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# TWO-DIMENSIONAL HEAT TRANSFER AND THERMAL STRESS ANALYSIS IN THE FLOAT GLASS PROCESS

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## TWO-DIMENSIONAL HEAT TRANSFER AND THERMAL STRESS ANALYSIS IN THE FLOAT GLASS PROCESS

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Thesis

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## ABSTRACT

We consider the glass manufacturing process where the glass floats on a tin layer through a furnace and the temperature of the glass changes from 1100°C at the entrance to 600°C at the exit from the furnace.

Two float glass systems, a pure-layer and a multi-layer system, are considered. For each system asymptotic analysis is performed on the governing equations and corresponding boundary conditions. The small parameter is the ratio of the glass height to its length. The asymptotic analysis results in a simpler heat transfer model that is subsequently solved numerically.

Further, analysis of thermal stresses in the glass ribbon is performed under plane strain assumption, so that the strain (but not stress) transversal to the axis of the ribbon vanish. No-stress boundary conditions are imposed on the remaining parts of the boundary of the ribbon. The asymptotic analysis is performed on thermal stresses up to and including third order terms in order to obtain a solution valid up to first order in the small parameter.

Once the thermal stresses are determined, we optimize the temperature of the air to minimize the longitudinal thermal stresses while the temperature of the glass is fixed at 1100°C at the entrance and 600°C and at the exit from the furnace.

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"The only way to discover the limits of the possible is to go beyond them into the impossible." Arthur C. Clarke

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# "Najvaznija stvar za cijeli zivot je izbor zanimanja, a najcesce je to plod slucajnosti." Blaise Pascal

"Obrazovanje se ne sastoji od toga koliko ste zapamtili ili koliko znate. Sastoji se od toga da razlikujete koliko znate, a koliko ne." Anatole France

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## CHAPTER I

## INTRODUCTION

#### 1.1 Glass history

Glass is a material that has been exploited by humans for many millenia. Probably, it is the oldest man-made material still being used and improved up to this day. New applications as well as new technologies for process improvement are being suggested continuously. Glass can be defined as "a large class of materials with highly variable mechanical and optical properties that solidify from the molten state without crystallization". Glass is generally hard, brittle, and transparent or translucent, and is considered to be supercooled liquid rather than true solid [1]. In other literature, the authors claim that glass is neither a liquid nor a solid since it has properties of both liquids and solids [2, 3].

The oldest glass finds date to 7000 B.C., in the Neolithic period. Glass was first used in Egypt for decorative objects before 3000 B.C., mainly as a colored glaze on stone, pottery, and beads. The art of making glass was perfected about 1500 B.C. in Egypt and the Near East. The entire history of glass is characterized by the development of production methods. A small group of master craftsmen controlled the glass industry and this industry was among the last to become modernized [4]. The two most important properties of glass are the resistance to corrosion due to many oxide molecules in the system and its brittleness owing to the fact that a fracture, once it starts, travels through glass easily.

#### 1.2 Float glass process

This research will focus on float glass. In particular we are interested in determining the temperature profile and thermally induced stresses in a glass sheet. Float glass is a term for perfectly flat, clear, tinted glass manufactured from a process introduced by Sir Pilkington in the United Kingdom in the 1950's [5]. Float glass has been one of the most advantageous improvements seen in the glass industry during the past 50 years. More than 90 % of the flat glass produced in the Western world is made by the float glass process. Float glass is used for smaller windows in domestic housing and its usage in commercial buildings is also increasing. This type of glass provides an attractive and easy to maintain exterior surface [5].

In the float glass process the raw materials are kept in silos until they are needed for production. Through a computerized process, the exact amount of raw material from each silo is put on the scale until the batch makeup is complete. Finegrained ingredients such as silica sand, soda ash, limestone, dolomite, alumina, and others, closely controlled for quality, are properly weighed and mixed into batches. Table 1.1 lists the possible ingredients and their percentages for the batch mix [5]. All of the ingredients provide a major contribution to the final product. For example, limestone improves chemical durability and increases viscosity in the lower tempera-

Raw Material	Percentage						
Silica sand	72.6						
Soda Ash	13.0						
Limestone	8.4						
Dolomite	4.0						
Alumina $(Al_2O_3)$	1.0						
Others	1.0						

Table 1.1: Raw materials.

ture ranges. The batched raw materials pass from a mixing silo to a five-chambered furnace where they melt at a temperature of 1500°C (see Figure 1.1). The temperature is then increased to about 1600°C, so that even the hardest grains can melt. The molten glass is poured onto a bath of molten tin at a temperature of approximately 1500°C where the glass spreads out in the same way that oil spreads out if poured onto water (see Figure 1.4). Note that the molten glass does not spread indefinitely over the surface of clean molten tin [6]. The stream of molten glass at 1500°C has an initial thickness several times greater than the equilibrium value, consequently the molten glass spreads outwards.

This spreading phenomenon can be explained by the surface tension of glass, tin, and the glass-tin interface [7]. As the temperature/concentration increases, the surface tension decreases. The molecules at the surface of a molten glass experience a different environment from the molecules in the interior of the molten glass. In the interior the molecules surround each other, while on the contrary the molecules at the air-glass interface attract only the molecules from below and beside them, and not from above. This causes the surface of a glass to contract and act like a "skin". Therefore, surface tension can be thought of as the force required to penetrate this skin. The stronger the forces between the glass molecules, the greater the surface tension [1, 4].

A typical furnace is 53.34 m long, 9.14 m wide, 1.22 m deep, and it holds about 1600 tons of molten glass. The atmosphere inside the furnace is 94 % nitrogen with 6 % hydrogen to prevent oxidation of the tin and staining of the glass. The tin bath holds approximately 150 tons of molten tin, with a value of 1.5 million dollars. The tin is 5.28 to 7.82 cm deep. The tin bath is 47.55 m long by 6.10 m wide at the front and 4.27 m wide at the end [4]. The flat surface of the molten tin gives the glass, at the tin-glass interface, a very smooth and undistorted surface as it cools. The molten glass forms a ribbon with a width of approximately 3210 mm which is normally between 3 and 25 mm thick [1]. The thicknesses and widths of the ribbon are controlled by the use of rollers. The effect of the rollers is described by the stretching model that was introduced by Narayanaswamy [7]. It was shown in [6, 7] that the magnitude of the longitudinal tractive force in the forming process is determined by the relative speeds of the molten glass stream and the final ribbon. Also, it is known from [6] that any attempt to produce thinner ribbons by increasing the speed of the ribbon will generally produce a proportionally greater decrease in width than the thickness. Increasing the speed early in the production process decreases the

ribbon width more than it decreases the thickness. For that reason, the rollers must be under constant control. This continuous ribbon of solid glass moves out of the float bath when the temperature of the glass is about 600°C (see Figure 1.1). At this point, the glass has cooled down sufficiently to pass to an annealing chamber, called the lehr. The lehr is approximately 106 m long and it is the cooling chamber where electrically heated air is used to cool the glass uniformly. The glass enters the lehr at approximately 600°C and exits at approximately 280°C [4]. Annealing removes stresses induced during the cooling process. Additionally, it allows the glass to reach a more stable state resulting in a higher density, viscosity, and refractive index.

Most materials expand slightly as the temperature rises, and contract as the temperature falls. In this manner, we can observe that as the temperature decreases, the atomic structure of glass becomes more dense. In glass this behavior is a large part due to the strong silica-oxide bond [4].

Figure 1.1 illustrates the side view of the furnace with a layer of molten glass. Likewise, it shows the temperatures as well as the beginning stages of the melting process. After the glass ingredients are mixed, they are put into the furnace where at 1500°C most of the ingredients melt; however, at 1600°C the glass is thinner, since the remaining ingredients are further melted and refined. The uniform composition of glass is achieved through a homogenization process. In this process all the melted and refined ingredients are being composed into one unit. This means that the molecules, ordered or disordered, must be rigidly bound since the glass is slowly starting to cool. Underneath the molten glass we can see the circular motion of the molten tin, which



Figure 1.1: Side view of the furnace with a layer of molten glass.

helps the molten glass move out from the furnace and into the cooling chamber [5]. Also in Figure 1.1, we identify the region of interest for this thesis (shown in the rectangular box) as well as the corresponding axes where x represents the length, z the height, and y the width of the molten glass.

Figure 1.2 represents a very simple side view of the pure-layer glass system in the region of interest, shown in Figure 1.1. This system consists of one layer of molten glass having the height  $\hat{H}$  where

$$\widehat{H} = H + Ah(\widehat{x}, \widehat{y}).$$

*H* represents the height of the perfectly flat glass, *A* is the amplitude of the nonplanar correction, and  $h(\hat{x}, \hat{y})$  represents the nonplanar correction to the surface shape.



Figure 1.2: Side view of a pure-layer glass system.

Figure 1.3 represents a very simple drawing of the side view of the furnace in the region of interest, shown in Figure 1.1. Here, the molten glass is represented as three layers. For illustrative purposes, regions I and III are pure-layers of glass and region II is taken as contaminated layer of glass. These contaminants are some impurities/defects like air bubbles or even some ingredients that haven't melted. In this thesis, the contaminated layer or region II is assumed to be between two purelayers, but later in this thesis, we shall vary the location of the contaminated layer and analyze the impact it might have on the temperature profile. Also, in Figure 1.3 we distinguish the corresponding thicknesses of regions I, II, and III as  $\hat{H}_1$ ,  $\hat{H}_2$ , and



Figure 1.3: Side view of a multi-layer glass system.

 $\widehat{H}_3$ , where

$$\begin{aligned} \hat{H}_{1} &= H_{1} + Ac_{1}(\hat{x}, \hat{y}), \\ \hat{H}_{2} &= H_{2} + Ac_{2}(\hat{x}, \hat{y}), \\ \hat{H}_{3} &= H_{3} + Ac_{3}(\hat{x}, \hat{y}) = H + Ah(\hat{x}, \hat{y}) \end{aligned}$$

 $H_1$ ,  $H_2$ , and  $H_3$  represent the heights of the perfectly flat surface of regions I, II, and III, respectively; A is the amplitude of the nonplanar correction to the surface shape; and  $c_1(\hat{x}, \hat{y})$ ,  $c_2(\hat{x}, \hat{y})$ , and  $c_3(\hat{x}, \hat{y})$  represent the nonplanar surfaces of regions I, II, and III, respectively. Note that  $c_3(\hat{x}, \hat{y}) = h(\hat{x}, \hat{y})$ .

Figure 1.4 shows the top view of the furnace. Initially, the glass spreads out at higher temperatures [6] (at about 1500°C ) due to the surface tension; however, in the cooling zone (at about 1000°C ) the molten glass forms a uniform ribbon. In this



Figure 1.4: Top view of the molten glass in the furnace.

thesis for mathematical simplification, we focus our analysis on the center-line region of the glass sheet.

## 1.3 Literature review for the heat transfer problem

Charnock [6] used classical physics to describe the glass ribbon forming process. The requirement was to produce a flat glass ribbon of uniform thickness that is free of discrete faults (for optical purposes). The glass ribbon goes through the heating phase where the viscosity of the glass is reduced, so that the surface irregularities and the variation in thickness can be removed. The molten glass moves along the bath with the help of rollers. These rollers determine the thickness of the molten glass as described in section 1.2. Since the molten glass cools as it moves along the bath, the viscosity increases. It is important to remember that the viscosity of glass increases continuously as temperature decreases. The viscosity level and the lateral spread of the molten glass are determined by the longitudinal traction force per unit area and the gravitational and surface tension forces, respectively. However, the temperature level must be kept as uniform as possible in order to produce a glass ribbon of uniform thickness without any surface irregularities. The temperature of the molten glass is approximately 600°C when it leaves the bath. In this experiment a very interesting point was made, namely increasing the speed of the molten glass by rotating the rollers faster, increases the thickness of the glass rather than the width. In our model, the temperature profile of the glass ribbon is also around 600°C, however, we neglect the rollers.

Prieto, Diaz, and Egusquiza [8] considered an analysis of the fluid-dynamic and thermal behavior of the tin bath from both theoretical and experimental viewpoints. They studied the flow inside the tin bath and the use of baffles in the tin bath to reduce the reverse flow of the tin, (Figure 1.1). From this paper, we were able to find the temperature profile of the tin using the given experimental data. Prieto et al. established a numerical model for the tin bath using a three-dimensional approach, with variable temperature and negligible viscous dissipation effects in the energy equation. Their governing equations were the continuity, momentum, and energy equations. In order to solve the problem numerically, the authors decided to use the effective-viscosity turbulent flow model which was originally described by Boussinesq. For their governing equations, the following boundary conditions in the tin were used: the velocity at the tin bath walls was equal to zero; no stress distribution at the surface of the tin where the tin is not covered by the glass; this tin surface was assumed to be under radiation from the side walls, the top wall, the cooling surface, and the electric plates; the temperature of the tin particles that are in contact with the glass ribbon have the same temperature distribution as the glass ribbon itself. Finally, the PHOENICS code was used to solve the governing equations. All the simulations were performed for the bath both with and without baffles and using glass ribbon thicknesses of 2 mm and 3 mm for each case.

For the 3 mm thick glass ribbon and the bath without baffles, the authors found that the velocity of the reverse flow is the same as the velocity of the tin in contact with the glass ribbon. Furthermore, the tin velocity vectors under the glass ribbon have the same direction and almost the same value as the velocity vectors of the glass ribbon. The temperature of the tin is found to be higher at the right end of the bath (moving from left to right). However, the temperature difference is not big, it only ranges from 100°C - 200°C. Since the actual temperatures of the tin are high, it can be concluded that there is no temperature difference, thus the temperature is constant. For the 3 mm thick glass ribbon and the bath with the baffles, velocity was calculated and it is found to be lower than for the bath without baffles. Thus, the baffles can be used to diminish the reverse flow of the tin. Here, the tin temperature at the right end of the bath was similar to the temperature of the bath without the baffles. At the left side of the bath with baffles, the temperature was lower than the left side of the bath without the baffles. This difference occurred because the glass ribbon is more spread at the beginning of the float glass process.

For the 2 mm thick glass ribbon and the bath without the baffles, the authors discovered that the tin velocity increased as the thickness decreased. The tempera-

tures, however, were very similar to the temperatures calculated for the 3 mm thick glass ribbon and the bath with the baffles. For the 2 mm thick glass ribbon and the bath with the baffles, the velocity of the reverse flow increased as the thickness decreased. However, the temperature calculated showed some difference compared to the bath without baffles.

In this paper, it was found that the velocities of the tin flow depend on the actual glass ribbon thickness and not on the baffles. However, the use of baffles actually diminished the reverse flow as well as the temperatures at the right end of the bath. Furthermore, the authors considered the temperature profile of the tin and did not calculate the temperature of the glass ribbon.

In our model, we assume a presence of molten tin without any baffles. Further, we are interested in the temperature profile of the glass ribbon and not the temperature profile of the tin. The latter profile is an input to our model. Our model will also take into consideration the thermal stresses in the x- and z-direction as well as the minimization of thermal stresses in the x-direction by controlling the temperature of the air.

The paper by T.J. Wang, H. Zhang, G. Zhang, and T. Yuan [9] investigated the mechanism of formation of a tin profile in the glass using a one-dimensional finite difference computer model of the penetration of tin in float glass. Wang et al. found that the tin profile changed as the oxygen concentration in the atmosphere increased and so the diffusion of the tin particles in the glass ribbon increased. When the glass ribbon passed through the lehr, the tin profile remained the same. However, the tin profile changed when the ribbon passed through a lehr in a  $SO_2$  atmosphere since tin ions in the surface layer of glass changed from bivalent to a tetravalent state. Thus, the tin profile with a satellite peak (big hump) is formed. The formation of the satellite peak can be related to the iron concentration in glass. The authors concluded that the satellite peak can be formed in a tin bath as well as in an annealing lehr where  $SO_2$  atmosphere is present. This implies that the peak can be formed in the glass ribbon at the exit of a lehr. Further, it was observed that the float glass has two groups of tin profiles. One is a tin profile without the peak and it can be described by the diffusion equation. The other is the tin profile with the peak and it cannot be described by the diffusion equation. Furthermore, they have shown that as the  $SO_2$  injection varied, the location and height of the peak varied also. Likewise, it was found that the ribbon temperature, which was determined by a one-dimensional thermal conduction equation, was changing while the ribbon was passing through a gap between the bath and lehr, and through the lehr after departing from the tin bath.

In this thesis, we assume that the tin and glass don't mix. Further, we do not consider the effects of oxygen concentration in the air. Thus, the temperature of the air solely helps to cool the glass ribbon.

Mann, Field, and Viskanta [10] demonstrated a method to determine the thermal conductivity and the true specific heat of the float glass. The data to calculate the thermal conductivity of glass was found in the literature. In order to develop the model for the energy equation, the authors assumed the validity of Planck's law (as temperature increases, the radiance also increases) and Kirchhoff's law (the energy of radiation absorbed, reflected, and transmitted through the material must equal the energy hitting the material), one-dimensional conduction, convection, and radiation heat transfer through the thickness of the glass ribbon. Furthermore, the glass ribbon was assumed to be isotropic, homogeneous, and able to absorb and emit thermal radiation. The solution of the energy equation was obtained through numerical simulations [11, 12]. In [10], the model was time dependent. In contrast, our model is treated as a steady-state problem since we considered a moving coordinate system in which no parameters are changing. In [10], experimental and test procedures were conducted by measuring the initial temperature distribution after the glass was heated as well as cooled. The authors also calculated the specific heat in the liquid, solid, and transition region of the float glass. They used the data from Primenko [13], who calculated the heat content in several different glasses, and found that the temperature ranged from 300°C to 1500°C. This temperature range is very similar to our region of interest. An empirical equation from Sharp and Ginther [14] was used to extrapolate the data ranging from 300°C to room temperature. For temperatures above 300°C, cubic spline fit interpolation was used. Furthermore, they found that the specific heat remained constant in the liquid region, reached a maximum then decreased to a liquid equilibrium value in the transition region, increased slowly during the heating process in the solid region and then increased rapidly once it reached the transition region. Likewise, in the cooling process, the following observations for the specific heat were made: the specific heat decreased in the liquid region and had

approximately the same value as the specific heat found in the heating process in the solid region. In the liquid region, no maximum value for specific heat was obtained for their calculations. In the transition region, the specific heat depended on the temperature and the rate of heating; thus, it was not easy to determine the specific heat. In this paper, the thermal conductivity was defined as the heat transferred due to the lattice vibration. In addition, the thermal conductivity of float glass was expressed as a linear function of temperature. The thermal conductivity for thick plates (0.676 - 1.17 cm) was calculated by comparing the predicted temperatures from the energy model to the dynamic temperature data. On the other hand, the thermal conductivity for thin plates was not calculated since the temperature measurements were inconsistent. The specific heat played a big role in determining the thermal conductivity. It was discovered that a 10 % uncertainty in determining the specific heat could potentially result in 20 % uncertainty in determining the thermal conductivity. Finally, their measurement of the thermal conductivity and the specific heat of the float glass were compared to the data available in the literature. The specific heat from different tests showed remarkable consistency; however, the thermal conductivity did not show satisfactory results due to the temperature uncertainties. There were no actual temperature results for the glass ribbon to compare to ours.

Our assumptions compared to the author's in [10] differ in that we consider two-dimensional conduction and neglect convection due to the already formed glass ribbon. Further, internal radiation is neglected due to a thin domain approximation for the glass. Lentes and Siedow [15] considered three-dimensional radiative heat transfer in glass cooling processes where the temperature was  $\leq 600^{\circ}$ C. This temperature range is lower than that considered in this thesis. Lentes and Siedow used an improved diffusion approximation which takes into consideration the geometry of the solid and is as efficient as the Rosseland approximation which uses the formal solution of the radiative transfer equation. The Rosseland approximation (usually used for optically thick glass) describes the radiation as a correction of the heat conduction. This improved diffusion approximation is a numerical simulation of the radiative heat transfer; furthermore, it is more efficient and accurate than the Rosseland approximation. The results from the improved diffusion approximation for three-dimensional radiative heat transfer were compared to the exact solution of the one-dimensional cylindrical glass plate as well as the Rosseland approximation.

[15]'s three-dimensional solution was compared to the orthotropic diffusion approximation. The glass plate had a uniform temperature distribution of 600°C. The temperature surrounding the glass plate was taken to be 20°C and the measurements were taken after 1 s and 10 s at the axis of symmetry. Additionally, the heat transfer by convection was neglected. The comparison of temperature (ranging from 535°C - 580°C) with high absorption coefficient and at 10 s indicated that the two approximation methods were in good standing with the exact solution. In the case where the absorption coefficient was smaller, the difference between the Rosseland approximation and the exact solution was obvious. On the other hand, the improved diffusion approximation agreed well with the exact solution. The comparison between a three-dimensional problem and the orthotropic diffusion approximation was performed. The same assumptions were made as in the first comparison. However, the temperature distribution (ranged from 480°C - 600°C) in the thin glass plate was calculated along the radius by the quasi-exact solution, the Rosseland approximation, and the improved diffusion approximation. The results showed that the improved diffusion approximation agreed the best with the quasi-exact solution. It was expected for the Rosseland approximation to perform poorly since they were considering a thin glass plate. In our model, we solve the heat transfer as well as the thermal stress problem analytically first, then we examine the problem numerically.

An integrated mathematical model of the float glass process has been developed by Kamihori, Iga, Kakihara, and Mase [16]. The authors analyzed three models that influence and affect each other: the glass ribbon forming model, the float bath heat transfer model, and the molten tin flow model. Glass of 2 mm thickness was considered for all three models. In the glass ribbon forming model, which was not considered in this thesis, the governing equations were a two-dimensional problem which was previously considered by Iga and Mase [17]. Furthermore, numerical simulations were performed using the finite-element method. The authors calculated the longitudinal and lateral temperature distribution of the glass ribbon in order to collect information about the viscous fluid flow of the entire glass ribbon.

In the float bath heat transfer model, a three-dimensional radiative heat transfer as well as convective heat transfer of the glass ribbon was calculated. It is understood that nitrogen/hydrogen gas has been pumped into the bath to prevent oxidation of the molten tin. The furnace was divided into different sections and the nitrogen/hydrogen gas is assumed to be completely mixed within each section of the bath, thus representing consistency of temperature in each section. Further, the authors constructed the energy balance equation for the surface that is not covered by the glass ribbon and the surface of molten tin only. This equation contained components such as the radiative heat energy on the surface, the heat energy from the gas by heat convection, and the heat transmission from the atmosphere. In addition, the radiative heat energy component consisted of the Stefan-Boltzmann constant, emissivity, temperature etc. The authors examined the energy balance equation for the glass ribbon surface and the bare surface of molten tin. This equation was expressed as the sum of radiative heat, convective heat from the gas, heat from the molten tin, and convective heat energy by glass flow.

For the molten tin flow model, which we did not examine in this thesis, 3-D turbulent flow and heat transfer have been calculated using the  $k - \epsilon$  mathematical turbulent model that consisted of the continuity, momentum, and energy equations. Only the heat distribution in the tin was considered. The authors compared the integrated model with the results of the actual float glass process. The authors observed that the colder the molten tin beside the molten glass, the more of the convective heat energy of the glass ribbon and the molten tin under the glass ribbon was transferred to the colder molten tin. Hence, a convex lateral temperature distribution for the glass ribbon was established. The work demonstrated that the temperature in the center of the ribbon is higher than the temperature on the edges of the ribbon.

#### 1.4 Literature review for the thermal stress problem

We used the displacement equations from [18] to solve for thermal stresses by making the assumption of a plane strain problem. Once displacements are determined, thermal stresses are calculated along the length and thickness of the glass ribbon.

The paper by Ootao and Tanigawa [19], considered transient thermal stresses of an orthotropic functionally graded thick strip due to nonuniform heating in the width direction. The authors analyzed the two-dimensional temperature profile using Laplace and finite sine transformations. They also determined the exact solution for temperature using a simply supported strip and assuming plane strain. In their analysis, Ootao and Tanigawa assumed that the strip is at zero temperature initially, then it is heated from the lower to upper surfaces. As a result, the thermal conductivity is expressed as an exponential function. Also, the specific heat and density are kept constant. In contrast for our case, the initial temperature is kept at 1100°C and the glass is cooled until it reaches 600°C. Once the temperature profile is calculated, the authors expressed the displacement-strain relation and then the stress-strain relation. Rewriting the stresses in terms of displacements and applying the boundary conditions led to the solution for the thermal stress problem. After obtaining the analytical solution for the temperature and stress, they verified their results numerically. They ran different cases on nonhomogeneity of the thermal conductivity, coefficient of the linear thermal expansion, and the elastic stiffness constant. They calculated

numerically the thermal stresses at the midpoint of the strip in the thickness direction. Our assumptions compared to the author's in [19] differ in that we consider an isotropic thin glass ribbon and we also use a different method for solving the heat transfer problem. We consider the thermal conductivity to be a constant compared to an exponential function assumed in [19]. However, we make the same assumption of nonuniform heating and a plane strain problem.

## 1.5 Physics of the problem

In general, heat transfer occurs because of a temperature difference at the system boundaries. In general, there are three different modes of heat transfer: conduction, convection, and radiation [20, 21]. These three types of heat transfer can all occur either simultaneously or separately. They are defined as follows:

Conduction is transfer of heat from one part of a body to another part without displacement of the particles of the body [20]. This process is governed by the Fourier's law. In our case, conduction represents a heat transfer or temperature difference inside the glass ribbon through the thickness and along the length.

Convection is known as transfer of heat from one point to another within a fluid (gas or liquid) by mixing of one portion of the fluid with another [20].

The third mode is radiation, which is the transfer of heat from one body to another not necessarily in contact with it, by means of wave motion [20]. During the heating or cooling of glass, the radiative heat transfer may dominate over conduction and convection. Thermal radiation is generally a global phenomenon, meaning heat can be felt regardless of the distance from the actual source. On the other hand, conduction and convection are local phenomenon, meaning heat can only be felt in a limited region [15].

An analysis of thermal stresses with some constraints will also be conducted. Glass has a thermal conductivity, which is generally low compared to metals. The properties of glass vary and they can be regulated by modifying the composition, production techniques, or both. In any glass, the mechanical, chemical, optical, and thermal properties cannot occur separately. A change in temperature causes an expansion or contraction of glass. Further, if the temperature increases, the width of the glass ribbon increases and the height decreases. On the other hand, if the temperature decreases, the glass ribbon is expected to contract along all its directions [22].

Thermal stresses are induced by non-uniform heating [18]. From Hooke's law, we can conclude that the higher the Young's modulus E and the thermal expansion coefficient  $\alpha$ , the higher the thermal stresses [23]. Since thermal stresses increase as the temperature gradients increase [18], the probability for glass breakage is higher for large temperature gradients.

Here, we will conduct a study of displacements/stresses for the float glass problem to give us a better view of the glass behavior under the influence of temperature.
#### 1.6 Thesis overview

In this thesis, we investigate a two-dimensional heat transfer model in both a pure- and a multi-layer glass system, as well as the thermal stress distribution and displacements in each system, disregarding flow of the glass. The region of interest for this analysis is the part of the furnace where the temperature is between 1100°C at the entrance and 600°C at the exit from the furnace (see Figure 1.1) and the glass ribbon has already been formed. Thermal stresses can be observed and potentially avoided in this region.

A pure-layer glass system is a homogeneous glass ribbon. The multi-layer system studied is a glass ribbon consisting of one contaminated layer either placed between, beneath (next to the tin layer), or above (next to the air) a pure-layer (see Figure 1.3). The contaminated layer is assumed to be about a third of the actual molten glass thickness. Further, we vary the location of the contaminated layer and the thermal conductivity to determine if the temperature changes. From the actual process of glass making, we know that the thermal conductivity of the contaminated layer is 10 % of the thermal conductivity of the pure-layer [24]. In this thesis, the thermal conductivity of the pure-layer is taken to be 1  $\frac{W}{mK}$  [5], [9].

We neglect internal radiation since we are dealing with a thin sheet approximation [6]. The heat capacity, coefficient of thermal conductivity, and the glass density vary with temperature T; therefore, they are defined as functions of temperature. However, after expanding the temperature T asymptotically, the heat capacity, thermal conductivity, and density are kept constant at the leading order. Our analytical approach for temperature distribution consists of developing asymptotic expansions of the dependent system variables in the limit of small aspect ratio height to the length of the glass ribbon. An asymptotic expansion is applied to the governing two-dimensional heat transfer equation and corresponding boundary conditions. We analyze the temperature profile T for the glass ribbon up to and including the first order correction.

Considering the pure-layer glass system, we use two boundary conditions since we have a second order governing heat equation. One boundary condition describes the temperature at the molten tin-glass interface and the other boundary condition describes the temperature at the molten glass-air interface.

Next, we analyze a multi-layer glass system, see Figure 1.3. Here, we have a coupled set of three governing heat equations with six boundary conditions. Two of the governing heat equations resemble the governing heat equation of the pure-layer glass system. However, the governing heat equation for the contaminated layer differs from the pure-layer system governing equation only by the values of density, heat capacity, and thermal conductivity, in general. For simplicity, we only vary the thermal conductivity when considering contaminated layer. It is known that the density of the contaminated glass (apparent density) is less than the density of the pure-layer glass (true density) if molten glass contains bubbles. However, the apparent density can be greater than the true density in cases where particles of unmelted batches formed during cooling. Sand particles are the most likely representation of the un-

melted batches [3]. The prescribed boundary conditions describe the temperature at the tin-glass interface (bottom boundary condition), glass-air interface (top boundary condition), and two pure-contaminated layer interfaces (four boundary conditions). These four boundary conditions are expressed as continuity of temperature and heat flux at each interface.

After the temperature profile has been calculated, it is used to calculate the displacements and thermal stresses that occur in the glass ribbon. For the purpose of this thesis, we consider a three-dimensional thermal stress problem by imposing the plane strain condition along the width of the glass ribbon. Further, no-stress boundary conditions are applied along the length and height of the glass ribbon. Then thermal stresses are computed from the equations of mechanical equilibrium. Since the differential equations are written in terms of displacements, we obtain expressions for the displacements in the glass ribbon as a byproduct of our analysis. Finally, we optimize the temperature of the air to minimize the stresses along the length of the glass ribbon.

In summary, in this thesis:

- an analytical model of the glass ribbon temperature profile valid along the center-line and away from the glass ribbon edges is developed;
- a flexible mathematical formulation for studying contaminated layers, both interior to the glass ribbon and at the glass/tin and glass/air interface is developed;

- an analytical model for the thermal stresses incorporating the key effects (the longitudinal temperature gradient, surface heat transfer, and velocity of the glass ribbon) is developed;
- an optimization procedure to determine the temperature of the air to minimize the thermal stresses in the *x*-direction is developed;

### CHAPTER II

# FORMULATION OF THE MATHEMATICAL MODEL FOR THE HEAT TRANSFER PROBLEM

The formulation of the proposed model includes the discussion of the heat transfer and the thermal stress analysis inside the float glass in the region where the temperature of the glass is between 1100°C and 600°C from left to right. Specifically, we shall analyze the heat transfer through the thickness, the z-direction, of the molten glass since heat propagates faster through the thickness (approximately 3 mm) than along the length, the x-direction (approximately 60 m). Hence, our analysis focuses on the center line portion of the glass ribbon and away from the lateral edges of the ribbon and hence we neglect the width of the glass ribbon or the y-direction.

For this thesis, the glass is treated as either a pure-layer system, the pure system, or a multi-layer system, the contaminated system. We have developed governing heat equations and corresponding boundary conditions for both systems in order to calculate the temperature distribution within the glass ribbon. In both systems, the following assumption is made:  $H \ll W \ll L$ , where H is the height, W is the width, and L is the length of the glass ribbon in the cooling zone, see Figure 1.1. 2.1 Governing equations and boundary conditions for the pure-layer system

Let us first consider the heat transfer problem for the pure-layer glass system. The pure-layer glass system can be thought of as a single pure-layer glass floating on the molten tin. Thus, the governing heat equation in a dimensional form is represented as

$$\widehat{\rho}(\widehat{T})\widehat{C}_{p}(\widehat{T})\left[\widehat{T}_{\widehat{t}}+V\widehat{T}_{\widehat{x}}\right] = \left[\widehat{k}(\widehat{T})\widehat{T}_{\widehat{x}}\right]_{\widehat{x}} + \left[\widehat{k}(\widehat{T})\widehat{T}_{\widehat{y}}\right]_{\widehat{y}} + \left[\widehat{k}(\widehat{T})\widehat{T}_{\widehat{z}}\right]_{\widehat{z}} - \widehat{Q}(\widehat{T}), \quad (2.1)$$

where

$$\widehat{Q}(\widehat{T}) = \epsilon_{mglass} \sigma_{SB} \left[ \widehat{T}^4 - \widehat{T}^4_{tin} \right].$$
(2.2)

In the governing equation (2.1),  $\hat{x}$ ,  $\hat{y}$ , and  $\hat{z}$  are rectangular coordinates. Here,  $\hat{\rho}$  is the glass density,  $\hat{C}_p$  is the heat capacity,  $\hat{k}$  is the thermal conductivity, and  $\hat{Q}$  is the radiative heat transfer or the emitted energy. All of the above stated variables are functions of temperature  $\hat{T}$ . The variable V, which is considered a constant in this thesis, represents the velocity at which the glass ribbon is being pulled;  $\hat{t}$  is time,  $\hat{x}$ is length,  $\hat{y}$  is width, and  $\hat{z}$  is height of the glass ribbon.  $\epsilon_{mglass}$  is the emissivity of the glass,  $\sigma_{SB}$  is the Stefan-Boltzman constant. Refer to Table A.1 for a detailed description of every variable and their corresponding values.

In general, the heat equation describes the temperature in a given region over time. However, for easier calculations we neglect the term,  $\hat{T}_i$ , in equation (2.1) since we consider a continuous process along the center line of the glass ribbon, that has reached a steady-state configuration. Further, we introduce a moving coordinate system along the length direction. We also neglect the internal radiation term  $\widehat{Q}(\widehat{T})$ , since we have a very thin domain.

The conduction term,

$$\left[\hat{k}(\widehat{T})\widehat{T}_{\hat{x}}\right]_{\hat{x}} + \left[\hat{k}(\widehat{T})\widehat{T}_{\hat{y}}\right]_{\hat{y}} + \left[\hat{k}(\widehat{T})\widehat{T}_{\hat{z}}\right]_{\hat{z}}, \qquad (2.3)$$

in the governing heat equation (2.1) describes the heat diffusion in the molten glass in the  $\hat{x}$ ,  $\hat{y}$ , and  $\hat{z}$ -directions.

Two boundary conditions in the  $\hat{z}$ -direction for the pure system are required since we are dealing with a governing heat equation of second order. These boundary conditions must be prescribed at known locations, tin and air, of the independent variable  $\hat{z}$ . The two boundaries for this system are: the tin-glass interface ( $\hat{z} = 0$ ), which is referred to as the bottom boundary, and the air-glass interface ( $\hat{z} = \hat{H}$ ), which is referred to as the top boundary.

The top boundary condition at  $\hat{z} = \hat{H}$  is

$$-\hat{k}(T)\left(\nabla \widehat{T} \cdot \mathbf{n_1}\right) = h_a\left[\widehat{T} - \widehat{T}_{air}\right] + \widehat{s}, \qquad (2.4)$$

where

$$\hat{s} = \epsilon_{mair} \sigma_{SB} \left[ \hat{T}^4 - \hat{T}^4_{air} \right].$$
(2.5)

The bottom boundary condition at  $\hat{z} = 0$  is

$$-\hat{k}(T)\left(\nabla\widehat{T}\cdot\mathbf{n_2}\right) = h_{tin}\left[\widehat{T}-\widehat{T}_{tin}\right] + \widehat{S},\qquad(2.6)$$

where

$$\widehat{S} = \epsilon_{mtin} \sigma_{SB} \left[ \widehat{T}^4 - \widehat{T}^4_{tin} \right].$$
(2.7)

Here, the terms

$$-\hat{k}(\hat{T})\left(\nabla\hat{T}\cdot\mathbf{n_1}\right) \tag{2.8}$$

and

$$-\hat{k}(\widehat{T})\left(\nabla\widehat{T}\cdot\mathbf{n_2}\right) \tag{2.9}$$

are known as the heat flux, which is defined as the heat flow per surface area normal to the air and tin surfaces, respectively. The top surface of the glass ribbon is assumed to be sinusoidal [6]. Hence, the normal  $\mathbf{n_1}$  is

$$\mathbf{n_1} = \frac{\nabla \left( \hat{z} - \hat{h}(x, y) \right)}{\|\nabla \left( \hat{z} - \hat{h}(x, y) \right)\|}$$

$$= \frac{\left\langle -\frac{\partial \hat{h}}{\partial \hat{x}}, -\frac{\partial \hat{h}}{\partial \hat{y}}, 1 \right\rangle}{\sqrt{\left( -\frac{\partial \hat{h}}{\partial \hat{x}} \right)^2 + \left( -\frac{\partial \hat{h}}{\partial \hat{y}} \right)^2 + 1}},$$
(2.10)

where

$$\widehat{H} = H_3 + Ah(x, y).$$

Substituting equation (2.10) into equation (2.4) and simplifying, we obtain

$$-\hat{k}(\hat{T}) \begin{bmatrix} \widehat{T}_{\hat{x}} \frac{-\frac{\partial \hat{h}}{\partial \hat{x}}}{\sqrt{(-\frac{\partial \hat{h}}{\partial \hat{x}})^2 + (-\frac{\partial \hat{h}}{\partial \hat{y}})^2 + 1}} + \widehat{T}_{\hat{y}} \frac{-\frac{\partial \hat{h}}{\partial \hat{y}}}{\sqrt{(-\frac{\partial \hat{h}}{\partial \hat{x}})^2 + (-\frac{\partial \hat{h}}{\partial \hat{y}})^2 + 1}} + (2.11) \\ \widehat{T}_{\hat{z}} \frac{1}{\sqrt{(-\frac{\partial \hat{h}}{\partial \hat{x}})^2 + (-\frac{\partial \hat{h}}{\partial \hat{y}})^2 + 1}} \end{bmatrix} = h_{air} \left(\widehat{T} - \widehat{T}_{air}\right) - \hat{s}.$$

The contact surface between the molten glass and the tin is perfectly flat. Here, the normal to the surface is in the negative z-direction and the normal  $\mathbf{n_2}$  is,

$$\mathbf{n_2} = \langle 0, 0, -1 \rangle, \qquad (2.12)$$

which simplifies the bottom boundary condition, equation (2.6), to

$$\hat{k}(\hat{T})\hat{T}_{\hat{z}} = h_{tin}\left(\hat{T} - \hat{T}_{tin}\right) + \hat{S}.$$
(2.13)

The convection terms from the boundary conditions, equations (2.11) and (2.13),  $h_{air}\left(\hat{T}-\hat{T}_{air}\right)$  and  $h_{tin}\left(\hat{T}-\hat{T}_{tin}\right)$ , represent the heat transfer from the air/tin to the moving molten glass, where  $h_{air}$  and  $h_{tin}$  are heat transfer coefficients for air and tin, respectively. The heat transfer coefficient depends on the properties of the material; therefore,  $h_{air}$  and  $h_{tin}$  have different values. Further, the terms  $\hat{s}$  and  $\hat{S}$ describe the radiation at the boundaries, air and tin, or the emitted energy at these boundaries.

#### 2.2 Nondimensionalization for the pure-layer system

The scalings in Table 2.1 are used to nondimensionalize the governing heat equation (2.1) and boundary conditions, equations (2.11) and (2.13). Using these scalings, we find that the governing heat equation (2.1) becomes

$$\rho_r \rho(T) C_{pr} C_p(T) \frac{T_H V}{L} \left[ \frac{L}{V \tau} T_t + T_x \right] =$$

$$k_r \frac{T_H}{L^2} \left[ k(T) T_x \right]_x + k_r \frac{T_H}{W^2} \left[ k(T) T_y \right]_y + k_r \frac{T_H}{H^2} \left[ k(T) T_z \right]_z - Q_r Q(T),$$
(2.14)

where  $T_H$  is the reference value for temperature and it can be found in Appendix A, and

$$Q_r = \epsilon_m \frac{\sigma_{SB} T_H^4}{H}.$$
 (2.15)

Independent/Dependent variables	Scale	Units
$\widehat{T}$	$T_H T$	K
$\hat{x}$	Lx	m
$\hat{y}$	Wy	m
ź	Hz	m
$\hat{t}$	au t	s
$\hat{k}$	$k_r k(T)$	$\frac{W}{mK}$
$\widehat{C}_p(\widehat{T})$	$C_{pr}C_p(T)$	$\frac{J}{kgK}$
$\hat{ ho}(\widehat{T})$	$\rho_r \rho(T)$	$\frac{\mathrm{kg}}{\mathrm{m}^3}$
$\widehat{Q}(\widehat{T})$	$Q_r Q(T)$	$\frac{W}{m^3}$

Table 2.1: Nondimensionalization table

Further in this thesis as in [9],  $\widehat{C}_p(\widehat{T})$  and  $\widehat{\rho}(\widehat{T})$  are considered to be constants at leading order [9]. Since we expect much faster heat transfer through the thickness than through the length or the width of the glass, we isolate the conduction term in the z-direction in equation (2.14). We rewrite equation (2.14) to find

$$\frac{\rho_r C_{pr} H^2 V \rho(T) C_p(T)}{k_r L} \left[ \frac{L}{V \tau} T_t + T_x \right] =$$

$$\frac{H^2}{L^2} \left[ k(T) T_x \right]_x + \frac{H^2}{W^2} \left[ k(T) T_y \right]_y + \left[ k(T) T_z \right]_z - \frac{\sigma_{SB} T_H^3 H}{k_r} Q(T).$$
(2.16)

From equation (2.16), we calculate the values of the nondimensional groups, so that we can make the decision which terms are "small" and therefore choose the small aspect ratio, height H to length L of the glass ribbon, which we shall refer to as  $\epsilon$ . Thus,

$$\epsilon = \frac{H}{L}.\tag{2.17}$$

Typically the small aspect ratio is taken to be  $\epsilon = 0.00005$ . From Table A.1 we find

$$\frac{\rho_r C_{pr} H^2 V}{k_r L} = 0.5034, \qquad (2.18)$$

$$\frac{H}{L} = 0.00005, \qquad (2.19)$$

$$\frac{H^2}{L^2} = 0.000000025, \qquad (2.20)$$

$$\frac{H^2}{W^2} = 0.000001, \qquad (2.21)$$

$$\frac{\sigma_{SB}T_H^3H}{k_r} = 1.47, \qquad (2.22)$$

$$\frac{L}{V\tau} \frac{\rho_r C_{pr} H^2 V}{k_r L} = 1.$$
 (2.23)

where the values were taken from the literature [6, 7, 8, 9].

The time scale is defined as

$$\tau = \frac{\rho_r C_{pr} H^2}{k_r}.$$
(2.24)

Other nondimensional groups are:

Peclet number  $\overline{P}$ :

$$\epsilon P = \overline{P} = \frac{\rho_r C_{pr} H^2 V}{k_r L},\tag{2.25}$$

Radiation number  $\overline{R}$ :

$$\epsilon R = \overline{R} = \frac{\sigma_{SB} T_H^3 H}{k_r}.$$
(2.26)

Note that P and R are O(1). Using these scalings, equations (2.18) - (2.26), we find that the steady-state heat transfer governing equation (2.1) for the pure-layer system takes the following nondimensional form

$$\epsilon P \rho(T) C_p(T) T_x = \epsilon^2 \left[ k(T) T_x \right]_x + \epsilon^2 \left( \overline{A} \right)^2 \left[ k(T) T_y \right]_y + \left[ k(T) T_z \right]_z + \epsilon R Q(T).$$
(2.27)

Recall the conduction term in the y-direction from equations (2.16) and (2.21). The ratio  $\frac{H^2}{W^2}$  suggests another small parameter to be introduced along the y-direction since our asymptotic analysis is performed along the z-direction.

Using the scalings, equations (2.18) - (2.26), we determine another relation

$$\frac{H}{W} = \frac{H}{L}\frac{L}{W} = \epsilon \overline{A}, \qquad (2.28)$$

where we assume  $\overline{A} = \frac{L}{W} = O(1)$ . Thus, from equation (2.28), we have that

$$\frac{H}{W} = O(\epsilon). \tag{2.29}$$

With these assumptions the y-variable appears first at higher order,  $O(\epsilon^2)$ .

To simplify our model, we make the following assumptions for the heat transfer governing equation. First, the higher the thermal conductivity and/or the smaller the thickness of the molten glass, the smaller the nondimensional ratio  $\overline{P}$  which is referred as <u>convection</u>. Thus,  $\overline{P}$  is small if the conduction term dominates over the convection, which is the case here. Also, a higher thermal conductivity leads to a higher radiation term.

We neglect the radiation term RQ(T) since we deal with a thin approximation. The three-dimensional heat transfer governing equation that needs to be solved is

$$\epsilon P \rho(T) C_p(T) T_x = \epsilon^2 \left[ k(T) T_x \right]_x + \epsilon^2 \left[ k(T) T_y \right]_y + \left[ k(T) T_z \right]_z.$$
(2.30)

It is also necessary to nondimensionalize the boundary conditions, equations (2.11) and (2.13). Using the same scalings, given by equations (2.18) - (2.26), and (2.29), we find nondimensional forms for the top and bottom boundary conditions.

The top of the molten glass is assumed to have the shape of a sine function [6]. Assuming  $\hat{z} = zH_3$ , we nondimensionalize  $\hat{h}(x, y) = \hat{z} = H_3 + Ah(x, y)$  to obtain

$$z = 1 + \frac{A}{H_3}h(x, y),$$

where A is the amplitude of the nonplanar correction to the surface shape, and  $H = H_3$  is the height of the glass ribbon. We assume  $\frac{A}{H_3} = O(\epsilon)$ , and find the nondimensional form of the upper boundary to be

$$z = 1 + \epsilon h(x, y). \tag{2.31}$$

Further, we find

$$\frac{\partial \hat{h}}{\partial \hat{x}} = \frac{A}{L} \frac{\partial h}{\partial x} = \epsilon \frac{\partial h}{\partial x}, \qquad (2.32)$$

$$\frac{\partial \hat{h}}{\partial \hat{y}} = \frac{A}{L} \frac{\partial h}{\partial y} = \epsilon \frac{\partial h}{\partial y}, \qquad (2.33)$$

where  $\frac{A}{L} = O(\epsilon)$ ,  $A \ll L$ .

From the dimensional form of the top boundary condition, equation (2.11), we cal-

culate the nondimensional top boundary condition at  $z = 1 + \epsilon h(x, y)$  to be

$$k(T) \left\{ -\epsilon^{3} T_{x} \frac{\frac{\partial h}{\partial x}}{\sqrt{\left[-\epsilon \frac{\partial h}{\partial x}\right]^{2} + \left[-\epsilon \frac{\partial h}{\partial y}\right]^{2} + 1}} - (2.34) \right.$$

$$\left. \epsilon^{3} T_{y} \frac{\frac{\partial h}{\partial x}}{\sqrt{\left[-\epsilon \frac{\partial h}{\partial x}\right]^{2} + \left[-\epsilon \frac{\partial h}{\partial y}\right]^{2} + 1}} + \left. T_{z} \frac{1}{\sqrt{\left[-\epsilon \frac{\partial h}{\partial x}\right]^{2} + \left[-\epsilon \frac{\partial h}{\partial y}\right]^{2} + 1}}} \right\} = -\epsilon B_{air} \left(T - T_{air}\right) - \epsilon R_{air} \epsilon_{mair} \left(T^{4} - T_{air}^{4}\right),$$

where the Biot number for air  $\overline{B}_{air}$  and the radiation number for air  $\overline{R}_{air}$  are assumed to be  $O(\epsilon)$ . Here, we obtain

$$\epsilon B_{air} = \overline{B}_{air} = \frac{h_{air}H}{k_r},\tag{2.35}$$

$$\epsilon R_{air} = \overline{R}_{air} = \epsilon R. \tag{2.36}$$

From the dimensional form of the bottom boundary condition, equation (2.13), the nondimensional bottom boundary condition at z = 0 is

$$k(T)T_z = \epsilon B_{tin} \left[T - T_{tin}\right] + \epsilon R_{tin} \epsilon_{mtin} \left[T^4 - T_{tin}^4\right], \qquad (2.37)$$

where the Biot number for tin  $\overline{B}_{tin}$  and the radiation number for tin  $\overline{R}_{tin}$  is assumed to be  $O(\epsilon)$  and hence

$$\epsilon B_{tin} = \overline{B}_{tin} = \frac{h_{tin}H}{k_r},\tag{2.38}$$

$$\epsilon R_{tin} = \overline{R}_{tin} = \epsilon R. \tag{2.39}$$

From equations (2.35) and (2.38), we observe that  $\overline{B}_{air}$  and  $\overline{B}_{tin}$  are small since  $h_{air}$ and  $h_{tin}$  are assumed to be small. Also, from equations (2.36) and (2.39),  $\overline{R}_{air}$  and  $\overline{R}_{tin}$ , we observe are small as well. Note further that  $\sigma_{SB}$  and  $T_H$  are treated as constants in this thesis (refer to Table A.1 for their exact values).

#### 2.3 Governing equations and boundary conditions for the multi-layer system

Let us now define the governing heat equations for the multi-layer system. The general dimensional form of the governing equation at each layer for the multi-layer system is represented as

$$\widehat{\rho}(\widehat{T}^{ml})\widehat{C}_{p}(\widehat{T}^{ml})\left[\widehat{T}_{\hat{t}}^{ml} + V\widehat{T}_{\hat{x}}^{ml}\right] = (2.40)$$

$$\left[\widehat{k}^{ml}(\widehat{T}^{ml})\widehat{T}_{\hat{x}}^{ml}\right]_{\hat{x}} + \left[\widehat{k}^{ml}(\widehat{T}^{ml})\widehat{T}_{\hat{y}}^{ml}\right]_{\hat{y}} + \left[\widehat{k}^{ml}(\widehat{T}^{ml})\widehat{T}_{\hat{z}}^{ml}\right]_{\hat{z}} - \widehat{Q}^{ml}(\widehat{T}^{ml}),$$

where  $\widehat{Q}^{ml}(\widehat{T}^{ml}) = \widehat{Q}(\widehat{T})$  from equation (2.2).

The multi-layer system will be modeled as a three layer system consisting of a contaminated layer between two pure-layers (see Figure 1.3 in Chapter 1). Therefore, the governing equation (2.40) is applied to each layer.

The first region (labeled as region I in Figure 1.3 in Chapter 1), is the bottom purelayer. Thus, the governing heat equation for region I resembles that of the pure-layer system:

$$\widehat{\rho}(\widehat{T}^{mlI})\widehat{C}_{p}(\widehat{T}^{mlI})\left[\widehat{T}_{\widehat{t}}^{mlI} + V\widehat{T}_{\widehat{x}}^{mlI}\right] = (2.41)$$

$$\left[\widehat{k}(\widehat{T}^{mlI})\widehat{T}_{\widehat{x}}^{mlI}\right]_{\widehat{x}} + \left[\widehat{k}(\widehat{T}^{mlI})\widehat{T}_{\widehat{y}}^{mlI}\right]_{\widehat{y}} + \left[\widehat{k}(\widehat{T}^{mlI})\widehat{T}_{\widehat{z}}^{mlI}\right]_{\widehat{z}} - \widehat{Q}(\widehat{T}^{mlI}).$$

The second region (labeled as region II in Figure 1.3 in Chapter 1), is the middle layer or the contaminated layer. Its governing equation is

$$\begin{split} \widehat{\rho}^{mlII}(\widehat{T}^{mlII})\widehat{C}_{p}^{mlII}(T^{mlII}) \left[\widehat{T}_{\hat{t}}^{mlII} + V\widehat{T}_{\hat{x}}^{mlII}\right] &= \\ & \left[\widehat{k}^{mlII}(\widehat{T}^{mlII})\widehat{T}_{\hat{x}}^{mlII}\right]_{\hat{x}} + \\ & \left[\widehat{k}^{mlII}(\widehat{T}^{mlII})\widehat{T}_{\hat{y}}^{mlII}\right]_{\hat{y}} + \\ & \left[\widehat{k}^{mlII}(\widehat{T}^{mlII})\widehat{T}_{\hat{z}}^{mlII}\right]_{\hat{z}} - \widehat{Q}^{mlII}(\widehat{T}^{mlII}). \end{split}$$

$$\end{split}$$

$$(2.42)$$

The third region (labeled as region III in Figure 1.3 in Chapter 1), is the top pure-layer and its governing equation is represented also as

$$\widehat{\rho}(\widehat{T}^{mlIII})\widehat{C}_{p}(\widehat{T}^{mlIII})\left[\widehat{T}_{\hat{t}}^{mlIII} + V\widehat{T}_{\hat{x}}^{mlIII}\right] = (2.43)$$

$$\left[\widehat{k}(\widehat{T}^{mlIII})\widehat{T}_{\hat{x}}^{mlIII}\right]_{\hat{x}} + \left[\widehat{k}(\widehat{T}^{mlIII})\widehat{T}_{\hat{y}}^{mlIII}\right]_{\hat{y}} + \left[\widehat{k}(\widehat{T}^{mlIII})\widehat{T}_{\hat{x}}^{mlIII}\right]_{\hat{x}} - \widehat{Q}(\widehat{T}^{mlIII}).$$

These three governing heat equations (2.41) - (2.43) have the same form as the governing equation (2.1); however, they differ in their values for density, heat capacity,

and thermal conductivity since we are dealing with different material. Also, we need to remember that we are analyzing a steady-state problem, thus  $\hat{T}_{\hat{t}}^{mlI} = \hat{T}_{\hat{t}}^{mlII} = \hat{T}_{\hat{t}}^{mlIII} = \hat{T}_{\hat{t}}^{mlIII} = 0$  in all three regions.

In order to solve the set of three governing heat equations of second order in the multi-layer system, we need to have six boundary conditions in the z-direction. In region I, we find that the bottom boundary condition at  $\hat{z} = 0$  is described by equation (2.6). Thus,

$$-\hat{k}(\widehat{T}^{mlI})\left(\nabla\widehat{T}^{mlI}\cdot\mathbf{n_2}\right) = h_{tin}\left[\widehat{T}^{mlI}-\widehat{T}^{mlI}_{tin}\right] + \widehat{S}.$$
 (2.44)

In region III, we find that the top boundary condition at  $\hat{z} = \hat{H}_3 + Ac_3(x, y)$  is described by equation (2.4), which corresponds to the top boundary condition of the pure system:

$$-\hat{k}(\widehat{T}^{mlIII})\left(\nabla\widehat{T}^{mlIII}\cdot\mathbf{n_1}\right) = h_a\left[\widehat{T}^{mlIII}-\widehat{T}_a^{mlIII}\right]+\hat{s}.$$
 (2.45)

The other four boundary conditions come from having perfect thermal contact between region I and II and between region II and III. Thus, we expect to have continuity of temperature and continuity of heat flux at these boundaries.

Continuity of temperature between region I and II and region II and III at  $\hat{z} = \hat{H}_1 + Ac_1(x, y)$  and  $\hat{z} = \hat{H}_2 + Ac_2(x, y)$ , respectively gives us

$$\widehat{T}^{mlI} = \widehat{T}^{mlII}, \qquad (2.46)$$

$$\widehat{T}^{mlII} = \widehat{T}^{mlIII}. \tag{2.47}$$

Continuity of heat flux between region I and II and region II and III at  $\hat{z} = \hat{H}_1 + Ac_1(\hat{x}, \hat{y})$  and  $\hat{z} = \hat{H}_2 + Ac_2(\hat{x}, \hat{y})$ , respectively gives us

$$\hat{k}(\nabla \widehat{T}^{mlI} \cdot \mathbf{n_3}) = \hat{k}^{mlII}(\nabla \widehat{T}^{mlII} \cdot \mathbf{n_4}), \qquad (2.48)$$

$$\hat{k}^{mlII}(\nabla \widehat{T}^{mlII} \cdot \mathbf{n_5}) = \hat{k}(\nabla \widehat{T}^{mlIII} \cdot \mathbf{n_6}), \qquad (2.49)$$

where

$$\mathbf{n_3} = \frac{\left\langle -\frac{\partial \hat{c_1}}{\partial \hat{\mathbf{x}}} - \frac{\partial \hat{c_1}}{\partial \hat{\mathbf{y}}}, \mathbf{1} \right\rangle}{\sqrt{\left(-\frac{\partial \hat{c_1}}{\partial \hat{\mathbf{x}}}\right)^2 + \left(-\frac{\partial \hat{c_1}}{\partial \hat{\mathbf{y}}}\right)^2 + \mathbf{1}}},\tag{2.50}$$

$$\mathbf{n_4} = \frac{\left\langle -\frac{\partial \hat{\mathbf{c}_2}}{\partial \hat{\mathbf{x}}}, -\frac{\partial \hat{\mathbf{c}_2}}{\partial \hat{\mathbf{y}}}, \mathbf{1} \right\rangle}{\sqrt{\left(-\frac{\partial \hat{\mathbf{c}_2}}{\partial \hat{\mathbf{x}}}\right)^2 + \left(-\frac{\partial \hat{\mathbf{c}_2}}{\partial \hat{\mathbf{y}}}\right)^2 + \mathbf{1}}},$$
(2.51)

$$\mathbf{n}_{5} = \frac{\left\langle -\frac{\partial \hat{c_{1}}}{\partial \hat{\mathbf{x}}}, -\frac{\partial \hat{c_{1}}}{\partial \hat{\mathbf{y}}}, 1 \right\rangle}{\sqrt{\left(-\frac{\partial \hat{c_{1}}}{\partial \hat{\mathbf{x}}}\right)^{2} + \left(-\frac{\partial \hat{c_{1}}}{\partial \hat{\mathbf{y}}}\right)^{2} + 1}},$$
(2.52)

$$\mathbf{n_6} = \frac{\left\langle -\frac{\partial \hat{\mathbf{c}_2}}{\partial \hat{\mathbf{x}}}, -\frac{\partial \hat{\mathbf{c}_2}}{\partial \hat{\mathbf{y}}}, \mathbf{1} \right\rangle}{\sqrt{\left(-\frac{\partial \hat{\mathbf{c}_2}}{\partial \hat{\mathbf{x}}}\right)^2 + \left(-\frac{\partial \hat{\mathbf{c}_2}}{\partial \hat{\mathbf{y}}}\right)^2 + \mathbf{1}}}.$$
(2.53)

It is also important to point out that the interfaces between region I and II as well as region II and III are assumed to be flat at leading order for simplicity.

#### 2.4 Nondimensionalization for the multi-layer system

The next step is the nondimensionalization process of the multi-layer system. The same scalings as for the pure-layer system are used (see Table 2.1); however, the superscripts mlI, mlII, and mlIII are used to indicate the three layers. Further, the radiation terms  $\epsilon R^{mlI}Q^{mlI}(T^{mlI})$ ,  $\epsilon R^{mlII}Q^{mlII}(T^{mlII})$ , and  $\epsilon R^{mlIII}Q^{mlIII}(T^{mlII})$  are neglected since we are dealing with thin plates. Further,  $T_t^{mlI} = T_t^{mlII} = T_t^{mlIII} = 0$ , since we are considering the steady-state problem. Therefore, the nondimensional multi-layer governing heat equations are:

for region I

$$\epsilon P \rho(T^{mlI}) C_p(T^{mlI}) T_x^{mlI} =$$

$$\epsilon^2 \left[ k(T^{mlI}) T_x^{mlI} \right]_x +$$

$$\epsilon^2 \left[ k(T^{mlI}) T_y^{mlI} \right]_y +$$

$$\left[ k(T^{mlI}) T_z^{mlI} \right]_z,$$
(2.54)

for region II

$$\epsilon P^{mlII} \rho^{mlII} (T^{mlII}) C_p^{mlII} (T^{mlII}) T_x^{mlII} = \qquad (2.55)$$

$$\epsilon^2 \left[ k^{mlII} (T^{mlII}) T_x^{mlII} \right]_x + \epsilon^2 \left[ k^{mlII} (T^{mlII}) T_y^{mlII} \right]_y + \left[ k^{mlII} (T^{mlII}) T_z^{mlII} \right]_z,$$

and for region III

$$\epsilon P \rho(T^{mlIII}) C_p(T^{mlIII}) T_x^{mlIII} =$$

$$\epsilon^2 \left[ k(T^{mlIII}) T_x^{mlIII} \right]_x +$$

$$\epsilon^2 \left[ k(T^{mlIII}) T_y^{mlIII} \right]_y +$$

$$\left[ k(T^{mlIII}) T_z^{mlIII} \right]_z.$$
(2.56)

This set of three coupled second order governing heat equations (2.54) - (2.56) requires six boundary conditions in the z-direction. As in the pure-layer system, the z coordinate needs to be nondimensionalized since the top of the molten glass is not flat. Thus, letting  $\hat{z} = \hat{H}_3 + Ac_3(\hat{x}, \hat{y})$ , we obtain

$$z = 1 + \frac{A}{H_3}c_3(x,y),$$

where A is the amplitude of the nonplanar correction to the surface shape. We take  $\frac{A}{\hat{H}_3} = O(\epsilon)$  where  $A \ll \hat{H}_3$ , and find the nondimensional form of the upper boundary to be

$$z = 1 + \epsilon c_3(x, y). \tag{2.57}$$

Notice that  $c_3(x, y)$  in the multi-layer system is represented as h(x, y) in the purelayer system. Thus,  $c_3(x, y) = h(x, y)$ .

The bottom boundary condition at z = 0 is

$$k(T^{mlI})T_z^{mlI} = \epsilon B_{tin} \left[ T^{mlI} - T_{tin} \right] + \epsilon R_{tin} \epsilon_{mtin} \left[ (T^{mlI})^4 - T_{tin}^4 \right].$$
(2.58)

The top boundary condition at  $z = 1 + \epsilon c_3(x, y)$  is

$$k(T^{mlIII}) \begin{cases} -\epsilon^{3} T_{x}^{mlIII} \frac{\frac{\partial c_{3}}{\partial x}}{\sqrt{\left[-\epsilon \frac{\partial c_{3}}{\partial x}\right]^{2} + \left[-\epsilon \frac{\partial c_{3}}{\partial y}\right]^{2} + 1}} & - (2.59) \\ \epsilon^{3} T_{y}^{mlIII} \frac{\frac{\partial c_{3}}{\partial x}}{\sqrt{\left[-\epsilon \frac{\partial c_{3}}{\partial x}\right]^{2} + \left[-\epsilon \frac{\partial c_{3}}{\partial y}\right]^{2} + 1}} & + \\ T_{z}^{mlIII} \frac{1}{\sqrt{\left[-\epsilon \frac{\partial c_{3}}{\partial x}\right]^{2} + \left[-\epsilon \frac{\partial c_{3}}{\partial y}\right]^{2} + 1}}} \\ -\epsilon B_{air} \left(T^{mlIII} - T_{air}^{mlIII}\right) - \epsilon R_{air} \epsilon_{mair} \left((T^{mlIII})^{4} - T_{air}^{4}\right). \end{cases}$$

For the continuity conditions, we determine the appropriate thickness for each region. Namely, between region I and II, and region II and III, we find  $\hat{z} = \hat{H}_1 + Ac_1(\hat{x}, \hat{y})$ and  $\hat{z} = \hat{H}_2 + Ac_2(\hat{x}, \hat{y})$ , respectively, leading to the nondimensional expression for z:  $z = \frac{H_1}{H_3} + \epsilon c_1(x, y)$  and  $z = \frac{H_2}{H_3} + \epsilon c_2(x, y)$ .

The boundary conditions for the continuity of temperature and continuity of heat flux at  $z = \frac{H_1}{H_3} + \epsilon c_1(x, y)$  and  $z = \frac{H_2}{H_3} + \epsilon c_2(x, y)$  respectively, are

$$T^{mlI} = T^{mlII}, (2.60)$$

$$T^{mlII} = T^{mlIII} \tag{2.61}$$

and

$$k_{r}^{mlI}k(T^{mlI}) \begin{cases} -\epsilon^{3}T_{x}^{mlI} \frac{\frac{\partial c_{2}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{2}}{\partial y}\right]^{2} + 1}} & - \quad (2.62) \\ \epsilon^{3}T_{y}^{mlI} \frac{\frac{\partial c_{2}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{2}}{\partial y}\right]^{2} + 1}} & + \\ T_{z}^{mlI} \frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{2}}{\partial y}\right]^{2} + 1}}} & = \\ k_{r}^{mlII}k^{mlII}(T^{mlII}) \begin{cases} -\epsilon^{3}T_{x}^{mlII} \frac{\frac{\partial c_{2}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{2}}{\partial y}\right]^{2} + 1}}} & - \\ \epsilon^{3}T_{y}^{mlII} \frac{\frac{\partial c_{2}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{2}}{\partial y}\right]^{2} + 1}}} & + \\ T_{z}^{mlII} \frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{2}}{\partial y}\right]^{2} + 1}}} \end{cases}$$

and

$$k_{r}^{mlII}k^{mlII}(T^{mlII}) \begin{cases} -\epsilon^{3}T_{x}^{mlII} \frac{\frac{\partial c_{1}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{1}}{\partial y}\right]^{2} + 1}} & - \quad (2.63) \\ \epsilon^{3}T_{y}^{mlII} \frac{\frac{\partial c_{1}}{\partial y}}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{1}}{\partial y}\right]^{2} + 1}} & + \\ T_{z}^{mlII} \frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{1}}{\partial y}\right]^{2} + 1}}} & = \\ k_{r}^{mlIII}k(T^{mlIII}) \begin{cases} -\epsilon^{3}T_{x}^{mlIII} \frac{\frac{\partial c_{1}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{1}}{\partial y}\right]^{2} + 1}}} & - \\ \epsilon^{3}T_{y}^{mlIII} \frac{\frac{\partial c_{1}}{\partial y}}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{1}}{\partial y}\right]^{2} + 1}}} & + \\ T_{z}^{mlIII} \frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{1}}{\partial y}\right]^{2} + 1}}} \end{cases}$$

All governing equations and boundary conditions listed in this chapter are written in general form. This means that we included all three dimensions, the length x, width y, and height z. Now we are going to summarize the assumptions for our model and write the governing equations together with boundary conditions in their non-dimensional form.

It is important to note that in our model we are considering a steady-state problem since the float glass does not change its properties while cooling. Also, the width y of the glass ribbon is neglected since the temperature is assumed to be constant across the y-direction of the glass ribbon. Furthermore, the interface between the tin and the glass is assumed to be perfectly flat, whereas, the interface between the air and the glass is assumed to be sinusoidal.

The governing equation for the pure-layer system can be written as

$$\epsilon P \rho(T) C_p(T) T_x = \epsilon^2 \left[ k(T) T_x \right]_x + \left[ k(T) T_z \right]_z \tag{2.64}$$

subject to the bottom and top boundary conditions at z = 0 and  $z = 1 + \epsilon h(x)$ , respectively

$$k(T)T_z = \epsilon B_{tin} \left[T - T_{tin}\right] + \epsilon R_{tin} \epsilon_{mtin} \left[T^4 - T_{tin}^4\right], \qquad (2.65)$$

$$k(T) \left\{ -\epsilon^{3} T_{x} \frac{\frac{\partial h}{\partial x}}{\sqrt{\left[-\epsilon \frac{\partial h}{\partial x}\right]^{2} + 1}} + (2.66) \right.$$
$$T_{z} \frac{1}{\sqrt{\left[-\epsilon \frac{\partial h}{\partial x}\right]^{2} + 1}} = -\epsilon B_{air} \left(T - T_{air}\right) - \epsilon R_{air} \epsilon_{mair} \left(T^{4} - T_{air}^{4}\right).$$

The governing equation for the multi-layer system can be written as for region I

$$\epsilon P \rho(T^{mlI}) C_p(T^{mlI}) T_x^{mlI} =$$

$$\epsilon^2 \left[ k(T^{mlI}) T_x^{mlI} \right]_x + \left[ k(T^{mlI}) T_z^{mlI} \right]_z, \qquad (2.67)$$

for region II

$$\epsilon P^{mlII} \rho^{mlII} (T^{mlII}) C_p^{mlII} (T^{mlII}) T_x^{mlII} = (2.68)$$
  
$$\epsilon^2 \left[ k^{mlII} (T^{mlII}) T_x^{mlII} \right]_x + \left[ k^{mlII} (T^{mlII}) T_z^{mlII} \right]_z,$$

and for region III

$$\epsilon P \rho(T^{mlIII}) C_p(T^{mlIII}) T_x^{mlIII} =$$

$$\epsilon^2 \left[ k(T^{mlIII}) T_x^{mlIII} \right]_x + \left[ k(T^{mlIII}) T_z^{mlIII} \right]_z$$
(2.69)

subject to the bottom and top boundary condition at z = 0 and  $z = 1 + \epsilon c_3(x)$ , respectively

$$k(T^{mlI})T_z^{mlI} = \epsilon B_{tin} \left[ T^{mlI} - T_{tin} \right] + \epsilon R_{tin} \epsilon_{mtin} \left[ (T^{mlI})^4 - T_{tin}^4 \right], \qquad (2.70)$$

$$k(T^{mlIII}) \begin{cases} -\epsilon^{3} T_{x}^{mlIII} \frac{\frac{\partial c_{3}}{\partial x}}{\sqrt{\left[-\epsilon \frac{\partial c_{3}}{\partial x}\right]^{2}+1}} & - \qquad (2.71) \\ T_{z}^{mlIII} \frac{1}{\sqrt{\left[-\epsilon \frac{\partial c_{3}}{\partial x}\right]^{2}+1}} \end{cases} = \\ -\epsilon B_{air} \left(T^{mlIII} - T_{air}^{mlIII}\right) - \epsilon R_{air} \epsilon_{mair} \left((T^{mlIII})^{4} - T_{air}^{4}\right). \end{cases}$$

For the multi-layer system, the interface between the pure-layer and the contaminated-layer is further assumed to be flat. Thus, the normal on every interface is defined as

$$\mathbf{n_3} = \langle 0, 0, -1 \rangle, \qquad (2.72)$$

$$\mathbf{n_4} = \langle 0, 0, 1 \rangle, \qquad (2.73)$$

$$\mathbf{n_5} = \langle 0, 0, -1 \rangle, \qquad (2.74)$$

$$\mathbf{n_6} = \langle 0, 0, 1 \rangle \,. \tag{2.75}$$

Also, the temperature and the heat fluxes between regions I and II and regions II and III at  $z = \frac{H_1}{H_3} + \epsilon c_1(x, y)$  and  $z = \frac{H_2}{H_3} + \epsilon c_2(x, y)$ , respectively, are continuous.

$$T^{mlI} = T^{mlII}, (2.76)$$

$$T^{mlII} = T^{mlIII}, (2.77)$$

and

$$k_{r}^{mlI}k(T^{mlI})\left\{-\epsilon^{3}T_{x}^{mlI}\frac{\frac{\partial c_{2}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2}+1}} - (2.78)\right.$$

$$T_{z}^{mlI}\frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2}+1}}\right\} = k_{r}^{mlII}k^{mlII}(T^{mlII})\left\{-\epsilon^{3}T_{x}^{mlII}\frac{\frac{\partial c_{2}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2}+1}} - T_{z}^{mlII}\frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2}+1}}\right\},$$

and

$$k_{r}^{mlII}k^{mlII}(T^{mlII}) \begin{cases} -\epsilon^{3}T_{x}^{mlII} \frac{\frac{\partial c_{1}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2}+1}} & - \end{cases}$$
(2.79)  
$$T_{z}^{mlII} \frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2}+1}} \end{cases} = k_{r}^{mlIII}k(T^{mlIII}) \begin{cases} -\epsilon^{3}T_{x}^{mlIII} \frac{\frac{\partial c_{1}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2}+1}} & - \\ T_{z}^{mlIII} \frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2}+1}} \end{cases}$$

#### CHAPTER III

## SOLUTION PROCEDURE FOR THE TWO-DIMENSIONAL HEAT TRANSFER PROBLEM

In this model, we shall study the leading order and up to and including the first order problem,  $O(\epsilon)$ , for the temperature T in both the pure-layer and multi-layer systems. Therefore, it is necessary to consider an asymptotic expansion of temperature T. The following asymptotic sequence shall be used based on the term comparison from Chapter 2.

 $1, \epsilon, \epsilon^2, \dots$ 

The asymptotic expansion for T is

$$T(x,z) = T_0(x,z) + \epsilon T_1(x,z) + \epsilon^2 T_2(x,z) + O(\epsilon^3).$$
(3.1)

Using the asymptotic expansion given by equation (3.1) together with the Taylor series expansion, we determine expressions for the thermal conductivity k(T), density  $\rho(T)$ , and specific heat  $C_p(T)$ :

$$k(T) = k(T_0 + (\epsilon T_1 + \epsilon^2 T_2))$$

$$= k(T_0) + \epsilon T_1 k'(T_0) + \epsilon^2 \left( T_2 k'(T_0) + T_1^2 \frac{k''(T_0)}{2} \right) + O(\epsilon^3),$$
(3.2)

$$\rho(T) = \rho(T_0 + (\epsilon T_1 + \epsilon^2 T_2))$$

$$= \rho(T_0) + \epsilon T_1 \rho'(T_0) + \epsilon^2 \left( T_2 \rho'(T_0) + T_1^2 \frac{\rho''(T_0)}{2} \right) + O(\epsilon^3),$$
(3.3)

$$C_{p}(T) = C_{p}(T_{0} + (\epsilon T_{1} + \epsilon^{2} T_{2}))$$

$$= C_{p}(T_{0}) + \epsilon T_{1}C'_{p}(T_{0}) + \epsilon^{2} \left(T_{2}C'_{p}(T_{0}) + T_{1}^{2}\frac{C''_{p}(T_{0})}{2}\right) + O(\epsilon^{3}).$$
(3.4)

Substituting equations (3.2) - (3.4) into the governing heat equation (2.64), we obtain

$$\epsilon P \left[ \rho(T_0) + \epsilon T_1 \rho'(T_0) + \epsilon^2 \left( T_2 \rho'(T_0) + T_1^2 \frac{\rho''(T_0)}{2} \right) \right] \quad (3.5)$$

$$\left[ C_p(T_0) + \epsilon T_1 C'_p(T_0) + \epsilon^2 \left( T_2 C'_p(T_0) + T_1^2 \frac{C''_p(T_0)}{2} \right) \right]$$

$$\left[ T_0 + \epsilon T_1 + \epsilon^2 T_2 \right]_x$$

$$= \epsilon^2 \left[ \left( k(T_0) + \epsilon T_1 k'(T_0) + \epsilon^2 \left( T_2 k'(T_0) + T_1^2 \frac{k''(T_0)}{2} \right) \right) \left( T_0 + \epsilon T_1 + \epsilon^2 T_2 \right)_x \right]_x$$

$$+ \left[ k(T_0) + \epsilon T_1 k'(T_0) + \epsilon^2 \left( T_2 k'(T_0) + \epsilon^2 \left( T_2 k'(T_0) + T_1^2 \frac{k''(T_0)}{2} \right) \right) \right]_z.$$

It is also necessary to consider asymptotic expansions of the boundary conditions. Therefore, we expand the following terms when z = 0 and  $z = 1 + \epsilon h(x)$  using Taylor series.

At z = 0:

$$k(T) = k(T_0(0) + (\epsilon T_1(0) + \epsilon^2 T_2(0)))$$

$$= k(T_0(0)) + \epsilon T_1(0)k'(T_0(0))$$

$$+ \epsilon^2 \left( T_2(0)k'(T_0(0)) + T_1^2(0)\frac{k''(T_0(0))}{2} \right) + O(\epsilon^3),$$
(3.6)

$$\rho(T) = \rho(T_0(0) + (\epsilon T_1(0) + \epsilon^2 T_2(0)))$$

$$= \rho(T_0(0)) + \epsilon T_1(0)\rho'(T_0(0))$$

$$+ \epsilon^2 \left( T_2(0)\rho'(T_0(0)) + T_1^2(0)\frac{\rho''(T_0(0))}{2} \right) + O(\epsilon^3),$$

$$C_p(T) = C_p(T_0(0) + (\epsilon T_1(0) + \epsilon^2 T_2(0)))$$

$$= C_p(T_0(0)) + \epsilon T_1(0)C'_p(T_0(0))$$

$$+ \epsilon^2 \left( T_2(0)C'_p(T_0(0)) + T_1^2(0)\frac{C''_p(T_0(0))}{2} \right) + O(\epsilon^3).$$
(3.7)
(3.7)

At  $z = 1 + \epsilon h(x)$ :

$$k(T) = k(T_0 + (\epsilon T_1 + \epsilon^2 T_2))$$

$$= k(T_0(1)) + \epsilon [hT_{0z}(1) + T_1(1)] k'(T_0(1))$$

$$+ \epsilon^2 \left[ \frac{h^2}{2} T_{0zz}(1) + hT_{1z}(1) + \frac{1}{2} (hT_{0z}(1) + T_1(1))^2 \right] k''(T_0(1)),$$
(3.9)

$$\rho(T) = \rho(T_0 + (\epsilon T_1 + \epsilon^2 T_2))$$

$$= \rho(T_0(1)) + \epsilon [hT_{0z}(1) + T_1(1)] \rho'(T_0(1))$$

$$+ \epsilon^2 \left[ \frac{h^2}{2} T_{0zz}(1) + hT_{1z}(1) + \frac{1}{2} (hT_{0z}(1) + T_1(1))^2 \right] \rho''(T_0(1)),$$
(3.10)

$$C_{p}(T) = C_{p}(T_{0} + (\epsilon T_{1} + \epsilon^{2} T_{2}))$$

$$= C_{p}(T_{0}(1)) + \epsilon \left[hT_{0z}(1) + T_{1}(1)\right] C_{p}'(T_{0}(1))$$

$$+ \epsilon^{2} \left[\frac{h^{2}}{2}T_{0zz}(1) + hT_{1z}(1) + \frac{1}{2} \left(hT_{0z}(1) + T_{1}(1)\right)^{2}\right] C_{p}''(T_{0}(1)),$$
(3.11)

$$(T_0 + \epsilon T_1 + \epsilon^2 T_2) = T_0(1) + \epsilon [hT_{0z}(1) + T_1(1)]$$

$$+ \epsilon^2 \left[ \frac{h^2}{2} T_{0zz}(1) + hT_{1z}(1) + T_2(1) \right] + O(\epsilon^3),$$

$$(3.12)$$

$$(T_0 + \epsilon T_1 + \epsilon^2 T_2)_z = T_{0z}(1) + \epsilon [hT_{0z}(1) + T_1(1)]_z$$

$$+ \epsilon^2 \left[\frac{h^2}{2} T_{0zz}(1) + hT_{1z}(1) + T_2(1)\right]_z + O(\epsilon^3),$$
(3.13)

$$(T_0 + \epsilon T_1 + \epsilon^2 T_2)_x = T_{0x}(1) + \epsilon [hT_{0z}(1) + T_1(1)]_x$$

$$+ \epsilon^2 \left[\frac{h^2}{2} T_{0zz}(1) + hT_{1z}(1) + T_2(1)\right]_x + O(\epsilon^3),$$
(3.14)

$$(T_0 + (\epsilon T_1 + \epsilon^2 T_2))^4 = T_0^4(1) + \epsilon 4 T_0^3(1) [h T_{0z}(1) T_1(1)]$$

$$+ \epsilon^2 [6T_0^2(1) (h T_{0z}(1) + T_1(1))]$$

$$+ \epsilon^2 \left[ 4T_0^3(1) \left( \frac{h^2}{2} T_{0zz}(1) + h T_{1z}(1) + T_2(1) \right) \right].$$

$$(3.15)$$

Substituting equations (3.6) into the bottom boundary condition (2.65) and equations (3.9) - (3.16) into the top boundary condition (2.66), we find the expression for the bottom boundary condition at z = 0 to be

$$\left[k(T_{0}) + \epsilon T_{1}k'(T_{0}) + \epsilon^{2} \left(T_{2}k'(T_{0}) + T_{1}^{2} \frac{k''(T_{0})}{2}\right)\right] \left[T_{0} + \epsilon T_{1} + \epsilon^{2}T_{2}\right]_{z} = (3.16)$$

$$\epsilon B_{tin} \left[T_{0} + \epsilon T_{1} + \epsilon^{2}T_{2} - T_{tin}\right] + \epsilon R_{tin}\epsilon_{mtin} \left[(T_{0} + (\epsilon T_{1} + \epsilon^{2}T_{2}))^{4} - T_{tin}^{4}\right],$$

and the top boundary condition at z = 1 to be

$$[k(T_{0}(1)) + \epsilon [hT_{0z}(1) + T_{1}(1)] k'(T_{0}(1)) + (3.17)$$

$$\epsilon^{2} \left[ \frac{h^{2}}{2} T_{0zz}(1) + hT_{1z}(1) + \frac{1}{2} (hT_{0z}(1) + T_{1}(1))^{2} \right] k''(T_{0}(1)) \right]$$

$$\left\{ \left[ T_{0z}(1) + \epsilon [hT_{0z}(1) + T_{1}(1)]_{z} + \epsilon^{2} \left[ \frac{h^{2}}{2} T_{0zz}(1) + hT_{1z}(1) + T_{2}(1) \right]_{z} \right]$$

$$\frac{1}{\sqrt{(-\epsilon \frac{\partial h}{\partial x})^{2} + 1}} - O(\epsilon^{3}) \right\} =$$

$$-\epsilon B_{air} [T_{0}(1) + \epsilon (hT_{0z}(1) + T_{1}(1)) - T_{air}] -$$

$$\epsilon R_{air} \epsilon_{mair} \left[ (T_{0} + (\epsilon T_{1} + \epsilon^{2} T_{2}))^{4} - T_{air}^{4} \right].$$

Notice also that we need to apply the binomial expansion on

$$\frac{1}{\sqrt{1 + \left(-\epsilon \frac{\partial h}{\partial x}\right)^2}}$$

to obtain

$$\frac{1}{\sqrt{1 + \left[\left(-\epsilon\frac{\partial h}{\partial x}\right)^2\right]}} = \left[1 - \epsilon^2 \frac{1}{2} \left(\left(\frac{\partial h}{\partial x}\right)^2\right) + O(\epsilon^4)\right].$$
(3.18)

Thus, the top boundary condition at z = 1 becomes

$$[k(T_{0}(1)) + \epsilon [hT_{0z}(1) + T_{1}(1)] k'(T_{0}(1)) + (3.19)$$

$$\epsilon^{2} \left[ \frac{h^{2}}{2} T_{0zz}(1) + hT_{1z}(1) + \frac{1}{2} (hT_{0z}(1) + T_{1}(1))^{2} \right] k''(T_{0}(1)) \right]$$

$$\left\{ \left[ T_{0z}(1) + \epsilon [hT_{0z}(1) + T_{1}(1)]_{z} + \epsilon^{2} \left[ \frac{h^{2}}{2} T_{0zz}(1) + hT_{1z}(1) + T_{2}(1) \right]_{z} \right]$$

$$\left[ 1 - \epsilon^{2} \frac{1}{2} \left( \left( \frac{\partial h}{\partial x} \right)^{2} \right) + O(\epsilon^{4}) \right] - O(\epsilon^{3}) \right\} =$$

$$-\epsilon B_{air} [T_{0}(1) + \epsilon (hT_{0z}(1) + T_{1}(1)) - T_{air}] -$$

$$\epsilon R_{air} \epsilon_{mair} \left[ (T_{0} + (\epsilon T_{1} + \epsilon^{2} T_{2}))^{4} - T_{air}^{4} \right].$$

To summarize, the governing equation for the pure system is given by equation (3.5) subject to the top and bottom boundary conditions given by equations (3.16) and (3.19), respectively.

Let us now consider the non-dimensional governing heat equations (2.67) - (2.69) for the multi-layer system. As stated at the beginning of this chapter, the same approach and hence the same asymptotic expansion is used, so

$$T^{mlI}(x,z) = T_0^{mlI}(x) + \epsilon T_1^{mlI}(x,z) + \epsilon^2 T_2^{mlI}(x,z) + O(\epsilon^3), \qquad (3.20)$$

$$T^{mlII}(x,z) = T_0^{mlII}(x) + \epsilon T_1^{mlII}(x,z) + \epsilon^2 T_2^{mlII}(x,z) + O(\epsilon^3), \qquad (3.21)$$

$$T^{mlIII}(x,z) = T_0^{mlIII}(x) + \epsilon T_1^{mlIII}(x,z) + \epsilon^2 T_2^{mlIII}(x,z) + O(\epsilon^3).$$
(3.22)

The non-dimensional governing heat equations are:

$$\epsilon P \rho (T_0^{mlI} + \epsilon T_1^{mlI} + \epsilon^2 T_2^{mlI}) C_p (T_0^{mlI} + \epsilon T_1^{mlI} + \epsilon^2 T_2^{mlI})$$
(3.23)  
$$\left[ T_0^{mlI} + \epsilon T_1^{mlI} + \epsilon^2 T_2^{mlI} \right]_x =$$
  
$$\epsilon^2 \left[ k (T_0^{mlI} + \epsilon T_1^{mlI} + \epsilon^2 T_2^{mlI}) \left[ T_0^{mlI} + \epsilon T_1^{mlI} + \epsilon^2 T_2^{mlI} \right]_x \right]_x +$$
  
$$\left[ k (T_0^{mlI} + \epsilon T_1^{mlI} + \epsilon^2 T_2^{mlI}) \left[ T_0^{mlI} + \epsilon T_1^{mlI} + \epsilon^2 T_2^{mlI} \right]_z \right]_z,$$

$$\epsilon P^{mlII} \rho^{mlII} (T_0^{mlII} + \epsilon T_1^{mlII} + \epsilon^2 T_2^{mlII}) C_p^{mlII} (T_0^{mlII} + \epsilon T_1^{mlII} + \epsilon^2 T_2^{mlII}) \quad (3.24)$$

$$\begin{bmatrix} T_0^{mlII} + \epsilon T_1^{mlII} + \epsilon^2 T_2^{mlII} \end{bmatrix}_x =$$

$$\epsilon^2 \left[ k^{mlII} (T_0^{mlII} + \epsilon T_1^{mlII} + \epsilon^2 T_2^{mlII}) \left[ T_0^{mlII} + \epsilon T_1^{mlII} + \epsilon^2 T_2^{mlII} \right]_x \right]_x +$$

$$\begin{bmatrix} k^{mlII} (T_0^{mlII} + \epsilon T_1^{mlII} + \epsilon^2 T_2^{mlII}) \left[ T_0^{mlII} + \epsilon T_1^{mlII} + \epsilon^2 T_2^{mlII} \right]_z \right]_z,$$

$$\epsilon P \rho (T_0^{mlIII} + \epsilon T_1^{mlIII} + \epsilon^2 T_2^{mlIII}) C_p (T_0^{mlIII} + \epsilon T_1^{mlIII} + \epsilon^2 T_2^{mlIII}) \qquad (3.25)$$

$$[T_0^{mlIII} + \epsilon T_1^{mlIII} + \epsilon^2 T_2^{mlIII}]_x =$$

$$\epsilon^2 \left[ k (T_0^{mlIII} + \epsilon T_1^{mlIII} + \epsilon^2 T_2^{mlIII}) \left[ T_0^{mlIII} + \epsilon T_1^{mlIII} + \epsilon^2 T_2^{mlIII} \right]_x \right]_x +$$

$$[k (T_0^{mlIII} + \epsilon T_1^{mlIII} + \epsilon^2 T_2^{mlIII}) \left[ T_0^{mlIII} + \epsilon T_1^{mlIII} + \epsilon^2 T_2^{mlIII} \right]_z \right]_z.$$

These coupled governing heat equations (3.23) - (3.25) are subject to six boundary conditions.

The bottom boundary condition at z = 0 is

$$\begin{bmatrix} k(T_0^{mlI}) + \epsilon T_1^{mlI} k'(T_0^{mlI}) + \epsilon^2 \left( T_2^{mlI} k'(T_0^{mlI}) + (T_1^{mlI})^2 \frac{k''(T_0^{mlI})}{2} \right) \end{bmatrix}$$
(3.26)  
$$\begin{bmatrix} T_0^{mlI} + \epsilon T_1^{mlI} + \epsilon^2 T_2^{mlI} \end{bmatrix}_z =$$
  
$$\epsilon B_{tin} \begin{bmatrix} T_0^{mlI} + \epsilon T_1^{mlI} + \epsilon^2 T_2^{mlI} - T_{tin} \end{bmatrix} +$$
  
$$\epsilon R_{tin} \epsilon_{mtin} \begin{bmatrix} (T_0^{mlI} + (\epsilon T_1^{mlI} + \epsilon^2 T_2^{mlI}))^4 - T_{tin}^4 \end{bmatrix},$$

where  $T_{tin}$  is the same temperature profile of the tin as in the pure-layer system and similarly  $B_{tin}$ ,  $R_{tin}$ , and  $\epsilon_{mtin}$  are as before. The top boundary condition at z = 1 is

$$\begin{bmatrix} k(T_0^{mlIII}) + \epsilon \left[ hT_{0z}^{mlIII} + T_1^{mlIII} \right] k'(T_0^{mlIII}) \quad (3.27) \\ + \epsilon^2 \left[ \frac{h^2}{2} T_{0zz}^{mlIII} + hT_{1z}^{mlIII} + \frac{1}{2} \left( hT_{0z}^{mlIII} + T_1^{mlIII} \right)^2 \right] k''(T_0) \end{bmatrix} \\ \left\{ \begin{bmatrix} T_{0z}^{mlIII} + \epsilon \left[ hT_{0z}^{mlIII} + T_1^{mlIII} \right]_z + \epsilon^2 \left[ \frac{h^2}{2} T_{0zz}^{mlIII} + hT_{1z}^{mlIII} + T_2^{mlIII} \right]_z \right] \\ \frac{1}{\sqrt{\left( -\epsilon \frac{\partial c_3}{\partial x} \right)^2 + 1}} - O(\epsilon^3) \right\} \\ = -\epsilon B_{air} \left[ T_0^{mlIII} + \epsilon \left( hT_{0z}^{mlIII} + T_1^{mlIII} \right) - T_{air} \right] \\ -\epsilon R_{air} \epsilon_{mair} \left[ (T_0^{mlIII} + (\epsilon T_1^{mlIII} + \epsilon^2 T_2^{mlIII}))^4 - T_{air}^4 \right], \end{cases}$$

where  $T_{air}$  is the same temperature profile of the air as in the pure-layer system and similarly  $B_{air}$ ,  $R_{air}$ , and  $\epsilon_{mair}$  are as before.

As in equation (3.18), the binomial expansion gives

$$\frac{1}{\sqrt{1 + \left[\left(-\epsilon\frac{\partial c_3}{\partial x}\right)^2\right]}} = \left[1 - \epsilon^2 \frac{1}{2} \left(\left(\frac{\partial c_3}{\partial x}\right)^2\right) + O(\epsilon^4)\right].$$
(3.28)

Continuity of temperature between regions I and II, and regions II and III at  $z = \frac{H_1}{H_3}$ and  $z = \frac{H_2}{H_3}$ , respectively yields

$$T_0^{mlI} + \epsilon T_1^{mlI} + \epsilon^2 T_2^{mlI} = T_0^{mlII} + \epsilon T_1^{mlII} + \epsilon^2 T_2^{mlII}, \qquad (3.29)$$

$$T_0^{mlII} + \epsilon T_1^{mlII} + \epsilon^2 T_2^{mlII} = T_0^{mlIII} + \epsilon T_1^{mlIII} + \epsilon^2 T_2^{mlIII}, \qquad (3.30)$$

continuity of heat flux between region I and II, and region II and III at  $z = \frac{H_1}{H_3}$  and  $z = \frac{H_2}{H_3}$ , respectively are

$$\begin{aligned} &k(T_{0}^{mlI} + \epsilon T_{1}^{mlI} + \epsilon^{2}T_{2}^{mlI}) \qquad (3.31) \\ \begin{cases} -\epsilon^{3}(T_{0}^{mlI} + \epsilon T_{1}^{mlI} + \epsilon^{2}T_{2}^{mlI})_{x} \frac{\frac{\partial c_{2}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2} + \left[-\epsilon\frac{\partial c_{2}}{\partial y}\right]^{2} + 1}} & - \\ & (T_{0}^{mlI} + T_{1}^{mlI} + \epsilon^{2}T_{2}^{mlI})_{z} \frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2} + 1}} \\ & k_{r}^{*}k^{mlII}(T_{0}^{mlII} + \epsilon T_{1}^{mlII} + \epsilon^{2}T_{2}^{mlII}) \\ & \left\{ -\epsilon^{3}(T_{0}^{mlII} + \epsilon T_{1}^{mlII} + \epsilon^{2}T_{2}^{mlII})_{x} \frac{\frac{\partial c_{2}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2} + 1}} \right. \\ & (T_{0}^{mlII} + \epsilon T_{1}^{mlII} + \epsilon^{2}T_{2}^{mlII})_{z} \frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{2}}{\partial x}\right]^{2} + 1}} \\ \end{cases}$$

where

$$k_r^* = \frac{k_r^{mlI}}{k_r^{mlII}},\tag{3.32}$$

$$\frac{1}{\sqrt{1 + \left[\left(-\epsilon\frac{\partial c_2}{\partial x}\right)^2\right]}} = \left[1 - \frac{\epsilon^2}{2}\left(\left(\frac{\partial c_2}{\partial x}\right)^2 + \right) + O(\epsilon^4)\right],\tag{3.33}$$

and

$$k^{mlII}(T_{0}^{mlII} + \epsilon T_{1}^{mlII} + \epsilon^{2}T_{2}^{mlII}) \qquad (3.34)$$

$$\left\{ -\epsilon^{3}(T_{0}^{mlII} + \epsilon T_{1}^{mlII} + \epsilon^{2}T_{2}^{mlII})_{x} \frac{\frac{\partial c_{1}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2} + 1}} - (T_{0}^{mlII} + \epsilon T_{1}^{mlII} + \epsilon^{2}T_{2}^{mlII})_{z} \frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2} + 1}}} \right\} = k_{r}^{**}k(T_{0}^{mlIII} + \epsilon T_{1}^{mlIII} + \epsilon^{2}T_{2}^{mlIII}) \\ \left\{ -\epsilon^{3}(T_{0}^{mlIII} + \epsilon T_{1}^{mlIII} + \epsilon^{2}T_{2}^{mlIII})_{x} \frac{\frac{\partial c_{1}}{\partial x}}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2} + 1}} - (T_{0}^{mlIII} + \epsilon T_{1}^{mlIII} + \epsilon^{2}T_{2}^{mlIII})_{z} \frac{1}{\sqrt{\left[-\epsilon\frac{\partial c_{1}}{\partial x}\right]^{2} + 1}} \right\},$$

where

$$k_r^{**} = \frac{k_r^{mlIII}}{k_r^{mlII}},$$
(3.35)

$$\frac{1}{\sqrt{1 + \left[\left(-\epsilon \frac{\partial c_1}{\partial x}\right)^2 + \right]}} = \left[1 - \frac{\epsilon^2}{2} \left(\left(\frac{\partial c_1}{\partial x}\right)^2\right) + O(\epsilon^4)\right].$$
(3.36)

## 3.1 Leading order problem for the pure-layer system

From equation (3.5), we collect the leading order terms. The governing heat equation at O(1) for the pure-layer system becomes

$$[k(T_0)T_{0z}]_z = 0. (3.37)$$

This O(1) governing heat equation, (3.37), is subject to bottom and top boundary conditions at z = 0 and z = 1, respectively

$$k(T_0(z=0))T_{0z}(z=0) = 0, (3.38)$$

$$k(T_0(z=1))T_{0z}(z=1) = 0.$$
(3.39)

Integrating equation (3.37) with respect to z and applying boundary conditions (3.38) and (3.39), we find the solution at leading order,

$$T_0(x) = T_0. (3.40)$$

(Refer to Appendix C for an extension of the problem to three dimensions. An equivalent expression for  $T_0$  is then determined in x and y. Hence,  $T_0$  is a solution of a partial differential equation in a three-dimensional case).

From equation (3.40), we observe that the temperature  $T_0$  is independent of z, thus it is a function of x only. Since the exact temperature  $T_0$  at leading order could not be determined, we go to the next order, the first correction, to calculate  $T_0$ .

#### 3.2 First order problem for the pure-layer system

Collecting the first order terms,  $O(\epsilon)$ , from the governing heat equation (3.5), we write the governing heat equation for the pure-layer system as

$$P\rho(T_0)C_p(T_0)T_{0x} = k(T_0)T_{1zz} + T_1k'(T_0)T_{0zz},$$
(3.41)
subject to the following bottom and top boundary conditions, at z = 0 and z = 1,

$$k(T_0(0))T_{1z}(0) + T_{0z}(0)T_1(0)k'(T_0(0)) =$$

$$B_{tin} \left[T_0(0) - T_{tin}\right] + \epsilon_{mtin}R_{tin} \left[T_0^4(0) - T_{tin}^4\right],$$

$$k(T_0(1))T_{1z}(1) = -B_{air} \left[T_0(1) - T_{air}\right] - \epsilon_{mair}R_{air} \left[T_0^4(1) - T_{air}^4\right].$$
(3.42)
$$(3.43)$$

Notice that

$$T_{0z}T_1k'(T_0) = 0,$$
  
 $T_1k'(T_0)T_{0zz} = 0,$   
 $T_0(x) = T_0,$ 

from the leading order problem. Thus, the governing heat equation (3.41) at  $O(\epsilon)$  simplifies to

$$P\rho(T_0)C_p(T_0)T_{0x} = k(T_0)T_{1zz}, (3.44)$$

where the left hand side is independent of z since  $T_0(x)$  is independent of z. Let

$$C(x) = P\rho(T_0)C_p(T_0)T_{0x},$$
(3.45)

which is a function of x only. Therefore, the  $O(\epsilon)$  governing heat equation (3.44) can be rewritten as

$$C(x) = k(T_0)T_{1zz}.$$
(3.46)

Notice that the governing equation for  $T_1$  depends on the leading order temperature  $T_0$ .

Integrating equation (3.46) twice with respect to z, we obtain an expression for  $T_1$  which is parabolic in z (through the thickness)

$$T_1 = \frac{1}{k(T_0)} \left[ \frac{C(x)}{2} z^2 + D(x) z + E(x) \right], \qquad (3.47)$$

where D(x) and E(x) are unknown functions of x. Here, we can observe that the temperature  $T_1$  is a function of x and z,  $T_1 = T_1(x, z)$ .

After integrating the governing heat equation (3.46) once, we find

$$k(T_0)T_{1z} = C(x)z + D(x).$$
(3.48)

After applying boundary conditions at z = 0 and z = 1 to equation (3.48) and subtracting them, we determine D(x) to be

$$D(x) = B_{tin} \left[ T_0 - T_{tin} \right] + \epsilon_{mtin} R_{tin} \left[ T_0^4 - T_{tin}^4 \right], \qquad (3.49)$$

as well as the following ordinary differential equation for  $T_0$ 

$$T_{0x} = \frac{1}{P\rho(T_0)C_p(T_0)} \left\{ -B_{air} \left[ T_0 - T_{air} \right] - \epsilon_{mair} R_{air} \left[ T_0^4 - T_{air}^4 \right] - (3.50) \right. \\ \left. B_{tin} \left[ T_0 - T_{tin} \right] - \epsilon_{mtin} R_{tin} \left[ T_0^4 - T_{tin}^4 \right] \right\}.$$

(Refer to Appendix C for an extension to the three-dimensional problem. An equivalent partial differential equation in x and y is found).

Equation (3.50) is a non-linear, first order differential equation. It will be solved numerically using the fourth order Runge-Kutta method. Knowing  $T_{0x}$ , we can fully write an expression for equation (3.45) as

$$C(x) = -B_{air}(T_0 - T_{air}) - \epsilon_{mair}R_{air}(T_0^4 - T_{air}^4)$$

$$- B_{tin}(T_0 - T_{tin}) - \epsilon_{mtin}R_{tin}(T_0^4 - T_{tin}^4).$$
(3.51)

As it can be seen, we need to go to  $O(\epsilon)$  to calculate  $T_0$ . However, we did not fully determine  $T_1$  at  $O(\epsilon)$ , thus we need to go to the second order,  $O(\epsilon^2)$ , to find an expression for E(x). Once, E(x) is determined, the complete temperature profile for  $T_1$  will be known.

3.3 Higher order problem for the pure-layer system

Recall that

$$T_1(x,z) = \frac{1}{k(T_0)} \left[ \frac{C(x)}{2} z^2 + D(x)z + E(x) \right], \qquad (3.52)$$

where C(x) is defined by equations (3.45) and (3.51), D(x) is defined by equation (3.49), and E(x) is unknown.

Hence the temperature profile is

$$T(x,z) = T_0(x) + \epsilon \left[ \frac{1}{k(T_0)} \left( \frac{C(x)}{2} z^2 + D(x) z + E(x) \right) \right] + O(\epsilon^2).$$
(3.53)

From the governing heat equation (3.5), we write the  $O(\epsilon^2)$  governing heat equation for the pure-layer system as

$$P\left[\rho(T_0)C_p(T_0)T_{1x} + T_{0x}T_1C'_p(T_0)\rho(T_0) + T_{0x}T_1\rho'(T_0)C_p(T_0)\right] = (3.54)$$
$$[k(T_0)T_{0x}]_x + k(T_0)T_{2zz} + T_{1zz}T_{1z}k'(T_0),$$

subject to the bottom and top boundary conditions at z = 0 and z = 1, respectively. This yields

$$k(T_0)T_{2z} = \left[\frac{B_{tin}}{k(T_0)} + \epsilon_{mtin}R_{tin}\frac{4T_0^3}{k(T_0)}\right]E(x)$$
(3.55)

and

$$k(T_0)T_{2z} = \left[\frac{C(x)}{2} + D(x)\right] \left[\frac{-B_a}{k(T_0)} - R_a \epsilon_{ma} \frac{4T_0^3}{k(T_0)}\right] - h(x)C(x) \quad (3.56)$$

$$+ \left[\left(\frac{C(x)}{2}\right)^2 + C(x)D(x) + (D(x))^2\right] \left[\frac{-k'(T_0)}{(k(T_0))^2} - \frac{k''(T_0)}{2(k(T_0))^2}\right]$$

$$+ \left\{\left(\frac{-B_a}{k(T_0)} - R_a \epsilon_{ma} \frac{4T_0^3}{k(T_0)}\right)$$

$$- \left(\frac{C(x)}{2} + D(x)\right) \left(\frac{-k'(T_0)}{(k(T_0))^2} - \frac{k''(T_0)}{2(k(T_0))^2}\right)\right\} E(x)$$

$$= K_1(x) + K_2(x)E(x),$$

where  $h(x) = a \sin(\frac{2\pi x}{\lambda})$  was taken from Charnock's paper [6] and it describes the uneven top surface of the glass ribbon. Also, it is assumed that the amplitude a is considerably smaller than the wavelength  $\lambda$ ,  $a \ll \lambda$ .

Substituting expressions for  $T_1$ ,  $T_{1z}$ ,  $T_{1zz}$  and  $T_{1x}$  in equation (3.54) and integrating with respect to z, we observe that an expression for  $T_2$  is cubic in the variable z.

Applying the bottom and top boundary conditions at z = 0 and z = 1, respectively, we determine that E(x) satisfies

$$\frac{\partial E}{\partial x} + \left[-U_1(x)\right] E(x) = U_2(x), \qquad (3.57)$$

where  $U_1(x)$  and  $U_2(x)$  are known functions.

We obtain a linear, first order ordinary differential equation. Equation (3.57) is solved using the fourth order Runge-Kutta method. It seems like the differential equation in (3.57) can be solved analytically since it is linear, first order; however,  $U_1(x)$  and  $U_2(x)$  are very complicated expressions, so it is easier to solve using the Runge-Kutta method. Further, we also need to determine a boundary condition for E(x) by analyzing  $T_1(x, z)$ . From our assumptions, we know the temperature at the beginning of our region of interest.

Consider

$$\frac{C(x=0)}{2}z^2 + D(x=0)z + E(x=0) = 0.$$
(3.58)

Analyzing expressions C(x = 0) and D(x = 0), we noticed that they are very small; thus, by linear independence, C(x = 0) and D(x = 0) can be neglected. This produces a boundary condition, E(x = 0) = 0, for equation (3.57).

In the case that C(x = 0) and D(x = 0) have larger values, the left boundary condition (at 1100°C) would not be satisfied and a boundary layer analysis would have to be applied at the left boundary condition. If C(x = 0) and D(x = 0) have larger values our current analysis would fail.

Once E(x) is solved numerically, the temperature profile

$$T(x,z) = T_0(x) + \epsilon T_1(x,z) + O(\epsilon^2)$$
(3.59)

where

$$T_1(x,z) = \left[\frac{C(x)}{2}z^2 + D(x)z + E(x)\right],$$
(3.60)

is completely determined.

## 3.4 Leading order problem for the multi-layer system

Here, we shall define and solve the leading order problem,  $O(\epsilon)$ , for the multilayer system. The same assumptions and approaches for determining the leading order problem, i.e. Taylor series expansion, are used as in the pure-layer system. Therefore, the O(1) governing heat equations for the multi-layer system become

$$\left[k(T_0^{mlI})T_{0z}^{mlI}\right]_z = 0, (3.61)$$

$$\left[k^{mlII}(T_0^{mlII})T_{0z}^{mlII}\right]_z = 0, (3.62)$$

$$\left[k(T_0^{mlIII})T_{0z}^{mlIII}\right]_z = 0, (3.63)$$

subject to the following boundary conditions

top boundary condition at z = 1

$$k(T_0^{mlIII})T_{0z}^{mlIII} = 0, (3.64)$$

bottom boundary condition at z = 0

$$k(T_0^{mlI})T_{0z}^{mlI} = 0, (3.65)$$

continuity of temperature between regions I and II and regions II and III at  $z = \frac{H_1}{H_3}$ and  $z = \frac{H_2}{H_3}$ , respectively

$$T_0^{mlI} = T_0^{mlII}, (3.66)$$

$$T_0^{mlII} = T_0^{mlIII}, (3.67)$$

continuity of heat flux between regions I and II and regions II and III at  $z = \frac{H_1}{H_3}$  and  $z = \frac{H_2}{H_3}$ , respectively

$$k(T_0^{mlI})T_{1z}^{mlI} = k_r^* k^{mlII} (T_0^{mlII})T_{1z}^{mlII}, aga{3.68}$$

$$k^{mlII}(T_0^{mlII})T_{1z}^{mlII} = k_r^{**}k(T_0^{mlIII})T_{1z}^{mlIII}, \qquad (3.69)$$

where  $k_r^*$  and  $k_r^{**}$  are defined in equations (3.32) and (3.35). Integrating equations (3.61) - (3.63) with respect to z and applying boundary conditions at prescribed locations, we find the following solutions at leading order:

$$e(x) = k_r^* k^{mlII} (T_0^{mlI}) T_{0z}^{mlI}, \qquad (3.70)$$

$$f(x) = k_r^{**} k(T_0^{mlII}) T_{0z}^{mlII}, \qquad (3.71)$$

$$g(x) = \frac{1}{k_r^*} k(T_0^{mlIII}) T_{0z}^{mlIII}.$$
(3.72)

From equations (3.70) - (3.72), we observe that the temperature between the layers, specifically between region I and II, and region II and III, is the same. This means

$$T_0^{mlI} = T_0^{mlII} = T_0^{mlIII} = T_0 = T_0(x), (3.73)$$

which is similar behavior to the pure-layer system. Therefore,  $T_0$  must be determined at the next order.

## 3.5 First order problem for the multi-layer system

Collecting the first order terms,  $O(\epsilon)$ , from the governing heat equations (3.23) - (3.25), we write the  $O(\epsilon)$  governing heat equations for region I, II, and III respectively

$$P\rho(T_{0}^{mlI})C_{p}(T_{0}^{mlI})T_{0x}^{mlI} = (3.74)$$

$$k(T_{0}^{mlI})T_{1zz}^{mlI} + T_{1}^{mlI}k'(T_{0}^{mlI})T_{0zz}^{mlI},$$

$$P^{mlII}\rho^{mlII}(T_{0}^{mlII})C_{p}^{mlII}(T_{0}^{mlII})T_{0x}^{mlII} = (3.75)$$

$$k^{mlII}(T_{0}^{mlII})T_{1zz}^{mlII} + T_{1}^{mlII}k'^{mlII}(T_{0}^{mlII})T_{0zz}^{mlII},$$

$$P\rho(T_{0}^{mlIII})C_{p}(T_{0}^{mlIII})T_{0zz}^{mlIII} = (3.76)$$

$$k(T_{0}^{mlIII})T_{1zz}^{mlIII} + T_{1}^{mlIII}k'(T_{0}^{mlIII})T_{0zz}^{mlIII},$$

subject to the following boundary conditions:

bottom boundary condition at z = 0

$$k(T_0^{mlI})T_{1z}^{mlI} = B_{tin} \left[ T_0^{mlI} - T_{tin} \right] + R_{tin}\epsilon_{mtin} \left[ (T_0^{mlI})^4 - T_{tin}^4 \right], \qquad (3.77)$$

top boundary condition at z = 1

$$k(T_0^{mlIII})T_{1z}^{mlIII} = -B_{air} \left[ T_0^{mlIII} - T_{air} \right] - R_{air} \epsilon_{mair} \left[ (T_0^{mlIII})^4 - T_{air}^4 \right], \quad (3.78)$$

continuity of temperature between region I and II and region II and III at  $z = \frac{H_1}{H_3}$ and  $z = \frac{H_2}{H_3}$ , respectively

$$T_1^{mlI} = T_1^{mlII}, (3.79)$$

$$T_1^{mlII} = T_1^{mlIII}, (3.80)$$

continuity of heat flux between region I and II and region II and III at  $z = \frac{H_1}{H_3}$  and  $z = \frac{H_2}{H_3}$ , respectively

$$k(T_0^{mlI})T_{1z}^{mlI} = k_r^* k^{mlII} (T_0^{mlII})T_{1z}^{mlII}, \qquad (3.81)$$

$$k(T_0^{mlII})T_{1z}^{mlII} = k_r^{**}k^{mlIII}(T_0^{mlIII})T_{1z}^{mlIII}.$$
(3.82)

Notice that

$$T_{0z}^{mlI} = 0,$$
 (3.83)  
 $T_{0z}^{mlIII} = 0,$   
 $T_{0z}^{mlIII} = 0,$ 

from the leading order problem in each region. Thus, the governing heat equations (3.74) - (3.76) simplify to

$$P\rho(T_0^{mlI})C_p(T_0^{mlI})T_{0x}^{mlI} = k(T_0^{mlI})T_{1zz}^{mlI}, \qquad (3.84)$$

$$P^{mlII}\rho^{mlII}(T_0^{mlII})C_p^{mlII}(T_0^{mlII})T_{0x}^{mlII} = k^{mlII}(T_0^{mlII})T_{1zz}^{mlII}, \qquad (3.85)$$

$$P\rho(T_0^{mlIII})C_p(T_0^{mlIII})T_{0x}^{mlIII} = k(T_0^{mlIII})T_{1zz}^{mlIII}, \qquad (3.86)$$

where the left hand side of these three equations (3.84) - (3.86) is independent of z. Thus, by following the same convention as in the pure-layer system, we define

$$w_1(x) = w_1 = P\rho(T_0^{mlI})C_p(T_0^{mlI})T_{0x}^{mlI}, \qquad (3.87)$$

$$w_2(x) = w_2 = P^{mlII} \rho^{mlII} (T_0^{mlII}) C_p^{mlII} (T_0^{mlII}) T_{0x}^{mlII}, \qquad (3.88)$$

$$w_3(x) = w_3 = P\rho(T_0^{mlIII})C_p(T_0^{mlIII})T_{0x}^{mlIII}.$$
(3.89)

Substituting equations (3.87) - (3.89) into the first order governing heat equations (3.84) - (3.86) and integrating with respect to z, we find

$$k(T_0^{mlI})T_{1z}^{mlI} = w_1 z + d_1(x), (3.90)$$

$$k^{mlII}(T_0^{mlII})T_{1z}^{mlII} = w_2 z + d_2(x), (3.91)$$

$$k(T_0^{mlIII})T_{1z}^{mlIII} = w_3 z + d_3(x).$$
(3.92)

Applying the top boundary condition at z = 1, equation (3.78), to the region III governing heat equation (3.92), we obtain an expression for the arbitrary function of integration,  $d_3(x)$  to be

$$d_3(x) = d_3 = -B_{air} \left[ T_0^{mlIII} - T_{air} \right] - R_{air} \epsilon_{mair} \left[ (T_0^{mlIII})^4 - T_{air}^4 \right] - w_3.$$
(3.93)

Applying the bottom boundary condition at z = 0, equation (3.77), to the region I governing heat equation (3.90), we determine an expression for the arbitrary function of integration,  $d_1(x)$  to be

$$d_1(x) = d_1 = B_{tin} \left[ T_0^{mlI} - T_{tin} \right] + R_{tin} \epsilon_{mtin} \left[ (T_0^{mlI})^4 - T_{tin}^4 \right].$$
(3.94)

Using the heat flux continuity conditions at  $z = \frac{H_1}{H_3}$  and  $z = \frac{H_2}{H_3}$ , equations (3.81) and (3.82), and applying them to equations (3.90) and (3.92), we obtain expressions for the arbitrary function of integration in region II,  $d_2(x)$ , from equations (3.91).

At 
$$z = \frac{H_1}{H_3}$$

$$d_2(x) = \frac{H_1}{H_3 k_r^{**}} w_1 - \frac{H_1}{H_3} w_2 + \frac{1}{k_r^{**}} \left[ B_{tin} (T_0^{ml} - T_{tin}) + \epsilon_{mtin} R_{tin} ((T^{ml})_0^4 - T_{tin}^4) \right]$$
(3.95)

and at  $z = \frac{H_2}{H_3}$ 

$$d_{2}(x) = \frac{-H_{2}}{H_{3}}w_{2} + w_{3}\left(\frac{H_{2}}{H_{3}k_{r}^{*}} - \frac{1}{k_{r}^{*}}\right)$$

$$- \frac{1}{k_{r}^{*}}\left(B_{air}(T_{0}^{ml} - T_{air}) + \epsilon_{mair}R_{air}((T^{ml})_{0}^{4} - T_{air}^{4})\right).$$
(3.96)

Setting equations (3.95) and (3.96) equal, we are able to produce the governing heat equation for the contaminated-layer temperature  $T_0^{ml}$ . The governing heat equation for the contaminated-layer system becomes

$$T_{0x}^{ml} = \frac{\frac{1}{k_r^*} \left\{ B_{air} \left[ T_0^{ml} - T_{air} \right] + \epsilon_{mair} R_{air} \left[ (T_0^{ml})^4 - T_{air}^4 \right] \right\}}{\frac{H_1 - H_2}{H_3} P \rho(T_0^{ml}) C_p(T_0^{ml}) + \beta M} + \frac{\frac{1}{k_r^{**}} \left\{ B_{tin} \left[ T_0^{ml} - T_{tin} \right] + \epsilon_{mtin} R_{tin} \left[ (T_0^{ml})^4 - T_{tin}^4 \right] \right\}}{\frac{H_1 - H_2}{H_3} P \rho(T_0^{ml}) C_p(T_0^{ml}) + \beta M},$$
(3.97)

where

$$\beta = P\rho(T_0)C_p(T_0),$$
 (3.98)

$$M = \frac{1}{k_r^{**}} \frac{k_r^* H_2 - H_3 k_r^{**}}{k_r^{**} k_r^* H_3} - \frac{H_1}{H_3 k_r^{**}}.$$
(3.99)

Recall that  $T_{air}$  and  $T_{tin}$  are the same air and tin temperature profiles as in the pure system. Also, from the leading order solution recall that  $T_0^{mlII} = T_0^{mlIII} = T_0$ , thus

$$T_{0x} = \frac{\frac{1}{k_r^*} \{ B_{air} [T_0 - T_{air}] + \epsilon_{mair} R_{air} [T_0^4 - T_{air}^4] \}}{\frac{H_1 - H_2}{H_3} P \rho(T_0) C_p(T_0) + \beta M}$$

$$+ \frac{\frac{1}{k_r^{**}} \{ B_{tin} [T_0 - T_{tin}] + \epsilon_{mtin} R_{tin} [T_0^4 - T_{tin}^4] \}}{\frac{H_1 - H_2}{H_3} P \rho(T_0) C_p(T_0) + \beta M},$$
(3.100)

where M and  $\beta$  are defined in equations (3.99) and (3.98).

This is a non-linear, first order differential equation that describes the temperature distribution in the x-direction, through the length, of the glass ribbon. Because of

the non-linearity, this equation will be solved numerically using the Runge-Kutta method. It is important to note that if  $k_r^* = k_r^{**} = 1$  in the multi-layer differential equation (3.100) at  $O(\epsilon)$ , then equation (3.100) becomes equation (3.50) from the pure-layer system at  $O(\epsilon)$ . Therefore, the contaminated layer does not exist and the two differential equations give the same temperature profile.

Also, notice that by integrating equations (3.90)-(3.92) with respect to z, we obtain the same parabolic profile for  $T_1$  as in the pure-layer. Hence,

$$T_1^{mlI}(x,z) = \frac{1}{k^{mlI}(T_0)} \left[ \frac{w_1(x)}{2} z^2 + d_1(x) z + n_1(x) \right], \qquad (3.101)$$

$$T_1^{mlII}(x,z) = \frac{1}{k^{mlII}(T_0)} \left[ \frac{w_2(x)}{2} z^2 + d_2(x) z + n_2(x) \right], \qquad (3.102)$$

$$T_1^{mlIII}(x,z) = \frac{1}{k^{mlIII}(T_0)} \left[ \frac{w_3(x)}{2} z^2 + d_3(x) z + n_3(x) \right].$$
(3.103)

By substituting expressions for  $w_2(x)$  and  $d_2(x)$ , calculating  $n_1(x)$  at next order and assuming  $k^{mlII}(T_0) = k(T_0) = 1$ , we obtain the same temperature profile for  $T_1^{ml}$  as in the pure-layer, equation (3.47).

Combining the temperature profile  $T_1^{ml}$  for regions I - III, we can represent  $T_1^{ml}$  as a piecewise defined function

$$T_1^{ml}(x,z) = \begin{cases} \frac{1}{k^{mlI}(T_0)} \left[ \frac{w_1(x)}{2} z^2 + d_1(x) z + n_1(x) \right] & \text{if } 0 \le z \le H_1, \\ \frac{1}{k^{mlII}(T_0)} \left[ \frac{w_2(x)}{2} z^2 + d_2(x) z + n_2(x) \right] & \text{if } H_1 < z < H_2, \\ \frac{1}{k^{mlIII}(T_0)} \left[ \frac{w_3(x)}{2} z^2 + d_3(x) z + n_3(x) \right] & \text{if } H_2 \le z \le H_3. \end{cases}$$
(3.104)

# 3.6 Higher order problem for the multi-layer system

In this higher order problem, we need to calculate an expression for  $n_2(x)$  in  $T_1^{ml}(x, z)$ . The same approach and assumptions were followed as in the pure-layer system. Hence, the temperature profile can be written as

$$T^{mlII}(x,z) = T_0^{mlII}(x) + \epsilon T_1^{mlII}(x,z) + O(\epsilon^2)$$

$$= T_0^{mlII}(x) + \epsilon \frac{1}{k^{mlII}(T_0)} \left[ \frac{w_2(x)}{2} z^2 + d_2(x)z + n_2(x) \right]$$

$$+ O(\epsilon^2).$$
(3.105)

Collecting the  $O(\epsilon^2)$  terms from governing equations (3.23) - (3.25) and boundary conditions, we find

$$P\rho(T_{0}^{mll})C_{p}(T_{0}^{mll})T_{1x}^{mlI} + PT_{0x}^{mlI}T_{1}^{mlI}C_{p}'(T_{0}^{mlI})\rho(T_{0}^{mlI}) + (3.106)$$

$$PT_{0x}^{mlI}T_{1}^{mlI}\rho'(T_{0}^{mlI})C_{p}(T_{0}^{mlI}) = [k(T_{0}^{mlI})T_{0x}^{mlI}]_{x} + k(T_{0}^{mlI})T_{2zz}^{mlI} + T_{1zz}^{mlI}T_{1z}^{mlI}k'(T_{0}^{mlI}),$$

$$P\rho(T_{0}^{mlII})C_{p}(T_{0}^{mlII})T_{1x}^{mlII} + PT_{0x}^{mlII}T_{1}^{mlII}C_{p}'(T_{0}^{mlII})\rho(T_{0}^{mlII}) + (3.107)$$

$$PT_{0x}^{mlII}T_{1}^{mlII}\rho'(T_{0}^{mlII})C_{p}(T_{0}^{mlII}) = [k(T_{0}^{mlII})T_{0x}^{mlII}]_{x} + k(T_{0}^{mlII})T_{2zz}^{mlII} + T_{1zz}^{mlII}T_{1z}^{mlII}k'(T_{0}^{mlII}),$$

$$P\rho(T_{0}^{mlIII})C_{p}(T_{0}^{mlIII})T_{1x}^{mlIII} + PT_{0x}^{mlIII}T_{1}^{mlIII}C_{p}'(T_{0}^{mlIII})\rho(T_{0}^{mlIII}) + (3.108)$$

$$PT_{0x}^{mlIII}T_{1}^{mlIII}\rho'(T_{0}^{mlIII})C_{p}(T_{0}^{mlIII}) = [k(T_{0}^{mlIII})T_{0x}^{mlIII}]_{x} + k(T_{0}^{mlIII})T_{2zz}^{mlIII}T_{1z}^{mlIII}K'(T_{0}^{mlIII}) + (3.108)$$

subject to the bottom and top boundary conditions at z = 0 and z = 1, respectively

$$k(T_0^{mlI})T_{2z}^{mlI} + T_1^{mlI}k'(T_0^{mlI})T_{1z}^{mlI} +$$

$$\left(T_2^{mlI}k'(T_0^{mlI}) + (T_1^{mlI})^2 \frac{k''(T_0^{mlI})}{2}\right)T_{0z}^{mlI} =$$

$$B_{tin} \left[T_1^{mlI} - T_{tin}\right] +$$

$$R_{tin}\epsilon_{mtin} \left[2(T_0^{mlI})^3T_1^{mlI} - T_{tin}^4\right]$$
(3.109)

and

$$k(T_0^{mlIII}) \left[ \frac{h^2}{2} T_{0zz}^{mlIII} + hT_{1z}^{mlIII} + T_2^{mlIII} \right]_z + (3.110)$$

$$\left[ hT_{0z}^{mlIII} + T_1^{mlIII} \right] k'(T_0^{mlIII}) \left[ hT_{0z}^{mlIII} + T_1^{mlIII} \right]_z + \left[ \frac{h^2}{2} T_{0zz}^{mlIII} + hT_{1z}^{mlIII} + \frac{1}{2} \left( hT_{0z}^{mlIII} + T_1^{mlIII} \right)^2 \right] k''(T_0^{mlIII}) T_{0zz}^{mlIII} = -B_{air} \left[ \left( hT_{0z}^{mlIII} + T_1^{mlIII} \right) - T_{air} \right] - R_{air} \epsilon_{mair} \left[ 4 (T_0^{mlIII})^3 \left( hT_{0z}^{mlIII} + T_1^{mlIII} \right) - T_{tin}^4 \right],$$

where  $h(x) = a \sin(\frac{2\pi x}{\lambda})$  as before.

Continuity of temperature and heat flux at  $z = \frac{H_1}{H_3}$  and  $z = \frac{H_2}{H_3}$  also apply. Please refer to equations (3.29) - (3.31), and (3.34) for the exact form of the boundary conditions.

Recall from the assumptions we made and the solution from the leading order problem, which simplify the governing heat equations (3.23) - (3.25) and boundary conditions (3.29), (3.30), (3.31), (3.34), (3.109), and (3.110)

$$k'(T_0^{mlII}) = k''(T_0^{mlII}) = 0, \qquad (3.111)$$

$$C'_p(T_0^{mlII}) = C''_p(T_0^{mlII}) = 0, \qquad (3.111)$$

$$\rho'(T_0^{mlII}) = \rho''(T_0^{mlII}) = 0, \qquad T_{0z}^{mlII} = T_{0zz}^{mlII} = 0, \qquad T_{0z}^{mlII} = T_{0zz}^{mlII} = 0, \qquad T_{0z}^{mlIII} = T_{0zz}^{mlIII} = 0, \qquad T_{0z}^{mlIII} = T_{0zz}^{mlIII} = 0.$$

Applying the bottom and top boundary conditions at z = 0 and z = 1 as well as the continuity conditions  $z = \frac{H_1}{H_3}$  and  $z = \frac{H_2}{H_3}$ , we determine an expression for  $n_2(x)$  and hence an expression for  $T_1^{mlII}(x, z)$ . The expressions found are quite lengthy, thus we omit them. Refer to the numerical simulation and results section of this thesis for more details about the calculations.

## CHAPTER IV

# FORMULATION OF THE THERMAL STRESS AND DISPLACEMENT PROBLEM

Next we determine three-dimensional thermal stresses and displacements in the glass ribbon that appear due to nonuniform heating. Note that all the displacements and thermal stresses will be determined via a combination of asymptotic analysis and numerical simulations.

From [18], we obtain the basic equations of equilibrium in terms of displacements. For simplicity we assume a plane strain model since the heating of the glass is uniform across the width of the ribbon. This requires us to assume that there is no displacement along the  $\hat{y}$ -direction, thus,  $\hat{\epsilon}_{\hat{y}} = 0$ . However, the stresses in the  $\hat{y}$ -direction,  $\hat{\sigma}_{\hat{y}}$ , do not vanish in order to maintain the plane strain condition. Note further that the body forces are neglected and we assume that, except for the lateral surfaces, the boundary remains stress-free.

In what follows, variable subscripts denote derivatives with respect to that variable and scalar subscripts denote the order at which the asymptotic expansion is being solved.

# 4.1 Governing equations and boundary conditions for the thermal stress system

We begin with the general equilibrium stress equations

$$\frac{\partial \hat{\sigma}_{\hat{x}}}{\partial \hat{x}} + \frac{\partial \hat{\tau}_{\hat{x}\hat{y}}}{\partial \hat{y}} + \frac{\partial \hat{\tau}_{\hat{x}\hat{z}}}{\partial \hat{z}} = 0, \qquad (4.1)$$

$$\frac{\partial \hat{\sigma}_{\hat{x}}}{\partial \hat{x}} = \frac{\partial \hat{\tau}_{\hat{x}\hat{x}}}{\partial \hat{z}} = 0,$$

$$\frac{\partial \hat{\sigma}_{\hat{y}}}{\partial \hat{y}} + \frac{\partial \hat{\tau}_{\hat{x}\hat{y}}}{\partial \hat{x}} + \frac{\partial \hat{\tau}_{\hat{y}\hat{z}}}{\partial \hat{z}} = 0,$$

$$\frac{\partial \hat{\sigma}_{\hat{z}}}{\partial \hat{z}} + \frac{\partial \hat{\tau}_{\hat{x}\hat{z}}}{\partial \hat{x}} + \frac{\partial \hat{\tau}_{\hat{y}\hat{z}}}{\partial \hat{y}} = 0$$
(4.2)

and constitutive conditions

$$\hat{\epsilon}_{\hat{x}} - \alpha \left( \widehat{T} - T_{ref} \right) = \frac{1}{E} \left[ \hat{\sigma}_{\hat{x}} - \nu (\hat{\sigma}_{\hat{y}} + \hat{\sigma}_{\hat{z}}) \right], \qquad (4.3)$$

$$\hat{\epsilon}_{\hat{y}} - \alpha \left( \widehat{T} - T_{ref} \right) = \frac{1}{E} \left[ \hat{\sigma}_{\hat{y}} - \nu (\hat{\sigma}_{\hat{x}} + \hat{\sigma}_{\hat{z}}) \right], \qquad (4.4)$$
$$\hat{\epsilon}_{\hat{z}} - \alpha \left( \widehat{T} - T_{ref} \right) = \frac{1}{E} \left[ \hat{\sigma}_{\hat{z}} - \nu (\hat{\sigma}_{\hat{x}} + \hat{\sigma}_{\hat{y}}) \right].$$

Adding the set of equations (4.3), we find

$$\hat{e} = \frac{1}{E} (1 - 2\nu)\widehat{\Theta} + 3\alpha \left(\widehat{T} - T_{ref}\right), \qquad (4.5)$$

where  $\hat{e}$  is the sum of strains,  $\Theta$  the sum of the principal stresses,  $\alpha$  thermal expansion coefficient, E the Young's modulus,  $\lambda$  and  $\nu$  the Lame constants, G the shear modulus, and  $T_{ref}$  is the reference temperature. Further,

$$\hat{e} = \hat{\epsilon}_{\hat{x}} + \hat{\epsilon}_{\hat{y}} + \hat{\epsilon}_{\hat{z}}, \qquad (4.6)$$

$$\widehat{\Theta} = \hat{\sigma}_{\hat{x}} + \hat{\sigma}_{\hat{y}} + \hat{\sigma}_{\hat{z}}, \qquad (4.7)$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)},$$
(4.8)

$$G = \frac{E}{2(1+\nu)}.$$
 (4.9)

Recall we made an assumption about solving a plane strain problem along the  $\hat{y}$ direction, i.e.

$$\hat{u} = \hat{u}(x, z), \tag{4.10}$$

$$\hat{v} = 0, \qquad (4.11)$$

$$\hat{w} = \hat{w}(x, z) \tag{4.12}$$

hence,

$$\hat{\epsilon}_{\hat{y}} = 0, \qquad (4.13)$$

$$\hat{\gamma}_{\hat{x}\hat{y}} = \frac{\partial\hat{u}}{\partial\hat{y}} + \frac{\partial\hat{v}}{\partial\hat{x}} = 0, \qquad (4.14)$$

$$\hat{\gamma}_{\hat{z}\hat{y}} = \frac{\partial \hat{w}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{z}} = 0.$$

Here,  $\hat{u}$ ,  $\hat{v}$ , and  $\hat{w}$  are displacements.

However, the shear strain  $\hat{\gamma}$  in  $\hat{x}$  and  $\hat{z}$  is still present. Therefore,

$$\hat{\gamma}_{\hat{x}\hat{z}} = \frac{\partial \hat{u}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{x}}.$$
(4.15)

Using equation (4.5) and solving for stresses from the set of equations (4.3), we determine expressions for the thermal stresses,  $\hat{\sigma}_{\hat{x}}$  and  $\hat{\sigma}_{\hat{z}}$ , in terms of strains, and thus displacements. They are written as

$$\hat{\sigma}_{\hat{x}} = \lambda \hat{e} + 2G\hat{\epsilon}_{\hat{x}} - \frac{\alpha E\left(\hat{T} - T_{ref}\right)}{1 - 2\nu}, \qquad (4.16)$$
$$\hat{\sigma}_{\hat{z}} = \lambda \hat{e} + 2G\hat{\epsilon}_{\hat{z}} - \frac{\alpha E\left(\hat{T} - T_{ref}\right)}{1 - 2\nu}.$$

From equation (4.3) and since  $\hat{\epsilon}_{\hat{y}} = 0$ , we find

$$\hat{\sigma}_{\hat{y}} = \nu(\hat{\sigma}_{\hat{x}} + \hat{\sigma}_{\hat{z}}). \tag{4.17}$$

Substituting equations (4.16) and (4.17) into the stress equilibrium equations (4.1), we determine the governing equations in the dimensional form for the three-dimensional problem:

$$\begin{aligned} (\lambda + G)\frac{\partial \hat{e}}{\partial \hat{x}} + G\nabla^2 \hat{u} - \frac{\alpha E}{(1 - 2\nu)} \frac{\partial \left(\hat{T} - T_{ref}\right)}{\partial \hat{x}} &= 0, \end{aligned} \tag{4.18} \\ (\lambda + G)\frac{\partial \hat{e}}{\partial \hat{y}} + G\nabla^2 \hat{v} - \frac{\alpha E}{(1 - 2\nu)} \frac{\partial \left(\hat{T} - T_{ref}\right)}{\partial \hat{y}} &= 0, \end{aligned} \\ (\lambda + G)\frac{\partial \hat{e}}{\partial \hat{z}} + G\nabla^2 \hat{w} - \frac{\alpha E}{(1 - 2\nu)} \frac{\partial \left(\hat{T} - T_{ref}\right)}{\partial \hat{z}} &= 0. \end{aligned}$$

Boundary conditions without any surface forces can also be derived from equations (4.16) and (4.17) in the dimensional form:

,

$$\frac{\alpha E\left(\widehat{T}-T_{ref}\right)}{1-2\nu}l = \lambda \hat{e}l + G\left[\frac{\partial \hat{u}}{\partial \hat{x}}l + \frac{\partial \hat{u}}{\partial \hat{y}}m + \frac{\partial \hat{u}}{\partial \hat{z}}n\right] \qquad (4.19)$$

$$+ G\left[\frac{\partial \hat{u}}{\partial \hat{x}}l + \frac{\partial \hat{v}}{\partial \hat{x}}m + \frac{\partial \hat{w}}{\partial \hat{x}}n\right],$$

$$\frac{\alpha E\left(\widehat{T}-T_{ref}\right)}{1-2\nu}m = \lambda \hat{e}m + G\left[\frac{\partial \hat{v}}{\partial \hat{x}}l + \frac{\partial \hat{v}}{\partial \hat{y}}m + \frac{\partial \hat{v}}{\partial \hat{z}}n\right]$$

$$+ G\left[\frac{\partial \hat{u}}{\partial \hat{y}}l + \frac{\partial \hat{v}}{\partial \hat{y}}m + \frac{\partial \hat{w}}{\partial \hat{y}}n\right],$$

$$\frac{\alpha E\left(\widehat{T}-T_{ref}\right)}{1-2\nu}n = \lambda \hat{e}n + G\left[\frac{\partial \hat{w}}{\partial \hat{x}}l + \frac{\partial \hat{w}}{\partial \hat{y}}m + \frac{\partial \hat{w}}{\partial \hat{z}}n\right]$$

$$+ G\left[\frac{\partial \hat{u}}{\partial \hat{z}}l + \frac{\partial \hat{v}}{\partial \hat{z}}m + \frac{\partial \hat{w}}{\partial \hat{z}}n\right]$$

where l, m, and n are the components of the normal in the  $\hat{x}\hat{y}\hat{z}$ -system and correspond to the  $\hat{x}$ ,  $\hat{y}$ , and  $\hat{z}$  direction.

Making simplifications to the above governing and boundary condition equations in dimensional form, equations (4.18) and (4.19), and using the plane strain assumption,

equation (4.13), we find the governing equations pertinent to our model

$$(\lambda + G) \left[ \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{w}}{\partial \hat{x} \partial \hat{z}} \right] + G \left[ \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{z}^2} \right] - \frac{\alpha E}{(1 - 2\nu)} \frac{\partial \left( \hat{T} - T_{ref} \right)}{\partial \hat{x}} = 0, \quad (4.20)$$

$$(\lambda + G)\left[\frac{\partial^2 \hat{u}}{\partial \hat{x} \partial \hat{z}} + \frac{\partial^2 \hat{w}}{\partial \hat{x}^2}\right] + G\left[\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{w}}{\partial \hat{z}^2}\right] - \frac{\alpha E}{(1 - 2\nu)}\frac{\partial\left(1 - I_{ref}\right)}{\partial \hat{z}} = 0. \quad (4.21)$$

subject to the boundary conditions

at  $\hat{z} = 0, H$ :

$$\frac{\partial \hat{u}}{\partial \hat{z}} = -\frac{\partial \hat{w}}{\partial \hat{x}}, \qquad (4.22)$$
$$\frac{\alpha E\left(\hat{T} - T_{ref}\right)}{1 - 2\nu} = \lambda \frac{\partial \hat{u}}{\partial \hat{x}} + (\lambda + 2G) \frac{\partial \hat{w}}{\partial \hat{z}}$$

at  $\hat{x} = 0, L$ :

$$\frac{\partial \hat{u}}{\partial \hat{z}} = -\frac{\partial \hat{w}}{\partial \hat{x}}, \qquad (4.23)$$

$$\frac{\alpha E\left(\widehat{T} - T_{ref}\right)}{1 - 2\nu} = \lambda \frac{\partial \hat{w}}{\partial \hat{z}} + (\lambda + 2G) \frac{\partial \hat{u}}{\partial \hat{x}}.$$

#### 4.2 Nondimensionalization for the thermal stress system

The scalings in Table 4.1 are used to nondimensionalize the governing thermal stress equations (4.20) and (4.21) and the boundary conditions, equations (4.22) and (4.23). We assume that the reference temperature  $T_{ref}$  is equal to the temperature at the exit of the furnace, 873 K. Using the aspect ratio, equation (2.17), and the scalings in Tables 2.1 and 4.1, we find that the governing thermal stress equations,

Independent/Dependent variables	Scale	Units
$\hat{u}$	Lu	m
$\hat{w}$	Hw	mm

Table 4.1: Nondimensionalization table

(4.20) and (4.21), become

$$\epsilon^{2} \left(\lambda + 2G\right) \frac{\partial^{2} u}{\partial x^{2}} + \epsilon^{2} \left(\lambda + G\right) \frac{\partial^{2} w}{\partial x \partial z} + G \frac{\partial^{2} u}{\partial z^{2}} - \epsilon^{2} \frac{\alpha E T_{H}}{(1 - 2\nu)} \frac{\partial \left(T - T_{ref}\right)}{\partial x} = 0, \quad (4.24)$$
$$\left(\lambda + 2G\right) \frac{\partial^{2} w}{\partial z^{2}} + \left(\lambda + G\right) \frac{\partial^{2} u}{\partial x \partial z} + \epsilon^{2} G \frac{\partial^{2} w}{\partial x^{2}} - \frac{\alpha E T_{H}}{(1 - 2\nu)} \frac{\partial \left(T - T_{ref}\right)}{\partial z} = 0 \quad (4.25)$$

subject to the boundary conditions

at z = 0, 1:

$$\frac{\partial u}{\partial z} = -\epsilon^2 \frac{\partial w}{\partial x}, \tag{4.26}$$

$$\frac{\alpha ET_H}{1-2\nu} \left(T - T_{ref}\right) = \lambda \frac{\partial u}{\partial x} + \left(\lambda + 2G\right) \frac{\partial w}{\partial z}$$
(4.27)

at x = 0, 1:

$$\frac{\partial u}{\partial z} = -\epsilon^2 \frac{\partial w}{\partial x},\tag{4.28}$$

$$\frac{\alpha ET_H}{1-2\nu} \left(T - T_{ref}\right) = \lambda \frac{\partial w}{\partial z} + \left(\lambda + 2G\right) \frac{\partial u}{\partial x}.$$
(4.29)

From the heat transfer analysis, there is a boundary layer at the beginning and end of the region of interest. Therefore, the second boundary condition in x, equation (4.29), will not be used. Only the shear condition, equation (4.28), will be applied at x = 0, 1. We performed an asymptotic analysis on temperature T, equation (3.1); now the displacements and the thermal stresses will be expanded asymptotically as well. Therefore, the displacement u in the x-direction and the displacement w in the z-direction can be written as

$$u(x,z) = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + O(\epsilon^4)$$
(4.30)

and

$$w(x,z) = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + O(\epsilon^4), \qquad (4.31)$$

respectively.

The thermal stress equations in x-, y-, and z-direction can be written as

$$\sigma_x = (\lambda + 2G)u_x + \lambda w_z - \frac{\alpha ET_H}{1 - 2\nu} (T - T_{ref}), \qquad (4.32)$$

$$\sigma_y = \nu \left( \sigma_x + \sigma_z \right) - \alpha E \left( T - T_{ref} \right), \qquad (4.33)$$

$$\sigma_z = (\lambda + 2G)w_z + \lambda u_x - \frac{\alpha ET_H}{1 - 2\nu} \left(T - T_{ref}\right).$$
(4.34)

We formulated the governing displacement equations subject to the corresponding boundary conditions as well as the thermal stress governing equations.

# CHAPTER V

# SOLUTION PROCEDURE OF THE THERMAL STRESS AND DISPLACEMENT PROBLEM

Here, we will study the displacements u in x- and w in z-direction as well as the thermal stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$  up to and including the third order problem,  $O(\epsilon^3)$  in order to determine the thermal stresses at O(1) and  $O(\epsilon)$ . The asymptotic sequence has already been obtained in Chapter 4, equations (4.30) - (4.34).

Let us recall the nondimensional equilibrium equations in terms of displacements we obtained in Chapter 4

$$\epsilon^{2} \left(\lambda + 2G\right) \frac{\partial^{2} u}{\partial x^{2}} + \epsilon^{2} \left(\lambda + G\right) \frac{\partial^{2} w}{\partial x \partial z} + G \frac{\partial^{2} u}{\partial z^{2}} - \epsilon^{2} \frac{\alpha E T_{H}}{1 - 2\nu} \frac{\partial \left(T - T_{ref}\right)}{\partial x} = 0, \quad (5.1)$$

$$(\lambda + 2G)\frac{\partial^2 w}{\partial z^2} + (\lambda + G)\frac{\partial^2 u}{\partial x \partial z} + \epsilon^2 G \frac{\partial^2 w}{\partial x^2} - \frac{\alpha E T_H}{1 - 2\nu} \frac{\partial \left(T - T_{ref}\right)}{\partial z} = 0 \qquad (5.2)$$

subject to the boundary conditions

at z = 0, 1:

$$\frac{\partial u}{\partial z} = -\epsilon^2 \frac{\partial w}{\partial x},\tag{5.3}$$

$$\frac{\alpha E T_H}{1 - 2\nu} \left( T - T_{ref} \right) = \lambda \frac{\partial u}{\partial x} + \left( \lambda + 2G \right) \frac{\partial w}{\partial z}$$
(5.4)

at x = 0, 1:

$$\frac{\partial u}{\partial z} = -\epsilon^2 \frac{\partial w}{\partial x} \tag{5.5}$$

and

$$\sigma_x = (\lambda + 2G)u_x + \lambda w_z - \frac{\alpha ET_H}{1 - 2\nu} (T - T_{ref}), \qquad (5.6)$$

$$\sigma_y = \nu \left( \sigma_x + \sigma_z \right) - \alpha E \left( T - T_{ref} \right), \qquad (5.7)$$

$$\sigma_z = (\lambda + 2G)w_z + \lambda u_x - \frac{\alpha ET_H}{1 - 2\nu} (T - T_{ref}).$$
(5.8)

# 5.1 Leading order problem for the displacements

From the governing equilibrium equations (5.1) and (5.2) and the corresponding boundary condition equations (5.3) - (5.5), we collect the leading order terms to set up the O(1) problem. The governing equilibrium equations in terms of the displacement at O(1) then becomes,

$$Gu_{0zz} = 0, (5.9)$$

$$(\lambda + 2G)w_{0zz} + (\lambda + G)u_{0xz} = 0 (5.10)$$

subject to the boundary conditions

at z = 0, 1

$$u_{0z} = 0,$$
 (5.11)

$$\frac{\alpha E T_H}{1 - 2\nu} \left( T_0 - T_{ref} \right) = \lambda u_{0x} + (\lambda + 2G) w_{0z}$$
(5.12)

and at x = 0, 1

$$u_{0z} = 0.$$
 (5.13)

Integrating equations (5.9) and (5.10) with respect to z and applying boundary condition (5.11) - (5.13), we find the displacement in x- and z-direction to be

$$u_0 = u_0(x) = p_0(x),$$
 (5.14)

$$w_{0} = w_{0}(x, z) = \frac{\alpha E T_{H}}{(\lambda + 2G)(1 - 2\nu)} (T_{0}(x) - T_{ref}) z \qquad (5.15)$$
$$- \frac{\lambda}{\lambda + 2G} p_{0}'(x) z + h_{0}(x).$$

We observe that  $u_0$  is only a function of x and that  $w_0$  depends on the unknown functions  $p_0(x)$  and  $h_0(x)$ . Thus, we were not able to fully solve for the displacements and we must go to higher orders.

### 5.2 First order problem for the displacements

In this section, we solve for the displacements at  $O(\epsilon)$ . From the governing equilibrium equations (5.1) and (5.2) and the corresponding boundary condition equations (5.3) - (5.5), we collect the first order terms to set up the  $O(\epsilon)$  problem. The governing equilibrium equations in terms of the displacement at  $O(\epsilon)$  then become

$$Gu_{1zz} = 0, \qquad (5.16)$$

$$(\lambda + 2G)w_{1zz} - \frac{\alpha ET_H}{1 - 2\nu}T_{1z} = 0$$
(5.17)

subject to the boundary conditions

at z = 0, 1

$$u_{1z} = 0,$$
 (5.18)

$$\frac{\alpha ET_H}{1-2\nu}T_1 = \lambda u_{1x} + (\lambda + 2G)w_{1z}, \qquad (5.19)$$

and at x = 0, 1

$$u_{1z} = 0. (5.20)$$

Recall from Chapter 3 the expression for  $T_1$ , equation (3.47).

Integrating the governing equations (5.16) and (5.17) with respect to z and applying boundary conditions (5.18) - (5.20), we find expressions for the displacements in the x- and z-direction,

$$u_1 = u_1(x) = p_1(x),$$
 (5.21)

$$w_{1} = w_{1}(x, z) = \frac{\alpha ET_{H}}{(\lambda + 2G)(1 - 2\nu)} \left[ C(x)\frac{z^{3}}{6} + D(x)\frac{z^{2}}{2} + E(x)z \right] \quad (5.22)$$
$$- \frac{\lambda}{\lambda + 2G}p'_{1}(x)z + j_{1}(x).$$

Notice again that the only information we obtained from this order is that  $u_1$  is a function of x,  $p_1(x)$ .  $w_1$  depends on the unknown functions  $p_1(x)$  and  $j_1(x)$ . Having worked through  $O(\epsilon)$  analysis, we observe that the answers to O(1) problem will be determined at  $O(\epsilon^2)$  and answers to  $O(\epsilon)$  at  $O(\epsilon^3)$ . Therefore, we set up higher order problems.

#### 5.3 Second order problem for the displacements

In this section, we solve for the displacements at  $O(\epsilon^2)$ . From the governing equilibrium equations (5.1) and (5.2) and the corresponding boundary condition equations (5.3) - (5.5), we collect the second order terms to set up the  $O(\epsilon^2)$  problem. The governing equilibrium equations in terms of the displacement at  $O(\epsilon^2)$  then become

$$(\lambda + 2G)u_{0xx} + (\lambda + G)w_{0xz} + Gu_{2zz} - \frac{\alpha ET_H}{1 - 2\nu}T_{0x} = 0, \qquad (5.23)$$

$$(\lambda + 2G)w_{2zz} + (\lambda + G)u_{2xz} + Gw_{0xx} - \frac{\alpha ET_H}{1 - 2\nu}T_{2z} = 0$$
 (5.24)

subject to the boundary conditions

at z = 0, 1

$$u_{2z} + w_{0x} = 0, (5.25)$$

$$\frac{\alpha ET_H}{1 - 2\nu} T_2 = \lambda u_{2x} + (\lambda + 2G)w_{2z}$$
(5.26)

and at x = 0, 1

$$u_{2z} + w_{0x} = 0. (5.27)$$

Substituting expressions for  $u_{0xx}$ ,  $w_{0xx}$ , and  $w_{0xz}$  from O(1), equations (5.14) and (5.15), into the governing equations (5.23) and (5.24), we obtain governing equations that can be easily solved. Then integrating these equations with respect to z and applying boundary conditions, equations (5.25) - (5.27), we find an expression for  $p''_0(x)$  which after integration produces an expression for  $p_0(x)$ , thus  $u_0$ . We find

$$u_0(x) = \frac{\alpha E T_H}{2(\lambda + G)(1 - 2\nu)} \int_0^x \left( T_0(x) - T_{ref} \right) \, dx + Bx \tag{5.28}$$

where B = 0 is chosen not to allow rotation with respect to the x-axis at x = 60 m. We can also construct the solution for  $w_0$  as

$$w_{0}(x,z) = \frac{\alpha ET_{H}}{(\lambda + 2G)(1 - 2\nu)} (T_{0}(x) - T_{ref}) z$$

$$- \frac{\lambda \alpha ET_{H}}{2(\lambda + 2G)(\lambda + G)(1 - 2\nu)} (T_{0}(x) - T_{ref}) z + h_{0}(x)$$
(5.29)

where  $h_0(x)$  is still undetermined. As stated previously, we needed to go to  $O(\epsilon^2)$  to find displacements  $u_0(x)$  and  $w_0(x, z)$ .

#### 5.4 Third order problem for the displacements

In this section we will set up the  $O(\epsilon^3)$  problem, but obtain answers for the  $O(\epsilon)$  problem. From the governing equilibrium equations (5.1) and (5.2) and the corresponding boundary condition equations (5.3) - (5.5), we collect the third order terms to set up the  $O(\epsilon^3)$  problem. The governing equilibrium equations in terms of the displacement at  $O(\epsilon^3)$  then become

$$(\lambda + 2G)u_{1xx} + (\lambda + G)w_{1xz} + Gu_{3zz} - \frac{\alpha ET_H}{1 - 2\nu}T_{1x} = 0, \qquad (5.30)$$

$$(\lambda + 2G)w_{3zz} + (\lambda + G)u_{3xz} + Gw_{1xx} - \frac{\alpha ET_H}{1 - 2\nu}T_{3z} = 0$$
(5.31)

subject to boundary conditions

at z = 0, 1

$$u_{3z} + w_{1x} = 0, (5.32)$$

$$\frac{\alpha ET_H}{1 - 2\nu} T_3 = \lambda u_{3x} + (\lambda + 2G) w_{3z}$$
(5.33)

and at x = 0, 1

$$u_{3z} + w_{1x} = 0. (5.34)$$

We also assume that  $T_3 = 0$  for simplicity purposes.

Substituting expressions for  $u_{1xx}$ ,  $w_{1xx}$ , and  $w_{1xz}$  from O(1), equations (5.21) and (5.22), and  $T_{1x}$  from equation (3.47) into the governing equations (5.30) and (5.31),

we obtain governing equations that are ready to solve. Integrating these equations with respect to z and applying boundary conditions, equations (5.32) - (5.34), we find an expression for  $p_1''(x)$  which after integration produces an expression for  $p_1(x)$ , thus  $u_1$ . We find

$$u_1(x) = \frac{\alpha E T_H}{2(\lambda + G)(1 - 2\nu)} \int_0^1 \left(\frac{1}{6}C(x) + \frac{1}{2}D(x) + E(x)\right) dx , \quad (5.35)$$

$$w_{1}(x,z) = \frac{\alpha ET_{H}}{(\lambda + 2G)(1 - 2\nu)} \left[ \frac{1}{6} C(x) z^{3} + \frac{1}{2} D(x) z^{2} + E(x) z \right]$$
(5.36)  
$$- \frac{\lambda \alpha ET_{H}}{2(\lambda + 2G)(\lambda + G)(1 - 2\nu)} \left[ \frac{1}{6} C(x) + \frac{1}{2} D(x) + E(x) \right] z.$$

To summarize, we can represent the u and w displacements as

$$u(x) = \frac{\alpha ET_H}{2(\lambda + G)(1 - 2\nu)} \int_0^x (T_0(x) - T_{ref}) dx \qquad (5.37)$$
  
+  $\frac{\alpha ET_H}{2(\lambda + G)(1 - 2\nu)} \int_0^x \left(\frac{1}{6}C(x) + \frac{1}{2}D(x) + E(x)\right) dx ,$   
$$w(x, z) = \frac{\alpha ET_H}{(\lambda + 2G)(1 - 2\nu)} (T_0(x) - T_{ref}) z \qquad (5.38)$$
  
-  $\frac{\lambda \alpha ET_H}{2(\lambda + 2G)(\lambda + G)(1 - 2\nu)} (T_0(x) - T_{ref}) z$   
+  $h_0(x) + \frac{\alpha ET_H}{(\lambda + 2G)(1 - 2\nu)} \left[\frac{1}{6}C(x)z^3 + \frac{1}{2}D(x)z^2 + E(x)z\right]$   
-  $\frac{\lambda \alpha ET_H}{2(\lambda + 2G)(\lambda + G)(1 - 2\nu)} \left[\frac{1}{6}C(x) + \frac{1}{2}D(x) + E(x)\right] z$ 

where  $h_0(x)$  is an unknown function. We would have to go to higher orders to determine this function, but for simplicity of this thesis we decided to stop at  $O(\epsilon^3)$ . Hence,  $w_0$  is determined up to an unknown function of x.

The displacements calculated here represent the difference of displacements across the length and height of the glass ribbon. Note that the expression for u, equation (5.37), integrates temperature over small subdomains first, thus, we can say that the displacements accumulate.

#### 5.5 Leading order problem for the thermal stresses

Since we determined displacements at O(1) and  $O(\epsilon)$  previously in this chapter, we can define the thermal stresses at O(1) and  $O(\epsilon)$  as well. Recall the thermal stress governing equations (5.6) - (5.8). Thus,

$$\sigma_{0x} = (\lambda + 2G)u_{0x} + \lambda w_{0z} - \frac{\alpha ET_H}{1 - 2\nu} (T_0(x) - T_{ref}), \qquad (5.39)$$

$$\sigma_{0y} = \nu(\sigma_{0x} + \sigma_{0z}) - \alpha E \left( T_0(x) - T_{ref} \right), \qquad (5.40)$$

$$\sigma_{0z} = (\lambda + 2G)w_{0z} + \lambda u_{0x} - \frac{\alpha ET_H}{1 - 2\nu} (T_0(x) - T_{ref}).$$
 (5.41)

Substituting expressions for  $u_{0x}$  and  $w_{0z}$ , equations (5.28) and (5.29), into the thermal stress equations (5.39) - (5.41), we find

$$\sigma_{0x} = 0, \tag{5.42}$$

$$\sigma_{0y}(x) = -\alpha E \left( T_0(x) - T_{ref} \right), \qquad (5.43)$$

$$\sigma_{0z} = 0. \tag{5.44}$$

As we can see, only the imposed thermal stresses in the y-direction from the plane strain assumption survive at this order. We have to go to higher order to determine thermal stresses in the x- and z-directions.

# 5.6 First order problem for the thermal stresses

Collecting  $O(\epsilon)$  terms from the governing equations (5.6) - (5.8), we obtain

$$\sigma_{1x} = (\lambda + 2G)u_{1x} + \lambda w_{1z} - \frac{\alpha ET_H}{1 - 2\nu}T_1(x, z), \qquad (5.45)$$

$$\sigma_{1y} = \nu(\sigma_{1x} + \sigma_{1z}) - \alpha E T_H T_1(x, z), \qquad (5.46)$$

$$\sigma_{1z} = (\lambda + 2G)w_{1z} + \lambda u_{1x} - \frac{\alpha ET_H}{1 - 2\nu}T_1(x, z).$$
(5.47)

Substituting expressions for  $u_{1x}$  and  $w_{1z}$ , equations (5.35) and (5.36), into the thermal stress equations (5.39) - (5.41), we find

$$\sigma_{1x} = \frac{2G\alpha ET_H}{(\lambda + 2G)(1 - 2\nu)} \left[ \frac{1}{6}C(x) + \frac{1}{2}D(x) \right]$$
(5.48)  
$$- \frac{2G\alpha ET_H}{(\lambda + 2G)(1 - 2\nu)} \left[ \frac{1}{2}C(x)z^2 + D(x)z \right],$$
  
$$\sigma_{1y} = \frac{2G\nu\alpha ET_H}{(\lambda + 2G)(1 - 2\nu)} \left[ \frac{1}{6}C(x) + \frac{1}{2}D(x) \right]$$
(5.49)  
$$- \frac{2G\nu\alpha ET_H}{(\lambda + 2G)(1 - 2\nu)} \left[ \frac{1}{2}C(x)z^2 + D(x)z \right]$$
(5.49)  
$$- \alpha ET_H \left[ \frac{1}{2}C(x)z^2 + D(x)z + E(x) \right],$$

$$\sigma_{1z} = 0. \tag{5.50}$$

Combining O(1) and  $O(\epsilon)$  results, we determine

$$\sigma_{x} = \frac{2G\alpha ET_{H}}{(\lambda + 2G)(1 - 2\nu)} \left[ \frac{1}{6}C(x) + \frac{1}{2}D(x) \right]$$
(5.51)  

$$- \frac{2G\alpha ET_{H}}{(\lambda + 2G)(1 - 2\nu)} \left[ \frac{1}{2}C(x)z^{2} + D(x)z \right] ,$$
  

$$\sigma_{y} = -\alpha ET_{H}(T_{0}(x) - T_{ref}) + \frac{2G\nu\alpha ET_{H}}{(\lambda + 2G)(1 - 2\nu)} \left[ \frac{1}{6}C(x) + \frac{1}{2}D(x) \right]$$
(5.52)  

$$- \frac{2G\nu\alpha ET_{H}}{(\lambda + 2G)(1 - 2\nu)} \left[ \frac{1}{2}C(x)z^{2} + D(x)z \right]$$
  

$$- \alpha ET_{H} \left[ \frac{1}{2}C(x)z^{2} + D(x)z + E(x) \right] ,$$
  

$$\sigma_{z} = 0.$$
(5.53)

Notice the C(x), D(x), and E(x), equations (3.45), (3.49), (3.51), and (3.57), dependence in the above equations (5.51) and (5.52). From equations (3.45) and (3.51), we observe that C(x) is proportional to the longitudinal temperature gradient. C(x) also depends on the temperature of the air and tin, whereas D(x) only depends on the temperature of the tin. Thus, equations (3.51) and (3.49) influence the surface heat transfer in the system, which lead to the development of the thermal stresses. Note further, in the case that surface heat transfer is not present, i.e. C(x) = D(x) = 0, the longitudinal thermal stresses would not exist. Only the imposed thermal stresses due to plane strain assumption would survive.

For simplicity, thermal stresses in the contaminated layer are not considered. However, in order to determine stresses in the contaminated layer, one must change the material properties, i.e. Young's modulus, Poisson's ratio etc.

#### CHAPTER VI

#### HEAT TRANSFER RESULTS

In this section, we study the distribution of temperature in the glass ribbon for the one-layer and the multi-layer systems.

First, the non-linear, first order differential equation, (3.50), along the length x of the glass ribbon will be solved numerically by using the fourth order Runge-Kutta method. Here, the Peclet number P, Biot number for air and tin,  $B_{air}$  and  $B_{tin}$ , emissivity values for air and tin,  $\epsilon_{mair}$  and  $\epsilon_{mtin}$ , radiation number for air and tin,  $R_{air}$ and  $R_{tin}$  are considered as constants. Further, the thermal conductivity at leading order  $k(T_0)$ , heat capacity at leading order  $C_p(T_0)$ , and density at leading order  $\rho(T_0)$ are kept constant for both systems. Temperature of the air,  $T_{air}$ , and temperature of the tin,  $T_{tin}$  are taken as functions of x that were determined by curve fitting the data obtained from the literature [8, 9, 16, 25]. This data was nondimensionalized with the reference values for this thesis (60 m for the length x of the tin bath and 1373 K for the temperature T). After nondimensionalizing, polynomial interpolation was applied. It seemed reasonable to use cubic interpolation since it fit the individual points the best. Finally, we obtained the temperature profiles of the air and tin, respectively

$$T_{air}(x) = 0.342x^3 - 0.3175x^2 - 0.4792x + 0.9993, \tag{6.1}$$

$$T_{tin}(x) = -0.0654x^3 - 0.1363x^2 - 0.1695x + 1.109.$$
(6.2)

Once the temperature distribution  $T_0$  in the x-direction is determined, it is plotted together with the temperature of the air,  $T_{air}$ , and the temperature of the tin,  $T_{tin}$ , (Figure 6.2). To verify our calculations we used simple functions for which the analytical and numerical results are comparable. Table A.1 lists the nondimensional values for calculations used in the code. Furthermore, the final temperature profile including the correction, equation (3.52), was determined numerically as well.

For the multi-layer, recall the non-linear, first order differential equation, (3.100), along the x-direction that is solved numerically using the fourth order Runge-Kutta method as well. The same approach as in the one-layer system is applied. The one- and multi-layer results are plotted in the same graph to compare the temperatures between the layers. Note that, if the thermal conductivities for the contaminated layer  $k_r^* = k_r^{**} = 1$ , equation (3.100), then the governing heat equation (3.100) reduces to the pure-layer governing equation (3.50). First, the pure-layer results were plotted against the multi-layer results. In both systems, it was determined that the temperature at leading order,  $T_0$ , only depends on the length x of the glass ribbon. Recall  $k(T_0) = 1$  for the one-layer system, whereas,  $\frac{1}{1.1} \leq k^{ml}(T_0) \leq \frac{1}{0.9}$  for the multilayer system. Figure 6.1 represents the pure-layer temperature  $T_0$ , temperature of the air,  $T_{air}$ , and tin,  $T_{tin}$ . The glass temperature is between the temperature of the tin and air because heat conducts from the hotter tin to the cooler air. Using equation (3.50), we can explain this temperature profile. Analyzing the terms that represent the Newton's law of cooling, i.e.  $[T_0 - T_{air}]$  and  $[T_0 - T_{tin}]$ , we can say that  $-B_{air} [T_0 - T_{air}] <$ 0 and  $-\epsilon_{mair}R_{air} [T_0^4 - T_{air}^4] < 0$  since  $[T_0 - T_{air}] > 0$ . This observation leads to  $T_{0x} < 0$ , thus,  $T_0$  is a decreasing function. Note the temperature does not change along the height z since  $T_0$  is only a function of x. This figure also shows that the temperature of the glass ribbon ranges from exactly 1100°C to about 600°C. The pure-layer temperature shown in Figure 6.1, is the reference configuration for other analysis in this thesis. Parameters used in equation (3.50) are listed in Table A.1. Figure 6.2 represents the surface plot of the temperature  $T_0$  along the x-direction.



Figure 6.1: Pure-layer temperature  $T_0$ , air  $T_{air}$ , and tin  $T_{tin}$ .



Figure 6.2: Surface plot of pure-layer temperature  $T_0$ .

Considering equation (3.50), further analysis can be performed. From [24], we know that glass can be treated as a vertical sheet, thus ignoring the tin. Assume purelayer temperature is subjected to the same temperature on both sides. Therefore, replacing the temperature and the Biot numbers of the tin with the temperature and the Biot numbers of the air,  $T_{tin} = T_{air}$  and  $B_{tin} = B_{air}$ , we observe lower pure-layer temperature (Figure 6.3). Since  $T_{air} < T_0$ , the glass ribbon is cooled by the air from both sides. Hence, the pure-layer temperature  $T_0$  decreases more as x increases. From Figure 6.3 we observe that the reference state  $T_0$  is larger as we move towards the end of the furnace than the newly calculated  $T_0$  when  $T_{tin} = T_{air}$  and  $B_{tin} = B_{air}$ .

Figure 6.4 compares the pure-layer temperature  $T_0$  and the pure-layer temperature  $T_0$  when  $R_{air} = R_{tin} = 0$ . Clearly, radiation terms play a very important role in calculating the pure-layer temperature since radiation is responsible for cooling


Figure 6.3: Pure-layer temperature  $T_0$ :  $T_{tin} = T_{air}$  and  $B_{tin} = B_{air}$ .

the glass. Thus, including the radiation terms causes substantial temperature change within glass ribbon. Without radiation in the governing equation (3.50), the heat does not escape as quickly. Hence, it takes longer for the glass ribbon to cool.

In Figure 6.5, we vary values of  $B_{air}$  and  $B_{tin}$ . Recall the reference values for  $B_{air}$  and  $B_{tin}$ . Taking  $B_{air} = B_{tin} = 0$ , implies, equations (2.35) and (2.38), that  $h_{air}$  and  $h_{tin}$  are neglected; that is  $h_{air} = h_{tin} = 0$ . Since there is no convective heat transfer at the surface of the glass, the pure-layer temperature is expected to be higher. Taking  $B_{air} = B_{tin} = 0.0002$ , which is smaller than the reference values for  $B_{air}$  and  $B_{tin}$  we used to calculate (3.50), we observe lower pure-layer temperature than the reference state temperature.

However, taking  $B_{air} = B_{tin} = 0.03$  and  $B_{air} = B_{tin} = 0.0325$ , where both values are larger than the reference values, we obtain a higher pure-layer temperature than



Figure 6.4: Pure-layer temperature  $T_0$ :  $R_{air} = R_{tin} = 0$ .

the reference state temperature. This occurs because of larger values of  $h_{air}$  and  $h_{tin}$ . Larger values of  $h_{air}$  and  $h_{tin}$  lead to greater the heat exchange between the air and glass and between the tin and glass. Thus, the pure-layer temperature is lower.

Figures 6.6 - 6.9 compare the pure-layer and multi-layer temperatures. Analyzing the governing heat equations (3.50) and (3.100) analytically by assuming  $k(T_0) = 1$  for both layers, we obtain the same temperature distribution. In Figure 6.6, the contaminated layer is placed on the bottom, at the tin interface. In Figure 6.7, the contaminated layer is placed exactly in the middle of the two pure-layers. In Figure 6.9, the contaminated layer is placed on top of the two pure-layers at the air interface. The height of the layer is assumed to be 1 mm. Decreasing the thermal conductivity of the contaminated layer, equation (3.100), increases  $B_{air}$  and  $B_{tin}$ , equations (2.35) and (2.38). This leads to a greater heat transfer and we obtain lower



Figure 6.5: Pure-layer temperature  $T_0$ : varying Biot numbers.

temperature for the contaminated layer (about 595.6°C when contaminated layer is on the bottom; 597.3°C when in the middle, and 617.8°C when on the top). Increasing the thermal conductivity of the contaminated layer, decreases  $B_{air}$  and  $B_{tin}$ , leading to a lower heat transfer and hence, higher temperature of the contaminated layer (about 622.1°C when contaminated layer is on the bottom; 620°C when in the middle, and 598.9°C when on the top). Varying thermal conductivity corresponds to a temperature difference between the pure- and contaminated layer. Note that the pureand the contaminated layer exhibit a similar temperature profile at higher order, i.e. both are parabolic in z. Combining the analysis results shown in Figures 6.6 - 6.9, we observe that the higher the thermal conductivity, the lower the surface heat transfer, and the higher the temperature of the contaminated layer independent of its location. On the other hand, a decrease in the thermal conductivity results in an increase in



Figure 6.6: Pure-layer temperature  $T_0$ , contaminated layer is on the bottom with  $H_1 = 1$  mm.

the surface heat transfer and a decrease the temperature of the contaminated layer. Figure 6.8 is an enlarged view of Figure 6.7.

Now, consider the first order solution in the analysis as well. Thus,  $T = T_0 + \epsilon T_1 + O(\epsilon^2)$ . Since  $\epsilon$  is very small, the dominating term is  $T_0$ , hence we obtain the identical temperature profile as in Figure 6.1.

Figure 6.11 is the contour plot of the pure-layer temperature T in a cross section of the glass ribbon. The contours are represented as straight vertical lines since the  $O(\epsilon)$  correction is very small and the parabolic shape of T in z is hard to observe.

We can also represent the multi-layer temperature,  $T^{ml}$ . A contour plot of the multi-layer temperature is shown in Figure 6.12. We have seen graphical representation of the multi-layer in Figures 6.6 - 6.9. Thus, we can say that  $T^{ml}$ 



Figure 6.7: Pure-layer temperature  $T_0$ , contaminated layer is in the middle with  $H_2 = 2$  mm.

resembles very closely  $T_0$  and contours shown in Figure 6.12 resemble the contours of the pure-layer in Figure 6.11.

Figure 6.13 shows the  $O(\epsilon)$  temperature distribution  $T_1$  of pure- and multilayer system along the height z for x = 0 m. From equations (3.47) and (3.104), we know that  $T_1$  is parabolic in z and we also observe that behavior in this figure. As stated earlier in the thesis, there is no significant difference in the leading order temperatures between the two glass systems. However, we observe different behaviors for temperature  $T_1$ . Note that  $\epsilon T_1$  is of the order  $10^{-4}$ , thus it does not effect the complete temperature profile,  $T_0^{ml} + \epsilon T_1$ , significantly.



Figure 6.8: Enlarged view of Figure 6.7.



Figure 6.9: Pure-layer temperature  $T_0$ , contaminated layer is on the top with  $H_3 = 3$  mm.



Figure 6.10: Pure-layer temperature  $T_0 + \epsilon T_1$ ,  $\epsilon = 0.00005$ .



Figure 6.11: Contour plot of the pure-layer temperature  $T_0 + \epsilon T_1$ ,  $\epsilon = 0.00005$ .



Figure 6.12: Contour plot of the multi-layer temperature  $T_0^{ml} + \epsilon T_1^{ml}$ ,  $\epsilon = 0.00005$ .



Figure 6.13: Pure-layer  $\epsilon T_1$  and multi-layer temperature  $\epsilon T_1^{ml}$ ,  $\epsilon = 0.00005$ , thermal conductivity of the multi-layer  $k^{ml}(T_0) = \frac{1}{1.1}$ .

#### CHAPTER VII

### THERMAL STRESS, DISPLACEMENT, AND OPTIMIZATION RESULTS

In this section, we introduce the optimization process for the thermal stresses by minimizing thermal stresses of the glass ribbon in the x-direction. We optimize the temperature of the air since it is under our control and not the temperature of the tin. From literature [8, 9], we know that the temperature of the air has a profile of a cubic function. Therefore, we assume

$$T_{air}(x; a, b, c, d) = ax^3 + bx^2 + cx + d.$$
(7.1)

The goal is to determine the parameters a, b, c, and d in such a way that the thermal stresses of the glass ribbon are minimized. Also, the temperature of the glass ribbon should be closer to 600°C at the exit of the furnace. From literature [8, 9] we have very good estimate on what  $T_{air}$  should be; therefore, we have good initial guesses for the parameters a, b, c, and d.

Recall equations (3.49) and (3.51) for D(x) and C(x), respectively. Note that D(x) does not depend on  $T_{air}$ , but C(x) does. Thus, C(x; a, b, c, d). Furthermore, from equation (3.45), we know that C(x; a, b, c, d) is proportional to  $T_{0x}$ . Hence,  $T_0(x; a, b, c, d)$  also depends upon a, b, c, and d.

We want to minimize the longitudinal thermal stress,  $\sigma_{1x} = \sigma_x$  only since  $\sigma_{0x} = 0$ , equation (5.42).

Also,  $\sigma_{1x}$  is defined in terms of C(x; a, b, c, d) and it is a function of x and z. Thus,  $\sigma_{1x}(x, z; a, b, c, d)$  also depends upon a, b, c, and d. Therefore, for the thermal stress, equation (5.51), to be minimized we consider

$$\sigma_{1x}(x, z; a, b, c, d) = \frac{2G\alpha E}{(\lambda + 2G)(1 - 2\nu)} \left[ \frac{1}{6} C(x; a, b, c, d) + \frac{1}{2} D(x) \right]$$
(7.2)  
$$- \frac{2G\alpha E}{(\lambda + 2G)(1 - 2\nu)} \left[ \frac{1}{2} C(x; a, b, c, d) z^2 + D(x) z \right].$$

In order to minimize equation (7.2), we assume  $L^2$  space and take the following approach

$$\min \| \sigma_{1x}(x, z; a, b, c, d) \|^2 = \left[\sqrt{\langle \sigma_{1x}, \sigma_{1x} \rangle}\right]^2 = \int_0^1 \sigma_{1x}^2 \, dx. \tag{7.3}$$

Note that we could have selected other criterion to minimize.

We choose to optimize the thermal stress at the glass/air interface since the thermal stress is the largest at that location. Thus, by fixing variable z, z = 1, which corresponds to the glass/air interface, and integrating with respect to x, we obtain a function of a, b, c, and d only

$$F(a, b, c, d) = \int_0^1 \left[\sigma_{1x}(x, z; a, b, c, d)\right]^2 dx,$$
(7.4)

where  $\sigma_{1x}$  is defined by equation (7.2).

In order to fully determine  $\sigma_{1x}$ , we have to presolve  $\sigma_{1x}$  numerically. The trapezoidal rule was invoked from the Matlab library. In order to optimize the stresses, we decided on the following approach, which can be considered as the Lagrange multiplier approach

minimize 
$$\left[F(a, b, c, d) + \lambda \left(T_0(x=1) - \frac{873}{1373}\right)^2\right],$$
 (7.5)

where  $\lambda$  is the Lagrangian multiplier. Notice that we minimize the area under the curve  $\sigma_{1x}^2$ . In order to optimize equation (7.5), we have to have initial guesses for the parameters a, b, c, d, and  $\lambda$ . Initial guesses for the parameters a, b, c, and d are taken from [8, 9] and used to calculate  $T_0$  as in Chapter 3.  $\lambda$  is a free parameter and it is up to us to choose its value based on the results. Our written code in Matlab is adapted to use the built in minimization function, fminsearch. fminsearch is a multi-variable minimization function that is based on the Nelder-Mead Simplex algorithm. Once the values for a, b, c, and d are determined, we obtain a new  $T_{air}$  function

$$T_{air}(x) = -0.68061337684171x^3 - 0.14991607513187x^2$$
(7.6)  
+ 0.15213769916125x + 0.99402703828734.

Now, we will discuss the results of the displacement and the thermal stress calculations in the x-, y-, and z-direction. Recall the governing equations for displacement, (5.1) and (5.2), and corresponding boundary conditions, equations (5.3) - (5.5). The thermal stress equations, (5.6) - (5.8), are determined after the displacements are obtained. Notice that even though the displacements and thermal stresses are solved analytically, the results are presented numerically since some stress components are presolved numerically in advance. We invoke the trapezoidal rule to help us solve the thermal stress equations. Our two-dimensional heat transfer analysis would imply that the thermal stress analysis is also two-dimensional, since thermal

stresses depend on the temperature. Based on our plane strain assumption, we have to consider all three dimensions; particularly the imposed  $\sigma_y$  stress. Therefore, we are interested in examining the thermal stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$ . Thermal stresses in *x*-direction,  $\sigma_x$ , will be plotted against the length *x* and height *z* of the glass ribbon; whereas, thermal stresses in *y*-direction,  $\sigma_y$ , will be plotted against the length *x* of the glass ribbon. Note that the thermal stresses,  $\sigma_z$ , are zero.

Figure 7.1 represents the surface plot of the full thermal stresses  $\sigma_x$ . Notice the parabolic shape of the curve. This behavior comes from the presence of  $T_1$  in the expression for  $\sigma_{1x}$ . Also, due to prescribed displacement boundary conditions, the thermal stresses at O(1),  $\sigma_{0x}$ , do not appear. Note the compressive stresses at z = 0 mm, and tensile stresses at z = 3 mm. Further, observe the small values for the thermal stresses due to  $\sigma_{0x} = 0$ . Analyzing equation (7.2), we observe that at the tin interface, z = 0 mm, the second term vanishes. Also, from analysis we know that C(x) is the dominating term in the expression and it is negative along x. Therefore, we expect to see compressive stresses at and around z = 0 mm. The thicker the glass becomes and approaches z = 3 mm, the more dominating the term  $C(x)z^2 + D(x)z$  in equation (7.2) becomes; thus, creating tensile stresses at the top of the glass ribbon. To better observe the behavior at z = 0 mm and z = 3 mm Figure 7.2 shows the rotated view of Figure 7.1.

Figure 7.3 represents the imposed thermal stresses in y-direction,  $\sigma_y$ . Note the large compressive stresses, which are due to the plane strain assumptions. We also observe no z dependence in  $\sigma_y$ . Since  $\epsilon$  is very small  $\sigma_{0y}$ , is dominating.



Figure 7.1: Thermal stresses  $\sigma_{0x} + \epsilon \sigma_{1x}$ ,  $\epsilon = 0.00005$ .

Vertical and horizontal displacements up to and including the first order are also calculated. Figure 7.4 - 7.6 show the vertical displacement at different orders. The O(1) solution is represented in Figure 7.4. We observe that the values are of order  $10^{-3}$  mm and that there is no displacement at z = 0 mm and x = 60 m due to the prescribed boundary conditions and the assumption that there is no rotation at x = 60 m. Notice the displacements are the largest at z = 3 mm and x = 0 m. This behavior is expected since the glass expands the most at that location. Also, the displacements calculated here are accumulating through the thickness of the glass ribbon. Thus, we can say that Figures 7.4 - 7.6 represent the difference between displacements along the thickness.

Figure 7.5 is the surface plot of the first order solution. It is expected that the values are negligible,  $10^{-6}$ , due to the small aspect ratio  $\epsilon$ . The behavior of the



Figure 7.2: Rotated Figure 7.1.

graph around x = 0 m indicates a possible boundary layer which we mentioned in previous chapters. However, when the complete profile for the vertical displacement is put together, we observe the dominance of the leading order term. Thus, Figure 7.6 resembles the already discussed Figure 7.4. Increasing the velocity by which the glass is being pulled, decreases the difference between the vertical displacements (Figure 7.7).

Horizontal displacements are also studied. We observe a linear profile in xdirection and no z dependence in Figure 7.8. Also, the values of the displacement are much higher than for the vertical displacement shown in Figure 7.4. This is due to small aspect ratio  $\epsilon$ , equation (2.17). The displacement is the largest at x = 60 m since displacements accumulates as the glass ribbon is being pulled towards the exit of the furnace. Thus, increasing the velocity by which the glass is being pulled,



Figure 7.3: Thermal stresses  $\sigma_{0y} + \epsilon \sigma_{1y}$ ,  $\epsilon = 0.00005$ .

decreases the difference between the horizontal displacements (Figure 7.11). Figure 7.9 demonstrates the first order solution for horizontal displacements. Notice the much smaller values here,  $10^{-5}$ . Graphing the full horizontal displacement u, the leading order term dominates and we obtain a very similar graph, Figure 7.10, as in Figure 7.8.

As described in this chapter, longitudinal thermal stresses are optimized. Figures 7.12 and 7.13 show the optimized stresses. Notice that there are hardly any stresses at z = 1.5 mm. The stresses increase toward x = 60 m and z = 3 mm. Compare this to the longitudinal thermal stresses prior to optimization where the largest stresses are around x = 30 m and z = 3 mm. This change of location in x is due to the choice of the minimization process. Figure 7.13 is a contour plot of Figure 7.12 where we can confirm our calculations shown in Figure 7.12 by observing that



Figure 7.4: Vertical displacement  $w_0$ , velocity  $V = \frac{1}{9} \frac{\mathrm{m}}{\mathrm{s}}$ .

the largest tensile stresses are around x = 60 m and z = 3 mm. Notice that there are hardly any stresses 10 m < x < 35 m and  $\frac{1}{2}$  mm < z < 2 mm.

Figure 7.14 shows the already discussed optimized longitudinal thermal stresses. However, we want to show the optimized thermal stresses at the tin and air interface, z = 0 mm and z = 3 mm, respectively. Notice again the compressive and tensile stresses as explained for Figures 7.12 and 7.13.

Figure 7.15 shows compressive longitudinal thermal stresses and optimized longitudinal thermal stresses at z = 0 mm. The magnitude of the optimized thermal stresses is about the same compared to the thermal stresses prior to optimization.

In Figure 7.16, we compare the tensile longitudinal thermal stresses and the optimized longitudinal thermal stresses at z = 3 mm. The longitudinal thermal stresses are larger than the optimized thermal stresses by about 10 % due to the



Figure 7.5: Vertical displacement  $\epsilon w_1$ ,  $\epsilon = 0.00005$ , velocity  $V = \frac{1}{9} \frac{\text{m}}{\text{s}}$ .

optimization choice, the fact that we optimized the area under the curve  $\sigma_{1x}^2$ , and very good choice of parameters such as the Lagrangian multiplier  $\lambda$ , number of mesh points for x and z, M and N, respectively..

In Figure 7.17, we compare  $T_0$  and  $T_{air}$  with the optimized  $T_0$  and  $T_{air}$ . The temperature gradient of the optimized temperature  $T_0$  is not as decreasing at the entrance of the furnace as the temperature gradient of  $T_0$ . The steepest temperature gradient is shifted towards the exit of the furnace. Also, note that the temperature profile  $T_0$  follows the temperature profile  $T_{air}$  in terms of the shape of the curve. For this figure we observe that the temperatures are closer together around x = 0 m since the temperature is prescribed at that location, 1100°C.

Figure 7.18 demonstrates the contour plot of the pure-layer temperature when velocity by which the glass is being pulled longitudinally is increased. We observe



Figure 7.6: Vertical displacement  $w_0 + \epsilon w_1$ ,  $\epsilon = 0.00005$ , velocity  $V = \frac{1}{9} \frac{\text{m}}{\text{s}}$ .

higher temperature at x = 60 m by about 100°C than the reference temperature  $T_0$ at x = 60 m. This behavior is understandable since the glass is being pulled faster, thus it does not have enough time to cool down.

Further, we graph  $T_0$  for higher values of velocity together with  $T_{air}$  and  $T_{tin}$  (Figure 7.19). In order to lower the temperature of the glass while increasing velocity, we have to lower the temperature of the air since we have control over it.



Figure 7.7: Vertical displacement  $w_0 + \epsilon w_1$ ,  $\epsilon = 0.00005$ , velocity  $V = \frac{1}{3} \frac{\text{m}}{\text{s}}$ .



Figure 7.8: Horizontal displacement  $u_0$ , velocity  $V = \frac{1}{9} \frac{\text{m}}{\text{s}}$ .



Figure 7.9: Horizontal displacement  $\epsilon u_1$ ,  $\epsilon = 0.00005$ , velocity  $V = \frac{1}{9} \frac{\text{m}}{\text{s}}$ .



Figure 7.10: Horizontal displacement  $u_0 + \epsilon u_1$ ,  $\epsilon = 0.00005$ , velocity  $V = \frac{1}{9} \frac{\text{m}}{\text{s}}$ .



Figure 7.11: Horizontal displacement  $u_0 + \epsilon u_1$ ,  $\epsilon = 0.00005$ , velocity  $V = \frac{1}{3} \frac{\text{m}}{\text{s}}$ .



Figure 7.12: Optimized  $\epsilon \sigma_{1x}$ ,  $\epsilon = 0.00005$ .



Figure 7.13: Contour plot of the optimized  $\epsilon \sigma_{1x}$ ,  $\epsilon = 0.00005$ .



Figure 7.14: Optimized  $\epsilon \sigma_{1x}$ ,  $\epsilon = 0.00005$  for z = 0 mm, z = 3 mm.



Figure 7.15:  $\epsilon \sigma_{1x}$  vs. optimized  $\epsilon \sigma_{1x}$  for z = 0 mm,  $\epsilon = 0.00005$ .



Figure 7.16:  $\epsilon \sigma_{1x}$  vs. optimized  $\epsilon \sigma_{1x}$  for z = 3 mm,  $\epsilon = 0.00005$ .



Figure 7.17: Pure-layer temperature  $T_0, T_{air}$  vs. optimized pure-layer temperature  $T_0, T_{air}$ .



Figure 7.18: Contour plot of pure-layer  $T_0$ , velocity  $V = \frac{1}{3} \frac{\text{m}}{\text{s}}$ .



Figure 7.19: Pure-layer temperature  $T_0$ , velocity  $V = \frac{1}{3} \frac{\text{m}}{\text{s}}$ .

## CHAPTER VIII

## SUMMARY, CONCLUSION, AND FUTURE WORK

In this thesis we have developed an asymptotic model for the two-dimensional heat transfer and thermal stress development during the processing of float glass process. The glass ribbon is constrained by two boundaries in the vertical direction: the interface of the molten tin and glass (bottom) and the interface of the air and glass (top). We considered a very long and thin glass ribbon with a small aspect ratio,  $\epsilon = \frac{H}{L} << 1$ . The region of interest is the part of the furnace where the temperature of the glass ribbon changes from 1100°C to 600°C, from the entrance to the exit of the furnace.

First, we analyzed the one-layer system, which represents a pure-layer glass where we have one governing heat equation subject to corresponding boundary conditions at the top and bottom. Then, the three layer system, which represents the multi-layer glass where we have a set of three governing equations subject to corresponding boundary conditions.

Several assumptions are made:

• at the leading order the thermal conductivity  $k(T_0)$ , density  $\rho(T_0)$ , heat capacity  $C_p(T_0)$  are treated as constants where  $k(T_0) = 1$  for the pure-layer and  $\frac{1}{1.1} \leq k(T_0) \leq \frac{1}{0.9}$  for the contaminated layer;

- the interfaces between the contaminated layer and two pure-layers are considered to be flat;
- the interface of the air and glass ribbon is assumed to take the form of a sine function [6];
- the starting temperature for the glass ribbon is prescribed to be 1100°C.

We calculate the temperature profiles of the one- and multi-layer systems up to and including the first order problem. Both profiles at the first order, exhibit a parabolic shape through the thickness of the glass ribbon. However, in both cases, the temperature at leading order was determined to only be a function of the length of the glass ribbon.

We examined the temperature profiles by varying the parameters in the governing equation such as the Biot numbers and the radiation numbers for the air and tin. From the analysis, we observed that the presence of radiation terms, while keeping the same values for thermal conductivity, in the governing equation is very important for establishing the temperature profile of the glass ribbon. Without radiation, the glass ribbon would need a very long time to cool. Moreover, varying Biot numbers of the air and tin (thus varying the heat transfer coefficients) did not have a significant influence on the the temperature profile.

By varying the thermal conductivity  $k(T_0)$  of the contaminated layer, we were able to find a small temperature difference compared to the pure-layer. The location of the contaminated layer did not have a significant influence on the temperature profile.

Due to non-uniform heating in the glass system, thermal stresses develop. The equilibrium equations in terms of the displacement were used to solve for the thermal stresses subject to the displacement boundary conditions along the length and height of the glass ribbon. The system was treated as a plane strain problem along the width of the glass ribbon. This meant imposing the thermal stresses along the width as well.

We found that the stresses along the width have large compressive values due to the plane strain assumption. Also, they only depend on the length of the glass ribbon.

For the longitudinal stresses, the largest tensile stresses were observed at the glass/air interface, whereas the largest compressive stresses were found at the glass/tin interface due to non-uniform heating. Also, we created surface plots of thermal stresses along the length and thickness and we observed that the stresses do not exceed 0.003 Pa.

We determined vertical and horizontal displacements in the glass ribbon up to and including the first order. The vertical displacements are the largest at the glass/air interface and at the entrance of the furnace. Note that the expression for the vertical displacement was calculated up to an unknown function of x. Thus, we can not comment on the function behavior along the length, but rather along the thickness for a fixed length. Horizontal displacement has a linear profile along the length of the glass ribbon, thus, it is a function of x. The horizontal displacements are the largest at the exit of the furnace and the smallest at the entrance of the furnace due to the prescribed displacement boundary conditions. Graphing only the first order solution of the horizontal displacement, we observe the parabolic shape along the thickness due to the parabolic temperature presence. However, the dominating term in the full expression for the horizontal displacement is the leading order term.

Temperature, displacements, and thermal stresses are all linked to the analysis of C(x), equations (3.45) and (3.51). Note also from equation (3.45) that C(x)is proportional to the longitudinal temperature gradient. Thus, the larger C(x), the larger the temperature gradient and the thermal stresses. This analysis of C(x) also shows that the surface heat transfer across the air/glass and tin/glass interface influences the longitudinal thermal stresses. In particular, one needs to minimize this surface heat transfer, while still cooling the glass in order to minimize the thermal stresses in the longitudinal direction. Velocity V also has an influence on determining the thermal stresses since it influences the Peclet number P and hence C(x) and the temperature gradient. This means increasing velocity (i.e.  $V = \frac{1}{3} \frac{\text{m}}{\text{s}}$ ), increases the Peclet number P, C(x), and the displacements/thermal stresses.

The thermal stresses,  $\sigma_x$ , were minimized by taking into consideration the temperature of the air on the top of the glass ribbon only. We were able to redefine the temperature of the air by recalculating the coefficients of the function. The graphical representation showed the optimized temperature and stresses. We were also able to lower the temperature of the glass ribbon by some 8°C and bring it down to 600.9°C at the exit of the furnace. Therefore, the thermal stresses along the length of the glass ribbon were also minimized. It is important to mention that the final stress calculations must include a mechanical stress analysis as well. In this thesis, we did not consider the mechanical stresses. The size of thermal stresses is also very significant since the thermal stresses could have an effect on the glass response particularly if close to the yield strength.

Extending this research to the full three-dimensional problem should give us better and more applicable results since all three dimensions would be considered. For example, the pure-layer temperature at the leading order would become a function of x and y. This will make the problem harder to analyze since the governing equations will become partial differential equations. Refer to appendix C for an overview. Also, adding the width will allow us to examine the thermal stresses in more detail and the possibility of crack propagation inside the glass ribbon.

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APPENDICES

# APPENDIX A

## TABLES FOR THE HEAT TRANSFER PROBLEM

Symbol	Description	Units	Reference Value	Citation
$B_{tin}$	Biot number for tin	unitless	0.0024	[8]
$B_{air}$	Biot number for air	unitless	0.0024	[9]
$C_p(T)$	Heat capacity	$\frac{J}{kgK}$	-	[9]
$C_{pr}$	Reference value	$\frac{J}{kgK}$	1000	[8, 9]
	for heat capacity			
Н	Height of the	m	0.003	[6]
	glass ribbon for one-layer			
$H_3$	Height of the	m	0.003	[6]
	glass ribbon for multi-layer			
$H_2$	Height of the	m	0.002	-
	glass ribbon for multi-layer			
$H_1$	Height of the	m	0.001	-
	glass ribbon for multi-layer			

Table A.1: Table of variables for the heat transfer problem (part I)

Symbol	Description	Units	Reference Value	Citation
$h_{tin}$	Heat transfer	$\frac{W}{m^2K}$	0.7972678	[8]
	coefficient of the tin			
$h_{air}$	Heat transfer	$\frac{W}{m^2K}$	0.7972678	[9]
	coefficient of the air			
k(T)	Thermal conductivity	$\frac{W}{mK}$	-	[9]
$k_r$	Reference value for	$\frac{W}{mK}$	1	[8, 9]
	thermal conductivity			
kair	Thermal conductivity	$\frac{W}{mK}$	$60 \times 10^{-3}$	[9]
	of the air			

Table A.2: Table of variables for the heat transfer problem (part II)  $% \left( {{\rm{Table}}} \right) = \left( {{\rm{Table}}} \right)$
Symbol	Description	Units	Reference Value	Citation
L	Length of the glass ribbon	m	60	[8]
Р	Peclet number	unitless	0.0367	[8, 9]
Q(T)	Radiative	$\frac{W}{m^3}$	-	-
	heat transfer			
$Q_r$	Reference value for	$\frac{W}{m^3}$	830.1447	[8, 9]
	radiative heat transfer			
$R_{tin}$	Radiation number	unitless	0.4403	[8]
	for tin			
R <sub>air</sub>	Radiation number	unitless	0.4403	[9]
	for air			
$\widehat{S}$	Radiation	$\frac{W}{cm^2}$	_	-
$\hat{s}$	Radiation	$\frac{W}{cm^2}$	_	-
Т	Temperature	К	_	-
$T_H$	Reference value	К	1373	[6, 5]
	for temperature			
$T_{tin}$	Temperature	К	1013 - 1523	[8]
	of the tin			
$T_{air}$	Temperature of	К	723.3 - 1372	[9]
	the furnace (air)			

Table of variables for the heat transfer problem (part III)

$\mathbf{Symbol}$	Description	Units	Reference Values	Citation
t	Time	s	-	-
V	Velocity of the glass slab	$\frac{\mathrm{m}}{\mathrm{s}}$	$\frac{1}{9}$	[8]
W	Width of the glass ribbon	m	10	[8]
$\rho(T)$	Density	$\frac{\text{kg}}{\text{m}^3}$	-	-
$\rho_r$	Reference value	$\frac{\text{kg}}{\text{m}^3}$	2200	[9]
	for density			
$ ho_a$	Reference value	$\frac{\text{kg}}{\text{m}^3}$	1	[9]
	for density of the air			
$\epsilon_{mtin}$	Emissivity	unitless	0.08	[8]
	of the tin			
$\epsilon_{mair}$	Emissivity	unitless	0.39	[8]
	of the air			
μ	Viscosity of the air	$\frac{\text{kg}}{\text{ms}}$	$1.73 \times 10^{-5}$	[26]
$\sigma_{SB}$	Stefan Boltzmann constant	$\frac{W}{cm^2K^4}$	$5.67 \times 10^{-12}$	[8]

Table of variables for the heat transfer problem (part IV)

$\operatorname{Subscript}$	Description		
ref	reference value		
Н	hot		
air	air		
tin	tin		
SB	Stefan-Boltzman		

Table A.3: Subscript identifiers for the heat transfer problem

Table A.4: Superscript identifiers for the multi-layer heat transfer problem

$\operatorname{Superscript}$	Description		
mlI	multi-layer region I		
mlII	multi-layer region II		
mlIII	multi-layer region III		

## APPENDIX B

## TABLES FOR THE THERMAL STRESS PROBLEM

Symbol	Description	Units	Value	Citation
α	Thermal	$\frac{1}{K}$	0.09	[18]
	expansion coefficient			
ν	Poisson's ratio	unitless	0.17	[18], [27]
E	Young's modulus	Pa	$0.003843 \cdot 10^9$	[18], [27]
λ	Lame's constant	Pa	$8.4604 \cdot 10^5$	[18]
G	Modulus of rigidity	Pa	$1.6423 \cdot 10^{6}$	[18]
u	Displacement	m	-	[18]
	field in x-direction			
v	Displacement	length	-	[18]
	field in y-direction			
w	Displacement	mm	-	[18]
	field in z-direction			

Table B.1: Table of variables for the thermal stress problem

Symbol	Description	Units	Value	Citation
$\epsilon_x$	Strain in	unitless	-	[18]
	x-direction			
$\epsilon_y$	Strain in	unitless	-	[18]
	y-direction			
$\epsilon_z$	Strain in	unitless	-	[18]
	z-direction			
$\sigma_x$	Stress in	Pa	-	[18]
	x-direction			
$\sigma_y$	Stress in	Pa	-	[18]
	y-direction			
$\sigma_z$	Stress in	Pa	-	[18]
	z-direction			

Table B.2: Table of variables for the thermal stress problem

## APPENDIX C

## 3-D HEAT TRANSFER PROBLEM

In this appendix, we shall go through the derivation of the three-dimensional heat transfer problem. Here, the width of the glass ribbon or the y-direction shall be considered. Hence, the asymptotic expansion for the temperature profile is represented as

$$T(x, y, z) = T_0(x, y) + \epsilon T_1(x, y, z) + \epsilon^2 T_2(x, y, z) + O(\epsilon^3).$$
(C.1)

Recall the nondimensional governing equation for the one-layer system:

$$\frac{\rho_r C_{pr} H^2 V \rho(T) C_p(T)}{k_r L} \left[ \frac{L}{V \tau} T_t + T_x \right] =$$

$$\frac{H^2}{L^2} \left[ k(T) T_x \right]_x + \frac{H^2}{W^2} \left[ k(T) T_y \right]_y + \left[ k(T) T_z \right]_z - \frac{\sigma_{SB} T_H^3 H}{k_r} Q(T)$$
(C.2)

where equations (2.18)-(2.24) from Chapter II hold and

$$\epsilon = \frac{H}{L}, \tag{C.3}$$

$$\epsilon^{\frac{1}{2}}\overline{A} = \frac{H}{W}.$$
 (C.4)

Also, here in this three-dimensional analysis, we are considering a steady-state problem as well as neglecting the radiation term from the governing equation.

After applying the Taylor and Binomial expansions to the governing heat equation and boundary conditions, we can collect the orders. Thus, the O(1) governing equation is

$$[k(T_0)T_{0z}]_z = 0 (C.5)$$

subject to the boundary conditions at z = 0 and z = 1, respectively

$$k(T_0(z=0))T_{0z}(z=0) = 0,$$
 (C.6)

$$k(T_0(z=1))T_{0z}(z=1) = 0.$$
 (C.7)

Integrating equation (C.5) with respect to z and applying boundary conditions (C.6)-(C.7), we obtain the solution to the O(1) problem

$$T_0 = T_0(x, y).$$
 (C.8)

Clearly, the temperature profile at O(1) is a function of x and y, and it can be found by going to the higher order,  $O(\epsilon)$ .

The  $O(\epsilon)$  governing equation is

$$P\rho(T_0)C_p(T_0)T_{0x} - (\overline{A})^2 [k(T_0)T_{0y}]_y = k(T_0)T_{1zz}$$
(C.9)

subject to the boundary conditions at z = 0 and z = 1, respectively

$$k(T_0)T_{1z} = B_{tin}(T_0 - T_{tin}) + R_{tin}\epsilon_{mtin}(T_0^4 - T_{tin}^4),$$
(C.10)

$$k(T_0)T_{1z} = -B_a(T_0 - T_{air}) - R_{air}\epsilon_{mair}(T_0^4 - T_{air}^4).$$
(C.11)

Applying the boundary conditions (C.10) and (C.11) to the governing equation (C.9), we obtain the following partial differential equation to be solved for  $T_0$ 

$$P\rho(T_0)C_p(T_0)\left[T_{0x} - \left(\overline{A}\right)^2 k(T_0)T_{0yy}\right] = -B_{air}\left(T_0 - T_{air}\right) \qquad (C.12)$$
$$-R_{air}\epsilon_{mair}\left(T_0^4 - T_{air}^4\right) - B_{tin}\left(T_0 - T_{tin}\right) - R_{tin}\epsilon_{mtin}\left(T_0^4 - T_{tin}^4\right).$$

This partial differential equation is two-dimensional. Thus, it takes into consideration the width, y, of the glass ribbon. The first correction,  $O(\epsilon)$ , becomes a non-linear partial differential equation in terms of the length x, width y, and height z,  $T_1(x, y, z)$ , which would require more effort to solve and we would have to choose a different numerical analysis approach. Also, it would be parabolic in z.

We should also mention that the thermal stresses should change as well. Thermal stresses in x-direction should become functions of x and y, whereas thermal stresses in z-direction should be functions of x, y, and z. Therefore, the stress analysis would also be more complex, but more accurate. We could possibly track the crack propagation through the thickness z as well as the width y.