FIELD EMISSION AND SCATTERING FROM CONDUCTING NANOFIBERS

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Toma M. Marinov

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FIELD EMISSION AND SCATTERING FROM CONDUCTING NANOFIBERS

Toma M. Marinov

Dissertation

Approved:

Advisor
Dr. S. I. Hariharan

Committee Member
Dr. G. Young

Committee Member
Dr. N. Ida

Committee Member
Dr. C. Clemons

Committee Member
Dr. K. Kreider

Committee Member
Dr. D. Zywicki

Accepted:

Department Chair
Dr. J. A. De Abreu

Dean of the College
Dr. G. K. Haritos

Dean of the Graduate School
Dr. G. R. Newkome

Date

ABSTRACT

Field emission from conducting nanofibers has a significant importance due to its possible application in electronics like flat panel displays, x-ray machines, sensors, etc. The standard theoretical model describing field emission is the Fowler-Nordheim model, which is valid for bulk material, constant applied electric field and $O^{\circ}K$. A more general theoretical model is required in the realistic cases of arbitrary electromagnetic fields and arbitrary but finite temperature.

This work presents an asymptotic procedure for calculating field emission from nanofibers of finite length for static and dynamic fields at arbitrary finite temperature. It investigates the behavior of a nanofiber in the presence of electrostatic and EM fields. The resultant field potentials outside the system are obtained by employing the slender-body approximation ([1], [2], [3]) . The total external potential is used in conjunction with the the Wentzel-Krammers-Brillouin approximation [4] to estimate the tunneling probability of the electrons in the fiber due the total external field. Unlike the standard Fowler-Nordheim method [5], the current density of the field emission is obtained by using quantum wire density of states.

In addition, this work investigates radiative and scattering properties of conducting nanofibers for the purpose of nanoantenna applications . The results for the

distributions of the induced currents are compared to the results from the solution of Hallen's integral equation [6] and the corresponding radiation patterns are compared. The results are extended for the case of a broadside uniform array of N aligned fibers.

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CHAPTER I

INTRODUCTION

The advent of modern physics in the last century lead to better understanding of solid state phenomena both qualitatively and quantitatively. Quantum mechanics and solid state physics gave satisfactory explanation of the properties and behavior of semiconductors, which lead to the invention of the point contact transistor by John Bardeen and Walter Brittain in 1947 and the junction transistor by William Shockley four years later. The first integrated circuit (IC) and the planar technology followed soon. The subsequent development of the planar technology lead to decreasing the size of the integral elements and thus increasing the scale of integration, i.e. the number of transistors per IC.

Currently electronics is in the realm of the ultra large scale of integration and the size of the integral elements is on the nanoscale. This is the driving force behind the recent vigorous research in the area of nanoscale systems.

One of the possible areas of nanoscale research is the investigation of the properties of conductive nanofibers/ nanowires, which in this work we refer to simply as nanofibers. Nanofibers have the potential of being used in several different applications: they can serve as parts of integral elements for ICs or devices [7], [8], [9], [10], [11], [12], nanoantennas [13], [14], [15], [16], waveguides [17], photonic crystals [18], field emission electron guns in electron microscopes and x-ray machines [19], [20], nanolithography [21] or in flat panel displays [22], [23], [24], etc. This motivates the investigation of the behavior of nanofibers in the presence of electromagnetic fields. Such problems include scattering, radiative properties, electronic transport, field emission, etc. Even though these problems are interlinked and it is impossible to treat them separately, they represent different physical phenomena and the investigation of each of them requires employment of several different theoretical constructs. Another difficulty comes from the fact that the size of the systems in question is at a scale where classical phenomena are less present and quantum effects become more pronounced. This leads to an inherent ambiguity about choosing the correct model for description of the underlying physics of the system. Although some experimental data exists, we are only beginning to develop qualitative picture of the processes on nanoscale. In this work we focus on field emission from conductive nanofibers/nanowires, as well as some of their radiative properties.

Field emission is emission of electrons from a material due to an external field. The emitting body is called cathode. In order to observe field emission from cathodes with macroscopic dimensions, one needs strong electric fields. However, for small distances and small cathode dimensions, field emission is possible for small applied voltages.

Field emission is entirely a quantum mechanical effect, the emission current is a result of a quantum tunneling of electrons through a potential barrier. The first emission from metal was observed by Wood [25] in 1897. Schottky [26] made the first attempt to theoretically describe field emission in 1923 based on classical physics. His model failed to match the experimental results.

The next attempt belongs to Fowler and Nordheim [5] in 1928. They employed quantum mechanical concepts to describe the electrons tunneling through surface potential barrier. The original calculation employed a triangular potential barrier due to a constant external electric field from a semi-infinite piece of metal at $0°K$. Later, calculations were extended by other authors to square barriers [27], trapezoidal barriers [28], repulsive δ-function barrier [29], parabolic potential [30], etc.

Field emission (FE) from nanostructures has recently attracted attention due to its possible applications. The first reported FE from a nanofiber (carbon nanotube) was in 1995 by [31]. Subsequently, more experimental results were reported for FE from semiconductor and metallic nanowires [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], as well as some applications [44], [45], [46], [47].

It is our goal in this work to create a consistent and realistic mathematical model of field emission from nanofibers by going beyond the standard Fowler-Nordheim model [5]. As an input for the field emission calculation we need an analytic expression for the potential energy of the field. This requires the investigation of the electromagnetic scattering from a nanofiber of finite length for the usual static as well as the dynamic case. Also, we use the dynamic scattering results for the purpose of investigating the radiative properties of the nanofiber (or an array in the general case) due to the induced eddy currents by obtaining the current distribution in the fiber and obtaining the radiation pattern for a single/system of fibers. The results are compared to the corresponding results from the solution of Hallen's integral equation.

This work is structured in the following way: In Chapter II we discuss the foundations of the FE theory. We introduce a model for the calculation of FE current and compare it to the standard Fowler-Nordheim model. We also motivate the necessity for solving the EM scattering problem in Chapter III.

In Chapter III we investigate the EM scattering properties from nanofibers. We consider two cases - static and dynamic. Since the systems under investigation have a large aspect ratio (length to width), we employ the slender body approximation [1], [2]. For the static case we consider static electric field and solve for the resultant potential. The result is used to evaluate the potential energy, which is used as an input for the field emission calculations. For the dynamic case we consider both axial and transverse incidence of the electric field. The results from the axial case are used in a conjunction with the Lorentz-Lorenz gauge in order to determine the potential energy of the dynamic field, which just as in the static case is used as an input for the field emission calculations. The results from the case of transverse incidence are generalized in Chapter IV. Here we derive an expression allowing us to determine the current distribution in the fiber and the resulting radiation pattern of the fiber as a scatterer. Furthermore, we generalize the results for N aligned fibers.

In Chapter V we present the numerical results based on the work in Chapters II, III and IV. First we evaluate the field emission current for **Ni** nanofibers in the case of static incident field for four different values of ϵ. We compare our results to the results from the Fowler-Nordheim model as well as recent experimental results [48]. We also evaluate the field emission current due to axial dynamic field for three different frequencies and four different values of ϵ. Next, we present the current distributions and radiation patterns predicted by our model and compare them with the current distributions and radiation patterns resulting from numerical evaluation of the Hallen's delta gap case [49], and thus essentially comparing the properties of the fiber as a scatterer and radiator. Finally, we extend the results to the case of an array of N aligned fibers and investigate the array radiation pattern for several different numbers of elements, several frequencies and several different values of ϵ.

Additionally, there are two appendices: Appendix A and Appendix B. Appendix A contains intermediate results in the derivation of the electric and magnetic fields due to an electric and a magnetic dipole in the dynamic case in Chapter III. Appendix B contains the derivation of the asymptotic expansions of the integral operators occurring in both the static and the dynamic case in Chapter III.

CHAPTER II

FIELD EMISSION

Consider an electron in a metal in the presence of an external electric field. The work needed to overcome the potential barrier on the metal vacuum interface is $A = e\varphi$ (see Fig.(2.1)), where φ is the work function, which is specific for the material and e is the electric charge of the electron. According to classical physics, in order to leave the surface of the cathode, the energy of the electron has to be higher than the height of the potential barrier, i.e. in the classical case the barrier is completely nontransparent for electrons with energies lower than its height. Quantum mechanical laws, however, allow the particle to tunnel through the barrier instead of overcoming it. This is known as the tunneling effect, which is responsible for emission of electrons from the metal/vacuum interface.

The transparency of the barrier T, which is the probability of an electron with a certain energy to tunnel through a barrier with a specific height and shape is obtained by invoking 1-D WKB approximation. Here we are following closely the derivation in [4].

Consider an electron with energy W moving in constant potential V and with momentum given by:

$$p = \sqrt{2m_e(W - V)}. \tag{2.1}$$

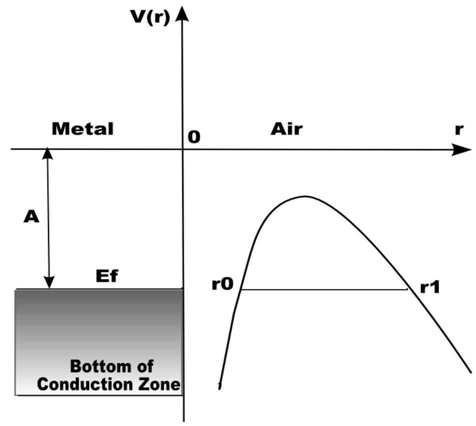

Figure 2.1: Energy diagram of the metal/air interface

The one dimensional Schrödinger equation in energy representation is:

$$\left[\frac{d^2}{dr^2} + \frac{1}{\hbar^2}p^2(r)\right]\Psi(r) = 0. \tag{2.2}$$

Here m_e is its mass, $\psi(r)$ is the wave function, r is the dimensional spatial coordinate, $\hbar$ is the Planck's constant. Therefore, ψ will propagate in space with a constant wavelength $\lambda = 2\pi\hbar/p$ and the phase shift per unit length $p/\hbar$ is constant, too.

Our next step is to assume that the potential V is not a constant anymore, but instead that it is slowly varying. Then ψ will behave like a plane wave locally, however the wavelength will be a function of the position, i.e.:

$$\lambda(r) = \frac{2\pi\hbar}{p(r)} = \frac{2\pi\hbar}{\sqrt{2m_e(W - V(r))}} \tag{2.3}$$

and

$$\left|\frac{d\lambda}{dr}\right| \ll 1. \tag{2.4}$$

If we write the solution of the Schrödinger equation in the form

$$\Psi(r) = \exp\left[\frac{is(r)}{\hbar}\right], \tag{2.5}$$

and insert this into the Schrödinger equation, we will obtain

$$-(s^{'})^2 + i\hbar s^{''} + p^2(r) = 0. \tag{2.6}$$

If we expand s in series in terms of $\hbar$,

$$s = s_0 + \hbar s_1 + \hbar^2 s_2 + ... \tag{2.7}$$

and make this substitution for s in eq.(2.2), we find:

$$-(s_0^{'})^2 + p^2(r) + (is_0^{''} - 2s_1^{'}s_0^{'})\hbar + O(\hbar^2) = 0. \tag{2.8}$$

Since $\hbar \sim 10^{-34} J.s$, the WKB approximation is widely valid, because any potential can be considered slowly varying in this limit. Keeping only the leading term we obtain:

$$s_0^{'} = \pm p(r), \tag{2.9}$$

or

$$s_0(r) = \pm \int_{r_0}^{r} p(r^{'})dr^{'}. \tag{2.10}$$

Therefore the leading order in the WKB approximation will result in

$$\Psi(r) = \Psi(r_0)\exp\left[\pm\frac{i}{\hbar}\int_{r_0}^{r} p(r^{'})dr^{'}\right]. \tag{2.11}$$

The transmission probability given by

$$T = \langle \Psi^*, \Psi \rangle|_{r=r_1} = |\Psi(r_0)|^2 \exp\left[-\frac{2}{\hbar} \int_{r_0}^{r_1} \sqrt{2m_e(V(r') - W)} dr^{'} \right] \tag{2.12}$$

and since $|\Psi(r_0)|^2 = 1$, for a slowly varying potential V, $T(W)$ becomes

$$T(W) = \exp\left[-\frac{2}{\hbar} \int_{r_0}^{r_1} \sqrt{2m_e(V(r') - W)} dr^{'} \right]. \tag{2.13}$$

where r_0 and r_1 represent the classical turning points, i.e. where $V(r) = W$.

The total current density J will be

$$J = -2e\sqrt{\frac{2E_f}{m_e}} \int_{-\infty}^{\infty} D(W)f(W)T(W)dW, \tag{2.14}$$

where E_f is the Fermi level of the material, $D(W)$ is the electron density of states. Also, the Fermi distribution function $f(W)$ is given by:

$$f(W) = \frac{1}{e^{\left(\frac{W-E_f}{k_B T_k}\right)} + 1}, \tag{2.15}$$

where k_B is the Boltzmann's constant, and T_k is the temperature in Kelvin.

In the case of the standard Fowler-Nordheim model it is assumed [5], [50], [51] that the:

- material has a free-electron structure (i.e free electron model is valid)
- electrons are in thermal equilibrium
- temperature in the material is $0°K$
- uniform constant electric field is uniform above the emitting surface (triangular potential barrier)

- tunneling probability can be calculated by the WKB approximation

The assumption for the constant electric field allows direct integration in eq.(2.13) and a simple analytic expression for the tunneling probability. The $0°K$ assumption and the free electron density of states simplify the expressions for $f(W)$ and $D(W)$, respectively [5]. As a result, the current density (eq.(2.14)) predicted by the Fowler-Nordheim model becomes a simple analytic expression.

Clearly, the Fowler-Nordheim theory is a crude model with built-in assumptions which are hardly valid in the case of field emission from nanofibers. A realistic field emission model for nanofibers should take into account several important factors. Generally, in applications, field emission occurs at room temperature and so a temperature dependent model is desired. The temperature dependence is reflected in the Fermi distribution function (eq.(2.15)).

Also, the electron density of states (DOS) in a nanofiber differs significantly from the free electron model, which is best suited for bulk material. This is due to the fact that the transverse size of the nanofiber is on the nanoscale and therefore the energy quantization rules forbid anisotropy unlike the case of bulk material. This means that a nanofiber exhibits the DOS of a 1D quantum system. Thus the DOS of a quantum wire [52], [53], [54] is:

$$D(W) = \frac{k}{2\pi} \left(\frac{2m}{\hbar^2} \right)^{1/2} W^{-1/2}. \tag{2.16}$$

In addition to these considerations, the model should give predictions for a broad class of applied fields (static and dynamic) and potential barriers which take into account both the incident and the scattered fields.

To our knowledge, currently there is no theoretical model involving dynamic fields, since there are several significant difficulties to be overcome. One of them is that in the dynamic case, the potential cannot be obtained directly, one first has to solve for the electric field and subsequently, relate the electric field and the potential through some kind of gauge conditions. Trying to solve for the electric field, however, might present a problem by itself, since there are singularities at both ends of the fiber. In this work we avoid the singularity problem by employing the slender body approximation in both the static and the dynamic cases. Additionally, in the dynamic case, we use the Lorentz-Lorenz gauge in order to relate the results for the electric field to the field potential.

Thus for a given incident field, we should be able to obtain an analytic expression for the potential due to the external field, which defines the shape and height of the potential barrier in eq.(2.13). For a given electron energy W the current density will depend on the transition probability $T(W)$, the number of electron sates $D(W)$ available at that energy and the probability $F(W)$ of these states being occupied.

In this case, unlike the Fowler-Nordheim case the total current density J (eq.(2.14)) cannot be evaluated explicitly. Here we propose a numerical evaluation technique of eq.(2.14) based on the following observations. Even though eq.(2.14) defines an improper integral, in reality both of the integration limits are finite. Detailed

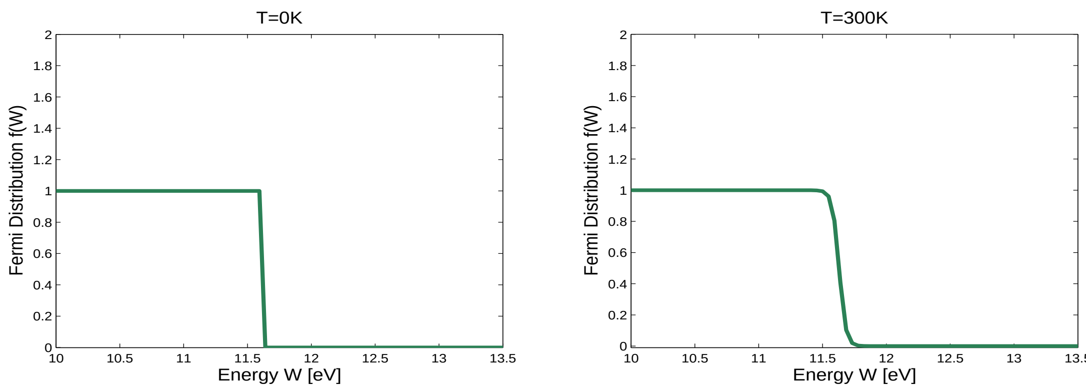

Figure 2.2: Fermi distribution $f(W)$ at 0°K and 300°K for $E_f = 11.83$ eV

analysis of the integrand shows that it is nonzero only for a small range of energies. This behavior is dictated by both $F(W)$ and $T(W)$.

At $0°K$, $F(W) = 1$ for $W \leq E_f$ and zero for $W > E_f$. At room temperature $(300°K)$, $F(W) \to 0$ rapidly for energies slightly above E_f (see Fig.(2.2)).

The physical interpretation of this fact is that there are no available electrons with energies higher than the Fermi energy, and thus essentially replacing the upper integration limit in eq.(2.14) with $E_f + \triangle E_f$, where $\triangle E_f$ can be defined in such a way that $f(E_f + \triangle E_f) = 1/2$ or any other reasonable value.

On the other hand $T(W)$ depends on the area enclosed by the potential barrier and W (see fig.2.1). For sufficiently sharp and thin barriers and appropriate W,

$$\int_{r_0}^{r_1} \sqrt{2m_e(V(r') - W)}dr' \sim \hbar \qquad \text{and} \qquad T(W) \neq 0, \tag{2.17}$$

and this truncates the lower integration limit. It is important noting that for a given system at given temperature, $T(W)$ would vary drastically for different incident fields. That implies that one and the same system subject to different external field should produce drastically different field emission currents. Therefore, the field emission calculation can be split in two separate problems:

1. Scattering: Given a conducting fiber and and incident static or dynamic electric field, the goal is to determine the scattered field as a function of type and magnitude of the incident field as well as the geometry and the physical properties of the material of the fiber.

2. Electron Tunneling Probability and Current Density: Given a fiber with known physical properties and geometry interacting with a given incident field, the goal is to calculate the tunneling probability and the current density of the field emission.

CHAPTER III

SCATTERING

In this chapter we investigate the scattering properties of a finite dimensional fiber due to both static and dynamic fields. In a realistic case a nanofiber is rotationally symmetric and has a large aspect ratio ($\frac{\max r}{L} << 1$). This is why we introduce the slender body approximation models [1], [2], [55], [56], [57], [58], [3] and modify them for our goals. The slender body approximation is applied to bodies of revolution with length L. A small parameter ϵ is defined through $\epsilon = \frac{\max r}{L} << 1$. The system is nondimensionalized by L, so that the new length is 1 and the maximum radius is ϵ. This method is applicable for a large class of geometries, as long as there are no sharp tips at both ends. However, for the purpose of modeling a realistic nanofiber geometry, we can choose a cylinder with spherical caps. For simplicity, but without loss of generality, it can be assumed that the axis of revolution is the z-axis. Thus the surface Γ of a body of revolution, given by $r = \epsilon\sqrt{S(z)}$. For the fiber geometry we choose a cylinder with unit length and spherical caps with radius ϵ (see Fig.(3.1)). Here we investigate the properties of the fiber as a scatterer in both the static and dynamic cases. We derive results using the slender body approximation technique for a general slender body geometry and then apply them for our geometry in particular (fig.3.1).

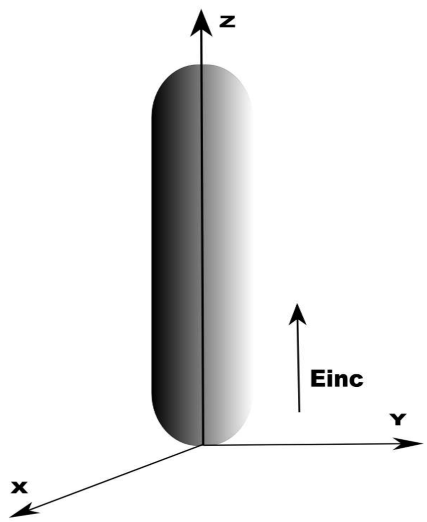

Figure 3.1: Fiber geometry

In the static case we represent the potential as a linear superposition of point sources with unknown distribution $f(z, \epsilon)$ along the rotational axis of the body (z-axis)[2]. The unknown distribution f must satisfy a linear integral equation. Our goal is to obtain a uniform asymptotic expansion of the integral operator and thus obtain a uniform solution for f.

In the dynamic case point charge representation is not sufficient. However [59] allows us to represent the scattered electric field as a linear superposition of both magnetic and electric dipoles with unknown densities $g(z, \epsilon)$ and $h(z, \epsilon)$ along the z-axis. One of the BC requires vanishing tangential components of the total electric field on the surface of the system which leads to a system of two integral equations for the unknown g and h. Again, our goal is to obtain uniform solutions for them by obtaining uniform asymptotic expansions of the integral operators involved.

The proposed technique is based on the assumption that the charge/dipole densities f, g and h also have an asymptotic series in ϵ with expansion coefficients which are smooth functions of z. In both cases (static and dynamic), the integral operators resulting from the BC are of the form

$$\int_{\alpha(\epsilon)}^{\alpha(\epsilon)} G_i(\xi, z) F(\xi, \epsilon) d\xi, \tag{3.1}$$

where $G_i(r, z)$ represent the operator kernels and F represent the charge/dipole densities f, g and h.

In both the static and dynamic cases we apply a similar procedure. First, the integral is represented as a sum of two new integrals by introducing an intermediate integration limit. Each of the resulting integrals undergoes a change of variables, expansion of the integrand and term regrouping. The goal is to represent each of the resultant integrals as asymptotic sum of linear operators [1], [2], [55], [56], [57], [58], [3], [60]. This could be achieved through additional expansions, regrouping or integration by parts.

The expansion resulting from such a procedure is not automatically guaranteed to be uniform. Clearly, the uniformity and nonsingularity of the asymptotic expansions of the integrals above would depend on the unknown integration limits $\alpha(\epsilon)$ and $\beta(\epsilon)$. In the limit $\epsilon \to 0$, i.e. for an infinitely long body of revolution, a uniform expansion does exist and there is no singularity. However, in the case of finite L, analysis shows that trouble occurs when $\alpha = 0$ and $\beta = 1$. These two observations suggest that α and β should be of the form

$$\alpha = \alpha_1 \epsilon^2 + \alpha_2 \epsilon^4 + ... = \sum_{n=1}^{\infty} \alpha_n \epsilon^{2n} \tag{3.2}$$

and

$$\beta = 1 - \beta_1 \epsilon^2 - \beta_2 \epsilon^4 - ... = 1 - \sum_{n=1}^{\infty} \beta_n \epsilon^{2n} \tag{3.3}$$

Part of the solution is determining the unknown integration limits $\alpha(\epsilon)$ and $\beta(\epsilon)$ given by eq.(3.2) and eq.(3.3). Choosing particular α_i's and β_i's will allow us to have an asymptotic expansion which is uniform even close to the end of the fiber. The resulting uniform expansion allows us to solve for the unknown f, g and h up to the desired correctional order and thus obtain an analytic result for the total electrostatic potential (static case) and the total electric field (dynamic case) outside the fiber. An important fact is that $\alpha(\epsilon)$ and $\beta(\epsilon)$ depend entirely on the geometry and they are easily obtained for a given $S(z)$. This would be convenient even when one is pursuing a numerical solution to the problems described above. In that case the unknown densities f, g and h can be obtained numerically, however knowledge of α and β is still necessary in order to avoid issues with singularity. Even though possible, such an approach is not justified, since the analytic results we obtain are derived for any general slender body geometry.

3.1 STATIC CASE

Consider a finite dimensional nanowire with a surface Γ and incident electric field with potential Φ^0. The electrostatic potentials inside and outside Γ must satisfy

Laplace's equations, i.e.:

$$\nabla^2 \Phi^0 = 0 \quad \text{in} \quad \mathbf{R^3} \setminus \Gamma, \tag{3.4}$$

$$\nabla^2 U = 0 \quad \text{in} \quad \mathbf{R^3} \setminus \Gamma. \tag{3.5}$$

Here Γ is a body of revolution, given by $r = \epsilon\sqrt{S(z)}$, where $S(z)$ is a function chosen appropriately for a given geometry and ϵ is the ratio of the largest diameter to the length.

The requirement for continuity of the electrostatic potential on the boundary implies that on Γ, U and Φ^0 must satisfy the following BC:

$$C = U + \Phi^0, \tag{3.6}$$

where C is constant. Also U must vanish at infinity, i.e. $U \to 0$ as $r \to \infty$.

U must be a solution to eq.(3.5) subject to eq.(3.6). However, analytical solutions exist only for the simplest geometries. Instead of attempting to solve eq.(3.5) directly, we take a different approach: we represent U as potential due to unknown charge distribution along the z-axis, i.e we seek $U(z,r)$ in the form ([2]):

$$U(z,r) = \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{f(\xi,\epsilon)}{\sqrt{(z-\xi)^2 + r^2}} d\xi, \tag{3.7}$$

where $0 < \alpha < \beta < 1$ are to be determined as part of the solution, as well as the unknown function $f(\xi,\epsilon)$. Here α and β are the same as in eq.(3.2) and eq.(3.3).

On the boundary,

$$U(z,r)\Big|_{r=\epsilon\sqrt{S(z)}} = \int_{\alpha}^{\beta} \frac{f(\xi,\epsilon)}{\sqrt{(z-\xi)^2 + \epsilon^2 S(z)}} d\xi. \tag{3.8}$$

Also, the total charge in the system is zero, i.e.

$$\int_{\alpha}^{\beta} f(\xi,\epsilon)d\xi = 0. \tag{3.9}$$

Thus the BC (3.6) becomes a linear integral equation. Our goal is to obtain a uniform asymptotic expansion of both sides of eq.(3.6), and hence a uniform asymptotic expansion of the integral representation of U in terms of powers of ϵ. We want the expansion terms of $f(z,\epsilon)$ with respect to ϵ to be smooth and so each of them to be analytic for $z \in [0,1]$ ([1], [2]). This leads to realizing the conditions necessary to determine the coefficients in α and β.

Consider the integral

$$I_s(z,\epsilon,F) = \int_{\alpha}^{\beta} \frac{F(\xi)}{\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}d\xi. \tag{3.10}$$

U will be exactly of the same form as $I_s(z,\epsilon,F)$, provided $f(\xi,\epsilon) = F(\xi)$. Usually asymptotic problems in the case of slender body geometry involve both ϵ^n and $\log\epsilon$ terms. Thus we expect $I_s(z,\epsilon,F)$ to have an asymptotic expansion of the form:

$$I_s(z,\epsilon,F) = \sum_{m=1}^{\infty}\sum_{j=0}^{\infty} \frac{\epsilon^{2j}}{(\log\epsilon)^m}(L_j + G_j\log\epsilon)F(z). \tag{3.11}$$

To show that, let

$$I_s(z,\epsilon,F) = \int_{\alpha}^{z} \frac{F(\xi)}{\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}d\xi + \int_{z}^{\beta} \frac{F(\xi)}{\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}d\xi. \tag{3.12}$$

Make the substitution $v = z-\xi$ in the first integral and $v = \xi - z$ in the second to find:

$$I_s(z,\epsilon,F) = \int_{\alpha}^{z-\alpha} \frac{F(z-v)}{\sqrt{v^2+\epsilon^2 S(z)}}dv + \int_{0}^{\beta-z} \frac{F(z+v)}{\sqrt{v^2+\epsilon^2 S(z)}}dv$$

$$= I_s^{-}(z,\epsilon,F) + I_s^{+}(z,\epsilon,F). \tag{3.13}$$

For $I^+(z,\epsilon,F)$ we have

$$I_s^+(z,\epsilon,F) = F(z)\int_0^{\beta-z}\frac{dv}{\sqrt{v^2+\epsilon^2 S(z)}}dv + \int_0^{\beta-z}\frac{F(z+v)-F(z)}{\sqrt{v^2+\epsilon^2 S(z)}}dv$$
$$= F(z)\int_0^{\beta-z}\frac{dv}{\sqrt{v^2+\epsilon^2 S(z)}}dv + \int_0^{\beta-z}\frac{F(z+v)-F(z)}{v}dv$$
$$+ \int_0^{\beta-z}\frac{F(z+v)-F(z)}{v}\left(\left(1+\frac{\epsilon^2 S(z)}{v^2}\right)^{-1/2}-1\right)dv = I_{1s}^+ + I_{2s}^+ + I_{3s}^+.$$

Similarly for $I^-(z,\epsilon,F)$

$$I_s^-(z,\epsilon,F) = F(z)\int_0^{z-\alpha}\frac{dv}{\sqrt{v^2+\epsilon^2 S(z)}}dv + \int_0^{z-\alpha}\frac{F(z-v)-F(z)}{\sqrt{v^2+\epsilon^2 S(z)}}dv$$
$$= F(z)\int_0^{z-\alpha}\frac{dv}{\sqrt{v^2+\epsilon^2 S(z)}}dv + \int_0^{z-\alpha}\frac{F(z-v)-F(z)}{v}dv$$
$$+ \int_0^{z-\alpha}\frac{F(z-v)-F(z)}{v}\left(\left(1+\frac{\epsilon^2 S(z)}{v^2}\right)^{-1/2}-1\right)dv = I_{1s}^- + I_{2s}^- + I_{3s}^-.$$

For each of the integrals I_{1s} , I_{2s}, I_{3s} as well as Φ^0 we need to find a uniform asymptotic expansion (see Appendix B: Static case).

Combining the expansions from eq.(B.1-B.21) above with eq.(3.11) results in

$$G_0F(z) = -F(z),$$
$$G_1F(z) = S(z)\frac{F''(z)}{4},$$
$$L_0F(z) = \int_0^{1-z}\frac{F(z+v)-F(z)}{v}dv + \int_0^{z}\frac{F(z-v)-F(z)}{v}dv + F(z)\log\left(\frac{4z(1-z)}{S(z)}\right).$$

:

Assume $f(z,\epsilon)$ is of the form

$$f(z,\epsilon) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\epsilon^n}{(\log\epsilon)^m}f_{nm}(z).$$

The BC (eq.(3.6)) becomes

$$\sum_{j=0}^{\infty} \Phi_j(z)\epsilon^{2j} S^j(z) - C = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{j=0}^{\infty} \frac{\epsilon^{2n+2j}}{(\log\epsilon)^m}(L_j + G_j \log\epsilon) f_{nm}(z),$$

where

$$\Phi_j(z) = \frac{1}{j!}\left[\left(\frac{\partial}{\partial r^2}\right)^j \Phi^0(z, r^2)\right]_{r^2=0}.$$

This can be rewritten as

$$\sum_{j=0}^{\infty} \Phi_j(z)\epsilon^{2j} S^j(z) - C = \sum_{j=0}^{\infty}\Bigg[\Bigg(\frac{\epsilon^{2j}}{\log\epsilon}(L_j + G_j\log\epsilon) f_{01}(z)$$
$$+ \frac{\epsilon^{2j+2}}{\log\epsilon}(L_j + G_j\log\epsilon) f_{11}(z) + \frac{\epsilon^{2j+4}}{\log\epsilon}(L_j + G_j\log\epsilon) f_{21}(z) + \ldots\ldots\ldots\Bigg)$$
$$+ \Bigg(\frac{\epsilon^{2j}}{(\log\epsilon)^2}(L_j + G_j\log\epsilon) f_{02}(z) + \frac{\epsilon^{2j+2}}{(\log\epsilon)^2}(L_j + G_j\log\epsilon) f_{12}(z)$$
$$+ \frac{\epsilon^{2j+4}}{(\log\epsilon)^2}(L_j + G_j\log\epsilon) f_{22}(z) + \ldots\ldots\Bigg) + \ldots\ldots\Bigg].$$

Hence

$$\Phi_0 - b = G_0 f_{01} \qquad \text{(terms } O(\epsilon^0))$$

$$\Phi_1 S = G_1 f_{01} + G_0 f_{11} \qquad \text{terms } O(\epsilon^2)$$

$$\Phi_2 S^2 = G_2 f_{01} + G_1 f_{11} + G_2 f_{21} \qquad \text{terms } O(\epsilon^4)$$

$$\vdots$$

$$0 = L_0 f_{01} + G_0 f_{02} \qquad \text{terms } O(\epsilon^0/\log\epsilon)$$

$$0 = L_1 f_{01} + L_0 f_{11} + G_1 f_{02} + G_0 f_{12} \qquad \text{terms } O(\epsilon^2/\log\epsilon)$$

$$0 = L_2 f_{01} + L_1 f_{11} + L_0 f_{21} + G_2 f_{02} + G_1 f_{12} + G_0 f_{22} \qquad \text{terms } O(\epsilon^4/\log\epsilon)$$

$$\vdots$$

Employing the fact that $G_0F(z) = -F(z)$, we have that

$$f_{n,m+1} = -(\Phi_n(z)S^n(z) - C\delta_{n0})\delta_{m0} + \sum_{p=0}^{n} L_{n-p}f_{pm} + \sum_{p=0}^{n-1} G_{n-p}f_{p,m+1} \quad n \geq 0, m \geq 0$$

and assuming that C is of the form

$$C = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\epsilon^n}{(\log\epsilon)^m} C_{nm},$$

$$f_{n,m+1} = \Phi_n(z)S^n(z)\delta_{m0} + C_{nm} + \sum_{p=0}^{n} L_{n-p}f_{pm} + \sum_{p=0}^{n-1} G_{n-p}f_{p,m+1} \quad n \geq 0, m \geq 0.$$

The second BC can be written in the form

$$\int_{\alpha}^{\beta} f(\xi,\epsilon)d\xi = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\epsilon^n}{(\log\epsilon)^m}\int_{\alpha}^{\beta} f_{nm}(\xi)d\xi = 0.$$

If we expand $\int_{\alpha}^{\beta} f_{nm}(\xi)d\xi$ in a Taylor series,

$$\int_{\alpha}^{\beta} f_{nm}(\xi)d\xi = \sum_{j=0}^{\infty} \frac{\epsilon^{2j}}{j!}\left(\frac{d^j}{d\epsilon^{2j}}\int_{\alpha}^{\beta} f_{nm}(\xi)d\xi\Big|_{\epsilon=0}\right), \tag{3.14}$$

and

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{j=0}^{\infty} \frac{\epsilon^{2n+2j}}{j!(\log\epsilon)^m}\left(\frac{d^j}{d\epsilon^{2j}}\int_{\alpha}^{\beta} f_{nm}(\xi)d\xi\right)\Big|_{\epsilon=0} = 0.$$

Suppose m is fixed. Then

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{\epsilon^{2n+2j}}{j!}\left(\frac{d^j}{d\epsilon^{2j}}\int_{\alpha}^{\beta} f_{nm}(\xi)d\xi\right)\Big|_{\epsilon=0} = 0,$$

and thus by collecting like terms in ϵ^{2n},

$$\int_{0}^{1} f_{nm}(\xi)d\xi = -\sum_{j=1}^{n}\left(\frac{d^j}{d\epsilon^{2j}}\int_{\alpha}^{\beta} f_{n-jm}(\xi)d\xi\right)\Big|_{\epsilon=0}.$$

By integrating both sides of $f_{n,m+1} = \Phi_n(z)S^n(z)\delta_{m0} + C_{nm} + \sum_{p=0}^{n} L_{n-p}f_{pm} + \sum_{p=0}^{n-1} G_{n-p}f_{p,m+1}$ we get

$$\int_0^1 f_{n,m+1}(\xi)d\xi = \int_0^1 \left[\Phi_n(z)S^n(z)\delta_{m0} + C_{nm} + \sum_{p=0}^{n} L_{n-p}f_{pm} + \sum_{p=0}^{n-1} G_{n-p}f_{p,m+1}\right] d\xi. \tag{3.15}$$

Combining this result with the expression from eq.(3.14) yields

$$C_{nm} = \int_0^1 \left[\Phi_n(z)S^n(z)\delta_{m0} - \sum_{p=0}^{n} L_{n-p}f_{pm} - \sum_{p=0}^{n-1} G_{n-p}f_{p,m+1}\right] d\xi$$
$$-\sum_{j=1}^{n}\left(\frac{d^j}{d\epsilon^{2j}}\int_\alpha^\beta f_{n-j,m+1}(\xi)d\xi\right)\Big|_{\epsilon=0},$$

which after substitution in eq.(3.15) becomes

$$f_{n,m+1} = \int_0^1 \left[\Phi_n(z)S^n(z)\delta_{m0} - \sum_{p=0}^{n} L_{n-p}f_{pm} - \sum_{p=0}^{n-1} G_{n-p}f_{p,m+1}\right] d\xi$$
$$-\sum_{j=1}^{n}\left(\frac{d^j}{d\epsilon^{2j}}\int_\alpha^\beta f_{n-j,m+1}(\xi)d\xi\right)\Big|_{\epsilon=0} - \Phi_n(z)S^n(z)\delta_{m0} + \sum_{p=0}^{n} L_{n-p}f_{pm} + \sum_{p=0}^{n-1} G_{n-p}f_{p,m+1}.$$

For the system's geometry we consider a cylinder with unit length and spherical caps with radius ϵ (see fig.(3.1)), which yields $S(z)$ to be:

$$\begin{aligned} r^2 &= 2\epsilon z - z^2 = \epsilon^2 S_1(z) && \text{for } 0 \le z < \epsilon \\ r^2 &= \epsilon^2 && \text{for } \epsilon \le z \le 1-\epsilon \\ r^2 &= 2z(1-\epsilon) - z^2 + 2\epsilon - 1 = \epsilon^2 S_2(z). && \text{for } 1-\epsilon < z \le 1. \end{aligned} \tag{3.16}$$

Finally, we choose

$$\Phi(z, r^2, \epsilon) = Mz \tag{3.17}$$

and so

$$f_{01} = \int_0^1 \Phi_0(\xi) d\xi - \Phi_0(z) = M\left(\frac{1}{2} - z\right),$$
$$f_{02} = M\left(2z - 1 + \frac{1-2z}{2}\log\frac{4z(1-z)}{S(z)} - \int_0^1 \frac{1-2\xi}{2}\log\frac{4\xi(1-\xi)}{S(\xi)}d\xi\right),$$
$$f_{11} = M\frac{\alpha_1 + \beta_1}{2} = M\frac{c_1 - d_1}{8}. \tag{3.18}$$

Using the results for the expansion coefficients c_i and d_i from Section 3.4, we obtain

$$c_1 = S_1'(0) = \frac{2}{\epsilon} \qquad c_2 = \frac{S_1''(0)}{2} = -\frac{2}{\epsilon^2} \qquad c_3 = 0$$
$$d_1 = -S_2'(1) = 2/\epsilon \qquad d_2 = \frac{S_2''(1)}{2} = -\frac{2}{\epsilon^2} \qquad d_3 = 0$$

and since

$$\alpha = c_1\left(\frac{\epsilon}{2}\right)^2 - c_1 c_2\left(\frac{\epsilon}{2}\right)^4 + c_1(c_1 c_3 + 2c_2^2)\left(\frac{\epsilon}{2}\right)^6 + O(\epsilon^8),$$
$$\beta = 1 - d_1\left(\frac{\epsilon}{2}\right)^2 + d_1 d_2\left(\frac{\epsilon}{2}\right)^4 - d_1(d_1 d_3 + 2d_2^2)\left(\frac{\epsilon}{2}\right)^6 + O(\epsilon^8),$$

for our particular choice of geometry

$$\alpha = \epsilon + O(\epsilon^8),$$
$$\beta = 1 - \epsilon + O(\epsilon^8). \tag{3.19}$$

Also, for the given geometry

$$\int_0^1 \frac{1-2\xi}{2}\log\frac{4\xi(1-\xi)}{S(\xi)}d\xi = 0,$$
$$f_{11} = M\frac{\alpha_1 + \beta_1}{2} = M\frac{c_1 - d_1}{8} = 0, \tag{3.20}$$

and thus

$$f(z,\epsilon) = \frac{1}{\log\epsilon}\left(\frac{1}{2} - z + \frac{1}{\log\epsilon}\left(2z - 1 + \frac{1-2z}{2}\log\frac{4z(1-z)}{S(z)}\right)\right). \tag{3.21}$$

Hence,

$$U(z,r) = C_1 \int_{\epsilon}^{1-\epsilon} \frac{1/2 - \xi + C_1\left(2\xi - 1 + (1/2-\xi)\log 4\xi(1-\xi)\right)}{\sqrt{(z-\xi)^2 + r^2}} d\xi, \tag{3.22}$$

with $C_1 = \frac{1}{\log \epsilon}$.

3.2 DYNAMIC CASE: AXIAL INCIDENCE

Now suppose we have a incident plane wave in the z-direction, i.e.

$$\mathbf{E_{inc}} = e^{i(kz-\omega t)}\mathbf{i_x}, \tag{3.23}$$

$$\mathbf{H_{inc}} = \frac{1}{\eta} e^{i(kz-\omega t)}\mathbf{i_y}. \tag{3.24}$$

Then outside Γ the electric and magnetic fields will satisfy the free space source-free Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 0, \tag{3.25a}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{3.25b}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \tag{3.25c}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}, \tag{3.25d}$$

where $\mathbf{E} = \mathbf{E_{inc}} + \mathbf{E_s}$. Taking into account that $k = \omega\sqrt{\mu\epsilon}$ as well as

$$\eta = \frac{\omega\mu}{k} = \frac{k}{\omega\epsilon}, \tag{3.26}$$

the last two equations take the form

$$\nabla \times \mathbf{E} = i\mu\omega\mathbf{H} = ik\eta\mathbf{H}, \tag{3.27a}$$

$$\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} = -i\frac{k}{\eta}\mathbf{E}, \tag{3.27b}$$

and $e^{-i\omega t}$ is suppressed. By manipulation of the last two equations, it follows that the total external electric field must satisfy the Helmholtz equation

$$\nabla^2 \mathbf{E} + \gamma^2 \mathbf{E} = 0, \tag{3.28}$$

where $\gamma = k$.

Since the tangential components of the electric field on the surface of the body vanish, the BC is:

$$\mathbf{n} \times \mathbf{E_{inc}} = -\mathbf{n} \times \mathbf{E_s}. \tag{3.29}$$

Also, the total electric field must satisfy the Sommerfeld radiation condition:

$$\lim_{R \to \infty} (\mathbf{R} \times (\nabla \times \mathbf{E}) + ikR\mathbf{E}) = 0, \tag{3.30}$$

where $\mathbf{R} = x\mathbf{i_x} + y\mathbf{i_y} + z\mathbf{i_z}$ and $R = \sqrt{x^2 + y^2 + z^2}$.

Solving the Helmholtz equation (3.28) subject to the BC (eq.(3.29)) and (eq.(3.30)) is not trivial except for simple geometries. Therefore we will take a different approach. We will represent the scattered electric field as a field generated by electric and magnetic dipoles (due to the Schelkunoff Equivalence Principle [59])with unknown distributions $g(z, \epsilon)$ and $h(z, \epsilon)$ [3] along the z-axis in the interval $[\alpha, \beta]$. Here the expressions for α and β are the same as eq.(3.2) and eq.(3.3) and the arguments for the expansion coefficients α_n and β_n are identical to the ones discussed above.

The magnetic vector potential $\mathbf{A}_e$ due to an electric dipole $\mathbf{p}$ and the vector potential $\mathbf{A}_m$ due to a magnetic dipole $\mathbf{m}$ are given by [61]

$$\mathbf{A}_e = -ik\mathbf{p}\frac{e^{ikR}}{R}, \tag{3.31}$$

$$\mathbf{A}_m = ik(\mathbf{i_R} \times \mathbf{m})\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right), \tag{3.32}$$

where $\mathbf{p}$ is the electric dipole moment, $\mathbf{m}$ is an magnetic dipole moment $\mathbf{R} = x\mathbf{i_x} + y\mathbf{i_y} + z\mathbf{i_z}$, $R = \sqrt{x^2 + y^2 + z^2}$ and $\mathbf{i_R} = \frac{\mathbf{R}}{R}$.

Since the electric field $\mathbf{E}$, the magnetic field $\mathbf{B}$ and the magnetic vector potential A are related through

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{3.33}$$

$$\mathbf{E} = \frac{i}{k}\nabla \times \mathbf{B}, \tag{3.34}$$

and since for any scalar ϕ and vector A, $\nabla \times (\phi\mathbf{A}) = \phi\nabla \times \mathbf{A} + \nabla\phi \times \mathbf{A}$,

$$\mathbf{B}_e = \nabla \times \mathbf{A}_e = -ik\nabla \times \left(\frac{e^{ikR}}{R}\mathbf{p}\right) = -ik\frac{e^{ikR}}{R}\nabla \times \mathbf{p} - ik\nabla\left(\frac{e^{ikR}}{R}\right) \times \mathbf{p}. \tag{3.35}$$

Also, since $\nabla \times \mathbf{p} = 0$, the expression for $\mathbf{B}_e$ becomes

$$\begin{aligned}\mathbf{B}_e &= -ik\nabla\left(\frac{e^{ikR}}{R}\right) \times \mathbf{p} = -ik\left(-\frac{\mathbf{i_R}}{R^2}e^{ikR} + \frac{ik\mathbf{i_R}}{R}e^{ikR}\right) \times \mathbf{p} \\ &= k^2\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right)(\mathbf{i_R} \times \mathbf{p}).\end{aligned} \tag{3.36}$$

The electric field due to a single electric dipole is given by

$$\mathbf{E}_e = \frac{i}{k}\nabla \times \mathbf{B}_e = \frac{i}{k}\nabla \times \nabla \times \mathbf{A}_e = \nabla \times \nabla \times \left(\mathbf{p}\frac{e^{ikR}}{R}\right). \tag{3.37}$$

For an arbitrary vector $\mathbf{V}$, $\nabla \times \nabla \times \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2\mathbf{V}$. Hence

$$\begin{aligned}\mathbf{E}_e &= \nabla \times \nabla \times \left(\mathbf{p}\frac{e^{ikR}}{R}\right) = \nabla\left(\nabla \cdot \left(\mathbf{p}\frac{e^{ikR}}{R}\right)\right) - \nabla^2\left(\mathbf{p}\frac{e^{ikR}}{R}\right) \\ &= \nabla\left(\frac{e^{ikR}}{R}(\nabla \cdot \mathbf{p}) + \mathbf{p} \cdot \nabla\left(\frac{e^{ikR}}{R}\right)\right) - \mathbf{p}\nabla \cdot \nabla\left(\frac{e^{ikR}}{R}\right).\end{aligned} \tag{3.38}$$

Using the identities

$$\nabla \cdot \mathbf{p} = 0, \tag{3.39}$$

$$\nabla\left(\frac{e^{ikR}}{R}\right) = ik\frac{e^{ikR}}{R}\mathbf{i_R} - \frac{e^{ikR}}{R^2}\mathbf{i_R}, \tag{3.40}$$

$$\nabla\left(\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\right) = \mathbf{i_R}e^{ikR}\left(-\frac{k^2}{R} - \frac{2ik}{R^2} + \frac{2}{R^3}\right), \tag{3.41}$$

allows us to express the electric field $\mathbf{E}_e$ (A.11) in the form

$$\begin{aligned}\mathbf{E}_e &= \nabla\left(\mathbf{p}\cdot\left(ik\frac{e^{ikR}}{R}\mathbf{i_R} - \frac{e^{ikR}}{R^2}\mathbf{i_R}\right)\right) - \mathbf{p}\nabla\cdot\left(ik\frac{e^{ikR}}{R}\mathbf{i_R} - \frac{e^{ikR}}{R^2}\mathbf{i_R}\right)\\ &= (\mathbf{p}\cdot\mathbf{i_R})\nabla\left(\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\right) + \frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\nabla(\mathbf{p}\cdot\mathbf{i_R})\\ &\quad - \mathbf{p}\left(\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\nabla\cdot\mathbf{p} + \mathbf{i_R}\cdot\nabla\left(\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\right)\right).\end{aligned} \tag{3.42}$$

Also, since

$$\nabla\cdot\mathbf{i_R} = \nabla\cdot\frac{\mathbf{R}}{R} = \frac{1}{R}\nabla\cdot\mathbf{R} + \mathbf{R}\cdot\nabla\left(\frac{1}{R}\right) = \frac{3}{R} - \frac{\mathbf{R}\cdot\mathbf{i_R}}{R} = \frac{2}{R}, \tag{3.43}$$

$$\begin{aligned}\nabla(\mathbf{p}\cdot\mathbf{i_R}) &= \nabla\left(\frac{p_1x + p_2y + p_3z}{R}\right) = \frac{\mathbf{p}}{R} + (p_1x + p_2y + p_3z)\nabla\left(\frac{1}{R}\right)\\ &= \frac{\mathbf{p}}{R} - (\mathbf{i_R}\cdot\mathbf{p})\frac{\mathbf{i_R}}{R},\end{aligned} \tag{3.44}$$

the expression for the electric field $\mathbf{E}_e$ (A.12) becomes

$$\begin{aligned}\mathbf{E}_e &= \mathbf{i_R}(\mathbf{p}\cdot\mathbf{i_R})e^{ikR}\left(-\frac{k^2}{R} - \frac{2ik}{R^2} + \frac{2}{R^3}\right) + \left(\frac{\mathbf{p}}{R} - (\mathbf{i_R}\cdot\mathbf{p})\frac{\mathbf{i_R}}{R}\right)\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\\ &= \mathbf{i_R}(\mathbf{p}\cdot\mathbf{i_R})e^{ikR}\left(-\frac{k^2}{R} - \frac{3ik}{R^2} + \frac{3}{R^3}\right) + \mathbf{p}e^{ikR}\left(\frac{k^2}{R} + \frac{ik}{R^2} - \frac{1}{R^3}\right).\end{aligned} \tag{3.45}$$

If we choose $\mathbf{p} = \mathbf{i_x}$, i.e. $\mathbf{p} = (1, 0, 0)$ the electric field is given by:

$$\mathbf{E}_e = \frac{x^2\mathbf{i_x} + xy\mathbf{i_y} + xz\mathbf{i_z}}{R^2}e^{ikR}\left(-\frac{k^2}{R} - \frac{3ik}{R^2} + \frac{3}{R^3}\right) + \mathbf{i_x}e^{ikR}\left(\frac{k^2}{R} + \frac{ik}{R^2} - \frac{1}{R^3}\right). \tag{3.46}$$

Transforming the expression for $\mathbf{E}_e$ into cylindrical coordinates using the following relations

$$x = r\cos\theta \qquad \mathbf{i_x} = \cos\theta\mathbf{i_r} - \sin\theta\mathbf{i}_\theta$$

$$y = r\sin\theta \qquad \mathbf{i_y} = \sin\theta\mathbf{i_r} + \cos\theta\mathbf{i}_\theta$$

and taking $R = (r^2 + (z-\xi)^2)^{1/2}$ leads to (A.13):

$$\begin{aligned}
\mathbf{E}_e = & \frac{(r^2\cos\theta\mathbf{i_r} + rz\cos\theta\mathbf{i_z})}{R^2} e^{ikR}\left(-\frac{k^2}{R} - \frac{3ik}{R^2} + \frac{3}{R^3}\right) \\
& + (\cos\theta\mathbf{i_r} - \sin\theta\mathbf{i}_\theta)e^{ikR}\left(\frac{k^2}{R} + \frac{ik}{R^2} - \frac{1}{R^3}\right) \\
& = k^2\mathbf{i_r}\cos\theta\frac{1}{R^3}\left(-2r^2 f_1 - (z-\xi)^2 f_2\right) + \\
& + ik^2\mathbf{i}_\theta\sin\theta\frac{1}{R}f_2 + ik^2\mathbf{i_z}\cos\theta,\frac{1}{R^3}rz(f_2 - 2f_1),
\end{aligned} \tag{3.47}$$

where $\quad f_1 = e^{ikR}\left(\frac{1}{kR} + \frac{i}{k^2R^2}\right) \quad$ and $\quad f_2 = e^{ikR}\left(i - \frac{1}{kR} - \frac{i}{k^2R^2}\right).$

For magnetic induction $\mathbf{B}_e$ we have

$$\begin{aligned}
\mathbf{B}_e = & k^2\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right)(\mathbf{i_R}\times\mathbf{p}) = \frac{x\mathbf{i_x} + y\mathbf{i_y} + z\mathbf{i_z}}{R} \times \mathbf{i_x}k^2\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right) \\
= & \left(-\frac{y}{R}\mathbf{i_z} + \frac{z}{R}\mathbf{i_y}\right)k^2\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right).
\end{aligned} \tag{3.48}$$

Therefore $\mathbf{B}_e$ becomes (A.15)

$$\mathbf{B}_e = \left(-\frac{y}{R}\mathbf{i_z} + \frac{z}{R}\mathbf{i_y}\right)k^2\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right) = k^3\frac{1}{R}f_1(z\sin\theta\mathbf{i_r} + z\cos\theta\mathbf{i}_\theta - r\sin\theta\mathbf{i_z}). \tag{3.49}$$

Now we can calculate the electric field $\mathbf{E}_m$ and the magnetic induction $\mathbf{B}_m$ due to a single magnetic dipole $\mathbf{m}$. Recall that the vector potential $\mathbf{A}_m$ due to the magnetic dipole $\mathbf{m}$ is

$$\mathbf{A}_m = ik(\mathbf{i_R} \times \mathbf{m})\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right) \tag{3.50}$$

and so the corresponding magnetic induction $\mathbf{B}_m$ is (A.16)

$$\begin{aligned}\mathbf{B}_m = \nabla \times \mathbf{A}_m &= ik\nabla \times \left((R\mathbf{i_R} \times \mathbf{m})\frac{e^{ikR}}{R^2}\left(1 - \frac{1}{ikR}\right)\right)\\ &= -2\mathbf{m}\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right) + \nabla\left(\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right)\right) \times (R\mathbf{i_R} \times \mathbf{m}).\end{aligned} \tag{3.51}$$

Since $\nabla \times (R\mathbf{i_R} \times \mathbf{m}) = -2\mathbf{m}$,

$$\nabla\left(\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right)\right) = \left(ik - \frac{1}{R}\right)\left(ik\mathbf{i_R}\frac{e^{ikR}}{R^2} - 2\mathbf{i_R}\frac{e^{ikR}}{R^3}\right) + \mathbf{i_R}\frac{e^{ikR}}{R^4} \tag{3.52}$$

and $\mathbf{i_R} \times (R\mathbf{i_R} \times \mathbf{m}) = R(-\mathbf{m} + \mathbf{i_R}(\mathbf{i_R} \cdot \mathbf{m}))$, the expression for $\mathbf{B}_m$ is (A.17)

$$\begin{aligned}\mathbf{B}_m = \nabla \times \mathbf{A} &= -2\mathbf{m}\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right) + \nabla\left(\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right)\right) \times (R\mathbf{i_R} \times \mathbf{m})\\ &= \mathbf{m}\frac{e^{ikR}}{R^2}\left(ik + k^2R - \frac{1}{R}\right) + \mathbf{i_R}(\mathbf{i_R}.\mathbf{m})\frac{e^{ikR}}{R^2}\left(-k^2R - 3ik + \frac{3}{R}\right).\end{aligned} \tag{3.53}$$

Note that the expression for the magnetic field for the magnetic dipole is the same as the expression for the electric field for the electric dipole if we replace $\mathbf{p}$ with $\mathbf{m}$. Employing the fact that $\mathbf{E}_m = \frac{i}{k}\nabla \times \mathbf{B}_m$,

$$\begin{aligned}\mathbf{E}_m &= \frac{i}{k}\nabla \times \left(\mathbf{m}\frac{e^{ikR}}{R^2}\left(ik + k^2R - \frac{1}{R}\right) + \mathbf{i_R}(\mathbf{i_R} \cdot \mathbf{m})\frac{e^{ikR}}{R^2}\left(-k^2R - 3ik + \frac{3}{R}\right)\right)\\ &= \frac{i}{k}\nabla\left(\frac{e^{ikR}}{R^2}\left(ik + k^2R - \frac{1}{R}\right)\right) \times \mathbf{m} + \frac{i}{k}\left(\frac{e^{ikR}}{R^2}\left(ik + k^2R - \frac{1}{R}\right)\right)\nabla \times \mathbf{m}\\ &+ \frac{i}{k}\nabla\left(\frac{e^{ikR}}{R^2}\left(-k^2R - 3ik + \frac{3}{R}\right)(\mathbf{i_R} \cdot \mathbf{m})\right) \times \mathbf{i_R}\\ &+ \frac{i}{k}\left(\frac{e^{ikR}}{R^2}\left(-k^2R - 3ik + \frac{3}{R}\right)(\mathbf{i_R} \cdot \mathbf{m})\right)\nabla \times \mathbf{i_R}.\end{aligned} \tag{3.54}$$

Therefore for the electric field $\mathbf{E}_m$ (A.19) we have:

$$\mathbf{E}_m = \frac{i}{k}\nabla\left(\frac{e^{ikR}}{R^2}\left(ik + k^2R - \frac{1}{R}\right)\right) \times \mathbf{m} - \frac{i}{k}\nabla\left(\frac{e^{ikR}}{R^2}\left(k^2R + 3ik - \frac{3}{R}\right)(\mathbf{i_R}\cdot\mathbf{m})\right) \times$$
$$\times\, \mathbf{i_R} = \frac{i}{k}\frac{e^{ikR}}{R^2}\left(-k^2 + ik^3R\right)(\mathbf{i_R}\times\mathbf{m}) = k^2\frac{e^{ikR}}{R}\left(\frac{1}{ikR} - 1\right)(\mathbf{i_R}\times\mathbf{m}). \qquad (3.55)$$

The expression for the electric field for the magnetic dipole is the same as the negative of the expression for the magnetic field for the electric dipole if we replace $\mathbf{p}$ with $\mathbf{m}$. Since we choose $\mathbf{m}$ to be $\mathbf{m} = (0, 1, 0)$,

$$\begin{aligned}\mathbf{i_R}\times\mathbf{m} &= \frac{x\mathbf{i_x} + y\mathbf{i_y} + z\mathbf{i_z}}{R}\times\mathbf{i_y} = \frac{x}{R}\mathbf{i_z} - \frac{z}{R}\mathbf{i_x} \\ &= \frac{r}{R}\cos\theta\mathbf{i_z} - \frac{z}{R}(\cos\theta\mathbf{i_r} - \sin\theta\mathbf{i_\theta}).\end{aligned} \qquad (3.56)$$

Converting the results for $\mathbf{E}_m$ in cylindrical coordinates (A.20) yields

$$\begin{aligned}\mathbf{E}_m &= k^2\frac{e^{ikR}}{R^2}\left(\frac{1}{ikR} - 1\right)(r\cos\theta\mathbf{i_z} - z\cos\theta\mathbf{i_r} + z\sin\theta\mathbf{i_\theta}) \\ &= -k^3\frac{1}{R}f_1(r\cos\theta\mathbf{i_z} - z\cos\theta\mathbf{i_r} + z\sin\theta\mathbf{i_\theta}).\end{aligned} \qquad (3.57)$$

Since

$$\mathbf{i_R}(\mathbf{i_R}.\mathbf{m}) = \frac{xy\mathbf{i_x} + y^2\mathbf{i_y} + zy\mathbf{i_z}}{R^2} = \frac{r^2}{R^2}\sin\theta\mathbf{i_r} + \frac{rz}{R^2}\sin\theta\mathbf{i_z}, \qquad (3.58)$$

the expression for $\mathbf{B}_m$ in cylindrical coordinates (A.21) becomes

$$\begin{aligned}\mathbf{B}_m &= \mathbf{m}\frac{e^{ikR}}{R^2}\left(ik + k^2R - \frac{1}{R}\right) + \mathbf{i_R}(\mathbf{i_R}\cdot\mathbf{m})\frac{e^{ikR}}{R^2}\left(-k^2R - 3ik + \frac{3}{R}\right) \\ &= \sin\theta\mathbf{i_r}\frac{e^{ikR}}{R^3}\left(ikR + k^2R^2 - 1 - r^2k^2 - \frac{3ikr^2}{R} + \frac{3r^2}{R^2}\right) \\ &+ \cos\theta\mathbf{i_\theta}\frac{e^{ikR}}{R}\left(k^2 + \frac{ik}{R} - \frac{1}{R^2}\right) - rz\sin\theta\mathbf{i_z}\frac{e^{ikR}}{R^3}\left(k^2 + \frac{3ik}{R} + \frac{3}{R^2}\right).\end{aligned} \qquad (3.59)$$

Finally, since (A.22)

$$e^{ikR}\Big(ikR + k^2R^2 - 1 - r^2k^2 - \frac{3ikr^2}{R} + \frac{3r^2}{R^2}\Big) = \frac{k^2}{i}(2r^2 f_1 + (z-\xi)^2 f_2) \qquad (3.60)$$

as well as $e^{ikR}\Big(k^2 + \frac{ik}{R} - \frac{1}{R^2}\Big) = \frac{k^2}{i} f_2$ and $e^{ikR}\Big(k^2 + \frac{3ik}{R} + \frac{3}{R^2}\Big) = \frac{k^2}{i}(f_2 - 2f_1)$,

the expression for $\mathbf{B}_m$ becomes:

$$\mathbf{B}_m = \frac{k^2}{i}\left[\mathbf{i_r}\sin\theta\frac{1}{R^3}(2r^2 f_1 + (z-\xi)^2 f_2) + \mathbf{i}_\theta\frac{1}{R}\cos\theta f_2 - \mathbf{i_z} rz\sin\theta\frac{1}{R^3}(f_2 - 2f_1)\right]. \qquad (3.61)$$

We want to represent the scattered electric field $\mathbf{E_s}$ as a linear superposition of fields due to electric and magnetic dipoles along the z-axis with unknown densities g and h . This allows us to write the expression for the scattered electric field $\mathbf{E_s}$ as:

$$\mathbf{E_s} = \int_\alpha^\beta \Big(-\frac{\cos\theta}{R^3}(2r^2 f_1 + (z-\xi)^2 f_2)\mathbf{i_r} + \frac{\sin\theta}{R} f_2 \mathbf{i}_\theta + \frac{\cos\theta}{R^3} r(z-\xi)(f_2 - 2f_1)\mathbf{i_z}\Big)$$
$$\times\, g(\xi,\epsilon)d\xi + \int_\alpha^\beta \frac{f_1}{R}\left(-(z-\xi)\cos\theta\mathbf{i_r} + (z-\xi)\sin\theta\mathbf{i}_\theta + r\cos\theta\mathbf{i_z}\right) h(\xi,\epsilon)d\xi. \qquad (3.62)$$

Since the tangential components of the electric field on the surface of the body vanish, the BC become

$$\mathbf{n}\times\mathbf{E_{inc}} = -\mathbf{n}\times\mathbf{E_s}. \qquad (3.63)$$

Let $\phi = r - \epsilon\sqrt{S(z)}$. Then

$$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{i_r} + \frac{\partial\phi}{\partial z}\mathbf{i_z} \qquad (3.64)$$

and the unit vector normal to the surface is

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{\sqrt{1 + \frac{\epsilon^2}{4}\frac{S'^2(z)}{S(z)}}}\left(\mathbf{i_r} - \frac{\epsilon}{2}\frac{S'(z)}{\sqrt{S(z)}}\mathbf{i_z}\right) = a\mathbf{i_r} + b\mathbf{i_z} \qquad (3.65)$$

Therefore

$$\begin{aligned}\mathbf{n}\times\mathbf{E_{inc}} &= (a\mathbf{i_r}+b\mathbf{i_z})\times(e^{ikz}(\cos\theta\mathbf{i_r}-\sin\theta\mathbf{i_\theta}))\\ &= e^{ikz}(b\sin\theta\mathbf{i_r}+b\cos\theta\mathbf{i_\theta}-a\sin\theta\mathbf{i_z})\end{aligned} \tag{3.66}$$

and

$$\begin{aligned}&\mathbf{n}\times\mathbf{E_s} = (a\mathbf{i_r}+b\mathbf{i_z})\times\\ &\times\left[\int_\alpha^\beta\left(-\frac{\cos\theta}{R^3}(2r^2f_1+(z-\xi)^2f_2)\mathbf{i_r}+\frac{\sin\theta}{R}f_2\mathbf{i_\theta}+\frac{\cos\theta}{R^3}r(z-\xi)(f_2-2f_1)\mathbf{i_z}\right)\right.\\ &\left.\times\, g(\xi,\epsilon)d\xi+\int_\alpha^\beta\frac{f_1}{R}\left(-(z-\xi)\cos\theta\mathbf{i_r}+(z-\xi)\sin\theta\mathbf{i_\theta}+r\cos\theta\mathbf{i_z}\right)h(\xi,\epsilon)d\xi\right]\\ &=\int_\alpha^\beta a\left(\frac{\sin\theta}{R}f_2\mathbf{i_z}-\frac{\cos\theta}{R^3}r(z-\xi)(f_2-2f_1)\mathbf{i_\theta}\right)g(\xi,\epsilon)d\xi\\ &+\int_\alpha^\beta a\frac{f_1}{R}\left((z-\xi)\sin\theta\mathbf{i_z}-r\cos\theta\mathbf{i_\theta}\right)h(\xi,\epsilon)d\xi\\ &+\int_\alpha^\beta b\left(-\frac{\cos\theta}{R^3}(2r^2f_1+(z-\xi)^2f_2)\mathbf{i_\theta}-\frac{\sin\theta}{R}f_2\mathbf{i_r}\right)g(\xi,\epsilon)d\xi\\ &+\int_\alpha^\beta b\frac{f_1}{R}\left(-(z-\xi)\cos\theta\mathbf{i_\theta}-(z-\xi)\sin\theta\mathbf{i_r}\right)h(\xi,\epsilon)d\xi.\end{aligned} \tag{3.67}$$

By linear independence

$$e^{ikz}=\int_\alpha^\beta\frac{f_2}{R}g(\xi,\epsilon)d\xi+\int_\alpha^\beta\frac{f_1}{R}(z-\xi)h(\xi,\epsilon)d\xi \tag{3.68}$$

and

$$\begin{aligned}be^{ikz} &= \int_\alpha^\beta a\frac{1}{R^3}r(z-\xi)(f_2-2f_1)g(\xi,\epsilon)d\xi+\int_\alpha^\beta a\frac{f_1}{R}rh(\xi,\epsilon)d\xi\\ &+\int_\alpha^\beta b\frac{1}{R^3}(2r^2f_1+(z-\xi)^2f_2)g(\xi,\epsilon)d\xi+\int_\alpha^\beta b\frac{f_1}{R}(z-\xi)h(\xi,\epsilon)d\xi.\end{aligned} \tag{3.69}$$

The last equation can be rewritten as

$$-\frac{\epsilon}{2}\frac{S^{'}(z)}{\sqrt{S(z)}}e^{ikz} = \int_{\alpha}^{\beta}\left(r(z-\xi)(f_2-2f_1) - \frac{\epsilon}{2}\frac{S^{'}(z)}{\sqrt{S(z)}}(2r^2 f_1 + (z-\xi)^2 f_2)\right)\frac{g(\xi,\epsilon)}{R^3}d\xi$$
$$+\int_{\alpha}^{\beta}\left(r - \frac{\epsilon}{2}\frac{S^{'}(z)}{\sqrt{S(z)}}(z-\xi)\right)\frac{f_1}{R}h(\xi,\epsilon)d\xi, \tag{3.70}$$

and since $r = \epsilon\sqrt{S(z)}$, it becomes

$$S^{'}(z)e^{ikz} = -\int_{\alpha}^{\beta}\left(2S(z)(z-\xi)(f_2-2f_1) - S^{'}(z)(2\epsilon^2 S(z)f_1 + (z-\xi)^2 f_2)\right)$$
$$\times\frac{g(\xi,\epsilon)}{R^3}d\xi - \int_{\alpha}^{\beta}\left(2S(z) - S^{'}(z)(z-\xi)\right)\frac{f_1}{R}h(\xi,\epsilon)d\xi. \tag{3.71}$$

We are going to show that $S^{'}(z)e^{ikz} = -E_1 - E_2$, where E_1 and E_2 are given by

$$E_1 = -\frac{2i}{\epsilon^2 k^2}\left(\frac{d}{dz}\int_{\alpha}^{\beta} e^{ikR}\frac{(z-\xi)}{R}h(\xi,\epsilon)d\xi - ik\int_{\alpha}^{\beta} e^{ikR}h(\xi,\epsilon)d\xi\right) \tag{3.72}$$

and

$$E_2 = -\frac{S^{'}(z)}{k}\int_{\alpha}^{\beta}\frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi + iS^{'}(z)\int_{\alpha}^{\beta}\frac{e^{ikR}}{R}g(\xi,\epsilon)d\xi$$
$$-\frac{2S(z)}{k}\frac{d}{dz}\int_{\alpha}^{\beta}\frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi. \tag{3.73}$$

Using the fact that

$$\frac{dR}{dz} = \frac{2(z-\xi)+\epsilon^2 S'(z)}{2R}, \tag{3.74}$$

$$\frac{d}{dz}e^{ikR}\frac{(z-\xi)}{R} = \frac{e^{ikR}}{R} + ik(z-\xi)\frac{e^{ikR}}{R}\frac{2(z-\xi)+\epsilon^2 S'(z)}{2R} - (z-\xi)\frac{e^{ikR}}{R^2}\frac{2(z-\xi)+\epsilon^2 S'(z)}{2R} \tag{3.75}$$

and

$$\frac{d}{dz}e^{ikR}\left(\frac{1}{R^2}+\frac{i}{kR^3}\right) = \frac{e^{ikR}}{2}\left(ik\left(\frac{1}{R^3}+\frac{i}{kR^4}\right) + \left(-\frac{2}{R^4} - \frac{3i}{kR^5}\right)\right)(2(z-\xi)+\epsilon^2 S^{'}(z)), \tag{3.76}$$

the expression for E_1 can be rewritten as

$$E_1 = -\frac{2i}{\epsilon^2 k^2}\int_\alpha^\beta \frac{e^{ikR}}{R}\Big(1 + ik(z-\xi)\frac{2(z-\xi)+\epsilon^2 S(z)}{2R} - (z-\xi)\frac{2(z-\xi)+\epsilon^2 S(z)}{2R^2} - 2ik\frac{R^2}{2R}\Big)h(\xi,\epsilon)d\xi. \tag{3.77}$$

Therefore, by (A.23)

$$\begin{aligned} E_1 &= -\frac{i}{\epsilon^2 k^2}\int_\alpha^\beta \frac{e^{ikR}}{R}\Big(\frac{2(z-\xi)^2 + 2\epsilon^2 S(z) - 2(z-\xi)^2 - (z-\xi)\epsilon^2 S'(z)}{R^2} \\ &+ ik\frac{2(z-\xi)^2 + (z-\xi)\epsilon^2 S'(z) - 2\epsilon^2 S(z) - 2(z-\xi)^2}{R}\Big)h(\xi,\epsilon)d\xi \\ &= \frac{i}{k^2}\int_\alpha^\beta \frac{e^{ikR}}{R}\Big(-\frac{ik}{R} + \frac{1}{R^2}\Big)((z-\xi)S'(z) - 2S(z))h(\xi,\epsilon)d\xi \\ &= \int_\alpha^\beta \frac{f_1}{R}((z-\xi)S'(z) - 2S(z))h(\xi,\epsilon)d\xi. \end{aligned} \tag{3.78}$$

For E_2 we have (A.24)

$$\begin{aligned} E_2 &= -S'(z)\int_\alpha^\beta e^{ikR}\left(\frac{1}{kR^2} + \frac{i}{k^2R^3} - \frac{i}{R}\right)g(\xi,\epsilon)d\xi \\ &- S(z)\int_\alpha^\beta e^{ikR}\left(\frac{i}{R^3} - \frac{1}{kR^4} - \frac{2}{kR^4} - \frac{3i}{k^2R^5}\right)(2(z-\xi) + \epsilon^2 S'(z))g(\xi,\epsilon)d\xi \\ &= \int_\alpha^\beta \Big(S'(z)(2\epsilon^2 S(z)f_1 + f_2(z-\xi)^2) - 2S(z)(z-\xi)(f_2 - 2f_1)\Big)\frac{g(\xi,\epsilon)}{R^3}d\xi. \end{aligned} \tag{3.79}$$

Therefore the last BC can be rewritten as

$$\begin{aligned} -\epsilon^2 S'(z)e^{ikR} &= \frac{2i}{k^2}\left\{\frac{d}{dz}\int_\alpha^\beta e^{ikR}\frac{(z-\xi)}{R}h(\xi,\epsilon)d\xi - ik\int_\alpha^\beta e^{ikR}h(\xi,\epsilon)d\xi\right\} \\ &+ \frac{\epsilon^2 S'(z)}{k}\int_\alpha^\beta \frac{e^{ikR}}{R^2}\left(1 + \frac{i}{kR}\right)g(\xi,\epsilon)d\xi - i\epsilon^2 S'(z)\int_\alpha^\beta \frac{e^{ikR}}{R}g(\xi,\epsilon)d\xi \\ &+ \frac{2\epsilon^2 S(z)}{k}\frac{d}{dz}\int_\alpha^\beta \frac{e^{ikR}}{R^2}\left(1 + \frac{i}{kR}\right)g(\xi,\epsilon)d\xi. \end{aligned} \tag{3.80}$$

Recall that the BC are

$$e^{ikz} = \int_{\alpha}^{\beta} \frac{f_2}{R} g(\xi,\epsilon) d\xi + \int_{\alpha}^{\beta} \frac{f_1}{R}(z-\xi) h(\xi,\epsilon) d\xi \tag{3.81}$$

and

$$\begin{aligned} S^{'}(z)e^{ikz} = &- \int_{\alpha}^{\beta} (2S(z)(z-\xi)(f_2-2f_1) - S^{'}(z)(2\epsilon^2 S(z) f_1 + (z-\xi)^2 f_2)) \\ &\times \frac{g(\xi,\epsilon)}{R^3} d\xi - \int_{\alpha}^{\beta} (2S(z) - S^{'}(z)(z-\xi)) \frac{f_1}{R} h(\xi,\epsilon) d\xi. \end{aligned} \tag{3.82}$$

Multiplying eq.(3.81) by $4S(z) + \epsilon^2 S'^2(z)$ and eq.(3.82) by $-\epsilon^2 S'(z)$ and adding the resultant equations yields

$$\begin{aligned} 4S(z)e^{ikz} = &\, 4S(z) \int_{\alpha}^{\beta} \frac{f_2}{R} g(\xi,\epsilon) d\xi + 4S(z) \int_{\alpha}^{\beta} \frac{f_1}{R}(z-\xi) h(\xi,\epsilon) d\xi \\ &+ \epsilon^2 S'^2(z) \int_{\alpha}^{\beta} \frac{f_2}{R} g(\xi,\epsilon) d\xi + \epsilon^2 S'(z) \int_{\alpha}^{\beta} \Big(2S(z)(z-\xi)(f_2-2f_1) \\ &- S^{'}(z)(2\epsilon^2 S(z) f_1 + (z-\xi)^2 f_2) \Big) \frac{g(\xi,\epsilon)}{R^3} d\xi + \epsilon^2 S'(z) \int_{\alpha}^{\beta} 2S(z) \frac{f_1}{R} h(\xi,\epsilon) d\xi, \end{aligned} \tag{3.83}$$

which after division by $2S(z)$ results in:

$$\begin{aligned} 2e^{ikz} = &\, 2 \int_{\alpha}^{\beta} \frac{f_2}{R} g(\xi,\epsilon) d\xi + 2 \int_{\alpha}^{\beta} \frac{f_1}{R}(z-\xi) h(\xi,\epsilon) d\xi \\ &+ \frac{\epsilon^2 S'^2(z)}{2S(z)} \int_{\alpha}^{\beta} \frac{f_2}{R} g(\xi,\epsilon) d\xi + \frac{\epsilon^2 S'(z)}{2S(z)} \int_{\alpha}^{\beta} \Big(2S(z)(z-\xi)(f_2-2f_1) \\ &- S^{'}(z)(2\epsilon^2 S(z) f_1 + (z-\xi)^2 f_2) \Big) \frac{g(\xi,\epsilon)}{R^3} d\xi + \epsilon^2 S'(z) \int_{\alpha}^{\beta} \frac{f_1}{R} h(\xi,\epsilon) d\xi. \end{aligned} \tag{3.84}$$

Furthermore, the last equation can be rewritten as (A.25)

$$
\begin{aligned}
2e^{ikz} &= 2\int_\alpha^\beta \frac{f_2}{R} g(\xi,\epsilon)d\xi + \int_\alpha^\beta \frac{f_1}{R}(2(z-\xi)+\epsilon^2 S'(z))h(\xi,\epsilon)d\xi \\
&+ \frac{\epsilon^2 S'^2(z)}{2S(z)}\int_\alpha^\beta \frac{f_2}{R} g(\xi,\epsilon)d\xi + \frac{\epsilon^2 S'(z)}{2S(z)}\int_\alpha^\beta \Big(2S(z)(z-\xi)(f_2-2f_1) \\
&- S^{'}(z)(2\epsilon^2 S(z) f_1 + (z-\xi)^2 f_2)\Big)\frac{g(\xi,\epsilon)}{R^3}d\xi \\
&= 2\int_\alpha^\beta \frac{f_2}{R} g(\xi,\epsilon)d\xi + 2\int_\alpha^\beta f_1 \frac{\partial R}{\partial z} h(\xi,\epsilon)d\xi + \frac{\epsilon^2 S'(z)}{2}\int_\alpha^\beta (f_2-2f_1)2\frac{\partial R}{\partial z}\frac{g(\xi,\epsilon)}{R^2}d\xi.
\end{aligned}
\tag{3.85}
$$

By using the identities

$$
\frac{1}{ik^2}\frac{d}{dz}\frac{e^{ikR}}{R} = \frac{1}{ik^2}e^{ikR}\Big(\frac{ik}{R}-\frac{1}{R^2}\Big)\frac{\partial R}{\partial z} = e^{ikR}\Big(\frac{1}{kR}+\frac{i}{k^2R^2}\Big)\frac{\partial R}{\partial z} = f_1\frac{\partial R}{\partial z}, \tag{3.86}
$$

$$
\frac{f_2}{R} = i\frac{e^{ikR}}{R} - \frac{1}{k}\frac{e^{ikR}}{R}\Big(1+\frac{i}{kR}\Big) \tag{3.87}
$$

and (A.26)

$$
\frac{d}{dz}\frac{e^{ikR}}{R^2}\Big(1+\frac{i}{kR}\Big) = \frac{k}{R^2}(f_2-2f_1)\frac{\partial R}{\partial z}, \tag{3.88}
$$

the BC finally become

$$
\begin{aligned}
-\epsilon^2 S^{'}(z)e^{ikR} &= \frac{2i}{k^2}\left\{\frac{d}{dz}\int_\alpha^\beta e^{ikR}\frac{(z-\xi)}{R}h(\xi,\epsilon)d\xi - ik\int_\alpha^\beta e^{ikR}h(\xi,\epsilon)d\xi\right\} \\
&+ \frac{\epsilon^2 S^{'}(z)}{k}\int_\alpha^\beta \frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi - i\epsilon^2 S^{'}(z)\int_\alpha^\beta \frac{e^{ikR}}{R}g(\xi,\epsilon)d\xi \\
&+ \frac{2\epsilon^2 S(z)}{k}\frac{d}{dz}\int_\alpha^\beta \frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi
\end{aligned}
\tag{3.89}
$$

and

$$
\begin{aligned}
2e^{ikz} &= \frac{2}{ik^2}\frac{d}{dz}\int_\alpha^\beta \frac{e^{ikR}}{R}h(\xi,\epsilon)d\xi + 2i\int_\alpha^\beta \frac{e^{ikR}}{R}g(\xi,\epsilon)d\xi \\
&- \frac{2}{k}\int_\alpha^\beta \frac{e^{ikR}}{R}\Big(1+\frac{i}{kR}\Big)g(\xi,\epsilon)d\xi + \frac{\epsilon^2 S'(z)}{k}\frac{d}{dz}\int_\alpha^\beta \frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi.
\end{aligned}
\tag{3.90}
$$

Let

$$I_0(z,\epsilon,F) = \int_\alpha^\beta \frac{e^{ikR}}{R} F(\xi) d\xi, \tag{3.91}$$

$$I_1(z,\epsilon,F) = \int_\alpha^\beta \frac{(\xi - z)e^{ikR}}{R} F(\xi) d\xi, \tag{3.92}$$

$$J(z,\epsilon,F) = \int_\alpha^\beta e^{ikR} F(\xi) d\xi, \tag{3.93}$$

and

$$I_2(z,\epsilon,F) = \int_\alpha^\beta \frac{e^{ikR}}{R} \left(1 + \frac{i}{kR}\right) F(\xi) d\xi, \tag{3.94}$$

where $R = \left((z-\xi)^2 + \epsilon^2 S(z)\right)^{1/2}$ on the surface of the system. Expressed in terms of $I_0(z,\epsilon,F)$, $I_1(z,\epsilon,F)$, $J(z,\epsilon,F)$ and $I_2(z,\epsilon,F)$, the BC become:

$$\begin{aligned} -\epsilon^2 S'(z) e^{ikz} = \frac{2i}{k^2}\left\{-\frac{d}{dz} I_1(z,\epsilon,h) - ikJ(z,\epsilon,h)\right\} + \frac{\epsilon^2 S'(z)}{k} I_2(z,\epsilon,g) \\ - i\epsilon^2 S'(z) I_0(z,\epsilon,g) + \frac{2\epsilon^2 S(z)}{k}\frac{d}{dz} I_2(z,\epsilon,g) \end{aligned} \tag{3.95}$$

and

$$2e^{ikz} = \frac{2}{ik^2}\frac{d}{dz} I_0(z,\epsilon,h) + 2iI_0(z,\epsilon,g) - \frac{2}{k} I_2(z,\epsilon,g) + \frac{\epsilon^2 S'(z)}{k}\frac{d}{dz} I_2(z,\epsilon,g). \tag{3.96}$$

Solving the BC in the form of integral equations (eq.(3.95) and eq.(3.96)) will give us the much needed density of electric and magnetic dipoles along the z-axis, i.e $g(z,\epsilon)$ and $h(z,\epsilon)$. Before we attempt that, however, we need to analyze the behavior of the integral operators $I_0(z,\epsilon,F)$, $I_1(z,\epsilon,F)$, $J(z,\epsilon,F)$ and $I_2(z,\epsilon,F)$. Recall that we imposed the requirement on the unknown densities $g(z,\epsilon)$ and $h(z,\epsilon)$ to have an asymptotic expansion in terms of powers of ϵ as well as analyticity of the expansion coefficients as functions of the z coordinate. Hence the behavior of the integral operators I_0, I_1, J and I_2 is closely related to the behavior of their kernels, i.e

$$\frac{e^{ikR}}{R}, \qquad \frac{(\xi - z)e^{ikR}}{R}, \qquad e^{ikR}, \qquad \frac{e^{ikR}}{R}\left(1 + \frac{i}{kR}\right), \tag{3.97}$$

respectively. One important observation is that for sufficiently high frequency/wavenumber k, all the kernels above become highly oscillatory. Also, in the case when $\alpha = 0$ and $\beta = 1$, the kernels of I_0 and I_2 become singular. To avoid this, we require no dipoles on the tips ($\alpha > 0$, $\beta < 1$) as well as α and β to have the expansion of the form $\alpha(\epsilon) = \sum_{n=1}^{\infty} \alpha_n \epsilon^{2n}$ and $\beta(\epsilon) = 1 - \sum_{n=1}^{\infty} \beta_n \epsilon^{2n}$. Such an expansion solves the problem with the singularities occurring at both ends. However, it does not automatically solve the problem with the uniformity of the asymptotic expansions of the four integral operators I_0, I_1, J and I_2. This can be achieved by choosing the coefficients α_i and β_i as discussed in Chapter 3.4, i.e. achieving uniformity of the expansions can simultaneously lead to determination of the integration limits α and β. Our next goal will be obtaining uniform asymptotic expansions of $I_0(z, \epsilon, F)$, $I_1(z, \epsilon, F)$, $J(z, \epsilon, F)$ and $I_2(z, \epsilon, F)$.

3.2.1 EXPANSION OF $I_0(z, \epsilon, F)$

Let us start with the integral

$$I_0(z, \epsilon, F) = \int_{\alpha}^{\beta} \frac{e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}}{\sqrt{(z-\xi)^2+\epsilon^2 S(z)}} F(\xi) d\xi. \tag{3.98}$$

We are looking for an expansion of $I_0(z,\epsilon,F)$ in the form [58]

$$I_0(z,\epsilon,F) \sim -\log\epsilon^2 F + G_1 F + \epsilon^2\log\epsilon^2 G_2 F + O(\epsilon^2), \tag{3.99}$$

where the operators G_1, G_2 and G_3 are to be determined from the following asymptotic expansion. Let

$$I_0 = I_0^- + I_0^+, \tag{3.100}$$

where

$$I_0^- = \int_\alpha^z \frac{e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}}{\sqrt{(z-\xi)^2+\epsilon^2 S(z)}} F(\xi)d\xi \tag{3.101}$$

and

$$I_0^+ = \int_z^\beta \frac{e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}}{\sqrt{(z-\xi)^2+\epsilon^2 S(z)}} F(\xi)d\xi. \tag{3.102}$$

Changing of variables $v = z-\xi$ for I_0^- and $v = \xi - z$ for I_0^+ leads to

$$I_0^- = \int_0^{z-\alpha} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} F(z-v)dv \tag{3.103}$$

and

$$I_0^+ = \int_0^{\beta-z} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} F(z+v)dv. \tag{3.104}$$

Since the Taylor expansion of $F(z+v)$ and $F(z-v)$ for small v are

$$F(z+v) = F(z) + vF'(z) + \frac{v^2}{2}F''(z) + \sum_{j=3}^{\infty}\frac{v^j F^{(j)}(z)}{j!},$$
$$F(z-v) = F(z) - vF'(z) + \frac{v^2}{2}F''(z) + \sum_{j=3}^{\infty}\frac{(-v)^j F^{(j)}(z)}{j!},$$

the expression becomes

$$\begin{aligned} I_0^+ = &\int_0^{\beta-z} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} (F(z) + vF'(z) + \frac{v^2}{2}F''(z))dv \\ &+ \int_0^{\beta-z} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} (F(z+v) - \sum_{j=0}^{2}\frac{v^j F^{(j)}(z)}{j!})dv. \end{aligned} \tag{3.105}$$

Also, since

$$\frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} = \frac{e^{ikv}}{v} + \epsilon^2 S(z)\left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3}\right) + O(\epsilon^4), \tag{3.106}$$

we obtain

$$\begin{aligned}
I_0^+ &= F(z)\int_0^{\beta-z} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}}dv + \int_0^{\beta-z} \frac{F(z+v)-F(z)}{v}e^{ikv}dv \\
&+ F^{'}(z)\int_0^{\beta-z}\left\{v\frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} - e^{ikv}\right\}dv + \frac{F^{''}(z)}{2}\int_0^{\beta-z}\left\{v^2\frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}}\right. \\
&\left. - ve^{ikv}\right\}dv + \frac{\epsilon^2 S(z)}{2}\int_0^{\beta-z} e^{ikv}\Big(F(z+v) - \sum_{j=0}^{2}\frac{v^j F^{(j)}(z)}{j!}\Big)\left(\frac{ik}{v^2} - \frac{1}{v^3}\right)dv + R \\
&= F(z)W_0^+ + W_1^+ + F^{'}(z)W_2^+ + \frac{F^{''}(z)}{2}W_3^+ + W_4^+ + R^+.
\end{aligned} \tag{3.107}$$

Combining the results for the integral operators W_0^+, W_1^+, W_3^+ and W_4^+ in the expansion of I_0^+ (and similarly W_0^-, W_1^-, W_3^- and W_4^- in I_0^-) from eq.(B.32)-eq.(B.100), for I_0 we get:

$$\begin{aligned}
I_0 \sim F(z)\Big\{ &-2\log\epsilon + \log(\frac{z(1-z)}{4S(z)}) + \int_0^{1-z}\frac{e^{iku}-1}{u}du + \int_0^{z}\frac{e^{iku}-1}{u}du\Big\} \\
&+ \epsilon^2\log\epsilon\frac{S(z)}{2}(k^2F(z) + F''(z)) + \int_0^{1-z}\frac{F(z+v)-F(z)}{v}e^{ikv}dv \\
&+ \int_0^{z}\frac{F(z-v)-F(z)}{v}e^{ikv}dv + O(\epsilon^2).
\end{aligned} \tag{3.108}$$

Let

$$V(z) = \log\left(\frac{z(1-z)}{4S(z)}\right) + \int_0^{1-z}\frac{e^{iku}-1}{u}du + \int_0^{z}\frac{e^{iku}-1}{u}du + 1. \tag{3.109}$$

Then

$$G_1F = (V(z)-1)F(z) + \int_0^{1-z}\frac{F(z+v)-F(z)}{v}e^{ikv}dv + \int_0^{z}\frac{F(z-v)-F(z)}{v}e^{ikv}dv \tag{3.110}$$

and

$$G_2F = \frac{S(z)}{4}(k^2F(z) + F''(z)) \tag{3.111}$$

are the operators in the expansion of

$$I_0(z, \epsilon, F) = -\log \epsilon^2 F + G_1F + \epsilon^2 \log \epsilon^2 G_2F + O(\epsilon^2). \tag{3.112}$$

3.2.2 EXPANSION OF $I_1(z, \epsilon, F)$

Recall that the second integral arising from the BC (eq.3.95 and eq.3.96)

$$I_1(z, \epsilon, F) = \int_\alpha^\beta \frac{(\xi - z)e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}}{\sqrt{(z-\xi)^2 + \epsilon^2 S(z)}} F(\xi)d\xi. \tag{3.113}$$

We expect an asymptotic expansion of $I_1(z, \epsilon, F)$ of the form [58]

$$I_1(z, \epsilon, F) = -\int_0^z e^{ik(z-\xi)}F(\xi)d\xi + \int_z^1 e^{ik(\xi-z)}F(\xi)d\xi + \epsilon^2 \log \epsilon^2 L_{11} + \epsilon^2 L_{12} + HOT. \tag{3.114}$$

Let

$$I_1 = I_1^- + I_1^+, \tag{3.115}$$

where

$$I_1^- = \int_\alpha^z \frac{(\xi - z)e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}}{\sqrt{(z-\xi)^2 + \epsilon^2 S(z)}} F(\xi)d\xi \tag{3.116}$$

and

$$I_1^+ = \int_z^\beta \frac{(\xi - z)e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}}{\sqrt{(z-\xi)^2 + \epsilon^2 S(z)}} F(\xi)d\xi. \tag{3.117}$$

Changing variables $v = z - \xi$ in I_1^- and $v = \xi - z$ in I_1^+ in conjunction with

$$F(\xi) = F(\xi) + F(z - v) - F(\xi) = F(\xi) + F(z - v) - (F(z) - vF'(z) + O), \tag{3.118}$$

$$F(\xi) = F(\xi) + F(z + v) - F(\xi) = F(\xi) + F(z + v) - (F(z) + vF'(z) + O), \tag{3.119}$$

leads to

$$I_1^- = -\int_0^{z-\alpha} \frac{ve^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} F(z-v)dv. \tag{3.120}$$

and

$$I_1^+ = \int_0^{\beta-z} \frac{ve^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} F(z+v)dv. \tag{3.121}$$

Combining the results from eq.(B.106)-eq.(B.120) leads to an asymptotic expansion for I_1 of the form eq.(3.114), where the linear operators L_{11} and L_{12} are given by:

$$L_{11}F = \frac{S(z)}{2} F'(z) \tag{3.122}$$

and

$$\begin{aligned} L_{12}F = \frac{S(z)}{2} \Bigg\{ & \Big(\frac{e^{ik(1-z)}}{1-z} - \frac{e^{ikz}}{z} \Big) F(z) + \Big(e^{ik(1-z)} + e^{ikz} - V(z) \Big) F'(z) \\ & + \int_0^{1-z} v \left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3} \right) \Big(F(z+v) - \sum_{j=0}^{1} \frac{v^j F^{(j)}(z)}{j!} \Big) dv \\ & - \int_0^{z} v \left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3} \right) \Big(F(z-v) - \sum_{j=0}^{1} \frac{(-v)^j F^{(j)}(z)}{j!} \Big) dv \Bigg\} \\ & + e^{ikz} F(0)\alpha_1 - e^{ik(1-z)} F(1)\beta_1. \end{aligned} \tag{3.123}$$

3.2.3 EXPANSION OF $J(z, \epsilon, F)$

The next integral to be expanded is

$$J(z,\epsilon,F) = \int_\alpha^\beta e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}} F(\xi)d\xi. \tag{3.124}$$

We expect $J(z, \epsilon, F)$ to be of the form [58]

$$J(z,\epsilon,F) = \int_0^z e^{ik(z-\xi)} F(\xi)d\xi + \int_z^1 e^{ik(\xi-z)} F(\xi)d\xi + \epsilon^2 \log \epsilon^2 J_1 + \epsilon^2 J_2 + HOT. \tag{3.125}$$

Let

$$J = J^- + J^+, \tag{3.126}$$

where

$$J^- = \int_\alpha^z e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}} F(\xi) d\xi \tag{3.127}$$

and

$$J^+ = \int_z^\beta e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}} F(\xi) d\xi. \tag{3.128}$$

Changing variables $v = z - \xi$ in J^-, $v = \xi - z$ in J^+ and

$$F(\xi) = F(z) + F(z-v) - F(z), \tag{3.129}$$

$$F(\xi) = F(z) + F(z+v) - F(z), \tag{3.130}$$

leads to

$$J^- = \int_0^{z-\alpha} e^{ik\sqrt{v^2+\epsilon^2 S(z)}} F(z-v) dv \tag{3.131}$$

and

$$J^+ = \int_0^{\beta-z} e^{ik\sqrt{v^2+\epsilon^2 S(z)}} F(z+v) dv. \tag{3.132}$$

Using the results from eq.(B.121)-eq.(B.134), for J we obtain

$$\begin{aligned} J = {} & \epsilon^2 \frac{S(z)ik}{2} \left(\log \frac{4z(1-z)}{S(z)} - \log \epsilon^2 + \int_0^{1-z} \frac{e^{iku}-1}{u} du + \int_0^z \frac{e^{iku}-1}{u} du \right) \\ & + \int_z^1 e^{ik(\xi-z)} F(\xi) d\xi + \int_0^z e^{ik(z-\xi)} F(\xi) d\xi - \epsilon^2 e^{ik(1-z)} F(1)\beta_1 - \epsilon^2 e^{ikz)} F(0)\alpha_1 \\ & + \epsilon^2 \frac{ikS(z)}{2} \left(\int_0^{1-z} \frac{e^{ikv}}{v} \Big(F(z+v) - F(z) \Big) dv + \int_0^z \frac{e^{ikv}}{v} \Big(F(z-v) - F(z) \Big) dv \right) \\ & + HOT. \end{aligned} \tag{3.133}$$

Comparing the expression for J with eq.(3.125) implies that for the operators J_1 and J_2 we get

$$J_1 F(z) = -ik\frac{S(z)}{2}F(z) \tag{3.134}$$

and

$$J_2 F(z) = ik\frac{S(z)}{2}\Bigg\{V(z)F(z) + \int_0^{1-z}\frac{e^{ikv}}{v}\Big(F(z+v)-F(z)\Big)dv + \int_0^{z}\frac{e^{ikv}}{v}\Big(F(z-v)-F(z)\Big)dv\Bigg\} - e^{ik(1-z)}F(1)\beta_1 - e^{ikz)}F(0)\alpha_1. \tag{3.135}$$

3.2.4 EXPANSION OF $I_2(z,\epsilon,F)$

Recall that

$$I_2 = \int_\alpha^\beta e^{ikR}\left(\frac{1}{R^2}+\frac{i}{kR^3}\right)F(\xi,\epsilon)d\xi = -\frac{i}{k}\int_\alpha^\beta e^{ikR}\left(\frac{ik}{R^2}-\frac{1}{R^3}\right)F(\xi,\epsilon)d\xi. \tag{3.136}$$

We are going to seek an asymptotic expansion for $I_2(z,\epsilon,F)$ of the form:

$$I_2(z,\epsilon,F) = \frac{1}{\epsilon^2}L_{20} + \frac{1}{\epsilon^2}(\epsilon^2\log\epsilon^2 L_{21} + \epsilon^2 L_{22} + O(\epsilon^4\log\epsilon^2)). \tag{3.137}$$

First, consider the expression:

$$\left(ike^{ikR} - \frac{\partial}{\partial\xi}\left((\xi-z)\frac{e^{ikR}}{R}\right)\right), \tag{3.138}$$

where $R = \big((z-\xi)^2+\epsilon^2 S(z)\big)^{1/2}$. Since

$$\frac{\partial R}{\partial \xi} = -\frac{z-\xi}{R}, \tag{3.139}$$

$$\left(ike^{ikR} - \frac{\partial}{\partial\xi}\left((\xi - z)\frac{e^{ikR}}{R}\right)\right) = \left(ike^{ikR} - \frac{e^{ikR}}{R} - (\xi - z)\frac{\partial}{\partial\xi}\frac{e^{ikR}}{R}\right)$$
$$= e^{ikR}\left(ik - \frac{1}{R} - (\xi - z)\left(\frac{ik}{R} - \frac{1}{R^2}\right)\frac{\partial R}{\partial\xi}\right)$$
$$= e^{ikR}\left(ik - \frac{1}{R}\right)\left(1 - \frac{\xi - z}{R}\frac{\partial R}{\partial\xi}\right) = e^{ikR}\left(ik - \frac{1}{R}\right)\left(\frac{R^2 - (\xi - z)^2}{R^2}\right)$$
$$= \epsilon^2 S(z)\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right), \tag{3.140}$$

and therefore

$$\frac{1}{\epsilon^2 S(z)}\left(ike^{ikR} - \frac{\partial}{\partial\xi}\left((\xi - z)\frac{e^{ikR}}{R}\right)\right) = \frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right) = e^{ikR}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right), \tag{3.141}$$

which will allow us to rewrite the integral

$$\int_\alpha^\beta \frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right) g(\xi, \epsilon)d\xi. \tag{3.142}$$

The kernel of the integral above is a Helmholtz kernel as well as the kernels of the integrals

$$I^n = \int_\alpha^\beta e^{ikR}\sum_{j=1}^{n}\left(\frac{A_j}{R^{2j+1}} + i\frac{B_j}{R^{2j}}\right) F(\xi, \epsilon)d\xi. \tag{3.143}$$

provided A_j and B_j meet the following criteria(eq.(B.137)-eq.(B.167)):

$$B_1 = -kA_1 \qquad \text{for} \qquad n = 1,$$
$$A_j = B_{j+1}\frac{k(1 - 2j + n)}{(j - n)(1 + 2j)} \qquad \text{for} \qquad n > 1.$$

For $n = 1$ and $A_1 = 1/2$, $B_1 = -k/2$ and

$$
\begin{aligned}
I^1 &= \int_\alpha^\beta \Big(\frac{A_1}{R^3} + i\frac{B_1}{R^2}\Big) e^{ikR}(\beta-\xi)(\xi-\alpha)F(\xi)d\xi \\
&= \frac{1}{2}\int_\alpha^\beta \Big(\frac{1}{R^3} - \frac{ik}{R^2}\Big) e^{ikR}(\beta-\xi)(\xi-\alpha)F(\xi)d\xi \\
&= \frac{1}{2\epsilon^2 S(z)}\int_\alpha^\beta \left(\frac{\partial}{\partial\xi}\left((\xi-z)\frac{e^{ikR}}{R}\right) - ike^{ikR}\right)(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi \\
&= -\frac{1}{2\epsilon^2 S(z)}\left(\int_\alpha^\beta (\xi-z)\frac{e^{ikR}}{R}\frac{\partial}{\partial\xi}((\beta-\xi)(\xi-\alpha)\tilde{F}(\xi))d\xi\right. \\
&\left. + ik\int_\alpha^\beta e^{ikR}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi\right) = -\frac{1}{2\epsilon^2 S(z)}\Big(I_1\Big(z, \frac{\partial F}{\partial\xi}\Big) + ikJ_1(z,F)\Big).
\end{aligned}
\tag{3.144}
$$

On the other hand,

$$
I_2 = \int_\alpha^\beta e^{ikR}\left(\frac{1}{R^2} + \frac{i}{kR^3}\right)F(\xi,\epsilon)d\xi = -\frac{i}{k}\int_\alpha^\beta e^{ikR}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right)F(\xi,\epsilon)d\xi, \tag{3.145}
$$

and thus

$$
I_2 = \frac{2i}{k}I^1. \tag{3.146}
$$

Using the results from eq.(B.171)-eq.(B.179), and combining the results for for $I_1\Big(z,\epsilon,\frac{\partial F}{\partial\xi}\Big)$ and $J_1(z,F)$, we get

$$
\begin{aligned}
I_1\Big(z,\epsilon,\frac{\partial F}{\partial\xi}\Big) + ikJ_1(z,F) &= -2z(1-z)\tilde{F}(z) + \epsilon^2\log\epsilon^2 L_1 \\
&+ \epsilon^2(L_2 + 2\epsilon^2(\beta_1 z + \alpha_1(1-z))\tilde{F}(z) - e^{ik(1-z)}\beta_1\tilde{F}(1) \\
&- e^{ikz}\alpha_1\tilde{F}(0)) + ik(\epsilon^2\log\epsilon^2 J_1 + \epsilon^2 J_2) + HOT.
\end{aligned}
\tag{3.147}
$$

Finally, for the linear operators L_1, J_1, L_2 and J_2,

$$
\begin{aligned}
L_1 + ikJ_1 &= \frac{S(z)}{2}\frac{d}{dz}F'(z) + ik(-ik\frac{S(z)}{2}F(z)) \\
&= \frac{S(z)}{2}\Big(\frac{d^2}{dz^2}z(1-z))\tilde{F}(z) + k^2 z(1-z))\tilde{F}(z)\Big)
\end{aligned}
\tag{3.148}
$$

and

$$L_2 + ikJ_2 = L_2\left(\frac{d}{dz}z(1-z)\tilde{F}(z)\right) + ikJ_2(z(1-z)\tilde{F}(z))$$
$$- e^{ik(1-z)}F(1)\beta_1 - e^{ikz)}F(0)\alpha_1 + 2(\beta_1 z + \alpha_1(1-z))\tilde{F}(z). \quad (3.149)$$

The results for the scattered field obtained from evaluation of eq.(3.62) can be compared to the explicit results for a simple geometry. For the case of prolate spheroid the far field explicit results given in [62], [63] and [64] are:

$$E^s_\theta = \frac{e^{ikR}}{kR}\sum_{n=1}^{\infty}\left(\alpha_{1n}\frac{\partial P^1_n(\cos\theta)}{\partial\theta} + \overline{\beta}_{1n}\frac{P^1_n(\cos\theta)}{\sin\theta}\right)\cos\phi, \quad (3.150)$$

$$E^s_\phi = -\frac{e^{ikR}}{kR}\sum_{n=1}^{\infty}\left(\overline{\beta}_{1n}\frac{\partial P^1_n(\cos\theta)}{\partial\theta} + \alpha_{1n}\frac{P^1_n(\cos\theta)}{\sin\theta}\right)\sin\phi, \quad (3.151)$$

where $P^1_n(\cos\theta)$ are Legendre functions,

$$\alpha_{11} = -\tfrac{2}{3}c^3\tfrac{P^1_1}{Q^1_1}\left(1 - \tfrac{1}{50}c^2\left(22 - 10\tfrac{Q^1_{-1}}{Q^1_1}\right)\right), \quad (3.152)$$

$$\alpha_{12} = \tfrac{1}{270}c^5\left(\tfrac{P^1_2}{Q^1_2} - 5\tfrac{P'^1_1}{Q'^1_1}\right), \quad (3.153)$$

$$\alpha_{13} = \tfrac{2}{675}c^5\tfrac{P^1_1}{Q^1_1}, \quad (3.154)$$

$$\bar{\beta}_{11} = \tfrac{2}{3}c^3\tfrac{P'^1_1}{Q'^1_1}\left(1 - \tfrac{1}{50}c^2\left(22 - 10\tfrac{Q'^1_{-1}}{Q'^1_1} - 40\tfrac{Q^1_0}{Q'^1_1}\right)\right), \quad (3.155)$$

$$\bar{\beta}_{12} = -\tfrac{1}{270}c^5\left(\tfrac{P'^1_2}{Q'^1_2} - 5\tfrac{P^1_1}{Q^1_1}\right), \quad (3.156)$$

$$\bar{\beta}_{13} = -\tfrac{2}{675}c^5\tfrac{P'^1_1}{Q'^1_1}. \quad (3.157)$$

$$(3.158)$$

Here

$$c = \frac{1}{2}kd,$$

the interfocal distance d is

$$d = \sqrt{1 - 4\epsilon^2}$$

and P_i^1 and Q_i^1 are Legendre functions of the first and second kind evaluated at $\xi_1 = 1/d$.

For small k, by considering only the terms involving c^3, we get

$$\alpha_{11} = -\frac{2}{3}c^3\frac{P_1^1}{Q_1^1},$$

$$\bar{\beta}_{11} = \frac{2}{3}c^3\frac{P_1'^1}{Q_1'^1}.$$

Also, since

$$P_1^1(\cos\theta) = -\sin\theta,$$

$$\frac{\partial}{\partial\theta}P_1^1(\cos\theta) = -\cos\theta,$$

the ϕ component scattered field in the axial direction becomes

$$E_\phi \sim \frac{e^{ikR}}{kR}(\alpha_{11} + \overline{\beta}_{11}\cos\theta)\sin\phi. \tag{3.159}$$

On the other hand, in terms of the slender body approximation, the prolate spheroid geometry is defined by:

$$S(z) = 4z(1-z),$$

where $r = \epsilon\sqrt{S(z)}$ and $0 \leq z \leq 1$.

For this particular geometry, the dipole densities g and h are

$$g(\epsilon, z) = (z-\alpha)(\beta - z)\Big\{2i\epsilon^2k^2e^{ikz} - 4i(\epsilon^4\log\epsilon^2)k^2e^{ikz} + O(\epsilon^4)\Big\},$$

$$h(\epsilon, z) = 2\epsilon^2k^3z(1-z)e^{ikz} + 4(\epsilon^4\log\epsilon^2)k^3z(1-z)e^{ikz} + O(\epsilon^4), \tag{3.160}$$

and α and β are given by:

$$\alpha = \epsilon^2 + \epsilon^4 + 2\epsilon^6 + O(\epsilon^8),$$

$$\beta = 1 - \epsilon^2 - \epsilon^4 - 2\epsilon^6 + O(\epsilon^8).$$

In the far field, i.e. $R = (r^2 + (z-\xi)^2)^{\frac{1}{2}} \to R = (r^2 + z^2)^{\frac{1}{2}} \to \infty$ and

$$(r^2 + (z-\xi)^2)^{1/2} = (r^2 + z^2 - 2z\xi + \xi^2)^{1/2} = (r^2 + z^2)^{1/2}\left(1 - \frac{2z\xi}{r^2+z^2} + \frac{\xi^2}{r^2+z^2}\right)^{1/2}$$
$$= (r^2 + z^2)^{1/2}\left(1 - \frac{z\xi}{r^2+z^2} + HOT\right) = R(1 - \frac{z\xi}{R} + HOT),$$

where $R = (r^2 + z^2)^{\frac{1}{2}}$ and thus we can expand e^{ikR} in the following way

$$e^{ik(r^2+(z-\xi)^2)^{1/2}} = e^{ikR}e^{-ikz\xi/R}. \tag{3.161}$$

By substituting eq.(3.161) into the expression for the scattered field $\mathbf{E_s}$ (eq.(3.62)), in the far field we get

$$\mathbf{E^s} = \frac{e^{ikR}}{kR}\Bigg\{ik\left(\int_\alpha^\beta e^{-ikz\xi/R}g(\xi,\epsilon)d\xi\right)\left[\frac{rz}{R^2}\cos\phi\mathbf{i_z} + \sin\phi\mathbf{i_\phi} - \frac{z^2}{R^2}\cos\phi\mathbf{i_r}\right] +$$
$$+\left(\int_\alpha^\beta e^{-ikz\xi/R}h(\xi,\epsilon)d\xi\right)\left[\frac{r}{R}\cos\phi\mathbf{i_z} + \frac{z}{R}\sin\phi\mathbf{i_\phi} - \frac{z}{R}\cos\phi\mathbf{i_r}\right]\Bigg\}. \tag{3.162}$$

Suppose we define p and m through

$$p = ik\int_\alpha^\beta e^{-ikz\xi/R}g(\xi,\epsilon)d\xi$$
$$m = \int_\alpha^\beta e^{-ikz\xi/R}h(\xi,\epsilon)d\xi \tag{3.163}$$

Then the ϕ component of the scattered field is

$$E_\phi \sim \frac{e^{ikR}}{kR}(p + m\cos\theta)\sin\phi \tag{3.164}$$

For small a

$$\int_{\epsilon^2}^{1-\epsilon^2} x(1-x)e^{ax}dx \sim \int_{\epsilon^2}^{1-\epsilon^2} x(1-x)(1+ax)dx$$
$$= \frac{1}{6} + \frac{a}{12} - \epsilon^4 - \frac{a\epsilon^4}{2} + \frac{2\epsilon^6}{3} + \frac{a\epsilon^6}{3}. \tag{3.165}$$

In our case $a = ik(1-\cos\theta)$, where $\cos\theta = z/R$ and $\mathbf{i_R} = (x\mathbf{i_x} + y\mathbf{i_y} + z\mathbf{i_z})/R$. In the axial direction $\theta = \pi$ and combining eq.(3.163) with eq.(3.160) and eq.(3.165) and taking the leading order results in :

$$p \sim -\frac{k^3\epsilon^2}{3} \tag{3.166}$$
$$m \sim \frac{k^3\epsilon^2}{3} \tag{3.167}$$

Fig.3.2 shows the radial dependence for the scattered field from prolate spheroid for small k in the far field from both the slender body calculation and [63], [62]. The comparison yields exact agreement between the two models.

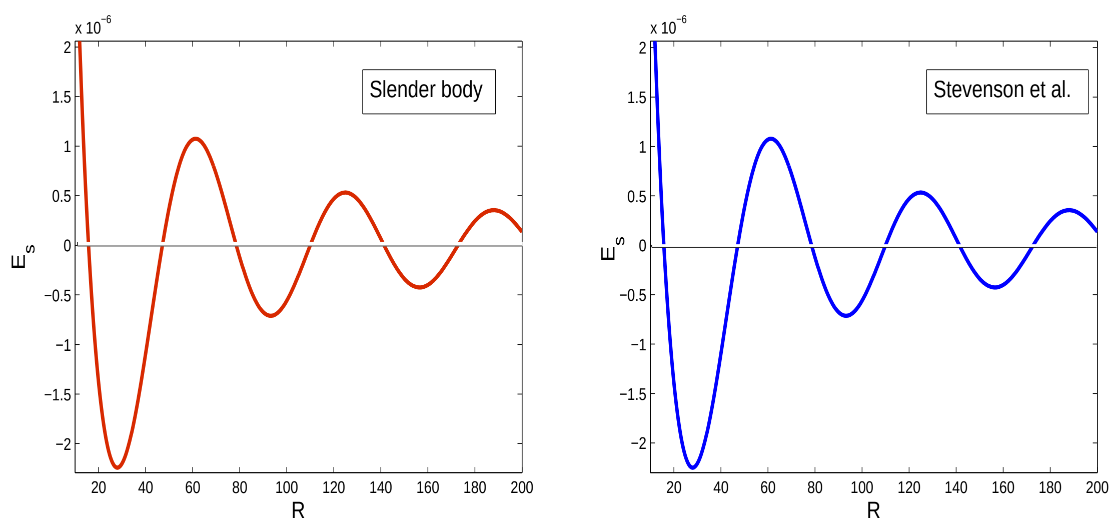

Figure 3.2: Electric field scattered from a prolate spheroid

3.2.5 CALCULATION OF h AND g: AXIAL CASE

By substitution of the asymptotic expansions for $I_0(z,\epsilon,F)$, $I_1(z,\epsilon,F)$, $I_2(z,\epsilon,F)$ and $J(z,\epsilon,F)$ into the BC (eq.(3.95) and eq.(3.96)) we get

$$\begin{aligned} 2e^{ikz} &= \frac{2}{ik^2}\frac{d}{dz}\Big[-\log\epsilon^2 h + G_1 h\Big] + 2i\Big[-\log\epsilon^2 g + G_1 g\Big] \\ &- \frac{2i}{k^2}\Big\{2-\epsilon^2 S'(z)\frac{d}{dz}\Big\}\Big[\frac{1}{\epsilon^2}\frac{z(1-z)}{S(z)}\tilde{g} + \log\epsilon^2 L_{21}\tilde{g} + L_{22}\tilde{g}\Big] \end{aligned} \tag{3.168}$$

and

$$\begin{aligned} -\epsilon^2 S'(z)e^{ikz} &= \frac{2i}{k^2}\Big[2h - \epsilon^2\log\epsilon^2\Big(\frac{d}{dz}L_1 + ikJ_1\Big)h - \epsilon^2\Big(\frac{d}{dz}L_2 + ikJ_2\Big)h\Big] \\ &- i\epsilon^2 S'(z)\Big[-\log\epsilon^2 g + G_1 g\Big] + \frac{2i\epsilon^2}{k^2}\Big\{S'(z) + 2S(z)\frac{d}{dz}\Big\}\Big[\frac{1}{\epsilon^2}\frac{z(1-z)}{S(z)}\tilde{g} \\ &+ \log\epsilon^2 L_{21}\tilde{g} + L_{22}\tilde{g}\Big], \end{aligned} \tag{3.169}$$

where

$$g(z,\epsilon) = (z-\alpha)(\beta-z)\tilde{g}(z,\epsilon), \tag{3.170}$$

i.e.

$$g(z,\epsilon) = (z-\alpha)(\beta-z)(\tilde{g}_0(z)\epsilon^2 + \tilde{g}_1(z)\epsilon^4\log\epsilon^2 + \tilde{g}_2(z)\epsilon^4 + ...) \tag{3.171}$$

and

$$h(z,\epsilon) = h_0(z)\epsilon^2 + h_1(z)\epsilon^4\log\epsilon^2 + h_2(z)\epsilon^4 + ... \tag{3.172}$$

In particular, by collecting terms of $O(1)$ from eq.(3.168) we get

$$2e^{ikz} = -\frac{4i}{k^2}\frac{z(1-z)}{S(z)}\tilde{g}_0(z) \tag{3.173}$$

and so the expression for $\tilde{g}_0(z)$ yields

$$\tilde{g}_0(z) = \frac{ik^2}{2}e^{ikz}\frac{S(z)}{z(1-z)}. \tag{3.174}$$

By collecting terms of $O(\epsilon^2)$ from eq.(3.169) we get

$$-S'(z)e^{ikz} = \frac{4i}{k^2}h_0 + \frac{2i}{k^2}\left(S'(z) + 2S(z)\frac{d}{dz}\right)\frac{z(1-z)}{S(z)}\tilde{g}_0(z) \tag{3.175}$$

and the expression for $h_0(z)$ is

$$h_0(z) = \frac{k^3}{2}S(z)e^{ikz}. \tag{3.176}$$

By collecting terms of $O(\epsilon^2 \log \epsilon^2)$ from eq.(3.168) we get

$$0 = -\frac{2}{ik^2}\frac{d}{dz}h_0 - 2ig_0 - \frac{2i}{k^2}\left(2\frac{z(1-z)}{S(z)}\tilde{g}_1 + L_{21}\tilde{g}_0\right), \tag{3.177}$$

$$0 = -\frac{d}{dz}h_0 + k^2 g_0 + 2\frac{z(1-z)}{S(z)}\tilde{g}_1 + 2L_{21}\tilde{g}_0, \tag{3.178}$$

and therefore

$$\begin{aligned} 2\frac{z(1-z)}{S(z)}\tilde{g}_1 = {} & \frac{ik^4}{2}S(z)e^{ikz} + \frac{k^3}{2}S'(z)e^{ikz} - \frac{ik^4}{2}e^{ikz}S(z) \\ & + \frac{1}{2}\left(\frac{ik^4}{2}e^{ikz}S(z) + \frac{d^2}{dz^2}\frac{ik^2}{2}e^{ikz}S(z)\right), \end{aligned} \tag{3.179}$$

and

$$2\frac{z(1-z)}{S(z)}\tilde{g}_1 = \frac{ik^2}{4}e^{ikz}S''(z). \tag{3.180}$$

Hence

$$\tilde{g}_1 = \frac{ik^2}{8}e^{ikz}S''(z)\frac{S(z)}{z(1-z)}. \tag{3.181}$$

By collecting terms of $O(\epsilon^4 \log \epsilon^2)$ from eq.(3.169) we get

$$0 = \frac{2i}{k^2}\Big(2h_1 - \Big(\frac{d}{dz}L_1 + ikJ_1\Big)h_0\Big) + iS'(z)g_0 + \frac{2i}{k^2}\Big\{S'(z) + 2S(z)\frac{d}{dz}\Big\}\Big[\frac{z(1-z)}{S(z)}\tilde{g}_1 + L_{21}\tilde{g}_0\Big] \tag{3.182}$$

or

$$h_1 = \frac{1}{2}\Big(\frac{d}{dz}L_1 + ikJ_1\Big)h_0 - \frac{k^2}{4}S'(z)g_0 - \frac{1}{2}\Big\{S'(z) + 2S(z)\frac{d}{dz}\Big\}\Big[\frac{z(1-z)}{S(z)}\tilde{g}_1 + L_{21}\tilde{g}_0\Big], \tag{3.183}$$

leading to

$$\begin{aligned} h_1 = &\frac{1}{2}\frac{d}{dz}\frac{S(z)}{2}\frac{d}{dz}\left(\frac{k^3}{2}S(z)e^{ikz}\right) + \frac{k^5}{4}S^2(z)e^{ikz} - \frac{ik^4}{8}S'(z)S(z)e^{ikz} \\ &- \frac{1}{2}\Big\{S'(z) + 2S(z)\frac{d}{dz}\Big\}\Big[\frac{ik^2}{8}e^{ikz}S''(z) - \frac{1}{4}\Big(\frac{ik^4}{2}e^{ikz}S(z) + \frac{ik^2}{2}\frac{d^2}{dz^2}e^{ikz}S(z)\Big)\Big] \end{aligned} \tag{3.184}$$

and

$$\begin{aligned} h_1 = &\frac{k^3}{8}\frac{d}{dz}S(z)\frac{d}{dz}S(z)e^{ikz} + \frac{k^5}{4}S^2(z)e^{ikz} - \frac{ik^4}{8}S'(z)S(z)e^{ikz} \\ &- \frac{1}{2}\Big\{S'(z) + 2S(z)\frac{d}{dz}\Big\}\frac{k^3}{4}e^{ikz}S'(z). \end{aligned} \tag{3.185}$$

Finally, for h_1, we get

$$h_1 = -\frac{k^3}{8}S(z)S''(z)e^{ikz}. \tag{3.186}$$

For the particular choice of geometry (fig.3.1, eq.(3.1)), $S'(z) = S''(z) = 0$ for $\alpha \leq z \leq \beta$

$$\begin{aligned} \tilde{g}_0(z) &= \frac{ik^2}{2}e^{ikz}\frac{S(z)}{z(1-z)}, \\ h_0(z) &= \frac{k^3}{2}S(z)e^{ikz}, \\ \tilde{g}_1 &= 0, \\ h_1 &= 0. \end{aligned} \tag{3.187}$$

So far, we have obtained the solution to the scattering problem in the axial dynamic case in terms of the electric field. However, for the field emission calculations, we need an analytic expression for the potential due to the total electric field (incident and scattered).

We know that

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{3.188}$$

and therefore

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \nabla \times \left(-\frac{\partial \mathbf{A}}{\partial t}\right), \tag{3.189}$$

and therefore **E** can be chosen to be

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V. \tag{3.190}$$

On the other hand

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{A} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \tag{3.191}$$

and hence

$$\nabla \times \nabla \times \mathbf{A} = \varepsilon\mu \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla V\right), \tag{3.192}$$

and finally

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\varepsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla\left(-\varepsilon\mu \frac{\partial \mathbf{V}}{\partial t}\right). \tag{3.193}$$

A has to satisfy the wave equation, a condition which can be fulfilled by choosing

$$\nabla \cdot \mathbf{A} = -\varepsilon\mu \frac{\partial V}{\partial t} = -\frac{1}{c^2}\frac{\partial V}{\partial t}, \tag{3.194}$$

which is known as the Lorentz-Lorenz gauge.

Since we assume that the incident field is time-harmonic, the potential has to be time-harmonic as well, i.e.

$$\nabla \cdot \mathbf{A} = \frac{i\omega}{c^2} V = \frac{ik}{c} V, \tag{3.195}$$

and thus

$$V_e = -\frac{ic}{k} \nabla \cdot \mathbf{A_e}, \tag{3.196}$$

$$V_m = -\frac{ic}{k} \nabla \cdot \mathbf{A_m}, \tag{3.197}$$

where

$$\mathbf{A}_e = -ik\mathbf{p}\frac{e^{ikR}}{R}, \tag{3.198}$$

$$\mathbf{A}_m = ik(\mathbf{i_R} \times \mathbf{m})\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right). \tag{3.199}$$

Employing the following vector identities

$$\nabla \cdot (\alpha \mathbf{A}) = \alpha \nabla \cdot \mathbf{A} + (\nabla \alpha) \cdot \mathbf{A} \tag{3.200}$$

and

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = -\mathbf{A}.(\nabla \times \mathbf{B}) + (\nabla \times \mathbf{A}) \cdot \mathbf{B} \tag{3.201}$$

leads to

$$\nabla \cdot \mathbf{A_e} = -ikxe^{ikR}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right) = kr\cos\theta e^{ikR}\left(\frac{k}{R^2} + \frac{i}{R^3}\right) \tag{3.202}$$

and similarly, for $\nabla \cdot \mathbf{A_m}$

$$\nabla \cdot \mathbf{A_m} = ik\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right)\nabla \cdot (\mathbf{i_R} \times \mathbf{m}) + ik\left(\nabla \frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right)\right) \cdot (\mathbf{i_R} \times \mathbf{m}), \tag{3.203}$$

which leads to

$$\nabla \cdot \mathbf{A_m} = ik\left(ik\frac{e^{ikR}}{R} - 2\frac{e^{ikR}}{R^2} + \frac{2}{ik}\frac{e^{ikR}}{R^2}\right)\mathbf{i_R}\cdot(\mathbf{i_R}\times\mathbf{m}). \tag{3.204}$$

But $\mathbf{i_R}\cdot(\mathbf{i_R}\times\mathbf{m}) = 0$, and thus

$$\nabla \cdot \mathbf{A_m} = 0. \tag{3.205}$$

Hence

$$V = cr\cos\theta\int_\alpha^\beta e^{ikR}\left(\frac{k}{R^2} + \frac{i}{R^3}\right)g(\xi,\epsilon)d\xi. \tag{3.206}$$

3.3 DYNAMIC CASE: TRANSVERSE INCIDENCE

Choose the incident field to be

$$\mathbf{E_{inc}} = e^{ikr\cos\theta}\mathbf{i_z}, \tag{3.207}$$

and the scattered field to be

$$\begin{aligned}\mathbf{E_s} = &\int_\alpha^\beta\left(-\frac{\cos\theta}{R^3}(2r^2f_1 + (z-\xi)^2f_2)\mathbf{i_r} + \frac{\sin\theta}{R}f_2\mathbf{i_\theta} + \frac{\cos\theta}{R^3}r(z-\xi)(f_2-2f_1)\mathbf{i_z}\right)\\ &\times G(\theta,\xi,\epsilon)d\xi + \int_\alpha^\beta\frac{f_1}{R}\left(-(z-\xi)\cos\theta\mathbf{i_r} + (z-\xi)\sin\theta\mathbf{i_\theta} + r\cos\theta\mathbf{i_z}\right)H(\theta,\xi,\epsilon)d\xi.\end{aligned} \tag{3.208}$$

Since the tangential components of the electric field on the surface of the body vanish, the BC become

$$\mathbf{n}\times\mathbf{E_{inc}} = -\mathbf{n}\times\mathbf{E_s} \tag{3.209}$$

and

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{\sqrt{1+\frac{\epsilon^2}{4}\frac{S'^2(z)}{S(z)}}}\left(\mathbf{i_r} - \frac{\epsilon}{2}\frac{S'(z)}{\sqrt{S(z)}}\mathbf{i_z}\right) = a\mathbf{i_r} + b\mathbf{i_z}, \tag{3.210}$$

$$\mathbf{n} \times \mathbf{E_{inc}} = (a\mathbf{i_r} + b\mathbf{i_z}) \times (e^{ikr\cos\theta}(\mathbf{i_z}) = -ae^{ikr\cos\theta}\mathbf{i}_\theta, \tag{3.211}$$

and

$$\begin{aligned}
&\mathbf{n} \times \mathbf{E_s} = (a\mathbf{i_r} + b\mathbf{i_z})\times \\
&\times \int_\alpha^\beta \left(-\frac{\cos\theta}{R^3}(2r^2 f_1 + (z-\xi)^2 f_2)\mathbf{i_r} + \frac{\sin\theta}{R} f_2 \mathbf{i}_\theta + \frac{\cos\theta}{R^3} r(z-\xi)(f_2 - 2f_1)\mathbf{i_z}\right) \times \\
&\times G(\theta,\xi,\epsilon)d\xi + \int_\alpha^\beta \frac{f_1}{R}\left(-(z-\xi)\cos\theta\mathbf{i_r} + (z-\xi)\sin\theta\mathbf{i}_\theta + r\cos\theta\mathbf{i_z}\right) H(\theta,\xi,\epsilon)d\xi = \\
&= \int_\alpha^\beta a\left(\frac{\sin\theta}{R} f_2\mathbf{i_z} - \frac{\cos\theta}{R^3} r(z-\xi)(f_2-2f_1)\mathbf{i}_\theta\right) G(\theta,\xi,\epsilon)d\xi + \\
&+ \int_\alpha^\beta a\frac{f_1}{R}\left((z-\xi)\sin\theta\mathbf{i_z} - r\cos\theta\mathbf{i}_\theta\right) H(\theta,\xi,\epsilon)d\xi + \\
&+ \int_\alpha^\beta b\left(-\frac{\cos\theta}{R^3}(2r^2 f_1 + (z-\xi)^2 f_2)\mathbf{i}_\theta - \frac{\sin\theta}{R} f_2\mathbf{i_r}\right) G(\theta,\xi,\epsilon)d\xi + \\
&+ \int_\alpha^\beta b\frac{f_1}{R}\left(-(z-\xi)\cos\theta\mathbf{i}_\theta - (z-\xi)\sin\theta\mathbf{i_r}\right) H(\theta,\xi,\epsilon)d\xi.
\end{aligned} \tag{3.212}$$

By linear independence,

$$0 = \int_\alpha^\beta \frac{f_2\sin\theta}{R} G(\theta,\xi,\epsilon)d\xi + \int_\alpha^\beta \frac{f_1\sin\theta}{R}(z-\xi)H(\theta\xi,\epsilon)d\xi \tag{3.213}$$

and

$$\begin{aligned}
e^{ikr\cos\theta} &= \int_\alpha^\beta a\left(-\frac{\cos\theta}{R^3} r(z-\xi)(f_2-2f_1)\right) G(\theta,\xi,\epsilon)d\xi + \int_\alpha^\beta a\frac{f_1}{R}(-r\cos\theta) \\
&\times H(\theta,\xi,\epsilon)d\xi + \int_\alpha^\beta b\left(-\frac{\cos\theta}{R^3}(2r^2 f_1 + (z-\xi)^2 f_2)\right) G(\theta,\xi,\epsilon)d\xi \\
&+ \int_\alpha^\beta b\frac{f_1}{R}\left(-(z-\xi)\cos\theta\right) H(\theta,\xi,\epsilon)d\xi,
\end{aligned} \tag{3.214}$$

which can be rewritten as

$$0 = \int_\alpha^\beta f_2 G(\theta,\xi,\epsilon)d\xi + \int_\alpha^\beta f_1(z-\xi)H(\theta,\xi,\epsilon)d\xi, \tag{3.215}$$

and

$$e^{ikr\cos\theta} = -\cos\theta\int_\alpha^\beta \frac{1}{R^3}\left(r(z-\xi)(f_2-2f_1)+b(2r^2f_1+(z-\xi)^2f_2)\right)G(\theta,\xi,\epsilon)d\xi$$
$$-\cos\theta\int_\alpha^\beta \frac{f_1}{R}\left(r+b(z-\xi)\right)H(\theta,\xi,\epsilon)d\xi. \tag{3.216}$$

Assume $H(\theta,\xi,\epsilon)=\frac{e^{ik\epsilon\sqrt{S}\cos\theta}}{\cos\theta}h(\xi,\epsilon)$ and $G(\theta,\xi,\epsilon)=\frac{e^{ik\epsilon\sqrt{S}\cos\theta}}{\cos\theta}g(\xi,\epsilon)$. Hence

$$0=\int_\alpha^\beta f_2 g(\xi,\epsilon)d\xi+\int_\alpha^\beta f_1(z-\xi)h(\xi,\epsilon)d\xi \tag{3.217}$$

and

$$-1=\int_\alpha^\beta \frac{1}{R^3}\left(r(z-\xi)(f_2-2f_1)+b(2r^2f_1+(z-\xi)^2f_2)\right)g(\xi,\epsilon)d\xi$$
$$\int_\alpha^\beta \frac{f_1}{R}\left(r+b(z-\xi)\right)h(\xi,\epsilon)d\xi. \tag{3.218}$$

Using the fact that $r=\epsilon\sqrt{S(z)}$, the last equation can be rewritten as

$$-1=\int_\alpha^\beta(\epsilon\sqrt{S(z)}(z-\xi)(f_2-2f_1)-\frac{\epsilon}{2}\frac{S'(z)}{\sqrt{S(z)}}(2\epsilon^2S(z)f_1+(z-\xi)^2f_2))\frac{g(\xi,\epsilon)}{R^3}d\xi$$
$$+\int_\alpha^\beta(\epsilon\sqrt{S(z)}-\frac{\epsilon}{2}\frac{S'(z)}{\sqrt{S(z)}}(z-\xi))\frac{f_1}{R}h(\xi,\epsilon)d\xi, \tag{3.219}$$

and since $r=\epsilon\sqrt{S(z)}$, it becomes

$$-\frac{2}{\epsilon}\sqrt{S(z)}=\int_\alpha^\beta(2S(z)(z-\xi)(f_2-2f_1)-S'(z)(2\epsilon^2S(z)f_1+(z-\xi)^2f_2))\frac{g(\xi,\epsilon)}{R^3}d\xi$$
$$+\int_\alpha^\beta(2S(z)-S'(z)(z-\xi))\frac{f_1}{R}h(\xi,\epsilon)d\xi=-E_1-E_2. \tag{3.220}$$

But

$$E_1=-\frac{2i}{\epsilon^2k^2}\left\{\frac{d}{dz}\int_\alpha^\beta e^{ikR}\frac{(z-\xi)}{R}h(\xi,\epsilon)d\xi-ik\int_\alpha^\beta e^{ikR}h(\xi,\epsilon)d\xi\right\} \tag{3.221}$$

and

$$E_2 = -\frac{S^{'}(z)}{k}\int_{\alpha}^{\beta}\frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi + iS^{'}(z)\int_{\alpha}^{\beta}\frac{e^{ikR}}{R}g(\xi,\epsilon)d\xi$$
$$-\frac{2S(z)}{k}\frac{d}{dz}\int_{\alpha}^{\beta}\frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi. \tag{3.222}$$

Therefore the second BC can be rewritten as

$$-2\epsilon\sqrt{S(z)} = \frac{2i}{k^2}\left\{\frac{d}{dz}\int_{\alpha}^{\beta}e^{ikR}\frac{(z-\xi)}{R}h(\xi,\epsilon)d\xi - ik\int_{\alpha}^{\beta}e^{ikR}h(\xi,\epsilon)d\xi\right\}$$
$$+\frac{\epsilon^2 S^{'}(z)}{k}\int_{\alpha}^{\beta}\frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi - i\epsilon^2 S^{'}(z)\int_{\alpha}^{\beta}\frac{e^{ikR}}{R}g(\xi,\epsilon)d\xi$$
$$+\frac{2\epsilon^2 S(z)}{k}\frac{d}{dz}\int_{\alpha}^{\beta}\frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi. \tag{3.223}$$

Recall that the BC are

$$0 = \int_{\alpha}^{\beta}\frac{f_2}{R}g(\xi,\epsilon)d\xi + \int_{\alpha}^{\beta}\frac{f_1}{R}(z-\xi)h(\xi,\epsilon)d\xi \tag{3.224}$$

and

$$\frac{2}{\epsilon}\sqrt{S(z)} = -\int_{\alpha}^{\beta}(2S(z)(z-\xi)(f_2-2f_1) - S^{'}(z)(2\epsilon^2 S(z)f_1 + (z-\xi)^2 f_2))\frac{g(\xi,\epsilon)}{R^3}d\xi$$
$$-\int_{\alpha}^{\beta}(2S(z) - S^{'}(z)(z-\xi))\frac{f_1}{R}h(\xi,\epsilon)d\xi. \tag{3.225}$$

Multiplying eq.(3.224) by $4S(z)+\epsilon^2 S'^2(z)$ and eq.(3.225) by $-\epsilon^2 S'(z)$ and adding the resultant equations yields:

$$-2\epsilon S'(z)\sqrt{S(z)} = 4S(z)\int_{\alpha}^{\beta}\frac{f_2}{R}g(\xi,\epsilon)d\xi + 4S(z)\int_{\alpha}^{\beta}\frac{f_1}{R}(z-\xi)h(\xi,\epsilon)d\xi$$
$$+\epsilon^2 S'^2(z)\int_{\alpha}^{\beta}\frac{f_2}{R}g(\xi,\epsilon)d\xi + \epsilon^2 S'(z)\int_{\alpha}^{\beta}\Big(2S(z)(z-\xi)(f_2-2f_1)$$
$$-S^{'}(z)(2\epsilon^2 S(z)f_1 + (z-\xi)^2 f_2)\Big)\frac{g(\xi,\epsilon)}{R^3}d\xi + \epsilon^2 S'(z)\int_{\alpha}^{\beta}2S(z)\frac{f_1}{R}h(\xi,\epsilon)d\xi. \tag{3.226}$$

Division by $2S(z)$ yields:

$$
\begin{aligned}
-\epsilon\frac{S'(z)}{\sqrt{S(z)}} &= 2\int_\alpha^\beta \frac{f_2}{R}g(\xi,\epsilon)d\xi + 2\int_\alpha^\beta \frac{f_1}{R}(z-\xi)h(\xi,\epsilon)d\xi \\
&\quad + \frac{\epsilon^2 S'^2(z)}{2S(z)}\int_\alpha^\beta \frac{f_2}{R}g(\xi,\epsilon)d\xi + \frac{\epsilon^2 S'(z)}{2S(z)}\int_\alpha^\beta \Big(2S(z)(z-\xi)(f_2-2f_1) \\
&\quad - S^{'}(z)(2\epsilon^2 S(z)f_1 + (z-\xi)^2 f_2)\Big)\frac{g(\xi,\epsilon)}{R^3}d\xi + \epsilon^2 S'(z)\int_\alpha^\beta \frac{f_1}{R}h(\xi,\epsilon)d\xi
\end{aligned}
\tag{3.227}
$$

and

$$
\begin{aligned}
-\epsilon\frac{S'(z)}{\sqrt{S(z)}} &= 2\int_\alpha^\beta \frac{f_2}{R}g(\xi,\epsilon)d\xi + 2\int_\alpha^\beta f_1\frac{\partial R}{\partial z}h(\xi,\epsilon)d\xi \\
&\quad + \frac{\epsilon^2 S'(z)}{2}\int_\alpha^\beta (f_2-2f_1)2\frac{\partial R}{\partial z}\frac{g(\xi,\epsilon)}{R^2}d\xi.
\end{aligned}
\tag{3.228}
$$

But

$$
\frac{1}{ik^2}\frac{d}{dz}\frac{e^{ikR}}{R} = \frac{1}{ik^2}e^{ikR}\Big(\frac{ik}{R}-\frac{1}{R^2}\Big)\frac{\partial R}{\partial z} = e^{ikR}\Big(\frac{1}{kR}+\frac{i}{k^2R^2}\Big)\frac{\partial R}{\partial z} = f_1\frac{\partial R}{\partial z} \tag{3.229}
$$

and

$$
\frac{f_2}{R} = i\frac{e^{ikR}}{R} - \frac{1}{k}\frac{e^{ikR}}{R}\Big(1+\frac{i}{kR}\Big), \tag{3.230}
$$

and

$$
\begin{aligned}
&\frac{d}{dz}\frac{e^{ikR}}{R^2}\Big(1+\frac{i}{kR}\Big) = e^{ikR}\left[ik\Big(\frac{1}{R^2}+\frac{i}{kR^3}\Big)+\Big(-\frac{2}{R^3}-\frac{3i}{kR^4}\Big)\right]\frac{\partial R}{\partial z} \\
&= k\frac{e^{ikR}}{R^2}\left(i-\frac{3}{kR}-\frac{3i}{k^2R^2}\right)\frac{\partial R}{\partial z} = k\frac{e^{ikR}}{R^2}(f_2-2f_1)\frac{\partial R}{\partial z}.
\end{aligned}
\tag{3.231}
$$

Thus the BC become

$$
\begin{aligned}
-2\epsilon\sqrt{S(z)} = &\frac{2i}{k^2}\left\{\frac{d}{dz}\int_\alpha^\beta e^{ikR}\frac{(z-\xi)}{R}h(\xi,\epsilon)d\xi - ik\int_\alpha^\beta e^{ikR}h(\xi,\epsilon)d\xi\right\} \\
&+ \frac{\epsilon^2 S^{'}(z)}{k}\int_\alpha^\beta \frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi - i\epsilon^2 S^{'}(z)\int_\alpha^\beta \frac{e^{ikR}}{R}g(\xi,\epsilon)d\xi \\
&+ \frac{2\epsilon^2 S(z)}{k}\frac{d}{dz}\int_\alpha^\beta \frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi
\end{aligned}
\tag{3.232}
$$

and

$$
\begin{aligned}
-\epsilon\frac{S'(z)}{\sqrt{S(z)}} = &\frac{2}{ik^2}\frac{d}{dz}\int_\alpha^\beta \frac{e^{ikR}}{R}h(\xi,\epsilon)d\xi + 2i\int_\alpha^\beta \frac{e^{ikR}}{R}g(\xi,\epsilon)d\xi \\
&- \frac{2}{k}\int_\alpha^\beta \frac{e^{ikR}}{R}\Big(1+\frac{i}{kR}\Big)g(\xi,\epsilon)d\xi + \frac{\epsilon^2 S(z)}{k}\frac{d}{dz}\int_\alpha^\beta \frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi.
\end{aligned}
\tag{3.233}
$$

By substituting I_0, I_1, I_2 and J with their asymptotic expansions, for the BC we obtain

$$
\begin{aligned}
-2\epsilon\sqrt{S(z)} = &\frac{2i}{k^2}\Big[2h - \epsilon^2\log\epsilon^2\Big(\frac{d}{dz}L_1 + ikJ_1\Big)h - \epsilon^2\Big(\frac{d}{dz}L_2 + ikJ_2\Big)h\Big] \\
&- i\epsilon^2 S'(z)\Big[-\log\epsilon^2 g + G_1 g\Big] + \frac{2i\epsilon^2}{k^2}\Big\{S'(z) + 2S(z)\frac{d}{dz}\Big\} \\
&\times\Big[\frac{1}{\epsilon^2}\frac{z(1-z)}{S(z)}\tilde{g} + \log\epsilon^2 L_{21}\tilde{g} + L_{22}\tilde{g}\Big]
\end{aligned}
\tag{3.234}
$$

and

$$
\begin{aligned}
-\epsilon\frac{S'(z)}{\sqrt{S(z)}} = &\frac{2}{ik^2}\frac{d}{dz}\Big[-\log\epsilon^2 h + G_1 h\Big] + 2i\Big[-\log\epsilon^2 g + G_1 g\Big] \\
&- \frac{2i}{k^2}\Big\{2 - \epsilon^2 S'(z)\frac{d}{dz}\Big\}\Big[\frac{1}{\epsilon^2}\frac{z(1-z)}{S(z)}\tilde{g} + \log\epsilon^2 L_{21}\tilde{g} + L_{22}\tilde{g}\Big],
\end{aligned}
\tag{3.235}
$$

where

$$
g(z,\epsilon) = (z-\alpha)(\beta-z)(\tilde{g}_0(z)\epsilon^3 + \tilde{g}_1(z)\epsilon^5\log\epsilon^2 + \tilde{g}_2(z)\epsilon^5 + \ldots.),
\tag{3.236}
$$

and

$$h(z,\epsilon) = h_0(z)\epsilon + h_1(z)\epsilon^3 \log \epsilon^2 + h_2(z)\epsilon^3 + ... \tag{3.237}$$

3.3.1 CALCULATION OF h AND g: TRANSVERSE CASE

Collecting $O(\epsilon)$ from eq.(3.234) leads to

$$-2\sqrt{S(z)} = \frac{4i}{k^2} h_0(z) \tag{3.238}$$

and

$$h_0(z) = \frac{ik^2}{2}\sqrt{S(z)}. \tag{3.239}$$

Collecting $O(\epsilon)$ from eq.(3.235),

$$-\frac{S'(z)}{\sqrt{S(z)}} = \frac{2}{ik^2}\frac{d}{dz}G_1 h_0 - \frac{4}{ik^2}\frac{z(1-z)}{S(z)}\tilde{g}_0(z) \tag{3.240}$$

and therefore

$$\tilde{g}_0(z) = \frac{ik^2}{4}\frac{\sqrt{S(z)}S'(z)}{z(1-z)}. \tag{3.241}$$

Collecting $O(\epsilon^3 \log \epsilon^2)$ from eq.(3.234)

$$0 = \frac{2i}{k^2}\Big[2h_1 - \Big(\frac{d}{dz}L_1 + ikJ_1\Big)h_0\Big], \tag{3.242}$$

$$h_1 = \frac{1}{2}\Big(\frac{d}{dz}L_1 + ikJ_1\Big)h_0 = \frac{ik^2}{4}\Big(\frac{d}{dz}L_1 + ikJ_1\Big)\sqrt{S(z)}. \tag{3.243}$$

Using the fact that $L_1 F = \frac{S(z)}{2}F'$ and $J_1 F = -\frac{ikS(z)}{2}F$

$$\begin{aligned} h_1 &= \frac{ik^2}{8}\Big(\frac{d}{dz}(S(z)(\sqrt{S(z)})') + k^2 S(z)^{3/2}\Big) = \frac{ik^2}{8}\Big(\frac{1}{2}\frac{d}{dz}(\sqrt{S}S') + k^2 S(z)^{3/2}\Big) \\ &= \frac{ik^2}{8}\Big(\frac{\sqrt{S}S''}{2} + \frac{(S')^2}{4\sqrt{S}} + k^2 S(z)^{3/2}\Big). \end{aligned} \tag{3.244}$$

Collecting $O(\epsilon^3 \log \epsilon^2)$ from eq.(3.235),

$$0 = \frac{2}{ik^2}\frac{d}{dz}\Big[G_1 h_1\Big] - \frac{4i}{k^2}\Big[\frac{z(1-z)}{S(z)}\tilde{g}_1 + L_{21}\tilde{g}_0\Big], \tag{3.245}$$

and

$$\tilde{g}_1 = \frac{S(z)}{z(1-z)}\Big[\frac{1}{2}\frac{d}{dz}\Big[G_1 h_1\Big] - L_{21}\tilde{g}_0\Big], \tag{3.246}$$

where the operator L_{21} is defined as

$$L_{21}\tilde{g}_0 = -\frac{ik^2}{8S}\Bigg[L_2(\frac{d}{dz}\sqrt{S}S') + ikJ_2(\sqrt{S}S') + 2(\beta_1 z + \alpha_1(1-z))\frac{\sqrt{S(z)}S'(z)}{z(1-z)}\Bigg]. \tag{3.247}$$

The operator G_1 has the form

$$G_1 h_1(z) = (V(z)-1)h_1(z) + \int_0^{1-z} e^{ikv}\frac{h_1(z+v)-h_1(z)}{v}dv + \int_0^{z} e^{ikv}\frac{h_1(z-v)-h_1(z)}{v}dv, \tag{3.248}$$

and so

$$\frac{d}{dz}\Big[G_1 h_1(z)\Big] = \frac{d}{dz}\Big[(V(z)-1)h_1(z)\Big] + \frac{e^{ikz}}{z}(h_1(0)-h_1(z)) - \frac{e^{ik(1-z)}}{1-z}(h_1(1)-h_1(z)) = \frac{d}{dz}\Big[(V(z)-1)h_1(z)\Big] + h_1(z)\Big(\frac{e^{ik(1-z)}}{1-z} - \frac{e^{ikz}}{z}\Big), \tag{3.249}$$

where

$$\frac{d}{dz}\Big[(V(z)-1)h_1(z)\Big] = \frac{d}{dz}\Bigg[\Big(\int_0^z \frac{e^{iku}-1}{u}du + \int_0^{1-z}\frac{e^{iku}-1}{u}du + \log\frac{4z(1-z)}{S(z)}\Big)h_1(z)\Bigg] = \Bigg(\frac{e^{ikz}-1}{z} - \frac{e^{ik(1-z)}-1}{1-z} + \frac{d}{dz}\Big(\log\frac{4z(1-z)}{S(z)}\Big)\Bigg)h_1(z) + \Bigg(\int_0^z \frac{e^{iku}-1}{u}du + \int_0^{1-z}\frac{e^{iku}-1}{u}du + \log\frac{4z(1-z)}{S(z)}\Bigg)h_1'(z), \tag{3.250}$$

$$\frac{d}{dz}\Big(\log\frac{4z(1-z)}{S(z)}\Big) = \frac{S(z)}{4z(1-z)}\Big(\frac{4(1-z)-4z}{S(z)} - \frac{4z(1-z)S'}{S^2}\Big), \tag{3.251}$$

and for our choice of geometry (eq.(3.1)), $S'(z) = S''(z) = 0$ for $\alpha \leq z \leq \beta$

$$g_0(\epsilon, z) = g_1(\epsilon, z) = 0, \tag{3.252}$$

$$h_0(\epsilon, z) = \frac{ik^2}{2}\sqrt{S(z)}, \tag{3.253}$$

$$h_1(\epsilon, z) = \frac{ik^4}{8}S(z)^{3/2}. \tag{3.254}$$

3.4 CALCULATION OF α AND β

So far, we have assumed that the charge distribution $f(z,\epsilon)$ in the static case and the electric and magnetic dipoles of unknown densities $g(z,\epsilon)$ and $h(z,\epsilon)$ in the dynamic case, are situated between $\alpha(\epsilon)$ and $\beta(\epsilon)$. We have also assumed the analyticity of $S(z)$ and particularly around the edges

$$S(z) = \sum_{n=1}^{\infty} c_n z^n \qquad \text{with} \qquad c_n = \frac{S^{(n)}(0)}{n!}, \tag{3.255}$$

$$S(z) = \sum_{n=1}^{\infty} d_n (1-z)^n \qquad \text{with} \qquad d_n = \frac{(-1)^n S^{(n)}(1)}{n!}. \tag{3.256}$$

The integrals above have a uniform asymptotic expansion if the expressions

$$u = \left[(z-\alpha(\epsilon))^2 + \epsilon^2 S(z)\right]^{\frac{1}{2}}, \tag{3.257}$$

$$w = \left[(\beta(\epsilon)-z)^2 + \epsilon^2 S(z)\right]^{\frac{1}{2}}. \tag{3.258}$$

have a uniform expansions of the form

$$u = \sum_{k=0}^{\infty} \epsilon^{2k} u_k(z), \tag{3.259}$$

$$w = \sum_{k=0}^{\infty} \epsilon^{2k} w_k(z), \tag{3.260}$$

where

$$u_k(z) = \frac{1}{k!}\frac{d^k}{d\epsilon^{2k}}\left[(z-\alpha(\epsilon))^2+\epsilon^2 S(z)\right]^{\frac{1}{2}}\Big|_{\epsilon=0} \tag{3.261}$$

and

$$w_k(z) = \frac{1}{k!}\frac{d^k}{d\epsilon^{2k}}\left[(\beta(\epsilon)-z)^2+\epsilon^2 S(z)\right]^{\frac{1}{2}}\Big|_{\epsilon=0}. \tag{3.262}$$

It is clear, however, that the u_k's are going to be singular except for certain values of α_i and w_k's are going to be singular except for certain values of β_i. By choosing appropriate α_i and β_i, we can remove the singularities in eq.(3.261) and eq.(3.262), respectively, and thus make them regular [1]. This will allow us to determine $\alpha(\epsilon)$ and $\beta(\epsilon)$ in such a way, that the asymptotic expansions for the integrals I_0, I_1, I_2 and J are uniform. To see this, let's look at the expansion for u.

For $k=0$

$$u_0 = z. \tag{3.263}$$

For $k=1$

$$u_1 = \frac{-2(z-\alpha)\frac{d\alpha}{d\epsilon^2}+S(z)}{2\left[(z-\alpha(\epsilon))^2+\epsilon^2 S(z)\right]^{\frac{1}{2}}}\Bigg|_{\epsilon=0} = -\alpha_1+\frac{S(z)}{2z}. \tag{3.264}$$

For $k=2$

$$\begin{aligned} u_2 &= \frac{1}{2}\frac{(\frac{d\alpha}{d\epsilon^2})^2-(z-\alpha)\frac{d^2\alpha}{d\epsilon^4}}{u}\Bigg|_{\epsilon=0} - \frac{1}{2}\frac{\left(-2(z-\alpha)\frac{d\alpha}{d\epsilon^2}+S(z)\right)^2}{4u^3}\Bigg|_{\epsilon=0} \\ &= \frac{1}{2}\left(\frac{\alpha_1^2-2\alpha_2 z}{z}-\frac{(-2z\alpha_1+S(z))^2}{4z^3}\right) = \frac{1}{2}\left(-2\alpha_2+\frac{\alpha_1^2}{z}-\frac{1}{z^3}\Big(-\alpha_1 z+\frac{S(z)}{2}\Big)^2\right), \end{aligned} \tag{3.265}$$

which can be rewritten as

$$u_2 = -\alpha_2+\frac{1}{2z}\left(\alpha_1^2-\left(\alpha_1-\frac{S(z)}{2z}\right)^2\right). \tag{3.266}$$

Recall that

$$S(z) = c_1 z + c_2 z^2 + c_3 z^3 + ..., \tag{3.267}$$

which yields

$$\frac{S(z)}{z} = c_1 + c_2 z + c_3 z^2 + ... \qquad \text{and so} \qquad \lim_{z\to 0} \frac{S(z)}{z} = c_1. \tag{3.268}$$

Obviously u_0 and u_1 are regular, however u_2 has a singular part, which vanishes if we make the choice

$$\alpha_1^2 - \left(\alpha_1 - \frac{S(z)}{2z}\right)^2 = 0, \tag{3.269}$$

which can be written as

$$\alpha_1^2 - \left(\alpha_1 - \frac{c_1}{2}\right)^2 = 0, \tag{3.270}$$

or

$$\frac{c_1}{2}\left(2\alpha_1 - \frac{c_1}{2}\right) = 0, \tag{3.271}$$

and therefore

$$\alpha_1 = \frac{c_1}{4}. \tag{3.272}$$

For $k = 3$

$$\begin{aligned} u_3 = \frac{1}{6}\Bigg[& \frac{3\frac{d\alpha}{d\epsilon^2}\frac{d^2\alpha}{d\epsilon^4} - (z-\alpha)\frac{d^3\alpha}{d\epsilon^6}}{u} - \frac{(\frac{d\alpha}{d\epsilon^2})^2 - (z-\alpha)\frac{d^2\alpha}{d\epsilon^4}}{u^2}\frac{du}{d\epsilon^2} \\ & - \frac{2(-2(z-\alpha)\frac{d\alpha}{d\epsilon^2} + S(z))(2\left(\frac{d\alpha}{d\epsilon^2}\right)^2 - 2(z-\alpha)\frac{d^2\alpha}{d\epsilon^4})}{4u^3} + 3\frac{\left(-2(z-\alpha)\frac{d\alpha}{d\epsilon^2} + S(z)\right)^2}{4u^4}\frac{du}{d\epsilon^2}\Bigg] \\ = \frac{1}{6}\Bigg[& \frac{6\alpha_1\alpha_2 - 6z\alpha_3}{z} - 3\frac{(\alpha_1^2 - 2z\alpha_2)(-2z\alpha_1 + S(z))}{2z^3} - \frac{(\alpha_1^2 - z\alpha_2)(-2z\alpha_1 + S(z))}{z^3} \\ & + 3\frac{(-2z\alpha_1 + S(z))^3}{8z^5}\Bigg], \end{aligned} \tag{3.273}$$

so that

$$u_3 = \frac{1}{6}\left[\frac{6\alpha_1\alpha_2 - 6z\alpha_3}{z} - 3\frac{(\alpha_1^2 - 2z\alpha_2)(-2z\alpha_1 + S(z))}{2z^3} + 3\frac{(-2z\alpha_1 + S(z))^3}{8z^5}\right]$$
$$= \frac{\alpha_1\alpha_2 - z\alpha_3}{z} - \frac{(\alpha_1^2 - 2z\alpha_2)(-2z\alpha_1 + S(z))}{4z^3} + \frac{(-2z\alpha_1 + S(z))^3}{16z^5}, \tag{3.274}$$

which can be rewritten as

$$u_3 = -\alpha_3 + \frac{\alpha_1\alpha_2}{z} - \frac{1}{2z^2}(\alpha_1^2 - 2z\alpha_2)\left(-\alpha_1 + \frac{S(z)}{2z}\right) + \frac{1}{2z^2}\left(-\alpha_1 + \frac{S(z)}{2z}\right)^3. \tag{3.275}$$

The singularity in u_3 can be removed by choosing

$$\alpha_1\alpha_2 - \frac{1}{2}\frac{c_2}{2}\alpha_1^2 - 2\alpha_1\alpha_2 = 0, \tag{3.276}$$

leading to

$$\alpha_2 = -\frac{1}{16}c_1c_2, \tag{3.277}$$

and so on. By apply similar arguments about the expansion coefficients of w, we get

$$\alpha = c_1\left(\frac{\epsilon}{2}\right)^2 - c_1c_2\left(\frac{\epsilon}{2}\right)^4 + c_1(c_1c_3 + 2c_2^2)\left(\frac{\epsilon}{2}\right)^6 + O(\epsilon^8),$$
$$\beta = 1 - d_1\left(\frac{\epsilon}{2}\right)^2 + d_1d_2\left(\frac{\epsilon}{2}\right)^4 - d_1(d_1d_3 + 2d_2^2)\left(\frac{\epsilon}{2}\right)^6 + O(\epsilon^8),$$

which for our particular choice of geometry yields

$$c_1 = S_1'(0) = \frac{2}{\epsilon} \qquad c_2 = \frac{S_1''(0)}{2} = -\frac{2}{\epsilon^2} \qquad c_3 = 0$$
$$d_1 = -S_2'(1) = \frac{2}{\epsilon} \qquad d_2 = \frac{S_2''(1)}{2} = -\frac{2}{\epsilon^2} \qquad d_3 = 0,$$

and so

$$\alpha = \epsilon + O(\epsilon^8),$$

$$\beta = 1 - \epsilon + O(\epsilon^8). \tag{3.278}$$

CHAPTER IV

SCATTERING AND RADIATIVE PROPERTIES

In this chapter, we investigate the scattering and radiative properties of nanofibers. In Chapter III we obtained results for the scattered electric field in both the axial and the transverse cases. As pointed out before, the results from the axial case will be used only for estimation of the field potential in the field emission calculation. Here we investigate the transverse incidence results and extend them for the case when the fiber is at an arbitrary distance c from the origin along the z-axis. These results are used to obtain the current distribution in the fiber and the corresponding radiation pattern. Finally, we investigate the antenna array properties of N fibers aligned along the z-axis.

We start with a fiber situated between c and $c+1$ on the z-axis (fig.4.1). Just as before, the incident field is

$$\mathbf{E_{inc}} = e^{ikr\cos\theta}\mathbf{i_z} \tag{4.1}$$

The resultant scattered electric field $\mathbf{E_s}$ as a linear superposition of fields due to electric and magnetic dipoles along the z-axis with unknown densities g and h in between α_c and β_c, where

$$\alpha_c = c + \alpha_1\epsilon^2 + \alpha_2\epsilon^4 + .., \tag{4.2}$$

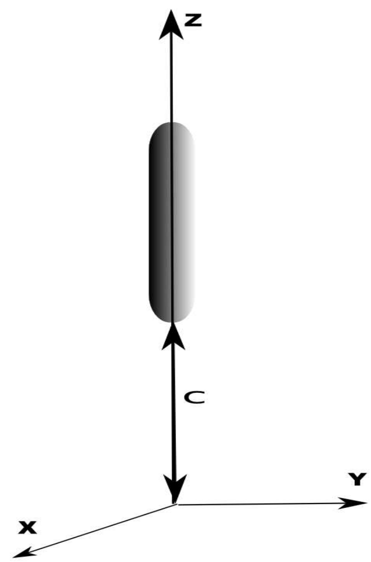

Figure 4.1: Fiber at an arbitrary distance c from the origin

$$\beta_c = c + 1 - \beta_1\epsilon^2 - \beta_2\epsilon^4 + .. \tag{4.3}$$

The scattered electric field $\mathbf{E_s}$ is:

$$\mathbf{E_s} = \int_{\alpha_c}^{\beta_c} \left(-\frac{\cos\theta}{R^3}(2r^2 f_1 + (z-\xi)^2 f_2)\mathbf{i_r} + \frac{\sin\theta}{R} f_2 \mathbf{i}_\theta + \frac{\cos\theta}{R^3} r(z-\xi)(f_2 - 2f_1)\mathbf{i_z} \right) \times G(\theta,\xi,\epsilon)d\xi + \int_{\alpha_c}^{\beta_c} \frac{f_1}{R} \left(-(z-\xi)\cos\theta\mathbf{i_r} + (z-\xi)\sin\theta\mathbf{i}_\theta + r\cos\theta\mathbf{i_z}\right) H(\theta,\xi,\epsilon)d\xi. \tag{4.4}$$

The tangential components of the electric field on the surface of the body ($r = \epsilon\sqrt{S(z)}$) vanish, i.e.

$$\mathbf{n} \times \mathbf{E_{inc}} = -\mathbf{n} \times \mathbf{E_s}, \tag{4.5}$$

and so the BC are

$$0=\int_{\alpha_c}^{\beta_c}\frac{f_2\sin\theta}{R}G(\theta,\xi,\epsilon)d\xi+\int_{\alpha_c}^{\beta_c}\frac{f_1\sin\theta}{R}(z-\xi)H(\theta\xi,\epsilon)d\xi, \tag{4.6}$$

and

$$\begin{aligned}e^{ikr\cos\theta} &= \int_{\alpha_c}^{\beta_c} a\left(-\frac{\cos\theta}{R^3}r(z-\xi)(f_2-2f_1)\right)G(\theta,\xi,\epsilon)d\xi \\ &- \int_{\alpha_c}^{\beta_c} a\frac{f_1}{R}r\cos\theta H(\theta,\xi,\epsilon)d\xi-\int_{\alpha_c}^{\beta_c} b\frac{\cos\theta}{R^3}(2r^2f_1+(z-\xi)^2f_2)G(\theta,\xi,\epsilon)d\xi \\ &+ \int_{\alpha_c}^{\beta_c} b\frac{f_1}{R}\left(-(z-\xi)\cos\theta\right)H(\theta,\xi,\epsilon)d\xi, \end{aligned} \tag{4.7}$$

which by assuming that $H(\theta,\xi,\epsilon)=\frac{e^{ik\epsilon\sqrt{S}\cos\theta}}{\cos\theta}h(\xi,\epsilon)$ and $G(\theta,\xi,\epsilon)=\frac{e^{ik\epsilon\sqrt{S}\cos\theta}}{\cos\theta}g(\xi,\epsilon)$ and after manipulation can be rewritten as

$$0=\int_{\alpha_c}^{\beta_c}\frac{f_2}{R}g(\xi,\epsilon)d\xi+\int_{\alpha_c}^{\beta_c}\frac{f_1}{R}(z-\xi)h(\xi,\epsilon)d\xi \tag{4.8}$$

and

$$\begin{aligned}\frac{2}{\epsilon}\sqrt{S(z)} &= -\int_{\alpha_c}^{\beta_c}(2S_2(z)(z-\xi)(f_2-2f_1)-S_2^{'}(z)(2\epsilon^2S_2(z)f_1+(z-\xi)^2f_2))\frac{g(\xi,\epsilon)}{R^3}d\xi \\ &- \int_{\alpha_c}^{\beta_c}(2S_2(z)-S_2^{'}(z)(z-\xi))\frac{f_1}{R}h(\xi,\epsilon)d\xi, \end{aligned} \tag{4.9}$$

i.e.

$$\begin{aligned}-2\epsilon\sqrt{S(z)} &= \frac{2i}{k^2}\left\{\frac{d}{dz}\int_{\alpha_c}^{\beta_c} e^{ikR}\frac{(z-\xi)}{R}h(\xi,\epsilon)d\xi-ik\int_{\alpha}^{\beta} e^{ikR}h(\xi,\epsilon)d\xi\right\} \\ &+ \frac{\epsilon^2S^{'}(z)}{k}\int_{\alpha_c}^{\beta_c}\frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi-i\epsilon^2S^{'}(z)\int_{\alpha_c}^{\beta_c}\frac{e^{ikR}}{R}g(\xi,\epsilon)d\xi \\ &+ \frac{2\epsilon^2S(z)}{k}\frac{d}{dz}\int_{\alpha_c}^{\beta_c}\frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi \end{aligned} \tag{4.10}$$

and

$$-\epsilon\frac{S'(z)}{\sqrt{S(z)}} = \frac{2}{ik^2}\frac{d}{dz}\int_{\alpha_c}^{\beta_c}\frac{e^{ikR}}{R}h(\xi,\epsilon)d\xi + 2i\int_{\alpha_c}^{\beta_c}\frac{e^{ikR}}{R}g(\xi,\epsilon)d\xi - \frac{2}{k}\int_{\alpha_c}^{\beta_c}\frac{e^{ikR}}{R}\Big(1+\frac{i}{kR}\Big)$$

$$\times\, g(\xi,\epsilon)d\xi + \frac{\epsilon^2 S(z)}{k}\frac{d}{dz}\int_{\alpha_c}^{\beta_c}\frac{e^{ikR}}{R^2}\left(1+\frac{i}{kR}\right)g(\xi,\epsilon)d\xi, \tag{4.11}$$

which after substitution of the asymptotic expansions for $I_{c0}(z,\epsilon,F)$, $I_{c1}(z,\epsilon,F)$, $I_{c2}(z,\epsilon,F)$ and $J_c(z,\epsilon,F)$ become

$$-\epsilon\frac{S'(z)}{\sqrt{S(z)}} = \frac{2}{ik^2}\frac{d}{dz}\Big[-\log\epsilon^2 h + G_{c1}h\Big] + 2i\Big[-\log\epsilon^2 g + G_{c1}g\Big]$$

$$-\frac{2i}{k^2}\Big\{2-\epsilon^2 S'(z)\frac{d}{dz}\Big\}\Big[\frac{1}{\epsilon^2}\frac{(z-c)(c+1-z)}{S_2(z)}\tilde{g} + \log\epsilon^2 L_{c21}\tilde{g} + L_{c22}\tilde{g}\Big] \tag{4.12}$$

and

$$-2\epsilon\sqrt{S(z)} = \frac{2i}{k^2}\Big[2h - \epsilon^2\log\epsilon^2\Big(\frac{d}{dz}L_{c1}+ikJ_1\Big)h - \epsilon^2\Big(\frac{d}{dz}L_{c2}+ikJ_{c2}\Big)h\Big]$$

$$-\, i\epsilon^2 S_2'(z)\Big[-\log\epsilon^2 g + G_{c1}g\Big] + \frac{2i\epsilon^2}{k^2}\Big\{S_2'(z) + 2S(z)\frac{d}{dz}\Big\}$$

$$\times\Big[\frac{1}{\epsilon^2}\frac{(z-c)(c+1-z)}{S(z)}\tilde{g} + \log\epsilon^2 L_{c21}\tilde{g} + L_{c22}\tilde{g}\Big], \tag{4.13}$$

where

$$g(z,\epsilon) = (z-\alpha_c)(\beta_c - z)\Big(\tilde{g}_0(z)\epsilon^3 + \tilde{g}_1(z)\epsilon^5\log\epsilon^2 + \tilde{g}_2(z)\epsilon^5 + \ldots.\Big) \tag{4.14}$$

and

$$h(z,\epsilon) = h_0(z)\epsilon + h_1(z)\epsilon^3\log\epsilon^2 + h_2(z)\epsilon^3 + \ldots. \tag{4.15}$$

The linear operators G_{c1}, $V_c(z)$, L_{c21} and L_{c22} are given by

$$G_{c1}F = (V_c(z)-1)F(z) + \int_0^{c+1-z}\frac{F(z+v)-F(z)}{v}e^{ikv}dv$$

$$+\int_0^{z-c}\frac{F(z-v)-F(z)}{v}e^{ikv}dv \tag{4.16}$$

and

$$V_c(z) = \log\left(\frac{(z-c)(c+1-z)}{4S(z)}\right) + \int_0^{c+1-z} \frac{e^{iku}-1}{u}du + \int_0^{z-c} \frac{e^{iku}-1}{u}du + 1, \tag{4.17}$$

$$L_{c21} = -\frac{1}{4}\Big(\frac{d^2}{dz^2}(z-c)(c+1-z))\tilde{F}(z) + k^2(z-c)(c+1-z))\tilde{F}(z)\Big), \tag{4.18}$$

and

$$\begin{aligned} L_{c22} = -\frac{1}{S_2(z)}\Bigg(& L_2\left(\frac{d}{dz}(z-c)(c+1-z)\tilde{F}(z)\right) + ikJ_2(z-c)(c+1-z)\tilde{F}(z)) \\ & - e^{ik(c+1-z)}F(c+1)\beta_1 - e^{ik(z-c)}F(b)\alpha_1 + 2(\beta_1(z-c) + \alpha_1(c+1-z))\tilde{F}(z)\Bigg). \end{aligned} \tag{4.19}$$

By collecting $O(\epsilon)$ and $O(\epsilon^3 \log \epsilon^2)$ from eq.(4.12) and eq.(4.13), we get

$$\begin{aligned} h_0(z) &= \frac{ik^2}{2}\sqrt{S(z)}, \\ \tilde{g}_0(z) &= \frac{ik^2}{4}\frac{\sqrt{S(z)}S'(z)}{(z-c)(c+1-z)}, \\ h_1 &= \frac{ik^2}{8}\Big(\frac{\sqrt{S}S''}{2} + \frac{(S')^2}{4\sqrt{S}} + k^2 S(z)^{3/2}\Big), \\ \tilde{g}_1 &= \frac{S(z)}{(z-c)(c+1-z)}\Big[\frac{1}{2}\frac{d}{dz}\Big[G_1 h_1\Big] - L_{21}\tilde{g}_0\Big]. \end{aligned} \tag{4.20}$$

For our choice of geometry, i.e.

$$\begin{aligned} r^2 &= 2\epsilon(z-c) - (z-c)^2 = \epsilon^2 S_1(z) && \text{for } c < z < c+\epsilon \\ r^2 &= \epsilon^2 && \text{for } c+\epsilon < z < c+1-\epsilon \\ r^2 &= 2(z-c)(1-\epsilon) - (z-c)^2 + 2\epsilon - 1 = \epsilon^2 S_2(z) && \text{for } c+1-\epsilon < z < c+1 \end{aligned} \tag{4.21}$$

between the integration limits α_c and β_c, $\tilde{g}_0 = \tilde{g}_1 = 0$ and

$$h(z,\epsilon) = h_0(z)\epsilon + h_1(z)\epsilon^3 \log \epsilon^2 = \frac{ik^2}{2}\epsilon + \frac{ik^4}{8}\epsilon^3 \log \epsilon^2 + O(\epsilon^3). \tag{4.22}$$

Scattering and antenna properties are closely related: An incident electric field on the surface of a conducting fiber induces surface current, which radiates and generates the scattered field. In the case of a fiber acting as an antenna, the incident field is produced by the antenna feed. In the case of a fiber acting as a scatterer, the induced current is simply an Eddy current. Here we are going to assume that the fiber is a scatterer and that the induced current distribution is due to incident transverse plane wave (eq.(4.1)).

Recall that the magnetic vector potential due to a single magnetic dipole at the origin is $\mathbf{A}_m = ik(\mathbf{i_R} \times \mathbf{m})\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right)$ and we choose $\mathbf{m}$ to be $\mathbf{m} = (0,1,0)$, so for a magnetic dipole at a distance ξ from the origin

$$\begin{aligned} \mathbf{i_R} \times \mathbf{m} &= \frac{x\mathbf{i_x} + y\mathbf{i_y} + (z-\xi)\mathbf{i_z}}{R} \times \mathbf{i_y} = \frac{x}{R}\mathbf{i_z} - \frac{z-\xi}{R}\mathbf{i_x} \\ &= \frac{r}{R}\cos\theta \mathbf{i_z} - \frac{z-\xi}{R}(\cos\theta \mathbf{i_r} - \sin\theta \mathbf{i}_\theta). \end{aligned} \tag{4.23}$$

Therefore the z component of the magnetic vector potential A due to magnetic dipoles distributed with density $H(z,\epsilon,\theta)$ along the z-axis is given by

$$A_z = r\cos\theta \int_{\alpha_b}^{\beta_b} \frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right) H(\xi,\epsilon,\theta)d\xi, \tag{4.24}$$

where $R = \sqrt{(z-\xi)^2 + r^2}$.

On the surface of the fiber, between α_c and β_c, A_z becomes

$$A_z = \epsilon e^{ik\epsilon\cos\theta} \int_{\alpha_c}^{\beta_c} \frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right) h(\xi,\epsilon)d\xi, \tag{4.25}$$

where $R = \sqrt{(z-\xi)^2 + \epsilon^2 S(z)}$.

Furthermore, for wavelengths comparable to the size of the nanofiber, $k \sim 1$ and $e^{ik\epsilon\cos\theta} \sim O(1)$. Therefore

$$A_z \sim \epsilon \int_{\alpha_c}^{\beta_c} \frac{e^{ikR}}{R^2} \left(ik - \frac{1}{R} \right) h(\xi, \epsilon) d\xi. \tag{4.26}$$

On the other hand, for a thin center-fed antenna aligned along the z-axis, the current can be assumed to be θ independent and only on the surface and the magnetic vector potential is :

$$A_z = \frac{\mu}{4\pi} \int_c^{c+1} \frac{e^{-ikR}}{R} \mathcal{J}(\xi) d\xi. \tag{4.27}$$

In the limit $\epsilon \to 0$, i.e in the case of a very thin antenna, $\alpha_c \to c$ and $\beta_c \to c+1$. This corresponds to the case of a regular cylinder and allows us to compare the RHS of eq.(4.27) to the known magnetic vector potential from eq.(4.26). Such a comparison yields a Fredholm integral equation of the first type with the standard Helmholtz kernel.

We are going to solve for the unknown current density distribution $\mathcal{J}$ by using the Method of Moments, i.e. discretizing the length of the antenna and solving numerically. For a given current density distribution, the radiation pattern of a single fiber due to the induced currents is given by:

$$|f_e(\theta)| = \sin\theta \int_c^{c+1} \mathcal{J}(z') e^{ikz'\cos\theta} dz'. \tag{4.28}$$

Once we have the current density distribution $\mathcal{J}$ of a single fiber, we can align an arbitrary number of them (see fig.4.2) and investigate the array properties.

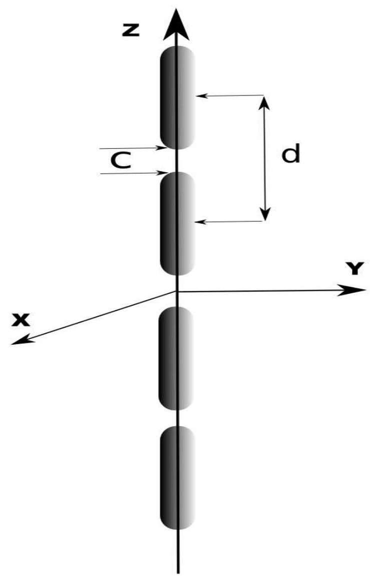

Figure 4.2: N cylinder problem

The total field is a linear superposition of the fields of each of them. If the current distribution is the same in each fiber (which is the case in antenna arrays), the total field is the field of a single fiber multiplied by an array factor, which depends on the number of array elements, the current phase difference between them, as well as their spatial separation. Since the incident electric field is a transverse plane wave, the induced current in all the fibers have the same current distribution and the same phase. Such arrays are called uniform broadside arrays. In the case of $2N$ fibers aligned symmetrically with respect to the origin, separated by a distance c in between, the array factor for them is [65]:

$$f_{array}(\theta) = \sum_{n=-N}^{N} e^{inkd\cos\theta}, \tag{4.29}$$

where $d = c + 1$ is the distance separating the centers of the fibers. Note that the array factor from eq.(4.29) depends only on the number of elements and the separation between them.

CHAPTER V

RESULTS AND DISCUSSION

In this chapter we present the results obtained using the derivations in Chapters II, III and IV. First, we investigate the theoretical field emission results in the static case and compare them to the Fowler-Nordheim model, as well as with recent experimental data [48] for field emission from **Ni** nanowires. Next, we investigate the theoretical field emission results in the dynamic case. Finally, we investigate the results for the current distributions in a fiber due to an incident transverse plane wave, and compare them with the results with the numerical results obtained from the solution of Hallen's equation, compare the radiation patterns and extend the results for the case of a uniform broadside array of $2N$ fibers.

5.1 FIELD EMISSION: STATIC CASE

In this section we demonstrate the results from the calculation for the FE current density from **Ni** nanofibers in the static case and compare it to the Fowler-Nordheim model as well as with recent experimental data [48].

For the the electrostatic energy of the external field, we derived and analytic expression (eq.(3.22)), using the slender body approximation. Recall that this approximation is applicable for bodies with rotational symmetry along one of their axis (z-axis in our case) and a large aspect ratio (here we investigate the cases $\epsilon = 0.1$, $\epsilon = 0.01$, $\epsilon = 0.001$ and $\epsilon = 0.0001$). We use the results from eq.(3.22) in conjunction with the 1D WKB approximation (eq.(2.13)), the quantum wire DOS (eq.(2.16)) in order to estimate the tunneling current density (eq.(2.14)) as described in Chapter II. We also take into account the fact that the electron leaving the surface of the cathode influences the shape of the potential barrier by introducing a mirror image term [61]

$$W_m \sim -\frac{1}{d}, \tag{5.1}$$

where d is the distance between the charge and the surface.

In the static case, we can compare our results to the Fowler-Nordheim model, in which the current density is given by

$$J = aE^2 e^{-b/E}, \tag{5.2}$$

where E is the electric field strength, $a = 1.56 \times 10^{-10}\beta^2/\phi$, $b = 6.83 \times 10^9 \phi^{3/2}/\beta$, $\phi = 5.15eV$ is the work function for **Ni** and $\beta = 1300$ is the field enhancement factor [48]. Figure 5.1 shows the current density J from **Ni** nanowires as a function of applied electric field for the Fowler-Nordheim case with a, b and β provided by [48].

Recent experimental work [48] reports FE current densities in the range 3.0×10^{-9} - 1.0×10^{-3}A$/cm^2$, i.e. 3.0×10^{-5} - 1.0×10A$/m^2$ with threshold electric field strength 4V$/\mu m$, i.e. 4×10^6V$/m$. Comparison of our theoretical results to the Fowler-Nordheim model and the experimental data leads to several important observation. One of them is that both the Fowler-Nordheim and our theoretical model predict tunneling (and resulting FE currents) at lower voltages, i.e 2.3-2.7 $\times$ 10^6V$/m$ versus

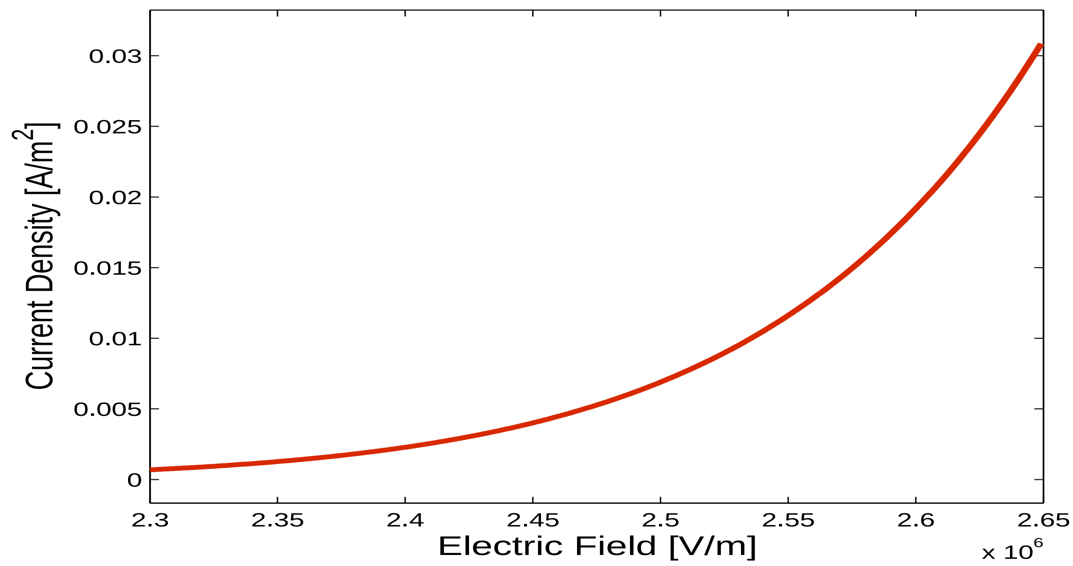

Figure 5.1: Current density: Fowler-Nordheim model

the experimentally determined threshold of $4 \times 10^6 \mathrm{V}/m$. There are several possible factors that might cause that difference: the electrostatic interaction of the tunneling electron with the surface charges of the fiber, the imperfect geometry of the fibers in the experiment, the nonuniform applied electric field, etc. Another important observation is that within the range of applied electric field intensities, our theoretical model predicts FE current densities which are two orders of magnitude closer to the experimental results than the Fowler-Nordheim model. Thus could be explained by the fact that the Fowler-Nordheim model assumes $0°K$ and DOS for a bulk material. Also, the Fowler-Nordheim model is derived for a flat surface and thus it is insensitive to the geometry (radius and length) of the fiber.

Figures 5.2, 5.3, 5.4 and 5.5 show the results for the current density J from **Ni** nanowires as a function of applied static electric field as calculated by the proposed model in Chapter II for the cases of $\epsilon = 0.1$, $\epsilon = 0.01$, $\epsilon = 0.001$ and $\epsilon = 0.0001$, respectively. Results show that as the fiber radius ϵ decreases, so does the FE current density.

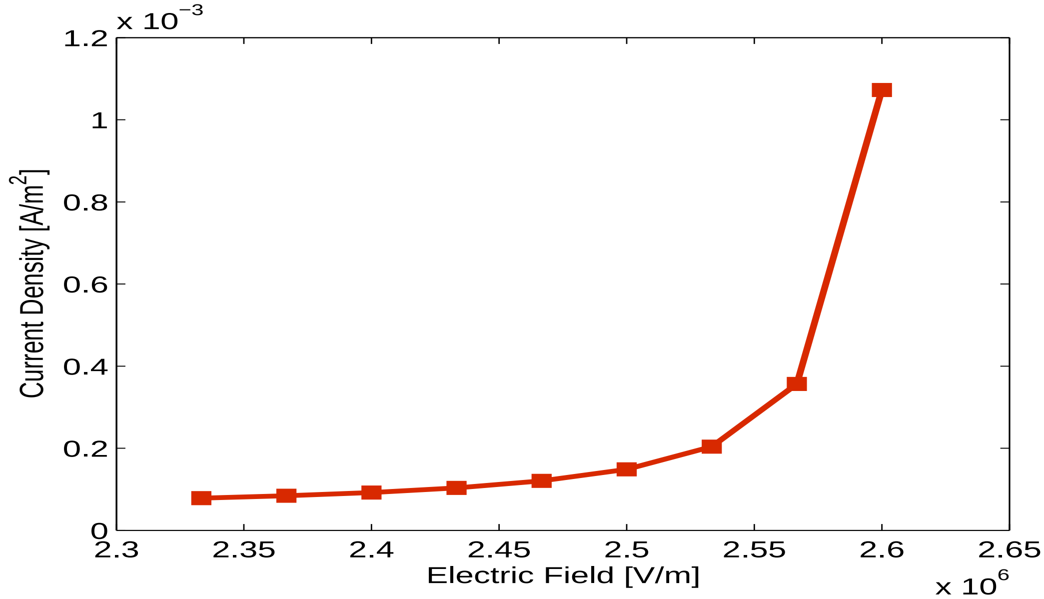

Figure 5.2: Current density: Static case, $\epsilon = 0.1$

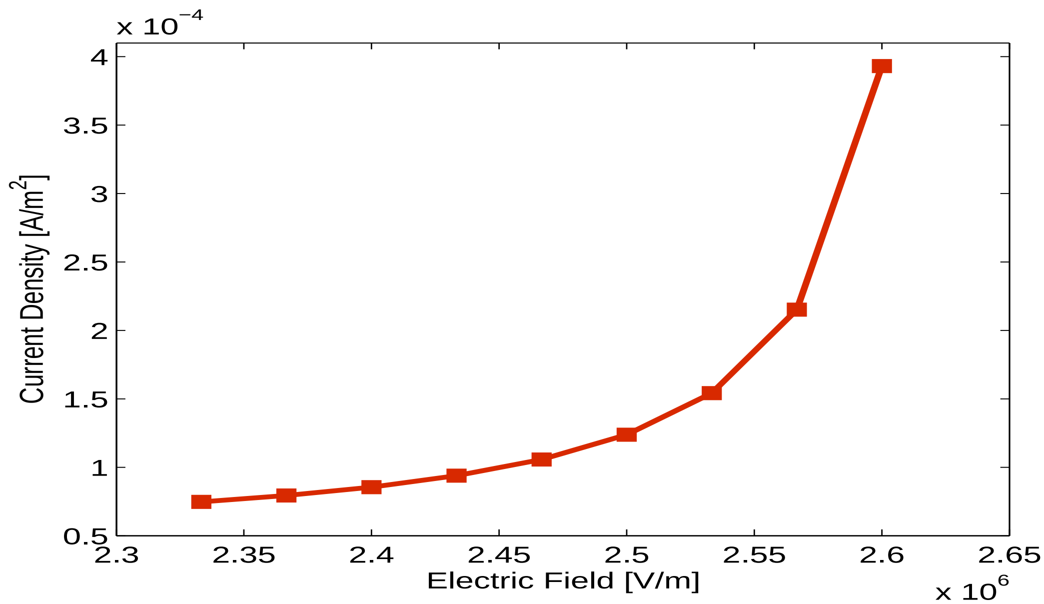

Figure 5.3: Current density: Static case, $\epsilon = 0.01$

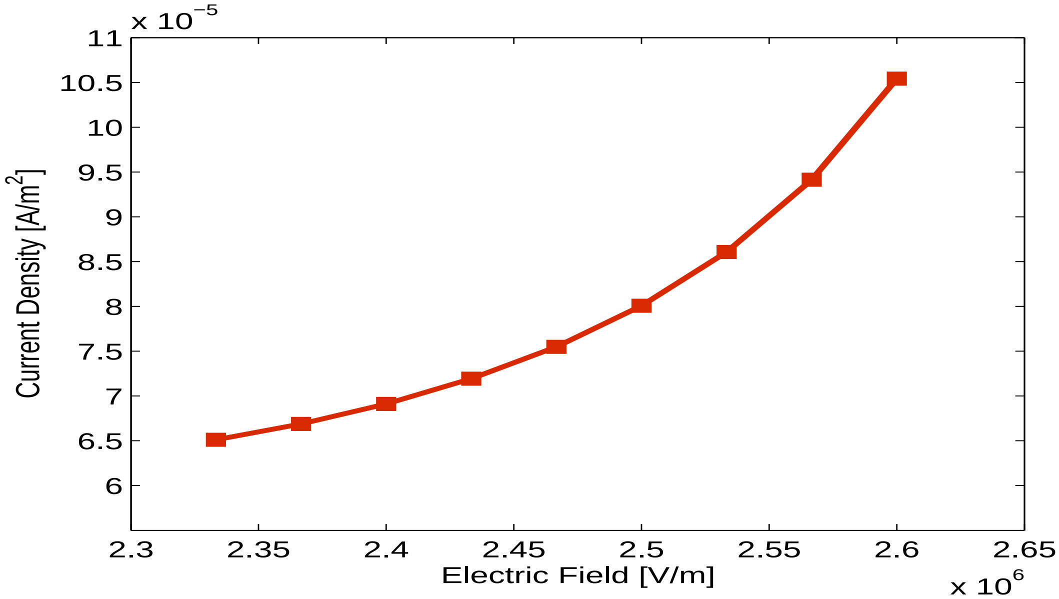

Figure 5.4: Current density: Static case $\epsilon = 0.001$

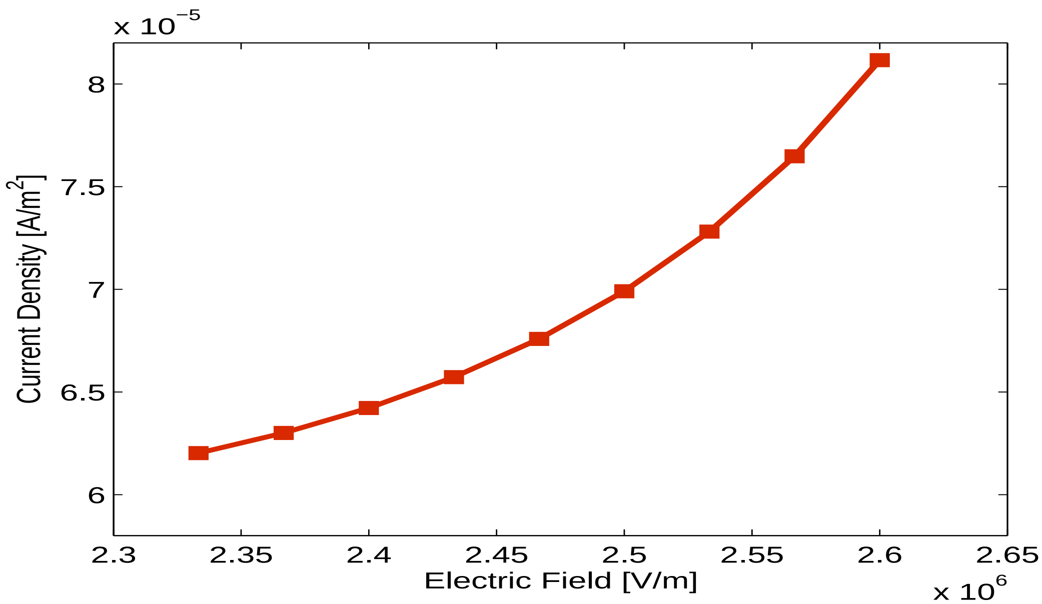

Figure 5.5: Current density: Static case $\epsilon = 0.0001$

5.2 FIELD EMISSION: DYNAMIC CASE

In this section we demonstrate the results from the field emission calculations for the dynamic case. We use the results for the potential V due to incident axial dynamic electric field derived in Chapter III in conjunction to the numerical method we developed in Chapter II for the calculation of the field emission. Analogously to the static case, we introduce a mirror image term to the expression for the total potential.

Figures 5.6, 5.7, 5.8 and 5.9 show the results for the current density J from **Ni** nanowires as a function of applied dynamic electric field (k=3.3, $\lambda = 1.904\mu m$) as calculated by the proposed model in Chapter II for the cases of $\epsilon = 0.1$, $\epsilon = 0.01$, $\epsilon = 0.001$ and $\epsilon = 0.0001$, respectively.

Figures 5.10, 5.11, 5.12 and 5.13 show the results for the current density J from **Ni** nanowires as a function of applied dynamic electric field (k=3.5, $\lambda = 1.795\mu m$) for the cases of $\epsilon = 0.1$, $\epsilon = 0.01$, $\epsilon = 0.001$ and $\epsilon = 0.0001$, respectively.

Figures 5.14, 5.15, 5.16 and 5.17 show the results for the current density J from **Ni** nanowires as a function of applied dynamic electric field (k=3.7,$\lambda = 1.698\mu m$) for the cases of $\epsilon = 0.1$, $\epsilon = 0.01$, $\epsilon = 0.001$ and $\epsilon = 0.0001$, respectively.

To our knowledge, there are no theoretical models or published experimental results we can refer to for comparison to our calculated values. However, our model predicts the following important results: Just like in the static case, as the system radius ϵ decreases, so does the FE current density for a given frequency. Also, for a fixed geometry (i.e. ϵ and L), the FE current density increases as the frequency of the incident electric field increases. Since we assumed realistic geometry for the nanofiber, as well as realistic conditions, both the static an dynamic FE results from our model could be experimentally validated.

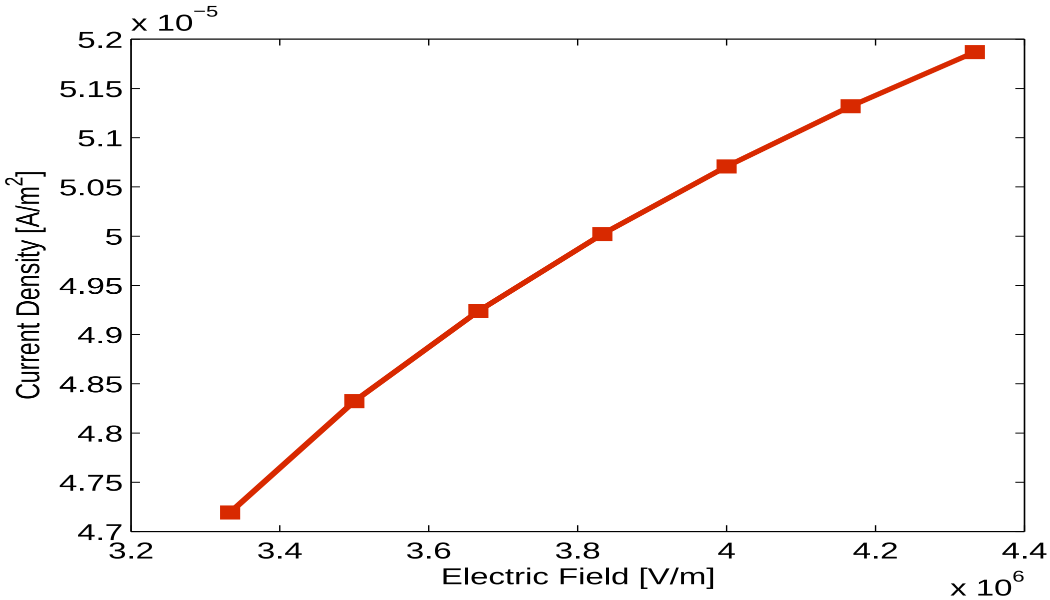

Figure 5.6: Current density: Dynamic case (k=3.3, $\lambda = 1.904\mu m$), $\epsilon = 0.1$

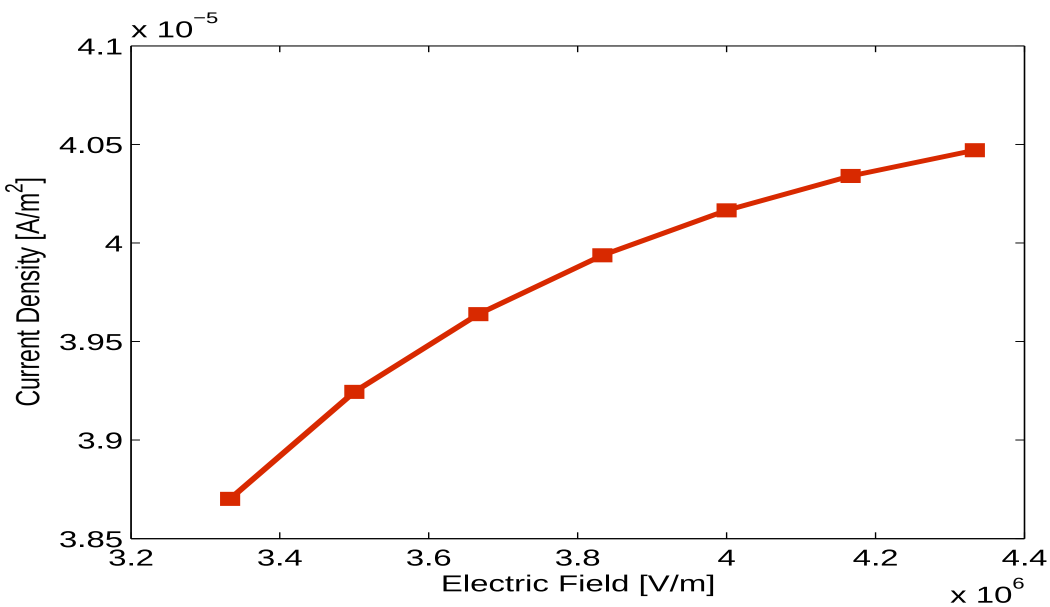

Figure 5.7: Current density: Dynamic case (k=3.3, $\lambda = 1.904\mu m$), $\epsilon = 0.01$

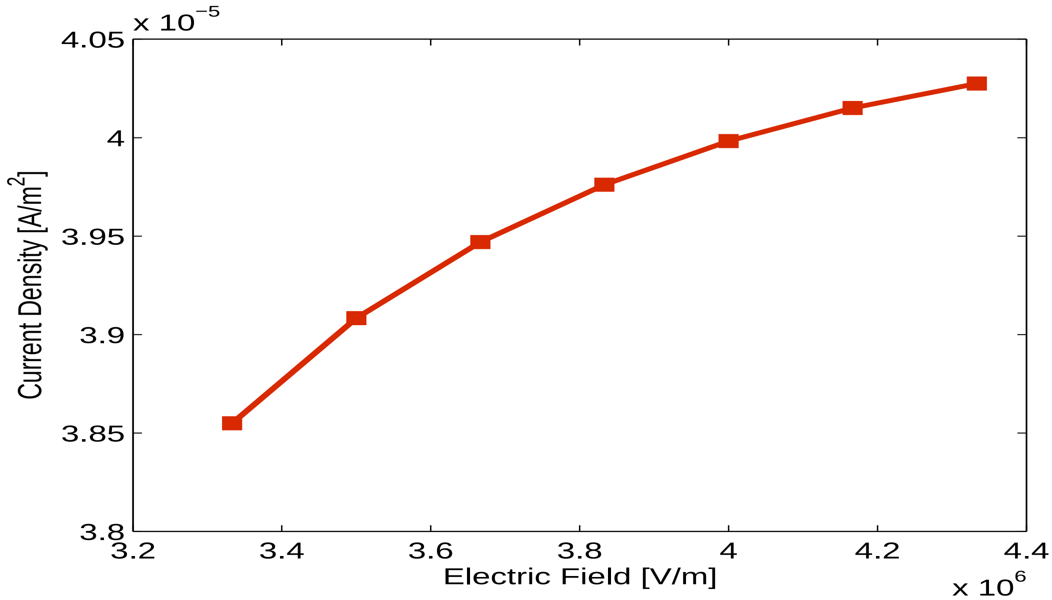

Figure 5.8: Current density: Dynamic case (k=3.3, $\lambda = 1.904\mu m$), $\epsilon = 0.001$

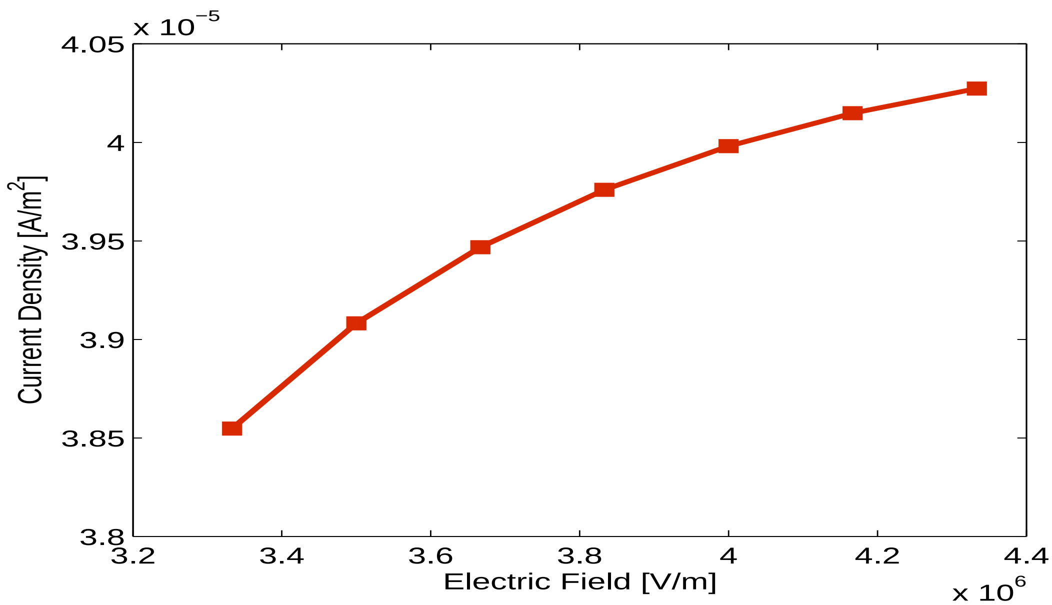

Figure 5.9: Current density: Dynamic case (k=3.3, $\lambda = 1.904\mu m$), $\epsilon = 0.0001$

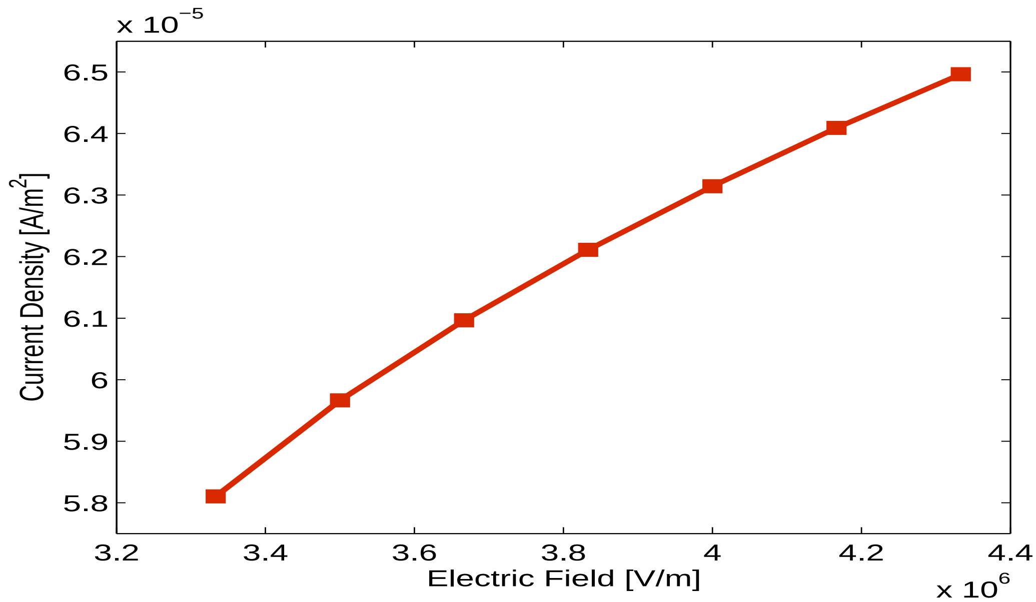

Figure 5.10: Current density: Dynamic case (k=3.5, $\lambda = 1.795\mu m$), $\epsilon = 0.1$

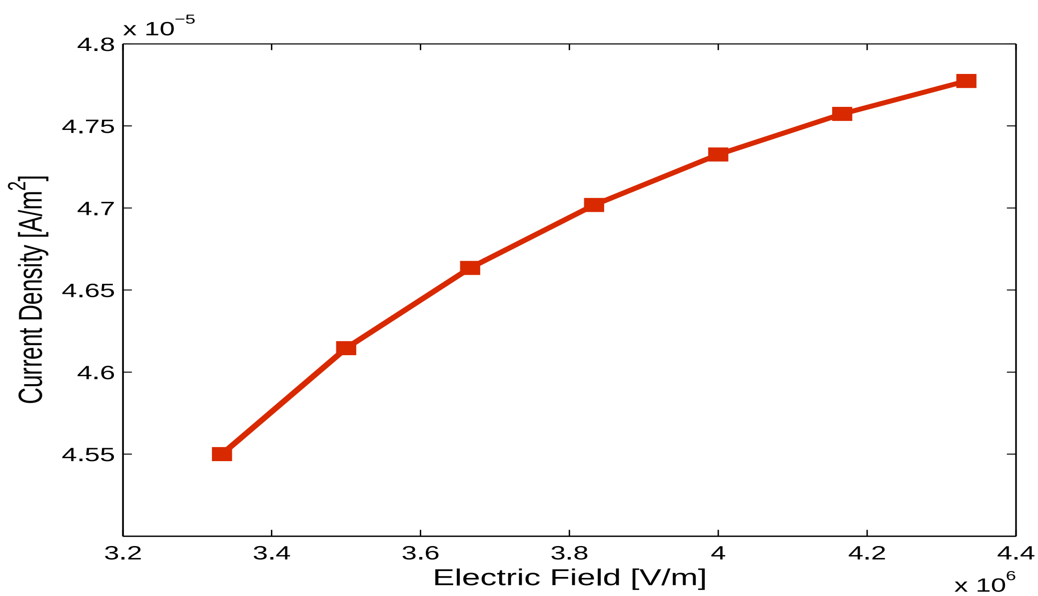

Figure 5.11: Current density: Dynamic case (k=3.5, $\lambda = 1.795\mu m$), $\epsilon = 0.01$

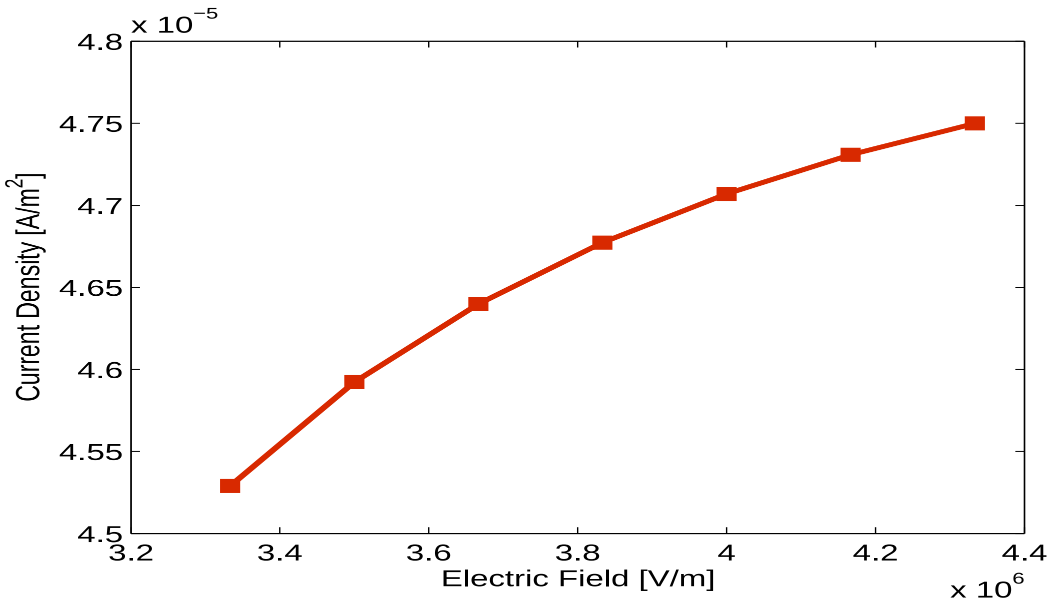

Figure 5.12: Current density: Dynamic case (k=3.5, $\lambda = 1.795\mu m$), $\epsilon = 0.001$

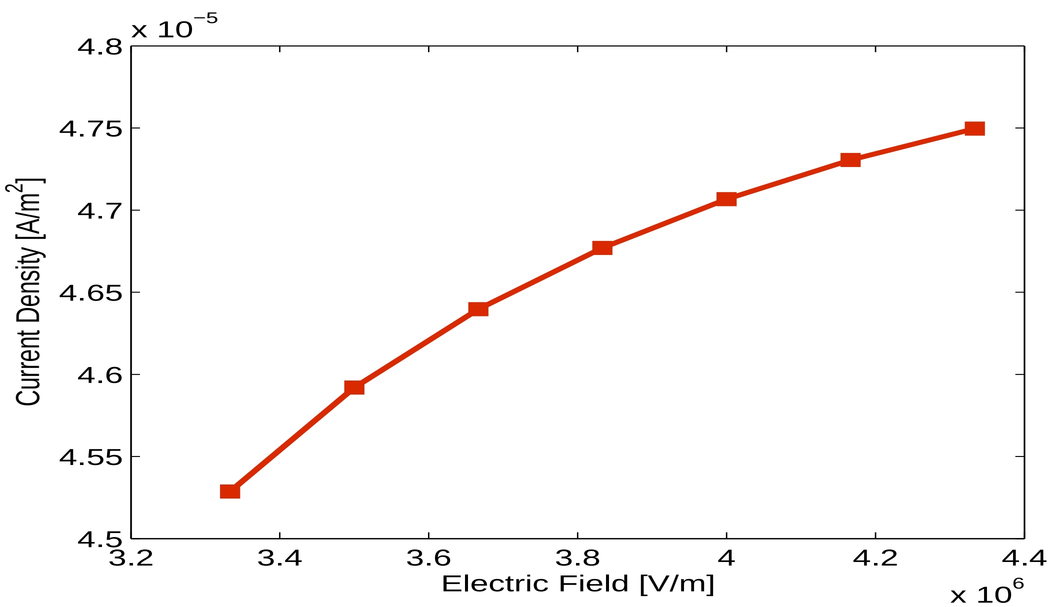

Figure 5.13: Current density: Dynamic case (k=3.5, $\lambda = 1.795\mu m$), $\epsilon = 0.0001$

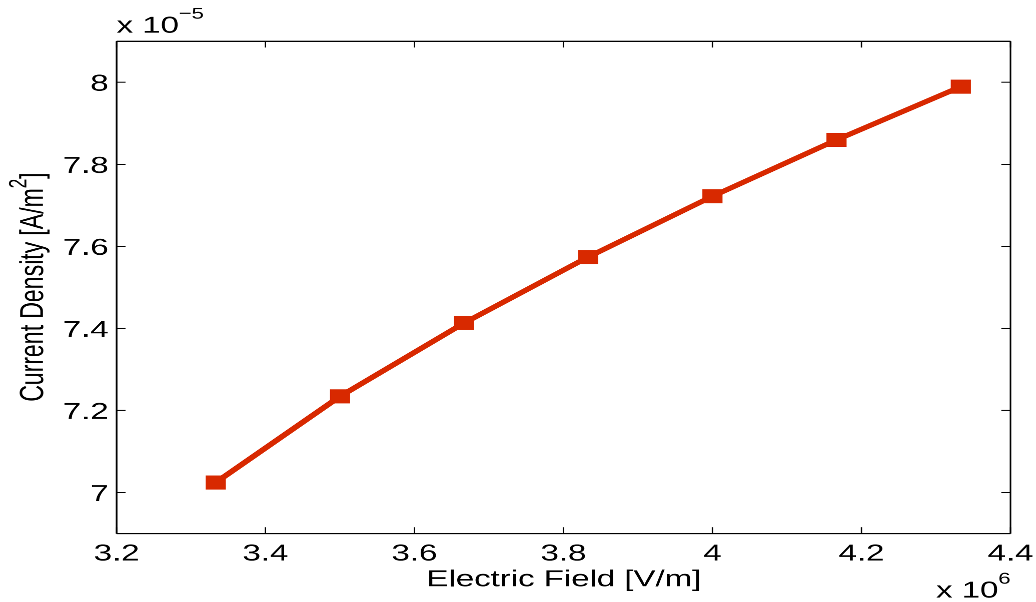

Figure 5.14: Current density: Dynamic case (k=3.7, $\lambda = 1.698\mu m$), $\epsilon = 0.1$

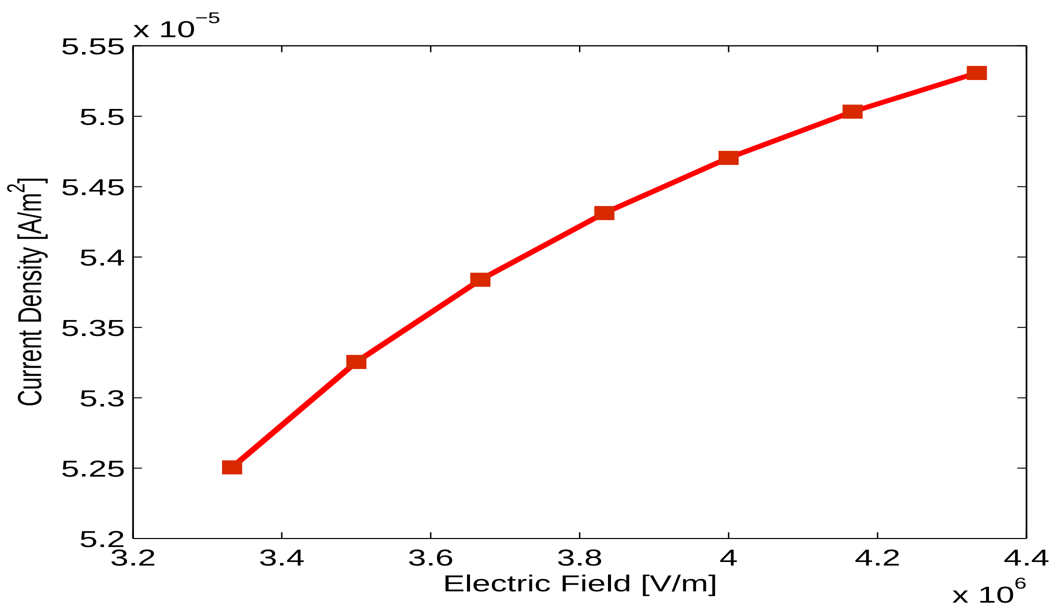

Figure 5.15: Current density: Dynamic case (k=3.7, $\lambda = 1.698\mu m$), $\epsilon = 0.01$

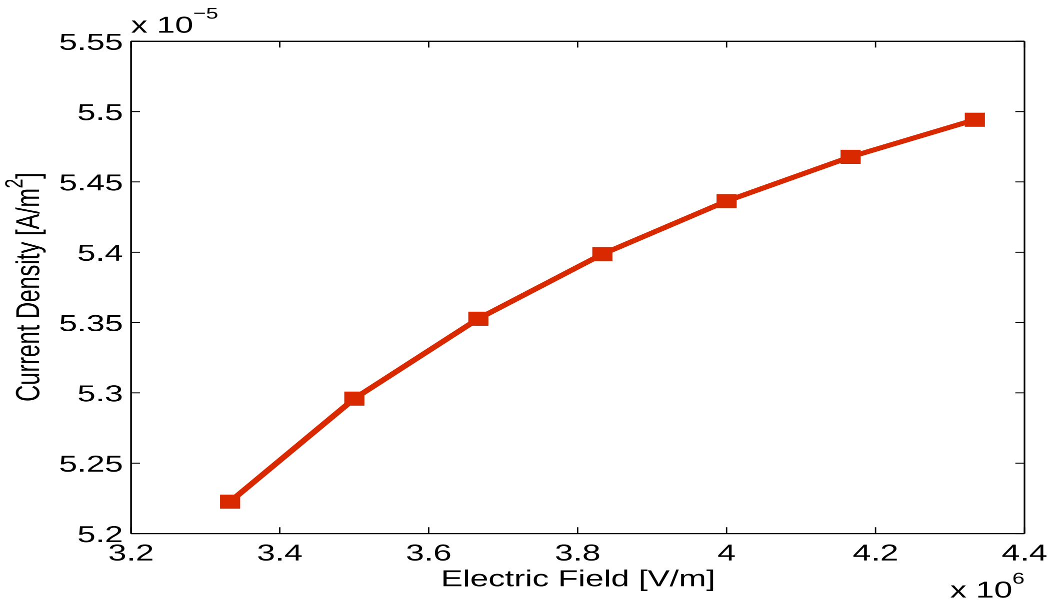

Figure 5.16: Current density: Dynamic case (k=3.7, $\lambda = 1.698\mu m$), $\epsilon = 0.001$

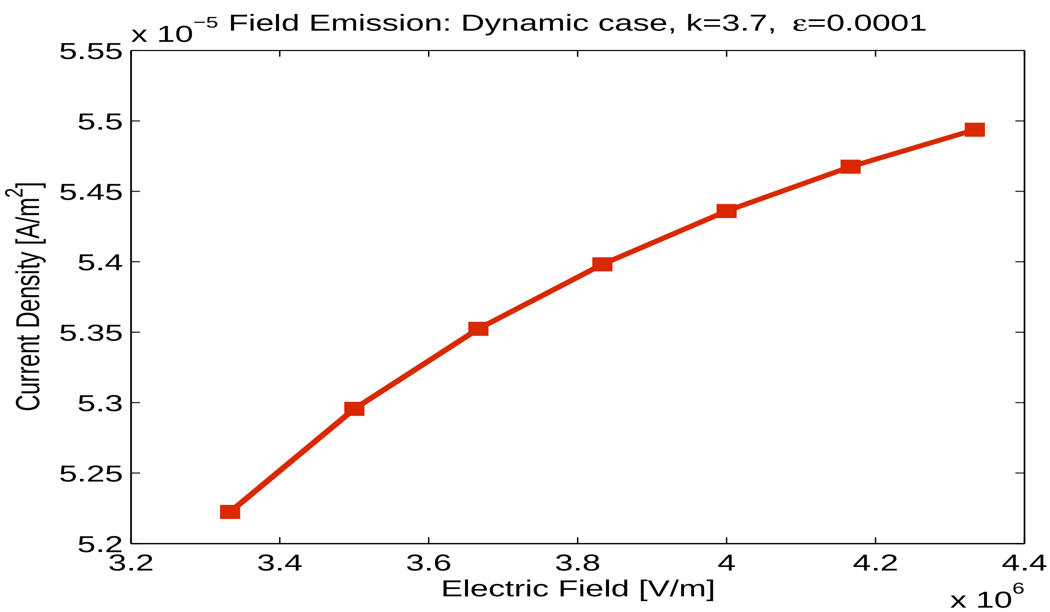

Figure 5.17: Current density: Dynamic case (k=3.7, $\lambda = 1.698\mu m$), $\epsilon = 0.0001$

5.3 SCATTERING AND RADIATIVE PROPERTIES

Next we present the results from the scattering and radiative calculations. Recall that in Chapter IV we obtained a method for calculation of the current distribution in the fiber due to transverse incident electric field. Here we present the current distribution results from our model for three different frequencies and three different fiber radii and compare them with the delta gap antenna feed results from the Hallen's equation for the same frequencies and fiber geometries.

Figures 5.18, 5.19, 5.22, 5.23, 5.26, 5.27, 5.30, 5.31, 5.34, 5.35, 5.38, 5.39, 5.42, 5.43, 5.46, 5.47, 5.50, and 5.51 represent the results for the current distributions due to transverse incident field $E_{inc} = e^{ikx}\mathbf{i_z}$ and Hallen's equation for a delta gap source for $k = 2\pi$, $k = 3\pi$ and $k = 4\pi$ and antenna radii $\epsilon = 0.01$, $\epsilon = 0.001$ and $\epsilon = 0.0001$.

Results show that currents induced by the transverse incident field and the delta gap antenna feed are of similar magnitudes. Another important observation is that away from the ends of the fiber, the current distribution is similar in both cases. Close to the edges, however, our model displays oscillatory behavior. There are several possible explanations for this phenomenon. One of them is that we use an approximate kernel in the numerical evaluation scheme, which is known to cause oscillations close to the ends of the antenna. This effect is discussed in detail in [66], [67], [68], [69], and others. The oscillatory behavior could also be explained with the fact that our model investigates the z-component of the induced Eddy current, which is not 'linear' unlike the current from the delta gap feed Hallen's case. In both cases the calculated currents are sufficiently large to induce experimentally measurable electromagnetic fields.

The data for the current distribution from our model as well as Hallen's equation allows us to obtain and compare the radiation patterns for both of the cases. Figures 5.20, 5.21, 5.24, 5.25, 5.28, 5.29, 5.32, 5.33, 5.36, 5.37, 5.40, 5.41, 5.44, 5.45, 5.48, 5.49, 5.52 and 5.53 represent the element radiation patterns corresponding to current distribution due to transverse incident field $E_{inc} = e^{ikx}\mathbf{i_z}$ and Hallen's equation for a delta gap source for $k = 2\pi$, $k = 3\pi$ and $k = 4\pi$ and antenna radii $\epsilon = 0.01$, $\epsilon = 0.001$ and $\epsilon = 0.0001$. Results show that the radiation patterns due to currents induced by transverse incident electric field show very high degree of similarity to the radiation patterns due to delta gap antenna feed, i.e. the properties of the fiber as a scatterer and a radiator are very similar. In both cases the radiation patters are typical for the one- and the half-wavelength antennas.

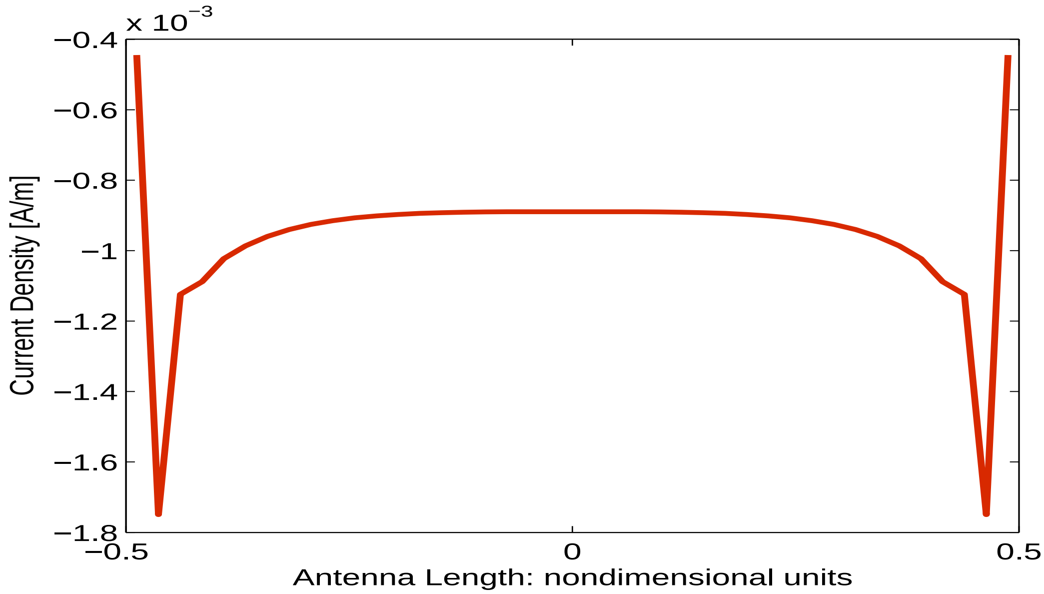

Figure 5.18: Current distribution: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 2\pi$,$(\lambda = 1\mu m)$, $\epsilon = 0.01$

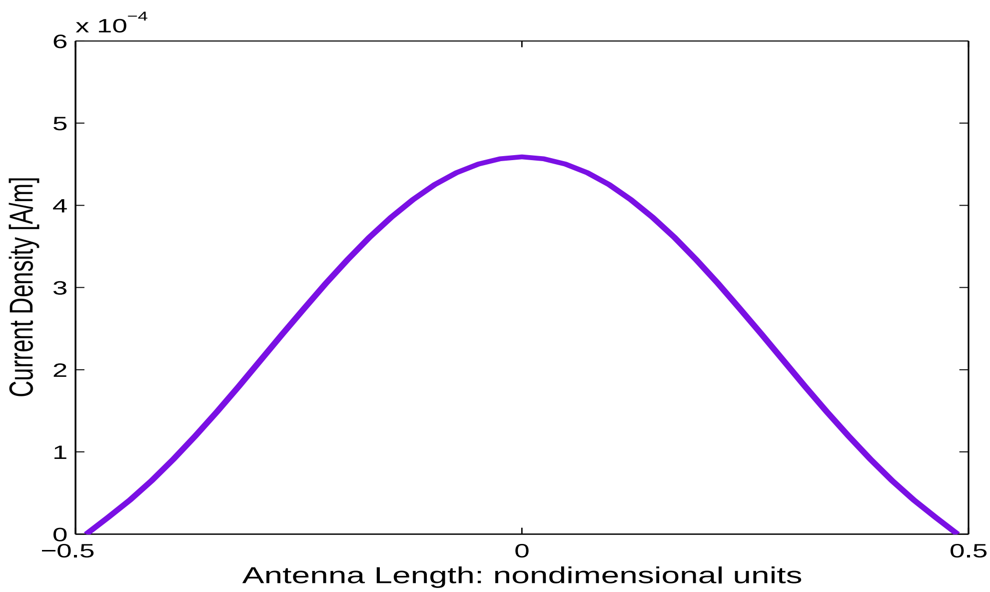

Figure 5.19: Current distribution: Hallen's, $k = 2\pi$,$(\lambda = 1\mu m)$, $\epsilon = 0.01$

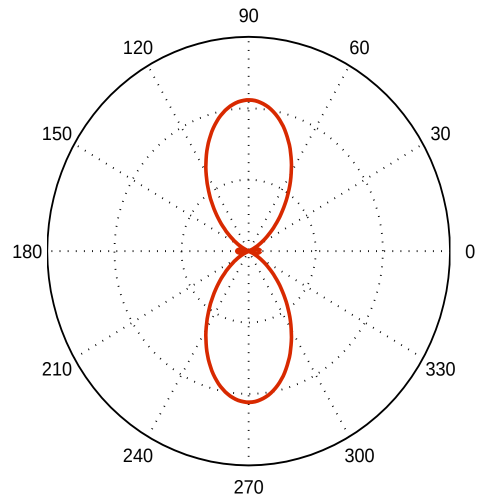

Figure 5.20: Radiation pattern: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 2\pi$, $(\lambda = 1\mu m)$, $\epsilon = 0.01$

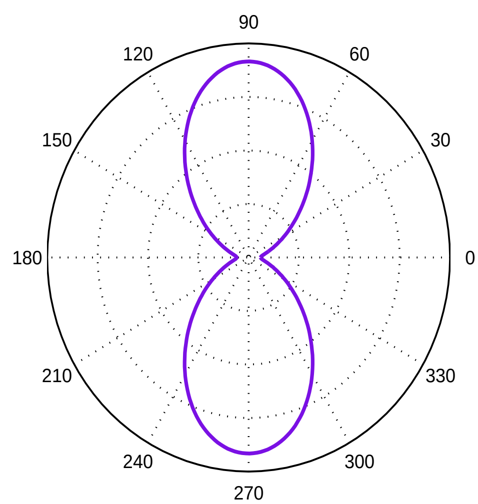

Figure 5.21: Radiation pattern: Hallen's, $k = 2\pi$, $(\lambda = 1\mu m)$, $\epsilon = 0.01$

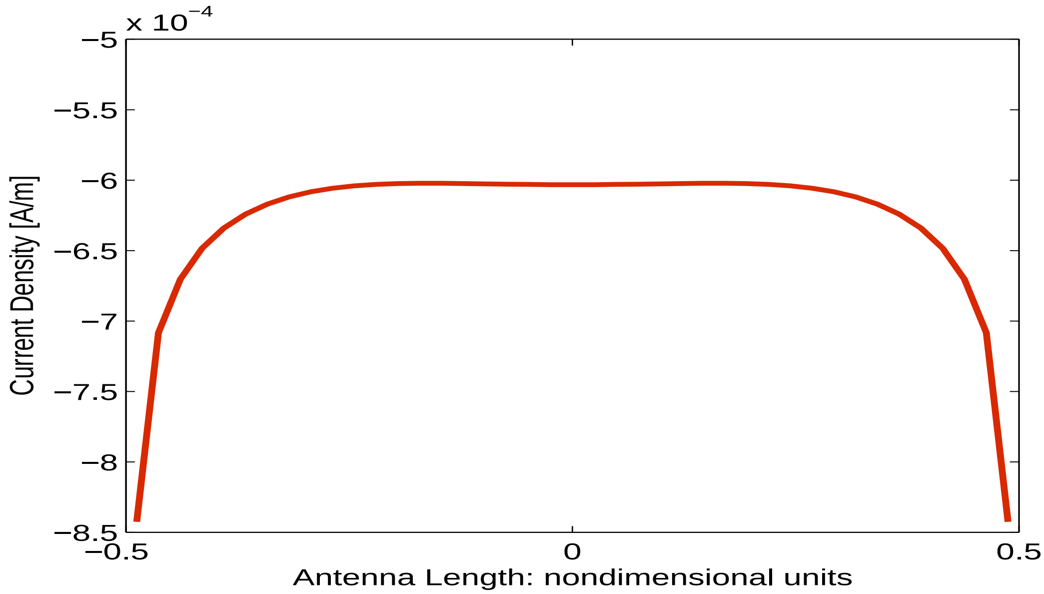

Figure 5.22: Current distribution: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 2\pi$,($\lambda = 1\mu m$), $\epsilon = 0.001$

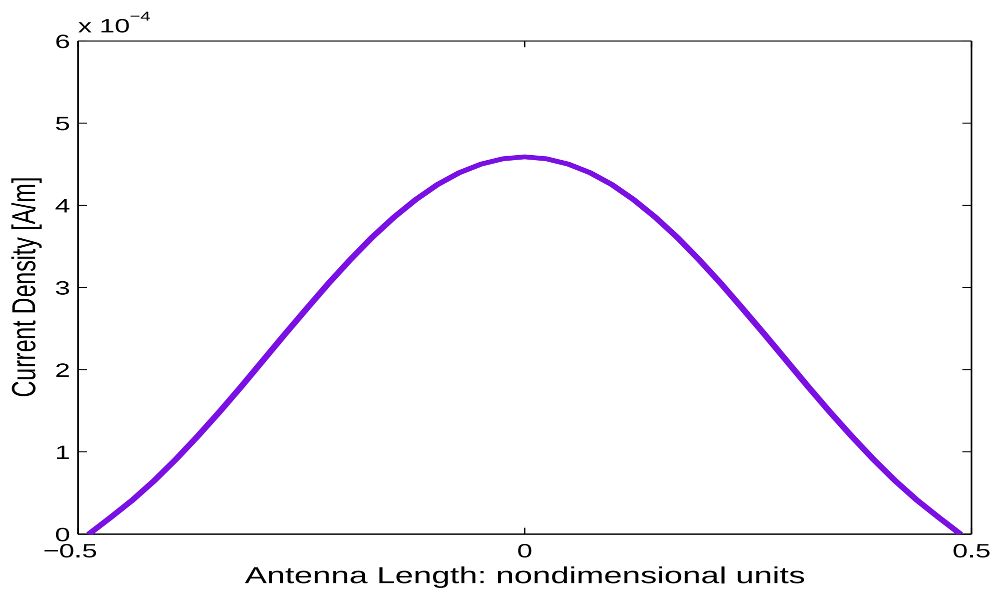

Figure 5.23: Current distribution: Hallen's, $k = 2\pi$,($\lambda = 1\mu m$), $\epsilon = 0.001$

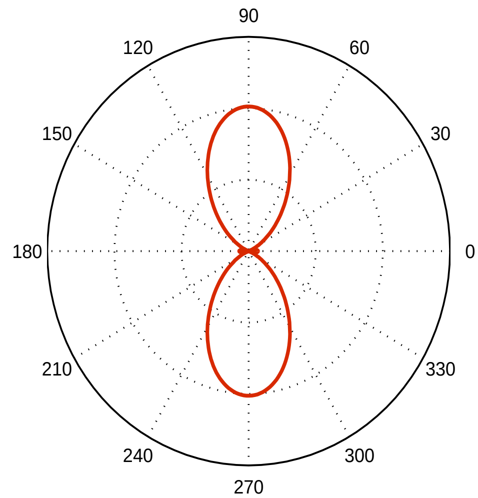

Figure 5.24: Radiation pattern: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 2\pi$, $(\lambda = 1\mu m)$, $\epsilon = 0.001$

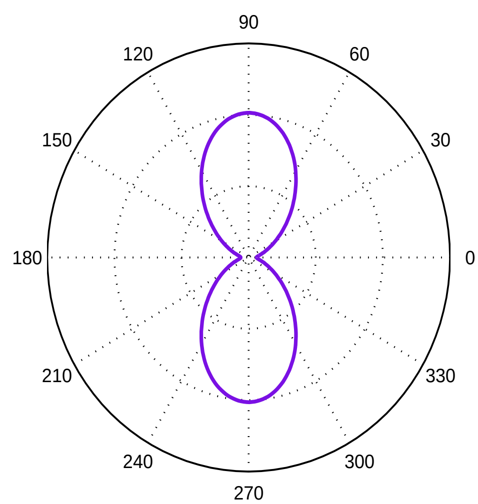

Figure 5.25: Radiation pattern: Hallen's, $k = 2\pi$, $(\lambda = 1\mu m)$, $\epsilon = 0.001$

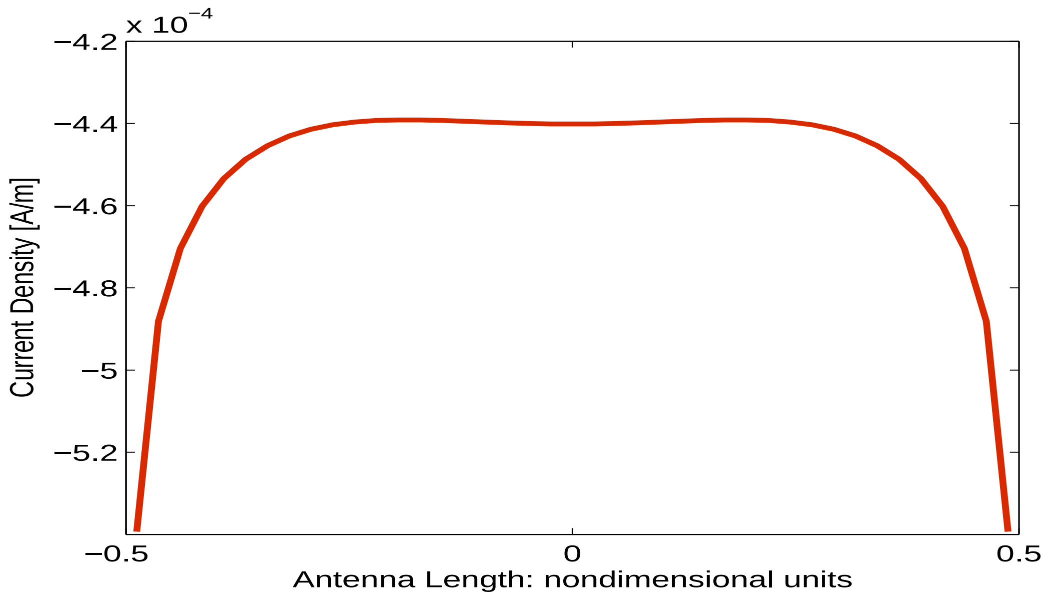

Figure 5.26: Current distribution: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 2\pi$, $(\lambda = 1\mu m)$, $\epsilon = 0.0001$

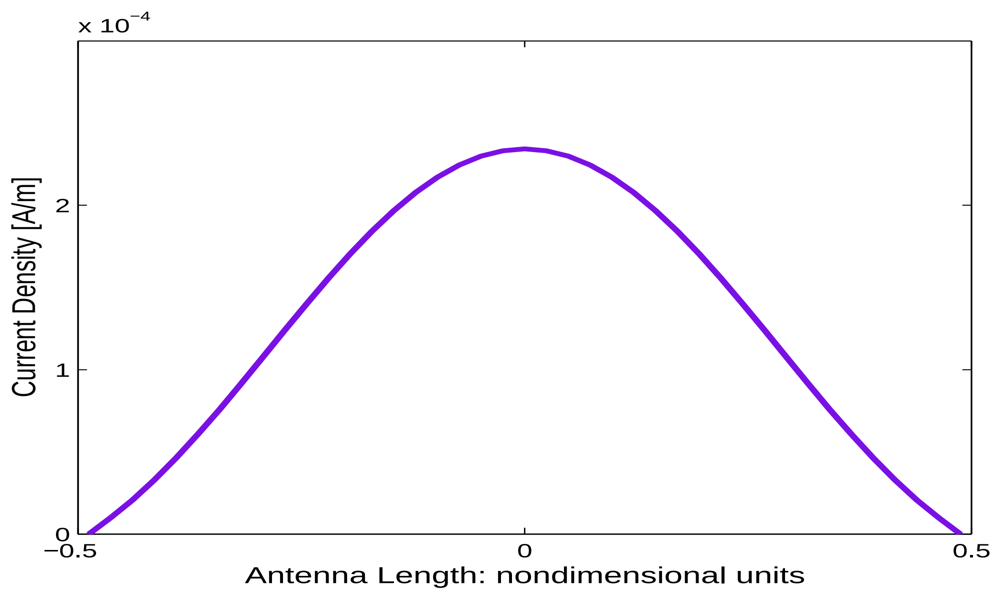

Figure 5.27: Current distribution: Hallen's, $k = 2\pi$, $(\lambda = 1\mu m)$, $\epsilon = 0.0001$

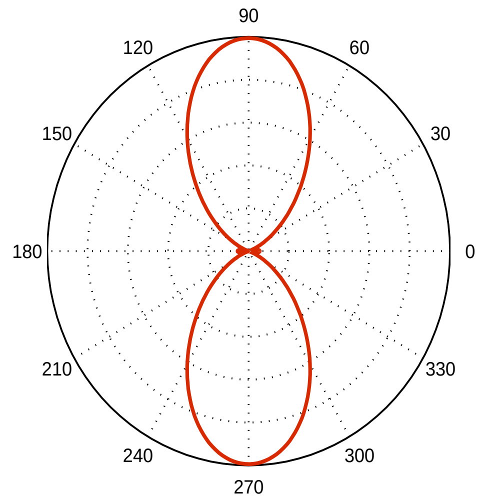

Figure 5.28: Radiation pattern: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 2\pi$, $(\lambda = 1\mu m)$, $\epsilon = 0.0001$

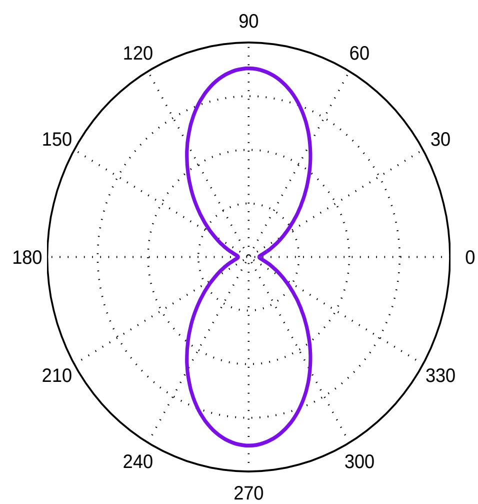

Figure 5.29: Radiation pattern: Hallen's, $k = 2\pi$, $(\lambda = 1\mu m)$, $\epsilon = 0.0001$

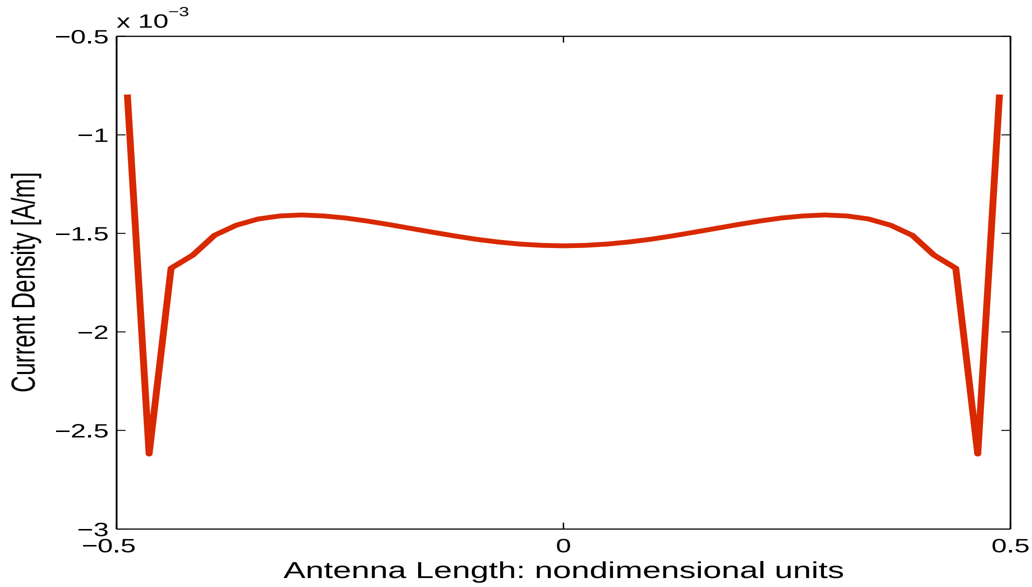

Figure 5.30: Current distribution: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.01$

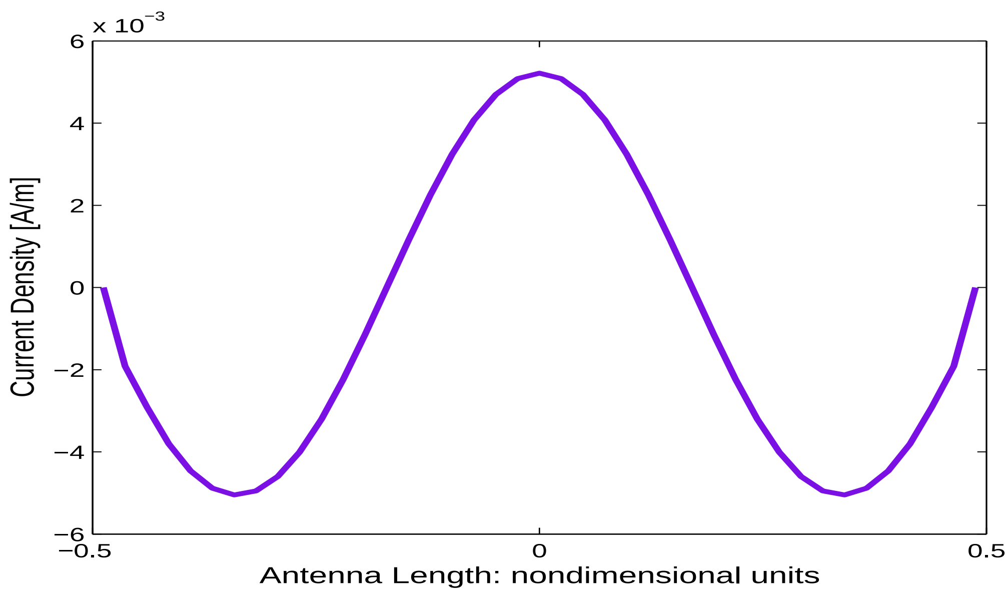

Figure 5.31: Current distribution: Hallen's, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.01$

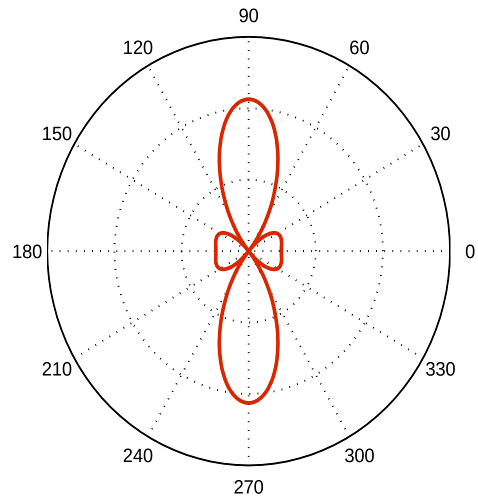

Figure 5.32: Radiation pattern: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.01$

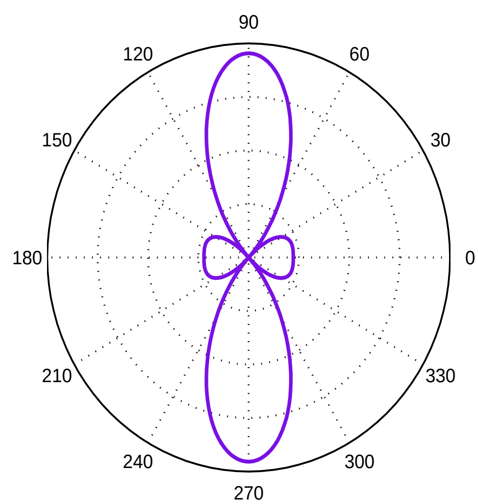

Figure 5.33: Radiation pattern: Hallen's, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.01$

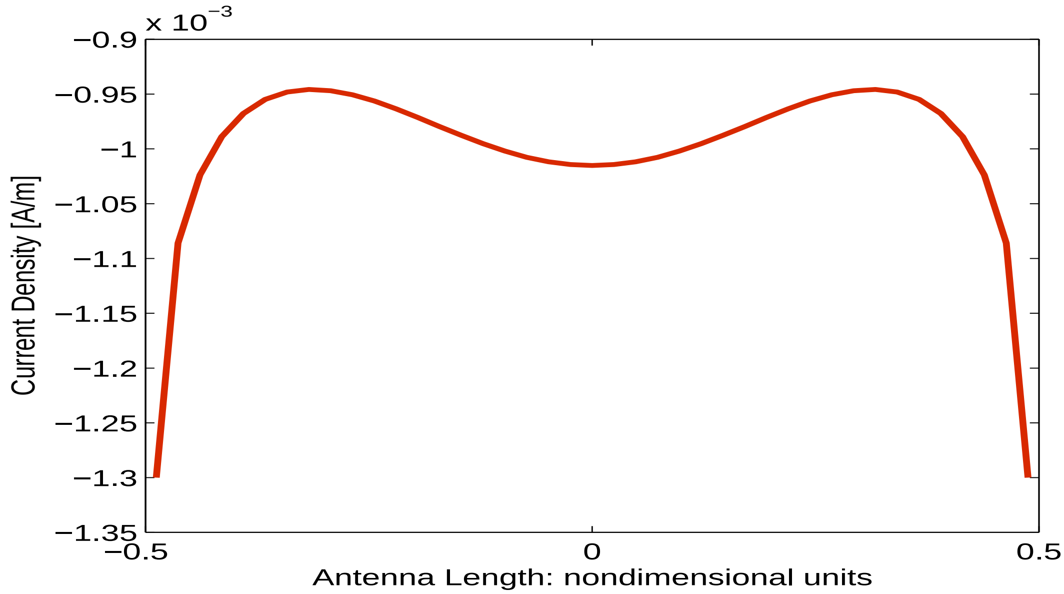

Figure 5.34: Current distribution: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.001$

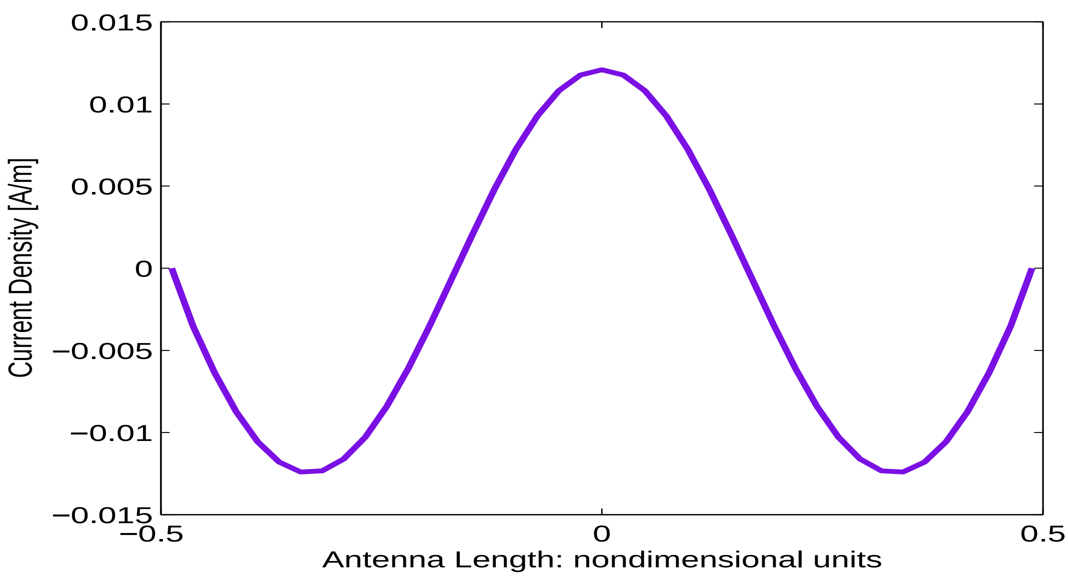

Figure 5.35: Current distribution: Hallen's, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.001$

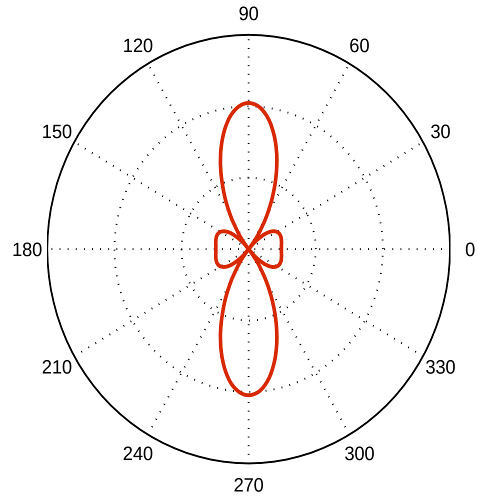

Figure 5.36: Radiation pattern: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.001$

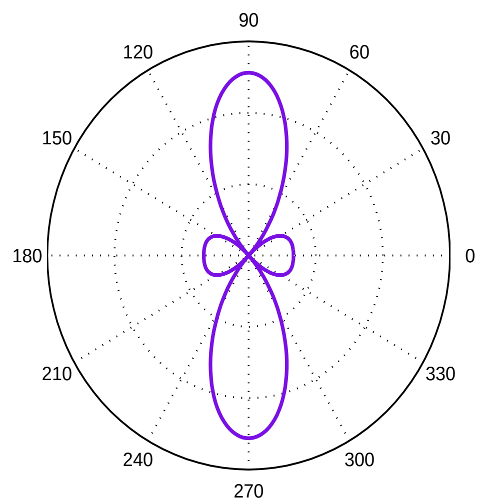

Figure 5.37: Radiation pattern: Hallen's, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.001$

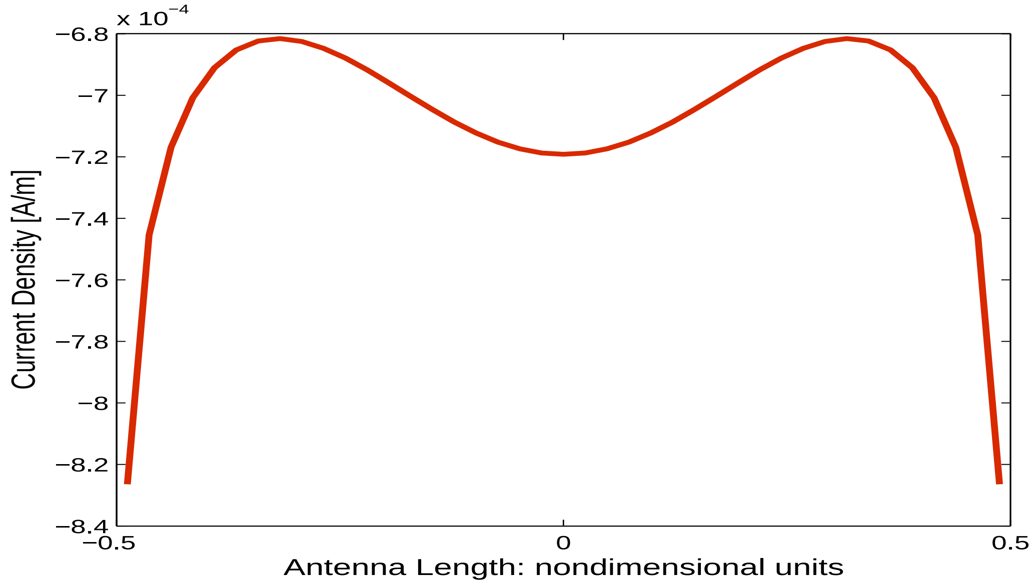

Figure 5.38: Current distribution: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.0001$

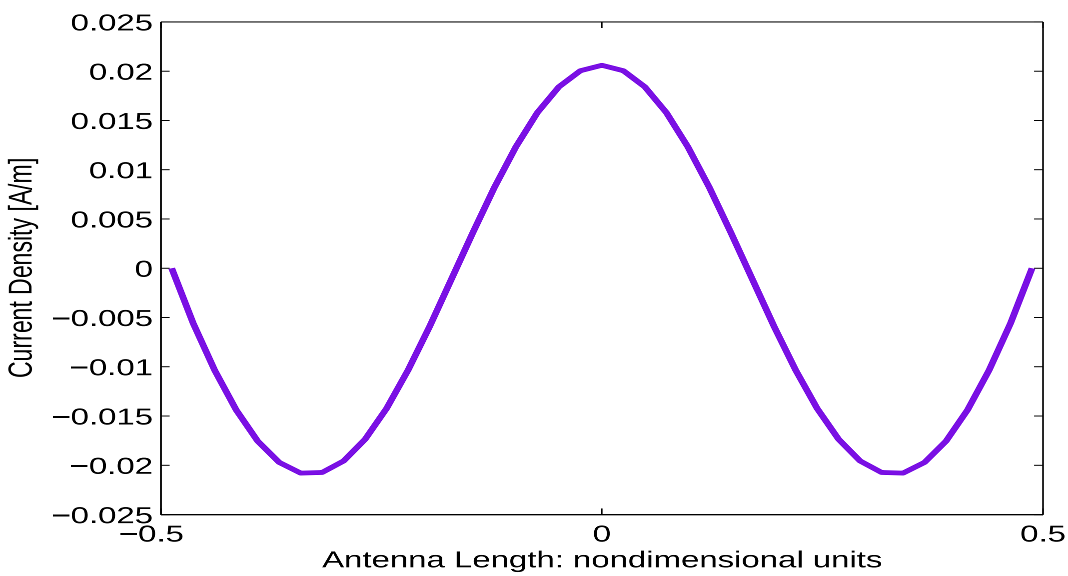

Figure 5.39: Current distribution: Hallen's, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.0001$

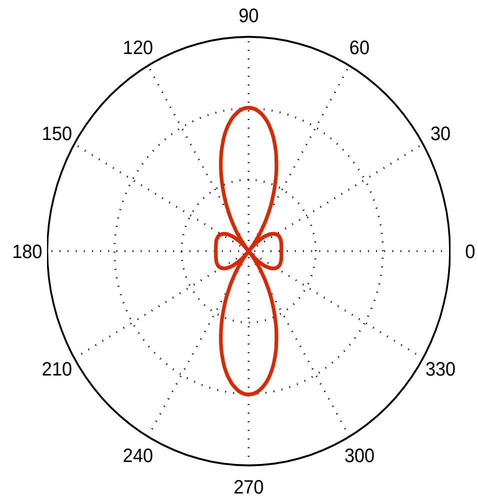

Figure 5.40: Radiation pattern: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.0001$

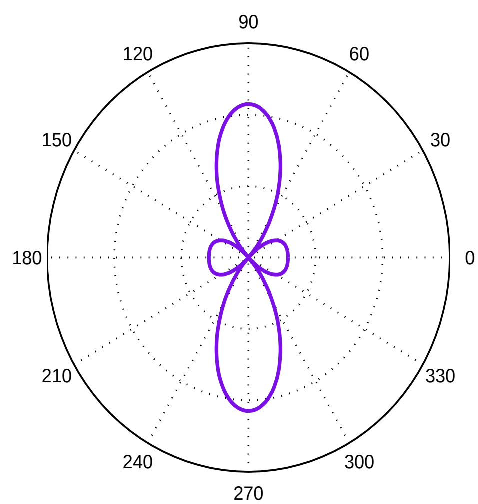

Figure 5.41: Radiation pattern: Hallen's, $k = 3\pi$, $(\lambda = 0.67\mu m)$, $\epsilon = 0.0001$

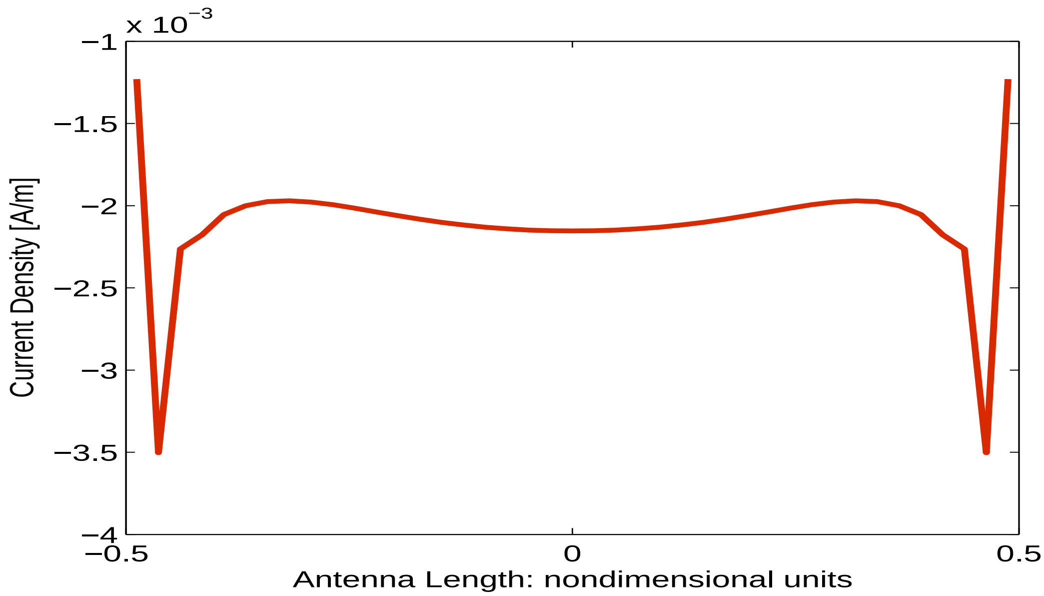

Figure 5.42: Current distribution: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 4\pi$, $(\lambda = 0.5\mu m)$, $\epsilon = 0.01$

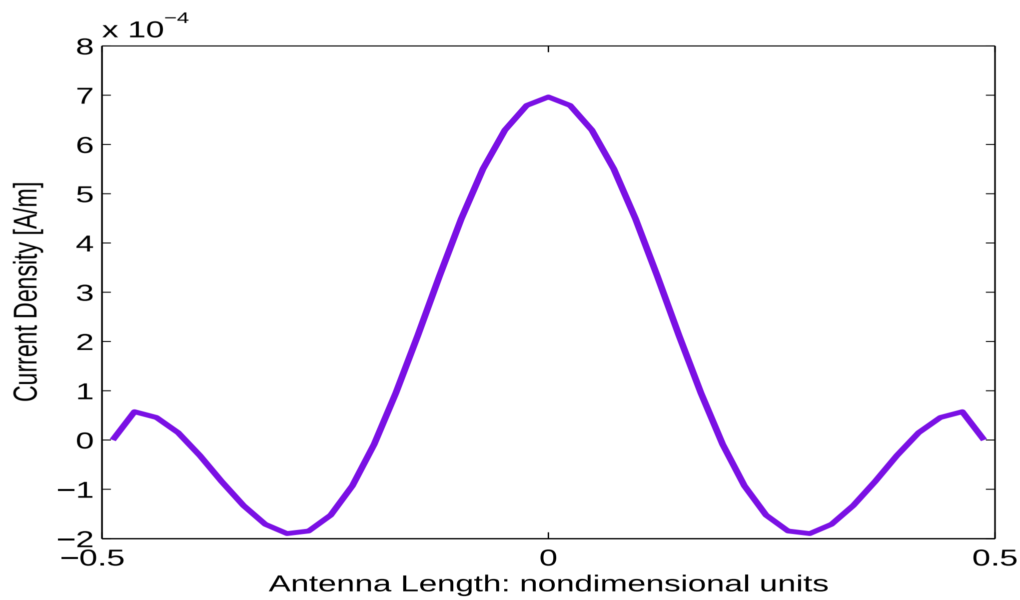

Figure 5.43: Current distribution: Hallen's, $k = 4\pi$, $(\lambda = 0.5\mu m)$, $\epsilon = 0.01$

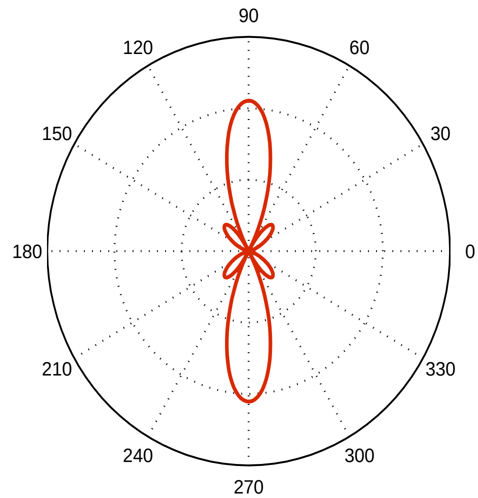

Figure 5.44: Radiation pattern: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 4\pi$, $(\lambda = 0.5\mu m)$, $\epsilon = 0.01$

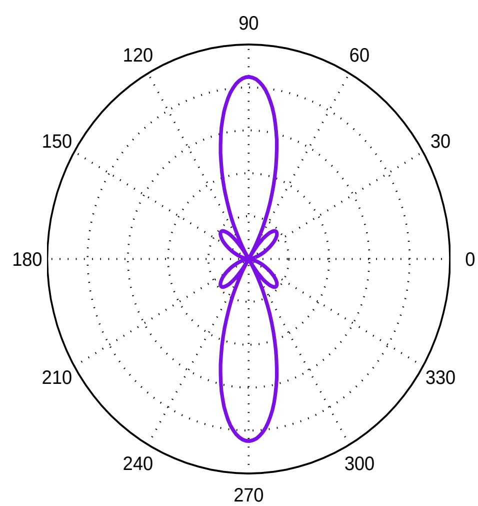

Figure 5.45: Radiation pattern: Hallen's, $k = 4\pi$, $(\lambda = 0.5\mu m)$, $\epsilon = 0.01$

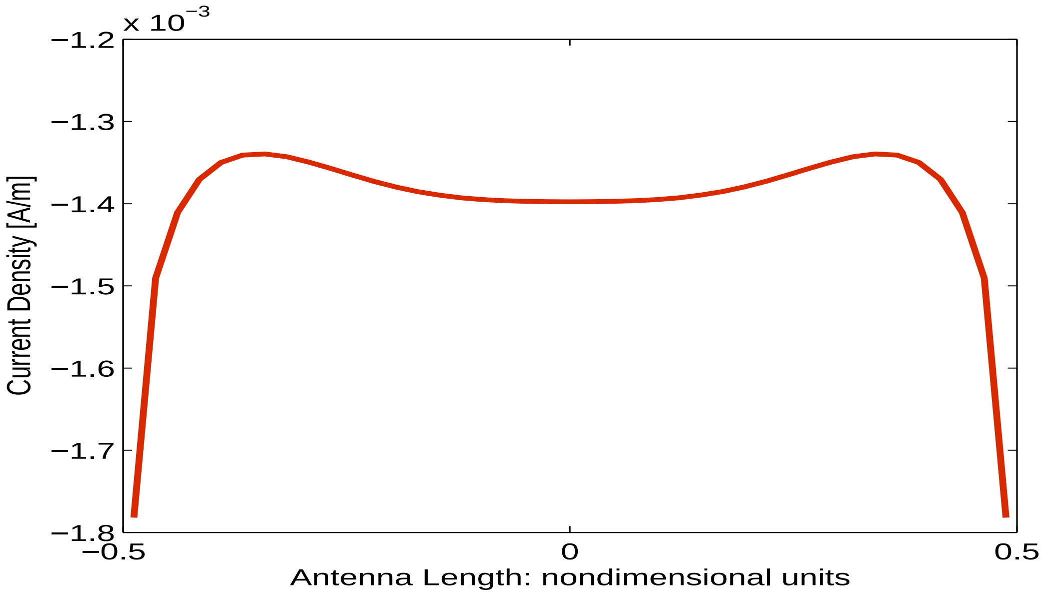

Figure 5.46: Current distribution: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 4\pi$, ($\lambda = 0.5\mu m$), $\epsilon = 0.001$

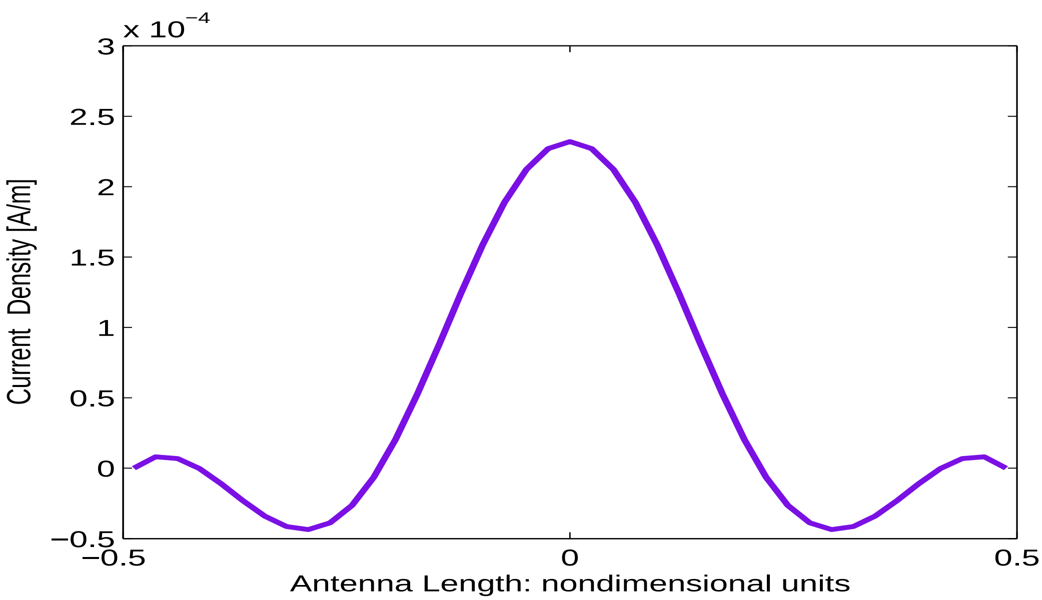

Figure 5.47: Current distribution: Hallen's, $k = 4\pi$, ($\lambda = 0.5\mu m$), $\epsilon = 0.001$

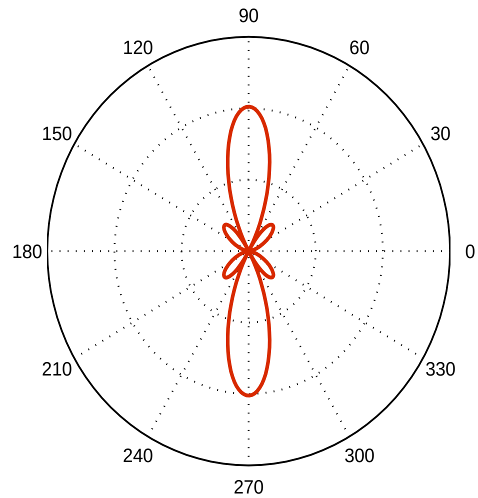

Figure 5.48: Radiation pattern: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 4\pi$, $(\lambda = 0.5\mu m)$, $\epsilon = 0.001$

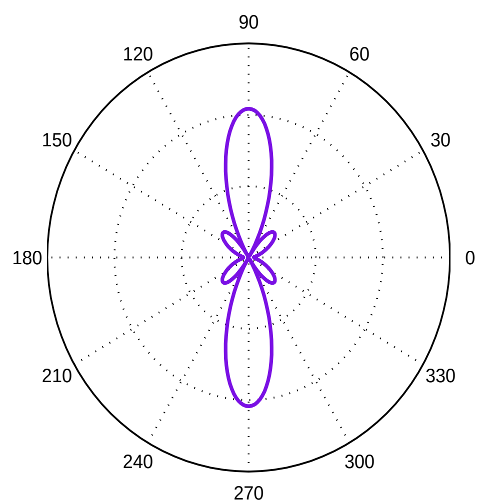

Figure 5.49: Radiation pattern: Hallen's, $k = 4\pi$, $(\lambda = 0.5\mu m)$, $\epsilon = 0.001$

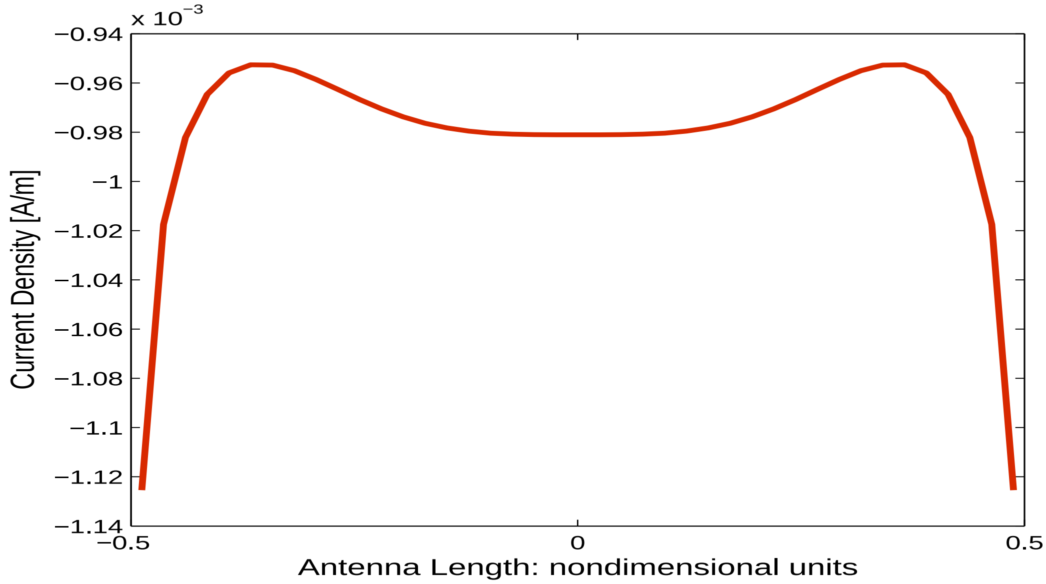

Figure 5.50: Current distribution: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 4\pi$, $(\lambda = 0.5\mu m)$, $\epsilon = 0.0001$

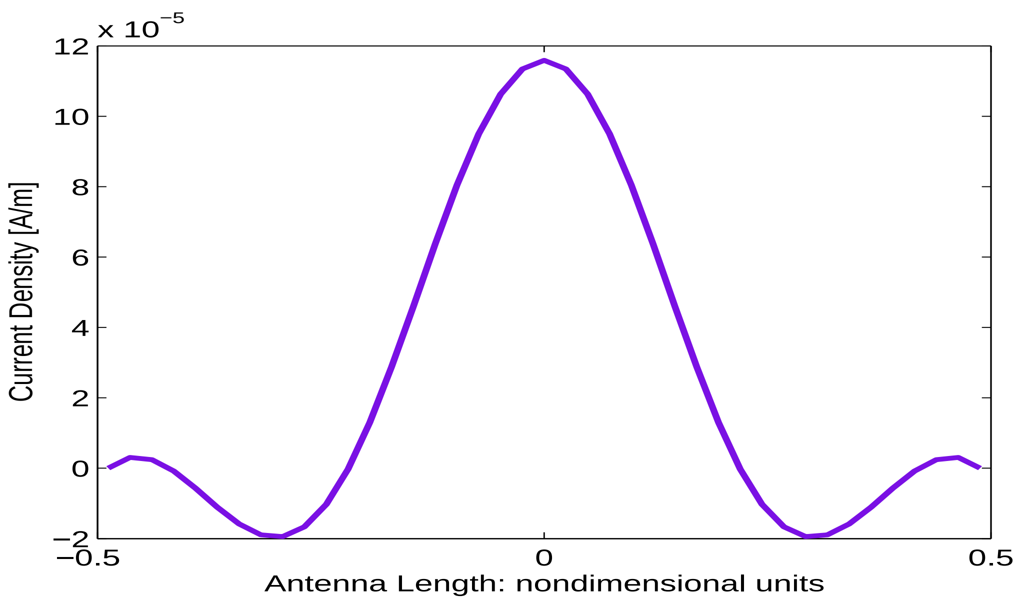

Figure 5.51: Current distribution: Hallen's, $k = 4\pi$, $(\lambda = 0.5\mu m)$, $\epsilon = 0.0001$

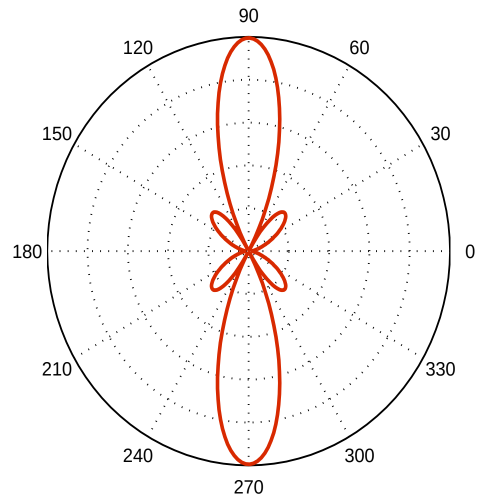

Figure 5.52: Radiation pattern: $\mathbf{E} = e^{ikx}\mathbf{i_z}$, $k = 4\pi$, $(\lambda = 0.5\mu m)$, $\epsilon = 0.0001$

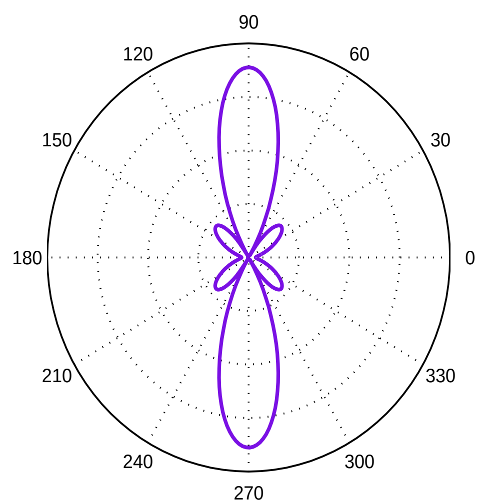

Figure 5.53: Radiation pattern: Hallen's, $k = 4\pi$, $(\lambda = 0.5\mu m)$, $\epsilon = 0.0001$

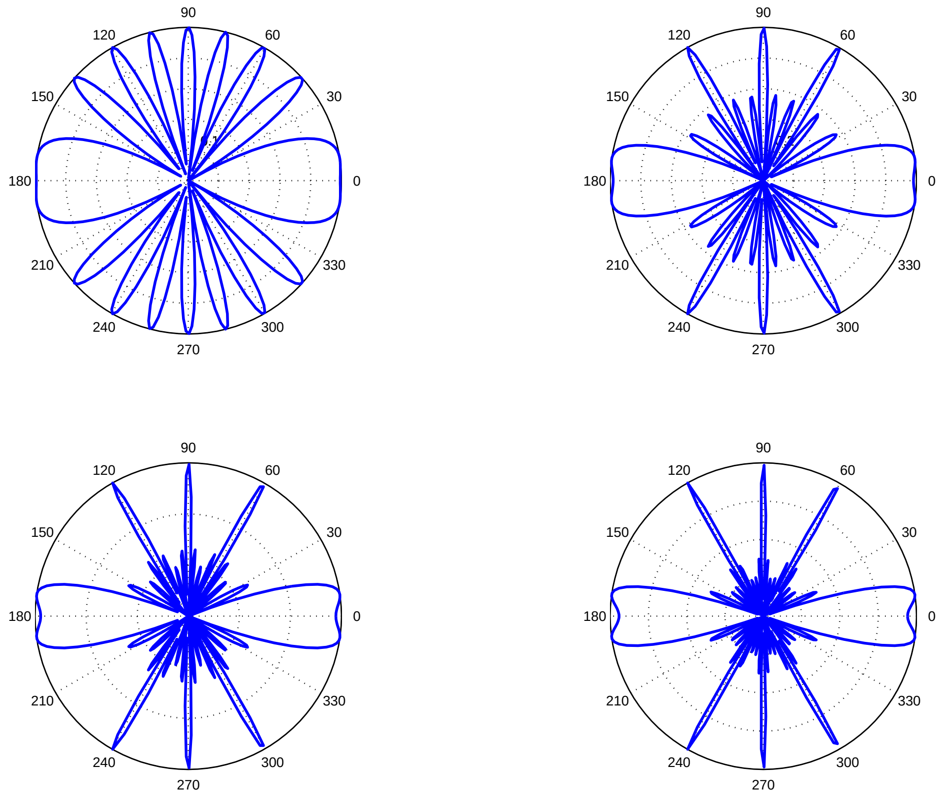

Figure 5.54: Array factor $f_{array}(\theta)$ for 2, 4, 6 and 8 elements, separated by $c = 0.01$, ($0.01\mu m$)

Our next step is to investigate the array factor for of a uniform broadside array as a function of interelement separation and different number of array elements. Figures 5.54, 5.55 and 5.56 show the array factors for the separations $c = 0.01$, $(0.01\mu m)$, $c = 0.1$, $(0.1\mu m)$ and $c = 1$, $(1\mu m)$ and for 2, 4, 6 and 8 array elements, respectively:

From figures 5.54, 5.55 and 5.56 it is evident that uniform broadside arrays are end firing, i.e. the largest lobes in the array factor are along the z-axis. With the

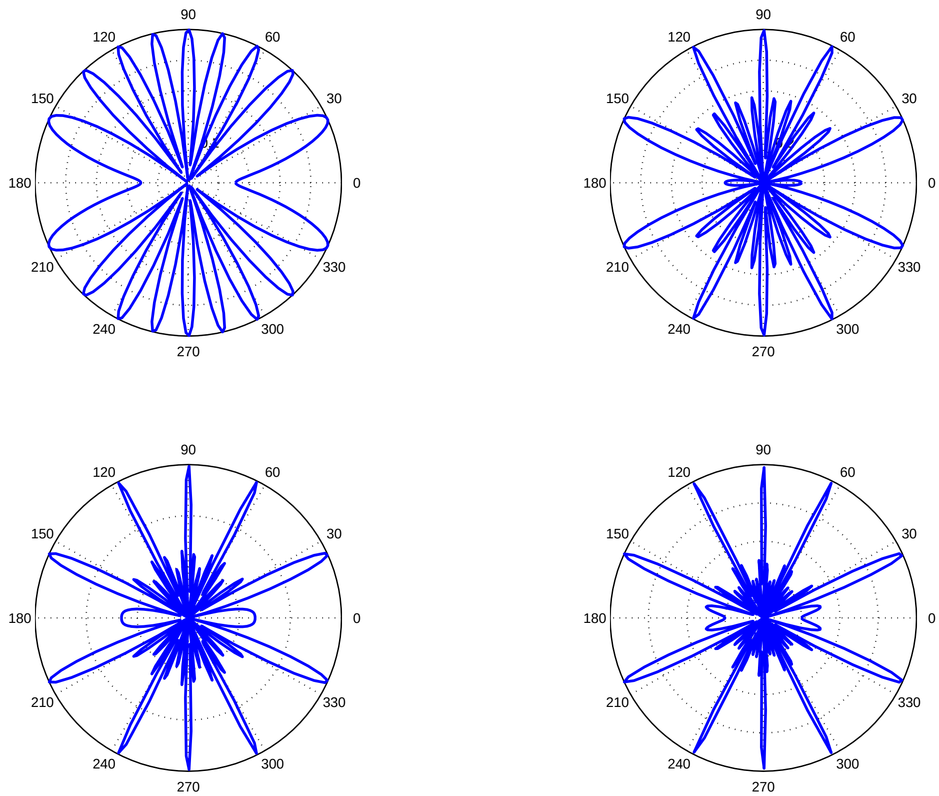

Figure 5.55: Array factor $f_{array}(\theta)$ for 2, 4, 6 and 8 elements, separated by $c = 0.1$, ($0.1\mu m$)

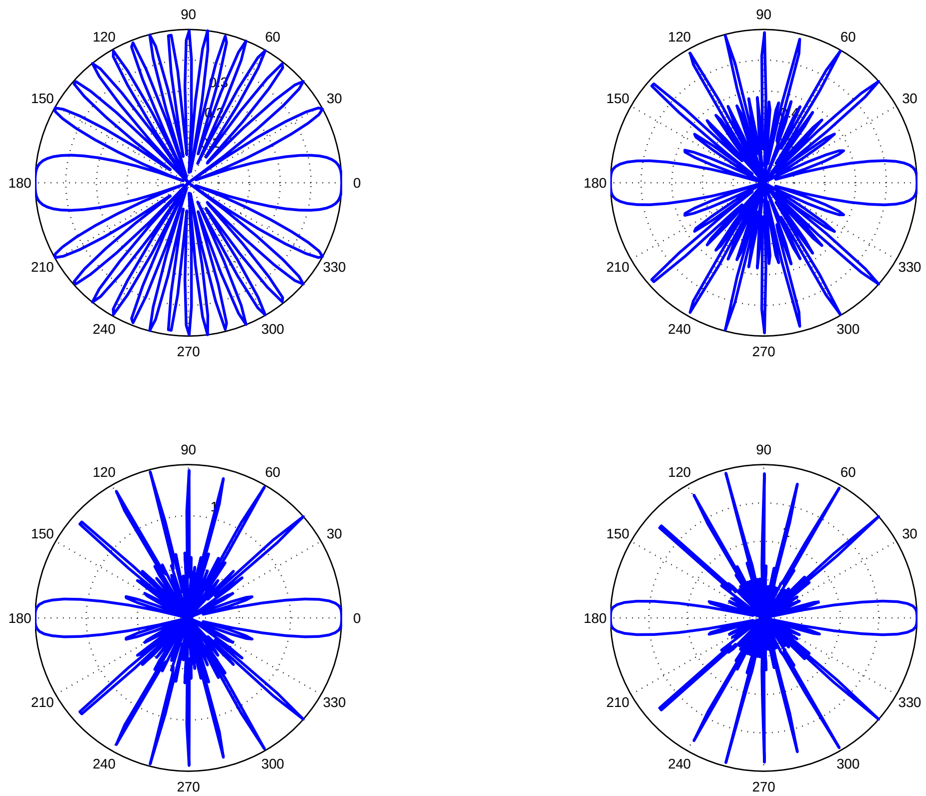

Figure 5.56: Array factor $f_{array}(\theta)$ for 2, 4, 6 and 8 elements, separated by $c = 1$, ($1 \mu m$)

increase of the number of elements in the array, the existing side lobes become less and less pronounced, while the end firing effect becomes more dominant. This fact is important, since the array factor might suppress radiation in directions in which the element radiation pattern reaches its maximum.

The total array radiation pattern is a product of the element radiation pattern and the array factor. Figures 5.57, 5.58, 5.59, 5.60, 5.61 and 5.62 show the total array radiation pattern for $k = 2\pi$, $k = 3\pi$, antenna radii $\epsilon = 0.01$, $\epsilon = 0.001$ and $\epsilon = 0.0001$, an interelement separations $c = 0.1$ and 2, 4, 6 and 8 elements, respectively.

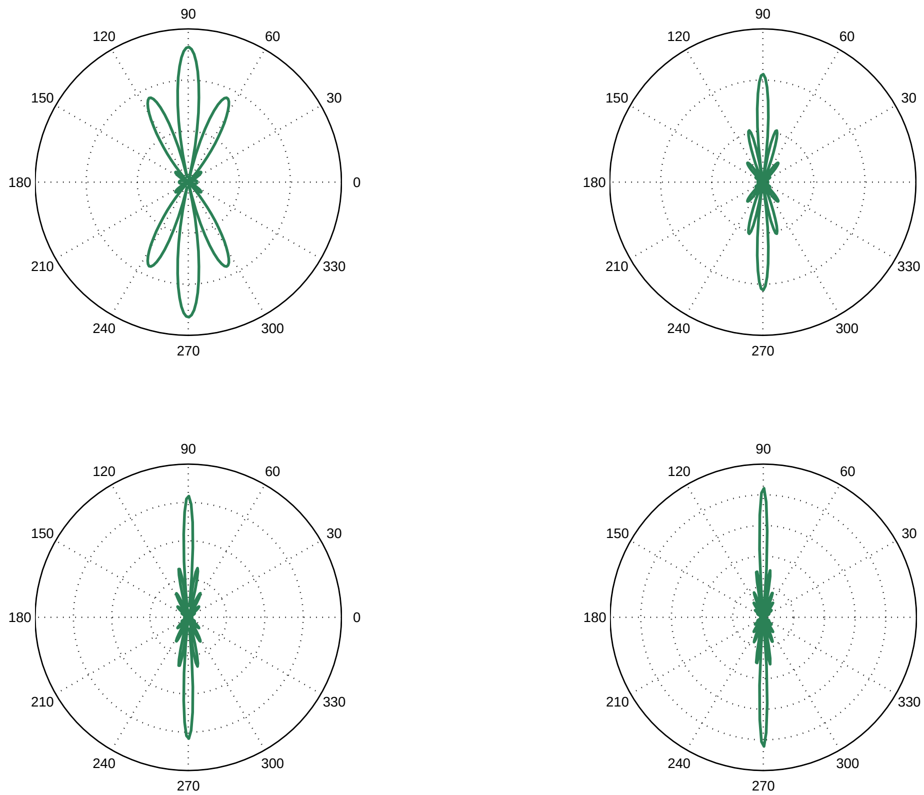

Figure 5.57: Array radiation pattern for 2, 4, 6 and 8 elements, $k = 2\pi$, $(\lambda = 1\mu m)$, $\epsilon = 0.01$ and separation $c = 0.1$, $(0.1\mu m)$

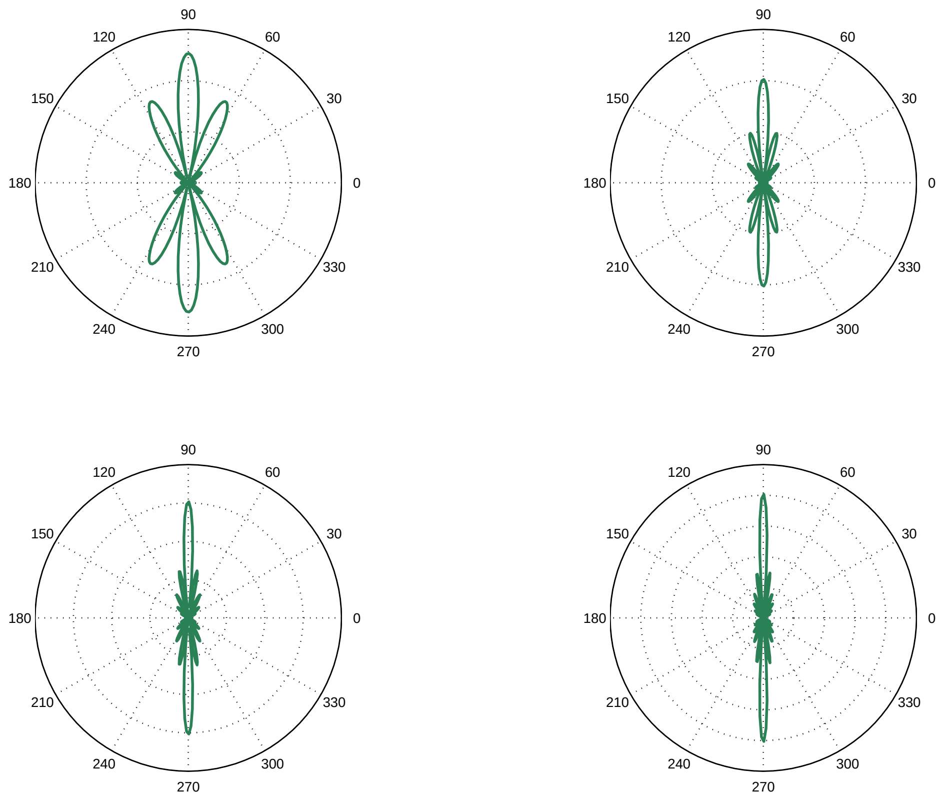

Figure 5.58: Array radiation pattern for 2, 4, 6 and 8 elements, $k = 2\pi$, ($\lambda = 1\mu m$), $\epsilon = 0.001$ and separation $c = 0.1$, ($0.1\mu m$)

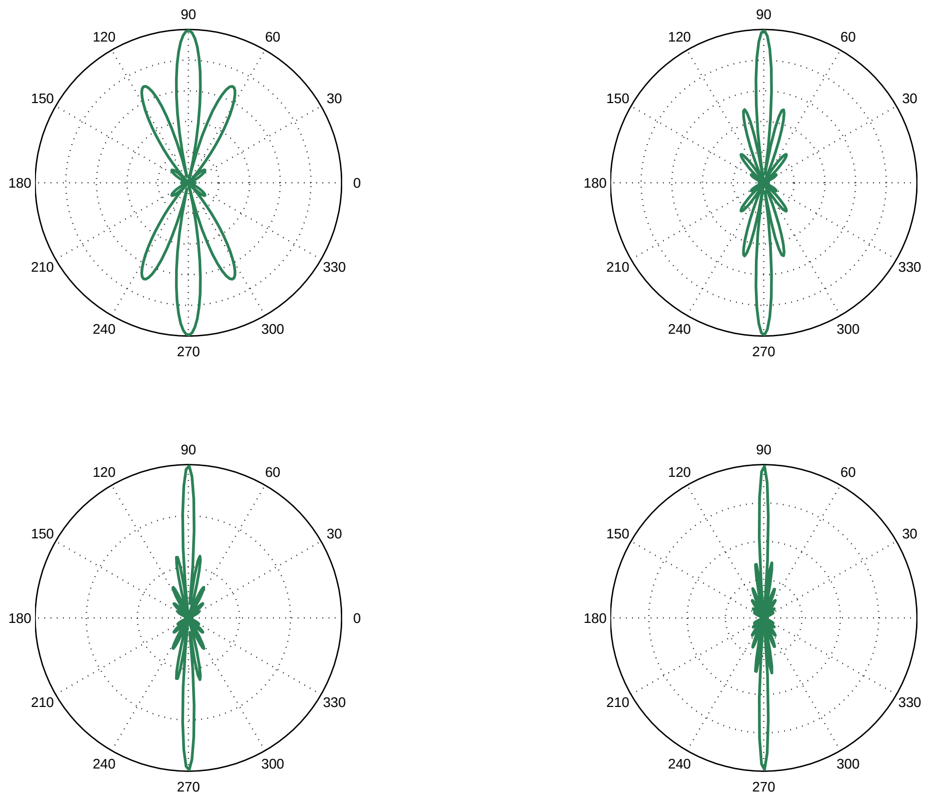

Figure 5.59: Array radiation pattern for 2, 4, 6 and 8 elements, $k = 2\pi$, $(\lambda = 1\mu m)$, $\epsilon = 0.0001$ and separation $c = 0.1$, $(0.1\mu m)$

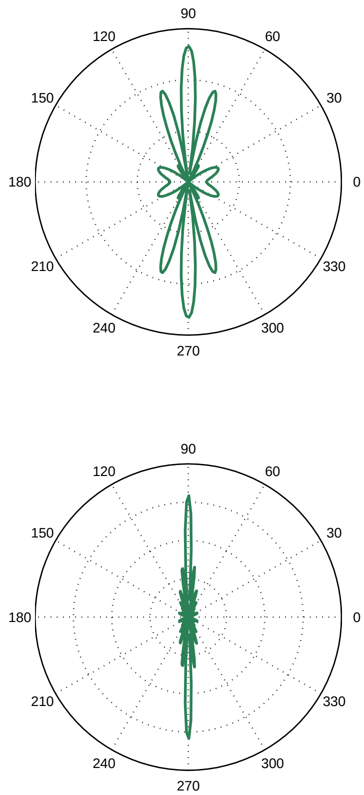

Figure 5.60: Array radiation pattern for 2, 4, 6 and 8 elements, $k = 3\pi$, ($\lambda = 0.67\mu m$), $\epsilon = 0.01$ and separation $c = 0.1$, ($0.1\mu m$)

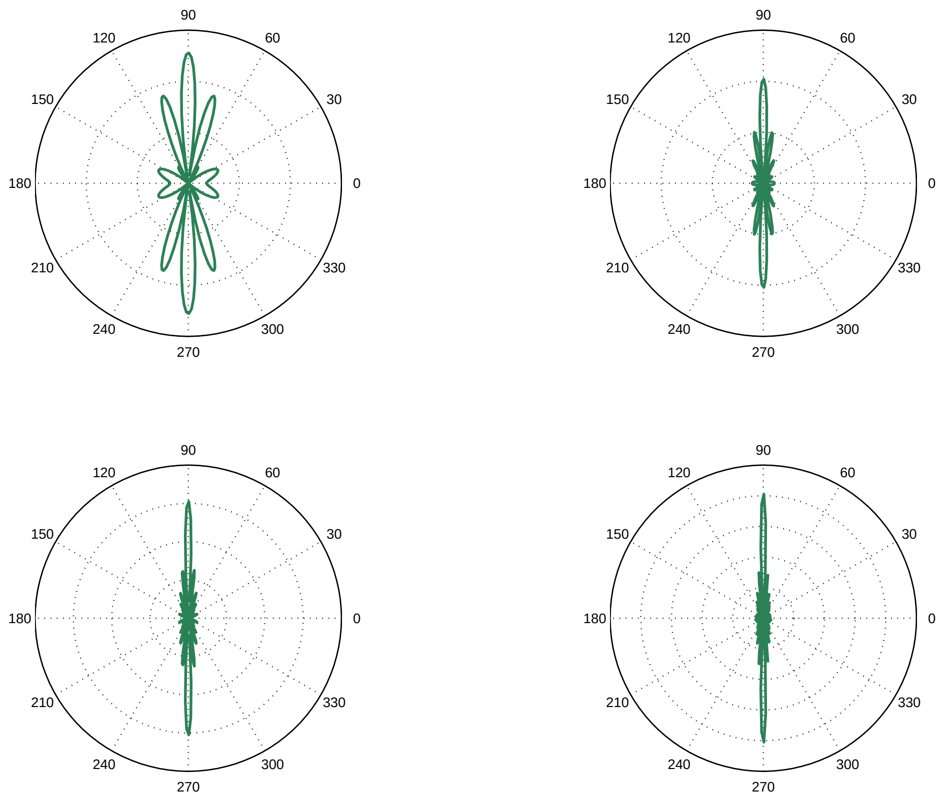

Figure 5.61: Array radiation pattern for 2, 4, 6 and 8 elements, $k = 3\pi$, ($\lambda = 0.67\mu m$), $\epsilon = 0.001$ and separation $c = 0.1$, ($0.1\mu m$)

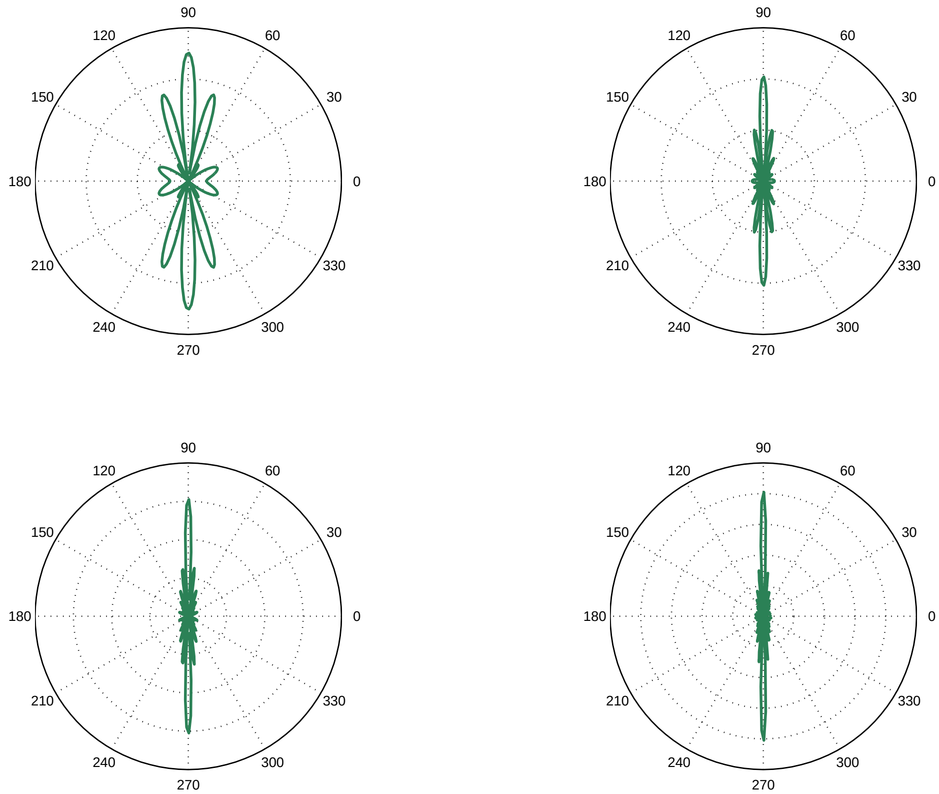

Figure 5.62: Array radiation pattern for 2, 4, 6 and 8 elements, $k = 3\pi$, ($\lambda = 0.67\mu m$), $\epsilon = 0.001$ and separation $c = 0.1$, ($0.1\mu m$)

Results show that radiation is suppressed almost everywhere except in planes perpendicular to the z-axis, an effect that becomes more pronounced as the number of array elements increases. This phenomenon can be explained by the fact that the array radiation pattern is a product of the element radiation pattern and the array factor of the broadside uniform array.

5.4 DISCUSSION

In this work we developed a technique for calculating field emission current from a conducting nanofiber of finite length due to both static and dynamic field in the axial direction. We also investigated the scattering/radiative properties of a single nanofiber and a uniform array of nanofibers due to transverse incident electric field. For the field emission model we employed the 1D WKB approximation, and the quantum wire density of states (Chapter II). For the tunneling probability calculation we needed an analytic expression for the potential due to axial incident electric field. We investigated two different cases: static and dynamic. For both of them we employed the slender body approximation, which assumes large aspect (length to width) ratio, rotational symmetry with respect to the 'long' (z-) axis and no sharp edges. This resulted in nondimensionalization and rescaling in such a way that the new length is unity and the maximum radius of the fiber is a small parameter ϵ.

In the static case we represented the electrostatic potential (solution to Laplace's equation) as a potential due to a superposition of charges on the z-axis with unknown charge distribution f between unknown integration limits α and β (eq.(3.2) and eq. (3.3)). The BC was simply a continuity requirement for the electrostatic potential on the surface of the fiber, leading to a Fredholm integral equation of the first kind. We proceeded by seeking a uniform asymptotic expansion of the integral. It was shown that uniformity could be achieved by choosing appropriate α_i and β_i in the α and β expansions and they depend only on the geometry. The unknown charge distribution

f was obtained in the form of an asymptotic expansion, which allowed us to evaluate the electrostatic potential and subsequently, calculate the the field emission current for different radii of the fiber. Comparison of the theoretical predictions of our model with theoretical predictions by the Fowler-Nordheim model as well as recent experimental data [48] showed that our model gives estimates for the FE current, which are significantly closer to the experimental results than the Fowler-Nordheim model. Also, the model shows that current density decreases as ϵ becomes smaller.

In the dynamic case we represented the scattered electric field (solution to Helmholtz's equation) due to an axial plane wave as a field due to a superposition of electric and magnetic dipoles on the z-axis with unknown distributions g and h between unknown integration limits α and β. The BC was a requirement for tangential components of the electric field to vanish on on the surface of the fiber, leading to two linear integral equations. We proceeded by seeking a uniform asymptotic expansion of the integrals. Just as before, uniformity was achieved by choosing appropriate α_i and β_i in the α and β expansions. In that case α and β were the same as in the static case. Once the unknown distributions g and h were obtained, we were able to obtain the electric potential by invoking the Lorentz-Lorenz gauge, which allowed us to calculate the the field emission current for different radii of the fiber and different frequencies of the incident field. Results show that for a fixed frequency, FE current density decreases with the decrease of the fiber radius ϵ, just as in the static case. Also, for a fixed radius ϵ, FE current density increases as the frequency of the incident field increases.

Additionally, we investigated the scattering properties of nanofibers in the presence of transverse incident electric field. Using a similar technique as the one in the axial dynamic case from above, we represented the scattered electric field as a field due to a superposition of electric and magnetic dipoles on the z-axis with unknown distributions g and h between the integration limits α and β. Just as before, we required vanishing tangential components of the electric field on the surface of the fiber. Expanding the integrals from the resulting integral equations and collecting terms at different orders allowed us to determine coefficients in the asymptotic expansion of g and h. This allowed us to determine the scattered field and obtain numerically the current distribution generating it. We compared the results to the current distribution obtained by solving Hallen's equation numerically for a delta gap antenna feed. Results show that not only are currents within the same order of magnitude for a given frequency and antenna radius, but current distributions display very similar behavior in the center of the antenna away from the edges. Furthermore, the comparison of the radiation patterns of the fiber as a scatterer (our model) and as an emitter (Hallen) demonstrated a very high degree of similarity. The results above have important practical implications. Since the magnitude of the calculated currents is sufficiently large to induce a measurable EM field, an experimental validation of our results is desired. There are several considerations for the preparation of such an experiment. One of them is the length of the fiber. In our calculations we assumed fiber length $L = 10^{-6}m$. Thus $\epsilon = 0.1$ corresponds to a fiber radius of $r = 100nm$. Such a fiber length makes measurements difficult. For a more realistic experimental setup, a longer fiber can be chosen. In order for the fiber radius to still be on the nanoscale, smaller values for ϵ are required (for example $L = 10^{-2}m$ and $r = 100nm$ implies $\epsilon = 0.00001$).

Finally, we investigated the behavior of an array of equally spaced aligned fibers in the presence of transverse incident electric field. Since the incident field was a plane wave, the current induced in every fiber had the same phase. As we discovered in Chapter IV, the array factor modifies the total array radiation pattern by selecting radiation lobes close to the xy-plane and suppressing the rest and thus increasing the gain of the array, an effect that becomes more pronounced as the number of array elements is increased. This fact has important practical implications, since increasing the number of elements in the broadside uniform array significantly increases the gain of the array.

There are several possible venues for future research. One of them is the investigation of the field emission of two or more (an array) of parallel nanofibers. In comparison to the single fiber case, one would expect a significant change in the field emission current due to the influence of the field scattered by neighboring fiber(s). Analytic expression for the scattered field, however, might be difficult to obtain due to the fact that the rotational symmetry is broken by the second fiber and the problem is not angularly independent. It is not clear at this point if this problem could be addressed by a modification to the slender body approximation, since that method relies on the rotational symmetry of the scattering object.

Another possible direction for future work is the case of modeling axial roughness on the fiber by introducing a periodic axial deviation from the cylindrical shape. The proposed model above could be used in this case (as long as $S(z)$ is smooth), even if the deviation from the cylindrical shape is large, by introducing $S(z)$ which describes the geometry of the new system.

Yet another interesting possibility is investigating the radiative and scattering properties of aligned parallel nanofibers. As we discussed earlier, broadside uniform arrays of nanofibers allow achieving high gains. Aligning several broadside uniform arrays in 2D arrays could possibly allow us to enhance the beam shaping capabilities of a single broadside array. Furthermore, varying the current phase between adjacent broadside arrays would possibly allow not only beam shaping, but scanning array capabilities as well. Additional investigation of the case of a 3D array alignment of nanofibers should be considered, since that configuration could possibly exhibit photonic crystal properties.

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APPENDICES

APPENDIX A

DERIVATION OF E_S AND B_S,

$$\sqrt{((1-z)-\sum_{n=1}^{\infty}\beta_n\epsilon^{2n})^2+\epsilon^2 S(z)} = (1-z)\sqrt{1+\frac{\epsilon^2(S(z)-2\beta_1(1-z))+O(\epsilon^4)}{(1-z)^2}}$$
$$= (1-z)(1+\frac{\epsilon^2(S(z)-2\beta_1(1-z))}{2(1-z)^2}+O(\epsilon^4))$$
$$= (1-z)+\epsilon^2\left(\frac{S(z)}{2(1-z)}-\beta_1\right)+O(\epsilon^4) \tag{A.1}$$

$$\log(\sqrt{(\beta-z)^2+\epsilon^2 S(z)}+\beta-z)$$
$$= \log\left(1-z-\sum_{n=1}^{\infty}\beta_n\epsilon^{2n}+\sqrt{((1-z)-\sum_{n=1}^{\infty}\beta_n\epsilon^{2n})^2+\epsilon^2 S(z)}\right)$$
$$= \log(1-z)+\log\left(1+\sqrt{1+\frac{\epsilon^2(S(z)-2\beta_1(1-z))+O(\epsilon^4)}{(1-z)^2}}-\frac{\sum_{n=1}^{\infty}\beta_n\epsilon^{2n}}{1-z}\right)$$
$$= \log(1-z)+\log(2+\frac{\epsilon^2(S(z)-4\beta_1(1-z))}{2(1-z)^2}+O(\epsilon^4))$$
$$= \log(1-z)+\log 2+\frac{\epsilon^2(S(z)-4\beta_1(1-z))}{4(1-z)^2}+O(\epsilon^4) \tag{A.2}$$

$$
\begin{aligned}
&\log\sqrt{(\beta - z)^2 + \epsilon^2 S(z)} \\
&= \frac{1}{2}\log\left[(1-z)^2 - 2(1-z)\left(\sum_{n=1}^{\infty}\beta_n\epsilon^{2n}\right) + \left(\sum_{n=1}^{\infty}\beta_n\epsilon^{2n}\right)^2 + \epsilon^2 S(z)\right] \\
&= \frac{1}{2}\log\left[(1-z)^2 + \epsilon^2(S(z) - 2(1-z)\beta_1) + O(\epsilon^4)\right] \\
&= \log(1-z) + \frac{1}{2}\log\left[1 + \frac{\epsilon^2(S(z) - 2(1-z)\beta_1) + O(\epsilon^4)}{(1-z)^2}\right] \\
&= \log(1-z) + \frac{1}{2}\left(\frac{S(z)}{(1-z)^2} - \frac{2\beta_1}{1-z}\right)\epsilon^2 + O(\epsilon^4)
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
&\sqrt{(z - \sum_{n=1}^{\infty}\alpha_n\epsilon^{2n})^2 + \epsilon^2 S(z)} = z\sqrt{1 + \frac{\epsilon^2(S(z) - 2\alpha_1 z) + O(\epsilon^4)}{(1-z)^2}} \\
&= (z(1 + \frac{\epsilon^2(S(z) - 2\alpha_1 z)}{2z^2} + O(\epsilon^4)) \\
&= z + \epsilon^2\left(\frac{S(z)}{2z} - \alpha_1\right) + O(\epsilon^4)
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
&\log(\sqrt{(z-\alpha)^2 + \epsilon^2 S(z)} + z - \alpha) \\
&= \log\left(z - \sum_{n=1}^{\infty}\alpha_n\epsilon^{2n} + \sqrt{(z - \sum_{n=1}^{\infty}\alpha_n\epsilon^{2n})^2 + \epsilon^2 S(z)}\right) \\
&= \log(z) + \log\left(1 + \sqrt{1 + \frac{\epsilon^2(S(z) - 2\alpha_1 z) + O(\epsilon^4)}{z^2}} - \frac{\sum_{n=1}^{\infty}\alpha_n\epsilon^{2n}}{z}\right) \\
&= \log(z) + \log(2 + \frac{\epsilon^2(S(z) - 4\alpha_1 z)}{2z^2} + O(\epsilon^4)) \\
&= \log(z) + \log 2 + \frac{\epsilon^2(S(z) - 4\alpha_1 z)}{4z^2} + O(\epsilon^4)
\end{aligned} \tag{A.5}
$$

$$\log\sqrt{(z-\alpha)^2+\epsilon^2 S(z)} =$$

$$= \frac{1}{2}\log\left[z^2 - 2z\left(\sum_{n=1}^{\infty}\alpha_n\epsilon^{2n}\right) + \left(\sum_{n=1}^{\infty}\alpha_n\epsilon^{2n}\right)^2 + \epsilon^2 S(z)\right] =$$

$$= \log(z) + \frac{1}{2}\left(\frac{S(z)}{z^2} - \frac{2\alpha_1}{z}\right)\epsilon^2 + O(\epsilon^4) \tag{A.6}$$

$$\mathbf{B}_e = -ik\nabla\left(\frac{e^{ikR}}{R}\right)\times\mathbf{p} = -ik\left(-\frac{\mathbf{n}}{R^2}e^{ikR} + \frac{ik\mathbf{n}}{R}e^{ikR}\right)\times\mathbf{p} =$$

$$= -ike^{ikR}\left(\frac{ik}{R} - \frac{1}{R^2}\right)(\mathbf{n}\times\mathbf{p}) =$$

$$= k^2\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right)(\mathbf{n}\times\mathbf{p}) \tag{A.7}$$

$$\nabla\times(R\mathbf{n}\times\mathbf{p}) = \begin{pmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (yp_3 - zp_2) & (zp_1 - xp_3) & (xp_2 - yp_1) \end{pmatrix} =$$

$$= (-2p_1\mathbf{i_x} - 2p_2\mathbf{i_y} - 2p_3\mathbf{i_z}) = -2\mathbf{p} \tag{A.8}$$

$$\nabla\left(\frac{e^{ikR}}{R^2}\left(1 - \frac{1}{ikR}\right)\right) = \left(1 - \frac{1}{ikR}\right)\nabla\left(\frac{e^{ikR}}{R^2}\right) + \frac{e^{ikR}}{R^2}\nabla\left(1 - \frac{1}{ikR}\right) =$$

$$= \left(1 - \frac{1}{ikR}\right)\left(ik\mathbf{n}\frac{e^{ikR}}{R^2} - 2\mathbf{n}\frac{e^{ikR}}{R^3}\right) + \mathbf{n}\frac{e^{ikR}}{ikR^4} \tag{A.9}$$

$$
\begin{aligned}
\mathbf{E}_e &= \nabla \times \nabla \times \left(\mathbf{p}\frac{e^{ikR}}{R}\right) = \nabla\left(\nabla \cdot \left(\mathbf{p}\frac{e^{ikR}}{R}\right)\right) - \nabla^2\left(\mathbf{p}\frac{e^{ikR}}{R}\right) \\
&= \nabla\left(\frac{e^{ikR}}{R}(\nabla \cdot \mathbf{p}) + \mathbf{p}\cdot\nabla\left(\frac{e^{ikR}}{R}\right)\right) - \mathbf{p}\nabla\cdot\nabla\left(\frac{e^{ikR}}{R}\right)
\end{aligned} \tag{A.10}
$$

$$
\begin{aligned}
\mathbf{E}_e &= \nabla\left(\mathbf{p}\cdot\left(ik\frac{e^{ikR}}{R}\mathbf{n} - \frac{e^{ikR}}{R^2}\mathbf{n}\right)\right) - \mathbf{p}\nabla.\left(ik\frac{e^{ikR}}{R}\mathbf{n} - \frac{e^{ikR}}{R^2}\mathbf{n}\right) \\
&= \nabla\left((\mathbf{p}\cdot\mathbf{n})\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\right) - \mathbf{p}\cdot\nabla\left(\mathbf{n}\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\right) \\
&= (\mathbf{p}\cdot\mathbf{n})\nabla\left(\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\right) + \frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\nabla(\mathbf{p}.\mathbf{n}) \\
&- \mathbf{p}\left(\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\nabla\cdot\mathbf{p} + \mathbf{n}\cdot\nabla\left(\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\right)\right)
\end{aligned} \tag{A.11}
$$

$$
\begin{aligned}
\mathbf{E}_e &= \mathbf{n}(\mathbf{p}\cdot\mathbf{n})e^{ikR}\left(-\frac{k^2}{R} - \frac{2ik}{R^2} + \frac{2}{R^3}\right) + \left(\frac{\mathbf{p}}{R} - (\mathbf{n}\cdot\mathbf{p})\frac{\mathbf{n}}{R}\right)\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right) \\
&= -2\frac{\mathbf{p}}{R}\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right) - \mathbf{p}e^{ikR}\left(-\frac{k^2}{R} - \frac{2ik}{R^2} + \frac{2}{R^3}\right) \\
&= \mathbf{n}(\mathbf{p}\cdot\mathbf{n})e^{ikR}\left(-\frac{k^2}{R} - \frac{3ik}{R^2} + \frac{3}{R^3}\right) + \mathbf{p}e^{ikR}\left(\frac{k^2}{R} + \frac{ik}{R^2} - \frac{1}{R^3}\right)
\end{aligned} \tag{A.12}
$$

$$
\begin{aligned}
\mathbf{E}_e &= \frac{(r^2\cos\theta\mathbf{i_r} + rz\cos\theta\mathbf{i_z})}{R^2} e^{ikR}\left(-\frac{k^2}{R} - \frac{3ik}{R^2} + \frac{3}{R^3}\right) \\
&+ (\cos\theta\mathbf{i_r} - \sin\theta\mathbf{i}_\theta)e^{ikR}\left(\frac{k^2}{R} + \frac{ik}{R^2} - \frac{1}{R^3}\right) \\
&= \mathbf{i_r}\cos\theta\frac{e^{ikR}}{R^3}\left(-k^2r^2 - \frac{3ikr^2}{R} + \frac{3r^2}{R^2} + k^2R^2 + ikR - 1\right) \\
&+ \mathbf{i}_\theta\sin\theta\frac{e^{ikR}}{R}\left(-k^2 - \frac{ik}{R} + \frac{1}{R^2}\right) + \mathbf{i_z}\cos\theta\frac{e^{ikR}}{R^3}rz\left(-k^2 - \frac{3ik}{R} + \frac{3}{R^2}\right) \\
&= \mathbf{i_r}\cos\theta\frac{e^{ikR}}{R^3}\left(\frac{-2ikr^2}{R} + \frac{2r^2}{R^2} + \frac{ik(z-\xi)^2}{R} - \frac{(z-\xi)^2}{R^2} + k^2(z-\xi)^2\right) \\
&+ \mathbf{i}_\theta\sin\theta\frac{e^{ikR}}{R}\left(-k^2 - \frac{ik}{R} + \frac{1}{R^2}\right) + \mathbf{i_z}\cos\theta\frac{e^{ikR}}{R^3}rz\left(-k^2 - \frac{3ik}{R} + \frac{3}{R^2}\right) \\
&= ik^2\mathbf{i_r}\cos\theta\frac{e^{ikR}}{R^3}\left(-2r^2\left(\frac{1}{kR} + \frac{i}{k^2R^2}\right) - (z-\xi)^2\left(-\frac{1}{kR} - \frac{i}{k^2R^2} + i\right)\right) \\
&+ ik^2\mathbf{i}_\theta\sin\theta\frac{e^{ikR}}{R}\left(i - \frac{1}{kR} - \frac{i}{k^2R^2}\right) + ik^2\mathbf{i_z}\cos\theta\frac{e^{ikR}}{R^3}rz\left(i - \frac{3}{kR} - \frac{3i}{k^2R^2}\right) \\
&= k^2\mathbf{i_r}\cos\theta\frac{e^{ikR}}{R^3}\left(-2r^2f_1 - (z-\xi)^2f_2\right) \\
&+ ik^2\mathbf{i}_\theta\sin\theta\frac{e^{ikR}}{R}f_2 + ik^2\mathbf{i_z}\cos\theta\frac{e^{ikR}}{R^3}rz(f_2 - 2f_1) \qquad \text{(A.13)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B}_e &= k^2\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right)(\mathbf{n}\times\mathbf{p})\frac{x\mathbf{i_x} + y\mathbf{i_y} + z\mathbf{i_z}}{R} \times \mathbf{i_x}k^2\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right) \\
&= \left(-\frac{y}{R}\mathbf{i_z} + \frac{z}{R}\mathbf{i_y}\right)k^2\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right) \qquad \text{(A.14)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B}_e &= \left(-\frac{y}{R}\mathbf{i_z} + \frac{z}{R}\mathbf{i_y}\right) k^2 \frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right) \\
&= \left(-\frac{r}{R}\sin\theta\mathbf{i_z} + \frac{z}{R}(\sin\theta\mathbf{i_r} + \cos\theta\mathbf{i}_\theta)\right) k^2 \frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right) \\
&= (-r\sin\theta\mathbf{i_z} + z(\sin\theta\mathbf{i_r} + \cos\theta\mathbf{i}_\theta))\, k^2 \frac{e^{ikR}}{R^2}\left(1 - \frac{1}{ikR}\right) \\
&= k^3 \frac{e^{ikR}}{R}\left(\frac{1}{kR} + \frac{i}{k^2R^2}\right)(z\sin\theta\mathbf{i_r} + z\cos\theta\mathbf{i}_\theta - r\sin\theta\mathbf{i_z}) \\
&= k^3 \frac{e^{ikR}}{R} f_1(z\sin\theta\mathbf{i_r} + z\cos\theta\mathbf{i}_\theta - r\sin\theta\mathbf{i_z}) \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B}_m &= \nabla \times \mathbf{A}_m = ik\nabla \times \left((R\mathbf{n}\times\mathbf{m})\frac{e^{ikR}}{R^2}\left(1 - \frac{1}{ikR}\right)\right) \\
&= \frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right)\nabla \times (R\mathbf{n}\times\mathbf{m}) + \nabla\left(\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right)\right) \times (R\mathbf{n}\times\mathbf{m}) \\
&= -2\mathbf{m}\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right) + \nabla\left(\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right)\right) \times (R\mathbf{n}\times\mathbf{m}) \qquad \text{(A.16)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B}_m &= \nabla \times \mathbf{A} = -2\mathbf{m}\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right) + \nabla\left(\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right)\right) \times (R\mathbf{n}\times\mathbf{m}) \\
&= -2\mathbf{m}\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right) + (-\mathbf{m} + \mathbf{n}(\mathbf{n}\cdot\mathbf{m}))\left(\left(ik - \frac{1}{R}\right)\left(ik\frac{e^{ikR}}{R} - 2\frac{e^{ikR}}{R^2}\right) + \frac{e^{ikR}}{R^3}\right) \\
&= -2\mathbf{m}\frac{e^{ikR}}{R^2}\left(ik - \frac{1}{R}\right) + (-\mathbf{m} + \mathbf{n}(\mathbf{n}\cdot\mathbf{m}))\frac{e^{ikR}}{R^2}\left(-k^2R - 3ik + \frac{3}{R}\right) \\
&= \mathbf{m}\frac{e^{ikR}}{R^2}\left(ik + k^2R - \frac{1}{R}\right) + \mathbf{n}(\mathbf{n}.\mathbf{m})\frac{e^{ikR}}{R^2}\left(-k^2R - 3ik + \frac{3}{R}\right) \qquad \text{(A.17)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}_m &= \frac{i}{k}\nabla\times\mathbf{B} \\
&= \frac{i}{k}\nabla\times\left(\mathbf{m}\frac{e^{ikR}}{R^2}\left(ik+k^2R-\frac{1}{R}\right)+\mathbf{n}(\mathbf{n}\cdot\mathbf{m})\frac{e^{ikR}}{R^2}\left(-k^2R-3ik+\frac{3}{R}\right)\right) \\
&= \frac{i}{k}\nabla\left(\frac{e^{ikR}}{R^2}\left(ik+k^2R-\frac{1}{R}\right)\right)\times\mathbf{m}+\frac{i}{k}\left(\frac{e^{ikR}}{R^2}\left(ik+k^2R-\frac{1}{R}\right)\right)\nabla\times\mathbf{m} \\
&+ \frac{i}{k}\nabla\left(\frac{e^{ikR}}{R^2}\left(-k^2R-3ik+\frac{3}{R}\right)(\mathbf{n}\cdot\mathbf{m})\right)\times\mathbf{n} \\
&+ \frac{i}{k}\left(\frac{e^{ikR}}{R^2}\left(-k^2R-3ik+\frac{3}{R}\right)(\mathbf{n}\cdot\mathbf{m})\right)\nabla\times\mathbf{n}
\end{aligned} \tag{A.18}
$$

$$
\begin{aligned}
\mathbf{E}_m &= \frac{i}{k}\nabla\left(\frac{e^{ikR}}{R^2}\left(ik+k^2R-\frac{1}{R}\right)\right)\times\mathbf{m} \\
&+ \frac{i}{k}\nabla\left(\frac{e^{ikR}}{R^2}\left(-k^2R-3ik+\frac{3}{R}\right)(\mathbf{n}\cdot\mathbf{m})\right)\times\mathbf{n} \\
&= \frac{i}{k}\frac{e^{ikR}}{R^2}\left(\left(ik-\frac{2}{R}\right)\left(ik+k^2R-\frac{1}{R}\right)+k^2+\frac{1}{R^2}\right)(\mathbf{n}\times\mathbf{m}) \\
&+ \frac{i}{k}\frac{e^{ikR}}{R^2}\left(k^2R+3ik-\frac{3}{R}\right)\frac{1}{R}(\mathbf{n}\times\mathbf{m}) \\
&= \frac{i}{k}\frac{e^{ikR}}{R^2}\left(-k^2+ik^3R\right)(\mathbf{n}\times\mathbf{m}) = k^2\frac{e^{ikR}}{R}\left(\frac{1}{ikR}-1\right)(\mathbf{n}\times\mathbf{m})
\end{aligned} \tag{A.19}
$$

$$
\begin{aligned}
\mathbf{E}_m &= k^2\frac{e^{ikR}}{R^2}\left(\frac{1}{ikR}-1\right)(r\cos\theta\mathbf{i_z}-z\cos\theta\mathbf{i_r}+z\sin\theta\mathbf{i}_\theta) \\
&= -k^3\frac{e^{ikR}}{R}\left(\frac{1}{kR}+\frac{i}{k^2R^2}\right)(r\cos\theta\mathbf{i_z}-z\cos\theta\mathbf{i_r}+z\sin\theta\mathbf{i}_\theta) \\
&= -k^3\frac{e^{ikR}}{R}f_1(r\cos\theta\mathbf{i_z}-z\cos\theta\mathbf{i_r}+z\sin\theta\mathbf{i}_\theta)
\end{aligned} \tag{A.20}
$$

$$\mathbf{B}_m = \mathbf{m}\frac{e^{ikR}}{R^2}\left(ik + k^2R - \frac{1}{R}\right) + \mathbf{n}(\mathbf{n}\cdot\mathbf{m})\frac{e^{ikR}}{R^2}\left(-k^2R - 3ik + \frac{3}{R}\right)$$
$$= (\sin\theta\mathbf{i_r} + \cos\theta\mathbf{i}_\theta)\frac{e^{ikR}}{R^2}\left(ik + k^2R - \frac{1}{R}\right)$$
$$+ (r^2\sin\theta\mathbf{i_r} + rz\sin\theta\mathbf{i_z})\frac{e^{ikR}}{R^4}\left(-k^2R - 3ik + \frac{3}{R}\right)$$
$$= \sin\theta\mathbf{i_r}\frac{e^{ikR}}{R^3}\left(ikR + k^2R^2 - 1 - r^2k^2 - \frac{3ikr^2}{R} + \frac{3r^2}{R^2}\right)$$
$$+ \cos\theta\mathbf{i}_\theta\frac{e^{ikR}}{R}\left(k^2 + \frac{ik}{R} - \frac{1}{R^2}\right) - rz\sin\theta\mathbf{i_z}\frac{e^{ikR}}{R^3}\left(k^2 + \frac{3ik}{R} + \frac{3}{R^2}\right) \tag{A.21}$$

$$ikR + k^2R^2 - 1 - r^2k^2 - \frac{3ikr^2}{R} + \frac{3r^2}{R^2}$$
$$= -\frac{2ikr^2}{R} + \frac{2r^2}{R^2} + k^2(z-\xi)^2 + \frac{ik(z-\xi)^2}{R} - \frac{(z-\xi)^2}{R^2}$$
$$= \frac{k^2}{i}\left[2r^2\left(\frac{1}{kR} + \frac{i}{k^2R^2}\right) + (z-\xi)^2\left(i - \frac{1}{kR} - \frac{i}{k^2R^2}\right)\right]$$
$$= \frac{k^2}{i}(2r^2f_1 + (z-\xi)^2f_2) \tag{A.22}$$

$$E_1 = -\frac{i}{\epsilon^2k^2}\int_\alpha^\beta \frac{e^{ikR}}{R}\{\frac{2(z-\xi)^2 + 2\epsilon^2S(z) - 2(z-\xi)^2 - (z-\xi)\epsilon^2S^{'}(z)}{R^2}$$
$$+ ik\frac{2(z-\xi)^2 + (z-\xi)\epsilon^2S^{'}(z) - 2\epsilon^2S(z) - 2(z-\xi)^2}{R}\}h(\xi,\epsilon)d\xi$$
$$= \frac{i}{k^2}\int_\alpha^\beta \frac{e^{ikR}}{R}\{-\frac{ik}{R} + \frac{1}{R^2}\}((z-\xi)S^{'}(z) - 2S(z))h(\xi,\epsilon)d\xi$$
$$= \int_\alpha^\beta \frac{f_1}{R}((z-\xi)S^{'}(z) - 2S(z))h(\xi,\epsilon)d\xi \tag{A.23}$$

$$
\begin{aligned}
E_2 &= -S^{'}(z)\int_{\alpha}^{\beta} e^{ikR}\left(\frac{1}{kR^2}+\frac{i}{k^2R^3}-\frac{i}{R}\right)g(\xi,\epsilon)d\xi \\
&- S(z)\int_{\alpha}^{\beta} e^{ikR}\left(\frac{i}{R^3}-\frac{1}{kR^4}-\frac{2}{kR^4}-\frac{3i}{k^2R^5}\right)(2(z-\xi)+\epsilon^2 S^{'}(z))g(\xi,\epsilon)d\xi \\
&= \int_{\alpha}^{\beta} S^{'}(z)\frac{e^{ikR}}{R^3}R^2\left(i-\frac{1}{kR}-\frac{i}{k^2R^2}\right)g(\xi,\epsilon)d\xi \\
&- \int_{\alpha}^{\beta} \frac{e^{ikR}}{R^3}S(z)\left(i-\frac{3}{kR}-\frac{3i}{k^2R^2}\right)(2(z-\xi)+\epsilon^2 S^{'}(z))g(\xi,\epsilon)d\xi \\
&= \int_{\alpha}^{\beta} S^{'}(z)\frac{e^{ikR}}{R^3}R^2 f_2 g(\xi,\epsilon)d\xi - \int_{\alpha}^{\beta}\frac{e^{ikR}}{R^3}S(z)(f_2-2f_1)(2(z-\xi)+\epsilon^2 S^{'}(z))g(\xi,\epsilon)d\xi \\
&= \int_{\alpha}^{\beta}\frac{e^{ikR}}{R^3}\Big(S^{'}(z)f_2((z-\xi)^2+\epsilon^2 S(z)) - S(z)(f_2-2f_1)(2(z-\xi)+\epsilon^2 S^{'}(z))\Big) \\
&\times g(\xi,\epsilon)d\xi = \int_{\alpha}^{\beta}\frac{e^{ikR}}{R^3}\Big(S^{'}(z)(2\epsilon^2 S(z)f_1+f_2(z-\xi)^2)-2S(z)(z-\xi)(f_2-2f_1)\Big) \\
&\times g(\xi,\epsilon)d\xi \qquad \text{(A.24)}
\end{aligned}
$$

$$
\begin{aligned}
2e^{ikz} &= 2\int_\alpha^\beta \frac{f_2}{R} g(\xi,\epsilon)d\xi + \int_\alpha^\beta \frac{f_1}{R}(2(z-\xi)+\epsilon^2 S'(z))h(\xi,\epsilon)d\xi \\
&+ \frac{\epsilon^2 S'^2(z)}{2S(z)}\int_\alpha^\beta \frac{f_2}{R} g(\xi,\epsilon)d\xi + \frac{\epsilon^2 S'(z)}{2S(z)}\int_\alpha^\beta \Big(2S(z)(z-\xi)(f_2-2f_1) \\
&- S^{'}(z)(2\epsilon^2 S(z) f_1 + (z-\xi)^2 f_2)\Big)\frac{g(\xi,\epsilon)}{R^3}d\xi \\
&= 2\int_\alpha^\beta \frac{f_2}{R} g(\xi,\epsilon)d\xi + 2\int_\alpha^\beta f_1 \frac{\partial R}{\partial z} h(\xi,\epsilon)d\xi \\
&+ \frac{\epsilon^2 S'(z)}{2S(z)}\int_\alpha^\beta \Big(S'(z) f_2((z-\xi)^2+\epsilon^2 S(z)) + 2S(z)(z-\xi)(f_2-2f_1) \\
&- S^{'}(z)(2\epsilon^2 S(z) f_1 + (z-\xi)^2 f_2)\Big)\frac{g(\xi,\epsilon)}{R^3}d\xi \\
&= 2\int_\alpha^\beta \frac{f_2}{R} g(\xi,\epsilon)d\xi + 2\int_\alpha^\beta f_1 \frac{\partial R}{\partial z} h(\xi,\epsilon)d\xi \\
&+ \frac{\epsilon^2 S'(z)}{2S(z)}\int_\alpha^\beta \Big(S(z)(f_2-2f_1)(2(z-\xi)+\epsilon^2 S'(z))\Big)\frac{g(\xi,\epsilon)}{R^3}d\xi \\
&= 2\int_\alpha^\beta \frac{f_2}{R} g(\xi,\epsilon)d\xi + 2\int_\alpha^\beta f_1 \frac{\partial R}{\partial z} h(\xi,\epsilon)d\xi \\
&+ \frac{\epsilon^2 S'(z)}{2}\int_\alpha^\beta (f_2-2f_1)2\frac{\partial R}{\partial z}\frac{g(\xi,\epsilon)}{R^2}d\xi
\end{aligned} \tag{A.25}
$$

$$
\begin{aligned}
&\frac{d}{dz}\frac{e^{ikR}}{R^2}\Big(1+\frac{i}{kR}\Big) = e^{ikR}\Bigg[ik\Big(\frac{1}{R^2}+\frac{i}{kR^3}\Big)+\Big(-\frac{2}{R^3}-\frac{3i}{kR^4}\Big)\Bigg]\frac{\partial R}{\partial z} \\
&= e^{ikR}\left(\frac{ik}{R^2}-\frac{3}{R^3}-\frac{3i}{kR^4}\right)\frac{\partial R}{\partial z} = k\frac{e^{ikR}}{R^2}\left(i-\frac{3}{kR}-\frac{3i}{k^2R^2}\right)\frac{\partial R}{\partial z} = k\frac{e^{ikR}}{R^2}(f_2-2f_1)\frac{\partial R}{\partial z}
\end{aligned} \tag{A.26}
$$

$$
\begin{aligned}
R^{+} &= (e^{ik(1-z)} + \epsilon^2 ik \left(\frac{S(z)}{2(1-z)} - \beta_1\right) e^{ik(1-z)} + O(\epsilon^3))\big[((1-z) \\
&+ \epsilon^2 \left(\frac{S(z)}{2(1-z)} - \beta_1\right) + O(\epsilon^3))(\log(1-z) + \log 2 + \epsilon^2 \frac{(S(z) - 4\beta_1(1-z))}{4(1-z)^2} + O(\epsilon^4)) \\
&- 1 + z + \epsilon^2 \beta_1 + (1-z) + \epsilon^2 \left(\frac{S(z)}{2(1-z)} - \beta_1\right) + O(\epsilon^3) - ((1-z) + \epsilon^2 \Big(\frac{S(z)}{2(1-z)} \\
&- \beta_1\Big) + O(\epsilon^3)) * (\log(1-z) + \log 2 + \epsilon^2 \frac{S(z) - 2\beta_1(1-z)}{2(1-z)^2} + O(\epsilon^4))\big] \\
&- (1 + \epsilon ik\sqrt{S(z)} - \epsilon^2 \frac{k^2 S(z)}{2} + O(\epsilon^3)) * [\epsilon\sqrt{S(z)}(1 - \log 2)] - R_1^{+} \\
&= \epsilon^2 e^{ik(1-z)} \left(-\frac{S(z)}{4(1-z)} + \beta_1\right) - \epsilon\sqrt{S(z)}(1 - \log 2) - \epsilon^2 ik S(z)(1 - \log 2) - R_1^{+} + HOT
\end{aligned}
\tag{A.27}
$$

$$
\begin{aligned}
\Delta_1^{+} &= ik \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} [u \log(u + \sqrt{u^2 - \epsilon^2 S(z)}) - \sqrt{u^2 - \epsilon^2 S(z)} + u \\
&- u \log 2u] du = \frac{ik}{4}(-2u^2 \log 2u + 3u(u - \sqrt{u^2 - \epsilon^2 S}) + (2u^2 + \epsilon^2 S(z)) \\
&\times \log(u + \sqrt{u^2 - \epsilon^2 S}))\Big|_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} = \frac{ik}{4}(-2((\beta - z)^2 + \epsilon^2 S(z)) \\
&\times \log 2\sqrt{(\beta-z)^2 + \epsilon^2 S(z)} + 3\sqrt{(\beta-z)^2 + \epsilon^2 S(z)}(\sqrt{(\beta-z)^2 + \epsilon^2 S(z)} - (\beta - z)) \\
&+ (2(\beta-z)^2 + 3\epsilon^2 S(z)) \log((\sqrt{(\beta-z)^2 + \epsilon^2 S(z)} + (\beta - z))) - \frac{ik}{4}\Big(- 2\epsilon^2 S(z) \\
&\times \log 2\epsilon\sqrt{S(z)} + 3\epsilon^2 S(z) + 3\epsilon^2 S(z) \log \epsilon\sqrt{S(z)}\Big) = \frac{ik}{4}(-2((\beta - z)^2 + \epsilon^2 S(z)) \\
&\times (\log(1-z) + \log 2 + \epsilon^2 \frac{1}{2}\left(\frac{S(z)}{(1-z)^2} - \frac{2\beta_1}{1-z}\right))) + 3((1-z) + \epsilon^2 \left(\frac{S(z)}{2(1-z)} - \beta_1\right) \\
&+ O(\epsilon^3)) * ((1-z) + \epsilon^2 \left(\frac{S(z)}{2(1-z)} - \beta_1\right) - (\beta - z)) + (2(\beta - z)^2 + 3\epsilon^2 S(z)) \\
&\times (\log(1-z) + \log 2 + \frac{\epsilon^2 (S(z) - 4\beta_1(1-z))}{4(1-z)^2} + O(\epsilon^4))) - \frac{ik}{4}(\epsilon^2 S(z) \log \epsilon\sqrt{S(z)} \\
&+ \epsilon^2 S(z)(3 - 2\log 2)) = \frac{ik}{4}\epsilon^2 S(z)(-2 - \log \epsilon\sqrt{S(z)} + \log 2 - \log(1-z))
\end{aligned}
\tag{A.28}
$$

$$
\begin{aligned}
R^- &= (e^{ikz} + \epsilon^2 ik\left(\frac{S(z)}{2z} - \beta_1\right)e^{ikz} + O(\epsilon^3))[(z + \epsilon^2\left(\frac{S(z)}{2z} - \alpha_1\right) + O(\epsilon^3))(\log z \\
&+ \log 2 + \epsilon^2\frac{(S(z) - 4\alpha_1 z)}{4z^2} + O(\epsilon^4)) - z + \epsilon^2\alpha_1 + z + \epsilon^2\left(\frac{S(z)}{2z} - \beta_1\right) + O(\epsilon^3) \\
&- (z + \epsilon^2\left(\frac{S(z)}{2z} - \alpha_1\right) + O(\epsilon^3)) * (\log z + \log 2 + \epsilon^2\frac{S(z) - 2\alpha_1 z}{2z^2} + O(\epsilon^4))] \\
&- (1 + \epsilon ik\sqrt{S(z)} - \epsilon^2\frac{k^2S(z)}{2} + O(\epsilon^3)) * [\epsilon\sqrt{S(z)}(1 - \log 2)] - R_1^- \\
&= \epsilon^2 e^{ikz}\left(-\frac{S(z)}{4z} + \alpha_1\right) - \epsilon\sqrt{S(z)}(1 - \log 2) - \epsilon^2 ikS(z)(1 - \log 2) - R_1^- + HOT
\end{aligned}
\tag{A.29}
$$

$$
\begin{aligned}
\Delta_1^- &= ik\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2S(z)}}[u\log(u + \sqrt{u^2 - \epsilon^2S(z)}) - \sqrt{u^2 - \epsilon^2S(z)} + u \\
&- u\log 2u]du = \frac{ik}{4}(-2u^2\log 2u + 3u(u - \sqrt{u^2 - \epsilon^2S}) + (2u^2 + \epsilon^2S(z)) \\
&\times \log(u + \sqrt{u^2 - \epsilon^2S}))|_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2S(z)}} = \frac{ik}{4}(-2((z-\alpha)^2 + \epsilon^2S(z)) \\
&\times \log 2\sqrt{(z-\alpha)^2 + \epsilon^2S(z)} + 3\sqrt{(z-\alpha)^2 + \epsilon^2S(z)}(\sqrt{(z-\alpha)^2 + \epsilon^2S(z)} - (z-\alpha)) \\
&+ (2(z-\alpha)^2 + 3\epsilon^2S(z))\log((\sqrt{(z-\alpha)^2 + \epsilon^2S(z)} + (z-\alpha))) - \frac{ik}{4}(-2\epsilon^2S(z) \\
&\times \log 2\epsilon\sqrt{S(z)} + 3\epsilon^2S(z) + 3\epsilon^2S(z)\log\epsilon\sqrt{S(z)}) = \frac{ik}{4}(-2((z-\alpha)^2 + \epsilon^2S(z))(\log z) \\
&+ \log 2 + \epsilon^2\frac{1}{2}\left(\frac{S(z)}{z^2} - \frac{2\alpha_1}{z}\right))) + 3(z + \epsilon^2\left(\frac{S(z)}{2z} - \alpha_1\right) + O(\epsilon^3)) * (z + \epsilon^2\Big(\frac{S(z)}{2z} \\
&- \alpha_1\Big) - (z-\alpha)) + (2(z-\alpha)^2 + 3\epsilon^2S(z))(\log z + \log 2 + \frac{\epsilon^2(S(z) - 4\alpha_1 z)}{4z^2} + O(\epsilon^4))) \\
&- \frac{ik}{4}(\epsilon^2S(z)\log\epsilon\sqrt{S(z)} + \epsilon^2S(z)(3 - 2\log 2)) = \frac{ik}{4}\epsilon^2S(z)(-2 - \log\epsilon\sqrt{S(z)} \\
&+ \log 2 - \log z)
\end{aligned}
\tag{A.30}
$$

APPENDIX B

ASYMPTOTIC EXPANSIONS OF INTEGRAL OPERATORS

B.1 STATIC CASE

In this section, we obtain a uniform asymptotic expansion for each of the integrals I_{1s} , I_{2s}, I_{3s}. $I_{1s} = I_{1s}^- + I_{1s}^+$ can be evaluated directly:

$$I_{1s}^+ = \int_0^{\beta - z} \frac{dv}{\sqrt{v^2 + \epsilon^2 S(z)}} dv = \log(\beta - z + \sqrt{(\beta - z)^2 + \epsilon^2 S}) - \log \epsilon \sqrt{S(z)} \quad \text{(B.1)}$$

$$I_{1s}^- = \int_0^{z - \alpha} \frac{dv}{\sqrt{v^2 + \epsilon^2 S(z)}} dv = \log(z - \alpha + \sqrt{(z - \alpha)^2 + \epsilon^2 S}) - \log \epsilon \sqrt{S(z)} \quad \text{(B.2)}$$

$$\log \epsilon \sqrt{S(z)} = \log \epsilon + \frac{1}{2} \log S(z). \quad \text{(B.3)}$$

Since (A.1, A.2, A.3 and A.4)

$$\Big(((1 - z) - \sum_{n=1}^{\infty} \beta_n \epsilon^{2n})^2 + \epsilon^2 S(z)\Big)^{1/2} = (1 - z) + \epsilon^2 \left(\frac{S(z)}{2(1 - z)} - \beta_1 \right) + O(\epsilon^4), \quad \text{(B.4)}$$

$$\log(\beta - z + \sqrt{(\beta - z)^2 + \epsilon^2 S(z)}) = \log 2(1 - z) + \frac{\epsilon^2 (S(z) - 4\beta_1(1 - z))}{4(1 - z)^2} + O(\epsilon^4), \quad \text{(B.5)}$$

$$\Big((z - \sum_{n=1}^{\infty} \alpha_n \epsilon^{2n})^2 + \epsilon^2 S(z)\Big)^{1/2} = z + \epsilon^2 \left(\frac{S(z)}{2z} - \alpha_1 \right) + O(\epsilon^4), \quad \text{(B.6)}$$

$$\log(z - \alpha + \sqrt{(z - \alpha)^2 + \epsilon^2 S(z)}) = \log 2z + \frac{\epsilon^2 (S(z) - 4\alpha_1 z)}{4z^2} + O(\epsilon^4), \quad \text{(B.7)}$$

the binomial expansion of $1/\sqrt{v^2+\epsilon^2 S(z)}$ yields

$$\frac{1}{(v^2+\epsilon^2 S(z))^{1/2}} = \frac{1}{v}\left(1+\frac{\epsilon^2 S(z)}{v^2}\right)^{-1/2} = \frac{1}{v}\left(1-\frac{1}{2}\frac{\epsilon^2 S(z)}{v^2}+\frac{3}{8}\frac{\epsilon^4 S^2(z)}{v^4}+...\right) \tag{B.8}$$

Adding and subtracting $\frac{1}{2}\frac{\epsilon^2 S(z)}{v^2}$ to $1/\sqrt{1+\frac{\epsilon^2 S(z)}{v^2}}-1$ we have

$$\begin{aligned}\left(1+\frac{\epsilon^2 S(z)}{v^2}\right)^{-1/2}-1 &= \left(1+\frac{\epsilon^2 S(z)}{v^2}\right)^{-1/2}-1-\frac{1}{2}\frac{\epsilon^2 S(z)}{v^2}+\frac{1}{2}\frac{\epsilon^2 S(z)}{v^2}\\ &= \left(1+\frac{\epsilon^2 S(z)}{v^2}\right)^{-1/2}-\sum_{j=0}^{1} a_j\left(\frac{\epsilon^2 S(z)}{v^2}\right)^{j}-\frac{1}{2}\frac{\epsilon^2 S(z)}{v^2}.\end{aligned} \tag{B.9}$$

Adding and subtracting $F(z)+vF'(z)$ to $F(z+v)-F(z)$ yields

$$\begin{aligned}F(z+v)-F(z) &= F(z+v)-F(z)+vF'(z)+\frac{v^2}{2}F''(z)-vF'(z)-\frac{v^2}{2}F''(z)\\ &= F(z+v)-\sum_{j=0}^{2}\frac{v^j F^{(j)}(z)}{j!}+vF'(z)+\frac{v^2}{2}F''(z).\end{aligned} \tag{B.10}$$

Thus the expression for I_{3s}^+ becomes

$$I_{3s}^+ = \int_0^{\beta - z} \frac{F(z+v) - F(z)}{v} \left(\left(1 + \frac{\epsilon^2 S(z)}{v^2} \right)^{-1/2} - 1 \right) dv$$

$$= \int_0^{\beta - z} \frac{F(z+v) - \sum_{j=0}^{2} \frac{v^j F^{(j)}(z)}{j!} + vF'(z) + \frac{v^2}{2} F''(z)}{v} \left(\left(1 + \frac{\epsilon^2 S(z)}{v^2} \right)^{-1/2} - 1 \right) dv$$

$$= \int_0^{\beta - z} \frac{F(z+v) - \sum_{j=0}^{2} \frac{v^j F^{(j)}(z)}{j!}}{v} \left(\left(1 + \frac{\epsilon^2 S(z)}{v^2} \right)^{-1/2} - 1 \right) dv$$

$$+ \int_0^{\beta - z} (F'(z) + \frac{v}{2} F''(z)) \left(\left(1 + \frac{\epsilon^2 S(z)}{v^2} \right)^{-1/2} - 1 \right) dv$$

$$= \int_0^{\beta - z} (F'(z) + \frac{v}{2} F''(z)) \left(\left(1 + \frac{\epsilon^2 S(z)}{v^2} \right)^{-1/2} - 1 \right) dv$$

$$- \frac{1}{2} \epsilon^2 S(z) \int_0^{\beta - z} \frac{1}{v^3} \Big(F(z+v) - \sum_{j=0}^{2} \frac{v^j F^{(j)}(z)}{j!} \Big) dv + \tag{B.11}$$

$$+ \int_0^{\beta - z} \frac{1}{v} \Big(F(z+v) - \sum_{j=0}^{2} \frac{v^j F^{(j)}(z)}{j!} \Big) \Big(\Big(1 + \frac{\epsilon^2 S(z)}{v^2} \Big)^{-1/2} - \sum_{j=0}^{1} a_j \left(\frac{\epsilon^2 S(z)}{v^2} \right)^j \Big) dv. \tag{B.12}$$

Also,I_{2s}^+ and I_{3s}^+ can be combined together to result in

$$I_{2s}^+ + I_{3s}^+ = \sum_{n=0}^{\infty} a_n (\epsilon^2 S(z))^n \int_0^{\beta - z} \frac{F(z+v) - \sum_{j=0}^{2n} \frac{v^j F^{(j)}(z)}{j!}}{v^{2n+1}} dv$$

$$+ \sum_{n=0}^{\infty} \frac{F^{(2n+1)}(z)}{(2n+1)!} \int_0^{\beta - z} v^{2n} \Big(\Big(1 + \frac{\epsilon^2 S(z)}{v^2} \Big)^{-1/2} - \sum_{j=0}^{n} a_j \left(\frac{\epsilon^2 S(z)}{v^2} \right)^j \Big) dv$$

$$+ \sum_{n=0}^{\infty} \frac{F^{(2n+2)}(z)}{(2n+2)!} \int_0^{\beta - z} v^{2n+1} \Big(\Big(1 + \frac{\epsilon^2 S(z)}{v^2} \Big)^{-1/2} - \sum_{j=0}^{n} a_j \left(\frac{\epsilon^2 S(z)}{v^2} \right)^j \Big) dv. \tag{B.13}$$

Similarly, I_{2s}^- and I_{3s}^- can be combined together as

$$I_{2s}^- + I_{3s}^- = \sum_{n=0}^{\infty} a_n(\epsilon^2 S(z))^n \int_0^{z-\alpha} \frac{F(z-v) - \sum_{j=0}^{2n} \frac{(-v)^j F^{(j)}(z)}{j!}}{v^{2n+1}} dv$$
$$- \sum_{n=0}^{\infty} \frac{F^{(2n+1)}(z)}{(2n+1)!} \int_0^{z-\alpha} v^{2n}\Big(\Big(1+\frac{\epsilon^2 S(z)}{v^2}\Big)^{-1/2} - \sum_{j=0}^{n} a_j \left(\frac{\epsilon^2 S(z)}{v^2}\right)^j\Big) dv$$
$$+ \sum_{n=0}^{\infty} \frac{F^{(2n+2)}(z)}{(2n+2)!} \int_0^{z-\alpha} v^{2n+1}\Big(\Big(1+\frac{\epsilon^2 S(z)}{v^2}\Big)^{-1/2} - \sum_{j=0}^{n} a_j \left(\frac{\epsilon^2 S(z)}{v^2}\right)^j\Big) dv. \quad \text{(B.14)}$$

Each of the terms above is of different order. To see that, let $v = \epsilon\sqrt{S(z)}u$. Then $dv = \epsilon\sqrt{S(z)}du$ and

$$\int_0^{\beta-z} v^{2n}\Big(\Big(1+\frac{\epsilon^2 S(z)}{v^2}\Big)^{-1/2} - \sum_{j=0}^{n} a_j \left(\frac{\epsilon^2 S(z)}{v^2}\right)^j\Big) dv$$
$$= \int_0^{\frac{\beta-z}{\epsilon\sqrt{S(z)}}} (\epsilon\sqrt{S(z)}u)^{2n}\Big(\Big(1+\frac{1}{u^2}\Big)^{-1/2} - \sum_{j=0}^{n} \frac{a_j}{(u^2)^j}\Big)\Big)\epsilon\sqrt{S(z)}du$$
$$= (\epsilon\sqrt{S(z)}u)^{2n+1} \int_0^{\frac{\beta-z}{\epsilon\sqrt{S(z)}}} \Big(\Big(1+\frac{1}{u^2}\Big)^{-1/2} - \sum_{j=0}^{n} \frac{a_j}{(u^2)^j}\Big)\Big)\epsilon\sqrt{S(z)}du$$
$$= O(\epsilon^{2n+1}), \quad \text{(B.15)}$$

$$\int_0^{\beta-z} v^{2n+1}\Big(\Big(1+\frac{\epsilon^2 S(z)}{v^2}\Big)^{-1/2} - \sum_{j=0}^{n} a_j \left(\frac{\epsilon^2 S(z)}{v^2}\right)^j\Big) dv$$
$$= \int_0^{\frac{\beta-z}{\epsilon\sqrt{S(z)}}} (\epsilon\sqrt{S(z)}u)^{2n+1}\Big(\Big(1+\frac{1}{u^2}\Big)^{-1/2} - \sum_{j=0}^{n} \frac{a_j}{(u^2)^j}\Big)\Big)\epsilon\sqrt{S(z)}du$$
$$= O(\epsilon^{2n+2}\log\epsilon), \quad \text{(B.16)}$$

$$\int_0^{\beta-z} \frac{1}{v^{2n+1}}\Big(F(z+v) - \sum_{j=0}^{2n} \frac{v^j F^{(j)}(z)}{j!}\Big) dv$$
$$\int_0^{\frac{\beta-z}{\epsilon\sqrt{S(z)}}} \frac{1}{(\epsilon\sqrt{S(z)}u)^{2n+1}}\Big(\sum_{j=3}^{\infty} \frac{(\epsilon\sqrt{S(z)}u)^j F^{(j)}(z)}{j!}\Big)\epsilon\sqrt{S(z)}du \sim O(1). \quad \text{(B.17)}$$

For $n = 0$,

$$\begin{aligned} &F'(z) \int_0^{\beta-z} \left(\left(1 + \frac{\epsilon^2 S(z)}{v^2}\right)^{-1/2} - 1 \right) dv \\ &= F'(z)(z - \beta - \epsilon\sqrt{S(z)} + \sqrt{(\beta - z)^2 + \epsilon^2 S(z)}) \\ &= F'(z)(-\epsilon\sqrt{S(z)} + \epsilon^2 \frac{S(z)}{2(1-z)} + O(\epsilon^4)), \end{aligned} \tag{B.18}$$

$$\begin{aligned} &\frac{F''(z)}{2} \int_0^{\beta-z} \left(\left(1 + \frac{\epsilon^2 S(z)}{v^2}\right)^{-1/2} - 1 \right) v dv \\ &= \frac{F''(z)}{4} \Big(- (\beta - z)^2 + ((\beta - z)\sqrt{(\beta - z)^2 + \epsilon^2 S(z)} \\ &+ \epsilon^2 S(z) \log(\sqrt{\epsilon^2 S(z))} - \epsilon^2 S(z) \log(\beta - z + \sqrt{(\beta - z)^2 + \epsilon^2 S(z)})) \Big) \\ &\frac{F''(z)}{4} \Big(\epsilon^2 \frac{S(z)}{2} + \epsilon^2 S(z) \log \epsilon\sqrt{S(z)} - \epsilon^2 S(z) \log(2(1-z)) + O(\epsilon^4) \Big). \end{aligned} \tag{B.19}$$

Similarly,

$$F'(z) \int_0^{z-\alpha} \left(\left(1 + \frac{\epsilon^2 S(z)}{v^2}\right)^{-1/2} - 1 \right) dv = F'(z)(-\epsilon\sqrt{S(z)} + \epsilon^2 \frac{S(z)}{2z} + O(\epsilon^4)) \tag{B.20}$$

and

$$\begin{aligned} &\frac{F''(z)}{2} \int_0^{z-\alpha} \left(\left(1 + \frac{\epsilon^2 S(z)}{v^2}\right)^{-1/2} - 1 \right) v dv \\ &\frac{F''(z)}{4} \Big(\epsilon^2 \frac{S(z)}{2} + \epsilon^2 S(z) \log \epsilon\sqrt{S(z)} - \epsilon^2 S(z) \log(2z) + O(\epsilon^4) \Big). \end{aligned} \tag{B.21}$$

B.2 DYNAMIC CASE

Expansion of $I_0(z,\epsilon,F)$:

Let us start with the integral

$$I_0(z,\epsilon,F) = \int_\alpha^\beta \frac{e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}}{\sqrt{(z-\xi)^2+\epsilon^2 S(z)}} F(\xi)d\xi. \tag{B.22}$$

We are looking for an expansion of $I_0(z,\epsilon,F)$ in the form [58]

$$I_0(z,\epsilon,F) \sim -\log\epsilon^2 F + G_1 F + \epsilon^2 \log\epsilon^2 G_2 F + O(\epsilon^2), \tag{B.23}$$

where the operators G_1, G_2 and G_3 are to be determined from the following asymptotic expansion. Let

$$I_0 = I_0^- + I_0^+, \tag{B.24}$$

where

$$I_0^- = \int_\alpha^z \frac{e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}}{\sqrt{(z-\xi)^2+\epsilon^2 S(z)}} F(\xi)d\xi \tag{B.25}$$

and

$$I_0^+ = \int_z^\beta \frac{e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}}}{\sqrt{(z-\xi)^2+\epsilon^2 S(z)}} F(\xi)d\xi. \tag{B.26}$$

Changing of variables $v = z-\xi$ for I_0^- and $v = \xi - z$ for I_0^+ leads to

$$I_0^- = \int_0^{z-\alpha} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} F(z-v)dv \tag{B.27}$$

and

$$I_0^+ = \int_0^{\beta-z} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} F(z+v)dv. \tag{B.28}$$

Since the Taylor expansion of $F(z+v)$ and $F(z-v)$ for small v are

$$F(z+v) = F(z) + vF^{'}(z) + \frac{v^2}{2}F^{''}(z) + \sum_{j=3}^{\infty} \frac{v^j F^{(j)}(z)}{j!},$$
$$F(z-v) = F(z) - vF^{'}(z) + \frac{v^2}{2}F^{''}(z) + \sum_{j=3}^{\infty} \frac{(-v)^j F^{(j)}(z)}{j!},$$

the expression becomes

$$\begin{aligned} I_0^+ &= \int_0^{\beta-z} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}}(F(z) + vF^{'}(z) + \frac{v^2}{2}F^{''}(z))dv \\ &+ \int_0^{\beta-z} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}}(F(z+v) - \sum_{j=0}^{2} \frac{v^j F^{(j)}(z)}{j!})dv. \end{aligned} \tag{B.29}$$

Also, since

$$\frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} = \frac{e^{ikv}}{v} + \epsilon^2 S(z)\left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3}\right) + O(\epsilon^4), \tag{B.30}$$

we obtain

$$\begin{aligned} I_0^+ &= F(z)\int_0^{\beta-z} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}}dv + \int_0^{\beta-z} \frac{F(z+v)-F(z)}{v}e^{ikv}dv \\ &+ F^{'}(z)\int_0^{\beta-z} \left\{ v\frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} - e^{ikv} \right\} dv + \frac{F^{''}(z)}{2}\int_0^{\beta-z} \left\{ v^2\frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} \right. \\ &\left. - ve^{ikv} \right\} dv + \frac{\epsilon^2 S(z)}{2}\int_0^{\beta-z} e^{ikv}\Big(F(z+v) - \sum_{j=0}^{2} \frac{v^j F^{(j)}(z)}{j!}\Big)\left(\frac{ik}{v^2} - \frac{1}{v^3}\right)dv + R \\ &= F(z)W_0^+ + W_1^+ + F^{'}(z)W_2^+ + \frac{F^{''}(z)}{2}W_3^+ + W_4^+ + R^+. \end{aligned} \tag{B.31}$$

Our next step is to investigate each of the integral operators W_0^+, W_1^+, W_3^+ and W_4^+ in the expansion of I_0^+ (and similarly W_0^-, W_1^-, W_3^- and W_4^- in I_0^-).

For

$$W_0^+ = \int_0^{\beta-z} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}}dv, \tag{B.32}$$

we consider the substitution $u = \sqrt{v^2 + \epsilon^2 S(z)}$. Then $v = \sqrt{u^2 - \epsilon^2 S(z)}$ and

$$dv = \frac{\sqrt{v^2 + \epsilon^2 S(z)}}{v} du = \frac{u}{v} du, \tag{B.33}$$

and W_0 becomes

$$\begin{aligned} W_0^+ &= \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \frac{e^{iku}}{u} \frac{u}{v} du = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \frac{e^{iku}}{\sqrt{u^2 - \epsilon^2 S(z)}} du \\ &= \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} e^{iku} \frac{d}{du} \log(u + \sqrt{u^2 - \epsilon^2 S(z)}) du, \end{aligned} \tag{B.34}$$

since

$$\frac{d}{du} \log(u + \sqrt{u^2 - \epsilon^2 S(z)}) = \frac{1}{\sqrt{u^2 - \epsilon^2 S(z)}}. \tag{B.35}$$

The expression for W_0^+ can be rewritten as:

$$\begin{aligned} W_0^+ &= \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} e^{iku} \frac{d}{du} \log(u + \sqrt{u^2 - \epsilon^2 S(z)}) du \\ &= \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \frac{e^{iku}}{u} du - \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \frac{e^{iku}}{u} du \\ &+ \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} e^{iku} \frac{d}{du} \log(u + \sqrt{u^2 - \epsilon^2 S(z)}) du \\ &= \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \frac{e^{iku}}{u} du \\ &+ \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} e^{iku} \frac{d}{du} (\log(u + \sqrt{u^2 - \epsilon^2 S(z)}) - \log 2u) du. \end{aligned} \tag{B.36}$$

If we integrate by parts the second integral and add and subtract

$\log\sqrt{(\beta-z)^2+\epsilon^2S(z)}-\log\epsilon\sqrt{S(z)}$, we obtain

$$W_0^+ = \log\sqrt{(\beta-z)^2+\epsilon^2S(z)} - \log\epsilon\sqrt{S(z)} + \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2S(z)}} \frac{e^{iku}-1}{u}du$$
$$+ e^{ik\sqrt{(\beta-z)^2+\epsilon^2S(z)}}\left[\log(\sqrt{(\beta-z)^2+\epsilon^2S(z)}+\beta-z) - \log(2\sqrt{(\beta-z)^2+\epsilon^2S(z)})\right]$$
$$+ e^{ik\epsilon\sqrt{S(z)}}\log 2 - ik\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2S(z)}} e^{iku}\left[\log(u+\sqrt{u^2-\epsilon^2S(z)}) - \log 2u\right]du.$$

(B.37)

Using the fact that $\alpha(\epsilon)=\sum_{n=1}^{\infty}\alpha_n\epsilon^{2n}$ and $\beta(\epsilon)=1-\sum_{n=1}^{\infty}\beta_n\epsilon^{2n}$, as well as

$$\log(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n};\quad \log(1+z) = z-\frac{z^2}{2}+...;\quad \sqrt{1+z}=1+\frac{1}{2}z-\frac{1}{8}z^2+...,$$

for the first term we get (byA.2)

$$\log\sqrt{(\beta-z)^2+\epsilon^2S(z)} = \log(1-z)+\frac{1}{2}\left(\frac{S(z)}{(1-z)^2}-\frac{2\beta_1}{1-z}\right)\epsilon^2+O(\epsilon^4).$$

The second term can be written as $\log\epsilon\sqrt{S(z)}=\log\epsilon+\frac{1}{2}\log S(z)$.

Also (byA.3),

$$\log(\sqrt{(\beta-z)^2+\epsilon^2S(z)}+\beta-z) = \log 2(1-z)+\frac{\epsilon^2(S(z)-4\beta_1(1-z))}{4(1-z)^2}+O(\epsilon^4).$$

Finally, let $F_1=e^{ik\epsilon\sqrt{S(z)}}$, $F_2=\sqrt{(\beta-z)^2+\epsilon^2S(z)}$ and $F_3=e^{ik\sqrt{(\beta-z)^2+\epsilon^2S(z)}}$ and use Taylor expansion in powers of ϵ. This leads to:

$$F_1 = 1+\epsilon ik\sqrt{S(z)}-\epsilon^2\frac{k^2S(z)}{2}+O(\epsilon^3)$$
$$F_2 = (1-z)+\epsilon^2\left(\frac{S(z)}{2(1-z)}-\beta_1\right)+O(\epsilon^3)$$
$$F_3 = e^{ik(1-z)}+\epsilon^2 ik\left(\frac{S(z)}{2(1-z)}-\beta_1\right)e^{ik(1-z)}+O(\epsilon^3).$$

The second and the last term in the expansion of W_0^+ still remain to be evaluated. Starting with the second term, let

$$T(\epsilon)^+ = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \frac{e^{iku}-1}{u} du. \tag{B.38}$$

Expanding $T(\epsilon)^+$ in a Taylor series yields

$$T(\epsilon)^+ = T(0)^+ + \epsilon T^{+'}(0) + \frac{1}{2}\epsilon^2 T^{+''}(0) + O(\epsilon^3). \tag{B.39}$$

Now let

$$T(\epsilon)^+ = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \frac{e^{iku}-1}{u} du = \int_{0}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \frac{e^{iku}-1}{u} du$$
$$- \int_0^{\epsilon\sqrt{S(z)}} \frac{e^{iku}-1}{u} du = T_1(\epsilon)^+ - T_2(\epsilon)^+. \tag{B.40}$$

The leading term in the expansion of $T_1(\epsilon)^+$ is

$$T_1(0)^+ = \int_0^{1-z} \frac{e^{iku}-1}{u} du. \tag{B.41}$$

Next term is derived by employing Leibnitz's rule, i.e:

$$\begin{aligned} T_1^{+'}(\epsilon) &= \frac{d}{d\epsilon} \int_0^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \frac{e^{iku}-1}{u} du \\ &= \frac{e^{ik\sqrt{(\beta-z)^2+\epsilon^2 S(z)}}-1}{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \frac{d}{d\epsilon}\sqrt{(\beta-z)^2+\epsilon^2 S(z)} \\ &= \frac{e^{ik\sqrt{(\beta-z)^2+\epsilon^2 S(z)}}-1}{(\beta-z)^2+\epsilon^2 S(z)} \Big((\beta-z)\frac{d\beta}{d\epsilon} + \epsilon S(z)\Big), \end{aligned} \tag{B.42}$$

and since $\frac{d\beta}{d\epsilon}\mid_{\epsilon=0} = 0$,

$$T_1^{+'}(0) = 0. \tag{B.43}$$

Also

$$T_1^{+''}(\epsilon) = \frac{d}{d\epsilon}\left(\frac{e^{ik\sqrt{(\beta-z)^2+\epsilon^2 S(z)}}-1}{(\beta-z)^2+\epsilon^2 S(z)}((\beta-z)\frac{d\beta}{d\epsilon}+\epsilon S(z))\right)$$

$$= \left(\frac{d}{d\epsilon}\frac{e^{ik\sqrt{(\beta-z)^2+\epsilon^2 S(z)}}-1}{(\beta-z)^2+\epsilon^2 S(z)}\right)\Big((\beta-z)\frac{d\beta}{d\epsilon}+\epsilon S(z)\Big)$$

$$+\frac{e^{ik\sqrt{(\beta-z)^2+\epsilon^2 S(z)}}-1}{(\beta-z)^2+\epsilon^2 S(z)}\Big((\beta-z)\frac{d^2\beta}{d\epsilon^2}+\left(\frac{d\beta}{d\epsilon}\right)^2+S(z)\Big), \tag{B.44}$$

and thus

$$T^{+''}(0) = \frac{e^{ik(1-z)}-1}{(1-z)^2}(S(z)-2\beta_1(1-z)). \tag{B.45}$$

For $T_2(\epsilon)^+$ we have a different approach. We expand the exponential function in the integrand in Taylor series. This yields:

$$T_2(\epsilon)^+ = \int_0^{\epsilon\sqrt{S(z)}} \frac{e^{iku}-1}{u}du = \int_0^{\epsilon\sqrt{S(z)}} \sum_{n=1}^{\infty}\frac{(ik)^n}{n!}u^{n-1}du$$

$$= ik\epsilon\sqrt{S(z)} - \frac{k^2}{4}\epsilon^2 S(z) + O(\epsilon^3). \tag{B.46}$$

Therefore

$$T(\epsilon)^+ = \int_0^{1-z}\frac{e^{iku}-1}{u}du$$

$$+\epsilon^2\frac{e^{ik(1-z)}-1}{2(1-z)^2}(S(z)-2\beta_1(1-z)) - ik\epsilon\sqrt{S(z)}+\frac{k^2}{4}\epsilon^2 S(z)+O(\epsilon^3). \tag{B.47}$$

The last term in the expansion of W_0^+ is

$$R^+ = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} e^{iku}\left[\log(u+\sqrt{u^2-\epsilon^2 S(z)})-\log 2u\right]du, \tag{B.48}$$

which is the remainder resultant from the integration by parts. Using the fact that

$$\log(u+\sqrt{u^2-\epsilon^2 S(z)})-\log 2u = \frac{d}{du}\Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)})-\sqrt{u^2-\epsilon^2 S(z)}$$

$$+u-u\log 2u\Big], \tag{B.49}$$

and integrating R^+ by parts leads to

$$R^+ = e^{iku}\Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)} + u - u\log 2u\Big]\Big|_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}}$$
$$-\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)} + u - u\log 2u\Big] de^{iku}, \tag{B.50}$$

which after integration by parts becomes

$$R^+ = e^{ik\sqrt{(\beta-z)^2+\epsilon^2 S(z)}}[\sqrt{(\beta-z)^2+\epsilon^2 S(z)}\log(\sqrt{(\beta-z)^2+\epsilon^2 S(z)}+\beta-z)$$
$$-(\beta-z)+\sqrt{(\beta-z)^2+\epsilon^2 S(z)} - \sqrt{(\beta-z)^2+\epsilon^2 S(z)}\log 2\sqrt{(\beta-z)^2+\epsilon^2 S(z)}]$$
$$-e^{ik\epsilon\sqrt{S(z)}}[\epsilon\sqrt{S(z)}\log(\epsilon\sqrt{S(z)}) + \epsilon\sqrt{S(z)} - \epsilon\sqrt{S(z)}\log 2\epsilon\sqrt{S(z)}]$$
$$-\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)} + u - u\log 2u\Big] de^{iku}. \tag{B.51}$$

Evaluating R^+ we obtain (eq.A.27)

$$R^+ = \epsilon^2 e^{ik(1-z)}\left(-\frac{S(z)}{4(1-z)}+\beta_1\right) - \epsilon\sqrt{S(z)}(1-\log 2)$$
$$-\epsilon^2 ikS(z)(1-\log 2) - R_1^+ + HOT. \tag{B.52}$$

For the new remainder R_1^+ we have

$$R_1^+ = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)} + u$$

$$- u\log 2u\Big] de^{iku} = ik\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} e^{iku}\Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)})$$

$$- \sqrt{u^2-\epsilon^2 S(z)} + u - u\log 2u\Big] du = ik\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} u^n$$

$$\times \Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)} + u - u\log 2u\Big] du$$

$$=ik\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)} + u - u\log 2u\Big] du$$

$$+ik\sum_{n=1}^{\infty}\frac{(ik)^n}{n!}\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} u^n\Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)}$$

$$+ u - u\log 2u\Big] du = \Delta_1^+ + \Delta_2^+. \tag{B.53}$$

The expression for Δ_1^+ eq.A.28 yields

$$\Delta_1^+ = ik\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)}$$

$$+ u - u\log 2u\Big] du = \frac{ik}{4}\Big(- 2u^2\log 2u + 3u(u - \sqrt{u^2-\epsilon^2 S})$$

$$+ (2u^2+\epsilon^2 S(z))\log(u+\sqrt{u^2-\epsilon^2 S})\Big)\Big|_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}}$$

$$= \frac{ik}{4}\epsilon^2 S(z)\Big(- 2 - \log\epsilon\sqrt{S(z)} + \log 2 - \log(1-z)\Big), \tag{B.54}$$

and $\Delta_2^+ \sim O(\epsilon^3)$. To verify that, we use the substitution familiar from the static case, i.e. $v = \epsilon\sqrt{S(z)}u$. Then $dv = \epsilon\sqrt{S(z)}du$ and

The next term in the expansion of I_0^+ is

$$W_1^+ = \int_0^{\beta-z}\frac{F(z+v)-F(z)}{v}e^{ikv}dv. \tag{B.55}$$

By expanding $W_1^+(\epsilon)$ in a Taylor series we get

$$W_1^+(\epsilon) = W_1^+(0) + \epsilon^2 (W_1^+)'(0) + \frac{1}{2}\epsilon^4 (W_1^+)''(0) + O(\epsilon^6). \tag{B.56}$$

Clearly,

$$W_1^+(0) = \int_0^{1-z} \frac{F(z+v) - F(z)}{v} e^{ikv} dv, \tag{B.57}$$

and $W_1^{+'}$ can be evaluated by using Leibnitz' theorem:

$$W_1^{+'} = \frac{d}{d\epsilon^2} \int_0^{\beta - z} \frac{F(z+v) - F(z)}{v} e^{ikv} dv = -e^{ik(\beta - z)} \frac{F(\beta) - F(z)}{(\beta - z)} (\beta_1 + 2\beta_2 \epsilon^2 + ...), \tag{B.58}$$

and hence

$$W_1^{+'}(0) = -e^{ik(1-z)} \frac{F(1) - F(z)}{(1-z)} \beta_1. \tag{B.59}$$

Now consider

$$W_2^+ = \int_0^{\beta - z} \left\{ \frac{v e^{ik\sqrt{v^2 + \epsilon^2 S(z)}}}{\sqrt{v^2 + \epsilon^2 S(z)}} - e^{ikv} \right\} dv, \tag{B.60}$$

and let $u^2 = v^2 + \epsilon^2 S(z)$. Then $vdv = udu$ and

$$\begin{aligned} W_2^+ &= \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta - z)^2 + \epsilon^2 S(z)}} e^{iku} du - \int_0^{\beta - z} e^{ikv} dv \\ &= \frac{1}{ik} \left\{ e^{ik\sqrt{(\beta - z)^2 + \epsilon^2 S(z)}} + 1 - e^{ik\epsilon\sqrt{S(z)}} - e^{ik(\beta - z)} \right\}. \end{aligned} \tag{B.61}$$

Since $e^{ik(\beta - z)} = e^{ik(1-z)} - \epsilon^2 ik\beta_1 e^{ik(1-z)} + O(\epsilon^3)$, W_2^+ becomes

$$\begin{aligned} W_2^+ &= \frac{1}{ik} \left\{ e^{ik\sqrt{(\beta - z)^2 + \epsilon^2 S(z)}} + 1 - e^{ik\epsilon\sqrt{S(z)}} - e^{ik(\beta - z)} \right\} \\ &= \epsilon^2 \left(\frac{S(z)}{2(1-z)} \right) e^{ik(1-z)} - \epsilon\sqrt{S(z)} - \epsilon^2 \frac{ik}{2} S(z). \end{aligned} \tag{B.62}$$

Also, consider

$$
\begin{aligned}
W_3^+ &= \int_0^{\beta-z} \left\{ \frac{v e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} - e^{ikv} \right\} v dv \\
&= \int_0^{\beta-z} \frac{v^2 e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} dv - \int_0^{\beta-z} e^{ikv} v dv \\
&= \int_0^{\beta-z} \frac{v}{ik} d\big(e^{ik\sqrt{v^2+\epsilon^2 S(z)}}\big) - \int_0^{\beta-z} \frac{v}{ik} d\big(e^{ikv}\big) \\
&= \frac{1}{ik} v \left(e^{ik\sqrt{v^2+\epsilon^2 S(z)}} - e^{ikv} \right)\Big|_0^{\beta-z} \\
&- \frac{1}{ik} \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \left(e^{iku} - e^{ik\sqrt{u^2-\epsilon^2 S(z)}} \right) u \frac{du}{\sqrt{u^2-\epsilon^2 S(z)}}.
\end{aligned} \tag{B.63}
$$

Since

$$
\begin{aligned}
e^{ik\sqrt{v^2+\epsilon^2 S(z)}} &= e^{ikv} + \epsilon^2 S(z) ik \frac{e^{ikv}}{2v} + O(\epsilon^4), \\
e^{ik\sqrt{u^2-\epsilon^2 S(z)}} &= e^{iku} - \epsilon^2 S(z) ik \frac{e^{iku}}{2u} + O(\epsilon^4),
\end{aligned}
$$

$$
W_3^+ = \epsilon^2 \frac{S(z)}{2} e^{ikv}\Big|_0^{\beta-z} - \epsilon^2 \frac{S(z)}{2} \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} \frac{e^{iku}}{u} u \frac{du}{\sqrt{u^2-\epsilon^2 S(z)}}. \tag{B.64}
$$

But

$$
W_0^+ = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2 S(z)}} e^{iku} \frac{du}{\sqrt{u^2-\epsilon^2 S(z)}}, \tag{B.65}
$$

and thus

$$
\begin{aligned}
W_3^+ &= \epsilon^2 \frac{S(z)}{2} e^{ikv}\Big|_0^{\beta-z} - \epsilon^2 \frac{S(z)}{2} W_0^+ \\
&= \epsilon^2 \frac{S(z)}{2} (e^{ik(1-z)} - 1 - W_0^+).
\end{aligned} \tag{B.66}
$$

Finally, the expression for W_4^+ is

$$
W_4^+ = \int_0^{\beta-z} e^{ikv} \left(F(z+v) - \sum_{j=0}^{2} \frac{v^j F^{(j)}(z)}{j!} \right) \left(\frac{ik}{v^2} - \frac{1}{v^3} \right) dv. \tag{B.67}
$$

By combining all the results above, the expression for I_0^+ becomes

$$
\begin{aligned}
I_0^+ &= F(z)\Bigg\{ \log(1-z) + \int_0^{1-z} \frac{e^{iku}-1}{u} du - \log\epsilon - \frac{1}{2}\log S(z) \\
&+ \epsilon^2 e^{ik(1-z)}\Big(\frac{S(z)}{4(1-z)^2} - \frac{\beta_1}{(1-z)}\Big) - ik\epsilon\sqrt{S(z)} + \frac{k^2}{4}\epsilon^2 S(z) \\
&+ \Big(1 + \epsilon ik\sqrt{S(z)} - \epsilon^2\frac{k^2 S(z)}{2}\Big)\log 2 \\
&- ik\Big(\epsilon^2 e^{ik(1-z)}\Big(-\frac{S(z)}{4(1-z)} + \beta_1\Big) - \epsilon\sqrt{S(z)}(1-\log 2) - \epsilon^2 ikS(z)(1-\log 2)\Big) \\
&+ ik\Big(\frac{ik}{4}\epsilon^2 S(z)(-2 - \log\epsilon\sqrt{S(z)} + \log 2 - \log(1-z))\Big)\Bigg\} \\
&+ \int_0^{1-z} \frac{F(z+v)-F(z)}{v} e^{ikv} dv - e^{ik(1-z)}\frac{F(1)-F(z)}{(1-z)}\beta_1 \\
&+ F'(z)\Bigg\{\epsilon^2\frac{S(z)}{2(1-z)}e^{ik(1-z)} - \epsilon\sqrt{S(z)} - \epsilon^2\frac{ik}{2}S(z)\Bigg\} \\
&+ \frac{F''(z)}{2}\Bigg\{\epsilon^2\frac{S(z)}{2}(e^{ik(1-z)}-1) - \epsilon^2\frac{S(z)}{2}\big(\log 2(1-z) - \log\epsilon\sqrt{S(z)}\big)\Bigg\} \\
&+ \frac{\epsilon^2 S(z)}{2}\int_0^{1-z} e^{ikv}\left(F(z+v) - \sum_{j=0}^{2}\frac{v^j F^{(j)}(z)}{j!}\right)\left(\frac{ik}{v^2} - \frac{1}{v^3}\right) dv + O(\epsilon^3). \qquad \text{(B.68)}
\end{aligned}
$$

Similarly, for I_0^- we have

$$
\begin{aligned}
I_0^- &= \int_0^{z-\alpha} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} F(z-v) dv \\
&= F(z)\int_0^{z-\alpha} \frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} dv + \int_0^{z-\alpha} \frac{F(z-v)-F(z)}{v} e^{ikv} dv \\
&- F^{'}(z)\int_0^{z-\alpha}\left\{ v\frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} - e^{ikv}\right\} dv + \frac{F^{''}(z)}{2}\int_0^{z-\alpha}\Bigg\{ v^2\frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} \\
&- ve^{ikv}\Bigg\} dv + \frac{\epsilon^2 S(z)}{2}\int_0^{z-\alpha} e^{ikv}\Big(F(z-v) - \sum_{j=0}^{2}\frac{(-v)^j F^{(j)}(z)}{j!}\Big)\left(\frac{ik}{v^2} - \frac{1}{v^3}\right) dv + R^- \\
&= F(z)W_0^- + W_1^- - F^{'}(z)W_2^- + \frac{F^{''}(z)}{2}W_3^- + W_4^- + R^- \qquad \text{(B.69)}
\end{aligned}
$$

and

$$W_0^- = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \frac{e^{iku}}{u}\frac{u}{v}du$$
$$= \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} e^{iku}\frac{d}{du}\log(u+\sqrt{u^2-\epsilon^2 S(z)})du. \tag{B.70}$$

Using the same procedure as in the case of W_0^+, we obtain the result

$$W_0^- = \log\sqrt{(z-\alpha)^2+\epsilon^2 S(z)} - \log\epsilon\sqrt{S(z)} + \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \frac{e^{iku}-1}{u}du$$
$$+ e^{ik\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}}\left[\log(\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}+z-\alpha) - \log(2\sqrt{(z-\alpha)^2+\epsilon^2 S(z)})\right]$$
$$+ e^{ik\epsilon\sqrt{S(z)}}\log 2 - ik\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} e^{iku}\left[\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \log 2u\right]du. \tag{B.71}$$

By (A.5 and A.6)

$$\log\sqrt{(z-\alpha)^2+\epsilon^2 S(z)} = \log(z) + \frac{1}{2}\left(\frac{S(z)}{z^2} - \frac{2\alpha_1}{z}\right)\epsilon^2 + O(\epsilon^4),$$
$$\log(\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}+z-\alpha) = \log(z) + \log 2 + \frac{\epsilon^2(S(z)-4\alpha_1 z)}{4z^2} + O(\epsilon^4).$$

Let us define $F_4 = \sqrt{(z-\alpha)^2+\epsilon^2 S(z)}$ and $F_5 = e^{ik\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}}$ and use Taylor expansion in powers of ϵ. This leads to:

$$F_4 = z + \epsilon^2\left(\frac{S(z)}{2z} - \alpha_1\right) + O(\epsilon^3),$$
$$F_5 = e^{ikz} + \epsilon^2 ik\left(\frac{S(z)}{2z} - \alpha_1\right)e^{ikz} + O(\epsilon^3).$$

We want to expand the second term in W_0^-

$$T(\epsilon)^- = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \frac{e^{iku}-1}{u}du \tag{B.72}$$

in a Taylor series in powers of ϵ, i.e.

$$T(\epsilon)^{+} = T(0)^{-} + \epsilon T^{-'}(0) + \frac{1}{2}\epsilon^2 T^{-''}(0) + O(\epsilon^3). \tag{B.73}$$

Also, we split $T(\epsilon)^{-}$ in two;

$$T(\epsilon)^{-} = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \frac{e^{iku}-1}{u}du = \int_{0}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \frac{e^{iku}-1}{u}du$$
$$- \int_{0}^{\epsilon\sqrt{S(z)}} \frac{e^{iku}-1}{u}du = T_1(\epsilon)^{-} - T_2(\epsilon)^{-}. \tag{B.74}$$

The leading term in the expansion of $T_1(\epsilon)^{-}$ is

$$T_1(0)^{-} = \int_0^z \frac{e^{iku}-1}{u}du. \tag{B.75}$$

Next term is derived by employing Leibniz's rule, i.e:

$$\begin{aligned} T_1^{-'}(\epsilon) &= \frac{d}{d\epsilon}\int_{0}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \frac{e^{iku}-1}{u}du \\ &= \frac{e^{ik\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}}-1}{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}}\frac{d}{d\epsilon}\sqrt{(z-\alpha)^2+\epsilon^2 S(z)} \\ &= \frac{e^{ik\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}}-1}{(z-\alpha)^2+\epsilon^2 S(z)}\Big(-(z-\alpha)\frac{d\alpha}{d\epsilon}+\epsilon S(z)\Big), \end{aligned} \tag{B.76}$$

and since $\frac{d\alpha}{d\epsilon}\mid_{\epsilon=0}= 0$

$$T_1^{-'}(0) = 0. \tag{B.77}$$

Also

$$\begin{aligned} T_1^{-''}(\epsilon) &= \frac{d}{d\epsilon}\left(\frac{e^{ik\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}}-1}{(z-\alpha)^2+\epsilon^2 S(z)}(-(z-\alpha)\frac{d\alpha}{d\epsilon}+\epsilon S(z))\right) \\ &= \left(\frac{d}{d\epsilon}\frac{e^{ik\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}}-1}{(z-\alpha)^2+\epsilon^2 S(z)}\right)\Big(-(z-\alpha)\frac{d\alpha}{d\epsilon}+\epsilon S(z)\Big) \\ &+ \frac{e^{ik\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}}-1}{(z-\alpha)^2+\epsilon^2 S(z)}\Big(-(z-\alpha)\frac{d^2\alpha}{d\epsilon^2}+\left(\frac{d\alpha}{d\epsilon}\right)^2+S(z)\Big), \end{aligned} \tag{B.78}$$

and thus

$$T^{-''}(0) = \frac{e^{ikz}-1}{z^2}(S(z) - 2\alpha_1 z). \tag{B.79}$$

For $T_2(\epsilon)^-$ we have the same expression as for $T_2(\epsilon)^+$, i.e.:

$$\begin{aligned} T_2(\epsilon)^- &= \int_0^{\epsilon\sqrt{S(z)}} \frac{e^{iku}-1}{u}du = \int_0^{\epsilon\sqrt{S(z)}} \sum_{n=1}^{\infty} \frac{(ik)^n}{n!}u^{n-1}du \\ &= ik\epsilon\sqrt{S(z)} - \frac{k^2}{4}\epsilon^2 S(z) + O(\epsilon^3). \end{aligned} \tag{B.80}$$

Therefore

$$T(\epsilon)^- = \int_0^z \frac{e^{iku}-1}{u}du + \epsilon^2 \frac{e^{ikz}-1}{2z^2}(S(z) - 2\alpha_1 z) - ik\epsilon\sqrt{S(z)} + \frac{k^2}{4}\epsilon^2 S(z) + O(\epsilon^3). \tag{B.81}$$

Analogous to the W_0^+ case, we can introduce R^- to be the last term in the expansion of W_0^- and integrate it by parts to obtain the expression:

$$\begin{aligned} R^- &= \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} e^{iku}\left[\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \log 2u\right] du \\ &= e^{ik\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}}\Big[\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}\log(\sqrt{(z-\alpha)^2+\epsilon^2 S(z)} + z - \alpha) \\ &- (z-\alpha) + \sqrt{(z-\alpha)^2+\epsilon^2 S(z)} - \sqrt{(z-\alpha)^2+\epsilon^2 S(z)}\log 2\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}\Big] \\ &- e^{ik\epsilon\sqrt{S(z)}}\left[\epsilon\sqrt{S(z)}\log(\epsilon\sqrt{S(z)}) + \epsilon\sqrt{S(z)} - \epsilon\sqrt{S(z)}\log 2\epsilon\sqrt{S(z)}\right] \\ &- \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \left[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)} + u - u\log 2u\right] de^{iku}, \end{aligned} \tag{B.82}$$

which after evaluation (eq.A.29) becomes

$$R^- = \epsilon^2 e^{ikz}\left(-\frac{S(z)}{4z} + \alpha_1\right) - \epsilon\sqrt{S(z)}(1-\log 2) - \epsilon^2 ikS(z)(1-\log 2) - R_1^-. \tag{B.83}$$

The new remainder R_1^- becomes

$$R_1^- = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)} + u$$

$$- u\log 2u\Big] de^{iku} = ik\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} e^{iku}\Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)})$$

$$- \sqrt{u^2-\epsilon^2 S(z)} + u - u\log 2u\Big] du = ik\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} u^n$$

$$\times \Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)} + u - u\log 2u\Big] du$$

$$= ik\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)} + u - u\log 2u\Big] du$$

$$+ ik\sum_{n=1}^{\infty}\frac{(ik)^n}{n!}\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} u^n\Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)} + u$$

$$- u\log 2u\Big] du = \Delta_1^- + \Delta_2^-. \quad \text{(B.84)}$$

The expression for Δ_1^- after evaluation (eq.A.30)

$$\Delta_1^- = ik\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \Big[u\log(u+\sqrt{u^2-\epsilon^2 S(z)}) - \sqrt{u^2-\epsilon^2 S(z)}$$

$$+ u - u\log 2u\Big] du = \frac{ik}{4}\Big(- 2u^2\log 2u + 3u(u-\sqrt{u^2-\epsilon^2 S})$$

$$+ (2u^2+\epsilon^2 S(z))\log(u+\sqrt{u^2-\epsilon^2 S})\Big)\Big|_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}}$$

$$= \frac{ik}{4}\epsilon^2 S(z)\Big(- 2 - \log\epsilon\sqrt{S(z)} + \log 2 - \log z\Big) \quad \text{(B.85)}$$

and

$$\Delta_2^- \sim O(\epsilon^3). \quad \text{(B.86)}$$

Let

$$W_1^- = \int_0^{z-\alpha}\frac{F(z-v)-F(z)}{v}e^{ikv}dv. \quad \text{(B.87)}$$

Expanding $W_1^-(\epsilon)$ in a Taylor series yields

$$W_1^-(\epsilon) = W_1^-(0) + \epsilon^2 (W_1^-)'(0) + \frac{1}{2}\epsilon^4 (W_1^-)''(0) + O(\epsilon^6), \tag{B.88}$$

where

$$W_1^-(0) = \int_0^z \frac{F(z-v) - F(z)}{v} e^{ikv} dv \tag{B.89}$$

and

$$W_1^{-\prime}(0) = e^{ik(z)} \frac{F(0) - F(z)}{z} \alpha_1. \tag{B.90}$$

Also consider

$$W_2^- = \int_0^{z-\alpha} \left\{ \frac{v e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2 + \epsilon^2 S(z)}} - e^{ikv} \right\} dv. \tag{B.91}$$

After the substitution $u^2 = v^2 + \epsilon^2 S(z)$ and direct integration

$$W_2^- = \frac{1}{ik} \left\{ e^{ik\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} + 1 - e^{ik\epsilon\sqrt{S(z)}} - e^{ik(z-\alpha)} \right\}. \tag{B.92}$$

Since

$$e^{ik(z-\alpha)} = e^{ikz} - \epsilon^2 ik\alpha_1 e^{ikz} + O(\epsilon^3), \tag{B.93}$$

W_2^- becomes

$$\begin{aligned} W_2^- &= \frac{1}{ik} \left\{ e^{ik\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} + 1 - e^{ik\epsilon\sqrt{S(z)}} - e^{ik(z-\alpha)} \right\} \\ &= \epsilon^2 \left(\frac{S(z)}{2z} \right) e^{ikz} - \epsilon\sqrt{S(z)} - \epsilon^2 \frac{ik}{2} S(z). \end{aligned} \tag{B.94}$$

For W_3^- we have

$$
\begin{aligned}
W_3^- &= \int_0^{z-\alpha} \left\{ \frac{v e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} - e^{ikv} \right\} v dv \\
&= \int_0^{z-\alpha} \frac{v^2 e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} dv - \int_0^{z-\alpha} e^{ikv} v dv \\
&= \int_0^{z-\alpha} \frac{v}{ik} d\big(e^{ik\sqrt{v^2+\epsilon^2 S(z)}}\big) - \int_0^{z-\alpha} \frac{v}{ik} d\big(e^{ikv}\big) \\
&= \frac{1}{ik} v \Big(e^{ik\sqrt{v^2+\epsilon^2 S(z)}} - e^{ikv} \Big)\Big|_0^{z-\alpha} \\
&\quad - \frac{1}{ik} \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \Big(e^{iku} - e^{ik\sqrt{u^2-\epsilon^2 S(z)}} \Big) u \frac{du}{\sqrt{u^2-\epsilon^2 S(z)}}. \qquad \text{(B.95)}
\end{aligned}
$$

Recall that

$$
\begin{aligned}
e^{ik\sqrt{v^2+\epsilon^2 S(z)}} &= e^{ikv} + \epsilon^2 S(z) ik \frac{e^{ikv}}{2v} + O(\epsilon^4) \\
e^{ik\sqrt{u^2-\epsilon^2 S(z)}} &= e^{iku} - \epsilon^2 S(z) ik \frac{e^{iku}}{2u} + O(\epsilon^4).
\end{aligned}
$$

The expression for W_3^- becomes

$$
\begin{aligned}
W_3^- &= \epsilon^2 \frac{S(z)}{2} e^{ikv}\Big|_0^{z-\alpha} - \epsilon^2 \frac{S(z)}{2} \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \frac{e^{iku}}{u} u \frac{du}{\sqrt{u^2-\epsilon^2 S(z)}} \\
&= \epsilon^2 \frac{S(z)}{2} (e^{ikz} - 1) - \epsilon^2 \frac{S(z)}{2} \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} e^{iku} d \log(u + \sqrt{u^2 - \epsilon^2 S(z)}), \qquad \text{(B.96)}
\end{aligned}
$$

which combined with the fact that

$$
W_0^- = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} e^{iku} \frac{du}{\sqrt{u^2-\epsilon^2 S(z)}} \qquad \text{(B.97)}
$$

yields

$$
\begin{aligned}
W_3^- &= \epsilon^2 \frac{S(z)}{2} e^{ikv}\Big|_0^{z-\alpha} - \epsilon^2 \frac{S(z)}{2} W_0^- \\
&= \epsilon^2 \frac{S(z)}{2} (e^{ikz} - 1 - W_0^-). \qquad \text{(B.98)}
\end{aligned}
$$

Recall that W_4^- is given by

$$W_4^- = \int_0^{z-\alpha} e^{ikv} \left(F(z-v) - \sum_{j=0}^{2} \frac{(-v)^j F^{(j)}(z)}{j!} \right) \left(\frac{ik}{v^2} - \frac{1}{v^3} \right) dv \tag{B.99}$$

By combining the results from the expansions for W_0^-, W_1^-, W_2^-, W_3^- and W_4^-, the expression for I_0^- becomes

$$\begin{aligned}
I_0^- = F(z)\Bigg\{ & \log z + \int_0^z \frac{e^{iku}-1}{u} du - \log\epsilon - \frac{1}{2}\log S(z) \\
& + \epsilon^2 e^{ikz}\Big(\frac{S(z)}{4z^2} - \frac{\alpha_1}{z}\Big) - ik\epsilon\sqrt{S(z)} + \frac{k^2}{4}\epsilon^2 S(z) \\
& + \Big(1 + \epsilon ik\sqrt{S(z)} - \epsilon^2 \frac{k^2 S(z)}{2}\Big)\log 2 \\
& - ik\Big(\epsilon^2 e^{ikz}\Big(- \frac{S(z)}{4z} + \alpha_1\Big) - \epsilon\sqrt{S(z)}(1-\log 2) - \epsilon^2 ik S(z)(1-\log 2)\Big) \\
& + ik\Big(\frac{ik}{4}\epsilon^2 S(z)(-2 - \log\epsilon\sqrt{S(z)} + \log 2 - \log z)\Big)\Bigg\} \\
& + \int_0^z \frac{F(z-v)-F(z)}{v} e^{ikv} dv - e^{ikz}\frac{F(0)-F(z)}{z}\alpha_1 \\
& - F'(z)\Bigg\{\epsilon^2 \frac{S(z)}{2z} e^{ikz} - \epsilon\sqrt{S(z)} - \epsilon^2 \frac{ik}{2} S(z)\Bigg\} \\
& + \frac{F''(z)}{2}\Bigg\{\epsilon^2 \frac{S(z)}{2}(e^{ikz}-1) - \epsilon^2 \frac{S(z)}{2}\big(\log 2z - \log\epsilon\sqrt{S(z)}\big)\Bigg\} \\
& + \frac{\epsilon^2 S(z)}{2}\int_0^z e^{ikv}\left(F(z-v) - \sum_{j=0}^{2}\frac{(-v)^j F^{(j)}(z)}{j!}\right)\left(\frac{ik}{v^2} - \frac{1}{v^3}\right) dv. + O(\epsilon^3).
\end{aligned} \tag{B.100}$$

Finally, the $O(1)$ terms in I_0 are:

$$F(z)\Bigg\{\log(1-z)-\frac{1}{2}\log S(z)+\int_0^{1-z}\frac{e^{iku}-1}{u}du-\log 2$$
$$+\log(z)-\frac{1}{2}\log S(z)+\int_0^{z}\frac{e^{iku}-1}{u}du-\log 2\Bigg\}$$
$$+\int_0^{1-z}\frac{F(z+v)-F(z)}{v}e^{ikv}dv+\int_0^{z}\frac{F(z-v)-F(z)}{v}e^{ikv}dv$$
$$=F(z)\{\log(\frac{z(1-z)}{4S(z)})+\int_0^{1-z}\frac{e^{iku}-1}{u}du+\int_0^{z}\frac{e^{iku}-1}{u}du\}$$
$$+\int_0^{1-z}\frac{F(z+v)-F(z)}{v}e^{ikv}dv+\int_0^{z}\frac{F(z-v)-F(z)}{v}e^{ikv}dv. \tag{B.101}$$

Let

$$V(z)=\log\left(\frac{z(1-z)}{4S(z)}\right)+\int_0^{1-z}\frac{e^{iku}-1}{u}du+\int_0^{z}\frac{e^{iku}-1}{u}du+1. \tag{B.102}$$

Then

$$G_1F=(V(z)-1)F(z)+\int_0^{1-z}\frac{F(z+v)-F(z)}{v}e^{ikv}dv+\int_0^{z}\frac{F(z-v)-F(z)}{v}e^{ikv}dv \tag{B.103}$$

and

$$G_2F=\frac{S(z)}{4}(k^2F(z)+F''(z)) \tag{B.104}$$

are the operators in the expansion of

$$I_0(z,\epsilon,F)=-\log\epsilon^2F+G_1F+\epsilon^2\log\epsilon^2G_2F+O(\epsilon^2). \tag{B.105}$$

Expansion of $I_1(z,\epsilon,F)$:

Recall that

$$I_1(z,\epsilon,F)=\int_\alpha^\beta\frac{(\xi-z)e^{ik\sqrt{(z-\xi)^2+\epsilon^2S(z)}}}{\sqrt{(z-\xi)^2+\epsilon^2S(z)}}F(\xi)d\xi. \tag{B.106}$$

and

$$I_1 = I_1^- + I_1^+, \tag{B.107}$$

where

$$I_1^- = -\int_0^{z-\alpha} \frac{ve^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} F(z-v)dv. \tag{B.108}$$

and

$$I_1^+ = \int_0^{\beta-z} \frac{ve^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} F(z+v)dv. \tag{B.109}$$

Also, since

$$v\frac{e^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} = v\frac{e^{ikv}}{v} + \frac{\epsilon^2 S(z)v}{2}\left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3}\right) + O(\epsilon^4), \tag{B.110}$$

the expression for I_1^- becomes

$$\begin{aligned} I_1^- &= -\int_0^{z-\alpha} \frac{ve^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} F(z-v)dv = -\int_0^{z-\alpha} \frac{ve^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} (F(z) - vF'(z))dv \\ &- \int_0^{z-\alpha} \frac{ve^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} \Big(F(z-v) - \sum_{j=0}^{1} \frac{(-v)^j F^{(j)}(z)}{j!}\Big)dv \\ &= -\int_0^{z-\alpha} \frac{ve^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} (F(z) - vF'(z))dv \\ &- \int_0^{z-\alpha} \left(v\frac{e^{ikv}}{v} + \frac{\epsilon^2 S(z)v}{2}\left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3}\right) + O(\epsilon^4)\right)\Big(F(z-v) - \sum_{j=0}^{1} \frac{(-v)^j F^{(j)}(z)}{j!}\Big)dv. \end{aligned} \tag{B.111}$$

Recombining the terms in I_1^- leads to

$$I_1^- = -F(z)\int_0^{z-\alpha}\left\{\frac{ve^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}} - e^{ikv}\right\}dv + F'(z)\int_0^{z-\alpha}\left\{\frac{ve^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}}\right.$$
$$\left. - e^{ikv}\right\}vdv - \int_0^{z-\alpha} v\frac{e^{ikv}}{v}F(z-v)dv$$
$$-\frac{\epsilon^2 S(z)}{2}\int_0^{z-\alpha} v\left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3}\right)\Big(F(z-v) - \sum_{j=0}^{1}\frac{(-v)^j F^{(j)}(z)}{j!}\Big)dv + HOT. \tag{B.112}$$

Using a Taylor expansion and the results of Leibnitz's theorem

$$\int_0^{z-\alpha} e^{ikv}F(z-v)dv = \int_0^{z} e^{ik(z-\xi)}F(\xi)dv + \epsilon^2\Big(-e^{ik(z-\alpha)}F(\alpha)\frac{d\alpha}{d\epsilon^2}\Big)\Big|_{\epsilon=0}$$
$$= \int_0^{z} e^{ik(z-\xi)}F(\xi)dv - \epsilon^2 e^{ikz}F(0)\alpha_1, \tag{B.113}$$

and finally

$$I_1^- = -F(z)W_2^- + F'(z)W_3^- - \int_0^{z} e^{ik(z-\xi)}F(\xi)d\xi + \epsilon^2 e^{ikz}F(0)\alpha_1$$
$$-\frac{\epsilon^2 S(z)}{2}\int_0^{z-\alpha} v\left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3}\right)\Big(F(z-v) - \sum_{j=0}^{1}\frac{(-v)^j F^{(j)}(z)}{j!}\Big)dv + HOT. \tag{B.114}$$

Similarly, for I_1^+ we get

$$I_1^+ = \int_0^{\beta-z}\frac{ve^{ik\sqrt{v^2+\epsilon^2 S(z)}}}{\sqrt{v^2+\epsilon^2 S(z)}}(F(z)+vF'(z))dv$$
$$+\int_0^{\beta-z}\left(v\frac{e^{ikv}}{v} + \frac{\epsilon^2 S(z)v}{2}\left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3}\right) + O(\epsilon^4)\right)\Big(F(z+v) - \sum_{j=0}^{1}\frac{v^j F^{(j)}(z)}{j!}\Big)dv \tag{B.115}$$

Using a Taylor expansion and Leibnitz's theorem yields

$$\int_0^{\beta - z} e^{ikv} F(z+v)dv = \int_z^1 e^{ik(\xi - z)} F(\xi)dv + \epsilon^2 \Big(e^{ik(\beta - z)} F(\beta) \frac{d\beta}{d\epsilon^2} \Big) \Big|_{\epsilon = 0}$$
$$= \int_z^1 e^{ik(\xi - z)} F(\xi)dv - \epsilon^2 e^{ik(1-z)} F(1)\beta_1. \tag{B.116}$$

Therefore

$$I_1^+ = F(z)W_2^+ + F'(z)W_3^+ + \int_z^1 e^{ik(\xi - z)} F(\xi)d\xi - \epsilon^2 e^{ik(1-z)} F(1)\beta_1$$
$$+ \frac{\epsilon^2 S(z)}{2} \int_0^{\beta - z} v \left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3} \right) \Big(F(z+v) - \sum_{j=0}^{1} \frac{v^j F^{(j)}(z)}{j!} \Big) dv + HOT. \tag{B.117}$$

After summing I_1^- and I_1^+ for I_1 we obtain:

$$I_1 = F(z)(W_2^+ - W_2^-) + F'(z)(W_3^+ + W_3^-) - \int_0^z e^{ik(z-\xi)} F(\xi)d\xi$$
$$+ \int_z^1 e^{ik(\xi - z)} F(\xi)d\xi + \epsilon^2 e^{ikz} F(0)\alpha_1 - \epsilon^2 e^{ik(1-z)} F(1)\beta_1$$
$$+ \frac{\epsilon^2 S(z)}{2} \Bigg(\int_0^{1-z} v \left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3} \right) \Big(F(z+v) - \sum_{j=0}^{1} \frac{v^j F^{(j)}(z)}{j!} \Big) dv$$
$$- \int_0^z v \left(\frac{ike^{ikv}}{v^2} - \frac{e^{ikv}}{v^3} \right) \Big(F(z-v) - \sum_{j=0}^{1} \frac{(-v)^j F^{(j)}(z)}{j!} \Big) dv \Bigg) \tag{B.118}$$

Also

$$W_2^+ - W_2^- = \epsilon^2 \frac{S(z)}{2} \Big(\frac{e^{ik(1-z)}}{1-z} - \frac{e^{ikz}}{z} \Big) + HOT. \tag{B.119}$$

and

$$W_3^+ + W_3^- = \epsilon^2 \frac{S(z)}{2} \Big(e^{ik(1-z)} + e^{ikz} - 2 - \log \frac{4z(1-z)}{S(z)} + \log \epsilon^2$$
$$- \int_0^{1-z} \frac{e^{iku} - 1}{u} du - \int_0^z \frac{e^{iku} - 1}{u} du \Big) + HOT$$
$$= \epsilon^2 \frac{S(z)}{2} \Big(e^{ik(1-z)} + e^{ikz} - V(z) + \log \epsilon^2 \Big) + HOT. \tag{B.120}$$

Expansion of $J(z, \epsilon, F)$:

Recall that

$$J(z,\epsilon,F) = \int_{\alpha}^{\beta} e^{ik\sqrt{(z-\xi)^2+\epsilon^2 S(z)}} F(\xi)d\xi. \tag{B.121}$$

and

$$J = J^- + J^+, \tag{B.122}$$

where

$$J^- = \int_0^{z-\alpha} e^{ik\sqrt{v^2+\epsilon^2 S(z)}} F(z-v)dv. \tag{B.123}$$

and

$$J^+ = \int_0^{\beta-z} e^{ik\sqrt{v^2+\epsilon^2 S(z)}} F(z+v)dv. \tag{B.124}$$

Also, since

$$e^{ik\sqrt{v^2+\epsilon^2 S(z)}} = e^{ikv} + \epsilon^2 S(z)\left(\frac{ike^{ikv}}{2v}\right) + O(\epsilon^4), \tag{B.125}$$

the expression for J^- becomes

$$\begin{aligned} J^- &= F(z)\int_0^{z-\alpha} \left(e^{ik\sqrt{v^2+\epsilon^2 S(z)}} - e^{ikv}\right)dv \\ &+ \int_0^{z-\alpha} e^{ikv}F(z-v)dv + \epsilon^2\frac{ikS(z)}{2}\int_0^{z-\alpha}\frac{e^{ikv}}{v}\Big(F(z-v)-F(z)\Big)dv. \end{aligned} \tag{B.126}$$

The first in J^- can be rewritten as

$$\int_0^{z-\alpha}\left(e^{ik\sqrt{v^2+\epsilon^2 S(z)}} - e^{ikv}\right)dv = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \frac{e^{iku} - e^{ik\sqrt{u^2-\epsilon^2 S(z)}}}{\sqrt{u^2-\epsilon^2 S(z)}} udu. \tag{B.127}$$

But $e^{iku} - e^{ik\sqrt{u^2-\epsilon^2 S(z)}} = \epsilon^2 \frac{S(z)ik}{2}\frac{e^{iku}}{u} + O(\epsilon^4)$ and therefore

$$\begin{aligned} \int_0^{z-\alpha}\left(e^{ik\sqrt{v^2+\epsilon^2 S(z)}} - e^{ikv}\right)dv &= \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \frac{e^{iku} - e^{ik\sqrt{u^2-\epsilon^2 S(z)}}}{\sqrt{u^2-\epsilon^2 S(z)}} udu \\ &= \epsilon^2\frac{S(z)ik}{2}\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(z-\alpha)^2+\epsilon^2 S(z)}} \frac{e^{iku}}{\sqrt{u^2-\epsilon^2 S(z)}} du = \epsilon^2\frac{S(z)ik}{2}W_0^-. \end{aligned} \tag{B.128}$$

Also

$$\int_0^{z-\alpha} e^{ikv}F(z-v)dv = \int_0^z e^{ik(z-\xi)}F(\xi)dv + \epsilon^2\Big(-e^{ik(z-\alpha)}F(\alpha)\frac{d\alpha}{d\epsilon^2}\Big)\Big|_{\epsilon=0}$$
$$= \int_0^z e^{ik(z-\xi)}F(\xi)dv - \epsilon^2 e^{ikz)}F(0)\alpha_1. \tag{B.129}$$

Similarly, for J^+ we get

$$J^+ = F(z)\int_0^{\beta-z}\Big(e^{ik\sqrt{v^2+\epsilon^2S(z)}} - e^{ikv}\Big)dv$$
$$+\int_0^{\beta-z} e^{ikv}F(z+v)dv + \epsilon^2\frac{ikS(z)}{2}\int_0^{\beta-z}\frac{e^{ikv}}{v}\Big(F(z+v)-F(z)\Big)dv, \tag{B.130}$$

where

$$\int_0^{\beta-z}\Big(e^{ik\sqrt{v^2+\epsilon^2S(z)}} - e^{ikv}\Big)dv = \int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2S(z)}}\frac{e^{iku}-e^{ik\sqrt{u^2-\epsilon^2S(z)}}}{\sqrt{u^2-\epsilon^2S(z)}}udu$$
$$= \epsilon^2\frac{S(z)ik}{2}\int_{\epsilon\sqrt{S(z)}}^{\sqrt{(\beta-z)^2+\epsilon^2S(z)}}\frac{e^{iku}}{\sqrt{u^2-\epsilon^2S(z)}}du = \epsilon^2\frac{S(z)ik}{2}W_0^+. \tag{B.131}$$

By using a Taylor expansion and Leibnitz's theorem

$$\int_0^{\beta-z} e^{ikv}F(z+v)dv = \int_z^1 e^{ik(\xi-z)}F(\xi)dv + \epsilon^2\Big(e^{ik(\beta-z)}F(\beta)\frac{d\beta}{d\epsilon^2}\Big)\Big|_{\epsilon=0}$$
$$= \int_z^1 e^{ik(\xi-z)}F(\xi)dv - \epsilon^2 e^{ik(1-z)}F(1)\beta_1, \tag{B.132}$$

and thus

$$J = \epsilon^2\frac{S(z)ik}{2}(W_0^+ + W_0^-) + \int_z^1 e^{ik(\xi-z)}F(\xi)d\xi + \int_0^z e^{ik(z-\xi)}F(\xi)d\xi$$
$$-\epsilon^2 e^{ik(1-z)}F(1)\beta_1 - \epsilon^2 e^{ikz)}F(0)\alpha_1 + \epsilon^2\frac{ikS(z)}{2}\Bigg(\int_0^{1-z}\frac{e^{ikv}}{v}\Big(F(z+v)-F(z)\Big)dv$$
$$+\int_0^z\frac{e^{ikv}}{v}\Big(F(z-v)-F(z)\Big)dv\Bigg) + HOT. \tag{B.133}$$

Using the expressions for W_0^+ and W_0^-, we obtain

$$J = \epsilon^2 \frac{S(z)ik}{2}\left(\log\frac{4z(1-z)}{S(z)} - \log\epsilon^2 + \int_0^{1-z}\frac{e^{iku}-1}{u}du + \int_0^{z}\frac{e^{iku}-1}{u}du\right)$$
$$+ \int_z^1 e^{ik(\xi-z)}F(\xi)d\xi + \int_0^z e^{ik(z-\xi)}F(\xi)d\xi - \epsilon^2 e^{ik(1-z)}F(1)\beta_1 - \epsilon^2 e^{ikz)}F(0)\alpha_1$$
$$+ \epsilon^2\frac{ikS(z)}{2}\left(\int_0^{1-z}\frac{e^{ikv}}{v}\Big(F(z+v)-F(z)\Big)dv + \int_0^{z}\frac{e^{ikv}}{v}\Big(F(z-v)-F(z)\Big)dv\right) + HOT. \tag{B.134}$$

Expansion of $I_2(z,\epsilon,F)$:

Recall that

$$I_2 = \int_\alpha^\beta e^{ikR}\left(\frac{1}{R^2}+\frac{i}{kR^3}\right)F(\xi,\epsilon)d\xi = -\frac{i}{k}\int_\alpha^\beta e^{ikR}\left(\frac{ik}{R^2}-\frac{1}{R^3}\right)F(\xi,\epsilon)d\xi. \tag{B.135}$$

The kernel of the integral above is a Helmholtz kernel as well as the kernels of the integrals

$$I^n = \int_\alpha^\beta e^{ikR}\sum_{j=1}^{n}\left(\frac{A_j}{R^{2j+1}}+i\frac{B_j}{R^{2j}}\right)F(\xi,\epsilon)d\xi. \tag{B.136}$$

provided A_j and B_j meet the following criteria:

$$B_1 = -kA_1 \qquad \text{for} \qquad n=1$$
$$A_j = B_{j+1}\frac{k(1-2j+n)}{(j-n)(1+2j)} \qquad \text{for} \qquad n>1$$

To see this,suppose

$$\psi = r^n e^{\pm in\theta}e^{ikR}\sum_{j=1}^{n}\left(\frac{A_j}{R^{2j+1}}+i\frac{B_j}{R^{2j}}\right) \tag{B.137}$$

where $R=[z^2+r^2]^{1/2}$.

We need to prove that ψ satisfies

$$\nabla^2\psi + k^2\psi = 0 \tag{B.138}$$

i.e.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0 \tag{B.139}$$

or

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0 \tag{B.140}$$

Suppose $\psi=\psi_1\psi_2$. Then

$$\frac{\partial\psi}{\partial r}=\frac{\partial\psi_1}{\partial r}\psi_2+\frac{\partial\psi_2}{\partial r}\psi_1 \tag{B.141}$$

$$\frac{\partial\psi}{\partial z}=\frac{\partial\psi_1}{\partial z}\psi_2+\frac{\partial\psi_2}{\partial z}\psi_1 \tag{B.142}$$

$$\frac{\partial^2\psi}{\partial r^2}=\frac{\partial^2\psi_1}{\partial r^2}\psi_2+2\frac{\partial\psi_1}{\partial r}\frac{\partial\psi_2}{\partial r}+\frac{\partial^2\psi_2}{\partial r}\psi_1 \tag{B.143}$$

$$\frac{\partial^2\psi}{\partial z^2}=\frac{\partial^2\psi_1}{\partial z^2}\psi_2+2\frac{\partial\psi_1}{\partial z}\frac{\partial\psi_2}{\partial z}+\frac{\partial^2\psi_2}{\partial z}\psi_1 \tag{B.144}$$

and

$$\frac{\partial^2\psi}{\partial\theta^2}=-n^2\psi_1\psi_2 \tag{B.145}$$

Also, let

$$\psi_1=r^n e^{\pm in\theta} \tag{B.146}$$

Then

$$\frac{\partial\psi_1}{\partial r}=\frac{n}{r}\psi_1 \tag{B.147}$$

Plugging this into

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0 \tag{B.148}$$

leads to

$$\Big(\frac{\partial^2\psi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_1}{\partial r}+\frac{\partial^2\psi_1}{\partial z^2}\Big)\psi_2+\Big(\frac{\partial^2\psi_2}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_2}{\partial r}+\frac{\partial^2\psi_2}{\partial z^2}\Big)\psi_1$$
$$+\Big(\frac{\partial\psi_1}{\partial r}\frac{\partial\psi_2}{\partial r}+\frac{\partial\psi_1}{\partial z}\frac{\partial\psi_2}{\partial z}\Big)+\Big(k^2-\frac{n^2}{r^2}\Big)\psi_1\psi_2=0 \tag{B.149}$$

and since

$$\frac{\partial^2\psi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_1}{\partial r}+\frac{\partial^2\psi_1}{\partial z^2}-\frac{n^2}{r^2}\psi_1=0 \tag{B.150}$$

$$\frac{\partial^2\psi_2}{\partial r^2}+\frac{2n+1}{r}\frac{\partial\psi_2}{\partial r}+\frac{\partial^2\psi_2}{\partial z^2}+k^2\psi_2=0 \tag{B.151}$$

Since

$$\psi_2=e^{ikR}\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+1}}+i\frac{B_j}{R^{2j}}\Big) \tag{B.152}$$

$$\frac{\partial\psi_2}{\partial r}=ik\frac{r}{R}e^{ikR}\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+1}}+i\frac{B_j}{R^{2j}}\Big)+\frac{r}{R}e^{ikR}\sum_{j=1}^{n}\Big(\frac{A_j(-2j-1)}{R^{2j+2}}+i\frac{B_j(-2j)}{R^{2j+1}}\Big)$$
$$=(re^{ikR})\Big(ik\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+2}}+i\frac{B_j}{R^{2j+1}}\Big)-\sum_{j=1}^{n}\Big(\frac{A_j(2j+1)}{R^{2j+3}}+i\frac{B_j(2j)}{R^{2j+2}}\Big)\Big) \tag{B.153}$$

$$\frac{\partial^2\psi_2}{\partial r^2}=\frac{\partial}{\partial r}(re^{ikR})\Big(ik\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+2}}+i\frac{B_j}{R^{2j+1}}\Big)-\sum_{j=1}^{n}\Big(\frac{A_j(2j+1)}{R^{2j+3}}+i\frac{B_j(2j)}{R^{2j+2}}\Big)\Big)$$
$$+(re^{ikR})\frac{\partial}{\partial r}\Big(ik\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+2}}+i\frac{B_j}{R^{2j+1}}\Big)-\sum_{j=1}^{n}\Big(\frac{A_j(2j+1)}{R^{2j+3}}+i\frac{B_j(2j)}{R^{2j+2}}\Big)\Big)$$
$$=(e^{ikR}+ikr^2\frac{e^{ikR}}{R})\Big(ik\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+2}}+i\frac{B_j}{R^{2j+1}}\Big)-\sum_{j=1}^{n}\Big(\frac{A_j(2j+1)}{R^{2j+3}}+i\frac{B_j(2j)}{R^{2j+2}}\Big)\Big)$$
$$+r^2\frac{e^{ikR}}{R}\Big(ik\sum_{j=1}^{n}\Big(\frac{A_j(-2j-2)}{R^{2j+3}}+i\frac{B_j(-2j-1)}{R^{2j+2}}\Big)-\sum_{j=1}^{n}\Big(\frac{A_j(-2j-3)(2j+1)}{R^{2j+4}}$$
$$+i\frac{B_j(-2j-2)(2j)}{R^{2j+3}}\Big)\Big) \tag{B.154}$$

and

$$\frac{\partial^2\psi_2}{\partial z^2} = (e^{ikR} + ikz^2\frac{e^{ikR}}{R})\Big(ik\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+2}} + i\frac{B_j}{R^{2j+1}}\Big) - \sum_{j=1}^{n}\Big(\frac{A_j(2j+1)}{R^{2j+3}} + i\frac{B_j(2j)}{R^{2j+2}}\Big)\Big)$$

$$+ z^2\frac{e^{ikR}}{R}\Big(ik\sum_{j=1}^{n}\Big(\frac{A_j(-2j-2)}{R^{2j+3}} + i\frac{B_j(-2j-1)}{R^{2j+2}}\Big) - \sum_{j=1}^{n}\Big(\frac{A_j(-2j-3)(2j+1)}{R^{2j+4}}$$

$$+ i\frac{B_j(-2j-2)(2j)}{R^{2j+3}}\Big)\Big) \tag{B.155}$$

Thus

$$e^{ikR}(2 + ik\frac{r^2+z^2}{R})\Bigg(ik\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+2}} + i\frac{B_j}{R^{2j+1}}\Big) - \sum_{j=1}^{n}\Big(\frac{A_j(2j+1)}{R^{2j+3}} + i\frac{B_j(2j)}{R^{2j+2}}\Big)\Bigg)$$

$$+ e^{ikR}\frac{r^2+z^2}{R}\Bigg(ik\sum_{j=1}^{n}\Big(\frac{A_j(-2j-2)}{R^{2j+3}} + i\frac{B_j(-2j-1)}{R^{2j+2}}\Big) - \sum_{j=1}^{n}\Big(\frac{A_j(-2j-3)(2j+1)}{R^{2j+4}}$$

$$+ i\frac{B_j(-2j-2)(2j)}{R^{2j+3}}\Big)\Bigg) + (2n+1)e^{ikR}\Bigg(ik\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+2}} + i\frac{B_j}{R^{2j+1}}\Big)$$

$$- \sum_{j=1}^{n}\Big(\frac{A_j(2j+1)}{R^{2j+3}} + i\frac{B_j(2j)}{R^{2j+2}}\Big)\Bigg) + k^2e^{ikR}\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+1}} + i\frac{B_j}{R^{2j}}\Big) = 0 \tag{B.156}$$

i.e.

$$(2+ikR)\Bigg(ik\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+2}} + i\frac{B_j}{R^{2j+1}}\Big) - \sum_{j=1}^{n}\Big(\frac{A_j(2j+1)}{R^{2j+3}} + i\frac{B_j(2j)}{R^{2j+2}}\Big)\Bigg)$$

$$+ R\Bigg(ik\sum_{j=1}^{n}\Big(\frac{A_j(-2j-2)}{R^{2j+3}} + i\frac{B_j(-2j-1)}{R^{2j+2}}\Big) - \sum_{j=1}^{n}\Big(\frac{A_j(-2j-3)(2j+1)}{R^{2j+4}}$$

$$+ i\frac{B_j(-2j-2)(2j)}{R^{2j+3}}\Big)\Bigg) + (2n+1)\Bigg(ik\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+2}} + i\frac{B_j}{R^{2j+1}}\Big)$$

$$- \sum_{j=1}^{n}\Big(\frac{A_j(2j+1)}{R^{2j+3}} + i\frac{B_j(2j)}{R^{2j+2}}\Big)\Bigg) + k^2\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+1}} + i\frac{B_j}{R^{2j}}\Big) = 0 \tag{B.157}$$

and finally

$$
\begin{aligned}
&(2+ikR)\Bigg(ik\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+2}}+i\frac{B_j}{R^{2j+1}}\Big)-\sum_{j=1}^{n}\Big(\frac{A_j(2j+1)}{R^{2j+3}}+i\frac{B_j(2j)}{R^{2j+2}}\Big)\Bigg)\\
&-R\Bigg(ik\sum_{j=1}^{n}\Big(\frac{A_j(2j+2)}{R^{2j+3}}+i\frac{B_j(2j+1)}{R^{2j+2}}\Big)-\sum_{j=1}^{n}\Big(\frac{A_j(2j+3)(2j+1)}{R^{2j+4}}\\
&+i\frac{B_j(2j+2)(2j)}{R^{2j+3}}\Big)\Bigg)+(2n+1)\Bigg(ik\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+2}}+i\frac{B_j}{R^{2j+1}}\Big)\\
&-\sum_{j=1}^{n}\Big(\frac{A_j(2j+1)}{R^{2j+3}}+i\frac{B_j(2j)}{R^{2j+2}}\Big)\Bigg)+k^2\sum_{j=1}^{n}\Big(\frac{A_j}{R^{2j+1}}+i\frac{B_j}{R^{2j}}\Big)=0 \qquad \text{(B.158)}
\end{aligned}
$$

Consider first the case of $n = 1$, i.e.

$$
\begin{aligned}
&(2+ikR)\Bigg(ik\Big(\frac{A_1}{R^4}+i\frac{B_1}{R^3}\Big)-\Big(\frac{3A_1}{R^5}+i\frac{2B_1}{R^4}\Big)\Bigg)\\
&-R\Bigg(ik\Big(\frac{4A_1}{R^5}+i\frac{3B_1}{R^4}\Big)-\Big(\frac{15A_1}{R^6}++i\frac{8B_1}{R^5}\Big)\Bigg)\\
&+3\Bigg(ik\Big(\frac{A_1}{R^4}+i\frac{B_1}{R^3}\Big)-\Big(\frac{3A_1}{R^5}+i\frac{2B_1}{R^4}\Big)\Bigg)+k^2\Big(\frac{A_1}{R^3}+i\frac{B_1}{R^2}\Big)=0 \qquad \text{(B.159)}
\end{aligned}
$$

leading to

$$B_1 = -kA_1 \qquad \text{(B.160)}$$

where A_1 can be arbitrary. For $n > 1$, we use

$$\begin{aligned}
&\left(2ik\sum_{j=1}^{n}\left(\frac{A_j}{R^{2j+2}}+i\frac{B_j}{R^{2j+1}}\right)-2\sum_{j=1}^{n}\left(\frac{A_j(2j+1)}{R^{2j+3}}+i\frac{B_j(2j)}{R^{2j+2}}\right)\right)\\
&-\left(k^2\sum_{j=1}^{n}\left(\frac{A_j}{R^{2j+1}}+i\frac{B_j}{R^{2j}}\right)+ik\sum_{j=1}^{n}\left(\frac{A_j(2j+1)}{R^{2j+2}}+i\frac{B_j(2j)}{R^{2j+1}}\right)\right)\\
&-\left(ik\sum_{j=1}^{n}\left(\frac{A_j(2j+2)}{R^{2j+2}}+i\frac{B_j(2j+1)}{R^{2j+1}}\right)-\sum_{j=1}^{n}\left(\frac{A_j(2j+3)(2j+1)}{R^{2j+3}}\right.\right.\\
&\left.\left.+i\frac{B_j(2j+2)(2j)}{R^{2j+2}}\right)\right)+(2n+1)\left(ik\sum_{j=1}^{n}\left(\frac{A_j}{R^{2j+2}}+i\frac{B_j}{R^{2j+1}}\right)\right.\\
&\left.-\sum_{j=1}^{n}\left(\frac{A_j(2j+1)}{R^{2j+3}}+i\frac{B_j(2j)}{R^{2j+2}}\right)\right)+k^2\sum_{j=1}^{n}\left(\frac{A_j}{R^{2j+1}}+i\frac{B_j}{R^{2j}}\right)=0 \qquad \text{(B.161)}
\end{aligned}$$

and collecting coefficients in front of $\frac{1}{R^{2j+2}}$ yields:

$$\begin{aligned}
&2ikA_j-4ijB_j-ik^2B_{j+1}-ik(2j+1)A_j-ik(2j+2)A_j+2ij(2j+2)B_j\\
&+(2n+1)(ikA_j-2ijB_j)+ik^2B_{j+1}=0 \qquad \text{(B.162)}
\end{aligned}$$

i.e.

$$2ikA_j(n-2j)=2ijB_j(2n-2j+1) \qquad \text{(B.163)}$$

and

$$B_j=A_j\frac{k(n-2j)}{j(2n-2j+1)} \qquad \text{(B.164)}$$

Collecting coefficients in front of $\frac{1}{R^{2j+3}}$ yields:

$$\begin{aligned}
&-2kB_{j+1}-2(2j+1)A_j-k^2A_{j+1}+2jkB_{j+1}+k(2j+1)B_{j+1}+(2j+1)(2j+3)A_j\\
&+(2n+1)(-kB_{j+1}-(2j+1)A_j)+k^2A_{j+1}=0 \qquad \text{(B.165)}
\end{aligned}$$

leading to

$$A_j(j-n)(1+2j)=B_{j+1}k(1-2j+n) \qquad \text{(B.166)}$$

and

$$A_j = B_{j+1} \frac{k(1-2j+n)}{(j-n)(1+2j)} \tag{B.167}$$

For $n = 1$ and $A_1 = 1/2$, $B_1 = -k/2$ and

$$\begin{aligned}
I^1 &= \int_\alpha^\beta \Big(\frac{A_1}{R^3} + i\frac{B_1}{R^2}\Big) e^{ikR}(\beta-\xi)(\xi-\alpha)F(\xi)d\xi \\
&= \frac{1}{2}\int_\alpha^\beta \Big(\frac{1}{R^3} - \frac{ik}{R^2}\Big) e^{ikR}(\beta-\xi)(\xi-\alpha)F(\xi)d\xi \\
&= \frac{1}{2\epsilon^2 S(z)} \int_\alpha^\beta \left(\frac{\partial}{\partial\xi}\left((\xi-z)\frac{e^{ikR}}{R}\right) - ike^{ikR}\right)(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi \\
&= -\frac{1}{2\epsilon^2 S(z)}\left(\int_\alpha^\beta (\xi-z)\frac{e^{ikR}}{R}\frac{\partial}{\partial\xi}((\beta-\xi)(\xi-\alpha)\tilde{F}(\xi))d\xi\right. \\
&\left. + ik\int_\alpha^\beta e^{ikR}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi\right) = -\frac{1}{2\epsilon^2 S(z)}\Big(I_1\Big(z, \frac{\partial F}{\partial\xi}\Big) + ikJ_1(z,F)\Big).
\end{aligned} \tag{B.168}$$

On the other hand,

$$I_2 = \int_\alpha^\beta e^{ikR}\left(\frac{1}{R^2} + \frac{i}{kR^3}\right)F(\xi,\epsilon)d\xi = -\frac{i}{k}\int_\alpha^\beta e^{ikR}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right)F(\xi,\epsilon)d\xi, \tag{B.169}$$

and thus

$$I_2 = \frac{2i}{k} I^1. \tag{B.170}$$

Let $F(\xi) = (\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)$. Using the fact that

$$e^{ikR}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right) = \frac{1}{\epsilon^2 S(z)}\left(ike^{ikR} - \frac{\partial}{\partial\xi}\left((\xi-z)\frac{e^{ikR}}{R},\right)\right) \tag{B.171}$$

the expression for I_2 becomes

$$
\begin{aligned}
I_2 &= -\frac{i}{k}\int_\alpha^\beta e^{ikR}\left(\frac{ik}{R^2}-\frac{1}{R^3}\right)F(\xi,\epsilon)d\xi \\
&= \frac{i}{k\epsilon^2 S(z)}\int_\alpha^\beta \left(\frac{\partial}{\partial\xi}\left((\xi-z)\frac{e^{ikR}}{R}\right)-ike^{ikR}\right)(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi \\
&= -\frac{i}{k\epsilon^2 S(z)}\Bigg(\int_\alpha^\beta (\xi-z)\frac{e^{ikR}}{R}\frac{\partial}{\partial\xi}((\beta-\xi)(\xi-\alpha)\tilde{F}(\xi))d\xi \\
&+ ik\int_\alpha^\beta e^{ikR}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi\Bigg) = -\frac{i}{k\epsilon^2 S(z)}\Big(I_1\Big(z,\frac{\partial F}{\partial\xi}\Big)+ikJ_1(z,F)\Big).
\end{aligned}
\tag{B.172}
$$

However

$$\frac{\partial}{\partial\xi}((\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)) = -(\xi-\alpha)\tilde{F}(\xi)+(\beta-\xi)\tilde{F}(\xi)+(\beta-\xi)(\xi-\alpha)\tilde{F}'(\xi). \tag{B.173}$$

Also

$$I_1(z,\epsilon,\tilde{F}) = -\int_0^z e^{ik(z-\xi)}F(\xi)d\xi + \int_z^1 e^{ik(\xi-z)}F(\xi)d\xi + \epsilon^2\log\epsilon^2 L_1 + \epsilon^2 L_2 + HOT. \tag{B.174}$$

But

$$
\begin{aligned}
&\int_0^z e^{ik(z-\xi)}\frac{\partial}{\partial\xi}((\beta-\xi)(\xi-\alpha)\tilde{F}(\xi))d\xi = e^{ik(z-\xi)}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)\Big|_0^z \\
&+ ik\int_0^z e^{ik(z-\xi)}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi = (\beta-z)(z-\alpha)\tilde{F}(z)+e^{ikz}\alpha\beta\tilde{F}(0) \\
&+ ik\int_0^z e^{ik(z-\xi)}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi
\end{aligned}
\tag{B.175}
$$

and

$$\int_z^1 e^{ik(\xi-z)}\frac{\partial}{\partial\xi}((\beta-\xi)(\xi-\alpha)\tilde{F}(\xi))d\xi = e^{ik(\xi-z)}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)\Big|_z^1$$

$$-ik\int_z^1 e^{ik(\xi-z)}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi = e^{ik(1-z)}(\beta-1)(1-\alpha)\tilde{F}(1)$$

$$-(\beta-z)(z-\alpha)\tilde{F}(z) - ik\int_z^1 e^{ik(\xi-z)}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi. \tag{B.176}$$

Thus

$$I_1\Big(z,\epsilon,\frac{\partial F}{\partial\xi}\Big) = e^{ik(1-z)}(\beta-1)(1-\alpha)\tilde{F}(1) - 2(\beta-z)(z-\alpha)\tilde{F}(z) - e^{ikz}\alpha\beta\tilde{F}(0)$$

$$-ik\int_0^z e^{ik(z-\xi)}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi - ik\int_z^1 e^{ik(\xi-z)}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi, \tag{B.177}$$

and since

$$2(\beta-z)(z-\alpha)\tilde{F}(z) = 2z(z-1)\tilde{F}(z) - 2\epsilon^2(\beta_1 z + \alpha_1(1-z))\tilde{F}(z) + O(\epsilon^4), \tag{B.178}$$

the expression for $I_1\Big(z,\epsilon,\frac{\partial F}{\partial\xi}\Big)$ becomes

$$I_1\Big(z,\epsilon,\frac{\partial F}{\partial\xi}\Big) = -2z(1-z)\tilde{F}(z) - ik\int_0^z e^{ik(z-\xi)}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi$$

$$-ik\int_z^1 e^{ik(\xi-z)}(\beta-\xi)(\xi-\alpha)\tilde{F}(\xi)d\xi + \epsilon^2\log\epsilon^2 L_1$$

$$+\epsilon^2(L_2 + 2\epsilon^2(\beta_1 z + \alpha_1(1-z))\tilde{F}(z) - e^{ik(1-z)}\beta_1\tilde{F}(1) - e^{ikz}\alpha_1\tilde{F}(0)) + HOT. \tag{B.179}$$