Adaptive Optics System Baseline Modeling for a USAF Quad Axis Telescope

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By

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ABSTRACT

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Atmospheric turbulence has afflicted accurate observations of celestial bodies since man first gazed upon the stars. In this past century, the technology of adaptive optics was invented to help compensate for the optical distortions that atmospheric turbulence causes. As part of that technology, artificial guide stars, wave front sensors, deformable mirrors, and other optical components were developed to correct these wave aberrations. The purpose of this study focuses on the modeling and configuration of an adaptive optics system that is appropriate for the John Bryan Observatory Quad Axis Telescope System (JBO-Q), which is funded by the United States Air Force. Scaling law modeling of site-specific atmospheric parameters using numerical weather data and laser propagation theory was used determination and optimization of some critical system specifications and threshold parameters for this baseline model.
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I. Introduction

Telescopes around the world come in many shapes and sizes and have been built on mountain tops, deserts, and even forest clearings. We now know that the environment which surrounds a telescope dramatically effects the telescope’s viewing capabilities even with modern day computing technology. With this in mind, it is important that each observatory should be carefully and uniquely designed to account for localized atmospheric effects to attain a nearly diffraction limited system. This process includes the application of an Adaptive Optics (AO) system. The purpose of an AO system is to measure the distortion caused by atmospheric turbulence and then to compensate for the distortion in order to obtain higher resolution images of the celestial body being observed. This technique requires many devices to work in unison to produce real-time image compensation. When developing an AO system for an existing telescope, it is critical to understand the magnitude of local atmospheric wavefront aberrations as these conditions will dramatically affect system performance. In essence, wavefront aberrations are caused by variations in the index of refraction along the optical path (looking path of the telescope). In similar terms, the varying index of refraction causes distortion of electromagnetic waves emanating from celestial bodies as they pass through the Earth’s atmosphere. This turbulent process is stochastic, and thus wavefronts must be corrected in real-time since the wave rapidly fluctuate in an unpredictable way. The magnitude of distortion may vary substantially within just a few milliseconds timeframe which implies that the AO system must compensate at an even faster rate.
The primary focus of this thesis is to accurately model the baseline system parameters required for an AO system that could be applied to the John Bryan Observatory (JBO) Quad Axis Telescope, which is currently funded by the Air Force Research Lab (AFRL) at Wright-Patterson Air Force Base (WPAFB), Ohio. The Quad Axis Telescope is a unique asset to the Air Force that exists in an environment uncommon to most other observatories since it operates in a humid, low altitude, atmospherically turbulent location. Furthermore, the intent of this thesis is to present an analysis of key system parameters such as the photon detection noise for the Shack-Hartmann wavefront sensor (SHWFS), the optimal size of the subapertures of the SHWFS, the optimal number of actuators for the deformable mirror component, and the necessary overall system framerate based off of Greenwood frequency calculations. Analysis of these parameters and other sub-parameters will show that the baseline requirements for an AO system for the JBO-Q site are achievable through the use of scaling law modeling and numerical weather data. The results of this preliminary modeling will enable AFRL to design an accurate, cost effective AO system to the Quad Axis Telescope in the future. The overall goal of this effort is to enhance the John Bryan Observatory’s Quad Axis Telescope’s ability to obtain higher resolution images of celestial objects.

In the chapters to come, I describe the structure and modeling of Earth’s atmosphere in regard to optical sensing, how a typical AO system works, how AO could be applied to the John Bryan Quad Axis Telescope, provide a detailed account on how baseline parameters are calculated, and offer an analysis on how parameters will affect the overall system performance. The next chapter begins with an overview of a typical AO system as this deepens the understanding of the challenges of the modeling to come.
II. Adaptive Optics System Overview

II.1 AO Introduction

There are two manners in which a telescope is able to collect imagery data of celestial objects without effects due to atmospheric turbulence. The first is to put a telescope outside of Earth’s atmosphere like those such as the Hubble Telescope or the James Webb Telescope. However, launching a telescope into orbit is both very expensive and technically intensive since it requires a multitude of environmental considerations. A space borne telescope must be able to endure an intense journey from Earth’s surface and also survive the harsh environment of space. alternative means are terrestrial observatory. Another type of AO is multi-conjugate AO which employs the use of multiple deformable mirrors (explained shortly) and multi-guide stars as wavefront references. This technique is more advanced and is used to correct turbulence 3-dimensionally. The benefit of such a technique is reduced anisoplanatism which increased the overall corrected field of view for the telescope [28].

As mentioned before, AO is a mature technology developed to improve the performance of optical systems by ameliorating the effect of wavefront distortions. The main advantages to using AO systems are that they can be applied to virtually any terrestrial telescope system and that acquisition of such a system is comparatively inexpensive compared to space borne telescopes. In fact, AO is so well developed that modern terrestrial observatory capabilities rival (and often exceed) those capabilities of space borne telescopes. Furthermore, AO systems are easier to access and thus maintain, which consequentially makes them more reliable than those in orbit in the long run. AO technology is utilized in a variety of commercial and government
applications including astronomical observation, laser communication, microscopy, optical fabrication, and retinal imaging systems [1, 2, 3, 4]. Its overall goal is to detect deformations of incoming light waves then deform a special mirror which compensates for wavefront aberrations. Before this wavefront correction is described in detail, is it necessary to understand the constituents of an AO system.

In the case of astronomical observation, atmospheric turbulence distorts the propagation of electromagnetic waves (in this case electro-optical) before it reaches and is collected by the telescope aperture. Note that before light waves (from celestial bodies) enter the atmosphere they can be approximated as planar waves (details later in chapter III). Ideally, planar waveforms are what AO systems strive to achieve as they represent perfect correction of turbulent wavefront. When this is achieved, telescopes operate much closer to their diffraction limited capability where any existing losses are primarily due to imperfect optics and improper design of the telescope itself. Turbulence affects both spatial and temporal accuracies of optical measurements and makes celestial observation more difficult. Atmospheric turbulence causes the index of refraction to vary rapidly along an optical path (path of observation) which causes the observed body to appear fuzzy. This dynamic fuzziness is due to temperature and pressure fluctuations in the atmosphere and produces observable changes in the body’s apparent motion and brightness. One can easily observe these distortions when seeing the stars twinkle at nighttime or by looking down a road on a hot summer day. Previous studies have proven that most optical distortion from turbulence occurs near ground level since this is where the atmosphere is most dense. Essentially, it is the heating or cooling of the ground (or sea) that primarily contributes to deviations in optical clarity near the surface of the earth [5]. Additionally, resonate heating or cooling of the observatory in which the telescope sits may also
significantly impact optical distortion. On a separate note, it is also important recognize that “active optics” differs from “adaptive optics” in the sense that active optics corrects for the geometry of primary mirrors at large time scales on telescopes.

II.2 Laser Guide Stars

In order to collect enough signal from a celestial target that is too dim to obtain higher resolution, it is necessary to use a guide star. The purpose of a guide star (also called “beacons”) is to provide a wavefront reference source in the optical path so that wavefront information can be collected by the telescope [5]. There are two types of guide stars that are used in modern practice, natural guide stars (NGS) and laser guide stars (LGS).

Generally, natural guide stars are utilized to help further resolve celestial objects within the optical path. An NGS is simply a bright star near the object of interest and within a narrow field of view to that object. Although it might be possible, it is not practical to use a fast-moving body such as a satellite in orbit to perform the function of a guide star. Since an NGS must be bright enough for the high-speed wavefront sensor, the brightness is often times is a major limiting factor for an AO system. If the NGS is too dim, then the wavefront sensor’s SNR is too low to make accurate measurements. Essentially this means that the sensor cannot get enough light to stimulate its photodetection array and thus distortion cannot be measured accurately. NGSs are typically used for calibration of sensors and, as mentioned before, for objects that are very close in proximity to the celestial object of interest. For an NGS to be useful, it needs to be within the isoplanatic angle so that the process of turbulence compensation can take place (described in Chapter III) [5]. As a result, NGSs do not often provide an astronomer with a means to dynamically correct for turbulence, thus “artificial” guide stars were developed and designed specifically to ensure sensor photon count requirements are satisfied.
An LGS performs the exact same function as an NGS but is much more flexible and is often times brighter, thus many observatories are fitted with them. LGSs come in three types: Rayleigh beacons, Sodium beacons, and Ultra-Violet (UV) beacons. UV (also known as Dye) beacons are rarer in practice since laser technology has not been commercially driven to develop cost-effective use of UV lasers at larger power outputs compared to Rayleigh and Sodium LGSs. The reason for this is that the use of toxic fluid gain media has been considered by some as disadvantageous [6]. Rayleigh beacon wavelengths span the optical frequencies but typically have “green” wavelengths between 500 - 540 nanometers (nm) and operate between 8 – 12 kilometers (km) in altitude as this region of the atmospheric yields the most laser backscatter (described shortly) [5]. These guide stars are named after British physicist, Lord Rayleigh, who deduced the statistical scattering property of light in Earth’s atmosphere, Rayleigh Scattering, which is defined as the elastic scattering of radiation which occurs when radiation is traveling through particles of much smaller size compared to the radiation wavelength [7]. This property will be mathematically described in chapter III and also explains why the sky corresponds to the color blue. Backscatter is a term used to describe how much laser power (in photons) is reflected by the atmosphere in the backward direction to initial propagation. The more backscatter that occurs for a specified wavelength, the more light that can be collected by the telescope aperture to compensate for optical distortion.

More recently, the Sodium LGS has been developed for AO application. Sodium LGSs are focused at creating a backscatter spot at 90 km in altitude and thus account for more atmospheric distortion. Consequentially, sodium LGSs are better (but also more expensive) reference sources comparatively to Rayleigh LGSs [5]. In contrast to Rayleigh LGSs, sodium LGSs exploit a gaseous layer of sodium in the mesosphere at 90 Km altitude to produce a resonance
fluorescence backscatter. This resonant backscatter is a direct result of the atomic transition states of the sodium atom which produce fine wavelength emissions of 589.1 - 589.2 nm and is a form of resonance scattering. The reason that many optical wavelengths are not commonly used for LGSs is a result of the inability to produce sufficient backscatter, thus making them obsolete in comparison to established LGSs. One technique used for many LGS systems is to pulse the laser and then range gate the returning signal. In essence, a range gate is intended to ensure that only light reflected from the desired atmospheric volume is collected by the telescope. This technique minimizes background noise and ensures the measured optical path is at a maximum by range gating to the highest altitude for efficient backscatter [5]. The range gating technique is shown in Figure 2.1.

![Figure 2.1](image)

Figure 2.1 An LGS focused at range \( z \) with a range gate of \( \Delta z \). Optical sensors at the telescope are shuttered so that they receive only information within a small timeframe centered at the laser return pulse [19].

To reiterate, if enough photons are collected to obtain a high enough SNR at the wavefront sensor, then we are able to begin the process of correcting for turbulence. It is also important to note that a Rayleigh LGS was analyzed for implementation on to the JBO-Q telescope in a previous study [18]. Information regarding parameters of the selected laser wavelength and
range gate optimization sizes can be found on Figure A.1, Figure A.2, and Table A.1 in the appendix. Therefore, this effort will not be focusing on the beacon wavelength selection since it is beyond the scope of this thesis but further information regarding can be found at reference [18]. The overall important thing to understand about LGSs is that laser light projected upward backscatters off of the atmospheric then travels back along the optical path (becoming distorted along the way) to ultimately be measured by a sensor for wavefront distortions. The most common sensor in use today for detection of backscatter is called the Shack-Hartmann Wavefront Sensor (SHWFS).

II.3. Shack-Hartmann Wavefront Sensor

The SHWFS was developed by Roland Shack by modifying Johannes Franz Hartmann’s aperture array design during the 1960’s in order to investigate the methods of wavefront distortion measurements for turbulence degraded images [17]. By using a set of lenslets (all of same focal length) to focus incoming light onto a CCD or CMOS detector array, as shown in Figure 2.2, the slopes of these waves can be measured.

Figure 2.2 Depicts two incident light waves (red and green) upon a lenslet (microlens) and the resultant spot on the sensor detector. Here, a 1-D lenslet is depicted but in practice 2-D lenslets are used to determine x-y slope measurements [14].
Each individual square lenslet focuses the light into a spot on a quad cell (for the purpose of this work), which is set of 4 pixels typically surrounded by a guard ring. If the focus spot is off-center then we know that the incoming wavefront has a “local tilt” (aberrated slope) or “phase distortion” and can thus calculate its tilt value from its position on the quad cell. The distribution function for intensity on the detector plane is:

\[ I(x, y) = I_o \left( \frac{\sin \frac{\pi dx}{\lambda z}}{\frac{\pi dx}{\lambda z}} \right)^2 \left( \frac{\sin \frac{\pi dy}{\lambda z}}{\frac{\pi dy}{\lambda z}} \right)^2 = I_o \left( \frac{\sin ax}{ax} \right)^2 \left( \frac{\sin ay}{ay} \right)^2 \]

Where \( I_o \) is the incident intensity, \( \lambda \) is the wavelength of the radiation measured, and \( z \) is the range to the measured object.

Many local tilts are measured simultaneously by the wavefront sensor and can be reconstructed into the overall shape of the incoming aberrated wave. The concept of measuring local tilt is depicted by Figure 2.3. Subsequently, phase distortion measurement signals are transmitted from the SHWFS to the AO control system which calculates the phase conjugate of the entire wavefront. In chapter III, residual phase error is considered which results from imperfect measurements taken by a SHWFS. Although residual phase error has multiple constituents, the error produced from the SHWFS is often referred to as “wavefront sensor error” (WFS error).
II.4. AO Control System

After the wavefront phase distortion has been properly measured by the SHWFS, the AO control system processes those measurements to calculate the overall shape of the total wave. Next, a wavefront reconstruction algorithm (which will not be outlined or analyzed for the purpose of this work) produces the wave’s approximate conjugate phase where correctness is dependent upon the density and accuracy of x-y tilt measurements taken [5]. Other sources of error during reconstruction occur due to system time delays.

Proper technique when coding the wavefront reconstruction algorithm should include the optimization of mapping SHWFS subapertures to the deformable mirror (DM) actuators. Although there are multiple methods that accomplish this (i.e. Fried, Hudgin, Southwell), the results of this model are independent of specific alignment geometry. Since the AO control system is the processing center for all incoming real-time data, it must ingest, process, and output signals rapidly which requires the use of parallel computing, but with the advancement of signal processing technology, capable processors are currently available and affordable in today’s market. Figure 2.4 shows the layout of a typical AO system.
II.5 Deformable Mirror

After the AO control system generates the wave phase conjugate of the incoming aberrated wave, the signals are mapped to the actuators on the deformable mirror. Actuators are piston-like components on the back of a reflective mirror that physically deform the mirror itself in order for the mirror to take shape of the wave phase conjugate. Upon conjugation, the process of wavefront compensation is complete if indeed accurate measurements were received, transmitted, and processed through the AO control system algorithm to the actuators (process depicted by Figure 2.5). Two important considerations when developing (or purchasing) a DM are that the number and geometrical configuration DM actuators result in different output performances. This work will consider the optimal number of actuators for the DM to be
Figure 2.5 Actuators warp DM in the wavefront reconstruction process to produce a non-aberrated wavefront [24].

implemented at site but will not consider the optimal geometry to mapping of SHWFS subapertures to actuators. The consequences of actuator configurations are labeled for each graphic in Figure 2.6.

Figure 2.6 Common DM actuator configurations [24].

As a result of varying configuration options, DMs may vary in design concept so it may be non-trivial in determining which type and configuration is best for the site of application. In
fact, many considerations may come from requiring a DM to communicate with previously purchased components. Types of DMs include continuous faceplate, liquid, segmented, bimorph, membrane, microelectro-mechanical, ferromagnetic, and magnetic mirrors. Although it is important to be aware of each type, the implications of each are not considered for this work. This work is focused solely on the continuous faceplate design concept for application since it is currently the most commonly used DM concept for AO.

DM components vary in material design as well as performance speed and resolution but generally consist of a reflective surface and actuators. The final outcome of DM wavefront reconstruction is to generate the wavefront conjugate so that the next incoming wave is corrected before it hits the telescope’s final photodetector, which in the case of an optical sensing, is an electro-optical (EO) camera.

![Figure 2.7 Depicts a typical actuator configuration [26]](image)

Furthermore, this entire process from measurement of the aberrated wave to wavefront compensation on the DM must take place before the atmospheric profile dramatically changes, thus overall system response rate must exceed the bare minimal threshold speeds for correction by a significant factor to avoid signal aliasing. Atmospheric turbulence changes at a rate called the Greenwood Frequency (defined in Chapter III) and is the external factor which (in general)
sets the minimum speed at which an AO system must run to perform proper correction [5]. On that note, is it always important to consider threshold operating frequencies at full resolution for all system components as the slowest component will likely be the system’s limiting factor. Often times, this will come down to pulsed laser power. Factors for determining overall system framerate are discussed in chapter IV.

II.6 Signal Detection

The final step of capturing high resolution images using an AO system is simply to collect the data with a camera. As mentioned above, the wave phase conjugate is produced at the DM using data from the SHWFS and compensates (approximately) for the next incoming distorted wave. In Figure 2.4 on page 11, we see this final detection to the right of the beam splitter after the wavefront is corrected off of the DM. Note that this corrected wavefront is the result of the entire AO system feedback loop. The most important specifications when choosing an EO camera for sensitive, high-speed measurements are typically the full resolution framerate, its frequency bandwidth, and its background noise as these three factors will determine the quality of the data that is collected. The analysis section of this work will help define the necessary EO camera framerate.

Now that a complete description of an AO system has been given, external and instrument factors will be discussed in Chapter III to provide an in depth understanding of how system performance is analytically calculated. First up in Chapter III is a closer look at Earth’s atmosphere which will provide a clearer picture of how light interacts with it.
III. Turbulence

III. Atmospheric Structure

The shape of Earth’s atmosphere is comparable to that of an onion in the sense that it has multiple layers where some layers are thicker than others. Earth’s atmospheric layers are the troposphere, the stratosphere, the mesosphere, the thermosphere, and the exosphere where the troposphere is lowest and the exosphere is highest in elevation. Likewise, density decreases with altitude thus the troposphere is most dense and the exosphere least dense. As a result, most atmospheric turbulence occurs in the troposphere and stratosphere, which combined, reaches up to 50 kilometers (km) in elevation. Although each of these layers have unique effects on turbulence, most atmospheric turbulence is considered negligible after 30 km and it is also within only 1 km of the ground that most optical modulation occurs [5]. Additionally, the laser to be used primarily for atmospheric characterization measurements (not AO) is expected to be integrated at the JBO-Q and projected at 10-15 Km altitudes. The remainder of this thesis focuses only the troposphere for optical effects due to atmospheric turbulence. It is within this tropospheric volume that atmospheric profile data has been collected. This atmospheric profile data was then used to simulate expected atmospheric turbulence information over the site using the LEEDR software tool (details in Chapter IV). Some of the significant parameters that contribute to these simulations include the time of day, climate, humidity, wind velocity, pressure, temperature, and geography. Together these correlated data are referred to as numerical weather prediction (NWP) data. Collection of NWP data is vital to site specific atmospheric characterization and enables calculation of local atmospheric parameters such as the Fried parameter and the Greenwood Frequency, which will be discussed in the sections to come.
III.2 Atmospheric Modeling Parameters

III.2.A Refractive Index Coefficients

Specifically, atmospheric and environmental factors such as the time of day has dramatic effects on the refractive index of atmospheric layers. In order to calculate these indices for a specific site, is it necessary to gather relevant atmospheric data as mentioned before (i.e. temperature, pressure, humidity, wind velocity, altitudes). Ultimately, we will integrate these individual refractive indices to obtain the Fried Coherence Parameter (also known as the turbulence parameter or as the atmospheric coherence length) by using an NWP data profile. An individual index of refraction coefficient for a specific atmospheric layer of arbitrary thickness is denoted as \( C_N^2 \).

\( C_N^2 \) values are dependent on the specific wavelength in a specified atmospheric layer. In the case of electro-optical sensing telescopes, we only consider refraction of visible wavelengths (390 nm to 700 nm). The refractivity of air is closely approximated by Equation 3.8 [5]. Here the equation is denoted by (1).

\[
N = (n - 1)10^6 = 77.6 \left( \frac{P}{T} \right) \tag{1}
\]

where \( P \) is pressure in millibars (mb) and \( T \) is temperature in Kelvin (K) [5]. Local pressure fluctuations are smoothed out at the speed of sound and are negligible compared to temperature fluctuations. Therefore, the refractive index fluctuations for vertical propagation are due mainly to temperature and can be expressed by

\[
\delta N = -77.6 \left( \frac{P}{T} \right) \delta T \tag{2}
\]
where the structure function of refractive index variations is defined as

$$D_N(r) = C_N^2 r^{\frac{2}{3}}$$

(3)

and $C_N = \frac{\delta N}{\delta t} C_T$ and $r$ is the distance between velocity components. A structure function is simply a probability density function and in the cause of the turbulence structure function it is the spatial variance in the difference of the refractive index as a function of separation. The turbulence structure function will be used in order to eventually compute the Fried coherence length. The value of $C_T$ is often measured by probing the atmosphere (i.e. collected as NWP data) [5]. Although a full description of turbulence velocity components will not be described in this work, the basic mechanic principles involve fluid dynamics based on the Kolmogorov model of currents circulating in whirlpool like forms called “eddies”. In short, the Kolmogorov Model is used to analyze complex, random phenomena and focused around modeling velocity of motion in a fluid. Turbulence occurs in this model when a critical Reynolds number is reached. The Reynolds number is defined as

$$Re = \frac{V_o L_o}{\nu_o}$$

(4)

where $V_o$ is the characteristic velocity, $L_o$ is the characteristic size of the flow, and $\nu_o$ is the kinematic viscosity of the fluid. Small Reynolds numbers correspond to laminar flow while large Reynolds numbers correspond to turbulent flow. In air, a scale of 15 meters, a velocity of 1 m/s, and a $\nu_o$ of 15 x $10^{-6}$ yields a value of 1 x $10^6$ which greatly exceeds the critical Reynolds number in air, thus the atmosphere is nearly always turbulent. To visualize turbulence, Figure
3.1 shows transition of energy through a medium until the critical Reynolds number is reached and the flow becomes turbulent.

![Diagram of Reynolds numbers and flow regions](image)

Figure 3.1: An illustration of Reynolds numbers which define turbulent flow regions [8].

Ultimately, solar heating produces disturbances over a large range of inertial scale sizes. During the daytime, convection cells on Earth’s surface produce turbulent fields while at night the mixing of air masses of different temperatures and altitudes is the key driver of turbulence. As turbulent decays, the kinetic energy is continually transferred to motions of smaller and smaller scales as an energy cascade until the Reynolds value drops below the point of criticality. At this stage, kinetic energy is dissipated into heat by molecular friction [5]. To analyze EM propagation through turbulence, the 3-D power spectrum is needed:

$$\phi(k)dk = 0.033C_n^2k^{-11/3}$$  \hspace{1cm} (5)
where k is the wavenumber. The limits of this spectrum are still debated since the Kolmogorov model assumes an incompressible medium and validity within an inertial range between the inner and outer scales. Specifically, this means that equation (5) is only valid for a defined range of velocities pertaining to Earth’s atmosphere [5].

III.2.B The Fried Coherence Parameter

Now that we have obtained the structure function of refractive index variation (equation 5), we now want to find out how it relates to the Fried Coherence Parameter which is a convenient measure of integrated turbulence strength and represents the coherence length for a given wavelength over all refractive-index fluctuations for the entire atmospheric in the optical path. To do so, we apply equation (3) to a 3-dimensional case considering the phase shift of an electro-magnetic wave propagating through a layer of thickness \( \delta h \) where it is a function of altitude (denoted by variable \( z \)) so that

\[
D_N(r) = C_N^2 (r^2 + z^2)^{\frac{1}{3}}, \quad r = |\mathbf{r}|
\]  

(6)

and

\[
D_\phi(r) = k^2 C_N^2 \delta h \int_{-\infty}^{\infty} \left[ (r^2 + z^2)^{\frac{1}{3}} - z^2 \right] dz
\]  

(7)

Upon evaluating the integral we find that,

\[
I = 2.914r^\frac{5}{3}
\]  

(8)

where \( I \) is simply the integral value and \( r \) is the refractive index and phase structure function at the output of a thin layer with Kolmogorov turbulence is then described by
\[ D_\varphi(r) = 2.914k^2C_N^2 r^{5/3} \delta h \] (9)

The coherence function at the layer output in terms of the refractive-index variations may be expressed as an exponential function eventually leading us to denote the continuous distribution of turbulence as

\[ B_0(r) = e^{-\frac{1}{2}[2.914k^2 \sec(\zeta)r^{5/3} \int_h C_N^2(h) dh]} \] (10)

Here, \( B_0 \) is the coherence function at the ground, \( \zeta \) is the angle from zenith and \( h \) is the atmospheric layer term. This expression is of fundamental importance in determining the effect of atmospheric turbulence on image structure [5]. Finally, this equation combined with the atmospheric transfer function denoted by \( B(f) \)

\[ B(f) = e^{-3.44(\frac{f}{r_0})^{5/3}} \] (11)

brings us to our final equation for \( r_0 \) in terms of the integrated turbulence.

\[ r_0 = [0.432k^2(\sec \zeta) \int C_N^2(h) dh]^{-2/5} \] (12)

where \( k \) is the wavenumber \( \left( k = \frac{2\pi}{\lambda} \right) \) and in units of rad/m. In essence, we are integrating all refractive indices over the optical path to obtain the Fried coherence parameter \( r_0 \). This parameter is a convenient measure of integrated turbulence strength and represents the coherence length for a given wavelength over all refractive-index fluctuations for the entire atmospheric in the optical path. Small values of \( r_0 \) correspond to strong turbulence while large values correspond to weak turbulence [5].
The Fried Coherence Parameter also describes the maximum aperture size which is not affected by turbulence effects. For example, if \( r_0 \) is larger than the aperture size (i.e. the coherence length is larger than the aperture diameter) then aberrated light is not detected by the aperture. Because \( r_0 \) has dimension of length in meters (m), it is often regarded as defining the “cell size” of atmospheric turbulence. It is also important to note that equation (12) shows that \( r_0 \) is wavelength dependent by a factor of 6/5 power of wavelength [5]. In a later chapter, we will see how this dependence is convenient when determining how longer wavelengths affect system performance.

### III.2.C Isoplanatic Angle

By definition, the isoplanatic angle is the angle for which the rms wavefront phase error increases by 1 radian. The isoplanatic angle depends primarily on the strength of the high-level turbulence where the difference between optical paths is greatest and is also proportional to \( \lambda^{6/5} \). Typically, the isoplanatic angle is on the order of 2 arcsecs at a wavelength of 0.5 micrometers. The guide star must be within this angle to ensure that the wavefront reference is within the same optical path as the object of interest, otherwise different optical wavefronts measured and object of interest is not resolved best [25]. Please reference the appendix for isoplanatic angle calculations. Note that these calculations are useful for future research for application of AO to the JBO-Q telescope but are not significant to results of this work.

### III.2.D Phase Error and Strehl Ratio

Imagine a spherical wave of light emanating from a faraway star and spreading out through space until finally it reaches the Earth’s outer atmosphere. At this point, the once spherical wave
has spread out so much that it is now approximately planar. From this point, this flat, planar wave then encounters countless layers of atmosphere distorting its shape both spatially and temporally. One way to visualize this is to think of a flat bedsheet as the planar wave of light which ripples as it falls through the atmosphere. This process is also known as phase distortion of a wave and in redundant terms, it is the optical distortion caused by atmospheric turbulence.

While the previous section mathematically describes the integrated coherence length of the turbulence, it does not describe how the wave itself geometrically and temporally “wobbles”. This “wobble” is synonymous to the spatial and temporal phase distortions of the wave. Spatial phase distortion is a measure of how much the same wave changes between two points at the same instant in time whereas temporal phase distortion is a measure of how much the same point on a wave changes over time. Specifically, temporal coherence is a manifestation of spectral purity meaning that waves that are temporally coherence are of identical wavelength [9]. Figure 3.2 depicts this concept.

![Figure 3.2](image)

Measurement of a single point on a temporally coherent wave at two different times

Measurement of two points on two spatially coherent waves at the same moment in time

Figure 3.2. Shows the difference between spatial phase measurements and temporal phase measurements.

While traveling through turbulent layers, phase changes are considered “external” since they occur outside the influence of a telescope system. Likewise, internal phase changes are controlled results of adaptive optical components in the telescope system such as mirrors, beam
splitters, wave plates, and lenses. For the purpose of wavefront measurement and compensation, all we really care about are the internal phase changes applied after measuring a distorted, incoming optical wave. When internal phase changes are applied, both external and instrument factors result in residual phase error. These factors are listed in Figure 3.3 on the page 23. Diagram 3.3 also shows which types of phase error external and instrument factors primarily affect. The instrument factors labeled in green text signify those that were considered for the scope of estimating phase error in this project.

The primary contributors to residual phase error are fitting error, temporal error, and photon error. Photon error is a term that will also be used synonymously used with wavefront sensor error. This work will focus on these three error aspects to baseline the overall performance. As seen in Figure 3.3, external conditions are factors such as the turbulence strength, background radiation, and beacon power while instrument factors include integration time, actuator spacing, and the telescope aperture. Although not explicitly listed, instrument factors can also be derived material imperfections or even may be quantum mechanical in nature due to the size of the measuring device. Although in more sophisticated models these errors have correlations but errors for this work are assumed to be independent. The total residual phase error is the summation of uncorrelated phase error components and described by the equation (13).

\[ \sigma_{\text{phase}}^2 = \sum_i \sigma_i^2 \] (13)

In practice, some individual phase errors are correlated values which can lead to an overestimation of residual phase error using uncorrelated values. Thus, in an AO system design, it is not necessarily a disadvantage to estimate residual phase error as uncorrelated phases since it often compensates for unrecognized error which ensures performance remains realistic [5].
Figure 3.3 Shows external and instrument factors that contribute to overall phase error. Instrument factors highlighted in green are those considered for the scope of this analysis [5].
Fitting error is a result of a wavefront reconstructor’s (i.e. a deformable mirror) inability to perfectly compensate for distortion. In other words, this means that the deformable mirror cannot perfectly form the distorted wave’s conjugate so that the incoming wave is not corrected into a perfectly planar wave. In order to perform nearly exact wavefront reconstruction, it would be necessary to have extremely high spatial frequency bandwidths which are simply impractical. Fitting error is mathematically defined as the difference between a turbulent wavefront and a wavefront corrector’s surface figure [5].

\[
\sigma_{\text{Fit}}^2 = \langle [\varphi_w(x) - \varphi_c(x)]^2 \rangle
\]  

(14)

In equation (14), \(x\) is the position vector and the right side is the ensemble average. For a wavefront in air with a Kolmogorov spectrum (power \(\propto l^{5/3}\) where \(l\) is the scaling size) the fitting error is approximated to be (see page 24)

\[
\sigma_{\text{Fit}}^2 = a_F \left(\frac{d}{r_0}\right)^{5/3}
\]  

(15)

where \(a_F\) is a constant depending on the segment geometry and control mode of the DM and \(d\) is the diameter of the SHWFS subapertures. For purposes of modeling, a continuous faceplate DM will be considered \((a_F = 0.28)\) which corresponds to shared actuators in a hexagonal segment geometry (reference Figure 2.6 on page 12) and as a result yields lower fitting error comparatively to other geometries [5].

Since the fitting error is only dependent on \(r_0\) and \(d\), it is obvious that as the ratio of \(\frac{d}{r_0}\) becomes smaller, the fitting error also diminishes. This essentially tells us that we want our subapertures small while \(r_0\) is large. Typically, \(r_0\) does not go above a value of 20 cm in Earth’s atmosphere but may be smaller than 1 cm in extremely poor viewing conditions. In contrast, it is
also important to consider that the smaller a subaperture is, the less light it collects, thus very small subapertures do not increase system performance but rather do the opposite since not enough signal can be collected comparatively to the device noise. As a result of these two considerations it has been found in practice that typically the optimal subaperture size is \( r_0 \), thus the ratio, \( \frac{d}{r_0} \), approaches 1. In section III.2.E, which estimates photon count at subaperture detectors, we will see more as to why larger WFS subaperture sizes increase system performance and how the balance of these two effects results in a WFS subaperture diameter length that approaches \( r_0 \) [5].

Residual phase error from temporal considerations is another large factor in the overall residual phase error. Residual temporal phase results from movement of perturbed layers by winds or telescope slewing movement. Inevitably the synchronicity of devices measuring and compensating for phase takes time and thus also results in some temporal error. In other words, temporal phase error is a direct result of multiple time delays which occur as the signal is measured, transmitted, and eventually conjugated. By definition, temporal error is described by the equation

\[
\sigma^2_t = \langle [\varphi_w(x,t) - \varphi_w(x,t + \Delta t)]^2 \rangle \tag{16}
\]

where \( \Delta t \) is the time delay. The mean-square temporal error has the same basic form as fitting error and is proportional to \( \left( \frac{\Delta t}{t_o} \right)^5 \) where \( t_o \) is the characteristic change time of the turbulence. This time is defined as the Greenwood Frequency which will be further explained in the next section [5].
The last primary component of residual phase error is photon error which is produced by random noise in the wavefront sensor. As mentioned before, this error is also referred to as wavefront sensor error and will be described mathematically in the following sections. Photon error consists of shot noise from the reference source and background radiation which combines with amplifier noise produced during detection of the reference signal. Shot noise is inversely proportional to the square root of the number of photons counted at detection and is dominant for small photon fluxes. The small amount of backscattered light available from LGSs or NGSs often causes wavefront sensors to operate under restricting conditions, thus photon error is typically a large component of the total residual error. The photon error may be reduced by increasing integration time but this consequentially increases temporal error, thus a tradeoff must be made to determine the optimum value. [5]

Shot noise and background radiation contribute to the overall signal-to-noise (SNR) ratio in the SHWFS. The residual wavefront sensor phase error produced from the SHWFS is defined mathematically by the equation

\[
\sigma_{WFS}^2 = \left( \frac{\pi^2 K_q}{\text{SNR}} \right)^2 \left[ 1.5^2 + \left( \frac{\theta d}{\lambda_s} \right)^2 \right] \text{if } d < r_o \text{ or}
\]

\[
\sigma_{WFS}^2 = \left( \frac{\pi^2 K_q}{\text{SNR}} \right)^2 \left[ 1.5^2 \left( \frac{d}{r_o} \right)^2 + \left( \frac{\theta d}{\lambda_s} \right)^2 \right] \text{if } d > r_o
\]

(17.1)

(17.2)

where \( K_q \) is the loss factor due to the gap between quadrant detector elements (= 1.3 to 1.5) and \( \theta \) is the angular subtense of the reference source in radians. These errors are the mean-square error in radians squared of phase per subaperture. Overall, SNR is proportional to \( \sqrt{n_p} \) which is the number of photons received at the detector [5]. The SNR in the above equations is defined as
\[ SNR = \frac{n_p}{\sqrt{n_p + \sigma_{SR}^2}} \]  

(18)

where \( \sigma_{SR}^2 \) is the combined shot and read noise of the sensor defined by

\[ \sigma_{SR}^2 = N_{pix} (n_B^2 + n_R) \]  

(19)

so that \( n_B \) is the number of background and noise photoelectrons, \( n_R \) is the read noise of the sensor and \( N_{pix} \) is the number of pixels of the detector (typically a quadcell).

When observing a celestial body, it is of utmost importance that the telescope system collects enough light from it so that the signal-to-noise (SNR) ratio is large enough for sufficient resolution. To increase SNR, background noise is often eliminated through the use of a range gated LGS. For a Rayleigh guide star, the following photon estimation will involve the Light Detection and Ranging (LIDAR) equation and specifications of a laser using a green wavelength. Before photon count is estimated, it is important to outline how the total phase error affects overall image quality which is measured by the Strehl ratio.

The Strehl ratio is a mathematical means to measure the quality of optical image formation and is used in situations where optical resolution is compromised due to lens aberrations or imaging through a turbulent atmosphere [10]. The Strehl ratio is defined as the ratio of the peak aberrated image intensity from a point source compared to the maximum attainable intensity using an ideal optical system which is only limited by diffraction over the system’s aperture [5]. Additionally, it is often expressed as the intensity at the image center from an on-axis source.
It is expected that wavefront errors will diffract light away from the center of the image and thereby reduce peak intensity. The diffraction limited intensity, \( I^* \), is computed with no aberrations as

\[
I^* = \pi^2 \left( \frac{Aa^2}{\lambda R^2} \right)^2
\]

(20)

where the ratio of the peak intensity \( I(P) \) to \( I^* \) is a useful measure of optical system performance at the Gaussian image plane. Here, \( a \) is the pupil radius, \( A \) is the intensity, and \( R \) is the spherical reference wavefront. For a system with aberration function \( \varphi(\rho, \theta) \) the Strehl ratio is given by the ratio of actual peak intensity \( I(P) \) to \( I^* \):

\[
S = \frac{I(P)}{I^*} = \frac{1}{\pi^2} \left| \int_0^1 \int_0^{2\pi} e^{i k \varphi(\rho, \theta)} \rho d\rho d\theta \right|^2.
\]

(21)

To compute the Strehl ratio using the equation above, it is necessary to know the aberration function \( \varphi(\rho, \theta) \) which is not common in the case of AO as a result of random errors due to turbulence [5]. Thus, if we expand the integral terms we obtain

\[
S = \frac{1}{\pi^2} \left| \int_0^1 \int_0^{2\pi} \left[ 1 + i k \varphi(\rho, \theta) + \frac{1}{2} [i k \varphi(\rho, \theta)]^2 + \cdots \right] \times \rho d\rho d\theta \right|^2
\]

(22)

in order to define the “aperture average” values of the wavefront errors over the pupil with respect to a reference sphere centered at focal point \( P \) so that,

\[
\text{avg}(\varphi^P) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \varphi(\rho, \theta) \rho d\rho d\theta.
\]

(23)

Neglecting higher order (less significant) terms, we find that
\[ S' = 1 - k^2 \left[ (\varphi_p^2) - \left( \bar{\varphi}_p \right)^2 \right] \]  

(24)

where the mean-square wavefront error is defined as

\[
(\Delta \varphi_p)^2 = (\varphi_p^2) - \left( \bar{\varphi}_p \right)^2
\]

where the standard deviation of the phase is \( \sigma_p = k \Delta \varphi_p \) and the Strehl ratio is approximated to be

\[ S = 1 - (\sigma_p)^2. \]  

(25)

This expression is only useful for very small phase errors up to about 0.6 rad (1/10 wave) rms and falls to zero at 1 rad of phase error. However, in 1947 Andre Marechal derived a similar expression for normalized intensity at the diffraction focus in presence of small aberrations. The Marechal approximation equation

\[ S \approx e^{-(\sigma_p)^2} \]  

(26)

is valid for phase errors up to 2 rad rms and is the most commonly used expression of the Strehl ratio. It is this equation that will be utilized for the modeling explained in chapter IV. Large wavefront errors, such as those caused by uncorrected atmospheric turbulence, produce radical changes in image structure and can be estimated better by the optical transfer function, which will not be discussed in this work [5].

III.2.E Greenwood Frequency

As mentioned previously, the Greenwood frequency is a common useful parameter for estimating the temporal phase error expected for wavefront compensation. To reiterate, temporal
errors are caused by the inability of AO systems to respond instantaneously to changes in the wavefront, thus any delay between measurement and correction of a wavefront disturbance results in temporal error. The main time delay is usually the integration time of the servo control loop during which the wavefront sensor collects photons from the reference source. Depending on turbulence strength, wind speed, and wavelength of measurement, the system delay constant is typically between 1 and 10 milliseconds. The frequency response of a closed-loop AO system is limited primarily by the sampling rate of the WFS (often between 1 – 10 KHz). To ensure stability of the feedback loop, the closed-loop bandwidth is often 1/10 of the sampling rate. Since other temporal errors are not considered in the scope of this project, the closed-loop bandwidth will be conservatively set to 1/20 of the sampling rate (recommended by Dr. Robert Johnson, PhD, Starfire Optical Range, Kirtland Air Force Base, New Mexico). The servo bandwidth determines the effective integration time during which photons are collected from the reference source and the optimum integration time is found by trading off measurement error (photon number vs temporal error). As most AO systems are photon starved, their optimal temporal bandwidth is usually lower than that of the wavefront disturbances which results in significant errors [5].

Daryl P. Greenwood first determined how to estimate temporal error caused by a finite correction bandwidth in 1977 by finding the residual wavefront error in terms of the ratio of the effective bandwidth of the turbulence to that of the correction servo. The servo response may be expressed by the complex function $G(f)$ where the power rejected by a low pass filter is described by equation 27 [5]:

$$\sigma_R^2 = \int_0^\infty |1 - G(f)|^2 F_\varphi(f) df$$  \hspace{1cm} (27)
where $G(f) = \frac{1}{2 + (f_c/f)^4}$ where $f_c = \frac{1}{2\pi RC}$ such that R is resistance in ohms and C is capacitance in farads. The asymptotic turbulence power spectrum

$$F_{\varphi}(f) = 0.0326 \sec(\zeta) k^2 \nu_{5/3} f^{-8/3}$$

(28)

which is the first power law of the 2-D Kolmogorov phase spectrum representing the absolute phase at a single point. In equation 28, $\nu$ is the turbulence weighted wind velocity over the optical path and $f^{-8/3}$is a Kolmogorov power law of phase for moderately frequencies.

Integrating equation (28) using equation (27), the wavefront temporal error, denoted as $\sigma_{TR}^2$, may be expressed in the form [5]

$$\sigma_{TR}^2 = \kappa \left( \frac{f_G}{f_s} \right)^{5/3}$$

(29)

where $f_s$ is the servo bandwidth of the system and $f_G$ is the Greenwood frequency. In the instance that a simple RC circuit is used and the bandwidth is defined at the half-power point, $\kappa = 1$. The Greenwood frequency is a characteristic frequency of atmospheric turbulence and given by the equation,

$$f_G = [0.102k^2 \int_0^\infty C_N^2(h) \nu^5(h) dh]^{3/5}.$$  

(30)

Equation (30) integrates over all indices of refraction in the optical path as a function of altitude $h$ while taking into account the wind velocity profiles at such altitudes. Additionally, equation (30) does not account for time delays in the feedback loop, nor for the averaging effects of the finite subaperture size but rather only considers the rate of changing atmosphere. In other words, it assumes that the wavefront is measured at a single instant of time, in WFS subapertures comparable in size of $r_0$. The wavefront error caused by time delays can be estimated separately
and added to the total residual temporal error. The effect of using subapertures significantly larger than \( r_0 \) reduces the temporal component of the error and increases the fitting error, thus equation (28) tends to overestimate temporal error when \( \frac{d}{r_0} > 1 \) [5].

For the Greenwood frequency computation, is it also plausible to include slew of the telescope which requires manipulation of the wind velocity profile by including angular velocity components as the telescope tracks an object in Earth’s orbit (such as a satellite). In essence, the Greenwood frequency equation provides one the analytic means to approximate the temporal error of the system which should be included in the overall residual phase error. More details as to how this error is included to baseline overall AO performance relating to framerate will be covered in Chapter VI.

**III.2.F Laser Propagation and Photon Count Estimation**

The process of using a light beam to probe a medium using the backscattered energy as a function of range is known as light detection and ranging (LIDAR). The use of lasers to create guide stars for AO is directly related to this remote sensing technology. The LIDAR equation defines the approximate energy detected at the receiver by knowing the scattering process of the light beam as it propagates through the medium. In the case of optical wavelengths through Earth’s atmosphere, this is predominately Rayleigh scattering. The LIDAR equation essentially multiplies various probabilities while using an initial photon emission count and added background noise to estimate the number of photons backscattered and then received. In addition, a pulsed laser and range gated detection are often used to better control the source of light returned. Assuming the range gate length \( \Delta z \) is small compared with the range \( z \) implies
n(z) and $T_A$ may be constant over the range gate distance so that the backscattering process is linear ($\sigma_B = \text{constant}$) and the LIDAR equation may be expressed symbolically as [5]:

$$ N(z) = \frac{E}{hc} (\sigma_B n(z) \Delta z \left( \frac{A_R}{4\pi z^2} \right) T_o T_A^2 \eta + N_B $$

(31)

where the variables of equation (31) are listed below [1].

$N(z) =$ Expected number of photons detected in range interval $\Delta z$

$E =$ Laser pulse energy, J

$\lambda =$ Optical wavelength of laser, m

$h =$ Planck’s constant, $6.626 \times 10^{-34}$, Js

$c =$ Velocity of light, $2.99 \times 10^8$, m/s

$\sigma_B =$ Effective backscatter cross-section, $m^2$

$n(z) =$ Number of density scatterers at range $z$, $m^{-3}$

$A_R =$ Area of receiving aperture, $m^2$

$\Delta z =$ Receiver range gate length, m

$z =$ range at center of range gate, m

$T_o =$ Transmission of optical components in transmit and receive paths

$T_A =$ One-way transmission of atmosphere between telescope and beacon

$\eta =$ Quantum efficiency of photon detector at wavelength $\lambda$

$N_B =$ Number of background and noise photoelectrons

It is also important to note here that all factors of optical efficiency have been grouped into a simple multiplicative term which indicates that some dependencies may be overlooked. The Rayleigh scattering process is characterized by its scattering cross-section, which is non-isotropic and depends on the polarization of the incident radiation, neither of which are considered in equation (31). Regardless, the scattering cross-section for Rayleigh scatter is given by:
\[
\sigma_B^R = \frac{\pi^2(n^2-1)^2}{N^2\lambda^4} \tag{32}
\]

where \(n\) is the refractive index of the medium, \(N\) is the atom density and \(\lambda\) is the wavelength of the laser. More specifically for a mixture of gases below 100 km in altitude, the Rayleigh backscattering cross section is [11]:

\[
\sigma_B^R = 5.45 \left[\frac{550}{\lambda}\right]^4 \times 10^{-32} \text{m}^2\text{sr}^{-1} \tag{33}
\]

The wavelength dependence \((\lambda^{-4})\) indicates that Rayleigh scattering is much more effective at shorter wavelengths and hence explains why the sky is blue. In layman terms, the sky is blue because all other colors scatter out. We also know that the product of the Rayleigh backscatter cross-section and the atmospheric density was estimated by C. Gardner to be [12]:

\[
\sigma_B^R n_R(z) = 3.6 \times 10^{-31} \frac{P(z)}{T(z)} \lambda_L^{-4.0117} \tag{34}
\]

where \(P(z)\) is the atmospheric pressure at range \(z\) in millibars (mb) and \(T(z)\) is the temperature at range \(z\) in Kelvin (K), and \(\lambda_L\) is the scattering wavelength [1]. By substituting in \(\sigma_B^R\) and using NWP data for \(P(z)\) and \(T(z)\) values allows us to solve for \(n_R(z)\) and find the approximate number of photons detected at the telescope aperture.

**III.2.G. Rytov Number**

The Rytov number, which is more commonly referred to as the Rytov approximation or Rytov variance is a fundamental scaling parameter for laser propagation through atmospheric turbulence and it is a statistical method for estimating scintillation along the propagation path.
Rylov numbers range from 0 to 1 and instances where it is greater than 0.2 are generally considered to be a result of strong scintillation. The Rylov variance is defined as

\[
\sigma_i^2 = 1.23 C_n^2 k^{7/6} L^{11/6}
\] (35)

where \( L \) is the propagation path length in meters, \( k \) is the wavenumber, and \( C_n^2 \) is a refractive index coefficient. The Rylov variance is the scintillation index calculated for a plane wave using the Kolmogorov spectrum where weak fluctuations correspond to Rylov variance \( \sigma_i^2 \ll 1 \).

Similarly, strong fluctuations imply \( \sigma_i^2 \gg 1 \) and moderate fluctuations imply \( \sigma_i^2 \approx 1 \). The Rylov approximation results in the log-amplitude variance (Rytov Number) for propagation through turbulence and given by the equation (36).

\[
\sigma_{\chi}^2 = 0.563 k_0^{7/6} \int_0^L C_n^2[h(z)] z^{5/6} \left( \frac{1-z}{L} \right)^{5/6} dz
\] (36)

Here \( k_0 \) is the wavenumber, \( z \) is vertical distance, \( L \) is the number of atmospheric layers propagated through and \( h(z) \) is a function of line of sight at range \( z \). It is important to note that Rylov theory fails to predict scintillation saturation which is a notable effect in propagation.

Regardless, many other beam control degradations have been noted to be governed by the Rylov number which include track scintillation error in active tracking schemes and hidden phase effects on least-squares reconstructors [21] [22].
IV. Adaptive Optics at the John Bryan Observatory Site

IV.1 Site Description

The John Bryan Observatory site is located at John Bryan State Park in Clifton, Ohio and serves primarily as an atmospheric turbulence research site for the United States Air Force (USAF) Air Force Research Lab Sensors Directorate (AFRL/RY).

This observatory was built by Kenneth E. Kissell in 1965 but has since become funded by the USAF. Since its construction, the facility has been retrofitted with modern computerized equipment to enhance astronomical research in support of the Air Force’s mission in Space Situational Awareness (SSA). The site exhibits a specialized, Quad-Axis, 0.61 meter, F17, Cassegrain telescope that has 207x pupil relay magnification and is referred to as the JBO-Q (shown in Figure 4.2 on the next page). This unique AF asset is capable of tracking objects through the zenith without rotation delay (no dead zone). As of June 2014, the telescope has demonstrated a closed-loop passive tracking capability to track a satellite in low Earth orbit (LEO) [18,19].
Since the passive tracking capability demonstration, research has refocused to apply an AO system in order to enhance its capabilities for space object identification (SOI). As a result, research for selection of an LGS has been accomplished by Nathan Figlewski which has shown that a Rayleigh beacon of 532 nm is a highly efficient initial LGS selection choice. Mr. Figlewski’s analysis was primarily based off of atmospheric propagation and scattering models and serves as some of the baseline analysis justifying the laser beacon wavelength chosen for this work. For additional reference to his analysis in regards to the Laser Guide Star Design for the JBO-Q site, please see source [18] in the bibliography.

The primary purpose of the current Rayleigh beacon system is to enable three-dimensional profile measurements of the refractive index structure parameter local to the humid, continental, low elevation site. It will provide additional capability to perform new studies in areas such as deep turbulence and other propagation research relevant to electro-optic ground-based sensing modalities [19].
IV.2 Scaling Law Modeling in Laser Propagation

The scaling laws are proportionality relations of any parameter associated with an object (or system) with its length scale. A simple example of this is how the volume of a cube relates to its surface area or side length. There are many common scaling relations that exist for effects of propagation through turbulence and most of which involve the ratio of a system parameter to an atmospheric parameter such as the ratio of $D/r_0$ (a dimensionless quantity). Common scaling relations are typically an asymptotic result of a more complicated integral expression. Treatment of light as a ray, scalar diffraction, performance limits, coherence, and propagation through random media are all examples of scaling law use in optics. Generally, scaling relations have limited applicability and regimes of validity, thus it is important to always check limiting cases when used for modeling [21,22].

Scaling laws used in propagation through turbulence include phase variance effects. With increasing turbulence strength, variance of atmospheric phase also increases. The atmospheric coherence parameter ($r_0$), Strehl ratio, the Marechal approximation, and the Rytov number are all scaling laws used in this work [22].

Scaling laws are analysis methods by which target irradiance properties are estimated through consideration of laser propagation metrics and function only under diverse parameter assumptions which stem from the system itself, the environment, and the target of interest. Scaling law methods are simply applicable approximations based on theoretical analysis or empirical observations and can help emphasize key dependencies of system performance. In a general sense, scaling laws are often used for concept generation and exploration, design trade studies, performance estimation and utility assessment [21].
To use a scaling law, one must consider a multitude of factors and establish analysis assumptions on the system and environment before proceeding. Note that methods vary when combining scaling law effects to quantify irradiance so it is important to vary the appropriate parameters depending upon interest. It is important to remain aware of assumptions underlying on analysis and understand that often times results will vary from other research analysis. The scaling laws used in this work imply optimism in terms of atmospheric turbulence and phase variance for baselining performance in this model. Additionally, they are limited in terms of accuracy as a result of access to atmospheric data relevant to the site [21,22].

IV.3 Laser Environmental Effects Definitions and Reference (LEEDR)

The NWP data analyzed for this project was originally collected for use by the LEEDR tool. LEEDR’s primary function is to output predictable atmospheric propagation parameters through analysis of local atmospheric profiles and is a government off-the-shelf (GOTS) software suite developed by the Air Force Institute of Technology (AFIT). LEEDR was designed with the intention of producing more realistic, higher resolution models of the atmosphere and its effects on optical propagation. The LEEDR software calculates path specific parameters such as transmittance, extinction, attenuation, and scattering. All of these parameters dramatically aid scientists when developing or modeling new optical components for potential integration. Table A.1 in the appendix shows atmospheric effects for a laser of 532 nm wavelength which is the wavelength used for WFS detection in this work. The NWP data was utilized in the Actuator Optimization Model (AOM) and the System Framerate Threshold Model (SFTM) which will be discussed later in Chapter IV.
For purpose of modeling an AO system for the JBO-Q, LEEDR was used to calculate a $C_n^2$ profile at resolution of 10 m using the Hufnagel-Valley 5/7 model using:

$$C_n^2(h) = 8.2 \times 10^{-26} W^2 h^{1.0} e^{-h} + 2.7 \times 10^{-16} e^{-h/1.5} + A e^{-h/1}$$  \hspace{1cm} (35)$$

where $h$ is the altitude in kilometers, $W$ is the root mean squared wind speed, and $A$ is a fitting term. The $C_n^2$ profiles using two data sets of NWP data are plotted in Fig. 4.3 [19]. The $C_n^2$ profiles and corresponding NWP wind velocity profiles were directly used to calculate the Fried parameter and Greenwood frequencies for the site at various angles of operation in both AOM and SFTM Models.

Figure 4.3 Modeled refractive index structure parameter using a Hufnagel-Valley 5/7 model and a Tatarskii model with NWP data for dates 15 July 2016 and 15 January 2016 [19][20].
IV.4 Adaptive Optics Site Modeling

IV.4.A System Framerate Threshold Model (SFTM) Overview

As stated before, the intention of the STFM is to approximate the bandwidth of the overall AO system by considering the maximum frequencies of atmospheric distortion likely to be experienced at site. As stated in Chapter II, the Greenwood frequency \( f_G \) is the external atmospheric factor that defines the rate at which the optical path changes and knowing this will enable us to define the system framerate bandwidth. To analytically solve for framerate, known \( f_G \) values are used along with a set requirement for the Strehl ratio \( S \geq 0.2 \) in this work to solve for the servo bandwidth described by equation (29).

Here, \( \sigma^2_{TR} \), which is the temporal resolution error, is defined by the Strehl ratio requirement mentioned above by using equation (25). The variable \( f_G \) was calculated using four profiles of NWP data which differ only by statistical weighting. Two sets of data (summer and winter) were split into two separate statistical percentiles (50% and 90%). The 50% cases average the data so that the resultant data can be interpreted as the average weather conditions to be experienced at the site for that season. In order words, 50% of the time weather conditions over the site will resemble this weighted profile or better. Similarly, the 90% data represents those instances where weather conditions are extreme and thus are less likely to occur. More precisely, \( C^2_n \) values are less than the given values 90% of the time, thus stronger \( C^2_n \) values are only experienced 10% of the time [24]. This NWP data is the only known available data with high enough resolution to justify any sort of analysis for site atmospheric characterization so to verify an actual seasonal average, more data must be recorded. Current plans for the JBO-Q include further collection of high-resolution atmospheric data that will yield a more accurate understanding of site turbulence parameters.
IV.4.B Analysis of SFTM

The Rytov Number, a common scaling law parameter for calculating the strength of atmospheric turbulence, was calculated over a number of elevation angles for each profile to gauge angles which strong turbulence is expected to occur and thus degrade AO capabilities. From analytic theory mentioned in the previous chapters, the green area in Fig. 4.4 represents angles of elevation in correlation with $r_0$ values that the JBO-Q will likely operate under. The four points marked in purple and red represent the threshold $r_0$ values (at angles) which will be used to evaluate system bandwidth since they are taken into account with Greenwood frequency calculations and then the overall system framerate. The points marked in Figure 4.4 were plotted to corresponding $r_0$ values as in Figure 4.5 on page 45. Since Rytov number values of larger than 0.2 indicate strong turbulence here, the profile data suggests that elevation angles below 40 degrees should generally not be operated during average weather conditions and below 53 degrees for extreme conditions.

![Rytov VS Elevation Angle](image)

Figure 4.4 Shows the relationship between atmospheric profiles and the Rytov numbers with respect to elevation angles. The marked points are correlated to $r_0$ values and thus calculated Greenwood frequencies.
We can see from Figures 4.4 & 4.5 that even though all points marked have a Rytov Number of approximately 0.2, an exact $r_0$ value is not explicitly shown. Regardless, the data profiles do suggest that $r_0$ values will fall within an appropriate range over which performance can be evaluated. The turbulence values circled in green will be used as performance evaluation measures in the AOM model which is covered in section IV.4.C. and IV.4.D. These two values were picked as they are the outermost limits of the probable $r_0$ range experienced at site based on the NWP data collected and the Rytov constraint.

Figure 4.5 Shows that there is not a directly proportional Rytov number value associated with an $r_0$ value.

Figure 4.6 was calculated using equation (28) and the NWP data from LEEDR using statistical percentiles to create four profile curves. The data in 4.6 suggests that the maximum Greenwood frequency generally would not exceed 145 Hz. Note that this is under the assumptions that weather conditions do not exceed the extreme 90% case shown. On that note, is it even more likely that the frequency range would be narrower since operating angles are likely to be higher (above 40 degrees), especially when considering performance expectations related to both figures 4.5 and 4.6. However, all angles of possible operation were taken into
account for this performance estimate to ensure the overall estimated system framerate is conservative rather than optimistic. Throughout this analysis, it is important to keep in mind that the limited number of conditions (i.e. day/night, spring/fall, wet/dry, etc.) available severely restricts calculation of highly probable performance expectations in both models (SFTM and AOM) and, in addition to that, scaling laws (developed analytically with underlying assumptions) are only useful to an extent. Based upon parameter assumptions and the overall goal of baselining an AO system for this site, these calculations should be sufficient to help further define the next AO research goals for the JBO-Q.

![Greenwood Frequency VS Elevation](image)

**Figure 4.6** Graphs expected Greenwood frequencies for four atmospheric profiles. The regions highlighted in orange emphasize the largest expected Greenwood frequency bandwidth needed to estimate appropriate performance.

The Greenwood frequency calculation is then used to calculate the servo bandwidth of the system using equation (29) by assuming a Strehl ratio requirement of 0.2 and then calculating the overall expected framerate. The framerate was selected to be 20 times the servo bandwidth.
as a rule of thumb for integral system control. This framerate ensures that overall aliasing of wavefronts does not occur in real-time measurements. This approximation is similar to bandwidth restrictions such as the Nyquist frequency. A more accurate calculation requires more detailed control system specifications and calculation of the system’s error rejection function (the error rejection function is beyond the scope of this thesis). The system framerate threshold is estimated to be 0.577 KHz while the highest expected system framerate (within data prediction) is estimated to be 2.153 KHz. Subtracting these two rates yields a bandwidth of approximately 1.576 KHz.

IV.4.C Actuator Optimization Model Overview

In order to evaluate the optimal number of actuators on the expected DM, it is necessary to estimate the Fried parameter \((r_0)\), the estimated light backscattered from the LGS at each subaperture on the SHWFS, and the SNR of the received light, and the total residual phase error from significant sources. The flow chart in Figure 4.6 outlines this process.

The algorithm developed calculates residual phase error as the actuator count increases (along with subaperture size), thus when the calculated residual phase error is at a minimum then the optimal size of subapertures and optimal number of actuators are determined. To calculate \(r_0\), the \(C_n^2\) profiles are derived (winter and summer) from non-weighted LEEDR data and input into equation (12).
AOM/SFTM Algorithm Flow Chart
1. Input LEEDR data (Cn profiles)
2. Calculate turbulence parameters (r0, fG, etc)
3. Insert parameters and laser power into scaling-law Equations
   (DM fitting error, subaperture measurement error, etc)
4. Find the minimum wavefront error
5. Calculate the required framerate

Figure 4.6. States the basic outlines the Actuator Optimization and System Framerate Threshold Model Algorithms.

The summer data set revealed an $r_0$ value of 0.76 cm while the winter data set revealed an $r_0$ value of 4.44 cm. These two data points were calculated using Equation (12) which does not include statistical weights as used in the SFTM. Additionally, these additional data point calculations are referenced in Figure 4.5 but not used in any further analysis except for validating the $r_0$ range calculated in the SFTM. These values simply help validate the $r_0$ range estimation calculated by the SFTM by being unweighted. In fact, since these values agree in range with the STFM, STFM $r_0$ values are used in the AOM tests which are explained shortly. Generally, it is expected that both of these $r_0$ values are a result of turbulent viewing conditions for the site which implies the AO system needed will require better performance capabilities to perform proper image compensation. Because $r_0$ values affect residual phase error (fitting and WFS), they contribute to determining the overall baseline DM optimal actuator range which we will see in section IV.4.D.

The amount of backscattered light is estimated using equation (29) at zenith for a Rayleigh beacon of 527 nm and 8 mJ per pulse (which are the specifications of the acquired beacon). Transmission coefficients, pressure and temperature at beacon height, quantum
efficiency, and background noise were inputs based on LEEDR calculations, sensor specifications, or simply expertise gained from field professionals. The total number of photons is then divided into the square subapertures of the SHWFS of increasing size (as index increases) simultaneous to calculating the SNR and the overall residual phase error based on its primary constituents, WFS error (equations (17.1, 17.2)) and fitting error (equation (15)). The total residual phase error for this model consists only of uncorrelated WFS error and fitting error which limits the model’s accuracy since errors such as anisoplanatic error, tilt error, tilt anisoplanatic error and temporal error are not explicitly included.

It is important to note that as subaperture size increases (meaning the amount light it collects also increases) so does the overall SNR which reduces WFS error. However, in contrast to WFS error, fitting error increases as the aperture size increases since it is a ratio of the aperture size to \( r_0 \) values. As a result, there is an optimal subaperture size that balances WFS error and fitting error which typically corresponds to a subaperture size near the calculated \( r_0 \) value meaning the ratio \( \frac{d}{r_0} \) approaches 1. The analysis section of the SFTM demonstrates this optimization result.

An additional consideration of fitting error is that the measured wavelength is actually different than that of the beacon wavelength. This is usually because the EO camera will collect the compensated image data in a different spectral band in order to preserve the LGS photon budget going to the SHWFS. To account for this wavelength dependence at the EO camera in the AOM’s fitting error calculation, the \( r_0 \) value of the scoring wavelength is scaled by a power of 6/5 and described the equation (35).

\[
\begin{align*}
    r_{0,scoring} &= \left( \frac{\lambda_{scoring}}{\lambda_{beacon}} \right)^{\frac{6}{5}} (r_{0,beacon}) \\
    \text{(35)}
\end{align*}
\]
Note that the DM corrects all incident optical frequencies identically as a result of the fact that they travel through the same optical path, thus no additional residual phase error must be considered from differing scoring and WFS wavelengths incident upon the DM. The total residual phase error in the model is defined by:

$$\sigma_{TRP}^2 = \sum_i \sigma_i^2 = \sigma_{fit}^2 + \sigma_{WFS}^2$$  \hfill (36)

When $\sigma_{TRP}^2$ (Total Residual Phase Error) is at a minimum, we have maximized the SNR in the SHWFS (hence minimized $\sigma_{WFS}^2$) while balancing the light collected and subaperture size to ultimately yield the optimal number of actuators for the DM. In all, the optimal number of DM actuators is highly dependent upon atmospheric turbulence ($r_0$), the backscatter collected by the telescope aperture, and overall residual phase error. In order to evaluate the overall resultant image quality, total residual phase error was plotted as the Strehl approximation against the total number of actuators. This analysis method enables performance evaluation with respect to phase error while considering image quality and DM complexity. Analysis of this model and approximated performance parameters will be discussed in the next section.

IV.4.D Analysis of AOM

As mentioned before, data from the SFTM served as the basis for analyzing AOM results. In particular, Rytov number correlations from the STFM were utilized to set the range of expected $r_0$ values (which were also associated with elevation angles) for the site. This process ensures that extreme scintillation magnitudes (most difficult for AO compensation) are ignored while also setting a baseline AO system performance expectation [13]. Note that a range of ~3 - 6 cm ($r_0$ values - refer to page 45, Figure 4.5) assumes the maximum amount of turbulence that will generally be experienced at site. By evaluating this range we can in turn evaluate the
highest performance required out of the AO system to obtain high resolution images. A 0.2 Strehl ratio requirement for these tests ensures that overall image quality is sufficient to obtain high resolution after compensation [5].

Figure 4.7 Test 1 - Test to determine optimal number of SHWFS subapertures and DM actuators where an 80 mJ/P 527 nm laser propagated at a 52 degree elevation angle in extreme conditions on a summer day in Dayton, Ohio where $r_0 = 2.99$ cm.

The four graphs above in Figure 4.7 represent the SNR, the individual phase variances, the total phase variance, and the Strehl ratio from a single backscattered laser pulse of 527 nm wavelength projected at a height of 10 Km. Note that laser beam quality or other beam control effects were not considered. This particular test quantified system specifications such as the pulsed laser power (80 mJ) and the elevation angle (pointed at zenith) with an assumed constant $r_0$ (beacon wavelength) of 2.99 cm. One concern for this test is that this $r_0$ value used was calculated for an elevation angle of 40 degrees, however, the laser backscatter calculated assumes an angle of 90 degrees. Note that a more comprehensive model would account for the number of backscattered photons at each zenith angle. Furthermore, anisoplanatism would be
considered different at each zenith angle. Thus, in essence, conservative estimates can be assumed for backscatter SNR at zenith while optimistic estimates can be assumed at angles near 40 degrees of elevation. The calculated results yield an SNR of 12.92, Strehl ratio of 0.215 and an optimal number of actuators of ~104. Also note that the optimal SHWFS subaperture size is calculated to be 5.3 cm which is about 1.77 times $r_0$. As a reminder, it is also important to verify that the calculated minimum total phase error (~1.54) is below 2 rad to ensure validity of the Marechal approximation curves which are graphed in the bottom right plot.

The second AOM test results in Figure 4.8 uses identical system parameters but a different atmospheric turbulence parameter where $r_0$ is now 4.75 cm. Compared to the first test, the larger $r_0$ value was used which increases our overall performance in terms of the ending image quality if all other calculating factors are held constant. Although the SNR decreases (top left graph), it remains significantly above a typical SNR threshold level (generally considered about 6-8). SNR decreases in the second test as a result of mapping the optimal actuator number to the SNR (test 1 optimizes at 103 and test 2 optimizes at 113). Additionally, the total phase variance decreases significantly which improves the overall Strehl ratio plot using the Marechal approximation to ~0.52, which is a 2.6x improvement factor. This modeling test suggests that a pulsed laser of 80 mJ with poor atmospheric conditions could perform at many operating angles; however, it is absolutely imperative to keep in mind that other error sources are not included in these calculations and in addition, the laser backscatter is greater as it is an estimate from zenith. These preliminary results are intended to provide an initial assessment (aka baseline) that enable future researchers to define more precise specifications for the AO system to be applied at the JBO-Q by laying the groundwork for AO system expected performance.
Figure 4.8 Test 2 – Test to determine optimal number of SHWFS subapertures and DM actuators where an 80 mJ/Pulse 527 nm laser propagated at a 40 degree elevation angle on an average winter day in Dayton, Ohio where r0 = 4.75 cm

The results from the third and final test are graphed in the figure above. These results are comparable to both tests 1 and 2. All parameters were identical to those used in test 2 with the exception of the pulsed laser energy. The pulsed laser energy was decreased to 31.2 mJ/pulse as this pulse power yielded a capability, in terms of image quality, similar to that of test 1. In other words, the image quality of test 3 was very similar to that of test 1 but the atmospheric turbulence parameters were consistent with test 2. The importance of this test is to show that the expected backscatter decreases and yields a lower Strehl Ratio (~0.2) and thus lower image quality when a more conservative r0 range value is used while all other considerations are consistent with test 2. In general, an observable trend is that the threshold Strehl ratio is achievable so if fewer subapertures and DM actuators are used. In contrast, too many subapertures and actuators yield worse performance using a less powerful beacon. Accuracy of the radiometric estimations which contribute to photon count are critical for system performance approximations.
Figure 4.9 Test 3 - Test to determine the optimal number of SHWFS subapertures and DM actuators where a 31.2 mJ/Pulse 527 nm laser propagated at a 40 degree angle on an average winter day in Dayton, Ohio where $r_0 = 4.75$. 

- Optimal # Actuators
- Overall WFS SNR
- Minimal Phase Error
- Meets Strehl ratio requirement
V. Conclusions and Future Work

V. 1 Results and Conclusions

V.1 SFTM Results and Conclusions

The results of the SFTM in chapter IV indicate that the JBO-Q site would require a DM, SHWFS, and LGS capable of running at 2.2 KHz in order to perform image compensation at most operational angles 90% of the time. The most difficult component to achieve this bandwidth with is the LGS, since a high powered pulse at this frequency results in a 176 W Rayleigh beacon (not an inexpensive endeavor). However, a 2.2 KHz system framerate suggests the site is definitely an applicable location for AO implementation since currently available commercial DM and EO camera bandwidths exceed that rate. Regardless, further study of DM properties such as physical size, desired resolution, and component configuration should be determined before purchase. Although the site appears to sustain relatively highly turbulent conditions during both summer and winter, the highest expected Greenwood frequency needed for correction would likely not exceed 145 Hz on 9 out of 10 days at operating angles above 30 degrees. Note that these assumptions are based on data which includes the bandwidth increase needed for telescope slew as the object is tracked. Additionally, it is important to keep in mind that the scaling laws and approximations used in this work limit the accuracy of this model but nonetheless suggest that image compensation most definitely plausible. In support of that, the calculated turbulence coherence lengths in the SFTM are of magnitudes similar to those that other observatories experience around the world. For example, Starfire Optical Range located at Kirtland Air Force Base in Albuquerque, New Mexico often experiences $r_0$ values between the ranges of 1-8 cm.
By using a conservative overall AO system framerate, the predicted maximum frequencies required at site are well under the maximum frequencies of commercially available DMs. One of these DM’s that fulfills specifications for the JBO-Q DM is the Thorlabs MEMS-Based DM which consists of a 144 actuator array and 3 decibel (dB) bandwidth of ~3.5 KHz that costs $18K [23]. The highest bandwidth calculated in the SFTM was 2.153 KHz which is an easily achievable goal if the STFM’s accuracy holds true. Note that the prior mentioned DM is not a suggestion for purchase but rather simply proof that affordable technology exists.

V.2 AOM Results and Conclusions

The results of the AOM suggest that system performance, in terms of the Strehl ratio, is dramatically improved by using a LGS, WFS, and DM. This is observed simply by considering the Strehl ratios when the number of actuators is close to 0. Essentially, the performance prior to implementing an AO system is extremely poor comparatively and thus according to Figures 4.7, 4.8, and 4.9, the “before” performance in Strehl is around 0.006. Even when a simple AO system is implemented, the overall image quality will improve by 2 full magnitudes. This model suggests the performance would increase, at minimum, by 3300% or ~33x.

In coordination with data used from the SFTM, the AOM results indicate that the overall complexity of a viable DM for site application is relatively low. Assuming that the majority of phase error was properly accounted for, the total number of actuators will range between 40 – 120 which we can see from the optimized actuator numbers on Figures 4.9 and 4.8 respectively. This range is not so much bound by the upper limit as that number primarily affected by the choice of components (primarily the LGS and SHWFS). In other words, DMs of higher complexity, bandwidth, and WFS and beacon noise will yield a larger optimal actuator number if provided enough signal (in addition to being more expensive). However, it is highly advisable
not to approach the lower limit since this deoptimizes the subapertures size results in significantly larger lengths (>2x times \( r_0 \)) than predicted \( r_0 \) values. Large subaperture sizes increase SNR but also result in larger DM fitting errors.

From the multiple test results, which were intended to be conservative and simple. It seems probable that the detection system will be able to receive sufficient signal from an 527 nm pulsed LGS at 80 mJ per pulse. However, this power is more expensive to achieve at higher pulse rates as suggested in the previous section and would imply 176 W laser, which is simply not a low cost endeavor. On that note, it is important to have similar framerate speeds for all AO components where higher bandwidths are generally advised. The reality is that generally cost is an important factor. If image resolution is the primary consideration for the telescope then a Strehl ratio of 0.2 should be the absolute minimum image quality threshold while the framerate is also slower and the laser power is no less than 40-50 mJ/pulse. This study suggests that an 8 mJ/pulse laser will not suffice for AO image compensation and one of at least 5x the strength would be needed to achieve high resolution images. The fact that photon budget is so limited (hence the reason for high power LGSs) is exactly the reason as to why astronomers, engineers, and physicists alike strive to achieve lasers of more and more power.

**V.3 Cost Analysis for Viable Components**

A commercial component cost estimate was considered for the LGS, WFS, and DM while keeping in mind the expected AO system baseline parameters calculated and stated in the analysis and conclusion sections. Once again, it is expected that implementation of these devices will increase the overall imaging performance of the telescope system by 3300% or better. A Rayleigh guidestar of appropriate wavelength and the suggested power requirements is expected to cost ~$125 K based on an estimate for a 500 W, 532 nm, single mode, fiber laser [29]. This
estimate was provided by Mr. Johnny Poon at IPG Photonics Corp. Note that 500 W is used since Rayleigh guide stars have around a 35% efficiency for the returning signal. To obtain 80 mJ/pulse return (~140 W), 28% of 500 W is needed, thus this estimate is pretty accurate and within scope of program budget and in addition will actually improve the performance somewhat more stated in the analysis and conclusion sections. The estimated cost of the DM is ~$18K and the justification for this cost can be found above on page 56 in the SFTM results and conclusions. Finally, a SHWFS with specifications of 2.2 KHz full resolution framerate and a maximum number of 140 subapertures was estimate for cost by Dr. Jason Schmidt from the MZA Associates Corporation. He stated that the cost estimate of an appropriate device satisfying these specifications (and other sub-parameters not mentioned) is approximately $26K.

V.4 Future Work

Follow-on research that would greatly contribute to pinning down the exact AO system parameters includes considering additional factors that contribute to phase error, photon error, and temporal error. Future work for determining a viable DM for purchase is optimization geometry of the DM actuators. Additionally, data collected for the site should be analyzed to validate the AOM’s and SFTM’s consistency as well as accuracy. Validation of these models will enable future researchers to focus on sources which contribute greatest to SNR and phase error at the detector. If these two factors are minimized then the overall telescope system may approach its diffraction limit. Comparison between the models to other techniques would also help verify and validate results. Overall, AO implemented at the JBO-Q would significantly contribute to space situational awareness (SSA) and space object identification (SOI) missions supported by AFRL. In essence, AO enables higher resolution of targets for tracking and characterization purposes. In conclusion, this thesis has analyzed possible options for baselining
an AO system specifically related to returned laser power at the SHWFS and optimization of the number of DM actuators and the DMs bandwidth. It is the recommendation of the author that further study of atmospheric parameters for the site occur and that the suggested estimated performance parameters of an applicable AO DM be further investigated before purchase to ensure that additional risk be avoided.
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Appendix

Figure A.1: Optimized range gate lengths for each wavelength. Red is the 1064 nm laser, green is the 532 nm laser, and purple is the 351 nm laser [18].

Figure A.2: The spectrum specific path attenuation as it relates to visible wavelengths [18].
Figure A.3 Highlights the minimum isoplanatic angle for likely angles of operation for the JBO-Q

<table>
<thead>
<tr>
<th>Path Transmittance</th>
<th>0.210761</th>
</tr>
</thead>
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<tr>
<td>Path Extinction (1/km)</td>
<td>0.127552</td>
</tr>
<tr>
<td>Path Attenuation (dB/km)</td>
<td>0.553951</td>
</tr>
<tr>
<td>Surface Visibility (km)</td>
<td>9.98</td>
</tr>
<tr>
<td>Slant Path Visibility (km)</td>
<td>32.76</td>
</tr>
</tbody>
</table>

Figure A.4. Path specific propagation parameters for a 532 nm laser [18].

**Actuator Optimization Model MATLAB Code**

% From relevant Cn2 profiles, using LEEDR values.
% Computes Fried parameter using tatarskii model using numeric weather data specific for
% Dayton Ohio from historic measurements in the winter and summer.
% Calculating Fried Parameter Using Numerical Weather Data and LEEDR
% Calculating Fried parameter (r0) using Eq 3.51 Hardy, page 92
% \[ r_0 = 0.423k^2*\secant(\zeta)\int(Cn^2(h))dh\]^-3/5 - Fried Parameter of atm [m]
lambda = 527E-9; % wavelength of backscattered laser
$\zeta = 0; \ % \text{assuming look angle straight up for these simulations}$

$k = (2\pi/\lambda); \ % \text{wave number}$

% Piecewise function for Cn², integral is the same as multiplying the altitude zone length (dh) times the Cn² [m⁻²/m³] taken from the % simulated turbulence data. The simulated data used measured weather data values to determine Cn² values. Data saved in file locations below

% Using Winter Data for 100K Range
% uiopen('C:\Users\RESON8\Documents\Graduate Study\Simulation Code\Thesis\LEEDR Sim\TurbValues')
% $r_0 = (0.423k^2secd(\zeta)*\text{sum(TurbWinter100K.Target.XData(1:end-1).*500)})^{3/5}$
% $r_0 = 0.0076 \ % \text{for winter simulation}$
% Using Summer Data for 100K Range
% uiopen('C:\Users\RESON8\Documents\Graduate Study\Simulation Code\Thesis\LEEDR Sim\TurbValues')
% $r_0 = (0.423k^2secd(\zeta)*\text{sum(TurbSummer100K.Target.XData(1:end-1).*500)})^{3/5}$
% $r_0 = 0.0443 \ % \text{for summer simulation}$

%% Determine Photons Received at Each Subaperture of the SHWFS
% LIDAR Equation 7.1, pg 222 in Hardy - calculation for power incident of receiver
% $N(z) = \left(\frac{E_0\lambda}{h\nu c}\right)\left(\frac{\sigma_b n(z)dz(A/(4\pi z^2))T_0 T_A^2 \nu}{\Sigma N_B}\right)$
% $E_0 = 10*0.008; \ % \text{Laser Pulse Energy (J)}$
% $\lambda_{\text{beacon}} = 527E-9; \ % \text{wavelength of expected laser}$
% $h = 2\pi*1.0545718E-34; \ % \text{planck's constant (m}^2\text{kg/s)}$
% $c = 2.99793458E8; \ % \text{speed of light (m/s)}$
% $r_P = 0.3048; \ % \text{radius of primary mirror (m)}$
% $A = \pi r_P^2; \ % \text{Area of primary mirror (m}^2\text{)}$
% $T = 216.2; \ % \text{Temp @ 10 Km altitude taken from LEEDR data (K)}$
% $P = 252.1; \ % \text{Pressure @ 10Km altitude taken from LEEDR data (mb)}$
% $\sigma_b = 5.45*(550/(527))^4*10E^{-32}; \ % \text{Eq 7.3 page 222 in Hardy, effective backscatter cross-section [m}^2\text{sr}^{-1}\text{]}$
% $n_z = (3.6*10^{-31}*(P/T)*\lambda_{\text{beacon}}^{-4.0117})/\sigma_b; \ % \text{Eq 7.4 page 222 in Hardy, number density of scatterers at range z [m}^{-3}\text{]}$
% $z = 10000; \ % \text{range at center of range gate [m]}$
% $\Delta z = 0.20*z; \ % \text{receiver range gate length [m], 10\% of beacon central altitude}$
% $T_0 = 0.95; \ % \text{Transmission of optical components in transmit and recieve paths, minor loss off of pri, sec, & DM mirrors, relay mirrors}$
% $T_A = 0.718529; \ % \text{One-way transmission of atmosphere between telescope and beacon, will vary based off atm conditions, used LEEDR sim data}$
% $\nu = 0.55; \ % \text{Quantum efficiency of photon detector, EO camera model - Mako G30B spec for 529 nm wavelength}$
% $N_B = 14; \ % \text{Number of background and noise photoelectrons per pixel, Mako G30B camera spec}$
% $\text{Photons Received at Pri} = (E_0\lambda_{\text{beacon}}/(h\nu c)\sigma_b n_z \Delta z (A/(4\pi z^2)) T_0 T_A^2 \nu + N_B \ % \text{# of photons collected at the primary mirror}$
% $\% \text{Laser Beacon Specs}$
% $\% \text{Laser pulse energy = 8 mJ}$
% $\% \text{pulse width} \approx 4\text{ish ns}$
\% rep rate = 200 Hz
\% notes: use 10\% range gate vs center altitude
\% Strictly for reference.... Calculates the Greenwood Frequency at constant wind velocity
\% Equation 9.54 in Hardy
\% k = 2*pi/\lambda; wavenumber
\% f_G = 0.102*k^2*\sec(zeta)*\int[\text{Constant}]^3/5
\% Here k is the wave number, \(zeta\) is the elevation angle, \(C_n\) and wind velocity "\(v\)" values are determined by LEEDR data.
\% For a single turbulent layer... \(f_G = 0.4278(v/r_0)\)
\% For curiosities sake...
\% \(v = 5, \% \text{[m/s]} \approx 11.5 \text{ mph}\)
\% \(f_g = 0.4278*(v/r_0)\)
\% Calculates the optimal number of actuators using the point of minimal phase error
\% \(T = 0.95; \% \text{Estimated transmission coefficient from primary mirror to sensor}\)
\% \(D = 0.6096; \% \text{Aperture diameter of Q telescope [m]}\)
\% \(Irr = T*\text{Photons}_\text{ReceivedatPri} ; \% \text{power captured by telescope (all subapertures)}\)
\% \(\text{[photons/m^2*pulse]}\)
\% \(N_{\text{Pix}} = 4; \% \text{The number of pixels over which light is processed, reference guard band}\)
\% \(\sigma_{2R} = N_{\text{Pix}}*(\text{NB}^2 + 3.3); \% \text{variance of sensor read noise [photoelectrons^2]}, \text{read noise per 4 pixels, ref pg 361 Hardy}\)
\% \((\approx 3.3) \text{ times the # of pixels per spot (\approx 4)}\)
\% \(\text{Good Silicon (visible) cameras have only a few photoelectrons of read noise.}\)
\% \(\% \text{DM:}\)
\% \(N_{\text{ActRng}} = \text{linspace(10, 1600, 2000)}; \% \text{range of number of actuators to study total number of actuators}\)
\% \(a_F = 0.28; \% \text{[rad^2]} \text{ DM fitting error coefficient, WF correctors.ppt slide 22 Page 343 Hardy}\)
\% \(\text{for Pyramid shaped subapt influence function}\)
\% \(\% \text{Essentially the influence coefficient that each actuator has all others}\)
\% \(\text{sig}_{2\text{Fit}} = \text{zeros(1, length(N_{\text{ActRng}}));}\)
\% \(\text{sig}_{2\text{WFS}} = \text{zeros(1, length(N_{\text{ActRng}}));}\)
\% \(\text{for idxA = 1 : length(N_{\text{ActRng}})}\)
\% \(\text{N_{\text{ActTot}} = N_{\text{ActRng}}(\text{idxA});}\)
\% \(\text{N_{ActSide} = sqrt(4*N_{\text{ActTot}}/\pi); \% \text{number of actuators per side, circle to square conversion}\)
\% \(\text{d(\text{idxA}) = D / N_{\text{ActSide}}; \% \text{subaperture diameter [m]}\}
\% \(\text{N}_p = \text{Irr } * \text{d(\text{idxA})}^2; \% \text{number of photons incident on a subaperture}\)
\% \(\% \text{Reference Scaling Procedure for wavelengths in Hardy, pg 315}\)
\% \(r_0\text{beacon} = 0.0475; \% \text{[m]} \text{Fried for beacon wavelength, calculated by LEEDR simulation}\)
\% \(\lambda_{\text{dcore}} = 551\times10^{-9}; \% \text{[m]} \text{Scoring wavelength used for sigamFit error, Arbitrarily chosen}\)
\% \(\text{astronomical band since EO camera not chosen yet}\)
\% \(\lambda_{\text{beam beacon}} = 527\times10^{-9}; \% \text{[m]} \text{Beacon wavelength used for sigawFS error}\)
\% \(r_0\text{scoring} = (\lambda_{\text{dcore}}^1.2*r_0\text{beacon})/(\lambda_{\text{beam beacon}}^1.2); \% \text{[m]} \text{scaled Fried parameter of EO camera sensing (aka "scoring") wavelength}\)
\% \(\text{sig}_{2\text{Fit}}(\text{idxA}) = a_F * (\text{d(\text{idxA})}/r_0\text{scoring})^{(5/3)}; \% \text{dimensionless DM fitting error where r0 should be the actuator size}\)
\% \(\% \text{sig}_{2\text{F}}^2 = a_F(d/r_0)^{5/3} [\text{rad}^2], \text{Eq 9.61, page 342 Hardy}\)
\% \(\text{WF correctors.ppt, slide 21}\)
SNR(idxA) = Np / sqrt(Np + sig2R); % SNR of WFS slope measurement, ref pg 361 Hardy
% This is SNR due to shot and read noise. Also, this assumes the
% beacon is a point, when it actually has some finite extent.
% HO_WFS.ppt, slide 22; also equation 2.3 in Hardy, pg 42
if r0beacon > d(idxA)
    sig2WFS(idxA) = 2*(1.3*pi^2./(4*SNR(idxA))).^2 * 1.5^2; % WFS measurement error
    % rad^2] Hardy Eqs 9.43, pg 351?
else % r0 <= d case
    sig2WFS(idxA) = 2*(1.3*pi^2./(4*SNR(idxA))).^2 * (1.5*d(idxA)/r0beacon)^2; % WFS
    % measurement error [rad^2]
end
end

sig2Tot = sig2Fit + sig2WFS; % sum the individual contributions of phase error
[Min, Idx] = min(sig2Tot); % find the minimum phase error
NActTotOpt = NActRng(Idx); % optimal number of actuators
dOpt = d(Idx)% optimal size of subapertures
dnumOpt = Idx % optimal number of subapertures

figure(1); clf;
% Plots of wavefront error:
subplot(2,2,1);
plot(NActRng, SNR);
xlabel('Total Number of Actuators');
ylabel('WFS SNR');
subplot(2,2,2);
plot(NActRng, sig2Fit, NActRng, sig2WFS);
xlabel('Total Number of Actuators');
ylabel('Phase Variance Terms [rad^2]');
legend('DM Fitting Error', 'WFS Noise Error', 'Location', 'NorthEast');
subplot(2,2,3);
plot(NActRng, sig2Tot, NActTotOpt, sig2Tot(Idx), 'rx');
xlabel('Total Number of Actuators');
ylabel('Total Phase Variance [rad^2]');
legend('Optimization Curve', 'Optimal Number of Acts', 'Location', 'NorthEast');

% Plots of Strehl ratio using Marechal approximation: Eq. 4.40 in Hardy, pg 115
StrehlFit = exp(-(-sig2Fit)); % terms are already squared
StrehlWFS = exp(-(-sig2WFS));
StrehlTot = exp(-(-sig2Tot));

figure(2); clf;
subplot(2,2,1);
plot(NActRng, StrehlFit, NActRng, StrehlWFS, NActRng, StrehlTot, ...
    NActRng(Idx), StrehlTot(Idx), 'rx');
xlabel('Total Number of Actuators');
ylabel('Strehl Ratios');
legend('DM Fitting Error', 'WFS Measurement Error', 'Overall Strehl Ratio', ...
    'Best Possible Strehl Ratio', 'Location', 'SouthEast');
System Framerate Threshold Model MATLAB Code

clear all;
load('WPAFB_AtmTurb', 'Atm', 'V', 'L', 'hp', 'vt', 'vp', 'el', ...
    'Seasons', 'timeOfDay', 'TurbPercentiles', 'RH');
NEl = numel(el);  % number of elevation angles
% NScreens = 100; % size(Altitude, 2);
wvl = 527e-9; % wavelength
k = 2*pi/wvl; % wavenumber
D = 24 * 0.0254; % Diameter of primary mirror
for idxP = 1 : 4    % loop over site profiles
    for idxE = 1 : NEl
        z = Atm(idxP,idxE).z; % z from each Atm is a column
        L = Atm(idxP,idxE).L;
        Cn2 = Atm(idxP,idxE).Cn2;
        v = squeeze(V(idxP,idxE,:));
        % r0SW = (0.423*k^2*sum(Atm(1).Cn2.*(1-z/L).^(5/3).*Atm(1).dz)).^(-3/5);
        r0SW(idxE,idxP) = (0.423*k^2*trapz(z, Cn2.*(1-z/L).^(5/3))).^(-3/5); % Hardy Eq. 3.51, pg 92
        RytovSW(idxE,idxP) = 0.563*k^(7/6)*trapz(z, Cn2.*z.^(5/6).*(1-z/L).^(5/6)); % Eq??????
        fG(idxE,idxP) = (0.102*k^2*trapz(z, Cn2.*v.^2)); % Hardy Eq. 9.54, pg 338
        theta0(idxE,idxP) = (2.91*k^2*trapz(z, Cn2.*z.^(5/3)))^(-3/5); % Hardy Eq. 3.103, pg 103
    end
end

% Calculating the minimum and maximum framerates for correction, based of
% Greenwood Frequency modeling at extremities of atmospheric data collection
% From Hardy, page 338 Servo Bandwidth, ~20 times the Max/Min Greenwood
% freq will be our framrates
% Greenwood Frequency measurements from graph
FgSMax9 = 143.2; % 90% Max Summer @ 30 deg ele
FgSMin9 = 63.58; % 90% Min Summer @ 90 deg ele
FgWMax9 = 134.9; % 90% Max Winter @ 30 deg ele
FgWMIn9 = 49.19; % 90% Min Winter @ 90 deg ele
FgSMax5 = 129.3; % 50% Max Summer @ 30 deg ele
FgSMin5 = 55.13; % 50% Min Summer @ 90 deg ele
FgWMax5 = 119.2; % 50% Max Winter @ 30 deg ele
FgWMIn5 = 38.38; % 50% Min winter @ 90 deg ele
% Assuming required Strehl ratio and known fitting & WFS errors, calculates
% phase error of temporal response, ref hardy pg 338.
% Strehl ratio temporal phase error: S = e^{-2*(sigTR)^2}
S = .20;
sigTRsquared = -log(S); % Inverse of the Marechal Approximation, S = e^{-2*sigTR^2}
kappa = 1; % Assuming bandwidth defined at half-power pointb(1/sqrt(2))
% Using equation 9.53 sigTR^2 = kappa*(Fgreenwood/Fservo)^{(5/3)}, solve for
% Fservo
FservoSMax9 = (kappa*FgSMax9)/(sigTRsquared^(3/5));
FservoSMin9 = (kappa*FgSMIn9)/(sigTRsquared^(3/5));
FservoWMax9 = (kappa*FgWMax9)/(sigTRsquared^(3/5));
FservoWMIn9 = (kappa*FgWMIn9)/(sigTRsquared^(3/5));
FservoSMax5 = (kappa*FgSMax5)/(sigTRsquared^(3/5));
FservoSMin5 = (kappa*FgSMin5)/(sigTRsquared^(3/5));
FservoWMax5 = (kappa*FgWMax5)/(sigTRsquared^(3/5));
FservoWMIn5 = (kappa*FgWMin5)/(sigTRsquared^(3/5));
% Note that frame rate is ~ 10 times the servo bandwidth
FrameRateSMax9 = 20*FservoSMax9
FrameRateSMin9 = 20*FservoSMin9
FrameRateWMax9 = 20*FservoWMax9
FrameRateWMIn9 = 20*FservoWMIn9
FrameRateSMax5 = 20*FservoSMax5
FrameRateSMin5 = 20*FservoSMin5
FrameRateWMax5 = 20*FservoWMax5
FrameRateWMIn5 = 20*FservoWMIn5
% %Plots Figures
figure(2); clf;
subplot(2,2,1);
ph = plot(el*180/pi, r0SW*1e2); % 30 to 90 degrees
grid on;
set(ph, {'Color'}, {[1 0 0]; [1 0 0]; [0 0 1]; [0 0 1]}, {'LineStyle'}, {'-'; '--'; '-'; '--'});
xlabel('Elevation Angle [deg]');
ylabel('Fried Parameter r_0 [cm]');
subplot(2,2,2);
ph = plot(el*180/pi, RytovSW);
grid on;
set(ph, {'Color'}, {[1 0 0]; [1 0 0]; [0 0 1]; [0 0 1]}, {'LineStyle'}, {'-'; '--'; '-'; '--'});
xlabel('Elevation Angle [deg]');
ylabel('Rytov Number');
legend('Summer 50%', 'Summer 90%', 'Winter 50%', 'Winter 90%', 'Location', 'Best');
subplot(2,2,3);
ph = plot(el*180/pi, theta0*1e6);
grid on;
set(ph, {'Color'}, {[1 0 0]; [1 0 0]; [0 0 1]; [0 0 1]}, {'LineStyle'}, {'--'; '--'; '--'; '--'});
xlabel('Elevation Angle [deg]');
ylabel('Isoplanatic Angle [\mu rad]');
subplot(2,2,4);
ph = plot(el*180/pi, fG);
grid on;
set(ph, {'Color'}, {[1 0 0]; [1 0 0]; [0 0 1]; [0 0 1]}, ...
    {'LineStyle'}, {'--'; '--'; '--'; '--'}, ... 
    {'LineWidth'}, {2; 2; 2; 2});
xlabel('Elevation Angle [deg]');
ylabel('Greenwood Frequency [Hz]');