Direction of Arrival Estimation using Wideband Spectral Subspace Projection

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Engineering

by

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ABSTRACT


Many areas such as Wireless Communication, Oil Mining, Radars, Sonar, and Seismic Exploration require direction of arrival estimation (DOA) of wideband sources. Most existing wideband DOA estimation algorithms decompose the wideband signals into several narrowband frequency bins, followed by either focusing or transforming to a reference frequency bin, before estimating the DOAs. The focusing based methods are iterative and their performance is affected by the choice of preliminary DOA estimates and the number of source DOAs to be estimated. The existing method requiring transformation to a reference frequency bin exhibits spurious peaks in the spatial spectrum and is not reliable in general.

In this thesis, a novel Wideband Spectral Subspace Projection (WSSP) approach is presented. WSSP exploits the properties of projected subspaces to estimate the wideband DOAs. The proposed method is non-iterative and it does not require any prior DOA estimates, focusing, beamforming or transformation to reference frequency bin. Theoretical small perturbation analysis has been conducted that confirms the ability of WSSP to produce large peaks at correct DOAs.

The validity of the proposed algorithm has been tested using a variety of typical wideband sources encountered in radar and wireless communication applications, including Chirp, QPSK and MC-CDMA. The performance of the proposed algorithm has been compared with those of previously existing algorithms via extensive simulation studies, in terms of bias and root mean square error (RMSE). The simulation results demonstrate that when compared to the existing methods, the performance of proposed method is accurate over a wide range of SNRs and it is not affected by the number of the source DOAs to be estimated.
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Dedicated to

My Parents and Uncle
Chapter 1: Introduction

1.1 Motivation

Direction of arrival estimation (DOA) has many applications particularly in Radar, Sonar, Seismic Exploration, Wireless Communication and in Defense. DOA has been used in radars for air traffic controlling, where elevation and azimuth angles are detected to locate the direction of airplanes and direct them for a safe landing. In sonar, noise produced by propellers and machinery is used to detect the direction of ships and submarines. In wireless communication, the information of direction of arrival can be used to estimate the multi-path channel accurately. In smart antenna, the information about direction of users can be used to direct power of base station in desired direction using adaptive filters. In defense, it is used to identify the direction of threat from the enemy sources. Most of the above discussed applications use wideband signals and hence accurate DOA algorithms for wideband sources are needed.

High resolution methods such as MUSIC, ESPRIT and Root-MUSIC were developed for narrowband signals. Most of the applications use wideband signals and hence development of wideband DOA algorithms are important. Most existing wideband DOA estimation algorithms decompose the signal into various narrowband frequencies to estimate the wideband DOAs. One of the early methods known as incoherent MUSIC performed narrowband MUSIC independently at several narrowband frequency bins and averaged the results to estimate the wideband DOAs [9]. This method is computationally expensive as
it requires DOA estimation at each frequency bin. In order to overcome this, coherent methods such as Coherent Signal Subspace (CSS) [7] [8] and Weighted Average of Signal Subspace (WAVES) [2] were developed, which involve focusing spectral domain correlations matrices at several narrowband frequencies to a reference frequency bin using unitary focusing matrices, where the focusing matrices are constructed using preliminary DOA estimates. Errors in estimation of preliminary DOA estimate may degrade the performance of these methods. A relatively recent algorithm [10] [11] aligns several source frequency components to a reference frequency bin to conduct Tests of Orthogonality of Projected Subspaces (TOPS). TOPS performs well at mid-level SNR, but tends to under-perform at high SNR levels and in noise-free case. Another disadvantage is that the TOPS pseudo-spectrum often exhibits spurious peaks at all SNR levels. Some of the spurious peaks are often stronger than the peaks at the true DOA locations and hence this method may generate false DOAs.

In this thesis, a novel Wideband Spectral Subspace Projection (WSSP) approach is proposed that exploits the inherent properties of projected subspaces to estimate the wideband DOAs. Among the key advantages of the proposed method is that it is non-iterative and does not require any prior DOA estimates, focusing, beamforming or transformation to reference frequency bin, as required by the existing algorithms. Closely spaced and disparate source DOAs are estimated simultaneously, without requiring refocusing or beamforming or iterations, as needed by many state-of-the art wideband DOA approaches. Theoretical small perturbation analysis has been conducted that confirms the ability of WSSP to produce significant peaks at correct DOAs.

The validity of the proposed WSSP algorithm has been tested using a variety of typical wideband sources encountered in radar and wireless communication applications, including Chirp, QPSK and MC-CDMA sources. The performance of the proposed algorithm has been compared with those of previously existing algorithms via extensive simulation studies, in terms of bias and root mean square error (RMSE). The simulation results demon-
strate that when compared to the existing methods, the performance of proposed method is accurate over a wide range of SNRs and it is not affected by the number of the source DOAs to be estimated.

## 1.2 Overview of the Thesis

The Thesis has been divided into following chapters. In Chapter 2, an overview of DOA estimation algorithms used for narrowband and wideband signal are discussed. In Chapter 3, the Wideband Spectral Subspace theory and algorithm are explained. In Chapter 4, Simulation results for Chirp, QPSK and MC-CDMA signals are reported along with comparison with other methods. Finally, in Chapter 5, the conclusion and future work are discussed.
Chapter 2: Overview of DOA Estimation Algorithms

The problem of DOA estimation can be divided into two categories depending on the bandwidth of the source signals in the frequency domain i.e. narrowband and wideband. Over the past four decades, many researchers have developed a large body of work on estimating DOAs of both narrowband and wideband signals [5] [4] [3] [1] [9] [7] [8] [10] [11] [2] [6].

This Chapter describes some of the major DOA estimation algorithms used for narrowband sources (see section 2.2) and wideband sources (see section 2.3). In section 2.1 the array configuration and signal model are described.

2.1 Signal and Array Model

Consider a uniform linear array (ULA) comprising of M sensors each separated by a distance of d. Let the l-th source signal be represented as \( s_l(t) \). Let’s consider L sources arriving from angle \( \theta_l, l = 1, \ldots, L \). The array output for \( m^{th} \) sensor is given in equation (2.1), where, \( \tau_{m,\theta_l} \) is the time delay of plane wave from direction \( \theta_l \) and \( n_m(t) \) is the noise.
at $m^{th}$ sensor element.

$$x_m(t) = \sum_{l=1}^{L} s_l(t - \tau_{m,\theta_l}) + n_m(t); \quad m = 1, 2, \ldots, M \quad (2.1)$$

For a uniform linear array (ULA),

$$\tau_{m,\theta_l} = \frac{(m-1)d}{c} \sin(\theta_l). \quad (2.2)$$

Then equation (2.1) can be rewritten as,

$$x_m(t) = \sum_{l=1}^{L} s_l(t - \frac{(m-1)d \sin(\theta_l)}{c}) + n_m(t). \quad (2.3)$$

### 2.2 Narrowband Algorithms

If the ratio of bandwidth of a source signal to its center frequency is very small, i.e., $\frac{\Delta f}{f_c} \ll 1$ then it is considered a narrowband signal. For a narrowband source at center frequency $= f_c$, equation (2.3) can be approximated as,

$$x_m(t) = \sum_{l=1}^{L} s_l(t)e^{-j2\pi v_m f_c \sin(\theta_l)} + n_m(t) \quad (2.4)$$
where, \( v_m = \frac{(m-1)d}{c} \), and \( c \) is velocity of light. The above equation can be written in matrix form as,

\[
\begin{align*}
x(t) &= A(\Theta)s(t) + n(t) \\
x(t) &= \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_M(t) \end{bmatrix}^T \\
s(t) &= \begin{bmatrix} s_1(t) & s_2(t) & \cdots & s_L(t) \end{bmatrix}^T \\
A(\Theta) &= \begin{bmatrix} a(\theta_1) & a(\theta_2) & \cdots & a(\theta_L) \end{bmatrix} \\
\Theta &= \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_L \end{bmatrix}^T \\
a(\theta) &= \begin{bmatrix} e^{-j2\pi f_c v_1 \sin(\theta)} & e^{-j2\pi f_c v_2 \sin(\theta)} & \cdots & e^{-j2\pi f_c v_M \sin(\theta)} \end{bmatrix}^T
\end{align*}
\]
Here $x(t)$ is the sensor array output, $A(\Theta)$ is the steering or manifold matrix, $s(t)$ is the signal vector and $n(t)$ is the noise vector.

### 2.2.1 Multiple Signal Classification (MUSIC)

This method [5] is based on minimizing the distance between signal subspace and steering vector $a(\theta)$. Considering equation (2.5), with no noise the sensor output is a linear combination of vectors in $A(\Theta)$. So if the signal subspace in which $x(t)$ lies can be estimated, then the DOAs can be determined by finding the distance between the steering vector $a(\theta)$ and the signal subspace. Assuming uncorrelated noise samples, the correlation matrix of the sensor output is formed as given below,

$$
R_x = E[x(t)x(t)^H] 
$$

$$
R_x = E[A(\Theta)s(t)s(t)^H A(\Theta)^H] + E[n(t)n(t)^H] 
$$

$$
R_x = A(\Theta)R_s A(\Theta)^H + \sigma^2 I 
$$

where, $R_s = E[s(t)s(t)^H]$ is the signal correlation matrix, which is diagonal as the sources are assumed to be uncorrelated with each other. The number of sources impinging on the ULA is assumed to be less than the number of array elements $M$ so the matrix $A(\Theta)R_s A(\Theta)^H$ will be singular as it’s rank will be equal to the number of sources. Therefore,

$$
|A(\Theta)R_s A(\Theta)^H| = |R_x - \sigma^2 I| = 0
$$

From equation (2.14) it can be seen that $\sigma^2$ is one of the eigenvalues of $R_x$ matrix. Since $A(\Theta)R_s A(\Theta)^H$ is positive semidefinite and its rank is equal to $L$, the $R_x$ matrix will have $M - L$ smallest eigenvalues equal to $\sigma^2$. When eigen-decomposition is performed on $R_x$, the subspace spanned by the eigenvectors corresponding to the $L$ largest eigenvalues will
represent signal subspace and $M - L$ eigenvectors corresponding to the $M - L$ smallest eigenvalues will represent noise subspace. Let the eigenvalues of $\mathbf{R}_x$ be denoted as, $\lambda_1, \lambda_2, \ldots, \lambda_M$ and the corresponding eigenvectors be represented by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_M$. The signal and noise subspaces are given by the range-space of the matrices in (2.16) and (2.17), respectively.

$$\lambda_1 > \lambda_2 > \cdots > \lambda_M$$

(2.15)

$$\mathbf{E}_S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_L \end{bmatrix}$$

(2.16)

$$\mathbf{E}_N = \begin{bmatrix} \mathbf{v}_{L+1} & \mathbf{v}_{L+2} & \cdots & \mathbf{v}_M \end{bmatrix}$$

(2.17)

The Euclidean distance between the steering vector $\mathbf{a}(\theta)$ for a hypothetical search angle $\theta$ and noise subspace is given by $|\mathbf{a}(\theta)^H \mathbf{E}_N|^2$. When $\theta$ is equal to one of the DOAs $\theta_1, \theta_2, \ldots, \theta_L$, the steering vector will be orthogonal to the noise subspace. The spatial spectrum for MUSIC is as given in equation (2.18). The MUSIC spectrum will have peaks when $\theta = \theta_1, \theta_2, \ldots, \theta_L$.

$$P_{MUSIC}(\theta) = \frac{1}{\mathbf{a}(\theta)^H \mathbf{E}_N \mathbf{E}_N^H \mathbf{a}(\theta)}$$

(2.18)

When compared to classical methods like beamforming, maximum likelihood and maximum entropy, the MUSIC method gives better results and reaches Cramer Rao bound asymptotically. This method can be applied to any sensor array geometry. The MUSIC method performs well at high SNR but may under-perform at low SNR [5].

### 2.2.2 Root-MUSIC

MUSIC method requires one-dimensional search of the spatial spectrum in (2.18) to determine the unknown DOAs. Barbell [1] developed the Root-MUSIC method that effectively
reduces the computational complexity of MUSIC by forming a polynomial to represent the spatial spectrum of MUSIC. For Root-MUSIC, instead of searching through all the angles, the roots of a polynomial can be used to determine the source DOAs. Consider the denominator of equation (2.18), as given below.

\[ S(\theta) = a(\theta)^H E_N E_N^H a(\theta) \] (2.19)

\[ B = E_N E_N^H \] (2.20)

\[ S(\theta) = \sum_{i=1}^{M} \sum_{k=1}^{M} e^{-j2\pi v_i f_c \sin(\theta)} B_{i+1,k+1} e^{-j2\pi v_k f_c \sin(\theta)} \] (2.21)

\[ S(\theta) = \sum_{l=-M+1}^{M-1} b_l e^{-j2\pi f_c ld \sin(\theta)} \] (2.22)

Equation (2.22) can be written it in the form of a polynomial as

\[ P(z) = \sum_{l=-M+1}^{M-1} b_l z^{-l} \] (2.23)

where, \( b_l \) is the sum of \( l^{th} \) diagonal of \( B \) matrix. The roots of the polynomial which are closer to the unit circle are used to estimate the DOAs. If \( z_1 \) is a root of the polynomial \( P(z) \) that is close to unit circle then the corresponding DOA is estimated as,

\[ \theta = \sin^{-1} \left( \frac{c}{2\pi df_c} \text{arg}(z_1) \right) \] (2.24)

where, \( \text{arg}(z_1) \) is the angle of root \( z_1 \). One of the important characteristics of this method when compared to the original MUSIC method is it’s ability to operate at relatively lower SNR, and another important feature of this method is that it can separate two closely spaced signals. The performance of Root-MUSIC and MUSIC can be seen in figure 2.2. It can be seen clearly that at SNR=13dB the root-MUSIC gives two roots closer to true angles while
MUSIC was unable to resolve the angles.

2.2.3 Estimation of Signal Parameter via Rotational Invariance Techniques (ESPRIT)

MUSIC [5] and Root-MUSIC [1] methods described above require information on the arrangement of array sensors and are computationally expensive. In the year 1989 Richard Roy in his PhD dissertation [4] exploited the rotational invariance property of the array of sensors, which required no knowledge about the array configuration [12]. In this method the sensor array is divided into two identical sub-arrays separated by a distance $\Delta$. Each sensor can have arbitrary phase response, gain, and polarization under the constraint that each sensor has an identical twin in the other sub-array.

Consider an array of $M$ sensors divided into two sub-arrays, each having $P$ sensors. If the sub-arrays overlap with each other then $M \leq 2P$, otherwise $M = 2P$. Let $\Delta$ be the
distance between the two sub-arrays. The output from the first sensor array is represented by \( x_1(t) \) and the output from the second sensor array is represented by \( x_2(t) \). Then the sub-array \( x_1(t) \) can be modeled as given in equation (2.25). As the second sub-array is displaced by a distance of \( \Delta \), \( x_2(t) \) can be modeled as given in equation (2.27),

\[
\begin{align*}
x_1(t) &= \left[ a(\theta_1) \ a(\theta_2) \ \ldots \ a(\theta_L) \right] s(t) + n_1(t) \\
x_1(t) &= A(\Theta)s(t) + n_1(t) \\
x_2(t) &= \left[ a(\theta_1)e^{-j2\pi f_c \Delta \sin(\theta_1)/c} \ a(\theta_2)e^{-j2\pi f_c \Delta \sin(\theta_2)/c} \ \ldots \ a(\theta_L)e^{-j2\pi f_c \Delta \sin(\theta_L)/c} \right] s(t) + n_2(t) \\
x_2(t) &= A(\Theta)\Phi s(t) + n_2(t)
\end{align*}
\]

where, \( \Phi \) is a diagonal matrix given by (2.29), and it is also known as rotation operator.

\[
diagonal(\Phi) = \begin{bmatrix}
e^{-j2\pi f_c \Delta \sin(\theta_1)/c} & e^{-j2\pi f_c \Delta \sin(\theta_2)/c} & \ldots & e^{-j2\pi f_c \Delta \sin(\theta_L)/c}
\end{bmatrix}
\] (2.29)

The outputs \( x_1(t) \) and \( x_2(t) \) are combined to form the equation given below,

\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A(\Theta) \\ A(\Theta)\Phi \end{bmatrix} s(t) + \begin{bmatrix} n_1(t) \\ n_2(t) \end{bmatrix}
\] (2.30)

If \( x(t) \) is noise free then the correlation matrix for \( x(t) \) is given as,

\[
R_x = E(x(t)x^H(t)) = \tilde{A}(\Theta)R_s\tilde{A}(\Theta)^H
\] (2.31)

where,

\[
\tilde{A} = \begin{bmatrix} A(\Theta) \\ A(\Theta)\Phi \end{bmatrix}
\] (2.32)
Since $R_s$ is a $L \times L$ matrix with a rank of $L$, $R_x$ will have a rank of $L$. Let $\tilde{E}_s$ be the signal subspace of $R_x$ having $L$ eigen-vectors. Since $\tilde{E}_s$ span the same space as $\tilde{A}(\Theta)$ there exists a non-singular matrix $T$ such that $\tilde{E}_s = \tilde{A}(\Theta)T$. The $\tilde{E}_s$ matrix can be decomposed as given below.

\[
\tilde{E}_s = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}(\Theta)T \\ \tilde{A}(\Theta)\Phi T \end{bmatrix}
\] (2.33)

It is known that the range space of $E_1$ and $E_2$ is equal to range space of $A(\Theta)$, therefore the range space of $E_1$ is equal to the range space of $E_2$. Hence there exists a matrix $\Psi$ such that $E_1 \Psi = E_2$.

\[
E_1 \Psi = E_2 \quad (2.34)
\]

\[
A(\Theta)T \Psi = A(\Theta)\Phi T \quad (2.35)
\]

\[
\Psi = (A(\Theta)T)^{-1}A(\Theta)\Phi T \quad (2.36)
\]

\[
\Psi = T^{-1}A(\Theta)^{-1}A(\Theta)\Phi T \quad (2.37)
\]

\[
\Psi = T^{-1}\Phi T \quad (2.38)
\]

Since $\Phi$ is a diagonal matrix the eigenvalues of $\Psi$ will give the information about DOA. So, the ESPRIT algorithm depends on the estimation of $\tilde{E}_s$ and there is no need for searching as in case of traditional MUSIC in Equation (2.18).

### 2.3 Wideband Algorithms

Most of the applications in wireless communication and radars use very wideband signals, unlike the narrowband sources appearing in sonar where the signal spectrum can be approximated by the center frequency. In the wideband case, the source signals occupy a
wide range of frequencies. The algorithms that were developed for narrowband signals can also be used for wideband signals but may give poor estimate of DOAs, especially at low SNRs because the information from the wide bandwidths of the sources is not utilized by the narrowband methods. In order to obtain better DOA estimates for wideband sources, all available frequency bins of the wideband signals should be used. Many researchers have developed algorithms to detect DOAs of wideband signals by first dividing the signal into various frequency bins using FFT and then finding the DOAs by combining the information from different frequency bins. In this section some of the existing wideband algorithms are discussed.

If Fourier transform is applied on equation (2.3), the \( m^{th} \) sensor array output at continuous frequency \( f \) will be given as,

\[
X_m(f) = \sum_{l=1}^{L} S_l(f) e^{-j2\pi f v_m \sin(\theta_l)} + N_m(f). \tag{2.39}
\]

In practice, discrete Fourier transform is used by taking FFT of the received signal samples. Therefore, discrete version of (2.39) will be used to formulate the problem. Consider \( f_k \) be the discrete frequency of the \( k^{th} \) frequency bin, then equation (2.39) can be rewritten as,

\[
X_m(f_k) = \sum_{l=1}^{L} S_l(f_k) e^{-j2\pi f_k v_m \sin(\theta_l)} + N_m(f_k) \tag{2.40}
\]

for \( k = 1, 2, \ldots K \). The array output can be written in matrix form as given below.

\[
x(f_k) = A(f_k, \Theta)S(f_k) + N(f_k) \tag{2.41}
\]
where,

\[
x(f_k) = \begin{bmatrix} X_1(f_k) & X_2(f_k) & \ldots & X_M(f_k) \end{bmatrix}^T
\] (2.42)

\[
A(f_k, \Theta) = \begin{bmatrix} a(f_k, \theta_1) & a(f_k, \theta_2) & \ldots & a(f_k, \theta_L) \end{bmatrix}
\] (2.43)

\[
a(f_k, \theta_l) = \begin{bmatrix} e^{-j2\pi f_k v_1 \sin(\theta_l)} & e^{-j2\pi f_k v_2 \sin(\theta_l)} & \ldots & e^{-j2\pi f_k v_M \sin(\theta_l)} \end{bmatrix}^T
\] (2.44)

\[
S(f_k) = \begin{bmatrix} S_1(f_k) & S_2(f_k) & \ldots & S_L(f_k) \end{bmatrix}^T
\] (2.45)

\[
N(f_k) = \begin{bmatrix} N_1(f_k) & N_2(f_k) & \ldots & N_M(f_k) \end{bmatrix}^T
\] (2.46)

### 2.3.1 Incoherent MUSIC (IMUSIC)

This algorithm [9] is based on narrowband MUSIC but applied at each frequency bin separately, and the results are then combined to estimate the DOAs for the wideband sources. In this algorithm first the data is segmented and FFT is applied at each sensor. Then correlation matrix is found at each frequency bin followed by eigen-decomposition performed at each bin to obtain the signal and noise subspaces, as explained in 2.2.1. Consider \( L \) wideband sources, \( K \) frequency bins, and let \( R_x(f_k) \) be the correlation matrix at frequency \( f_k \). Let \( v_m(f_k) \) be the \( m^{th} \) eigenvector at frequency \( f_k \). The spatial spectrum can be computed using one of the equations given below,

\[
J_1(\theta) = \frac{1}{K \sum_{k=1}^{K} \frac{1}{M-L} \sum_{m=L+1}^{M} ||a^H(f_k, \theta)v_m(f_k)||^2}
\] (2.47)

\[
J_2(\theta) = \frac{1}{\prod_{k=1}^{K} \frac{1}{M-L} \sum_{m=L+1}^{M} ||a^H(f_k, \theta)v_m(f_k)||^2}
\] (2.48)
2.3.2 Coherent Signal Subspace (CSS)

IMUSIC is computation intensive because of the need to perform eigen-decomposition at different frequency bins for combining the results to form the spatial spectrum. In the year 1984, Wang and Kaveh came up with the idea of coherent signal subspace [7] [8] which focuses the correlation matrices at different frequencies to a single center frequency. The focusing matrices are constructed with initial estimates of the DOAs. In [8] the signals were collected at $D$ non-overlapping intervals of $\Delta T$ duration. Let $X_i(f_k)$, be the sensor data collected for $i = 1, 2, \ldots, D$ and at $f_k$ frequency for $k = 1, 2, \ldots, K$. The correlation matrix is approximated as given below.

$$Cov(X_i(f_k)) \approx \frac{P_x(f_k)}{\Delta T} = \frac{1}{\Delta T} A(f_k, \theta) P_S(f_k) A^H(f_k, \theta) + \frac{\sigma^2}{\Delta T} P_N(f_k)$$ (2.49)

where,

$P_x(f_k) =$ Array Cross Spectral Density at frequency $f_k$

$P_S(f_k) =$ Cross Spectral Density of signals at frequency $f_k$

$P_N(f_k) =$ Noise Spectral Density at frequency $f_k$

$\sigma^2 =$ Noise power.

In this algorithm, FFT is applied at the sensor output to get the $X_i(f_k)$ at frequency $f_k$. Then correlation matrix is estimated as given below,

$$\hat{C}(X(f_k)) = \frac{1}{D} \sum_{i=1}^{D} X_i(f_k) X_i^H(f_k)$$ (2.50)
Periodogram or Capon’s method is typically used to estimate the initial DOA estimates.

Then a focusing matrix at frequency \( f_0 \) for correlation matrix at frequency \( f_k \) is constructed using the estimated angle, say \( \theta_0 \) as given in equation (2.51),

\[
T(f_k) = \begin{bmatrix}
a_1(f_0, \theta_0) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & a_M(f_0, \theta_0)
\end{bmatrix}
\tag{2.51}
\]

where, \( a_m(f_0, \theta_0) \) is the \( m \textsuperscript{th} \) element of \( \alpha(f_k, \theta_0) \) as given in equation (2.44) and \( T(f_k) \) is the diagonal matrix. When this diagonal focusing matrix is applied to the sensor data at frequency \( f_k \) the focused data is as given in equation (2.52)

\[
Y(f_k) = \sqrt{\Delta} T(f_k) X(f_k)
\tag{2.52}
\]

The sum of the correlation matrices for focused data for \( K \) frequencies are as given below.

\[
\sum_{k=1}^{K} C(Y(f_k)) = A(f_0, \theta_0) \left[ \sum_{k=1}^{K} P_s(f_k) \right] A^H(f_0, \theta_0) + \sigma^2 \sum_{k=1}^{K} T(f_k) \left[ P_N(f_k) \right] T^H(f_k)
\tag{2.53}
\]

Define the following correlation matrices after focusing,

\[
R = \sum_{k=1}^{K} C(Y(f_k)) \\
R_s = \sum_{k=1}^{K} P_s(f_k) \\
R_n = \sum_{k=1}^{K} T(f_k) \left[ P_N(f_k) \right] T^H(f_k)
\]

So equation (2.53) can be reduced to the form,

\[
R = A(f_0, \theta_0) R_s A^H(f_0, \theta_0) + \sigma^2 R_n
\tag{2.54}
\]
In [8] it has been proven that the matrix pencil \((\mathbf{R}, \mathbf{R}_n)\) will have \(M - L\) smallest eigenvalues equal to \(\sigma^2\) and the eigenvectors \(\mathbf{v}_{L+1}, \mathbf{v}_{L+2}, \ldots, \mathbf{v}_M\) corresponding to these eigenvalues will span the null space \(\mathbf{E}_N\).

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L \geq \sigma^2 \quad (2.55)
\]

\[
\lambda_{L+1} = \lambda_{L+2} = \cdots = \lambda_M = \sigma^2 \quad (2.56)
\]

\[
\mathbf{E}_N = \left[ \mathbf{v}_{L+1}, \mathbf{v}_{L+2}, \ldots, \mathbf{v}_M \right] \quad (2.57)
\]

The spatial spectrum can then be plotted as given in equation (2.58) and (2.59) by varying \(\theta\),

\[
P_{CSS}(\theta) = \frac{1}{\mathbf{a}(\theta, f_0)^H \mathbf{E}_N \mathbf{E}_N^H \mathbf{a}(\theta, f_0)} \quad (2.58)
\]

\[
P_{CSS}(\theta) = \frac{1}{\mathbf{a}(\theta, f_0)^H \mathbf{R}_n^{-1} \mathbf{a}(\theta, f_0) - \mathbf{a}(\theta, f_0)^H \mathbf{E}_s \mathbf{E}_s^H \mathbf{a}(\theta, f_0)} \quad (2.59)
\]

Unlike IMUSIC, the CSS method requires only one eigen-decomposition of the focused correlation matrix \(\mathbf{R}\). CSS is an iterative algorithm and the estimates from the previous iteration may be used to update the focusing matrices to further update the DOA estimates until convergence. Also, each step of focusing can estimate DOAs only in one direction as determined by the preliminary DOA estimate used for focusing. If there are disparate set of sources well separated from each other, then the number of iterations will increase. Typically, only one iteration step is used for each peak of the periodogram estimate. Therefore, when compared to IMUSIC, overall computational cost for CSS may still be less. Good initial DOA estimate is crucial for this algorithm to work. The CSS method works well for correlated sources unlike the IMUSIC approach which does not work in correlated scenario. With regards to robustness to noise, CSS works well at low SNR, whereas IMUSIC works best at high SNR.
2.3.3 Weighted Average of Signal Subspace (WAVES)

The CSS method focuses both signal and noise correlation matrices as can be seen in equation (2.53). In Weighted Subspace Fitting [6] weights are assigned to vectors in signal subspace to estimate the DOAs. This idea was extended to wideband source in [2] where asymptotically efficient estimate are found using equation (2.60), where $Q(f_k)$ is a diagonal weighting matrix whose diagonal elements are given in (2.61), $\lambda_l(f_k)$ is the $l^{th}$ eigenvalue at frequency $f_k$ and $\sigma^2$ is noise power. Here, $C_k = Q(f_k)E_s(f_k)A^\dagger(f_k, \theta)$, where $E_s(f_k)$ is the signal subspace at frequency $f_k$, and $\dagger$ denotes matrix pseudo-inverse given by, $A^\dagger(f_k, \theta) = (A^T(f_k, \theta)A(f_k, \theta))^{-1}A(f_k, \theta)^T$. Then,

$$\theta_{true} = \arg\min_{k=1}^{K} |A(f_k, \theta)C_k - E_s(f_k)Q(f_k)|_F^2.$$  

(2.60)

$$Q(f_k)_{l,l} = \frac{\lambda_l(f_k) - \sigma^2}{(\lambda_l(f_k)\sigma^2)^{1/2}}.$$  

(2.61)

In WAVES method the weighted subspace fitting idea was combined with CSS focusing matrix to come up with a universal signal subspace which can be used for DOA estimation. Multiplying $E_s(f_k)$ in equation (2.60) by focusing matrix as given by equation (2.51), will transform (2.61) as given below.

$$\theta_l = \arg\min_{k=1}^{K} |A(f_0, \theta)C_k - T(f_k)E_s(f_k)Q(f_k)|_F^2.$$  

(2.62)

For $L$ sources and $K$ frequency bins a new matrix $Z \in \mathbb{C}^{M \times LK}$ is constructed as given in equation (2.63). The matrix $Z$ has a rank of $L$ as proven in [2], but due to noise it will be
full rank. The signal subspace can be found by SVD of matrix $Z$,

$$Z_{M \times LK} = (LK)^{-1/2}[T(f_1)E_s(f_1)Q(f_1), \ldots, T(f_K)E_s(f_K)Q(f_K)]$$

$$SVD(Z_{M \times LK}) = \begin{bmatrix} E_s & E_n \end{bmatrix} \begin{bmatrix} \lambda_S & 0 \\ 0 & \lambda_N \end{bmatrix} \begin{bmatrix} W_s \\ W_n \end{bmatrix}$$

(2.64)

where, $E_s$ is universal signal subspace corresponding to $L$ principal singular values $\lambda_S$. This universal signal subspace can be used to estimate the angles of arrival. When compared to CSS, the WAVES method is computationally more expensive as it needs to perform eigen-decomposition of correlation matrices at all available frequency bins. Similar to CSS, this algorithm also depends on the initial angle estimation to form the focusing matrix.

### 2.3.4 Test of Orthogonality of Projected Subspace (TOPS)

CSS and WAVES methods described in sections 2.3.2 and 2.3.3, respectively, initial DOA estimates are needed in-order to apply focusing matrix at different frequency bins. The accuracy of the final DOA estimates produced by these methods depends on the initial DOA estimates used for focusing. The TOPS algorithm developed in [11] [10] does not require information about initial angle. In this algorithm the angles of arrival are estimated using the orthogonality of signal and noise subspace. TOPS depends on transformation matrices to transform the signal subspace at one frequency to another frequency. TOPS uses a diagonal transformation matrix $\Phi$, whose diagonal elements are given by,

$$\Phi(f_r, \theta_r)_{m,m} = e^{-j2\pi f_r v_m \sin(\theta_r)}, \quad m=1,2,\ldots,M$$

(2.65)
Considering an array manifold at frequency $f_i$ and angle $\theta_i$, the $m^{th}$ element of the array manifold is given by,

$$a(f_i, \theta_i)_m = e^{-j2\pi f_i v_m \sin(\theta_i)} \quad m = 1, 2, \ldots, M \quad (2.66)$$

Multiplying this array manifold with the transformation matrix the new array manifold is given by,

$$\Phi(f_r, \theta_r)m, m a(f_i, \theta_i)_m = e^{-j2\pi f_r v_m \sin(\theta_r)} e^{-j2\pi f_i v_m \sin(\theta_i)} \quad (2.67)$$

$$= e^{-j2\pi v_m (f_r \sin(\theta_r) + f_i \sin(\theta_i))} \quad (2.68)$$

$$= e^{-j2\pi (f_r + f_i) v_m (\frac{f_r \sin(\theta_r)}{f_r + f_i} + \frac{f_i \sin(\theta_i)}{f_r + f_i})} \quad (2.69)$$

$$= e^{-j2\pi (f_k) v_m (\sin(\theta_k))} \quad (2.70)$$

$$= a(f_k, \theta_k)_m \quad (2.71)$$

where, $f_k = (f_r + f_i)$ and $\sin(\theta_k) = \frac{f_r \sin(\theta_r)}{f_r + f_i} + \frac{f_i \sin(\theta_i)}{f_r + f_i}$. The array manifold has been transformed to frequency $f_k$ and angle $\theta_k$. See [11] for a proof of this frequency transformation concept. It should be noted that $\sin(\theta_k)$ is equal to $\sin(\theta_i)$ if $\theta_i = \theta_r$. Keeping this property in mind, consider $E_s(f_i)$ the signal subspace at frequency $f_i$ which is formed from eigen-decomposition of the correlation matrix $R(f_i)$. Next, consider a steering matrix $A(f_i, \theta)$ at frequency $f_i$. It is known that $E_s(f_i)$ and $A(f_i, \theta)$ have the same range span. Therefore, there exists a full rank square matrix $G_i$ such that,

$$E_s(f_i) = A(f_i, \Theta)G_i \quad (2.72)$$
Let, $\Delta f = f_r - f_i$ and consider a transformation matrix given by $\Phi(\Delta f, \phi)$. Multiplying equation (2.72) with $\Phi(\Delta f, \phi)$,

\[
\Phi(\Delta f, \phi)E_s(f_i) = \Phi(\Delta f, \phi)A(f_i, \Theta)G_i
\]

(2.73)

\[
\Phi(\Delta f, \phi)E_s(f_i) = A(f_r, \hat{\Theta})G_i
\]

(2.74)

The transformation matrix $\Phi(\Delta f, \phi)$ transforms $A(f_i, \Theta)$ to $A(f_r, \hat{\Theta})$ where, $\sin(\hat{\theta}) = \frac{\Delta f \sin(\phi)}{f_j} + \frac{f_i \sin(\theta_i)}{f_j}$. Hence, the range space of $\Phi(\Delta f, \phi)E_s(f_i)$ is equal to the range space of $A(f_j, \hat{\theta})$. This property is one of the key concepts derived in [10] [11] to formulate the TOPS algorithm which is outlined next. The signal subspace at the reference frequency, $E_s(f_0)$ is transformed into another frequency using,

\[
U_i(\phi) = \Phi(\Delta f_i, \phi)E_s(f_0)
\]

(2.75)

where, $\Delta f_i = f_i - f_0$ for $i = 1, \ldots, K - 1$. It should be noted that $f_0$ is the focusing or reference frequency and not the lowest frequency. Selection of the reference frequency has been discussed later. Let $E_n(f_i)$ be the noise subspace at frequency $f_i$. Then a matrix $D(\phi)$ was constructed as given below,

\[
D(\phi) = \begin{bmatrix}
U_1^H(\phi)E_n(f_1) & U_2^H(\phi)E_n(f_2) & \cdots & U_{K-1}^H(\phi)E_n(f_{K-1})
\end{bmatrix}
\]

(2.76)

It has been shown in [10] [11] that $D(\phi)$ loses its rank when $\phi = \theta_l$ where $\theta_l$ is one of the true DOAs, i.e., of the $l^{th}$ source. The value of hypothetical search angle $\phi$ is varied and
the spatial spectrum is estimated using the following equations,

\[ \theta_l = \arg \max \left( \frac{1}{\sigma_{\text{min}}(\phi)} \right) \]  

\[ P(\phi) = \left( \frac{1}{\sigma_{\text{min}}(\phi)} \right) \]  

where, \( \sigma_{\text{min}}(\phi) \) is the minimum singular value of matrix \( D(\phi) \) at angle \( \phi \).

**Performance Using Projected Matrices:** The preliminary TOPS algorithm as described above is highly dependent on the quality of estimated signal and noise subspaces. In practice, only estimated correlation matrices are available and noisy subspace estimates lead to performance degradation if the TOPS version given in (2.78) is used. In order to minimize estimation error, in the final version of TOPS, the signal subspace is projected onto the null subspace using,

\[ U'_i(\phi) = (I - P(f_i, \phi))U_i(\phi), \]  

where,

\[ P(f_i, \phi) = a(f_i, \phi)(a^H(f_i, \phi)a(f_i, \phi))^{-1}a^H(f_i, \phi) \]  

and \( a^H(f_i, \phi) \) is the steering vector defined in equation (2.44). Figure 2.3 shows the performance of TOPS algorithm by constructing \( D(\phi) \) in two ways: (a) without using projection matrix to form \( U_i \) as in (2.75) shown in dashed-black in the figure and (b) using projected subspace to form \( U'_i \) as in (2.79) shown in dashed-blue line. It can be observed that without the projection matrix, TOPS fails to detect the peaks at 9 and 12 degrees correctly. However with the use of Projection matrices, it successfully detects the peaks. The figure also
shows that TOPS using projection exhibits some spurious peaks at SNR=5dB. Figure 2.4 depicts the pseudo-spectrum of TOPS at infinite SNR, i.e., with no noise in the data. The dashed-black line in Fig. 2.4 shows the performance of TOPS without using projection matrix, and it can be seen that this version of TOPS is not able to estimate the DOAs correctly even in absence of noise. The dashed-blue line Fig. 2.4 shows the noise-free performance of the final version of TOPS that uses projection matrices, and this case the true DOAs are detected correctly. However, the pseudo-spectrum in this case exhibits spurious peaks which can be stronger than the true DOAs, biasing the results.

One of the drawbacks of the TOPS algorithm is that it requires eigenvalue calculation at every angle $\phi$ to form the pseudo-spectrum in (2.78), that can add to its computational cost.

**Frequency Selection in TOPS:** The subspace selection and reference frequency choice play important roles in the effectiveness of the projection based TOPS method. In [10], least noisy signal subspace $E_s(f_0)$ and least noisy noise subspaces, $E_n(f_k)$’s were selected by finding the frequency bins for which the difference between lowest signal eigenvalue, say $\sigma_{i,\text{min}}^s$ and the highest noise eigenvalue, say, $\sigma_{i,\text{max}}^n$ is maximum. Simulation studies indicate that this frequency bin selection approach is very effective in practice. In fact, arbitrary choice of subspace and reference frequencies may degrade the performance of TOPS.
Figure 2.3: TOPS performance with and without Projection at SNR=5dB

Figure 2.4: TOPS performance at SNR=Inf
Chapter 3: Wideband Spectral Subspace Projection (WSSP)

Most of the wideband algorithms discussed in Chapter 2 requires either focusing or transforming to reference frequency to generate the spatial spectrum. In this chapter, a novel non-iterative algorithm to estimate the wideband DOAs is developed that utilizes the properties of projected subspaces. A key advantage of the proposed approach is that prior estimates of the unknown DOAs are not needed. Furthermore, all DOAs are estimated in a single pass and no iterations are involved. The chapter is divided into the following sections. In section 3.1, the mathematical rationale for using wideband spectral subspace projection (WSSP) is discussed and construction of a matrix composed of projection of spectral noise subspaces onto hypothesized spectral signal subspace are presented. Choice of proper frequency bins for projection plays a key role in achieving desirable performance. In Section 3.2, selection of appropriate frequency bins is discussed, which improves the performance and reduces number of computations. In Section 3.4, the steps for implementing the WSSP algorithm are given.
3.1 Projection of Signal Subspace on to Noise Subspace

WSSP is a frequency domain algorithm. Similar to other existing frequency domain methods [7] [8] [11], the output of sensor array is decomposed into several narrowband bins using DFT. Let $R(f_k)$ denote the correlation matrix at frequency $f_k$. Similar to narrowband MUSIC the eigen-decomposition of $R(f_k)$ will give signal and noise subspaces at frequency $f_k$. $L$ largest eigenvalues of $R(f_k)$ correspond to the eigenvectors which span the signal subspace i.e., $E_s(f_k)$, and $M-L$ small eigenvalues correspond to eigenvectors spanning the noise subspace i.e., $E_n(f_k)$. Define the signal subspace projection matrix for $a(f_k, \theta)$,

$$P(f_k, \theta) = a(f_k, \theta)(a^H(f_k, \theta)a(f_k, \theta))^{-1}a^H(f_k, \theta)$$

(3.1)

where, $a(f_k, \theta)$ is the source manifold vector defined in (1.43). The projection operator, $P(f_k, \theta)$ projects any vector onto the signal subspace at frequency $f_k$ and hypothetical search angle $\theta$. Ideally, if $\theta \in \Theta$, i.e., the hypothetical search angle $\theta$ matches one of the true source angles and $f_k$ is one of the source frequency bins, then $P(f_k, \theta)$ would annihilate the noise subspace eigenvectors, i.e.,

$$P(f_k, \theta)E_n(f_k) = 0_{M \times M-L}.$$  

(3.2)

Equation (3.2) is a key equation and the development of the proposed WSSP algorithm is premised on this fundamental subspace projection concept. In practice, however, the noise subspace matrices $E_n(f_k)$ will be estimated from noisy data and the precise nulling due to projection in (3.2) will not hold. In that case, the lengths of the projected subspaces will be determined by the inner-products of the projected column vectors in $P(f_k, \theta)E_n(f_k)$ for different source spectral components $f_k$, as described next.

Consider matrix $Q(\theta)$ formed by concatenation of noise subspaces projected on to
signal subspace defined by $P(f_k, \theta)$’s:

$$Q(\theta) = \left[ P(f_1, \theta)E_n(f_1) \ P(f_2, \theta)E_n(f_2) \ \ldots \ P(f_K, \theta)E_n(f_K) \right]_{M \times (M-L)K} \quad (3.3)$$

where, $\theta$ is a hypothetical search angle and $K$ is the number of frequency bins. The noise subspace at frequency $f_k$ will have $M - L$ eigenvectors; therefore, matrix $Q(\theta)$ will contain $K(M - L)$ projected vectors. The $Q(\theta)$ matrix is expressed in expanded form as,

$$Q(\theta) = \left[ q_{1,1} \ \ldots \ q_{1,M-L} \ q_{2,1} \ \ldots \ q_{2,M-L} \ \ldots \ q_{K,1} \ \ldots \ q_{K,M-L} \right]_{M \times (M-L)K} \quad (3.4)$$

where, $q_{k,i}$ denotes the projected noise subspace vector of frequency $k$, and the $i$-th vector of noise subspace $E_n(f_k)$. Considering $Q(\theta)Q(\theta)^H$,

$$Q(\theta)Q(\theta)^H = \left[ q_{1,1}^H \ q_{1,2} \ q_{1,3} \ \ldots \ q_{K,M-L}^H \right] \left[ \begin{array}{c} q_{1,1}^H \\ q_{1,2}^H \\ q_{1,3}^H \\ \ldots \\ q_{K,M-L}^H \end{array} \right] \quad (3.5)$$

$$= q_{1,1}^H q_{1,1}^H + q_{1,2}^H q_{1,2}^H + q_{1,3}^H q_{1,3}^H + \ldots + q_{K,M-L}^H q_{K,M-L}^H \quad (3.6)$$

i.e., $Q(\theta)Q(\theta)^H$ is equal to the summation of individual outer-products of the columns in $Q(\theta)$. Since the dot-product of two vectors is equal to the trace of their outer products, i.e.,

$$q_{k,i}^H q_{k,i} = \text{trace}(q_{k,i}^H q_{k,i}^H),$$

the trace of $Q(\theta)Q(\theta)^H$ is equal to the sum of dot products of
the noise subspace projected onto the signal subspaces at hypothesized DOA θ, i.e.,

\[
\text{trace}(Q(\theta)Q(\theta)^H) = \sum_{k=1}^{K} \sum_{i=1}^{M-L} \text{trace}(q_{i,k}q_{i,k}^H) = \sum_{k=1}^{K} \sum_{i=1}^{M-L} q_{k,i}^H q_{k,i} \quad (3.7)
\]

When the hypothesized search angle θ equals any of the true DOAs, the signal subspace projection \( P(f_k, \theta) \) on to the noise subspaces \( E_n(f_k) \) are minimized, i.e., the dot product sum in (3.8) becomes small. This fact is utilized to estimate the unknown DOAs as,

\[
\hat{\theta} = \arg \max_{\theta} \frac{1}{\text{trace}(Q(\theta)Q(\theta)^H)} \quad (3.9)
\]

This is equivalent to estimating the peak locations of the pseudo-spectrum given below

\[
P(\theta) = \frac{1}{\text{trace}(Q(\theta)Q(\theta)^H)} \quad (3.10)
\]

by varying the hypothesized DOAs θ.

### 3.2 Frequency Selection

According to [10], and based on extensive simulation studies it is apparent that using all available frequency bins to estimate the unknown DOAs does not always yield acceptable performance. In practice, signal and noise subspaces are computed using noisy observation data, and hence, some frequency bins tend to be more noisy than the others. Simulation experience also indicates that incorrect choice of frequency bins may result in performance degradation. Furthermore, since the signal and/or noise eigenvectors need to be computed at each frequency bin, use of large number of frequency bins increases computational cost.
Therefore, it is very important to select frequency bins of the highest quality, *i.e.*, the least noisy bins. In these regards, [10] recommends selecting frequency bins for which the difference between the smallest signal eigenvalue ($\sigma_{\text{min}}^s$) and the largest noise eigenvalue ($\sigma_{\text{max}}^n$) is maximum. The difference $|\sigma_{\text{min}}^s - \sigma_{\text{max}}^n|$, with respect to SNR for a single frequency can be observed in Figure 3.1. It can be observed that as SNR increases, the spread between the signal and noise eigenvalues as given by, $|\sigma_{\text{min}}^s - \sigma_{\text{max}}^n|$ increases.

In this thesis, the frequency criteria used in TOPS has been adopted to select the noise subspaces of the least noisy frequency bins, except in this case there is no need for selecting a reference signal subspace. Since not all frequency bins are utilized by WSSP, it is not necessary to perform full eigen-decomposition at all frequency bins. As an initial step, only the eigenvalues at all frequency bins need to be calculated. Once the least noisy frequency bins are identified, full eigen-decomposition need to be performed only at those smaller subset of frequency bins. This manner of frequency selection will greatly reduce computational cost of WSSP because of the smaller number of vectors in $Q(\theta)$ to be processed, and since complete eigen-decomposition at all frequency bins is not needed.

The steps for selecting frequency in wideband signals are as follows,

1. Find eigenvalues of correlation matrix $R(f_k)$ for $k = 1, \ldots, K$.

2. Find the difference between the minimum signal eigenvalue and the maximum noise eigenvalue, *i.e.*, $|\sigma_{\text{min}}^s(f_k) - \sigma_{\text{max}}^n(f_k)|$ at frequency $f_k$ for $k = 1, \ldots, K$.

$$
\sigma^{sn}(f_k) = |\sigma_{\text{min}}^s(f_k) - \sigma_{\text{max}}^n(f_k)| \tag{3.11}
$$

where, $\sigma^{sn}(f_k)$ is the absolute difference at frequency $f_k$

3. Normalize $\sigma^{sn}(f_k)$ using the equation given below, where $\sigma^{sn}(f_u)$ is the maximum
Figure 3.1: Plot depicting relation between $|\sigma_{min}^s - \sigma_{max}^n|$ and SNR in AWGN channel for single Frequency

of $\sigma^{sn}(f_k)$ at frequency $f_k$ where $1 \leq k \leq K$

$$\alpha(f_k) = \sigma^{sn}(f_k)/|\sigma^{sn}(f_k)|_{max}$$ (3.12)

4. Choose threshold value $\beta$ where $0 < \beta < 1$.

5. Choose $K_{FS} \leq K$ least noisy frequency bins for which $\alpha(f_k) \geq \beta$ for $k = 1, 2, \ldots, K_{FS}$.

Figures 3.2, 3.3 and 3.4 show the frequency selection plots for MC-CDMA, QPSK, and Chirp sources, respectively.
Figure 3.2: Frequency selection at SNR=10dB and $\beta = 0.9$ for MC-CDMA

Figure 3.3: Frequency selection at SNR=10dB and $\beta = 0.85$ for QPSK

### 3.3 Error Analysis using Noise Subspace Projection

Consider an array of $M$ sensors impinged by $L$ sources from angle $\theta_l$ for $l = 1, 2, \ldots, L$. For a single frequency bin, let $E_n$ be the noise subspace and $P(\theta)$ be the projection matrix as given in Equation (3.1). According to (3.3), $Q(\theta)$ for a single frequency is:

$$Q(\theta) = P(\theta)E_n$$

(3.13)

$$E_n = \begin{bmatrix} v_{L+1} & v_{L+2} & \cdots & v_M \end{bmatrix}$$

(3.14)
Figure 3.4: Frequency selection at SNR=10dB and $\beta = 0.8$ for Chirp

If $\theta$ is equal to any of the true DOA $\theta_l$, $P(\theta)$ should be orthogonal to $E_n$, as noted in equation (3.2) and hence, $Q(\theta)$ will be a zeros matrix. In that case, the trace of $Q(\theta)Q(\theta)^H$ should also be zero. Suppose there is an error in estimation of noise subspace and that the estimated noise subspace $\hat{E}_n$ can be expressed as:

$$\hat{E}_n = \begin{bmatrix} v_{l+1} + \delta v_{l+1} & v_{l+2} + \delta v_{l+2} & \ldots & v_M + \delta v_M \end{bmatrix}$$  \hspace{1cm} (3.15)$$

If the projection matrix is applied on $\hat{E}_n$, the matrix $Q(\theta)$ will not be a zero matrix, as shown below:

$$Q(\theta) = P(\theta)\hat{E}_n$$  \hspace{1cm} (3.16)

$$Q(\theta) = P(\theta) \begin{bmatrix} v_{l+1} + \delta v_{l+1} & v_{l+2} + \delta v_{l+2} & \ldots & v_M + \delta v_M \end{bmatrix}$$  \hspace{1cm} (3.17)

$$Q(\theta) = \begin{bmatrix} 0 + \delta u_{l+1} & 0 + \delta u_{l+2} & \ldots & 0 + \delta u_M \end{bmatrix}$$  \hspace{1cm} (3.18)
where, $\delta u_i = P(\theta)\delta v_i$, for $i = L + 1, \ldots, M$ are projected error vectors, which will be small as long as $\delta v_i$’s are small. The matrix $Q(\theta)Q(\theta)^H$ can be expressed as:

$$Q(\theta)Q(\theta)^H = \delta u_{L+1}\delta u_{L+1}^H + \delta u_{L+2}\delta u_{L+2}^H + \cdots + \delta u_M\delta u_M^H. \quad (3.19)$$

The trace of $Q(\theta)Q(\theta)^H$ is the sum of squared 2-norm of the error vectors. Consider the overall $Error$ in the denominator of equation (3.10) is given by,

$$Error[\text{trace}(Q(\theta)Q(\theta)^H)] = \sum_{k=1}^{K} \sum_{i=1}^{M-L} \text{trace}(\delta u_{i,k}\delta u_{i,k}^H) \quad (3.20)$$

$$= \sum_{k=1}^{K} \sum_{i=1}^{M-L} \delta u_{k,i}^H\delta u_{k,i} \quad (3.21)$$

The error terms contain only squared $\delta$ terms that are negligible in value. Hence, $Error \to 0$ for small deviations in noise eigenvectors, which explains the good performance of WSSP as demonstrated in the simulation sections.

### 3.4 WSSP Algorithm Steps

The steps used for generation of the spatial pseudo-spectrum in (3.10) for WSSP are as follows:

1. Apply DFT at each sensor and estimate $\hat{R}(f_k)$ at frequency bins, $f_k$, for $k = 1, 2, \ldots, K$.

2. Compute eigenvalues of the $\hat{R}(f_k)$ matrices at all frequency-bins.

3. Select $K_{FS} \leq K$ least noisy frequency bins $k = 1, 2, \ldots, K_{FS}$ having the largest separations between the lowest signal eigenvalue and the highest noise eigenvalue.

4. Perform complete eigen-decomposition of $K_{FS}$ least noisy correlation matrices selected in step-3. Determine the noise subspace eigenvectors $E_n(f_k)$ at the selected
frequency bins $k = 1, 2, \ldots, K_{FS}$.

5. Form $Q(\theta)$ using equation (3.3), except use $K_{FS}$ least noisy noise subspaces instead of all $K$ bins.

6. Calculate the spatial pseudo-spectrum using the equation given below and estimate the DOAs from the peak locations of the pseudo spectrum,

$$P(\theta) = \frac{1}{\text{trace}(Q(\theta)Q(\theta)^H)}$$

(3.22)
Chapter 4: Simulation

The validity of the proposed Wideband Spectral Subspace Projection algorithm is tested and compared with various existing methods, such as, CSS, WAVES, TOPS. Simulations were performed on wideband Chirp, QPSK and MC-CDMA sources.

For the simulations, consider a ULA consisting of \( M = 16 \) elements separated by a distance \( d = \frac{\lambda_h}{2} \), \( \lambda_h = \frac{c}{f_h} \) where, \( c \) is the velocity of light and \( f_h \) is the highest source frequency. Wideband Chirp, QPSK and MC-CDMA sources having a bandwidth of 400MHz with 1GHz center frequency were generated. 4096 samples at each array element were collected for DOA estimation. The sampling frequency of 800MHz was used in each case and all processing was done in the baseband. The data was segmented into 64 non-overlapping blocks, followed by FFT on each array element and correlation matrices were estimated for each frequency bin.

This Chapter is divided into the following sections. In section 4.1, generation of Chirp, QPSK and MC-CDMA signals are discussed. In sections 4.2, 4.3 and 4.4, the DOA estimation results using WSSP for Chirp, QPSK and MC-CDMA sources are generated and compared with the performance of previously existing methods.

4.1 Signal Generation

The conventional way of generating delayed versions of a wideband signal is to convert the signal into frequency domain followed by multiplying by frequency domain phase shift as
given in equations (4.1) and (4.2). In this work, the source signals were generated using the
time-domain equations for the three types of sources discussed below. The array snapshots
received at the sensors with precise delays as defined in (2.2) were generated for individual
sources and summed according to (2.1). The time-domain snapshots are transformed to the
frequency domain using FFTs. Fourier transform pairs for a signal and delayed signal are
given below.

\[ F(s(t)) = S(f) \quad (4.1) \]
\[ F(s(t - t_o)) = S(f)e^{-j2\pi ft_o} \quad (4.2) \]

### 4.1.1 Chirp Signal Generation

Chirp is often used in Sonar, Radars and Wireless Communication. The frequency of Chirp
signal either increases or decreases with time; the frequency of chirp can increase linearly,
quadratically or exponentially. Three different chirps, namely Up-chirp, Down-chirp and
convex chirp, were simulated to generate the results. A general chirp signal equation is
expressed as follows,

\[ s(t) = \cos(2\pi \int f(t)dt) \quad (4.3) \]

where, \( f(t) \) is instantaneous frequency of chirp signal which either increases or decreases
with time. Consider \( f_l \) and \( f_h \) to be the lower and higher frequencies of the bandwidth,
respectively. \( T \) is sweeping time for chirp source to switch it’s frequency from \( f_l \) to \( f_h \).
The equation for generating linear up chirp signal is as given in equations (4.4) and (4.5),
where \( k_u \) is a constant defined in equation (4.6). It should be noted that when \( k_u \) is a positive
number the frequency of the function \( f(t) \) will increase with time. The spectrogram of an
Figure 4.1: Spectrogram of Up Chirp Signal whose frequency rises from 0 Hz to 300 Khz

up-chirp source generated using equation (4.5) is shown in Figure 4.1

\[
\begin{align*}
  f(t) &= f_i + k_u t \\
  s(t) &= \cos(2\pi(f_i t + \frac{1}{2}k_u t^2)) \\
  k_u &= \frac{(f_h - f_i)}{T}
\end{align*}
\]

Similarly, for generating down-chirp signal Equations (4.7), (4.8) and (4.9) were used. It is seen that as \(k_d\) is negative the frequency of \(f(t)\) will decrease as time increases. Figure 4.2 shows the spectrogram of a linear down-chirp generated using the following equations,

\[
\begin{align*}
  f(t) &= f_h + k_d t \\
  s(t) &= \cos(2\pi(f_h t + \frac{1}{2}k_d t^2)) \\
  k_d &= -\frac{(f_h - f_i)}{T}
\end{align*}
\]
Figure 4.2: Spectrogram of Down Chirp Signal whose frequency decreases from 30 Khz to 0 Hz

A convex quadratic chirp source used in simulation was generated using Equations (4.10), (4.11) and (4.12). Here, $k_c$ is a negative number. The function $f(t)$ decreases quadratically with respect to time as given in Equation (4.11). Figure 4.3 shows the spectrogram of a convex chirp signal.

\[
f(t) = f_h + k_c t^2 \quad (4.10)
\]
\[
s(t) = \cos(2\pi(f_h t + \frac{1}{3}k_c t^3)) \quad (4.11)
\]
\[
k_c = -\frac{(f_h - f_l)}{T^2} \quad (4.12)
\]
4.1.2 Quadrature Phase Shift Key (QPSK) Signal Generation

Quadrature Phase Shift Key is a passband modulation technique used in communication. In QPSK, the phase of carrier signal is varied with respect to the message data. QPSK signal is implemented using two carrier signals which are phase shifted by 90 degrees. Figure 4.5 shows the block diagram for generating QPSK signal. The steps for generating QPSK signals are as given below.

1. Initially an array of random binary bits is generated by the random generator block. The generated data is separated into even and odd data bits using a demultiplexer.

2. The Non-Returning to Zero (NRZ) encoder was used to generate 2-PAM signals for even and odd data bits. In order to generate a 2-PAM signal with bandwidth $BW = f_h - f_l$, a pulse with pulse-duration as given by Equation (4.13) is used.

$$T = \frac{2}{BW} \quad (4.13)$$
3. The generated 2-PAM signal for even and odd bits was then modulated using two sinusoidal carriers which are phase shifted by 90 degrees.

4. The two sinusoidal carriers were generated using a carrier generator and a 90 degree phase shifter as shown in block diagram. The operating frequency of the carrier generator is $f_c = \frac{f_l + f_h}{2}$.

5. The two modulated 2-PAM signals were added to generate the modulated QPSK signal.

The QPSK signal for $2N$ bit binary data can be generated using equation given below,

$$s(t) = \frac{1}{\sqrt{2}} \left( \sum_{n=1}^{N} (2b_o(t) - 1)p(t - nT) \cos(2\pi f_c t) + \sum_{n=1}^{N} (2b_e(t) - 1)p(t - nT) \sin(2\pi f_c t) \right) \tag{4.14}$$
where $T$ is pulse duration of $p(t)$. $b_o(t)$ and $b_e(t)$ are odd and even bits at time $t$ having a value of either 0 or 1. The constellation plot for QPSK signal generated using Equation (4.14) is as given in figure 4.4.

4.1.3 Multi-Carrier Code Division Multiple Access (MC-CDMA) Signal Generation

Multi-Carrier Code Division Multiple Access (MC-CDMA) is a scheme used in wireless communication. It is a combination of Code Division Multiple Access (CDMA) which is used in 2G/3G communication and OFDM which is used in 4G communication. MC-
CDMA uses a set of orthogonal frequencies called sub-carrier for carrying information. Multiple access in MC-CDMA is achieved by using similar set of sub carriers for all users, differing only in the way they are spread in frequency domain. Hadamard-Walsh spreading code is used to spread the signal in frequency domain. In MC-CDMA, all users occupy same bandwidth without interfering with each other. The block diagram 4.6 shows MC-CDMA generation for a single user. Consider $\Delta f$ to be the difference between two sub-carriers, $L$ be the length of Hadamard-Walsh code supporting $L$ users on the system, $f_l$ be the lowest frequency of signal and $f_h$ being the highest frequency. The steps for generating MC-CDMA are as follows.

1. Generate an array of random binary data using random data generator block.

2. A Non-Return to Zero (NRZ) Encoder was applied to the generated data to produce a 2-PAM signal. A pulse of pulse-width $T$ was used to generate the 2-PAM signal.
and \( T \) is given by,

\[
T = \frac{1}{\Delta f}
\]  

(4.15)

where, \( \Delta f = \frac{f_h - f_l}{L+1} \)

3. The generated 2-PAM signal is then channeled into \( L \) parallel lines.

4. The 2-PAM signal was spread in frequency domain by using Hadamard-Walsh code \([c_0, c_1 \ldots c_{L-1}]\) and a group of \( L \) sub-carrier oscillators as shown in the block diagram.

5. The output from the oscillators are added to generate the MC-CDMA signal.

The MC-CDMA signal for one user can be given as follows,

\[
s(t) = (2b(t) - 1) \sum_{l=0}^{L} c_l \cos(2\pi f_c t)p(t - nT)
\]  

(4.16)

\[
p(t) = \begin{cases} 
1, & 0 < t < T \\
0, & otherwise 
\end{cases}
\]

where \( T \) is the pulse duration of \( p(t) \) and \( b(t) \) is data bits at time \( t \) having two values, either 1 or 0.

\section*{4.2 Simulation Results for Chirp Sources}

For the two source case, one up chirp and a down chirp arriving from angles 9 and 12 degrees, respectively, were simulated. For the three source case, 3 chirp sources: an up chirp, a down chirp and a convex chirp signal arriving from 9, 12 and 25 degrees, respectively, were simulated. Focusing angle of 10.5 degree was used in forming the focusing matrices.
for implementing CSS and WAVES methods. For TOPS and WSSP, five least noisy frequency bins having the five largest $\sigma_{min}^s - \sigma_{max}^n$ values were selected out of 33 available frequency bins. For implementing TOPS, the least noisy frequency bin having the maximum difference for $\sigma_{min}^s - \sigma_{max}^n$ was chosen as $f_0$ to form the reference signal subspace $E_s(f_0)$.

![Figure 4.7: Chirp Spectrum at Sensor](image)

**Two Source Case - Chirp**

Figure 4.7 shows the spectrum of Chirp signal for 2 source case at one of the sensors. Figures 4.8, 4.9, 4.10 and 4.11 show spatial pseudo-spectrum generated using CSS, WAVES, TOPS and WSSP, respectively, at 10dB SNR. Figure 4.12 is a superimposed graph of all methods. For the two source case it was observed that CSS and WAVES have relatively more bias when compared to TOPS and WSSP. Although TOPS method was able to detect the DOAs correctly, it exhibits spurious peaks that may be stronger than the peaks at the
Figure 4.8: Coherent Signal Subspace (CSS) Psuedo Spectrum for 2 chirp sources arriving from 9 and 12 degrees at SNR=10dB

true DOAs, making it difficult to estimate the angle correctly.
Figure 4.9: Weighted Average of Signal Subspace (WAVES) Pseudo Spectrum for 2 chirp sources arriving from 9 and 12 degrees at SNR=10dB

Figure 4.10: Test of Orthogonality of Projected Subspace (TOPS) Pseudo Spectrum for 2 chirp sources arriving from 9 and 12 degrees at SNR=10dB
Figure 4.11: Wideband Spectral Subspace Projection (WSSP) Pseudo Spectrum for 2 chirp sources arriving from 9 and 12 degrees at SNR=10dB

Figure 4.12: Comparison of methods at SNR=10dB
Three Source Case - Chirp

Figures 4.13, 4.14, 4.15 and 4.16 show the spatial pseudo-spectrum of CSS, WAVES, TOPS and WSSP at 20dB SNR for 3 chirp signals. As seen in Figures 4.13 and 4.14, CSS and WAVES were unable to resolve the angles at 9 and 12 degrees; moreover, the third angle gives biased peak. Although TOPS method estimated all three angles correctly, it has a strong spurious peak which may be mistaken for source direction. Therefore, TOPS is not a reliable estimator in this case. It can be seen that only WSSP was successful in resolving all three angles without any spurious peaks. Similar performances were seen at other SNRs also.

Figure 4.13: Coherent Signal Subspace (CSS) Pseudo Spectrum for 3 chirp sources arriving from 9, 12 and 25 degrees at SNR=20dB
Figure 4.14: Weighted Average of Signal Subspace (WAVES) Pseudo Spectrum for 3 chirp sources arriving from 9, 12 and 25 degrees at SNR=20dB

Figure 4.15: Test of Orthogonality of Projected Subspace (TOPS) Psuedo Spectrum for 3 chirp sources arriving from 9, 12 and 25 degrees at SNR=20dB
Figure 4.16: Wideband Spectral Subspace Projection (WSSP) Pseudo Spectrum for 3 chirp sources arriving from 9, 12 and 25 degrees at SNR=20dB

Figure 4.17: Comparison of methods at SNR=20dB
**Bias and Root Mean Square Error for Chirp Source**

The bias and Root mean square error (RMSE) plots were generated for the 2 chirp sources discussed above, i.e., one Up-Chirp and a Down-Chirp, arriving from direction 9 and 12 degrees, respectively. 500 independent noise realizations were generated to estimate the bias and RMSE plots. Figures 4.18 and 4.19 show the bias plots; Figures 4.20 and 4.21 give the RMSE plots for 9 and 12 degrees, respectively. It can be seen from Figures 4.18, 4.19, 4.20 and 4.21 that TOPS method performs poorly when compared to CSS and WSSP. It should also be noted that the performance of CSS remains consistent throughout the SNR range, although it should be emphasized that a preliminary DOA estimate of 10.5° was in these runs that may not be available in practice. WSSP and TOPS do not require any prior estimates. The performance of WAVES is similar to that of CSS and is not included in the plots. The WSSP gives poor results when compared to CSS at 0dB SNR but as SNR increases the WSSP outperforms CSS.

![Bias of Chirp Angle−1 = 9 degrees](image)

**Figure 4.18:** Bias for DOA=9 degrees over 500 independent runs
Figure 4.19: Bias for DOA=12 degrees over 500 independent runs

Figure 4.20: RMSE for DOA=9 degrees over 500 independent runs

Figure 4.21: RMSE for DOA=12 degrees over 500 independent runs
4.3 Simulation for QPSK Signal

For two source case, two QPSK signals arriving from 9 and 12 degrees are considered. For three source case three signals arriving from 9, 12 and 25 degrees are considered. Focusing angle of 10.5 degree was used in forming the focusing matrices of CSS and WAVES. For TOPS and WSSP, three least noisy frequency bins having the three largest $\sigma_{min}^s - \sigma_{max}^n$ values were selected out of 33 available frequency bins. For implementing TOPS, the least noisy frequency bin having the maximum difference for $\sigma_{min}^s - \sigma_{max}^n$ was chosen as $f_0$ to form the reference signal subspace $E_s(f_0)$.

Figure 4.22 shows the spectrum of QPSK signal for 2 source case at one of the sensors. Figure 4.23, 4.24, 4.25 and 4.26 show the pseudo spectrum for CSS, WAVES, TOPS and WSSP, respectively, for SNR = 10dB. A focusing angle of 10.5 was used in CSS and WAVES methods. For TOPS and WSSP only 3 least noisy frequency bins were used to
generate the results. In case of TOPS, the least noisy frequency bin was selected as the reference bin. Figures show that all methods were able to resolve the sources at 9 and 12 degrees at this SNR level. It is seen in Figure 4.25, that even though TOPS has peaks at correct angles, it exhibits spurious peaks some of which are higher than the peaks at true DOAs, biasing the results.

Figure 4.23: Coherent Signal Subspace (CSS) Psuedo Spectrum for 2 QPSK sources arriving from 9 and 12 degrees at SNR=10dB
Figure 4.24: Weighted Average of Signal Subspace (WAVES) Psuedo Spectrum for 2 QPSK sources arriving from 9 and 12 degrees at SNR=10dB

Figure 4.25: Test of Orthogonality of Projected Subspace (TOPS) Psuedo Spectrum for 2 QPSK sources arriving from 9 and 12 degrees at SNR=10dB
Figure 4.26: Wideband Spectral Subspace Projection (WSSP) Psuedo Spectrum for 2 QPSK sources arriving from 9 and 12 degrees at SNR=10dB

Figure 4.27: Comparison of methods at SNR=10dB
Three Source Case - QPSK

Figures 4.28, 4.29, 4.30 and 4.31 show the pseudo spectrum of CSS, WAVES, TOPS and WSSP, respectively. A focusing angle of 10.5 was used in CSS and WAVES methods. For TOPS and WSSP only 3 least noisy frequency bins were used to generate the results. It is seen in Figure 4.28 and 4.29 that CSS and WAVES were unable to estimate the sources at 9 and 12 degrees. Similar to the two source case TOPS give three peaks at correct DOA locations of 9, 12 and 25 degrees, but exhibits spurious peaks that are stronger than the true DOAs, making TOPS unreliable for practical use. Figure 4.31 shows WSSP was successfully able to resolve all angles correctly.

Figure 4.28: Coherent Signal Subspace (CSS) Pseudo Spectrum for 3 QPSK sources arriving from 9, 12 and 25 degrees at SNR=10dB
Figure 4.29: Weighted Average of Signal Subspace (WAVES) Pseudo Spectrum for 3 QPSK sources arriving from 9, 12 and 25 degrees at SNR=10dB

Figure 4.30: Test of Orthogonality of Projected Subspace (TOPS) Pseudo Spectrum for 3 QPSK sources arriving from 9, 12 and 25 degrees at SNR=10dB
Figure 4.31: Wideband Spectral Subspace Projection (WSSP) Psuedo Spectrum for 3 QPSK sources arriving from 9,12 and 25 degrees at SNR=10dB

Figure 4.32: Comparison of methods at SNR=10dB
Bias and Root Mean Square Error for QPSK

The bias and Root mean square error (RMSE) plots were generated using two QPSK signals arriving from directions 9 and 12 degrees. 500 independent iterations were performed to estimate the DOAs for generating the bias and RMSE plots. Figures 4.33 and 4.34 show the bias plots; Figures 4.35 and 4.36 give the RMSE plots for 9 and 12 degrees respectively. Similar to the Chirp case, TOPS method performs poorly when compared to CSS and WSSP as shown in Figures 4.33, 4.34, 4.35 and 4.36, this is because some peaks in the pseudo spectrum were higher than true DOA as shown in . It is observed from figures that CSS performs better when compared to other methods, but it requires preliminary DOA estimates. It should be noted that the generated bias and RMSE plots were generated for 2 source case. Although the performance of CSS for 2 source case is better when compared to WSSP, the CSS performance may degrade as number of sources increases as observed in 4.32.

Figure 4.33: Bias for DOA=9 degrees over 500 independent runs
Bias of QPSK Angle−2 = 12 degrees

Figure 4.34: Bias for DOA=12 degrees over 500 independent runs

RMSE of QPSK Angle−1 = 9 degrees

Figure 4.35: Bias for DOA=9 degrees over 500 independent runs
4.4 Simulation of MC-CDMA Sources

For two source case, two signals arriving from 9 and 12 degrees were simulated. For three source case, three signals arriving from 9, 12 and 25 degrees are simulated. Focusing angle of 10.5 degree was used in forming the focusing matrices for CSS and WAVES method. For TOPS and WSSP, 5 least noisy frequency bins whose difference between $\sigma_{\text{min}}$ and $\sigma_{\text{max}}$ was maximum were selected out of 33 frequency bins. For implementing TOPS, the least noisy frequency bin having the maximum difference for $\sigma_{\text{min}} - \sigma_{\text{max}}$ was chosen as $f_0$ to form the reference signal subspace $E_s(f_0)$. Figure 4.37 shows the spectrum of MC-CDMA for 2 source case at one of the sensors.

Two Source Case - MC-CDMA

Figures 4.38, 4.39, 4.40 and 4.41 show the pseudo spectrum of CSS, WAVES, TOPS and WSSP, respectively. All the methods were able to resolve the two sources. TOPS method
Figure 4.37: MC-CDMA Spectrum at Sensor

did exhibit spurious peaks even in this case however the spurious peaks are weaker than the true angles of arrival in this case. After careful observation, it can be seen that the CSS and WAVES give biased DOA estimates.

**Three Source Case - MC-CDMA**

Figures 4.43, 4.44,4.45 and 4.46 show the pseudo spectrum of CSS, WAVES, TOPS and WSSP respectively at 20dB SNR. Figures 4.43 and 4.44 show that CSS and WAVES give bias estimates of DOAs. It is seen in Figure 4.44 that TOPS show spurious peaks, the WSSP on the other hand gives a smooth spatial pseudo spectrum.
Figure 4.38: Coherent Signal Subspace (CSS) Psuedo Spectrum for 2 MC-CDMA sources arriving from 9 and 12 degrees at SNR=10dB

Figure 4.39: Weighted Average of Signal Subspace (WAVES) Psuedo Spectrum for 2 MC-CDMA sources arriving from 9 and 12 degrees at SNR=10dB
Figure 4.40: Test of Orthogonality of Projected Subspace (TOPS) Psuedo Spectrum for 2 MC-CDMA sources arriving from 9 and 12 degrees at SNR=10dB

Figure 4.41: Wideband Spectral Subspace Projection (WSSP) Psuedo Spectrum for 2 MC-CDMA sources arriving from 9 and 12 degrees at SNR=10dB
Figure 4.42: Comparison of methods at SNR=10dB

Figure 4.43: Coherent Signal Subspace (CSS) Pseudo Spectrum for 2 chirp sources arriving from 9 and 12 degrees at SNR=10dB
Figure 4.44: Weighted Average of Signal Subspace (WAVES) Psuedo Spectrum for 2 chirp sources arriving from 9 and 12 degrees at SNR=10dB

Figure 4.45: Test of Orthogonality of Projected Subspace (TOPS) Psuedo Spectrum for 2 chirp sources arriving from 9 and 12 degrees at SNR=10dB
Figure 4.46: Wideband Spectral Subspace Projection (WSSP) Psuedo Spectrum for 2 chirp sources arriving from 9 and 12 degrees at SNR=10dB

Figure 4.47: Comparison of methods at SNR=10dB
Bias and Root Mean Square Error for MC-CDMA signal

The bias and Root mean square error (RMSE) plots were generated using 2 MC-CDMA signals arriving from 9 and 12 degrees. 500 independent realizations were performed to generate the bias and RMSE plots. Figures 4.48 and 4.49 give the bias plots; Figures 4.50 and 4.51 show the RMSE plots for 9 and 12 degrees respectively. It can be observed from the bias and RMSE plots that WSSP has less bias when compared to CSS and TOPS method. The performance of CSS and WSSP remains constant for extensive range of SNRs.
Figure 4.49: Bias for DOA=12 degrees over 500 independent runs

Figure 4.50: Bias for DOA=9 degrees over 500 independent runs


4.5 Comparison of TOPS and WSSP for \( d = \lambda_c/2 \)

In the following section performance comparison is made between TOPS and WSSP when the sensor elements are separated by \( d = \lambda_c/2 \), where, \( \lambda_c = c/f_c \) with \( f_c \) being the center frequency of the source spectrum, \( i.e., f_c = (f_l + f_h)/2 \). In the simulations above \( d \) was selected according the maximum frequency \( f_h \). All other parameters are same as before.

Chirp Sources

The figures below show the spatial pseudo-spectrum of TOPS and WSSP for 2, 3 and 4 sources at various SNRs. In each case, 5 least noisy frequency bins were selected to generate the results. TOPS exhibits significant spurious peak for the 2 and 4 source cases. WSSP estimated the DOAs successfully in all cases.
Figure 4.52: Chirp-2 Source TOPS and WSSP at SNR=10dB

Figure 4.53: Chirp-3 Source TOPS and WSSP at SNR=20dB
Figure 4.54: Chirp-4 Source TOPS and WSSP at SNR=30dB
QPSK Sources

The figures below show the spatial pseudo-spectrum for TOPS and WSSP 2, 3 and 4 QPSK sources. Three least noisy frequency bins were selected to generate the results in all cases. TOPS shows spurious peaks in all cases whereas, WSSP was able to generate spatial spectrum without any spurious peaks.

![QPSK-2 Source TOPS and WSSP at SNR=10dB](image)

Figure 4.55: QPSK-2 Source TOPS and WSSP at SNR=10dB
Figure 4.56: QPSK-3 Source TOPS and WSSP at SNR=20dB

Figure 4.57: QPSK-4 Source TOPS and WSSP at SNR=20dB
MC-CDMA Sources

The following figures display spatial pseudo-spectrum for 2, 3 and 4 MC-CDMA sources. It can clearly be seen that although there are no significant spurious peaks in MC-CDMA case, TOPS was unable to resolve angles in 3 source and 4 source cases. WSSP performed well in all cases.

Figure 4.58: MC-CDMA-2 Source TOPS and WSSP at SNR=10dB
Figure 4.59: MC-CDMA-3 Source TOPS and WSSP at SNR=20dB

Figure 4.60: MC-CDMA-4 Source TOPS and WSSP at SNR=30dB
Chapter 5: Concluding Remarks

This thesis presented a novel approach for estimating wideband DOAs by projecting noise subspaces on to hypothesized signal subspaces. The proposed method has several desirable properties. Firstly, unlike most existing wideband DOA processing approaches, the proposed algorithm does not require preliminary DOA estimates, focusing of spectral correlation matrices, translation to a reference frequency, beamforming, phase-shifters or true-time delay lines. Furthermore, the proposed method does not generate spurious peaks, as in case of the TOPS approach. All DOAs are estimated in a single pass without any iterations. It is shown with theory and simulations that by projecting the least-noisy noise subspaces on the signal subspaces at hypothetical DOAs, the proposed WSSP approach is effective over a wide range of SNRs. Error analysis of the proposed WSSP method justifies its effectiveness in producing unbiased estimates.

The thesis gives an overview of existing narrowband and wideband DOA estimation algorithms. Then the proposed algorithm is developed. The theoretical development is supported by extensive simulation studies using QPSK, Chirp and MC-CDMA signals. Results shows that the proposed new approach is more robust and accurate when compared to many existing algorithms.

**Comparison with TOPS:** The proposed WSSP method is in the same class as the existing TOPS method. Both methods perform DOA estimation in the frequency domain by processing a small number of least noisy frequency bins selected according to frequency selec-
tion criteria, and both methods use projected subspaces. However, there are key differences between the two methods. TOPS uses the least noisy signal subspace as the reference subspace, as well as the least noisy noise subspaces to estimate the DOAs, whereas WSSP uses only the least noisy noise subspaces and does not use any reference bin or signal subspace information. TOPS uses a diagonal matrix to translate the reference signal subspace to other least noisy frequency bins to perform test for orthogonality with the noise subspaces at those frequency bins (See equation (2.75)). WSSP does not require any frequency translation, reducing computation cost. TOPS method requires SVD to generate the final DOA estimates. The WSSP on other hand requires matrix multiplication and trace of a small matrix to estimate the DOAs. Furthermore, the TOPS algorithm projects the translated signal subspaces onto the null space of hypothetical search angles (see equations (2.79) and (2.80)). In case of WSSP, the noise subspaces are projected on to signal subspaces at hypothetical search angles (see equations 3.1, 3.2 and 3.3). TOPS uses loss of rank as the metric to form the TOPS pseudo-spectrum and uses the minimum eigenvalue of a matrix to determine it’s loss of rank (see equation (2.78)). WSSP forms the spatial pseudo-spectrum using the lengths of projected subspaces as metric which is calculated using the trace of a matrix (see equation 3.10). Finally, in its error analysis, TOPS uses \((I - P(f_i, \phi))\bar{E}_n = \bar{E}_n\), (see equation (58) in the TOPS paper [11]) which tends to accentuate the noise subspace that may explain the spurious peaks exhibited by the TOPS pseudo-spectrum. WSSP’s error analysis relies on annihilation of noise subspace, i.e., \(P(f_i, \phi))\bar{E}_n = 0\) (see equation (3.2)) that leads to smooth pseudo spectrum and accurate DOA estimates.

**Comparison with other Methods:** Existing methods such as CSS [7] [8] and WAVES [6], require prior information about preliminary DOAs to construct the focusing matrices. In WSSP no such information is required to generate the results. If a single CSS iteration is used, the computational complexity of the proposed method could potentially be more than that of CSS. However, the number of iterations required for CSS would increase as the
number of sources increases and if the sources are well separated. WSSP forms a single spatial spectrum and generates all DOA estimates in a single pass. The estimates generated by WSSP are also more accurate than all the other methods, in general.

**Future Work:** Performance of WSSP needs to be studied in non-Gaussian noise and in fading channel. Extension of WSSP to accommodate correlated, coherent and multi-path sources would make it useful in many fields, including wireless. Performance of WSSP needs to be studied for other wideband wireless technologies such as CDMA and OFDM. Performance of WSSP also need to be studied for antenna polarization. Reducing computational complexity of WSSP needs to be explored. Use of weighted noise subspaces and incorporation of signal subspaces may improve performance of WSSP and needs further study.
Bibliography


