sum_{t_i=r_j}\frac{\varepsilon}{2^j} \leq \sum_{j=1}^{\infty}\frac{\varepsilon}{2^j} = \varepsilon.$$

Notice that this summation above is a geometric series with a first term $a = \frac{\varepsilon}{2}$ and common ratio $r = \frac{1}{2}$. The sum of a geometric series, provided that $-1 < r < 1$ is given by $\frac{a}{1-r}$. In this case, $\frac{a}{1-r} = \frac{\varepsilon/2}{1-1/2} = \frac{\varepsilon/2}{1/2} = 2\frac{\varepsilon}{2} = \varepsilon$. Hence we have shown that

$$\left|\sum_{i=1}^{n} h(t_i)(x_i - x_{i-1}) - 0\right| \leq \varepsilon$$

as desired. Therefore, since we can choose ε arbitrarily, we conclude that Dirichlet's function is generalized Riemann integrable and that $\int_0^1 h(x)dx = 0$.

Chapter 5

Basic Properties of the Generalized Riemann Integral

In undergraduate calculus classes, students typically encounter many useful results of the Riemann integral which can be applied to evaluating the integral and finding solutions to problems. Similar results are also found in the theory of the generalized Riemann integral, and will be explored in this and the subsequent two chapters. In this chapter, we will begin by proving the uniqueness of the generalized Riemann integral. We will then proceed with proofs of the algebra of the generalized Riemann integral, which includes the constant multiple and linearity theorems. The aim of this chapter is not to give a comprehensive look into all of the properties of the generalized Riemann integral. The results presented here should be the most familiar ones from a calculus course and also prove to be the most useful. The interested reader is referred to Section 3 of *A Modern Theory of Integration* [2] for a more comprehensive look at the basic properties of the generalized Riemann integral.

First, we will look at the uniqueness of the integral. Knowing that a value given by the generalized Riemann integral is unique is important to both the computation and the understanding of the integral. If we have a function f defined on $[a, b]$ and we evaluate the integral over this interval, then the integral of this function on that particular interval can take no other value. For example, suppose that $f(x) = x$ and we wish to compute the integral of this function on $[0,2]$. The following theorem and subsequent proof assures us that this integral cannot take values of say, both 2 and 2.5 or any other value. This result is a necessary starting point for further discussion of the generalized Riemann integral.

Uniqueness Theorem. *If there exists a number M which satisfies the definition of the generalized Riemann integral, then this number is unique.*

Proof. We will do a proof by contradiction and assume that there exists an M_1 and an M_2 which both satisfy the definition of the generalized Riemann integral, and where $M_1 \neq M_2$. Let $\varepsilon = \frac{1}{3}|M_1 - M_2|$. Since M_1 satisfies the definition of the generalized Riemann integral, then there exists a gauge $\delta_1(t)$ such that if $\dot{P}$ is a $\delta_1(t)$-fine partition of I, then $\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - M_1\right| \leq \varepsilon$. Similarly, since M_2 satisfies the definition of the generalized Riemann integral, then there exists a gauge $\delta_2(t)$ such that if $\dot{P}$ is a $\delta_2(t)$-fine partition of I, then $\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - M_2\right| \leq \varepsilon$. Now let $\delta(t)$=min$\{\delta_1(t), \delta_2(t)\}$ so that $\delta(t)$ is a gauge on I. We can compute the minimum of $\delta_1(t)$ and $\delta_2(t)$ by using the equation $\min(\delta_1(t), \delta_2(t)) = \frac{\delta_1(t)+\delta_2(t)-|\delta_1(t)-\delta_2(t)|}{2}$. Furthermore, let $\dot{P}$ be a $\delta(t)$-fine partition of I. Because we let $\delta(t)$ equal the minimum of $\delta_1(t)$ and $\delta_2(t)$, then we know that the tagged partition $\dot{P}$ is both $\delta_1(t)$-fine and $\delta_2(t)$-fine. By the Triangle Inequality, we have

$$|M_1 - M_2| \leq \left|M_1 - \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})\right| + \left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - M_2\right| \leq \varepsilon + \varepsilon < |M_1 - M_2|.$$

So we have that $|M_1 - M_2| < |M_1 - M_2|$, which is a contradiction, so the theorem is proved. Q.E.D.

(This proof is based on the proof that Bartle gives in *A Modern Theory of Integration*, see [2], pp. 13-14.)

We now will discuss the algebra of the generalized Riemann integral. Perhaps not surprisingly, the main properties of the generalized Riemann integral are the same as for the Riemann integral. The proofs which are given here do differ slightly from those for the Riemann integral, but should be somewhat familiar and rely mainly on algebraic arguments. The first theorem assures us that the integral of a constant multiplied by a function on $[a,b]$ is equivalent to the constant multiplied by the integral of the function on $[a, b]$.

Constant Multiple Theorem. *If f is integrable on $I = [a, b]$ and $c \in \mathbb{R}$, then cf is integrable on $I = [a, b]$ and*

$$\int_a^b cf = c\int_a^b f.$$

Proof. If $c = 0$, then the result is trivial since $\int_a^b cf = c\int_a^b f = 0$. Therefore, suppose that $c \neq 0$. Let M denote the generalized Riemann integral of f. Let $\varepsilon > 0$ be given. Choose a gauge $\delta(t)$ on $I = [a, b]$ such that if the partition $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, ..., n\}$ is $\delta(t)$-fine, then

$$\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - M\right| \leq \frac{\varepsilon}{|c|}.$$

Using the definition of the generalized Riemann integral, we have

$$\begin{aligned}
\left|\sum_{i=1}^{n} cf(t_i)(x_i - x_{i-1}) - cM\right| &= \left|c\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - cM\right| \\
&= \left|c\left(\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - M\right)\right| \\
&= |c|\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - M\right| \\
&\leq |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.
\end{aligned}$$

Thus, cf is generalized Riemann integrable with value cM. Q.E.D.

(This proof is based on the proof that Bartle gives in *A Modern Theory of Integration*, see [2], p. 42.)

The next theorem is used quite regularly in calculus and assures us that the integral of a sum of two functions on $[a, b]$ is equal to the sum of the integrals of each of the functions on $[a, b]$.

Linearity Theorem. *If f and g are integrable on $I = [a, b]$, then $f + g$ is also* *integrable on* $I = [a,b]$ *and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Proof. Let A denote the generalized Riemann integral of f and let B denote the generalized Riemann integral of g. Let $\varepsilon > 0$ be given. Choose two gauges $\delta_1(t)$ on $I = [a,b]$ and $\delta_2(t)$ on $I = [a,b]$ such that if the partition $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, ..., n\}$ is $\delta_1(t)$-fine, then

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A \right| \leq \frac{1}{2}\varepsilon,$$

and if $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, ..., n\}$ is $\delta_2(t)$-fine, then

$$\left| \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) - B \right| \leq \frac{1}{2}\varepsilon.$$

Now let $\delta(t) = \min\{\delta_1(t), \delta_2(t)\}$ so that if the partition $\dot{P}$ is $\delta(t)$-fine, then it is both $\delta_1(t)$-fine and $\delta_2(t)$-fine. Using the definition of the generalized Riemann integral,

$$\begin{aligned}
\left| \sum_{i=1}^{n} (f+g)(t_i)(x_i - x_{i-1}) - [A+B] \right| &= \left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A \right. \\
&\qquad \left. + \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) - B \right| \\
&\leq \left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A \right| \\
&\qquad + \left| \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) - B \right| \\
&\leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.
\end{aligned}$$

Thus, $f+g$ is generalized Riemann integrable with value $A+B$. Q.E.D.

(This proof is based on the proof that Bartle gives in *A Modern Theory of Integration*, see [2], p. 42.)

From the two theorems above, we now know that $\alpha f + \beta g$ is integrable on $[a, b]$ provided that f and g are both integrable on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$. Furthermore,

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g.$$

This result can be extended to linear combinations with more terms by the use of induction. The proof of this fact will not be given here, but follows directly from the proofs of the theorems given above. Now, we will provide an example of the utility of the two above theorems.

Example. Suppose that $f(x) = 6x$ on $[1, 4]$ and let $g(x)$ be Dirichlet's function as studied in Chapter 4. We will now define Dirichlet's function on $[1, 4]$ instead of $[0, 1]$, but its integral is still 0 as the same argument from Chapter 4 holds. Suppose that we wish to find

$$\int_a^b (f(x) + 5g(x))dx = \int_1^4 (6x + 5g(x))dx.$$

Notice that this integral is quite cumbersome to evaluate if the results given above were not applicable. But since they are, we can rewrite this integral as

$$3\int_1^4 2xdx + 5\int_1^4 g(x)dx.$$

This integral can be evaluated rather easily as follows: From Chapter 4, we know that $\int_a^b 2xdx = b^2 - a^2$ and also that the integral of Dirichlet's function is 0. Hence,

$$3\int_1^4 2xdx + 5\int_1^4 g(x)dx = 3(4^2 - 1^2) + 5(0) = 3(15) - 0 = 45.$$

The next result which will be presented in this chapter is related to integrability of a function on a subinterval of an interval. The popular result from undergraduate calculus related to the Riemann integral is given below, but this time with a focus on the generalized Riemann integral. Essentially the following theorem states that if a function is generalized Riemann integrable on an interval $I = [a, b]$, then it is integrable on both $[a, c]$ and $[c, b]$ where $c \in (a, b)$. Furthermore, if a function is integrable on $[a, c]$ and $[c, b]$, then it is integrable on $[a, b]$ where $c \in (a, b)$.

Additivity Theorem. *Let $f : [a, b] \longrightarrow \mathbb{R}$ and let $c \in (a, b)$. Then f is generalized Riemann integrable on $[a, b]$ if and only if its restrictions to $[a, c]$ and $[c, b]$ are both generalized Riemann integrable. In this case, we have*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. ($\Leftarrow$) Suppose that we restrict the function f to the interval $[a, c]$. We will call this function f_1 and this interval I_1. Suppose that we then restrict the function f to the interval $[c, b]$. We will call this function f_2 and this interval I_2. Now suppose that the generalized Riemann integral of f_1 on I_1 is A_1 and that the generalized Riemann integral of f_2 on I_2 is A_2. Thus, given $\varepsilon > 0$, there exists a gauge δ_1 on I_1 and a gauge δ_2 on I_2 such that if $\dot{P}_1 = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., m\}$ is a δ_1-fine partition of I_1 and if $\dot{P}_2 = \{([x_{j-1}, x_j], t_j) : j = m + 1, ..., n\}$ is a δ_2-fine partition of I_2, then

$$\left| \sum_{i=1}^{m} f_1(t_i)(x_i - x_{i-1}) - A_1 \right| < \frac{1}{2}\varepsilon \quad \text{and} \quad \left| \sum_{j=m+1}^{n} f_2(t_j)(x_j - x_{j-1}) - A_2 \right| < \frac{1}{2}\varepsilon.$$

We now define a gauge on the interval $I = [a, b]$ as follows:

$$\delta(t) = \begin{cases} \min\{\delta_1(t), \frac{1}{2}(c - t)\} & t \in [a, c) \\ \min\{\delta_1(c), \delta_2(c)\} & t = c \\ \min\{\delta_2(t), \frac{1}{2}(t - c)\} & t \in (c, b]. \end{cases}$$

Let $\dot{P}$ be a partition of $I = [a, b]$ which is δ-fine. If this is the case, then the point c must be a tag of at least one subinterval in $\dot{P}$ per the definition of the gauge just given. Furthermore, we can force c to be the tag of two adjacent subintervals by the

Right-Left Lemma in Chapter 2, thus making c a partition point of $\dot{P}$. Let $\dot{P}_1$ be the partition of I_1 which consists of the partition points $\dot{P} \cap I_1$, and let $\dot{P}_2$ be the partition of I_2 consisting of the partition points $\dot{P} \cap I_2$. Hence,

$$\sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) = \sum_{i=1}^{m} f_1(t_i)(x_i - x_{i-1}) + \sum_{j=m+1}^{n} f_2(t_j)(x_j - x_{j-1}).$$

Since the partition $\dot{P}_1$ is δ_1-fine, and the partition $\dot{P}_2$ is δ_2-fine, we conclude that

$$\begin{aligned}\left|\sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) - (A_1 + A_2)\right| &\leq \left|\sum_{i=1}^{m} f_1(t_i)(x_i - x_{i-1}) - A_1\right| \\ &\qquad + \left|\sum_{j=m+1}^{n} f_2(t_j)(x_j - x_{j-1}) - A_2\right| \\ &\leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.\end{aligned}$$

Hence, f is integrable on $I = [a, b]$.

($\Rightarrow$) Conversely, suppose that f is integrable on $I = [a, b]$ and for each $\varepsilon > 0$, let η be a gauge which satisfies the Cauchy Criterion (see the end of this chapter for the Cauchy Criterion.) As above, let f_1 denote the restriction of f to I_1. Let η_1 be the restriction of η to I_1, and let $\dot{P}_1 = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., n\}$ and $\dot{Q}_1 = \{([x_{j-1}, x_j], \hat{t_j}) : j = 1, 2, 3, ..., m\}$ be partitions of I_1 which are η_1-fine. We can extend the partition of I_1 by adjoining partition points and tags from I_2. Hence, we can extend $\dot{P}_1$ and $\dot{Q}_1$ to partitions $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., q\}$ and $\dot{Q} = \{([x_{j-1}, x_j], \hat{t_j}) : j = 1, 2, 3, ..., r\}$ of I which are η-fine. Note that this construction is possible by using the result of Cousin's Lemma which assures us that given a gauge $\delta(t)$ on an interval, there exists a δ-fine partition of that interval. If we use the same partition points and tags in I_2 for both $\dot{P}$ and $\dot{Q}$, we can see that

$$\sum_{i=1}^{n} f_1(t_i)(x_i - x_{i-1}) - \sum_{j=1}^{m} f_1(\hat{t_j})(x_j - x_{j-1}) = \sum_{i=1}^{q} f(t_i)(x_i - x_{i-1}) - \sum_{j=1}^{r} f(\hat{t_j})(x_j - x_{j-1}).$$

Since both $\dot{P}$ and $\dot{Q}$ are both η-fine, we can conclude that

$$\left| \sum_{i=1}^{n} f_1(t_i)(x_i - x_{i-1}) - \sum_{j=1}^{m} f_1(\hat{t_j})(x_j - x_{j-1}) \right| \leq \varepsilon.$$

Hence, by the Cauchy Criterion, we can see that the restriction of f to f_1 on I_1 is integrable on I_1. In the same manner as above, we can also show that the restriction of f to f_2 on I_2 is integrable on I_2. Hence, the equality of the theorem holds from the given hypotheses. Q.E.D.

(This proof is based on the proof given in *A Modern Theory of Integration*, see [2], pp. 44-45.)

We will now present a corollary to this Additivity Theorem which follows easily from the result above. This corollary will be needed later in Chapter 7 when we look at the Substitution Theorem related to the generalized Riemann integral. This corollary assures us that if we know, for example, that $f(x) = 2x$ is integrable on $[0, 10]$, then $f(x) = 2x$ is also integrable on $[3, 7]$ since $[3, 7] \subseteq [0, 10]$.

Corollary to the Additivity Theorem. *If a function f is generalized Riemann integrable on an interval $[a, b]$ and if $[c, d] \subseteq [a, b]$, then the restriction of f to $[c, d]$ is integrable.*

Proof. Since f is integrable on $[a, b]$ and since $c \in [a, b]$, then we know that the restriction of the function f to $[c, b]$ is integrable by the Additivity Theorem given above. Now if $d \in [c, b]$, we can apply the Additivity Theorem again to see that f is integrable over the interval $[c, d]$. Q.E.D.

The final idea which will be presented in this chapter is one which is often encountered early on when exploring integration in a calculus class. The result is presented here as a definition and is a useful tool when computing the integral of certain functions.

Definition. If a function f is generalized Riemann integrable on an interval $[a, b]$, and $\alpha, \beta \in [a, b], \alpha < \beta$, we define the following:

$$\int_{\beta}^{\alpha} f = -\int_{\alpha}^{\beta} f \qquad \text{and} \qquad \int_{\alpha}^{\alpha} f = 0.$$

We conclude this chapter with an example which illustrates these two definitions and also the Additivity Theorem. We will use the function $f(x) = 2x$, as studied in Chapter 4.

Example. Suppose that we wish to integrate $f(x) = 2x$ over the interval $[0, 20]$. From the result in Chapter 4, we know that f is generalized Riemann integrable on any interval $[a, b]$ and hence integrable on $[0, 20]$. Suppose further that we choose $c = 10$ and wish to compute the integral of f over the intervals $[0, 10]$ and $[10, 20]$. The result given by the additivity theorem assures us that f is integrable on both $[0, 10]$ and $[10, 20]$ and also that

$$\int_0^{20} 2x dx = \int_0^{10} 2x dx + \int_{10}^{20} 2x dx = (10^2 - 0^2) + (20^2 - 10^2) = 100 + 300 = 400.$$

Notice that

$$\int_0^{20} 2x dx = 20^2 - 0^2 = 400,$$

and hence the solutions agree.

Now suppose that we wish to evaluate

$$\int_5^2 f(x) dx = \int_5^2 2x dx \quad \text{and} \quad \int_3^3 2x dx.$$

The definition above gives that

$$\int_5^2 2x dx = -\int_2^5 2x dx = -(5^2 - 2^2) = -21 \quad \text{and} \quad \int_3^3 2x dx = 0.$$

As noted above, the proof for the Additivity Theorem given earlier in this chapter relies on the Cauchy Criterion. The Cauchy Criterion can essentially be seen as an alternative definition of the generalized Riemann integral, and can be useful in certain cases. A formal proof is required to show that the definitions are in fact equivalent, but will be omitted here for the sake of readability. For a proof, the interested reader is referred to *A Modern Theory of Integration* [2], pp. 43-44. The Cauchy Criterion is given below.

Cauchy Criterion. *A function* $f : I \longrightarrow \mathbb{R}$ *is generalized Riemann integrable if and only if for any* $\varepsilon > 0$ *there exists a gauge* η *on* I *such that if* $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., n\}$ *and* $\dot{Q} = \{([x_{j-1}, x_j], \hat{t_j}) : j = 1, 2, 3, ..., m\}$ *are any partitions which are* η*-fine, then*

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - \sum_{j=1}^{m} f(\hat{t_j})(x_j - x_{j-1}) \right| \leq \varepsilon.$$

In the next chapter, we turn our attention to perhaps one of the most widely used theorems in undergraduate calculus classes: the Fundamental Theorem of Calculus. This theorem allows us to easily compute the integral of many functions without having to use the definition of the integral.

Chapter 6

The Fundamental Theorem of Calculus

Chapter 2 showed the issues that arise with the Fundamental Theorem of Calculus when we are working under the hypotheses of the definition of the Riemann integral. In the generalized Riemann integral, many of these restrictive hypotheses are removed and the Fundamental Theorem of Calculus holds in a much stronger sense. In fact, the proof of the Fundamental Theorem of Calculus based on the generalized Riemann integral is easier than the proof of the Fundamental Theorem of Calculus based on the Riemann integral [15]. Before we prove the Fundamental Theorem of Calculus as related to the generalized Riemann integral, we will first prove one lemma, known as the Straddle Lemma. Whereas the Mean Value Theorem is a useful tool in proving the Fundamental Theorem of Calculus related to the Riemann integral [17], the Straddle Lemma given below will be the tool that we use to prove the Fundamental Theorem of Calculus in relation to the generalized Riemann integral. This lemma is directly related to the definition of the derivative. Geometrically speaking, we begin with a function F defined on a closed interval $[a, b]$ and a point $t \in [a, b]$ where F is differentiable at t. We then choose two points, $u, v \in [a, b]$ which "straddle" t and establish the inequality given below, hence the name "Straddle" Lemma. In an intuitive sense, the result states that if the points u and v straddle the point t, then the slope of the chord between the points $(u, F(u))$ and $(v, F(v))$ is close to the slope of the tangent line at $(t, F(t))$, see [14].

Straddle Lemma. *Let* $F : [a, b] \longrightarrow \mathbb{R}$ *be differentiable at a point* $t \in [a, b]$*. Then for each* $\varepsilon > 0$ *there is a* $\delta(t) > 0$ *such that if* $u, v \in [a, b]$ *satisfy*

$$t - \delta(t) \leq u \leq t \leq v \leq t + \delta(t),$$

then

$$|F(v) - F(u) - F'(t)(v - u)| \leq \varepsilon(v - u).$$

Proof. Since F is differentiable at t, then given $\varepsilon > 0$ there exists $\delta(t) > 0$ such that if $0 < |z - t| \leq \delta(t), z \in [a, b]$, then

$$\left| \frac{F(z) - F(t)}{z - t} - F'(t) \right| \leq \varepsilon.$$

It then follows, after multiplying by $|z - t|$, that

$$|F(z) - F(t) - F'(t)(z - t)| \leq \varepsilon|z - t|$$

for all $z \in [a, b]$ such that $|z - t| \leq \delta(t)$. Now we choose $u \leq t$ and $v \geq t$ such that $t - \delta(t) \leq u \leq t \leq v \leq t + \delta(t)$ in order to satisfy the inequality given in the hypothesis of the lemma. From this it follows that $v - t \geq 0$ and $t - u \geq 0$. Now we add and subtract the term $F(t) - F'(t)t$, which gives

$$\begin{aligned} &|F(v) - F(u) - F'(t)(v - u)| \\ &= \Big| [F(v) - F(t) - F'(t)(v - t)] - [F(u) - F(t) - F'(t)(u - t)] \Big| \\ &\leq \Big| F(v) - F(t) - F'(t)(v - t) \Big| + \Big| F(u) - F(t) - F'(t)(u - t) \Big| \\ &\leq \varepsilon(v - t) + \varepsilon(t - u) = \varepsilon(v - u). \end{aligned}$$

Thus, the conclusion of the lemma is proved. Q.E.D.

(This proof is based on the one given in *A Modern Theory of Integration*, see [2], pp. 57-58.)

With the important result of the Straddle Lemma at our disposal, we are now ready to prove the Fundamental Theorem of Calculus. If we take a moment to look back at the Fundamental Theorem of Calculus as related to the Riemann integral (in Chapter 2), we stated that the conclusion only holds if the function F' is Riemann integrable. We went on to explain how there are many functions which are not Riemann integrable (Dirichlet's function being one of them). As a result, the Riemann integral is lacking in its power to integrate functions [12]. The Fundamental Theorem of Calculus given below is the one specifically for the generalized Riemann integral. Notice that the restrictive hypothesis of the function F' being Riemann integrable is removed. Hence, the theorem holds in a much stronger sense [13]. For the proof, the only tools we will need are the Straddle Lemma given above, the definition of the derivative, and the Triangle Inequality. The proof is remarkably straightforward and is easier than the proof for Riemann's integral.

The Fundamental Theorem of Calculus. *If $F : [a, b] \longrightarrow \mathbb{R}$ is differentiable on $[a, b]$ such that the derivative of F is f, then f is generalized Riemann integrable over $[a, b]$ and*

$$\int_a^b f = F(b) - F(a).$$

(See [14], p. 646).

Proof. Let $\varepsilon > 0$ be given. For $t \in [a, b]$, let $\delta(t) > 0$ be the value guaranteed by the Straddle Lemma for the given $\frac{\varepsilon}{b-a}$. Suppose that $\dot{P} = \{([x_{i-1}, x_i], t_i) : i = 1, 2, 3, ..., n\}$ is a δ-fine partition of $[a, b]$. Since x_{i-1} and x_i straddle the tag t_i, then by the Straddle Lemma,

$$|F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})| \leq \frac{\varepsilon}{b-a}(x_i - x_{i-1}).$$

Now in order for the conclusion of the theorem to hold, we must estimate the quantity $\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ in relation to the quantity $F(b) - F(a)$. To accomplish this, we will use the expression $F(b) - F(a) - \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$. We will also make use of the telescoping sum $F(b) - F(a) = \sum_{i=1}^{n}[F(x_i) - F(x_{i-1})]$. From this, we obtain

$$F(b) - F(a) - \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \Big[F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})\Big].$$

We can then use the Triangle Inequality to obtain the following:

$$\begin{aligned} \left|F(b) - F(a) - \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})\right| &= \left|\sum_{i=1}^{n} \big[F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})\big]\right| \\ &\leq \sum_{i=1}^{n} \left|F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})\right| \\ &\leq \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon. \end{aligned}$$

Notice that the last inequality above was obtained by making use of the result given by the Straddle Lemma. Since the choice of $\varepsilon > 0$ is arbitrary, we can conclude that f is generalized Riemann integrable and has value equal to $F(b) - F(a)$. Q.E.D.

(This proof is based on the one given in *A Modern Theory of Integration*, see [2], pp. 58-59.)

We will now present another version of the Fundamental Theorem of Calculus which will be useful for two of the proofs in the next chapter. This version of the theorem will not be proven here, but its proof is rather similar to the one just seen for the Fundamental Theorem of Calculus. Furthermore, the following theorem relies on the idea of a **c-primitive**. Essentially, a primitive can be thought of as an antiderivative. The precise definition of a c-primitive is given below.

Definition. Let $I = [a, b]$ and let $F, f : I \longrightarrow \mathbb{R}$. We say that F is a **c-primitive** of f on $I = [a, b]$ if F is continuous on $I = [a, b]$ and the set C of points in $I = [a, b]$ where either $F'(x)$ does not exist or where $F'(x)$ does not equal $f(x)$ is countable.

Fundamental Theorem of Calculus Version 2. *If* $f : [a, b] \longrightarrow \mathbb{R}$ *has a c-*

primitive F on $[a,b]$, then f is generalized Riemann integrable and

$$\int_a^b f = F(b) - F(a).$$

A Note About Differentiablity. When looking at the Fundamental Theorem of Calculus and the integrability of a function using the generalized Riemann integral, the only hypothesis imposed on the function is that it is differentiable on an interval $[a,b]$. What do we mean when we say that a function is differentiable? The definition of the derivative is as follows: Let f be defined on an interval $I = [a,b]$ and let $x_0 \in I$. The **derivative** of f at x_0, denoted by $f'(x_0)$ is given as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

provided that the limit exists or is infinite. If $f'(x_0)$ is finite, then we say that f is differentiable at x_0. If f is differentiable at every point of the set $I = [a,b]$, then we say that f is differentiable on $I = [a,b]$ (see [16], p. 272.) This notion of differentiability is the one used above in the Fundamental Theorem of Calculus.

Example 1. Notice that in Chapter 4, we spent considerable time finding the generalized Riemann integral of $f(x) = 2x$ since we only could rely directly on the definition of the integral. Now with the Fundamental Theorem of Calculus at our disposal, evaluating the integral of $f(x) = 2x$ is almost trivial. From Chapter 4, we know that an antiderivative of $f(x) = 2x$ is $F(x) = x^2$. Hence, by the Fundamental Theorem of Calculus, $\int_a^b 2x dx = b^2 - a^2$. As such, we can see the usefulness of the Fundamental Theorem of Calculus, especially in relation to evaluating integrals.

Example 2. For this example, we will present a series of functions which are not Riemann integrable but which are generalized Riemann integrable. A brief explanation will be given related to their integrability, but the results will not be shown formally. Some of these functions are ones which might be encountered in a calculus class. Establishing the integrability of these functions relies heavily on the Fundamental Theorem of Calculus as given in this chapter.

(a) First, consider the function

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & 0 < x \leq 1 \\ 0 & x = 0. \end{cases}$$

This function is unbounded on $[0, 1]$ and hence is not Riemann integrable. Now we see that $F(x) = 2\sqrt{x}$ for $0 \leq x \leq 1$ and hence $F(x)$ is continuous on $[0, 1]$. Now $F'(x) = \frac{1}{\sqrt{x}}$ for $0 < x \leq 1$; however, $F'(0)$ does not exist. Therefore, $F'(x) = f(x)$ except when $x = 0$. On account of this and because f is not Riemann integrable, the Fundamental Theorem of Calculus is not applicable. However, given $\varepsilon > 0$ and $t \in (0, 1]$, we can define a gauge $\delta(t) > 0$ such that the conclusion of the Straddle Lemma (as given above) holds and such that $\delta(0) = \frac{\varepsilon^2}{4}$. This gauge allows us to handle the unboundedness of the function. As a result, the Fundamental Theorem of Calculus Version 2 for the generalized Riemann integral as given above will hold and we find that the value of this integral is 2. For further explanation, see [2], pp. 59-60.

(b) The second function we will consider is

$$g(x) = \begin{cases} x\big|\cos(\frac{\pi}{x})\big| & 0 < x \leq 1 \\ 0 & x = 0. \end{cases}$$

This function is continuous on $[0, 1]$; however, it is not differentiable at every point in $[0, 1]$ on account of the absolute value. As a result, g is not Riemann integrable. But the set of points where g is not differentiable is countable. Hence $g(x)$ is the c-primitive for the function $g'(x)$ and thus $g'(x)$ will be generalized Riemann integrable and the Fundamental Theorem of Calculus Version 2 for the generalized Riemann integral will hold as explained previously in this chapter. For further discussion, see [2], pp. 69-70.

(c) The third function which we will consider is

$$h(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & 0 < x \leq 1 \\ 0 & x = 0. \end{cases}$$

The derivative $h'(x)$ of this function is defined at all points in $[0, 1]$; however this derivative is unbounded and hence is not Riemann integrable on $[0, 1]$. However, since the derivative exists at every point in $[0, 1]$, this derivative will be integrable using the generalized Riemann integral, and using the Fundamental Theorem of Calculus given above, we can find that $\int_0^1 h'(x)dx = \sin 1$. For further explanation, see [7].

(d) The last example which we will consider in this chapter is related to Dirichlet's function. Consider

$$q(x) = \begin{cases} q & x = \frac{p}{q}, \frac{p}{q} \in [0, 1], \frac{p}{q} \neq 0, \frac{p}{q} \text{ in lowest terms} \\ 0 & x \text{ is irrational in } [0, 1] \\ 0 & x = 0. \end{cases}$$

This function is not continuous at any point of $[0, 1]$ and is also unbounded on any subinterval of $[0, 1]$. Therefore, q is not Riemann integrable. However, we can show that this function is generalized Riemann integrable. First, we enumerate the nonzero rationals in $[0, 1]$ as $\{r_k = \frac{p_k}{q_k} : k \in \mathbb{N}\}$. Then, given $\varepsilon > 0$, we can define the gauge

$$\delta(t_i) = \begin{cases} \varepsilon / q_k 2^{k+1} & t_i = r_k \\ 1 & t_i = 0 \text{ or } t_i \text{ is irrational.} \end{cases}$$

We then proceed with a similar argument to what was used to integrate Dirichlet's function in Example 3 of Chapter 4. For further discussion, see [2], pp. 29-30.

In a typical undergraduate calculus class, there are two different parts of the Fundamental Theorem of Calculus which are typically presented: the one given here and the part related to the differentiation of indefinite integrals. We will not present this second part of the Fundamental Theorem as it is beyond the scope of this paper.

Although this second part is not given, much can be accomplished with just the first part of the theorem as presented here.

Chapter 7

Applications from Calculus

In this chapter, we will finish our discussion of useful results seen in calculus courses as they are applied to the generalized Riemann integral. We take a look at proofs of integration by parts and the substitution theorems. Upon completion of this chapter, all of the results of the Riemann integral seen in elementary calculus textbooks will have been presented, but with the focus on the generalized Riemann integral.

We will now look at integration by parts, which is a useful result related to evaluating integrals. Two different versions of integration by parts will be given here. The first is rather straightforward, while the second one requires slightly different hypotheses which leads to a different proof.

Integration by Parts Theorem I. *Let two functions, F and G, be differentiable on $I = [a, b]$. Then $F'G$ is generalized Riemann integrable if and only if FG' is generalized Riemann integrable. If this be the case, then*

$$\int_a^b F'G = FG\Big|_a^b - \int_a^b FG'.$$

Proof. Since F and G are both differentiable on $I = [a, b]$, the Product Rule from calculus asserts that $(FG)'$ exists on $I = [a, b]$ and that $(FG)' = FG' + F'G$. Also since F and G are both differentiable on $I = [a, b]$, the Fundamental Theorem of Calculus given in the previous chapter holds and implies that $(FG)'$ is generalized Riemann integrable on $I = [a, b]$. Since $(FG)'$ is generalized Riemann integrable, it follows from the Product Rule given above that $F'G$ is generalized Riemann integrable if and only if FG' is also generalized Riemann integrable. From the equation $(FG)' = FG' + F'G$, we can subtract FG' from both sides to obtain $(FG)' - FG' = F'G$. Now we integrate both sides to obtain

$$FG\Big|_a^b - \int_a^b FG' = \int_a^b F'G$$

and the theorem is proved. Q.E.D.

The second theorem related to integration by parts is only slightly more complicated and relies on the concept of the c-primitive as introduced in the previous chapter. Furthermore, the second version of the Fundamental Theorem of Calculus will be utilized.

Integration by Parts Theorem II. *If two functions f and g are generalized Riemann integrable and have c-primitives F and G on an interval $I = [a, b]$, then $Fg + fG$ has a c-primitive FG which is generalized Riemann integrable, and*

$$\int_a^b (Fg + fG) = FG\Big|_a^b.$$

Furthermore, Fg is generalized Riemann integrable if and only if fG is generalized Riemann integrable, in which case

$$\int_a^b Fg = FG\Big|_a^b - \int_a^b fG.$$

Proof. Since f and g have c-primitives on $I = [a, b]$, by the definition above, we know that F and G are continuous on $I = [a, b]$ and that there exist countable sets C_f and C_g of $I = [a, b]$ such that $F'(x) = f(x)$ for $x \in I - C_f$ and $G'(x) = g(x)$ for $x \in I - C_g$. Let $C = C_f \cup C_g$, so that C is a countable set (the union of two countable sets is countable.) The Product Rule from calculus implies that

$$(FG)'(x) = F(x)G'(x) + F'(x)G(x) = F(x)g(x) + f(x)G(x)$$

for $x \in I - C$ so that FG is a c-primitive for $Fg + fG$. The Fundamental Theorem of Calculus Version 2 given in the previous chapter asserts that $Fg + fG = (FG)'$ is generalized Riemann integrable and has integral $FG\Big|_a^b$. Thus, the first part of the theorem is proved. Also, from the linearity theorem given in Chapter 5, we have that Fg is generalized Riemann integrable if and only if fG is generalized Riemann integrable. Hence, we can write the first equation given in the theorem as

$$\int_a^b Fg + \int_a^b fG = FG\Big|_a^b.$$

Subtracting $\int_a^b fG$ from both sides gives that

$$\int_a^b Fg = FG\Big|_a^b - \int_a^b fG$$

and the second part of the theorem is proved. Q.E.D.

(Both of the preceding proofs are based upon the ones given in *A Modern Theory of Integration*, see [2], pp. 67, 187-188.)

We will now turn our attention to the substitution theorem which is a common tool in undergraduate calculus classes. In fact, there are many different versions of the substitution theorem which are useful, but our attention in this chapter will be focused upon the one most commonly utilized when studying integration in a calculus class. We will begin with an example to set the stage for the substitution theorem.

Example. Suppose that we wish to evaluate the following integral

$$\int_0^5 \frac{3x^2dx}{2+x^3}.$$

Notice that this integral cannot be evaluated directly using an elementary antiderivative, but we can still evaluate it rather easily. First, let $u = 2 + x^3$, and then if we take the derivative of u with respect to x, we have that $\frac{du}{dx} = 3x^2$ and hence $du = 3x^2 dx$. Then our integral becomes

$$\int_2^{127} \frac{du}{u}.$$

Using the Fundamental Theorem of Calculus and the antiderivative, we have that

$$\int_0^5 \frac{3x^2 dx}{2 + x^3} = \int_2^{127} \frac{du}{u} = \ln|u|\Big|_2^{127} = \ln(127) - \ln(2) = \ln(63.5).$$

Utilizing this example, the substitution theorem takes the general form

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

This theorem will now be formalized, and a proof will be given. Note that both the theorem and proof rely on the concept of the c-primitive which was introduced previously in this chapter.

Substitution Theorem. *Let $I = [a, b]$ and $J = [c, d]$ and suppose that:*

(i) *$f : J \longrightarrow \mathbb{R}$ has a c-primitive F on J,*

(ii) *$g : I \longrightarrow \mathbb{R}$ has a c-primitive G on I where $G(I) \subseteq J$,*

(iii) *G is a countable-to-one mapping of I into J.*

Then $(f \circ G) \cdot g$ is generalized Riemann integrable and f is generalized Riemann integrable. Moreover,

$$\int_a^b (f \circ G) \cdot g = \big(F \circ G\big)\Big|_a^b = \int_{G(a)}^{G(b)} f.$$

Proof. By hypothesis (i) and the definition of c-primitive, F is continuous on J and there exists a countable set $C \subset J$ such that $F'(u) = f(u)$ for all $u \in J - C$. By hypothesis (ii) and the definition of c-primitive, G is continuous on I and there exists a countable set $D \subset I$ such that $G'(x) = g(x)$ for all $x \in I - D$. Therefore

$F \circ G$ is continuous on I since the composition of continuous functions is continuous. Hypothesis (iii) implies that $G^{-1}(C)$ is a countable set in I, so $E = D \cup G^{-1}(C)$ is a countable set in I. The Chain Rule implies that

$$(F \circ G)'(x) = F'(G(x)) \cdot G'(x) = (f \circ G)(x) \cdot g(x)$$

for all $x \in I - E$. Hence, $F \circ G$ is a c-primitive of $(f \circ G) \cdot g$, such that $(f \circ G) \cdot g$ is generalized Riemann integrable on I by the Fundamental Theorem of Calculus Version 2 given in the previous chapter and

$$\int_a^b (f \circ G) \cdot g = (F \circ G)\Big|_a^b = F(G(b)) - F(G(a)).$$

From hypothesis (i), we know that f is generalized Riemann integrable on J by the Fundamental Theorem of Calculus Version 2. From hypothesis (ii), we know that $G(I)$ is a closed interval in J. The Corollary to the Additivity Theorem (see Chapter 5) implies that f is integrable on $G(I)$ and also on the closed interval with endpoints $G(a), G(b)$.

If $G(a) \leq G(b)$, then the Fundamental Theorem of Calculus Version 2 applied to the interval $[G(a), G(b)]$ implies that

$$\int_{G(a)}^{G(b)} f = F\Big|_{G(a)}^{G(b)} = F(G(b)) - F(G(a)).$$

If $G(b) < G(a)$, then we apply the Fundamental Theorem of Calculus Version 2 to the interval $[G(b), G(a)]$ to obtain

$$\int_{G(a)}^{G(b)} f = -\int_{G(b)}^{G(a)} f = -F\Big|_{G(b)}^{G(a)} = F(G(b)) - F(G(a)).$$

Hence the conclusion of the theorem holds. Q.E.D.

(This proof is based on the one given in *A Modern Theory of Integration*, see [2], pp. 210-211.)

The last major idea which will be explored in this paper is related to the incorporation of this new definition of the integral into existing calculus curricula. This is no easy task and many considerations must be made before a change can take place. The next chapter will explore these considerations and will give tips to instructors in relation to the use of the generalized Riemann integral in calculus classes.

Chapter 8

A Guide for Instructors

The aim of this chapter is to explain how the generalized Riemann integral can fit into a calculus or analysis curriculum. Although the goal of the entire paper is to present a theory which can be understood by both students and instructors alike, this particular section of the paper will explain the curriculum of presenting the integral and how it can satisfy different learning objectives. Many authors who have presented the generalized Riemann integral have suggested that the Riemann integral should be disposed of altogether and replaced by the generalized Riemann integral. However, since the definitions of both integrals are so similar, it would most likely be best to present the Riemann integral first and then explain how it can be generalized. Indeed, many topics in mathematics proceed in this manner. Logically, a solid introduction to the traditional Riemann integral can easily flow into the basics related to the generalized Riemann integral. The hope is that students will come away with a better understanding of the theory of integration as a whole, both at a specific and at a general level.

For those instructors who are teaching an elementary calculus course either at the high school or the college level, the generalized Riemann integral can be placed in the sequence directly after the Riemann integral is taught. However, the way in which the definition of the Riemann integral is taught differs from author to author. Many calculus textbook present the limit definition of the Riemann integral as follows: Let

f be a function which is defined on a closed interval $[a, b]$. If

$$\lim_{||P|| \to 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i$$

exists, then we say that f is integrable on $[a, b]$ and that this limit is equal to the definite integral of f from a to b written as

$$\int_a^b f(x)dx.$$

(This definition is taken from *Calculus*, see [17], p. 226.) In this definition, P represents the partition of the interval $[a, b]$, and we refer to $||P||$ as the **norm** or **mesh** of the partition. Now $||P||$ is the length of the longest subinterval in the partition P. Furthermore, Δx_i is the length of the i^{th} subinterval in the partition P. Lastly, $\bar{x}_i$ is called a sample point (or tag as we refer to it) in the interval $[x_{i-1}, x_i]$. Hence, in this definition, we are taking the sum of the areas of the rectangles under a curve where the height of the rectangle is given by $f(\bar{x}_i)$ and the length of the rectangle is given by $x_i - x_{i-1}$. We then evaluate the limit as the length of the longest subinterval in P approaches 0. If this limit exists, we say that this limit is equal to the definite integral of f on $[a, b]$.

This definition as given here is equivalent to the definition of the Riemann integral as presented in Chapter 2, but there are notational differences which might hinder a proper segue into the generalized Riemann integral. First, we prefer to write the definition using the ε-δ definition of the limit, rather than using the explicit limit notation. By introducing the Riemann integral using the ε-δ definition, students will not be burdened by having to learn a new notation as instruction progresses. Students are typically introduced to the ε-δ definition of the limit early on in a calculus course, so this should pose little difficulty. In fact, this is a good review of limits and will allow the students to see a direct application of the limit definition. Also, we can replace $\bar{x}_i$ with t_i, which are the tags in the subintervals of the partition. Lastly, we can denote Δx_i as $(x_i - x_{i-1})$, which is just the length of one of the

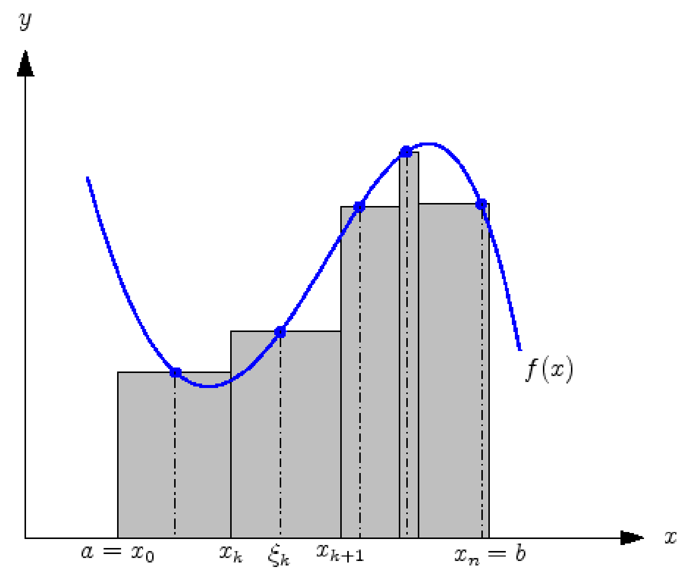

Figure 8.1: Partitioning an interval when using the Riemann integral. Note that we use $\bar{x}_i$ (or t_i) instead of ξ_k to represent the tags in a subinterval [10].

subintervals in the partition. Upon implementing these notational changes, we then have a definition of the Riemann integral which is the same, notationally, as the one given above in Chapter 2. It is recommended that this definition of the Riemann integral be presented to the students, rather than the one commonly encountered in calculus textbooks. Furthermore, Chapter 2 of this paper provides a simple example of how an interval can be partitioned, which is a useful instructional tool that can be incorporated in class, especially when the Riemann integral is first introduced. A graph, such as the one shown above in Figure 8.1, can also be used when explaining interval partitioning.

Many students upon encountering the definition of the integral for the first time are unaware of the technicalities which are present. Hence, instructors should spend the time to acquaint the students with the ε-δ definition and the process of computing an integral. Examples 1 and 2 from Chapter 4 can be used as examples of integral computation, or any other easily integrable function can also be used. If these issues are resolved early on, then the students will be better able to access the results of the generalized Riemann integral.

As we continue past Chapter 2 of this paper, there is an increased emphasis on theorems and proofs. We now discuss how these results may be used in calculus curricula. The proof of Cousin's Lemma in Chapter 3 is relatively straightforward and could be introduced to any student of mathematics who understands bisection of an interval. In fact, this proof lends itself well to a diagram and can be a good introduction to proof by contradiction. While many of the other proofs in this paper rely on subtle arguments that may be beyond the scope of some students, the proof of Cousin's Lemma is relatively elementary and is a good instructional tool for teachers of mathematics.

During instruction, and especially in mathematics instruction, examples are important tools that teachers can use in order to expose students to bigger-picture and more general results. Chapter 4 will be useful for these purposes since it presents several examples related to the computation of generalized Riemann integrals. These examples can become quite tedious and cumbersome, but rely mainly on inequalities and algebraic arguments. Also, the first two of these examples can be computed using the traditional Riemann integral and thus can provide a good segue into the generalized Riemann integral. There are also examples spread throughout the other chapters which are continuations of examples first introduced in Chapter 4. The hope is that by returning to these same examples the readers will better understand how the different parts of the theory fit together and how the theorems introduced have a useful and practical application. These examples will prove to be valuable to mathematics students.

Almost all of the theorems given throughout the paper are relatively easy to understand, and it is recommended that all of them be introduced to the students. Probably the most advanced result is the substitution theorem which is given at the end of Chapter 7. Some of the assumptions rely on c-primitives and countable-to-one mappings; ideas which will probably be foreign to most calculus and analysis students. Hence, this theorem and subsequent proof may be omitted, and more emphasis can be placed on the other results. While examples are not given for all of the theorems, instructors familiar with the results based on Riemann's integral can incorporate examples into their instruction. These examples can be found in any calculus textbook, such as [17].

Not all of the proofs presented here are suitable for undergraduate calculus classes. While some rely on basic algebra, others contain subtle arguments which might be better deferred until more instruction has been given. A simple, yet instructive proof that can be introduced to students is the proof of the constant multiple theorem as given in Chapter 5. The proof relies on a ε-δ argument and simple algebraic manipulation which students should be able to grasp. In relation to the nature of content in mathematics, many undergraduate calculus classes are mainly focused on calculation, rather than on theory and proofs. Most functions encountered in these courses are Riemann integrable and hence would be computationally identical if we used the generalized Riemann integral instead. However, some instructors may wish to introduce functions which are not Riemann integrable and explain the generalized Riemann integral. In these cases, the results of this paper can be useful.

For those who are instructors of undergraduate analysis courses, the generalized Riemann integral would be recommended material to present, especially when looking at pathological functions which are not Riemann integrable. Dirichlet's Function, which is presented in Chapter 4, would be a good example to show to the students. Not only will it introduce them to a pathological function which is not Riemann integrable, but it will also show them the power of the generalized Riemann integral. Most of the proofs are all accessible to the students, with the possible exception of the proof of the substitution theorem in Chapter 7. Most of the proofs will give the students good practice in writing ε-δ arguments, a concept repeatedly seen in analysis. The proof of the Straddle Lemma and even the Fundamental Theorem of Calculus from Chapter 6 could be left as exercises for the students. Perhaps the proof of the linearity theorem from Chapter 5 could also be left as an exercise, provided that the students understand the basic structure of the proof of the constant multiple theorem. The following outlines provide a possible course sequence, with regards to calculus and analysis. The items listed are the recommended ideas to present to the students, and instructors are encouraged to use their own discretion when developing their curricula.

Calculus:

- **Chapters 1-3.**
- **Chapter 4.** Focus on Example 3 and its explanation. For Examples 1 and 2, computation can be done using solely the Riemann integral.
- **Chapter 5.** Present all theorems, definitions, and examples. Proofs of the constant multiple theorem and linearity theorem can be incorporated. Other proofs can be omitted.
- **Chapter 6.** The straddle lemma, the proofs, and the note about differentiability can be omitted.
- **Chapter 7.** The substitution theorem and all proofs can be omitted. This chapter provides a good review of integration by parts.

Analysis:

- **Chapters 1-3.**
- **Chapter 4.** Focus on Example 3 and its explanation. For Examples 1 and 2, computation can be done using solely the Riemann integral.
- **Chapter 5.** Proof of the additivity theorem can be omitted.
- **Chapter 6.**
- **Chapter 7.** Proof of the substitution theorem can be omitted. This is a good review of substitution and integration by parts.

Chapter 9

Concluding Remarks

The material which has been presented in this paper is but a brief overview of the main ideas related to the generalized Riemann integral. This material should suffice for anyone who is interested in a new theory of the integral, but who does not wish to read a long exposition of all of the properties. This material also suffices for use in undergraduate calculus and analysis courses where a fresh look at integration might be useful and needed. For anyone interested in pursuing the generalized Riemann integral more in-depth, the books and articles (specifically [2], [9], [11], and [13]) in the references section at the end of this paper give a fairly comprehensive treatment of most aspects of the theory. The main ideas have been presented here in this paper as an introduction, but there is much more that can be learned from further study.

One of the ideas which has not been presented in this paper which is traditionally introduced in undergraduate calculus courses is the concept of the indefinite integral. Essentially, the indefinite integral is of the form $\int f(x)dx$ and hence does not contain upper or lower limits. This type of integration is concerned with finding an antiderivative, rather than computing the area under a function on an interval $[a, b]$, which we have focused on. The reason the indefinite integral is absent here is because it relies on the development of a theory of primitives (antiderivatives) before we can use the definition of the generalized Riemann integral. As such, the indefinite integral is a more complex idea which is beyond the scope of this paper.

Later on in calculus courses, students encounter improper integrals. One such type of improper integral concerns functions defined on infinite intervals, rendering infinite limits of integration. Riemann's definition is often extended and improper integrals can be computed using the following procedure:

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx \quad \text{and} \quad \int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx.$$

The generalized Riemann integral is able to handle these improper integrals without a change or extension to the definition. First, we must present a definition and then we will proceed with a discussion of the results related to this definition.

Definition. The **extended real number system** is the set $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ consisting of all of the real numbers with $-\infty$ and ∞ adjoined. We do not consider $-\infty$ and ∞ to be real numbers, however given a function f, we will define $f(\infty) = 0$ and $f(-\infty) = 0$.

In using this extended real number system, we can replace the interval $[a, b]$ with the interval $[a, \infty]$ in the definition of the generalized Riemann integral in Chapter 3, and the definition will still be valid. Most of the proofs given in this paper require slight modification when dealing with intervals of the type $[a, \infty]$, and interested readers are referred to [2], pp.263-264, to see the details of these modifications.

The hope is that students and instructors alike benefit from a highly powerful yet relatively simple definition of the integral as presented here. Although the popularity of this new integral pales in comparison to other integral definitions, perhaps the future will shed more light on the generalized Riemann integral.

Bibliography

[1] Andre, Nicole R., Susannah M. Engdahl, and Adam E. Parker, *An analysis of the first proofs of the Heine-Borel Theorem - Cousin's Proof*, *Convergence*, http://www.maa.org/press/periodicals/convergence/an-analysis-of-the-first-proofs-of-the-heine-borel-theorem-cousins-proof (Updated August 2013).

[2] Bartle, Robert G., *A Modern Theory of Integration*, Graduate Studies in Mathematics, vol. 32, American Mathematical Society, Providence, 2001.

[3] Bartle, Robert G., *Return to the Riemann integral*, American Mathematical Monthly **103** (1996), no. 8, 625-632.

[4] Dunham, William, *The Calculus Gallery: Masterpieces from Newton to Lebesgue*, Princeton University Press, Princeton, NJ, 2005.

[5] Harding, Simon and Paul Scott, *The history of the calculus*, Australian Mathematics Teacher **61** (2005), no. 2, 2-5.

[6] Henstock, Ralph, *A Riemann-type integral of Lebesgue power*, Canad. J. Math. **20** (1968), 79-87.

[7] Lamoreaux, Jack and Gerald Armstrong, *The fundamental theorem of calculus for gauge integrals*, Mathematics Magazine **71** (1998), no. 3, 208-212.

[8] Lange, Kenneth, *Optimization*, Second edition, Springer, New York, 2013.

[9] Lee, Peng Yee and Rudolf Výborný, *Integral: An Easy Approach After Kurzweil and Henstock*, Cambridge University Press, Cambridge, 2000.

[10] Mathematics Online, *Riemann Integral*, http://www.mathematics-online.org/kurse/kurs9/seite104.html (Updated October 2009).

[11] McLeod, Robert M., *The Generalized Riemann Integral*, The Carus Mathematical Monographs, Number twenty, Mathematical Association of America, 1980.

[12] Smithee, Alan, *The Integral Calculus*, http://www.classicalrealanalysis.com (2007).

[13] Swartz, Charles, *Introduction to Gauge Integrals*, World Scientific Pub. Co., Singapore, 2001.

[14] Swartz, Charles and Brian S. Thompson, *More on the fundamental theorem of calculus*, American Mathematical Monthly **95** (1988), no. 7, 644-648.

[15] Thomson, Brian S., *The natural integral on the real line*, Scientiae Mathematicae Japonicae **67** (2008), no. 1, 23-35.

[16] Thomson, Brian S., Judith B. Bruckner, and Andrew M. Bruckner, *Elementary Real Analysis*, Second edition, ClassicalRealAnalysis.com, 2008.

[17] Varberg, Dale, Edwin J. Purcell, and Steven E. Rigdon, *Calculus*, Ninth edition, Pearson Prentice Hall, Upper Saddle River, NJ, 2007.

[18] Výborný, Rudolf, *Some applications of Kurzweil-Henstock integration*, Math. Bohemica **118** (1993), no. 4, 425-441.

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