Extensions of Polar Format Scene Size Limits
to Squinted Geometries

A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Science in Engineering

by

Matthew S. Horvath
B.S.E.E, The Ohio State University, 2007

2012
Wright State University
I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Matthew S. Horvath ENTITLED Extensions of Polar Format Scene Size Limits to Squinted Geometries BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science in Engineering.

Brian D. Rigling, Ph.D.
Thesis Director

Kefu Xue, Ph.D.
Department Chair of Electrical Engineering

Committee on
Final Examination

Brian D. Rigling, Ph.D.

Mike Bryant, Ph.D.

Fred Garber, Ph.D.

Andrew T. Hsu, Ph.D.
Dean, Graduate School
ABSTRACT


The Polar Format Algorithm (PFA) is an often used algorithm to image synthetic aperture radar (SAR) phase history data. The algorithm relies on a far-field approximation wherein the curved wavefront of the transmitted pulses is approximated as a planar wavefront, introducing spatially variant phase errors in the phase history. While allowing for faster image formation compared to more exact imaging algorithms such as convolution backprojection, these phase errors lead to the distortion and defocus of point targets, degrading the quality of the resulting imaged scene.

Historically, a Taylor expansion has been used to approximate the phase errors leading to distortion and defocus based on the dominant second-order approximation to a differential range expression. This thesis extends the previous study of these errors for broadside imaging scenarios to squinted imaging scenarios. The complications of the squinted geometry require an additional Taylor expansion process to approximate linear and quadratic terms of the approximated phase error, yielding the distortion and defocus approximations respectively. These approximations are calculated based on both the conventional dominant polynomial error (DPE) described above and the true differential range error (DRE). The accuracy of these approximations is demonstrated, after which a bounding process is used to yield the scene size limits.
List of Figures

2.1 Geometry of the Differential Range Expression ............................................. 7
2.2 Squint Angle Convention ............................................................................. 7

3.1 Squinted Image Coordinate System ............................................................ 12
3.2 Squinted Spatial Frequency Coordinate System .......................................... 13
3.3 DPE vs. DRE Distortion Approximation Accuracy ........................................ 17

4.1 DPE Distortion Coefficients ........................................................................ 20
4.2 Distortion-Free Scene Size Limits from DPE Approximation ....................... 21
4.3 DPE Distortion Coefficients with Second-Order Corrections ....................... 22
4.4 Distortion-Free Scene Size Limits from DPE w/ Corrections Approximation .... 23
4.5 Distortion-Free Scene Size Limits from DRE and DPE ............................... 27
4.6 Distortion-Free Scene Size Comparison ..................................................... 29

6.1 Defocus Approximation Accuracy ............................................................... 38
6.2 Defocus Approximation Accuracy with Second-Order Corrections ............. 39
6.3 Defocus Free Scene Size ............................................................................ 42
6.4 Defocus Free Scene Size with Second-Order Corrections ............................ 44

A.1 Actual, DRE, and DPE Distortion, $\theta_s = -40^\circ$ ................................... 51
A.2 Actual, DRE, and DPE Distortion, $\theta_s = -35^\circ$ ................................... 51
A.3 Actual, DRE, and DPE Distortion, $\theta_s = -30^\circ$ ................................... 52
A.4 Actual, DRE, and DPE Distortion, $\theta_s = -25^\circ$ ................................... 52
A.5 Actual, DRE, and DPE Distortion, $\theta_s = -20^\circ$ ................................... 53
| B.8 | DPE Distortion Approximation Magnitude, $\theta_s = -5^\circ$ | 68 |
| B.9 | DPE Distortion Approximation Magnitude, $\theta_s = 0^\circ$ | 68 |
| B.10 | DPE Distortion Approximation Magnitude, $\theta_s = 5^\circ$ | 69 |
| B.11 | DPE Distortion Approximation Magnitude, $\theta_s = 10^\circ$ | 69 |
| B.12 | DPE Distortion Approximation Magnitude, $\theta_s = 15^\circ$ | 70 |
| B.13 | DPE Distortion Approximation Magnitude, $\theta_s = 20^\circ$ | 70 |
| B.14 | DPE Distortion Approximation Magnitude, $\theta_s = 25^\circ$ | 71 |
| B.15 | DPE Distortion Approximation Magnitude, $\theta_s = 30^\circ$ | 71 |
| B.16 | DPE Distortion Approximation Magnitude, $\theta_s = 35^\circ$ | 72 |
| B.17 | DPE Distortion Approximation Magnitude, $\theta_s = 40^\circ$ | 72 |
| B.18 | DRE Distortion Approximation Magnitude, $\theta_s = -40^\circ$ | 73 |
| B.19 | DRE Distortion Approximation Magnitude, $\theta_s = -35^\circ$ | 74 |
| B.20 | DRE Distortion Approximation Magnitude, $\theta_s = -30^\circ$ | 74 |
| B.21 | DRE Distortion Approximation Magnitude, $\theta_s = -25^\circ$ | 75 |
| B.22 | DRE Distortion Approximation Magnitude, $\theta_s = -20^\circ$ | 75 |
| B.23 | DRE Distortion Approximation Magnitude, $\theta_s = -15^\circ$ | 76 |
| B.24 | DRE Distortion Approximation Magnitude, $\theta_s = -10^\circ$ | 76 |
| B.25 | DRE Distortion Approximation Magnitude, $\theta_s = -5^\circ$ | 77 |
| B.26 | DRE Distortion Approximation Magnitude, $\theta_s = 0^\circ$ | 77 |
| B.27 | DRE Distortion Approximation Magnitude, $\theta_s = 5^\circ$ | 78 |
| B.28 | DRE Distortion Approximation Magnitude, $\theta_s = 10^\circ$ | 78 |
| B.29 | DRE Distortion Approximation Magnitude, $\theta_s = 15^\circ$ | 79 |
| B.30 | DRE Distortion Approximation Magnitude, $\theta_s = 20^\circ$ | 79 |
| B.31 | DRE Distortion Approximation Magnitude, $\theta_s = 25^\circ$ | 80 |
| B.32 | DRE Distortion Approximation Magnitude, $\theta_s = 30^\circ$ | 80 |
| B.33 | DRE Distortion Approximation Magnitude, $\theta_s = 35^\circ$ | 81 |
| B.34 | DRE Distortion Approximation Magnitude, $\theta_s = 40^\circ$ | 81 |
| B.35 | DPE with Corrections Distortion Approximation Magnitude, $\theta_s = -40^\circ$ | 82 |
| B.36 | DPE with Corrections Distortion Approximation Magnitude, $\theta_s = -30^\circ$ | 83 |
B.37 DPE with Corrections Distortion Approximation Magnitude, $\theta_s = -20^\circ$ ........ 83
B.38 DPE with Corrections Distortion Approximation Magnitude, $\theta_s = -10^\circ$ ........ 84
B.39 DPE with Corrections Distortion Approximation Magnitude, $\theta_s = 0^\circ$ ........ 84
B.40 DPE with Corrections Distortion Approximation Magnitude, $\theta_s = 10^\circ$ ........ 85
B.41 DPE with Corrections Distortion Approximation Magnitude, $\theta_s = 20^\circ$ ........ 85
B.42 DPE with Corrections Distortion Approximation Magnitude, $\theta_s = 30^\circ$ ........ 86
B.43 DPE with Corrections Distortion Approximation Magnitude, $\theta_s = 40^\circ$ ........ 86

C.1 DPE Distortion Bound, $\theta_s = -80^\circ$ ........................................ 87
C.2 DPE Distortion Bound, $\theta_s = -70^\circ$ ........................................ 88
C.3 DPE Distortion Bound, $\theta_s = -60^\circ$ ........................................ 88
C.4 DPE Distortion Bound, $\theta_s = -50^\circ$ ........................................ 89
C.5 DPE Distortion Bound, $\theta_s = -40^\circ$ ........................................ 89
C.6 DPE Distortion Bound, $\theta_s = -30^\circ$ ........................................ 90
C.7 DPE Distortion Bound, $\theta_s = -20^\circ$ ........................................ 90
C.8 DPE Distortion Bound, $\theta_s = -10^\circ$ ........................................ 91
C.9 DPE Distortion Bound, $\theta_s = 0^\circ$ ........................................ 91
C.10 DPE Distortion Bound, $\theta_s = 10^\circ$ ........................................ 92
C.11 DPE Distortion Bound, $\theta_s = 20^\circ$ ........................................ 92
C.12 DPE Distortion Bound, $\theta_s = 30^\circ$ ........................................ 93
C.13 DPE Distortion Bound, $\theta_s = 40^\circ$ ........................................ 93
C.14 DPE Distortion Bound, $\theta_s = 50^\circ$ ........................................ 94
C.15 DPE Distortion Bound, $\theta_s = 60^\circ$ ........................................ 94
C.16 DPE Distortion Bound, $\theta_s = 70^\circ$ ........................................ 95
C.17 DPE Distortion Bound, $\theta_s = 80^\circ$ ........................................ 95
C.18 DPE Distortion Bound with Corrections, $\theta_s = -80^\circ$ ................. 96
C.19 DPE Distortion Bound with Corrections, $\theta_s = -70^\circ$ ................. 97
C.20 DPE Distortion Bound with Corrections, $\theta_s = -60^\circ$ ................. 97
C.21 DPE Distortion Bound with Corrections, $\theta_s = -50^\circ$ ................. 98
C.51 DRE Distortion Bound, $\theta_s = 80^\circ$ .................................................. 113

D.1 DPE Maximum QPE Approximation, $\theta_s = -80^\circ$ ........................................ 114
D.2 DPE Maximum QPE Approximation, $\theta_s = -70^\circ$ ........................................ 114
D.3 DPE Maximum QPE Approximation, $\theta_s = -60^\circ$ ........................................ 115
D.4 DPE Maximum QPE Approximation, $\theta_s = -50^\circ$ ........................................ 115
D.5 DPE Maximum QPE Approximation, $\theta_s = -40^\circ$ ........................................ 115
D.6 DPE Maximum QPE Approximation, $\theta_s = -30^\circ$ ........................................ 115
D.7 DPE Maximum QPE Approximation, $\theta_s = -20^\circ$ ........................................ 116
D.8 DPE Maximum QPE Approximation, $\theta_s = -10^\circ$ ........................................ 116
D.9 DPE Maximum QPE Approximation, $\theta_s = 0^\circ$ ............................................. 116
D.10 DPE Maximum QPE Approximation, $\theta_s = 10^\circ$ .......................................... 116
D.11 DPE Maximum QPE Approximation, $\theta_s = 20^\circ$ .......................................... 117
D.12 DPE Maximum QPE Approximation, $\theta_s = 30^\circ$ .......................................... 117
D.13 DPE Maximum QPE Approximation, $\theta_s = 40^\circ$ .......................................... 117
D.14 DPE Maximum QPE Approximation, $\theta_s = 50^\circ$ .......................................... 117
D.15 DPE Maximum QPE Approximation, $\theta_s = 60^\circ$ .......................................... 118
D.16 DPE Maximum QPE Approximation, $\theta_s = 70^\circ$ .......................................... 118
D.17 DPE Maximum QPE Approximation, $\theta_s = 80^\circ$ .......................................... 118

E.1 DPE Defocus Bounds, $\theta_s = -80^\circ$ ................................................................. 119
E.2 DPE Defocus Bounds, $\theta_s = -70^\circ$ ................................................................. 120
E.3 DPE Defocus Bounds, $\theta_s = -60^\circ$ ................................................................. 120
E.4 DPE Defocus Bounds, $\theta_s = -50^\circ$ ................................................................. 121
E.5 DPE Defocus Bounds, $\theta_s = -40^\circ$ ................................................................. 121
E.6 DPE Defocus Bounds, $\theta_s = -30^\circ$ ................................................................. 122
E.7 DPE Defocus Bounds, $\theta_s = -20^\circ$ ................................................................. 122
E.8 DPE Defocus Bounds, $\theta_s = -10^\circ$ ................................................................. 123
E.9 DPE Defocus Bounds, $\theta_s = 0^\circ$ ...................................................................... 123
E.10 DPE Defocus Bounds, $\theta_s = 10^\circ$ ................................................................. 124
E.11 DPE Defocus Bounds, $\theta_s = 20^\circ$ ................................. 124
E.12 DPE Defocus Bounds, $\theta_s = 30^\circ$ ................................. 125
E.13 DPE Defocus Bounds, $\theta_s = 40^\circ$ ................................. 125
E.14 DPE Defocus Bounds, $\theta_s = 50^\circ$ ................................. 126
E.15 DPE Defocus Bounds, $\theta_s = 60^\circ$ ................................. 126
E.16 DPE Defocus Bounds, $\theta_s = 70^\circ$ ................................. 127
E.17 DPE Defocus Bounds, $\theta_s = 80^\circ$ ................................. 127
Acknowledgement

I would first like to thank my advisor, Dr. Brian Rigling, for his insight, patience, and guidance throughout the duration of this thesis. Secondly, I would like to thank the Center for Surveillance Research and all those involved with the program. Their efforts made it possible for me to return to academia and conduct this research. I would also like to thank Dr. Fred Garber for his contribution to my continued academic development and the many students with whom I worked and studied alongside in the Wright State University Sensors Exploitation Lab. Lastly, I would like to thank my parents for their support throughout my life.
Dedicated to My Parents,

Gary and Joan Horvath
Chapter 1

Introduction

1.1 Introduction

Synthetic aperture radar (SAR) imaging is a powerful tool that can be utilized where other conventional surveillance methods fail. It has a variety of applications including reconnaissance and surveillance for defense purposes, natural resource exploration, and environmental monitoring, among others. SAR systems generally create large datasets that need to be processed to form a final image. Processing this data can be computationally intensive, and applications may demand algorithms that can form images quickly. Generally, image resolution is inversely proportional, while scene size is proportional, to processing time. Fast image formation using exact methods usually requires coarse resolution or a small scene, which may not be acceptable. The goal and motivation of this research is to analyze algorithms that permit a large SAR dataset to be efficiently processed into a high-resolution image of a large scene.

The backprojection algorithm (BPA) [1] can serve as a baseline for performance relative to other SAR imaging algorithms. It results in accurately formed images for a vast variety of imaging scenarios. The tradeoff comes in its computational cost which is $O(N^3)$ for an $N \times N$ pixel image. It is noted that many “fast” BPA implementations exist, usually allowing for a specified amount of image quality degradation in exchange for faster image formation, pre-processing a large scene into smaller spatial sub-bands before image formation processing, or exploiting the advances in
The polar format algorithm (PFA) [6] is a long-standing and popular alternative to the BPA. The PFA allows the use of fast Fourier Transforms (FFTs), leading to a computational cost of $O(N^2 \log N)$ for an $N \times N$ pixel image. However, the PFA relies on a far-field approximation wherein the curved wavefront of the transmitted pulses is approximated as a planar wavefront, thereby introducing spatially variant phase errors and hence distortion and defocus in the PFA formed image. It will be shown that these phase errors can be approximated through analysis of a differential range term. The far-field approximation in the PFA kernel can be written using a first-order Taylor series expansion on the differential range. It follows that the higher order expansions of the differential range term are present in the signal, but not accounted for in the image formation algorithm, and this is what leads to distortion and defocus. Due to the factorial decay of the Taylor expansion, the second-order expansion of the differential range is considered the dominant error term. The defocus and distortion errors can be corrected, but this is not a trivial process. [7]. Historically, the corrections have been based on this dominant second-order error, meaning some of the phase error leading to distortion and defocus will remain after second-order corrections.

The goal of this thesis is to study the effects of the PFA linear phase approximation for squinted collection geometries. This entails approximating the phase errors leading to distortion and defocus before and after second-order corrections. Ultimately these approximations will be used to derive bounds on image dimensions such that the PFA algorithm can be used as is, without suffering image degradation caused by the far-field approximation. Additionally, by comparing the bounding results before and after second-order corrections, the scene size gained by applying the corrections can be determined. This process will be repeated for the conventional dominant polynomial error approach and a novel differential range error approach, which does not rely on a dominant error term.

This topic has been studied before, however for broadside imaging scenarios and without considering the second-order corrections. [8] As one moves away from the broadside geometry, an additional approximation is required to adequately approximate the phase errors leading to distortion and defocus due to the variation of a differential range term across the aperture. The contribution made in this thesis is taking the approach in [8] and extending it to squinted collection geometries.
both before and after second-order corrections. This approach is preferred over the one in [7], as it isolates the ideal PFA kernel terms from the error terms, which is important when considering the bounding problem.

1.2 Outline

This thesis is outlined as follows. The remainder of Chapter 1 presents a basic introduction into the types and effects of typical SAR phase errors, as well as the role of Taylor series expansions in helping to analyze the specific cases which apply to this thesis. Next, Chapter 2 compares the BPA and PFA imaging kernels, deriving the phase error introduced by the far-field approximation. This chapter also shows how the Taylor series expansion of the differential range can be used to derive an approximation of the phase error at a certain \([x, y]\) pixel in the image. Then, Chapters 3 and 4 discuss how this phase error leads to distortion in the final image, how the functions characterizing distortion can be approximated, and finally how these functions can be bounded to limit distortion in the resulting image, before and after correcting for the second-order phase errors. This discussion is repeated in Chapters 5 and 6 for defocus. Lastly, Chapter 7 presents a conclusion summarizing the work.

1.3 Phase Errors in SAR Imaging

In PFA imaging, the location of a point scatterer is encoded in the phase of the signal. In practice, many sources of error can disrupt this relationship, and it presents a fruitful area of study of practical interest to the SAR community. Majewski, Goodman, and Carrara present a comprehensive discussion of these errors. [9] The errors can be categorized as deterministic or random, variant or invariant, and linear, quadratic, higher-order polynomial, sinusoidal, or wideband. Each type of error will lead to a different effect in the image, all undesired. [10] Methods of mitigating or compensating for these errors also is, and has been, a fruitful area of study in SAR research.

It will be shown, that the far-field assumption leading to what is often referred to as the er-
ror due to wavefront or range curvature, is deterministic, spatially-varying, and low frequency. Spatially-varying refers to the variation in the phase error as a function of scatterer location in the scene. It is the spatially-varying quality that makes expressing these errors in closed form difficult and compensation a non-trivial process.

1.4 Role of Taylor Series Expansion

The phase error due to wavefront curvature is an ideal candidate for analysis using the Taylor series expansion. It is low frequency, varying less than than a wavelength over the course of the synthetic aperture, and therefore can be approximated well using a low-order polynomial. This has been known for some time and has been previously investigated by several authors, [8] [9] [7] [6], among others. However, none of these author’s formally proved the applicability of the Taylor Series expansion to SAR imaging which was more recently demonstrated in [11].
Chapter 2

SAR Imaging Kernels

2.1 Signal Model

For the sake of compactness, the following derivations consider a planar imaging geometry and restrict platform trajectories to lay in the same plane. This will also exclude any image effects due to lay over or height-of-focus errors in the simulation examples, and allow the error due to range curvature to be studied independently.

Considering the spotlight SAR geometry shown in Figure 2.1, the scene to be imaged is centered at the origin of a Cartesian coordinate system. A monostatic SAR system is considered, and the combined transmit/receive antenna platform is moving in the +y-direction. Its position can be expressed as the 2-D vector \( \vec{r}_a(\tau) = [x_a(\tau), y_a(\tau)] \), where \( \tau \) represents slow time.

A scatterer in the scene is placed at an arbitrary position \( \vec{r}_0 = [x, y] \). As the sensor travels along its synthetic aperture length, it periodically transmits pulses, which propagate to the scene, are reflected off any scatterers in the scene, and propagate back to the sensor. The propagation times of each pulse are assumed to be negligible, therefore the sensor platform can be interpreted as being stationary within a pulse. Assuming band-limited frequency domain samples, the output of the receiver due to a single point scatterer can then be modeled as a pulse delayed by the two-way propagation time, \( 2d_{a0}(\tau)/c \), where \( d_{a0} = \| \vec{r}_a(\tau) - \vec{r}_0 \| \) is the distance between the antenna position and scatterer position. It is noted that the receiver is designed such that a scatterer at the
scene center returns a pulse with zero phase. This location is commonly referred to as the central reference point (CRP). Therefore, the received signal can be represented as

\[ S(f, \tau) = \exp\{-j4\pi f (d_{a0}(\tau) - d_a(\tau))/c\} \]  

(2.1)

and

\[ d_{a0} = \|\vec{r}_a - \vec{r}_0\| = \sqrt{(x_a - x)^2 + (y_a - y)^2} \]  

(2.2)

where \(d_a = \|\vec{r}_a(\tau)\| = \sqrt{x_a^2 + y_a^2}\) represents the distance from the CRP to the antenna. The \(d_{a0}(\tau) - d_a(\tau)\) term in (2.1) is commonly referred to as the differential range

\[ \Delta R = \sqrt{(x_a - x)^2 + (y_a - y)^2} - \sqrt{x_a^2 + y_a^2} \]  

(2.3)

where the dependence on \(\tau\) has been suppressed.

Figure 2.2 illustrates the squint angle convention used in this thesis, which is described as the angle between the vector from the synthetic aperture center to the CRP and the vector perpendicular to the aperture at the aperture center. A broadside geometry is a specific case of the general squinted geometry with \(\theta_s = 0\). The squint angle is considered positive when the scene to be imaged lies ahead of the aperture (i.e., the antenna is looking forward to the scene) and negative when the antenna is looking backward to the scene. The derivation of the scene size bounds will be performed for an arbitrary squint angle, \(\theta_s\).
Figure 2.1: Illustration of the geometry used in the derivation of the differential range expression

Figure 2.2: Illustration of broadside, positive, and negative squint angle geometries
2.2 The Polar Format and Backprojection Algorithms

As an exact solution, the BPA provides a baseline for the comparison of SAR imaging algorithms. This is due to its implementation of the ideal matched filter for a point scatter at an arbitrary location, \( \mathbf{r}_0 = [x, y] \), within the scene

\[
I_{MF}(\mathbf{r}_0) = \frac{1}{N_p K} \sum_{n=1}^{N_p} \sum_{k=1}^{K} S(f_k, \tau_n) \exp \left\{ \frac{j 4\pi f_k \Delta R(\tau_n)}{c} \right\}, \tag{2.4}
\]

where \( N_p \) is the number of pulses sampled in slow time, and \( K \) is the number of frequency samples per pulse. The variables \( f_k \) and \( \tau_n \) represent samples in frequency and slow time, respectively. Instead of applying (2.4) to every pixel in the scene and thus requiring \( O(N^4) \) operations, conventional BPA implementations \cite{12} apply a range matched filter first via FFT and then backproject the range profile via 1-D interpolation to achieve \( O(N^3) \).

The PFA imaging kernel can be derived using a Taylor approximation of the differential range. \cite{8} Expanding \( \Delta R(\tau) \), with respect to an arbitrary scatterer location, \( \mathbf{r}_0 = [x, y] \), about the CRP, \( [x, y] = [0, 0] \), will yield the linear approximation using first-order terms:

\[
\Delta R^{(1)} = x \left( \frac{\partial d_{a0}}{\partial x} \bigg|_{x=0} \right) + y \left( \frac{\partial d_{a0}}{\partial y} \bigg|_{y=0} \right) = -x_a x - y_a y
\]

\[
\frac{x_a}{r_a^2} x - \frac{y_a}{r_a^2} y \tag{2.5}
\]

The superscript notation with the order in parantheses is used to denote the finite-order Taylor polynomial consisting of the n-th order terms of the Taylor approximation of the differential range expression throughout the thesis. For example, \( \Delta R^{(n)} \) represents the n-th order Taylor polynomial consisting of the n-th order terms of the Taylor approximation of the differential range expression.

The antenna position, \( \mathbf{r}_a = [x_a, y_a] \) can be written as \( \mathbf{r}_a = [r_a \cos(\theta_a), r_a \sin(\theta_a)] \), where the variable \( \theta_a \) refers to the instantaneous aperture angle and \( r_a \) is the range to the CRP. This results in

\[
\Delta R^{(1)} = -x \cos(\theta_a) - y \sin(\theta_a) \tag{2.6}
\]

The data can then be represented in \( k \)-space by assigning \( k_x = \frac{4\pi f}{c} \cos(\theta_a) \) and \( k_y = \frac{4\pi f}{c} \sin(\theta_a) \), leading to the form of the PFA image formation kernel. The data is first resampled from a polar
annulus to a uniform Cartesian grid allowing efficient image formation by a FFT:

\[ I_{PFA}(\vec{r}_0) = \frac{1}{N_xN_y} \sum_{k_x,k_y} S(k_x,k_y) \exp \{-j (xk_x + yk_y)\} \, . \]  

(2.7)

### 2.3 Error Due to Wavefront Curvature

It follows that the higher order Taylor expansion terms of the differential range expression contain phase terms present in the received signal but not accounted for in the image formation kernel. It is precisely these terms that cause errors when the PFA is used to form an image from SAR phase history data. The second-order terms of the Taylor approximation of the differential range expression can be calculated as

\[
\Delta R^{(2)} = \frac{\partial^2 \Delta R}{\partial x^2} \bigg|_{x=0} \frac{x^2}{2} + \frac{\partial^2 \Delta R}{\partial x \partial y} \bigg|_{x,y=0} xy + \frac{\partial^2 \Delta R}{\partial y^2} \bigg|_{y=0} \frac{y^2}{2}
\]

(2.8)

and the third-order terms of the Taylor approximation of the differential range expression can be calculated as

\[
\Delta R^{(3)} = \frac{\partial^2 \Delta R}{\partial x^3} \bigg|_{x=0} \frac{x^3}{6} + \frac{\partial^2 \Delta R}{\partial x^2 \partial y} \bigg|_{x,y=0} \frac{x^2y}{2} + \frac{\partial^2 \Delta R}{\partial y^2 \partial x} \bigg|_{y,x=0} \frac{y^2x}{2} + \frac{\partial^2 \Delta R}{\partial y^3} \bigg|_{y=0} \frac{y^3}{6}
\]

(2.10)

\[
= \left( \frac{x_a}{2r_a^3} - \frac{x_a^3}{2r_a^5} \right) x^3 + \left( \frac{y_a}{2r_a^3} - \frac{3x_a y_a}{2r_a^5} \right) x^2y
\]

(2.11)

\[
+ \left( \frac{x_a}{2r_a^3} - \frac{3x_a y_a^2}{2r_a^3} \right) y^2x + \left( \frac{3y_a}{6r_a^3} - \frac{3y_a^3}{6r_a^5} \right) y^3
\]

(2.12)

It has been shown that, given an approximation of the distortion and defocus seen at some \([x, y]\) coordinate, it can be removed in post-processing. [7] Due to the factorial decay of the Taylor expansion, the second-order terms in (2.9) have historically been considered the dominant error [6,8,9]. Assuming a correction based on (2.9) perfectly compensates for its associated phase error, (2.12) would be the residual approximation error. By deriving and bounding distortion and defocus ap-
proximations based on these dominant polynomial phase errors, the error-free scene size before and after second-order corrections can be determined. This approach is termed the dominant polynomial error (DPE) approach.

An alternative approach proposed here is to consider the exact differential range error. The total differential range error due to the first-order approximation in (2.5) can be written as

\[
\Delta \tilde{R}(1) = -\frac{x_a}{r_a} x - \frac{y_a}{r_a} y - \sqrt{(x - x_a)^2 + (y - y_a)^2} + \sqrt{x_a^2 + y_a^2}.
\]  

(2.13)

Assuming the second-order terms in (2.9) have been perfectly compensated, adding (2.9) in (2.13) will yield the residual error after second-order corrections:

\[
\Delta \tilde{R}(2) = -\frac{x_a}{r_a} x - \frac{y_a}{r_a} y + \frac{y_a^2}{2r_a^3} x^2 - \frac{x_a y_a}{r_a^3} x y + \frac{x_a^2}{2r_a^3} y^2 - \sqrt{(x - x_a)^2 + (y - y_a)^2} + \sqrt{x_a^2 + y_a^2}.
\]  

(2.14)

(2.15)

This approach is termed the differential range error (DRE) approach. By considering the differential range error directly, and not an approximation, it is by definition a more accurate characterization of the error, and should lead to more accurate expressions for distortion and defocus.

It requires noting that liberties were taken in reusing the superscript notation with the order in parantheses that was described in the previous section. \(\Delta \tilde{R}^{(n)}\) refers to the exact differential range error from the sum of terms through the \(n\)-th order of the Taylor approximation of the differential range expression. For example, \(\Delta \tilde{R}^{(2)}\) is the sum of the first and second order-terms of the differential range approximation minus the actual differential range.
Chapter 3

Approximating Distortion in Squinted PFA Imaging

3.1 Nature of Distortion

Distortion is most easily explained using the shift property of the Fourier transform, which states that linear phase terms in the frequency domain transform to constant offsets in the spatial domain.

\[ S(k_x, k_y)e^{-j[(x+\delta x)k_x+(y+\delta y)k_y]} \xrightarrow{F} s(x-\delta x, y-\delta y) \]  

(3.1)

Examining the PFA kernel in (2.7), this is precisely how the \([x, y]\) coordinate of a point target is encoded in the phase history. However, the higher order error terms in (2.9) and (2.13) will introduce an additional linear component, causing an unwanted translation and shifting targets from their true location in the imaged scene. Assuming these second-order phase errors have been perfectly compensated in post-processing, their distortion contribution will be negated, leaving the distortion due to (2.12) and (2.14). Therefore, by finding the linear components in the frequency space variables, \(k_x\) and \(k_y\), of these equations, the distortion before and after applying the second-order corrections for both the DPE and DRE approaches can be approximated.
3.2 Taylor Series Expansion with a Squinted Collection Geometry

The squinted collection geometry complicates the imaging process. Due to the tomographic paradigm and angular invariance between the spatial and spatial frequency domains [6], the resulting distortion and defocus appear in the squinted coordinate system illustrated in Figure 3.2 for a forward squint scenario, and the basis vectors describing the spatial \([x, y]\) and spatial frequency domains \([k_x, k_y]\) are equivalent. In SAR imaging, range resolution is gained through sampling in frequency, and cross-range resolution is gained by the movement of the aperture across its flight path. The antenna pointing vector from aperture center gives the \(k_x\) basis in the frequency domain. The frequency support in \(k\)-space is then defined by the bandwidth of the chirp about its center frequency in the \(k_x\) dimension. Taking the orthogonal vector gives the \(k_y\) basis. Motion along the aperture then gives the support in the \(k_y\) dimension. This coordinate system is illustrated Figure 3.2.
In broadside imaging, \( r_a(\tau) = \sqrt{x_a(\tau)^2 + y_a(\tau)^2} \) is nominally constant, and the variation across the aperture is due solely to the change in \( y_a \). However, for squinted collects, the variation in \( r_a(\tau) \) is not negligible, and this complication requires another first-order Taylor series expansion of the phase error about aperture center.

Assigning \( \Delta \hat{R} \) to be a dummy variable representing the result in (2.9), (2.12), (2.13), or (2.14), for the DPE, DPE with second-order corrections, DRE, or DRE with second-order corrections approximation respectively and performing another first-order Taylor series expansion on \( \Delta \hat{R} \) with respect to \( y_a/r_a \) yields
\[
\Delta \hat{R} \approx \Delta \hat{R}|_{\theta_c} + \frac{\partial \Delta \hat{R}}{\partial y_a|_{\theta_c}} \left( \frac{y_a}{r_a} - \frac{y_c}{r_c} \right) \quad (3.2)
\]
where \( (\cdot)|_{\theta_c} \) indicates evaluation at aperture center.

The squinted range distortion approximation \( (\delta x_s) \) is simply the error expression evaluated at aperture center. The squinted cross-range distortion approximation \( (\delta y_s) \) is the derivative of the error expression with respect to \( y_a/r_a \) evaluated at aperture center.

The squinted coordinate system can then be rotated back to the strictly broadside range/cross-range coordinate system using a 2-D rotation matrix whose argument is the negative of the squint
Defining the rotation matrix

\[
T(-\theta_s) = \begin{bmatrix}
\cos(-\theta_s) & -\sin(-\theta_s) \\
\sin(-\theta_s) & \cos(-\theta_s)
\end{bmatrix} = \begin{bmatrix}
\cos(\theta_s) & \sin(\theta_s) \\
-\sin(\theta_s) & \cos(\theta_s)
\end{bmatrix}
\]

then allows the distortion to be rotated from the squinted coordinate system into the global coordinate system

\[
\begin{bmatrix}
\delta x \\
\delta y
\end{bmatrix} = T(-\theta_s) \begin{bmatrix}
\delta x_s \\
\delta y_s
\end{bmatrix}
\]

(3.4)

This allows for the distortion approximations across all squint angles to be compared on a static grid shown in Figure 2.1, rather than one varying as a function of squint angle.

### 3.3 DPE Approach

Applying the approach in Section 3.2 to the second-order DPE, the squinted range distortion approximation results in the terms in (2.9) evaluated at aperture center

\[
\delta x_s^{DPE} = \frac{y_c^2}{2r_c^2} x^2 - \frac{x_c y_c}{r_c^2} x y + \frac{x_c^2 y_c^2}{2r_c^2 y^2},
\]

(3.5)

and the squinted cross-range distortion approximation can be obtained by differentiating (2.9) with respect to \(y_a/r_a\) and evaluating at aperture center to yield

\[
\delta y_s^{DPE} = \left(\frac{y_c}{r_c} - \frac{y_c^3}{2x_c r_c^2}\right) x^2 + \left(\frac{2y_c^2}{x_c r_c^2} - \frac{x_c}{r_c}\right) x y + \left(\frac{-3y_c}{2r_c^2}\right) y^2.
\]

(3.6)

### 3.4 DRE Approach

The squinted range distortion approximation results in the terms in (2.13) evaluated at aperture center

\[
\delta x_s^{DRE} = -x \cos \theta_s - y \sin \theta_s - \sqrt{(x - x_c)^2 + (y - y_c)^2} + \sqrt{x_c^2 + y_c^2},
\]

(3.7)
and the squinted cross-range distortion approximation can be obtained by differentiating (2.13) with respect to \( y_a/r_a \) and evaluating at aperture center to yield

\[
\delta y_{s}^{DRE} = x \tan \theta_s - y + \frac{(y - y_c)}{\sqrt{(x - x_c)^2 + (y - y_c)^2}} \frac{r_c}{x_c^2} \frac{y_c}{\cos^2 \theta_s}. \tag{3.8}
\]

### 3.5 DPE Approach: with Second-Order Corrections

Assuming the second-order phase error in (2.9) has been corrected in post-processing, the dominant polynomial error becomes (2.12). The corresponding squinted range distortion approximation results in the terms in (2.12) evaluated at aperture center,

\[
\delta x_{s}^{DPE} = \left( \frac{x_c}{2r_c^3} - \frac{x_c^3}{2r_c^5} \right) x^3 + \left( \frac{y_c}{2r_c^3} - \frac{3x_c^2y_c}{2r_c^5} \right) x^2 y
\]

\[
+ \left( \frac{x_c}{2r_c^3} - \frac{3x_c^3y_c^2}{2r_c^5} \right) y^2 x + \left( \frac{3y_c}{6x_c^3} - \frac{3y_c^3}{6r_c^5} \right) y^3, \tag{3.9}
\]

and the squinted cross-range distortion approximation can be obtained by differentiating (2.12) with respect to \( y_a/r_a \) and evaluating at aperture center, expressed here in polar coordinates as

\[
\delta y_{s}^{DPE} = \left( \frac{3c_{\theta_s}s_{\theta_s}}{2x_c^2} - \frac{5c_{\theta_s}^3s_{\theta_s}}{2x_c^2} \right) x^3 + \left( \frac{c_{\theta_s}^2 - 3c_{\theta_s}^4}{2x_c^2} + \frac{6c_{\theta_s}^2s_{\theta_s}^2 - s_{\theta_s}^2}{x_c^2} \right) x^2 y
\]

\[
+ \left( \frac{3c_{\theta_s}^3s_{\theta_s}}{x_c^2} + \frac{3c_{\theta_s}s_{\theta_s} - 9c_{\theta_s}s_{\theta_s}^3}{2x_c^2} \right) y^2 x + \left( \frac{c_{\theta_s}^2 - 9c_{\theta_s}^2s_{\theta_s}^2}{2x_c^2} + \frac{3s_{\theta_s}^4 - s_{\theta_s}^2}{x_c^2} \right) y^3. \tag{3.10}
\]

The shorthand notation \( c_{\theta_s} \) represents \( \cos \theta_s \), and \( s_{\theta_s} \) represents \( \sin \theta_s \).

### 3.6 DRE Approach: with Second-Order Corrections

Assuming the second-order effects in (2.9) are corrected in post processing, the dominant error term becomes (2.14). The corresponding squinted range distortion approximation results in the terms in
and the squinted cross-range distortion approximation can be obtained by differentiating (2.14) with respect to $y_a/r_a$ and evaluating at aperture center

$$
\delta y_s^{DRE^+} = x \tan \theta_s - y + \frac{(y - y_c)}{\sqrt{(x - x_c)^2 + (y - y_c)^2}} \frac{r_c}{x_c} + \frac{y_c}{\cos^2 \theta_s} + \left( \frac{y_c}{r_c^2} - \frac{y_c^3}{2x_c^2r_c^2} \right) x^2 + \left( \frac{2y_c^2}{x_cr_c^2} - \frac{x_c}{r_c^2} \right) xy + \left( \frac{-3y_c}{2r_c^2} \right) y^2.
$$

### 3.7 Comparison of Approaches

As hypothesized, the DRE approach leads to more accurate distortion approximations than the DPE approach under certain conditions, as demonstrated in Figure 3.3 for a $25^\circ$ degree squint angle with a stand-off range of 790 m. The plot shows the approximated distorted coordinates of point targets compared with the true distorted location resulting from numerical simulation. The results in (3.5) and (3.6) are plotted as squares, and the results in (3.7) and (3.8) are plotted as diamonds. However, as a target gets farther from the [0, 0] CRP coordinate, these approximations worsen.

The approximations also lose accuracy with greater squint angle. This is a logical result, as for greater squint angles, there will be more variation in the $y_a/r_a$ term due to the change in $r_a$ across the aperture. A linear approximation to a single point at the center of the aperture will not be able to capture this more complicated variation.

Appendix A presents more examples of these results, showing the DPE and DRE results plotted on top of the true distortion of simulated data both before and after applying the second-order corrections. Nonetheless, the approximations are still accurate around the scene center for all squint angles.
Figure 3.3: The DRE result does yield a better approximation as expected, however the DPE result is not far off.
Chapter 4

Bounding Distortion in Squinted PFA Imaging

Distortion-based bounds describe under what conditions the far field approximation holds, thereby allowing the PFA to be used for image formation without suffering unacceptable distortion in the image. Distortion is by far the most visible imaging error introduced by the PFA, and the distortion-free scene size of an uncorrected PFA image will generally be prohibitively small. Therefore, the second-order corrections will almost always be applied if a distortion-free image is required, making the scene size bounds after correction of more interest than the uncorrected bound. Also, by comparing the bounds resulting from before and after second-order correction approximations, the scene size gained by applying the corrections can be determined.

It is important to note, that it is assumed that the distortion correction perfectly compensates for its associated phase error. The distortion functions are approximations and may not perfectly model the actual distortion, and hence there will be some residual error left after applying the correction. This error is compounded by the approximation error of the next higher order term. For example, correcting for the distortion based on the second-order differential range expression in (2.9) uses the approximations given in (3.5) and (3.6), which do not predict the distortion exactly, and therefore, the correction based on this will leave some error. Plus, there will be additional error from the approximation based off the third-order approximation in (2.12).
Additionally, a trade-off has to be made on the direction in which to calculate the bound. The spatially-invariant distortion component across all squint angles is in the $y = -x \tan \theta_s$ direction. However, targets at these coordinates will experience no distortion as the assumed phase will match the actual received phase. These points represent the peak of the curved wavefront, therefore no wavefront curvature and consequently no phase error will be present. Normally, one would choose to bound in the direction leading to the greatest distortion, but the direction of the greatest distortion given by the approximations may not be where the distortion approximations most accurately represent the true distortion.

It can be seen in Appendices A and B that the approximations are accurate for targets close to the CRP where there will also be enough distortion to bound. Appendix A shows the approximated distortion plotted on top of simulated data showing the true distortion. Appendix B shows the magnitude of the distortion as a function of image coordinate, both before and after second-order corrections. Based on these results, the choice was made to bound the scene radius based on the distortion along the $y$ axis in the image. This choice presents a good trade-off between the magnitude of the distortion and the accuracy of the approximation as verified empirically in numerical simulation.

The bounding results will then give a heuristic or “rule of thumb” on a distortion-limited scene size when imaging with the PFA, before and after applying the second-order corrections.

### 4.1 Bounding DPE Distortion

The first step in calculating the bound is rotating the results in (3.5) and (3.6) into the strictly range/cross-range coordinate system as would be seen in a broadside imaging scenario. This is not necessary, but helps to create some consistency in analyzing the distortion across all squint angles.

Setting $x = 0$ in (3.5) and (3.6) and applying the rotation in (3.3) yields

$$\delta x = \left( \frac{c_\theta_x x_c^2}{2r_c^3} - \frac{3s_\theta_x y_c}{2r_c^2} \right) y^2 \quad \text{and} \quad \delta y = \left( \frac{-s_\theta_x x_c^2}{2r_c^3} - \frac{3c_\theta_x y_c}{2r_c^2} \right) y^2$$

(4.1)
Figure 4.1: Magnitude of quadratic distortion coefficients evaluated for $\theta_s \in (-90, 90)$ degrees

which, after realizing $y_c = -r_c \sin \theta_s$ and $x_c = r_c \cos \theta_s$ becomes

$$
\delta x = \left( \frac{c_{\theta_s}^3}{2r_c} - \frac{3s_{\theta_s}^2}{2r_c} \right) y^2 \quad \text{and} \quad \delta y = \left( \frac{-s_{\theta_s} c_{\theta_s}}{2r_c} + \frac{3c_{\theta_s} s_{\theta_s}}{2r_c} \right) y^2
$$

(4.2)

The magnitude of these coefficients is shown in Figure 4.1 for a 1-kilometer stand-off range measured from aperture center. It is evident that the $x$-distortion term is greater than the $y$-distortion term for all squint angles, and a target will be pushed out of a resolution cell in the $x$ direction before the $y$ direction. Therefore, the bound will be calculated for the $x$-distortion term in (4.2).

The calculation of the bound is as simple as solving a quadratic inequality. From Figure 4.1, the $x$-distortion term is a positive quadratic around zero. Extrapolating and replacing the $y^2$ in the $x$-distortion term of (4.2) with $r$ representing the scene radius, and limiting the distortion to an arbitrary number of square resolution cells ($N$) with width $\delta x$, results in the following inequality

$$
\left( \frac{c_{\theta_s}^3}{2r_c} - \frac{3s_{\theta_s}^2}{2r_c} \right) r^2 - N \delta x \leq 0.
$$

(4.3)

This can be solved with the quadratic equation to yield the scene size bound for a distortion-limited
The above process could be repeated for the $y$-distortion component in (4.2) if separate bounds are required for each distortion component. A plot of the result across all squint angles for a 1 km stand-off range scenario limiting distortion to a single 1-m resolution cell is shown in Figure 4.2.

### 4.2 Bounding DPE Distortion with Second-Order Corrections

Under the previously noted assumptions, the bound on the third-order DPE terms will give the distortion limited scene size after applying the second-order corrections. Comparing this bound with the previous is answering the question, “How much distortion free scene size does applying the correction generate?”

The process is similar to the second-order case. First, the approximations in (3.9) and (3.10) are rotated from the squinted coordinate system into the strictly range/cross-range coordinate system.
The magnitude of these coefficients is shown in Figure 4.3 for a 1-kilometer stand-off range measured from aperture center. Unlike the second-order case where the distortion was greater in one dimension than the other for all squint angles, the dimension of the image seeing the greatest distortion component varies as a function of squint angle. Rather than solving complicated trigonometric equations as a function of two variables, squint angle and stand-off range at aperture center, and using the greatest distortion term for each region, a bound will be calculated for each distortion component.

Restricting the distortion to lie within a number of resolution cells \((N)\) yields the following inequalities.

\[
|\delta x^{3rd}\Delta c| r^3 - N\delta_a \leq 0 \quad (4.7)
\]
\[
|\delta y^{3rd}\Delta c| r^3 - N\delta_a \leq 0 \quad (4.8)
\]
Figure 4.4: Maximum distortion-free scene size given by DPE with second-order corrections distortion approximations for $r_c = 1$ km and $\delta_a = 1$ m

As before, the distortion is being extrapolated to every pixel in the resulting scene while the bound actually only holds for the $x = 0$ direction, therefore the $y$ was replaced with an $r$ representing a scene radius.

It can be shown that substituting the inequality for an equals, will create cubics with one real root and two imaginary roots. The real root is the only one of significance to the bounding problem and leads to the following bound, where the right hand side is the root and is guaranteed to be real. The solution comes from the general form of the cubic equation yielding

$$|r| \leq \sqrt[3]{N\delta_a C}$$

(4.9)

where the C determines whether the bound is on the $x$ or $y$-distortion component

$$C = \begin{cases} \frac{c_{g_x} s_{g_x}^3 - c_{g_y} s_{g_y} s_{g_x}}{2r_c^2} + \frac{s_{g_x} - 9s_{g_y} s_{g_x}}{2r_c^2 c_{g_x}} + \frac{3s_{g_y}^5 - s_{g_x}^2}{r_c^2 c_{g_x}^2}, & \text{for } x\text{-distortion component} \\ \frac{s_{g_y}^2 - s_{g_x}^4}{2r_c^2} + \frac{c_{g_x} - 9s_{g_y} s_{g_x}}{2r_c^2 c_{g_x}} + \frac{3s_{g_y}^4 - s_{g_x}^2}{r_c^2 c_{g_x}^2}, & \text{for } y\text{-distortion component} \end{cases}$$

(4.10)

A plot of this result is given in Figure 4.4 for a 1-km stand-off range scenario limiting distortion to a single 1-m resolution cell.
4.3 Bounding DRE Distortion

The results in (3.7) and (3.8) are not as easy to bound as the strictly polynomial results in the previous section due to the existence of the original \( \Delta R \) term evaluated at the aperture center in the expressions. This represents the range to the point target at aperture center which will be denoted as

\[
r_t = \sqrt{(x - x_c)^2 + (y - y_c)^2}
\]

throughout the rest of the thesis. In order to calculate a closed form bound mathematically as opposed to numerically, another approximation will be used to reduce this term to a pure polynomial. This will allow the use of the general solutions of polynomials to calculate the bounds.

To keep the results at a similar level of accuracy compared to the others, a second-order approximation on \( r_t \) will be used. If a first-order approximation were used it would cancel out the first terms in the approximation. Unfortunately, this is the highest order Taylor approximation that can be used as it leads to a quartic inequality, which is the highest order polynomial with a general solution. Higher order approximations of \( r_t \) will result in polynomials of order greater than four, where no general solution exists.

Because the bound will be calculated in the \( x = 0 \) direction, the substitution \( x = 0 \) will be made in the \( r_t \) expression in order to reduce the complexity of the Taylor approximation calculation. This yields the following

\[
r_t|_{x=0} = \sqrt{x_c^2 + (y - y_c)^2} \quad (4.11)
\]

\[
r_t|^{(2)}_{x=0} = r_c - \frac{y_c}{r_c} y + \frac{x_c^2}{2r_c^3} y^2 \quad (4.12)
\]

Introducing (4.12) in (3.7) and (3.8), rotating to the broadside coordinate system, and setting \( x = 0 \) yields the following

\[
\delta x = \cos \theta_s \sin \theta_s y \quad (4.13)
\]

\[
\delta y = -\frac{2\theta_s}{r_c} y + \frac{\sin \theta_s x_c}{2r_c^3} y + \frac{\cos \theta_s y_c}{r_c} - \frac{\cos \theta_s y_c}{r_c} y + \frac{x_c}{2r_c^3} y^2 \quad (4.14)
\]

It is noted that the \( x \)-distortion component in (4.13) will be greater than the \( y \)-distortion com-
ponent in (4.14) for all squint angles and therefore, that is the expression that will be bounded.

Writing (4.13) as a function of $r_c$ and $\theta_s$, collecting the powers of $y$, restricting the bound to be within an arbitrary number of resolution cells, and multiplying through by the $r_t$ approximation in (4.12) yields the following inequality

$$\frac{-c_{\theta_s}^4}{4r_c^2}y^4 + \frac{-c_{\theta_s}^3 s_{\theta_s} - s_{\theta_s} \cos^2 \theta_s}{2r_c} y^3 + \frac{-c_{\theta_s}^3 - 3s_{\theta_s}^2}{2} y^2 + \left( -r_c s_{\theta_s} + \frac{r_c s_{\theta_s}}{c_{\theta_s}^2} - \frac{s_{\theta_s}^3 r_c}{c_{\theta_s}^2} \right) y \leq -N\delta_a (r_c - s_{\theta_s} y + \frac{c_{\theta_s}^2}{2r_c} y^2) \quad (4.15)$$

Noting that the coefficient for the first-order $y$ term on the left hand side reduces to approximately 0 leads to the following quartic equation

$$\left( \frac{-c_{\theta_s}^4}{4r_c^2} \right)y^4 + \left( \frac{-c_{\theta_s}^3 s_{\theta_s} - s_{\theta_s} \cos^2 \theta_s}{2r_c} \right)y^3 + \left( \frac{-c_{\theta_s}^3 - 3s_{\theta_s}^2}{2} \right)y^2 + \left( N\delta_a c_{\theta_s} \frac{2}{2r_c} \right)y + N\delta_a r_c = 0 \quad (4.16)$$

The solutions to this equation are not trivial, but can be found using the general form of the solution to a quartic equation of form $Ay^4 + By^3 + Cy^2 + Dy + E = 0$. Note, in this case unlike the previous, the bounds are not necessarily symmetric with respect to the scene center. They are for the broadside case, but with a squinted scenario the maximum $y$ value limiting distortion to an arbitrary number of resolution cells will be different for the positive and negative sides. The bounds are as follows

$$y_{\min} = \begin{cases} \frac{-B}{4A} - \frac{\sqrt{-a + \sqrt{a^2 - 4c}}}{2}, & \text{for } \theta_s = 0 \\ \frac{-B}{4A} + \frac{W - \sqrt{-(3a + 2y + \frac{2B}{W})}}{2}, & \text{for } \theta_s > 0 \\ \frac{-B}{4A} + \frac{-W - \sqrt{-(3a + 2y + \frac{2B}{W})}}{2}, & \text{for } \theta_s < 0 \end{cases} \quad (4.17)$$

$$y_{\max} = \begin{cases} \frac{-B}{4A} + \frac{\sqrt{-a + \sqrt{a^2 - 4c}}}{2}, & \text{for } \theta_s = 0 \\ \frac{-B}{4A} + \frac{W + \sqrt{-(3a + 2y + \frac{2B}{W})}}{2}, & \text{for } \theta_s > 0 \\ \frac{-B}{4A} + \frac{-W + \sqrt{-(3a + 2y + \frac{2B}{W})}}{2}, & \text{for } \theta_s < 0 \end{cases} \quad (4.18)$$

where $A, B, C, D, E$ are the coefficients in (4.16) in decreasing order, i.e. $A$ is the coefficient of the
$y^4$ term, $B$ the coefficient for the $y^3$ term, etc. The remaining variables can be calculated using the following equations.

$$\alpha = -\frac{3B^2}{8A^2} + \frac{C}{A}$$  \hspace{1cm} (4.19)
$$\beta = \frac{B^3}{8A^3} - \frac{BC}{2A^2} + \frac{D}{A}$$  \hspace{1cm} (4.20)
$$\gamma = -\frac{B^4}{256A^4} - \frac{CB^2}{16A^3} - \frac{BD}{4A^2} + \frac{E}{A}$$  \hspace{1cm} (4.21)
$$P = -\frac{\alpha^2}{12} - \gamma$$  \hspace{1cm} (4.22)
$$Q = -\frac{\alpha^3}{108} + \frac{\alpha\gamma}{3} - \frac{\beta^2}{8}$$  \hspace{1cm} (4.23)
$$R = -\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}$$  \hspace{1cm} (4.24)
$$U = \sqrt[3]{R}, \text{ Either value for } R \text{ will work}$$  \hspace{1cm} (4.25)
$$y = -\frac{5}{6}\alpha + U - \frac{P}{3U}$$  \hspace{1cm} (4.26)
$$W = \sqrt{\alpha + 2y}$$  \hspace{1cm} (4.27)

Obviously, this result is much less succinct than the previous approaches and despite all attempts was irreducible to a more compact form. A plot is included as Figure 4.5 for a 1 km standoff range limiting resolution to 1 m. It is noted that the diameter was the metric used here, due to the nonsymmetry of the DRE results in (4.17) and (4.18), therefore the radius predicted in (4.4) was doubled.
4.4 Bounding DRE Distortion with Second-Order Corrections

Sadly, this case shows the limitations of the DRE approach for bounding. As with the previous section, the corrected DRE approximations in (3.12) and (3.13) are intractable for bounding as is, therefore the additional approximation in (4.12) is introduced. As was mentioned, the second order approximation was the highest that could be used to result in a polynomial whose roots could be found using a general solution. However, because the corrected DRE approach approximation was based off the second-order approximation of the differential range term, the additional second order expansion on the differential range term in the approximation is nullifying the squinted $x$-distortion component.

To illustrate, again considering the $x = 0$ direction, (3.12) becomes

$$\delta x_s^{DRE+} = -\frac{y_c}{r_c} y + \frac{x_c^2}{r_c} \frac{y^2}{2} - \sqrt{x_c^2 + (y - y_c)^2} + \sqrt{x_c^2 + y_c^2}$$

(4.28)
and introducing the approximation in (4.12) and realizing $\sqrt{x_c^2 + y_c^2} = r_c$

$$\delta x_s^{DRE^+} = -\frac{y_c}{r_c} y + \frac{x_c^2}{r_c^2} \frac{y^2}{2} - \left( r_c - \frac{y_c}{r_c} y + \frac{x_c^2}{2r_c^2} y^2 \right) + r_c$$

(4.29)

$$= -\frac{y_c}{r_c} y + \frac{x_c^2}{r_c^3} \frac{y^2}{2} - r_c + \frac{y_c}{r_c} y - \frac{x_c^2}{2r_c^3} y^2 + r_c$$

(4.30)

$$= 0$$

(4.31)

The result would have been the same if (4.12) was calculated using an expansion with respect to both $x$ and $y$. Intuition suggests that at least a third-order expansion on the $r_t$ would be required to capture the necessary effects. However, this leads to an intractable bounding problem.

This result is not particularly bad if the $\delta x_s^{DRE^+}$ distortion was negligibly small, however that cannot be guaranteed for all scenarios. Therefore, introducing the most accurate Taylor approximation possible that will result in a bound by a general solution to a polynomial function is hamstringing the approximation. It will not accurately represent an already approximated solution making the bound worthless for practical application. An attempt could be made to bound a higher order quintic expression, however that problem is likely intractable, and not considered here.

### 4.5 Comparison and Analysis of Bounds and Approaches

It is evident from the distortion approximation derivations and bounding that the DPE approach is much preferred for several reasons; the required derivatives are straightforward to calculate and the resulting bounds were manageable expressions, unlike the DRE bound in (4.17) and (4.18). It is noted that choosing the $x = 0$ direction for bounding, did simplify the problem although this was not the motivation for the choice.

Given the discussion in Section 3.7, which ultimately concluded that although the DRE approximations are more accurate, the results are comparable for targets near the CRP. Limiting distortion to a small number of resolution cells is effectively keeping the approximations in this region, hence the similarity of the pre-correction DPE and DRE bounding results in Figure 4.5. Any and all accuracy gained by using the DRE approximation, is effectively eliminated by extrapolating the
dominant error dimension, therefore it is concluded that the DPE approach is the preferred method for deriving the heuristic scene-size limits.

The distortion-free scene size gained can be demonstrated from the results in (4.4) and (4.9). Figure 4.6 shows the distortion-free scene size gained by applying the second-order corrections for a 790 kilometer stand-off range limiting distortion to a single 1 m resolution cell. The minimum, hence limiting, component of the results in (4.9) was used to generate the maximum distortion free scene size after second-order corrections. Therefore, it is evident that correcting for the second order effects will lead to an increase in the distortion-limited scene size.

A comprehensive set of the bounding results for the 790 m stand-off range simulation is given in Appendix C.
Chapter 5

Approximating Defocus in Squinted PFA Imaging

5.1 Nature of Defocus

Unlike distortion, which had a very simple explanation, the defocus caused by the error due to wavefront curvature will require slightly more discussion. Examining the PFA kernel in (2.7), point targets are encoded in the phase history as complex exponentials with frequency proportional to their position in the scene. The Fourier transform would yield impulse responses in the spatial domain if it were not for the windowing inherent in the $k$-space sampling. This windowing causes these ideal impulse responses to spread, leading to what is commonly referred to as the Ideal Point Response (IPR) or Point Spread Function (PSF). This PSF is determined purely by the $k$-space extent of the sampled data.

While distortion was caused by erroneous linear terms in the spatial-frequency domain, defocus is caused by quadratic terms in the the spatial-frequency domain. The end effect is equivalent to convolving the PSF of a point target with the Fourier transform of the quadratic blurring kernel. This blurring kernel’s width in the spatial domain is determined by the peak amplitude of the quadratic in the frequency domain. [10] The amount of quadratic phase error tolerable is dependent on the application. Some authors have considered a maximum quadratic phase error of $\frac{\pi}{4}$ to be negligible.
while other have allowed a peak phase error of $\frac{\pi}{2}$ [8].

In order to approximate these quadratic phase error functions, the terms quadratic in the frequency space variables, $k_x$ and $k_y$, of the phase error terms are needed. Like the distortion case, the goal is to approximate the quadratic phase errors both before and after second-order corrections in order to figure out a rule of thumb estimate on the amount of scene size gained by applying the second-order corrections.

Again, a Taylor series approximation is used on the error term, except now it will require a second-order approximation. To illustrate, the phase error from the approximated or true differential range error is

\[ \Phi = \frac{4\pi f_c}{c} \Delta \hat{R} \] (5.1)

where $\Delta \hat{R}$ is a dummy variable representing the result in (2.9), (2.12), (2.13), or (2.14), for the DPE, DPE with second-order corrections, DRE, or DRE with second-order corrections approximation respectively. Performing another a second-order Taylor series expansion on $\Delta \hat{R}$ with respect to $y_a/r_a$ yields

\[ \Phi \approx \frac{4\pi f_c}{c} \left( \Delta \hat{R} \big|_{\theta_c} + \frac{\partial^2 \Delta \hat{R}}{\partial \left( \frac{y_a}{r_a} \right)^2} \bigg|_{\theta_c} \frac{1}{2} \left( \frac{y_a}{r_a} - \frac{y_c}{r_c} \right)^2 \right) \] (5.2)

where $(\cdot)$|_{\theta_c} indicates evaluation at aperture center.

Assigning $k_x \approx \frac{4\pi f_c}{c}$, there will be no squinted range defocus as there is no quadratic term in $k_x$. However, there is a quadratic component in $y_a/r_a$ and it is precisely this quadratic phase error that leads to defocus. The approximated quadratic phase error leading to defocus can then be written as

\[ \Phi_{QPE} \left( \frac{y_a}{r_a} \right) = \frac{2\pi f_c}{c} \frac{\partial^2 \Delta \hat{R}}{\partial \left( \frac{y_a}{r_a} \right)^2} \left( \frac{y_a}{r_a} - \frac{y_c}{r_c} \right)^2 \] (5.3)

where $f_c$ is the center frequency of the transmitted pulse.
5.2 DPE Approach

Recalling (2.9),

$$\Delta R^{(2)} = \frac{y_a^3}{2r_a^3} x^2 - \frac{x_a y_a}{r_a^2} x y + \frac{x_a^2}{2r_a^3} y^2$$

The approximated quadratic phase error using the DPE approach can be found by taking the second derivative of (2.9) with respect to $y_a/r_a$

$$\Phi_{QPE}^{DPE} \left( \frac{y_a}{r_a} \right) = \frac{2\pi f_c}{c} \left( \frac{\partial^2 a_1}{\partial \left( \frac{r_a}{r_c} \right)^2} + \frac{\partial^2 a_2}{\partial \left( \frac{r_a}{r_c} \right)^2} + \frac{\partial^2 a_3}{\partial \left( \frac{r_a}{r_c} \right)^2} \right) \left( \frac{y_a}{r_a} - \frac{y_c}{r_c} \right)^2. \quad (5.4)$$

Calculating the derivatives yields

$$\frac{\partial^2 a_1}{\partial \left( \frac{r_a}{r_c} \right)^2} = \frac{9c^2}{2x_c c_{\theta_a}} x^2 \quad (5.5)$$
$$\frac{\partial^2 a_2}{\partial \left( \frac{r_a}{r_c} \right)^2} = -\frac{9c^2}{2x_c c_{\theta_a}} x y \quad (5.6)$$
$$\frac{\partial^2 a_3}{\partial \left( \frac{r_a}{r_c} \right)^2} = \frac{2s^3_{\theta_a}}{x_c c_{\theta_a}} \frac{7s_{\theta_a}}{x_c} y^2 \quad (5.7)$$

5.3 DRE Approach

Recalling (2.13),

$$\Delta R^{(2)} = \frac{-x_a}{r_a} x - \frac{y_a}{r_a} y - \sqrt{(x - x_a)^2 + (y - y_a)^2} + \sqrt{x_a^2 + y_a^2}$$

The approximated quadratic phase error using the DRE approach can be found by taking the second derivative of the terms in (2.13) with respect to $y_a/r_a$

$$\Phi_{QPE}^{DRE} \left( \frac{y_a}{r_a} \right) = \frac{2\pi f_c}{c} \left( \frac{\partial^2 b_1}{\partial \left( \frac{r_a}{r_c} \right)^2} + \frac{\partial^2 b_2}{\partial \left( \frac{r_a}{r_c} \right)^2} + \frac{\partial^2 b_3}{\partial \left( \frac{r_a}{r_c} \right)^2} + \frac{\partial^2 b_4}{\partial \left( \frac{r_a}{r_c} \right)^2} \right) \left( \frac{y_a}{r_a} - \frac{y_c}{r_c} \right)^2. \quad (5.8)$$
Calculating the derivatives yields

\[
\frac{\partial^2 b_1}{\partial (\frac{y}{r_a})^2} = \frac{1}{c_0} x \\
\frac{\partial^2 b_2}{\partial (\frac{y}{r_a})^2} = 0 \\
\frac{\partial^2 b_3}{\partial (\frac{y}{r_a})^2} = -\frac{r_c^2}{c_0^2} \frac{(x - x_c)^2}{\sqrt{(x - x_c)^2 + (y - y_c)^2}} \\
\frac{\partial^2 b_4}{\partial (\frac{y}{r_a})^2} = \frac{x_c^2}{r_c c_0^3} 
\]

(5.9) - (5.12)

5.4 DPE Approach: with Second-Order Corrections

Assuming the second-order phase error in (2.9) has been corrected in post-processing, the dominant polynomial error becomes (2.12)

\[
\Delta R^{(3)} = \left( \frac{x_a}{2r_a^3} - \frac{x_a^3}{2r_a^5} \right) x^3 + \left( \frac{y_a}{2r_a^3} - \frac{3x_a^2 y_a}{2r_a^5} \right) x^2 y \\
+ \left( \frac{x_a}{2r_a^3} - \frac{3x_a y_a^2}{2r_a^5} \right) y^2 x + \left( \frac{y_a}{2r_a^3} - \frac{y_a^3}{2r_a^5} \right) y^3 
\]

The approximated quadratic phase error after second-order corrections using the DPE approach can then be found by taking the second derivative of (2.12) with respect to \(\frac{y_a}{r_a}\).

\[
\Phi_{DPE}^{(3)} \left( \frac{y_a}{r_a} \right) = \frac{2\pi f_c}{c} \left( \frac{\partial^2 c_1}{\partial (\frac{y}{r_a})^2} + \frac{\partial^2 c_2}{\partial (\frac{y}{r_a})^2} + \frac{\partial^2 c_3}{\partial (\frac{y}{r_a})^2} + \frac{\partial^2 c_4}{\partial (\frac{y}{r_a})^2} \right) \left( \frac{y_a}{r_a} - \frac{y_c}{r_c} \right)^2. 
\]

(5.13)
Calculating the derivatives yields

\[
\begin{align*}
\frac{\partial^2 c_1}{\partial \left( \frac{y_a}{r_a} \right)^2} &= \frac{25c_4^4_\theta - 29c_2^2_\theta + 6}{2x^2c_\theta} \quad (5.14) \\
\frac{\partial^2 c_2}{\partial \left( \frac{y_a}{r_a} \right)^2} &= \frac{75s_3^5 - 105s_3^3 + 32s_\theta x^2y}{2x^2c^4_\theta - 2x^2c_\theta} \quad (5.15) \\
\frac{\partial^2 c_3}{\partial \left( \frac{y_a}{r_a} \right)^2} &= \left( \frac{-75c_4^4_\theta + 78c_2^2_\theta - 12}{2x^2c^4_\theta} \right) y^2x \quad (5.16) \\
\frac{\partial^2 c_4}{\partial \left( \frac{y_a}{r_a} \right)^2} &= \left( \frac{13s_\theta - 25s_\theta^2}{2x^2c_\theta} \right) y^3 \quad (5.17)
\end{align*}
\]

### 5.5 DRE Approach: with Second-Order Corrections

Assuming the second-order effects in (2.9) are corrected for in post processing, the dominant error term becomes (2.14) which is repeated here

\[
\Delta R^{(3)} = -\frac{x_a}{r_a} y + \frac{y_a^2}{r_a^2} x^2 - \frac{x_a y_a}{r_a^3} xy + \frac{x_a^2}{2r^3_a} y^2 - \sqrt{(x - x_a)^2 + (y - y_a)^2} + \frac{x^2_a + y^2_a}{d_4}.
\]

The approximated quadratic phase error after second-order corrections using the DRE approach can then be found by taking the second derivative of (2.14) with respect to \( y_a/r_a \).

\[
\Phi_{QPE}^{DRE} \left( \frac{y_a}{r_a} \right) = \frac{2\pi f_c}{c} \left( \sum_{i=1}^{7} \frac{\partial^2 d_i}{\partial \left( \frac{y_a}{r_a} \right)^2} \right) \left( \frac{y_a}{r_a} - \frac{y_c}{r_c} \right)^2.
\]
Calculating the derivatives yields

\[
\frac{\partial^2 d_1}{\partial (\frac{\mu}{ra})^2} = \frac{1}{c_0^2} \frac{x}{x_c} \quad (5.19)
\]

\[
\frac{\partial^2 d_2}{\partial (\frac{\mu}{ra})^2} = 0 \quad (5.20)
\]

\[
\frac{\partial^2 d_3}{\partial (\frac{\mu}{ra})^2} = -\frac{r_c^2}{c_0^4 \sqrt{(x - x_c)^2 + (y - y_c)^2}} \quad (5.21)
\]

\[
\frac{\partial^2 d_4}{\partial (\frac{\mu}{ra})^2} = \frac{r_c^2 c_0^4}{x_c} \quad (5.22)
\]

\[
\frac{\partial^2 d_5}{\partial (\frac{\mu}{ra})^2} = \frac{9c_0^2 s_\theta}{2x_c c_\theta} - \frac{7}{x_c} x^2 \quad (5.23)
\]

\[
\frac{\partial^2 d_6}{\partial (\frac{\mu}{ra})^2} = -\frac{9c_0^2 s_\theta}{2x_c c_\theta} + 6x y \quad (5.24)
\]

\[
\frac{\partial^2 d_7}{\partial (\frac{\mu}{ra})^2} = \left( \frac{2s_\theta^3}{x_c c_\theta^2} - \frac{7s_\theta}{x_c} \right)y^2 \quad (5.25)
\]
Chapter 6

Bounding Defocus in Squinted PFA Imaging

Like distortion, bounding defocus is of obvious interest. It is answering the question, “Under what conditions does the far field approximation hold and allow the PFA to be used for image formation without suffering unacceptable defocus in the image?”

By bounding the defocus approximations before and after second-order corrections, the scene size gained by applying these corrections can be determined. Like the distortion case, this conclusion assumes the defocus correction perfectly compensates for its associated phase error. It does not and only approximates it. Regardless, this comparison will yield heuristic or “rule-of-thumb” bounds given the approximations are accurate, which will be demonstrated.

6.1 Additional Bounding Concerns

In the quest for a compact bound, other factors have to be considered. Recalling the form of the QPE approximations in (5.13) and (5.4)

$$\Phi_{QPE} \left( \frac{y_a}{r_a} \right) = \frac{2\pi f_c}{c} \frac{\partial^2 \Delta \hat{R}}{\partial (y_a^2)} \left( \frac{y_a}{r_a} - \frac{y_c}{r_c} \right)^2$$

(6.1)
where the second derivative can be assumed to be an arbitrary function of image position \([x, y]\) for purposes of the current discussion. This is a shifted quadratic in \(y_a/r_a\). Because \(y_a/r_a\) is not symmetric about the center point, the maximum phase error will be greater at one end of the aperture than the other. For negative squint angles \((y_a > 0)\), the QPE will be greatest at the most positive end of the aperture and for positive squint angles \((y_a < 0)\) at the most negative end of the aperture. This is not a problem, just another factor to consider when calculating the bound. The real problem lies in evaluating \((y_a/r_a - y_c/r_c)^2\) at the point of maximum QPE and having it yield a compact expression.

Despite attempts, no direct solution to this problem was discovered. Utilizing yet another Taylor expansion on \(y_a/r_a\) about \(y_c/r_c\) with respect to \(y_a\) yields the following

\[
\left( \frac{y_a}{r_a} \right)^{(1)} = \frac{y_c}{r_c} + \frac{\partial y_a}{\partial y_a} \bigg|_{\theta_s} (y_a - y_c) = \frac{y_c}{r_c} + \frac{c_s^2}{r_c} (y_a - y_c) \tag{6.2}
\]

Substituting this result into (6.1) eliminates the dependence on \(r_a\) and the function does become symmetric about the center of the aperture. Because the maximum quadratic phase error exists at the aperture extents, \(y_a = y_c \pm L_a/2\), where \(L_a\) is the length of the synthetic aperture, the \(y_a - y_c\) expression simplifies to a simple function of the aperture length, which was the desired result. The maximum QPE then becomes

\[
\Phi_{\text{max}}(x, y) = \frac{\pi f_c}{c} \frac{\partial^2 \hat{R}}{\partial (\frac{y_a}{r_a})^2} \frac{c_s^4 L_a^2}{2r_c^2} \tag{6.3}
\]

This is an excellent result and yields a satisfactorily accurate approximation as shown in Figures 6.1 and 6.2. Alternatively, the QPE approximations could have been derived with respect to \(y_a\) only, which was the initial approach to solving this problem \([13]\). This result is shown in Figure 6.1 for the DPE approach only. The expansion with respect to \(y_a/r_a\) with the additional approximation outperforms this initial approach in terms of approximation accuracy and also yields suitably accurate approximations compared to the expansion with respect to \(y_a/r_a\) without the additional complexity.

It is also noted that DRE approximations are comparable to the DPE approximations in terms of accuracy, which allows the simpler DPE approximations to be used for bounding, which is what
Figure 6.1: Comparison of second-order maximum QPE approximations compared with the true QPE was suggested by the distortion bounding results. Lastly, as with studying distortion, these approximations are most accurate around the CRP and also lose accuracy with greater squint angle, which is also evident in these plots. The true QPE was determined by using a least squares approach to fit the quadratic component of the true phase error in the numerical simulations.

Therefore, restricting the maximum QPE in (6.3) for a pixel at \([x, y]\) to less than \(\pi/2\) yields

\[
\Phi_{\text{max}}(x, y) = \frac{\pi f c}{c} \frac{\partial^2 \Delta R}{\partial (\Delta r_{\theta a})^2} \frac{c^4 L_a^2}{2 r_c^4} \leq \frac{\pi}{2}
\]

which is the expression that will be bounded.

Additionally, when considering the distortion bound problem, the first step was to rotate the results from a squinted geometry into a strictly broadside coordinate system. Looking at the spatially varying defocus approximations, they are a function of the broadside \([x, y]\) coordinate system. Examining the numerical results of the maximum quadratic phase errors given by the pre and post-correction defocus approximations in Appendix D, it is evident they are rotating as a function of
Figure 6.2: Comparison of maximum QPE approximations after second-order corrections compared with the true QPE
squint angle. Trying to identify a worst case error dimension in this coordinate system will be difficult, as it varies with squint angle.

By writing these as functions of squint angle, this complexity can be removed and a dominant dimension identified in the squinted coordinate system. Applying the transformation in (3.3) allows the equations in (5.4) and (5.13) to be written as functions of the squinted coordinate system

\[
x = \cos(\theta_s)x_s + \sin(\theta_s)y_s \\
y = -\sin(\theta_s)x_s + \cos(\theta_s)y_s
\]

(6.5) (6.6)

This eliminates the dependence of “worst-case” bounding dimension on squint angle and the expression to be bounded becomes,

\[
\Phi_{\text{max}}(x_s, y_s) = \frac{\pi f_c}{c} \frac{\partial^2 \Delta \hat{R}}{\partial (\frac{y_s}{r_a})^2} \frac{c_{\theta_s}^4 L_a^2}{2r_c^2} \leq \frac{\pi}{2}
\]

(6.7)

where the second derivatives given in the previous chapter have substituted (6.5) and (6.6) and are now written in terms of the squinted coordinate system.

Based on the difficulties encountered in arriving at succinct bounds using the DRE approach in the case of distortion and the comparable accuracy between the DPE and DRE QPE leading to defocus approximations, the DPE approximations will be used to bound defocus.

6.2 Bounding DPE Defocus

Examining the results in Appendix D, the image dimension presenting the limiting quadratic phase error is in the \( x = y \cot \theta_s \) direction. This is a convenient result as it corresponds to the \( x_s = 0 \) line in the squinted coordinate system.

Expressing the DPE second derivatives in (5.4) in polar coordinates, applying the rotation in (6.5) and (6.6), and setting \( x_s \) equal to zero yields

\[
\frac{\partial^2 \Delta \hat{R}}{\partial (\frac{y_s}{r_a})^2} = -\frac{c_{\theta_s}^2}{x_c c_{\theta_s}^2} \frac{1}{2} y_s^2
\]

(6.8)
which allows the bounding expression

\[
\Phi_{\text{max}} = \frac{\pi f_c - L_a^2 (c^2_{\theta_s} + \frac{1}{2} c^2_{\theta_s})}{\frac{2x_c r c^2}{2x_c r c^2}} y_s^2 \leq \frac{\pi}{2} \quad (6.9)
\]

In order to reduce the expression, algebraic manipulation on the above is performed utilizing the expression \( \delta_a = \frac{\lambda r_c}{2r_a x_c} \), \([9]\) yielding

\[
\Phi_{\text{max}} = -\frac{\pi \lambda (c_{\theta_s} + \frac{1}{2})}{8\delta_a^2 c_{\theta_s} x_c} y_s^2 \quad (6.10)
\]

where \( \delta_a \) is the cross-range resolution of the image. Then, bounding the magnitude of the maximum quadratic phase error and extrapolating the maximum quadratic phase error in the \( y_s \) dimension to every pixel with a scene radius \( r \) yields

\[
|\Phi_{\text{max}}(r)| \leq \frac{\pi}{2} \quad (6.11)
\]

which allows the final bound by the general solution to a quadratic equation

\[
r \leq 2\delta_a \sqrt{\frac{c_{\theta_s} x_c}{\lambda (c^2_{\theta_s} + \frac{1}{2})}} \quad (6.12)
\]

A series of plots demonstrating the validity of this result is included as Appendix E.

Figure 6.3 shows the maximum scene size predicted from this result. The plot was generated by fixing the stand-off range \( (r_c) \) to yield a 50m defocus-free scene size using the broadside result in \([8]\) for a resolution of 1 ft. The final bound in \((6.12)\) was then used to calculate the defocus-free scene size for each squint angle. Note, that the length of the synthetic aperture is varied to achieve the required resolution, so it was not constant for each squint angle scenario.
Figure 6.3: Maximum defocus-free scene size according to the DPE result with additional approximations.
6.3 Bounding DPE Defocus with Second-Order Corrections

Because the pre-correction QPE leading to defocus was bounded in the $x_s = 0$ dimension, the same choice will be made for the QPE after applying second-order corrections. Examining the results in Appendix D, it is evident that there is a trade-off here. While this choice generally captured the worst-case QPE before second-order corrections, after applying them it does a good job of capturing the worst-case defocus for larger squint angles, but a poorer job for small squint angles. Making matters worse, at $0^\circ$ squint the post second-order correction result in (5.13) yields no defocus in $x_s = 0$ dimension, hence the bound will suggest no defocus for a broadside scenario which is incorrect. The orthogonal image dimension should be used for smaller squint angles. Choosing a dominant bounding dimension that consistently captures the worst-case QPE across all squint angles is difficult due to the spatially-variant nature of the phase errors, but must be done to yield a single, compact heuristic bound, rather than a set of bounds for each scene region.

Expressing the DPE second derivatives in (5.13) in polar coordinates, applying the rotation in (6.5) and (6.6), and setting $x_s$ equal to zero yields

\[
\frac{\partial^2 \Delta R}{\partial (\frac{\omega_a}{c})^2} = \frac{3s_{\theta_s}}{2x_c^2 c_{\theta_s}} y_s^3
\]  

(6.13)

which yields the bounding expression

\[
\Phi_{\text{max}}(y_s) = \frac{4\pi f_c}{c} \frac{3L_a^2 s_{\theta_s} c_{\theta_s}^3}{16x_c^2 y_s^3} \leq \frac{\pi}{2}
\]  

(6.14)

In order to reduce the expression, algebraic manipulation on the above is performed utilizing the expression $\delta_a = \frac{\lambda c_{\theta_s} s_{\theta_s}}{2L_a c_{\theta_s}}$, [9] yielding

\[
\Phi_{\text{max}}(y_s) = -\frac{\pi \lambda c_{\theta_s} s_{\theta_s}}{16\delta_a x_c^2} y_s^3
\]  

(6.15)

where $\delta_a$ is the cross-range resolution of the image. Then, bounding the magnitude of the maximum quadratic phase error and extrapolating the maximum quadratic phase error in the $y_s$ dimension to
Figure 6.4: Maximum defocus-free scene size according to the DPE result with second-order corrections with additional approximations with pre-correction result for comparison. Every pixel within a scene radius $r$ yields

$$|\Phi_{\text{max}}(r)| \leq \frac{\pi}{2}$$

which allows the final bound by the general solution to a cubic equation

$$r \leq 2 \left( \frac{s_0^2 x_c^2}{\lambda c_0 s_{th_x}} \right)^{\frac{1}{3}}.$$

A series of plots demonstrating the validity of these bounds is included as Appendix E.

Figure 6.4 shows the maximum scene size predicted from this result along with the before second-order corrections result in (6.12) for comparison. The plot was generated by fixing the stand-off range ($r_c$) to yield a 50m defocus-free scene size using the broadside result in [8] for a resolution of 1 ft, allowing the length of the aperture to vary to meet this resolution requirement for each squint angle scenario. Again, it was noted that the result in (6.17) incorrectly predicts no
distortion for the broadside scenario, which is incorrect. However, for larger squint angles, applying the correction yields greater than triple the defocus-free scene size.

6.4 Comparison and Analysis of Bounds and Approaches

A couple points of note need to be made regarding the previous discussion. First, due to the generally intractability of the problem under study, many approximations have been made to arrive at the above results. It was demonstrated that regardless of these approximations, the maximum QPE approximations leading to defocus are still accurate and would lead to a valid “rule of thumb” result. The primary problem lies in identifying a bounding dimension that consistently captures the worst case defocus across all squint angles. While the bounding result did yield a good “rule of thumb” estimate, it relied on the heuristic assumption that the distortion at specific image coordinates applied to all coordinates within an image radius. This allowed a compact bound at the expense of a rigorous, but ultimately uncompact, solution. An alternative approach to bounding the defocus would be to create binary masks on formed images eliminating pixels that had greater than the allowable amount of QPE. While requiring numerical simulation, it is perhaps the only way to thoroughly capture the spatially-varying nature of the phase error.
Chapter 7

Conclusion

This thesis investigated the distortion and defocus of point targets caused by the error due to wavefront curvature in Polar Format imaging. It extended the broadside result in [8] to include squinted geometries and account for second-order corrections. This required utilizing a Taylor series expansion on either a dominant polynomial error term, which was based on another Taylor expansion of a differential range term, or the true differential range error.

This process resulted in accurate approximations for both the linear and quadratic phase error terms, which are responsible for distortion and defocus in the imaged scene, respectively. These approximations were calculated both before and after correcting for the second-order phase errors, using both the DPE and DRE approached.

It was found that the DPE approach led to approximations amenable to bounding due their polynomial form. The DRE approximations, although more accurate, had a more complicated mathematical form that required additional approximations to bound, if the expression was boundable at all. The bounding process then required identifying a dominant bounding dimension which had to both accurately approximate the true error and capture the worst case, hence limiting, error. This choice was made heuristically, by examining the results of numerical simulations. This process was made difficult due to the spatially-variant nature of the phase errors, as well as their variation with respect to squint angle and stand off range.

The final results give a heuristic or “rule of thumb” bound on imaged scene size such that these
distortion and defocus effects were limited in the resulting image, both before and after correcting for the second-order phase errors.
Bibliography


Appendix A

Distortion Approximation Accuracy

In order to test the accuracy of the approximations, SAR phase history was data synthesized for a grid of point targets for squint angles ranging from $-40^\circ$ to $40^\circ$ in $5^\circ$ increments at a stand-off range of 790m. Image formation was done with a linearly approximated BPA, in order to isolate for the range curvature error and not introduce interpolation or other unaccounted for errors that may skew the results. These resulting images will then characterize the actual distortion. The accuracy of the approximations can then be verified by plotting the approximated distorted locations over the true locations. The results are as follows.

Next, the second-order correction based off the results in (3.5) and (3.6) was applied for squint angles $-40^\circ$ to $40^\circ$ in $10^\circ$ increments. These images served as the test case to compare the actual distortion after second-order corrections to the DPE and DRE results with second-order corrections.
Figure A.1: Actual distortion, DPE approximation, and DRE approximation for a $-40^\circ$ squint angle scenario

Figure A.2: Actual distortion, DPE approximation, and DRE approximation for a $-35^\circ$ squint angle scenario
Figure A.3: Actual distortion, DPE approximation, and DRE approximation for a $-30^\circ$ squint angle scenario

Figure A.4: Actual distortion, DPE approximation, and DRE approximation for a $-25^\circ$ squint angle scenario
Figure A.5: Actual distortion, DPE approximation, and DRE approximation for a \(-20^\circ\) squint angle scenario.

Figure A.6: Actual distortion, DPE approximation, and DRE approximation for a \(-15^\circ\) squint angle scenario.
Figure A.7: Actual distortion, DPE approximation, and DRE approximation for a $-10^\circ$ squint angle scenario

Figure A.8: Actual distortion, DPE approximation, and DRE approximation for a $-5^\circ$ squint angle scenario
Figure A.9: Actual distortion, DPE approximation, and DRE approximation for a 0° squint angle scenario

Figure A.10: Actual distortion, DPE approximation, and DRE approximation for a 5° squint angle scenario
Figure A.11: Actual distortion, DPE approximation, and DRE approximation for a 10° squint angle scenario

Figure A.12: Actual distortion, DPE approximation, and DRE approximation for a 15° squint angle scenario
Figure A.13: Actual distortion, DPE approximation, and DRE approximation for a 20° squint angle scenario

Figure A.14: Actual distortion, DPE approximation, and DRE approximation for a 25° squint angle scenario
Figure A.15: Actual distortion, DPE approximation, and DRE approximation for a 30° squint angle scenario

Figure A.16: Actual distortion, DPE approximation, and DRE approximation for a 35° squint angle scenario
Figure A.17: Actual distortion, DPE approximation, and DRE approximation for a 40° squint angle scenario

Figure A.18: Actual distortion, DPE, and DRE approximations with second-order corrections for a −40° squint angle scenario
Figure A.19: Actual distortion, DPE, and DRE approximations with second-order corrections for a $-30^\circ$ squint angle scenario

Figure A.20: Actual distortion, DPE, and DRE approximations with second-order corrections for a $-20^\circ$ squint angle scenario
Figure A.21: Actual distortion, DPE, and DRE approximations with second-order corrections for a $-10^\circ$ squint angle scenario

Figure A.22: Actual distortion, DPE, and DRE approximations with second-order corrections for a $0^\circ$ squint angle scenario
Figure A.23: Actual distortion, DPE, and DRE approximations with second-order corrections for a 10° squint angle scenario

Figure A.24: Actual distortion, DPE, and DRE approximations with second-order corrections for a 20° squint angle scenario
Figure A.25: Actual distortion, DPE, and DRE approximations with second-order corrections for a $30^\circ$ squint angle scenario

Figure A.26: Actual distortion, DPE, and DRE approximations with second-order corrections for a $40^\circ$ squint angle scenario
Appendix B

Magnitude of Distortion Approximations

In order to identify a dominant bounding dimension, the magnitude of the $x$ and $y$-distortion components was plotted. The line through these images shows the zero distortion dimension representing the peak of the wavefront, therefore point targets on this line see no wavefront curvature.

B.1 DPE Distortion Approximation

Figure B.1: DPE approximated distortion components for a $-40^\circ$ squint angle scenario
Figure B.2: DPE approximated distortion components for a $-35^\circ$ squint angle scenario

Figure B.3: DPE approximated distortion components for a $-30^\circ$ squint angle scenario
Figure B.4: DPE approximated distortion components for a $-25^\circ$ squint angle scenario

Figure B.5: DPE approximated distortion components for a $-20^\circ$ squint angle scenario
Figure B.6: DPE approximated distortion components for a $-15^\circ$ squint angle scenario

Figure B.7: DPE approximated distortion components for a $-10^\circ$ squint angle scenario
Figure B.8: DPE approximated distortion components for a $-5^\circ$ squint angle scenario

Figure B.9: DPE approximated distortion components for a $0^\circ$ squint angle scenario
Figure B.10: DPE approximated distortion components for a 5° squint angle scenario

Figure B.11: DPE approximated distortion components for a 10° squint angle scenario
Figure B.12: DPE approximated distortion components for a $15^\circ$ squint angle scenario

Figure B.13: DPE approximated distortion components for a $20^\circ$ squint angle scenario
Figure B.14: DPE approximated distortion components for a 25° squint angle scenario

Figure B.15: DPE approximated distortion components for a 30° squint angle scenario
Figure B.16: DPE approximated distortion components for a 35° squint angle scenario

Figure B.17: DPE approximated distortion components for a 40° squint angle scenario
Figure B.18: DRE approximated distortion components for a $-40^\circ$ squint angle scenario
Figure B.19: DRE approximated distortion components for a $-35^\circ$ squint angle scenario

Figure B.20: DRE approximated distortion components for a $-30^\circ$ squint angle scenario
Figure B.21: DRE approximated distortion components for a $-25^\circ$ squint angle scenario

Figure B.22: DRE approximated distortion components for a $-20^\circ$ squint angle scenario
Figure B.23: DRE approximated distortion components for a $-15^\circ$ squint angle scenario

Figure B.24: DRE approximated distortion components for a $-10^\circ$ squint angle scenario
Figure B.25: DRE approximated distortion components for a $-5^\circ$ squint angle scenario

Figure B.26: DRE approximated distortion components for a $0^\circ$ squint angle scenario
Figure B.27: DRE approximated distortion components for a $5^\circ$ squint angle scenario

Figure B.28: DRE approximated distortion components for a $10^\circ$ squint angle scenario
Figure B.29: DRE approximated distortion components for a $15^\circ$ squint angle scenario

Figure B.30: DRE approximated distortion components for a $20^\circ$ squint angle scenario
Figure B.31: DRE approximated distortion components for a $25^\circ$ squint angle scenario

Figure B.32: DRE approximated distortion components for a $30^\circ$ squint angle scenario
Figure B.33: DRE approximated distortion components for a 35° squint angle scenario

Figure B.34: DRE approximated distortion components for a 40° squint angle scenario
B.3 DPE with Second-Order Corrections Distortion Approximation

Figure B.35: DPE with second-order corrections approximated distortion components for a $-40^\circ$ squint angle scenario
Figure B.36: DPE with second-order corrections approximated distortion components for a $-30^\circ$ squint angle scenario

Figure B.37: DPE with second-order corrections approximated distortion components for a $-20^\circ$ squint angle scenario
Figure B.38: DPE with second-order corrections approximated distortion components for a $-10^\circ$ squint angle scenario

Figure B.39: DPE with second-order corrections approximated distortion components for a $0^\circ$ squint angle scenario
Figure B.40: DPE with second-order corrections approximated distortion components for a 10° squint angle scenario

Figure B.41: DPE with second-order corrections approximated distortion components for a 20° squint angle scenario
Figure B.42: DPE with second-order corrections approximated distortion components for a 30° squint angle scenario

Figure B.43: DPE with second-order corrections approximated distortion components for a 40° squint angle scenario
Appendix C

Distortion Bound Results

This presents a conclusive demonstration of the distortion bounding results for a 790-m stand-off range.

C.1 DPE Distortion Bounds

Figure C.1: DPE distortion bounding result for a $-80^\circ$ squint angle scenario
Figure C.2: DPE distortion bounding result for a $-70^\circ$ squint angle scenario

Figure C.3: DPE distortion bounding result for a $-60^\circ$ squint angle scenario
Figure C.4: DPE distortion bounding result for a $-50^\circ$ squint angle scenario

Figure C.5: DPE distortion bounding result for a $-40^\circ$ squint angle scenario
Figure C.6: DPE distortion bounding result for a $-30^\circ$ squint angle scenario

Figure C.7: DPE distortion bounding result for a $-20^\circ$ squint angle scenario
Figure C.8: DPE distortion bounding result for a $-10^\circ$ squint angle scenario

Figure C.9: DPE distortion bounding result for a $0^\circ$ squint angle scenario
Figure C.10: DPE distortion bounding result for a 10° squint angle scenario

Figure C.11: DPE distortion bounding result for a 20° squint angle scenario
Figure C.12: DPE distortion bounding result for a 30° squint angle scenario

Figure C.13: DPE distortion bounding result for a 40° squint angle scenario
Figure C.14: DPE distortion bounding result for a 50° squint angle scenario

Figure C.15: DPE distortion bounding result for a 60° squint angle scenario
Figure C.16: DPE distortion bounding result for a $70^\circ$ squint angle scenario

Figure C.17: DPE distortion bounding result for a $80^\circ$ squint angle scenario
C.2 DPE Distortion with Second-Order Corrections Bound

Figure C.18: DPE distortion with second-order corrections bounding result for a $-80^\circ$ squint angle scenario
DPE with Second-Order Correction Distortion Bounds: $\theta_s = -70^\circ$

Figure C.19: DPE distortion with second-order corrections bounding result for a $-70^\circ$ squint angle scenario

DPE with Second-Order Correction Distortion Bounds: $\theta_s = -60^\circ$

Figure C.20: DPE distortion with second-order corrections bounding result for a $-60^\circ$ squint angle scenario
Figure C.21: DPE distortion with second-order corrections bounding result for a $-50^\circ$ squint angle scenario

Figure C.22: DPE distortion with second-order corrections bounding result for a $-40^\circ$ squint angle scenario
Figure C.23: DPE distortion with second-order corrections bounding result for a $-30^\circ$ squint angle scenario

Figure C.24: DPE distortion with second-order corrections bounding result for a $-20^\circ$ squint angle scenario
Figure C.25: DPE distortion with second-order corrections bounding result for a $-10^\circ$ squint angle scenario

Figure C.26: DPE distortion with second-order corrections bounding result for a $0^\circ$ squint angle scenario
Figure C.27: DPE distortion with second-order corrections bounding result for a $10^\circ$ squint angle scenario

Figure C.28: DPE distortion with second-order corrections bounding result for a $20^\circ$ squint angle scenario
Figure C.29: DPE distortion with second-order corrections bounding result for a $30^\circ$ squint angle scenario

Figure C.30: DPE distortion with second-order corrections bounding result for a $40^\circ$ squint angle scenario
Figure C.31: DPE distortion with second-order corrections bounding result for a $50^\circ$ squint angle scenario

Figure C.32: DPE distortion with second-order corrections bounding result for a $60^\circ$ squint angle scenario
Figure C.33: DPE distortion with second-order corrections bounding result for a $70^\circ$ squint angle scenario

Figure C.34: DPE distortion with second-order corrections bounding result for a $80^\circ$ squint angle scenario
C.3 DRE Distortion Bound

Figure C.35: DRE bounding result for a $-80^\circ$ squint angle scenario
Figure C.36: DRE bounding result for a $-70^\circ$ squint angle scenario

Figure C.37: DRE bounding result for a $-60^\circ$ squint angle scenario
Figure C.38: DRE bounding result for a $-50^\circ$ squint angle scenario

Figure C.39: DRE bounding result for a $-40^\circ$ squint angle scenario
Figure C.40: DRE bounding result for a $-30^\circ$ squint angle scenario

Figure C.41: DRE bounding result for a $-20^\circ$ squint angle scenario
Figure C.42: DRE bounding result for a $-10^\circ$ squint angle scenario

Figure C.43: DRE bounding result for a $0^\circ$ squint angle scenario
Figure C.44: DRE bounding result for a $10^\circ$ squint angle scenario

Figure C.45: DRE bounding result for a $20^\circ$ squint angle scenario
Figure C.46: DRE bounding result for a $30^\circ$ squint angle scenario

Figure C.47: DRE bounding result for a $40^\circ$ squint angle scenario
Figure C.48: DRE bounding result for a 50° squint angle scenario

Figure C.49: DRE bounding result for a 60° squint angle scenario
Figure C.50: DRE bounding result for a 70° squint angle scenario

Figure C.51: DRE bounding result for a 80° squint angle scenario
Appendix D

Maximum Quadratic Phase Error Plots

In order to identify a dominant dimension to bound defocus, the maximum quadratic phase error was plotted for each scene coordinate. The lines through these images show the choice of bounding dimension.

Figure D.1: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $-80^\circ$ squint angle scenario

Figure D.2: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $-70^\circ$ squint angle scenario
Figure D.3: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $-60^\circ$ squint angle scenario

Figure D.4: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $-50^\circ$ squint angle scenario

Figure D.5: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $-40^\circ$ squint angle scenario

Figure D.6: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $-30^\circ$ squint angle scenario
Figure D.7: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $-20^\circ$ squint angle scenario

Figure D.8: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $-10^\circ$ squint angle scenario

Figure D.9: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $0^\circ$ squint angle scenario

Figure D.10: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $10^\circ$ squint angle scenario
Figure D.11: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a 20° squint angle scenario

Figure D.12: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a 30° squint angle scenario

Figure D.13: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a 40° squint angle scenario

Figure D.14: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a 50° squint angle scenario
Figure D.15: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $60^\circ$ squint angle scenario.

Figure D.16: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $70^\circ$ squint angle scenario.

Figure D.17: Maximum quadratic phase error approximation for uncorrected and corrected DPE approaches at each scene pixel for a $80^\circ$ squint angle scenario.
Appendix E

Defocus Bound Results

This presents a conclusive demonstration of the defocus bounding results for a 790 m stand-off range.

Figure E.1: Results of bounding DPE approximated QPE before and after second-order corrections for a $-80^\circ$ squint angle scenario
Figure E.2: Results of bounding DPE approximated QPE before and after second-order corrections for a $-70^\circ$ squint angle scenario

Figure E.3: Results of bounding DPE approximated QPE before and after second-order corrections for a $-60^\circ$ squint angle scenario
Figure E.4: Results of bounding DPE approximated QPE before and after second-order corrections for a $-50^\circ$ squint angle scenario

Figure E.5: Results of bounding DPE approximated QPE before and after second-order corrections for a $-40^\circ$ squint angle scenario
Figure E.6: Results of bounding DPE approximated QPE before and after second-order corrections for a $-30^\circ$ squint angle scenario

Figure E.7: Results of bounding DPE approximated QPE before and after second-order corrections for a $-20^\circ$ squint angle scenario
Figure E.8: Results of bounding DPE approximated QPE before and after second-order corrections for a $-10^\circ$ squint angle scenario

Figure E.9: Results of bounding DPE approximated QPE before and after second-order corrections for a $0^\circ$ squint angle scenario
Figure E.10: Results of bounding DPE approximated QPE before and after second-order corrections for a $10^\circ$ squint angle scenario

Figure E.11: Results of bounding DPE approximated QPE before and after second-order corrections for a $20^\circ$ squint angle scenario
Figure E.12: Results of bounding DPE approximated QPE before and after second-order corrections for a $30^\circ$ squint angle scenario

Figure E.13: Results of bounding DPE approximated QPE before and after second-order corrections for a $40^\circ$ squint angle scenario
Figure E.14: Results of bounding DPE approximated QPE before and after second-order corrections for a $50^\circ$ squint angle scenario

Figure E.15: Results of bounding DPE approximated QPE before and after second-order corrections for a $60^\circ$ squint angle scenario
Figure E.16: Results of bounding DPE approximated QPE before and after second-order corrections for a 70° squint angle scenario

Figure E.17: Results of bounding DPE approximated QPE before and after second-order corrections for a 80° squint angle scenario