Performance Analysis of Radar Waveforms for Congested Spectrums

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Engineering

by

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ABSTRACT


With more users populating the RF spectrum and hence less available contiguous bandwidth, radar and communication waveforms are slowly forced to become more efficient at using their available frequencies. Two scenarios are considered: operation in a colored interference environment and operation in discontiguous spectral bands. Unconstrained algorithms for designing transmit waveforms and receive filters are evaluated, wherein varying a convex weight trades performance between spectral flatness and side lobe levels. An empirical study provides performance bounds for constrained radar waveform designs for an instantiation of the interference spectrum.

Closed-form predictions for integrated sidelobe ratio (ISLR) and peak-to-sidelobe ratio (PSLR) for radar waveforms designed to operate in discontiguous spectral bands are derived and validated against two spectrally-disjoint waveform designs. These spectrally-disjoint waveform designs must also consider constraints imposed by hardware, such as modulus and phase restrictions. In the final part of this thesis, four spectrally-disjoint waveform designs are subjected to hardware-in-the-loop tests. Experimental results are shown and compared to computer simulations.
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Dedicated to

Cooper Walter Frost
Chapter 1

Introduction

With more users populating the RF spectrum and hence less available contiguous bandwidth, radar and communication waveforms are slowly forced to become more efficient at using their available frequencies. In particular, VHF and UHF regions tend to contain many strong emitters and severely limit the available contiguous frequency bands for radar applications, such as foliage penetration (FOPEN). Many [1–12] have investigated and proposed tools for confronting this issue. These include the design of waveforms to avoid the interference and the design of receive filters to further reject noise while maintaining a clean system impulse response. Also included are waveforms and filters designed to avoid specific frequency bands, referred to here as spectrally-disjoint waveforms. The immaturity of radar waveform designs for congested spectrums serves as primary motivation for this thesis. The performance, and metrics for quantifying performance, of radar waveforms for congested spectrums is not well understood.

A waveform designed to avoid interference can be characterized by metrics like signal-to-interference-plus-noise ratio (SINR) for various assumed interference covariance matrices [13]. A waveform designed to avoid transmitting in specific bands, a spectrally-disjoint waveform, must be characterized using other metrics since interference is not driving the design, and thus, no such SINR can be calculated. Such metrics include average power levels in the undesired frequency bands, peak sidelobe levels, and integrated sidelobe levels.

Metrics such as these can be useful to adaptive radar systems making waveform design deci-
sions in congested frequency spectrums. An adaptive system might sense the spectrum and then make one of the following waveform design decisions: (a) Transmit between the sources of interference (choose the largest open band, current technology) (b) Transmit over the sources of interference (if they are not authorized users/sources in that band) (c) Transmit a spectrally-disjoint waveform, that occupies more than one of the available bands simultaneously (Higher range sidelobes may be detrimental to the radar’s performance).

1.1 Contribution

It is imperative that we understand the cost and capabilities of designing radar waveforms for congested spectrums. Radar systems must learn to become spectrally cooperative in environments with many RF users. In this thesis, we investigate both types of radar waveform designs for congested spectrums. Joint waveform/filter optimizations with direct ACS constraints have been previously considered by [14]. Constrained waveform and filter design methodologies that consider maximizing SINR while minimizing integrated or peak sidelobe ratios using penalty methods are considered in this thesis. In [15], performance bounds on generalized integrated sidelobe levels are derived. We introduce a Bernoulli model for the desired spectrum of a spectrally-disjoint waveform, then analyze the range sidelobe performance of these radar waveforms and derive closed form predictions for integrated sidelobe ratio (ISLR) and peak-to-sidelobe ratio (PSLR). Lastly, waveform and filter designs that are specified by usable and unusable frequencies for transmission are considered and subjected to hardware-in-the-loop tests.

1.2 Outline

The remainder of this thesis is outlined as follows. Chapter 2 presents previous work in radar waveform designs for congested spectrums. Chapter 3 considers unconstrained algorithms for jointly designing complex digital transmit waveforms and receive filters. Varying a convex weight trades
performance between signal-to-interference-plus-noise ratio (SINR) and sidelobe levels. Through an empirical study, performance bounds for constrained radar waveform design is established for an instantiation of the interference spectrum. Chapter 4 delivers closed-form predictions for integrated sidelobe ratio (ISLR) and peak-to-sidelobe ratio (PSLR) for spectrally-disjoint radar waveforms. This is done using stochastic models, and is validated against two spectrally-disjoint waveform designs. Chapter 5 investigates four spectrally-disjoint waveform designs. Experimental results are shown for multiple scenarios and validated using computer simulations. Lastly, Chapter 6 concludes with summary remarks about the performance of radar waveform designs for congested spectrums.
Chapter 2

Previous Work

Two scenarios of waveform design for congested spectrums are considered: operation in a colored interference environment and operation in discontiguous spectral bands. Although each scenario is closely related, they are fundamentally different problems that require different metrics for performance characterization.

2.1 Operation in Colored Interference

Waveform optimization for interference and clutter suppression has a mature and detailed history. A detailed review can be found in [14], but highlights pertaining more to spectral congestion are covered here.

In [14], a computationally expensive framework for waveform optimization in colored interference with direct cross-correlation sequence constraints is presented. A joint waveform/filter design exhibits promising results for waveform optimization in interference dominant environments, when computational burden is not a factor. A more computationally efficient design in [16] uses Fourier series phase perturbations to indirectly constrain the ACS while maximizing detection in colored interference. In [7], and extended in [8], small phase perturbations were applied to stepped-frequency and linear-frequency-modulated waveforms to generate frequency nulls at the known locations of interference. Others, most cited within [2], have approached congested spectrum waveform design
with frequency notching and interference suppression techniques on receive. These included the work of [10, 17, 18], where tones of interference are estimated and subtracted out. A few [5] have investigated PSD approximation methods for optimal waveform lookup techniques, when waveform optimizations cannot be computed online.

2.2 Operation in Discontiguous Spectral Bands

The following work considers designing a transmit waveform that minimizes transmit energy in specific frequency bands, rather than designing a waveform that is avoiding interference, while maintaining desirable envelope and sidelobe characteristics.

Spectrally-disjoint radar waveforms for ultra-wide bandwidth (UWB) synthetic aperture radars applied to FOPEN have been previously investigated by [2]. This transmit waveform design employs an iterative algorithm to minimize spectral energy in specified frequency bands, while the receive filter design solves a constrained minimization of two objective functions. One penalizes spectral energy in the previously specified frequency bands, while the other minimizes receive sidelobes caused by frequency filtering. This approach provided computationally inexpensive waveforms with good spectral properties, but at the expense of a relatively poor PSLR and ISLR.

Others have examined iterative algorithms to alternate between optimizing control of spectral power and integrated sidelobes [6], while [4] used approximations of an ISL cost function term for more computationally efficient solutions. A convex weighting scheme similar to what is used in this thesis is employed in [1]\(^1\), but uses the approximated ISL penalty term from [4]. Authors in [19] not only demonstrate a spectrally modulated, spectrally encoded radar waveform framework for spectral congestion, but deliver one of the first metrics to address performance of spectrally-disjoint radar waveforms. Bounds on generalized integrated sidelobe level (GISL) for spectrally-disjoint waveforms with arbitrary average-pass-to-stopband power ratio (APSPR) are delivered in [15]. The design in [11] generates a constant envelope waveform with a prescribed discrete Fourier transform magnitude with piece-wise linear phase. A waveform with piece-wise linear phase approximates a

\(^1\)The work in [1] was published in January 2011, two months after our IEEE Radar Conference article was submitted for publication.
linear frequency modulated chirp waveform (LFM) that has quadratic phase. The authors in [12] extended their algorithm in [20] to use an iterative algorithm to design another chirp-like waveform that has multiple frequency nulls. Four designs, [2,11,12,14] are subjected to analysis in this thesis and are summarized below.

2.2.1 Gradient Descent-Based Approach

First we consider the waveform and mismatch receive filter design from [2]. The transmit waveform is defined as, \( s(n) = \exp\{j\psi(n)\} \), for \( n = 0, \ldots, N - 1 \), where \( \psi(n) \) is the phase of the waveform at time \( n \). The unusable frequency bands are described in the matrix \( R \),

\[
R_{mn} = \sum_{p=1}^{N_b} w_p \begin{cases} \frac{\exp(\alpha f_{p2} \Delta t) - \exp(\alpha f_{p1} \Delta t)}{\alpha \Delta t} & m \neq n \\ f_{p2} - f_{p1} & m = n \end{cases}
\]  
(2.1)

where \( \alpha = 2\pi i (m - n) \) and \( N_b \) is the number of unusable frequency bands, with the \( p^{th} \) band between \( f_{p1} \) and \( f_{p2} \), and \( w_p \) is a weight relative to the other bands greater than zero. The cost function for the transmit waveform design is written as

\[
J_T = s^H R s.
\]  
(2.2)

The receive filter, \( h \), uses a convex weight to minimize range sidelobes and receiving energy from the specified frequency bands. The filter is computed analytically using

\[
h = \Phi^{-1} s (s^H \Phi^{-1} s)^{-1},
\]  
(2.4)
where

$$\Phi = \beta \Gamma + (1 - \beta) R + \lambda I, \quad (2.5)$$

$$\Gamma = S^H S.$$ 

and

$$S = \begin{bmatrix}
s(1) & 0 & \ldots & 0 \\
s(2) & s(1) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
s(N) & s(N-1) & \ldots & s(1) \\
0 & 0 & 0 & s(N)
\end{bmatrix}. $$

For stable inverse computations, the diagonal loading term, $\lambda$, is set to $10^{-5}$.

### 2.2.2 Alternating Projections

Next we consider the alternating projections algorithm outlined in [14]. Given desired time domain amplitudes, $z$, and a desired Fourier Transform Magnitude (FTM), $q$, find a $x$ with $|x| = z$ that minimizes the cost function $J$:

$$J(x) = |||x| - q||^2, \quad (2.6)$$

This discrete-time spectrum shaping problem becomes that of solving the optimization problem

$$\inf_{s \in U} J(s) \quad (2.7)$$

where the image and object domain constraint sets are defined

$$U := \{ x \in \mathbb{C}^N : |x| = z \} \quad \text{and}$$

$$\tilde{V} := \{ \tilde{v} \in \mathbb{C}^N : |\tilde{v}| = q \}.$$
Each entry of the vector $s$ is defined by $s(n) = z(n) \exp\{j\psi(n)\}$ where $\psi(n) \in \mathbb{R}$ for all $n$. Thus, the constrained optimization problem in (2.6) can be converted into an unconstrained problem by minimizing $J(s)$ with respect to $\psi = [\psi(1) \ldots \psi(N)]^T$ instead of $s$. Our final complex waveform, $s$, is close to achieving our desired FTM, but is constant modulus. Our receive filter, $h$, is a matched filter. The alternating projections approach is outlined in [14] and is repeated in Algorithm 1 for convenience.

Algorithm 1 Alternating Projections [14]

1: $\phi \leftarrow$ uniform random over $[0, 2\pi)$
2: $s \leftarrow z \odot \exp\{j\psi\}$
3: $J_{new} \leftarrow \| |F_s| - q \|^2$
4: while $|J_{old} - J_{new}| > \epsilon$ do
5: $\tilde{v} \leftarrow q \odot \exp\{j\angle F_s\}$
6: $v \leftarrow F^H \tilde{v}$
7: $\psi \leftarrow \angle v$
8: $s \leftarrow z \odot \exp\{j\psi\}$
9: $J_{old} = J_{new}$
10: $J_{new} = \| |F_s| - q \|^2$
11: end while

2.2.3 Piece-wise Linear Phase Waveform

The third algorithm [11] uses discrete values of a Fourier transform magnitude to develop a piece-wise linear phase waveform with a constant envelope. The piece-wise linear phase is to approximate a quadratic phase, like that of the linear frequency modulated (LFM) chirp waveform.

Given a magnitude function $P(\omega)$, the authors design a constant envelope signal $s(t)$ whose Fourier transform magnitude $|F(\omega)|$ approximately matches $P(\omega)$. The transmit waveform is specified to be constant modulus over the intervals $0 < t < T$, such that

$$s(t) = z \exp(j\psi(t)), \quad 0 < t < T,$$ (2.8)
where $\psi(t)$ is the phase function to be designed and $z$ is the amplitude constant. We can represent $s(t)$ using a Fourier series on the interval $(0, T)$,

$$s(t) = \sum_{n=-\infty}^{+\infty} a(n) \exp\{jn\omega_0 t\},$$

(2.9)

where $\omega_0 = 2\pi/T$, and the Fourier coefficients are

$$a(n) = \frac{1}{T} \int_{0}^{T} s(t) \exp\{-jn\omega_0 t\} dt = F(n\omega_0) = \frac{z}{T} \int_{0}^{T} \exp\{j(\psi(t) - n\omega_0 t)\} dt.$$

Since $z(n) = F(n\omega_0) = |F(n\omega_0)| \exp\{j\theta(n)\}$, a necessary condition for the Fourier transform magnitude of $s(t)$ to match the desired magnitude function $P(\omega)$, is the Fourier coefficients must have the form $z(n) = P(n\omega_0) \exp\{j\theta(n)\}$ for a suitable phase $\theta(n)$. The phase function is further restricted to be monotonically increasing and piece-wise linear. The authors also generate a similar waveform whose Fourier transform magnitude $|F(\omega)|$ exactly matches $P(\omega)$, and has a near constant envelope, but that will not be studied here.

### 2.2.4 Chirp-like Waveform with Notches

The final design [12] iteratively creates a chirp-like spectrally-disjoint waveform using multiple frequency nulls. The resulting waveform resembles the waveform at which it was initialized, $x(t)$, where we choose the LFM,

$$x(t) = \exp\{j(2\pi f_0 t + \beta t^2)\}$$

with instantaneous frequency $2\pi f_0 t + \beta t^2$. The authors describe the iterative algorithm using a discrete-time formulation and orthogonal projection matrices. The DTFT of our desired waveform, $s(n)$, is written as

$$S(f) = \sum_{n=0}^{N-1} s(n) \exp\{-j2\pi fn\}.$$
Given $K$ notch frequencies, $f_k$ for $1 \leq k \leq K$, the spectrum should satisfy $Y(f_k) = 0$, or equivalently, $C^H s = 0$, where $C$ is the $N \times K$ constraint matrix

$$C_{n,k} = \exp\{j2\pi f_k n\}$$

for $0 \leq n \leq N - 1$ and $1 \leq k \leq K$. The orthogonal projection onto the complement of the subspace is given by $Px$ where $P = I - C (C^H C)^{-1} C^H$ and $I$ is the identity matrix. Therefore, given an initial waveform $x$, $s = Px$ is the waveform closest to $x$ satisfying the null frequency constraints. To eliminate ill conditioning, the projection matrix can be written as

$$P = I - QQ^H$$

where $Q$ is the orthogonalization of the columns of $C$. The iterative algorithm is summarized in Algorithm 2.

**Algorithm 2 Chirp-like Waveform with Multiple Frequency Nulls [12]**

1: $s(n) \leftarrow z(n) \exp\{j (2\pi f_0 n + \beta n^2)\}, \quad 0 \leq n \leq N - 1$
2: while $k < M$ do
3: \hspace{1em} $s \leftarrow s - QQ^H s$
4: \hspace{1em} $s(n) \leftarrow z(n) \exp\{j \angle s(n)\}, \quad 0 \leq n \leq N - 1$
5: \hspace{1em} $k = k + 1$
6: end while

Where $M$ is the number of iterations specified, $z(n)$ is the desired waveform amplitude, $f_0$ is the initial frequency of the chirp and $\beta$ is the chirp parameter.
Chapter 3

Waveform Design Methodologies Among Colored Interference

We compare the performance of disjoint waveform/filter design to joint waveform/filter design, when optimizing SINR and constraining PSLR/ISLR. The matched filter receiver is included as a performance baseline. Radar waveforms with a low ISLR are usually desired to prevent masking weak targets, while waveforms with a high PSLR can generate false alarms. Similar constrained waveform optimization designs [14] have shown promising results for maintaining desirable sidelobe characteristics, but at computational expense. Here, we formulate a constrained waveform design problem that trades SINR performance and sidelobe levels through variation of a penalty weight, $\beta$.

3.1 Waveform Design Methodologies

We slightly modify the matrix in (2.1) to model the interference and noise environment. The interference-to-noise ratio (INR), dictated by the variance of the noise $\sigma^2$, instantiates an inter-
ference spectrum in the matrix $\mathbf{R}$:

$$
\mathbf{R}_{mn} = \sum_{p=1}^{N_b} \begin{cases} 
\frac{\exp(\alpha f_p \Delta t) - \exp(\alpha f_1 \Delta t)}{\alpha \Delta t} & m \neq n \\
(f_{p2} - f_{p1}) + \sigma^2 & m = n.
\end{cases}
$$

(3.1)

Prior to adding $\sigma^2$, $\mathbf{R}$ must be normalized by its maximum value. For the $p^{th}$ band of interference, $\text{INR}_p = 1/\sigma^2$. We select frequency bands, defined by $f_{p1}$ and $f_{p2}$, to specify an interference spectrum.

A matched filter design, disjoint design, and a joint waveform/filter design are considered for performance comparison. The first design, a matched filter approach, provides a baseline. The second approach utilizes a mismatched filter, to better trade SINR for lower sidelobes with separately designed waveform and receive filter. One might desire lower sidelobe levels than either design approach provides, and this motivates a joint waveform/filter design. Computational complexity increases with each step of the progression.

As mentioned, two design metrics are used for performance characterization. SINR and ISLR are defined respectively with the receive filter represented by $\mathbf{h}$:

$$
F_{\text{snr}}(s, h) = \frac{|s^H \mathbf{h}|^2}{\mathbf{h}^H \mathbf{R} \mathbf{h}},
$$

$$
F_{\text{islr}}(s, h) = \frac{\sum_{\Theta} |r_{s,h}(k)|^2}{\sum_{\Lambda} |r_{s,h}(k)|^2}.
$$

(3.3)

The cross-correlation sequence (XCS) of the waveform and filter at lag $k$ is computed as

$$
r_{s,h}(k) = \frac{1}{N} \sum_{n=0}^{N-1} s(n) h^*(n - k)
$$

(3.4)

while $\Theta := \{0 \leq k \leq r(0) - \tau\} \cap \{r(0) + \tau \leq k \leq 2N - 1\}$ defines the lags corresponding to the sidelobes of the XCS, $\Lambda := \{r(0) - \tau + 1 \leq k \leq r(0) + \tau - 1\}$ defines the lags for the mainlobe, and $\tau$ denotes half the width of the mainlobe, centered on $r(0)$. A second performance metric
measures PSLR and is defined as

$$F_{\text{pslr}}(s, h) = \left| \frac{r_{s,h}(0)}{r_{s,h}(l)} \right|^2$$

where, $r_{s,h}(l) = \max_{k \in \Theta} |r_{s,h}(k)|^2$. (3.5)

### 3.1.1 Matched Filter Design

The matched filter based design uses an optimization algorithm to design the transmit waveform $s(\hat{\psi}_{mf}) = e^{j\hat{\psi}_{mf}}$:

$$\hat{\psi}_{mf} = \arg \min_{\psi} F_{\text{snr}}(s, s) + \beta \left[ (1 - \gamma) F_{\text{islr}}(s, s) + \gamma F_{\text{pslr}}(s, s) \right].$$ (3.6)

The phase vector $\psi$ is initialized for each element on $U[0, 2\pi]$. Varying $\beta$ constrains the algorithm’s ability to maximize SINR, while controlling ISLR or PSLR. Choosing $\gamma = 0$ will penalize increased ISLR, while choosing $\gamma = 1$ will penalize increased PSLR.

### 3.1.2 Disjoint Mismatched Filter Design

The disjoint design also uses an optimization algorithm to design the transmit waveform $s(\hat{\psi}_{dd})$:

$$\hat{\psi}_{dd} = \arg \min_{\psi} F_{\text{snr}}(s, s) + \beta \left[ (1 - \gamma) F_{\text{islr}}(s, s) + \gamma F_{\text{pslr}}(s, s) \right].$$ (3.7)

The phase $\hat{\psi}_{dd}$ is identical to the matched filter designed phase, $\hat{\psi}_{mf}$, within numeric precision. After designing the transmit waveform, we design a receive filter $h$ with the transmit waveform fixed from the previous design:

$$\hat{h}_{dd} = \arg \min_{\psi, h} F_{\text{snr}}(s, h) + \beta \left[ (1 - \gamma) F_{\text{islr}}(s, h) + \gamma F_{\text{pslr}}(s, h) \right].$$ (3.8)

The receive filter $\hat{h}_{dd}$ is initialized with the phase only design, $\exp \left\{ j\hat{\psi}_{dd} \right\}$. While $\hat{h}_{dd}$ is initialized with a constant modulus, it is not restricted in the optimization to a phase-only design.
3.1.3 Joint Design Algorithm

The jointly designed waveform/filter pair, $s(\hat{\psi}_{jd})$ and $h_{jd}$, trades computational complexity for improved sidelobe suppression:

$$\hat{\psi}_{jd}, \hat{h}_{jd} = \arg \min_{\psi, h} F_{sn}(s, h) + \beta [(1 - \gamma)F_{isl}(s, h) + \gamma F_{psl}(s, h)] .$$  (3.9)

The transmit waveform and receive filter are initialized with the matched filter solution. We use the same weight in this design and the same ISLR, or PSLR, cost function term as the disjoint mismatched filter design.

3.2 Results

The simulation scenario used here contains results from 64 Monte Carlo trials, where the initial phase vectors for the matched filter and disjoint designs are randomly generated. Random initializations allow us to compute the average SINR, ISLR, and PSLR from diverse points on the non-convex cost function, rather than iteratively solving for a local minimum. Only the convex weights, $\beta$, that portray the most significant performance trades are shown. Each optimization problem was solved using the unconstrained minimization function in the MATLAB Optimization Toolbox. Interference covariance matrix $R$ was assumed known with $N = 64$.

3.2.1 Penalizing ISL

Each choice of $\beta$ can be viewed as corresponding to a specific constraint on ISLR. By varying $\beta$ in Monte Carlo simulations, we are able to plot average SINR-ISLR operating points for each INR, for each algorithm, and for this particular interference spectrum. Three INRs are considered for performance comparison plots. INR = -20dB as a noise dominant environment, INR = 0dB where neither interference nor noise are dominant, and INR = 20dB representative of an interference dominant environment. The interference power spectrum, corresponding to $R$ in (3.1), for the interference dominant scenario is outlined in blue in Figure 3.1.
Figure 3.1: Power Spectral Density (PSD) of the assumed interference

Figure 3.2: Varying $\beta$ trades SINR performance for improved ISL.

As shown in Figure 3.2, in the noise dominant case (INR = $-20$dB), almost all three algorithms perform equally, showing that there is little SINR to be gained over the matched filter. In
which case, $\beta$ should be selected close to one to minimize ISLR, or waveform optimization should be avoided. In the interference dominant case (INR = 20dB), the disjoint and joint designs can achieve 3dB and 5dB SINR gains respectively, over the matched filter for an ISLR of -13dB. This performance improvement is subject to an increase in computations. The matched filter average run time was 2.253 seconds, the disjoint-design was 3.097 seconds, and the joint-design was 6.321 seconds for this scenario.

![Power of transmit waveforms, INR = 20dB, small $\beta$](image)

Figure 3.3: Power of transmit waveforms, INR = 20dB, small $\beta$
In Figs. 3.3 and 3.4 respectively, the power spectrum and cross-correlation sequences are shown for a small value of $\beta$. A smaller choice of $\beta$ emphasizes SINR gain at the expense of ISLR. At this SINR-ISLR operating point, the ISLs of each design are almost equal, and the joint design can only achieve a negligible SINR gain.

As seen in Figs. 3.5 and 3.6, the designs can achieve much lower ISLRs by allowing more energy into the bands with interference, thus reducing SINR. As the SINR-ISLR operating point moves to lower ISLRs, larger values of $\beta$, the disjoint and joint designs have more SINR to gain. When the designs achieve an ISLR $= -13$dB, the disjoint design provides approximately 2.5dB of SINR gain, while the joint design provides approximately 5.5dB of SINR gain, over the matched filter solution.
Figure 3.5: Power of transmit waveforms, INR = 20dB, large $\beta$

Figure 3.6: XCS, INR = 20dB, large $\beta$
This SINR-ISLR performance characterization provides some insight as to what waveform designs are most effective for different interference spectrum instantiations. In general, we can see how waveform diversity, design, and optimization, can be beneficial radar waveform design in an interference dominant environment. Although some operating points provide promising results, many applications desire a lower PSLR, rather than ISLR, and this motivates an SINR-PSLR performance characterization.

3.2.2 Penalizing PSLR

The simulation scenario used here is exactly as outlined in the previous section, only now we explicitly trade PSLR for SINR.

Figure 3.7: Varying $\beta$ trades SINR performance for improved PSLR

Figure 3.7 shows similar relationships between the algorithms and interference. In the interference dominant case (INR = 20dB), the disjoint and joint designs still achieve a small SINR gain and PSLR reduction, but with an increase in computations. No SINR gain is achieved in the noise-dominant case (INR = −20dB), so $\beta$ should be large to minimize PSLR, or waveform optimization.
should be avoided. This would suggest that in a noise dominant environment, waveform diversity, design, and optimization provides little to no benefit. The matched-filter average run time was 1.207 seconds, the disjoint-design was 2.361 seconds, and the joint-design was 3.129 seconds for this scenario.

Figure 3.8: Power of transmit waveforms, INR = 20dB, small $\beta$
As seen in Figs. 3.8 and 3.9, emphasis remains on SINR for small values of $\beta$. As $\beta$ increases to directly minimize PSLR, each design is forced to compromise SINR. Figs. 3.10 and 3.11 show that for one $\beta$, the joint design achieves moderate sidelobe levels with minimal SINR degradation, while the other designs are forced to completely sacrifice SINR.
Figure 3.10: Power of transmit waveforms, INR = 20dB, large $\beta$

Figure 3.11: XCS, INR = 20dB, large $\beta$
Chapter 4

Sidelobe Predictions for
Spectrally-Disjoint Radar Waveforms

Spectrally-disjoint radar waveforms seek to minimize the amount of energy in a set of frequency bands within the bandwidth of the waveform. Such a waveform will suffer higher range sidelobes than a waveform with a contiguous bandwidth. As we continue to develop these waveforms, it becomes necessary to establish PSLR and ISLR predictions in order to objectively evaluate performance. Predictions such as these can aid in the decision making process of an adaptive radar.

4.1 Spectrum Model

If we redefine the transmit waveform as, \( s(n) = z(n) \exp \{ j \psi(n) \} \), for \( n = 0, \ldots, N - 1 \), where \( z(n) \) and \( \psi(n) \) are the amplitude and phase respectively, of the waveform at time \( n \), then the power spectral density (PSD), \( \phi(m) \), of that waveform can be written as the Fourier transform

\[
\phi(m) = \sum_{k = -(N-1)}^{N-1} r(k) \exp \left\{ -j \frac{2\pi mk}{2N-1} \right\} = \mathcal{F} \{ r(k) \},
\]
where \( r(k) \) is the ACS of the waveform at lag \( k \):

\[
    r(k) = \frac{1}{N} \sum_{n=0}^{N-1} s(n)s^*(n-k) = \mathcal{F}^{-1}\{\phi(m)\}
\]  

(4.2)

To establish analytic PSLR/ISLR predictions for a given usable bandwidth, we choose a Bernoulli process to model the usable and unusable frequencies. This means that each frequency of an \( M \)-point PSD, \( \phi(m) \), is either zero, meaning it is unusable, or one, meaning it is usable. This defines a probability mass function

\[
P\{\phi(m)\} = \begin{cases} 
    p, & \phi(m) = 1 \text{ ("usable"}) \\
    1 - p, & \phi(m) = 0 \text{ ("unusable"})
\end{cases}
\]

(4.3)

for \( m = 0, \ldots, M - 1 \), and we choose \( M = 2N - 1 \). This allows us to model the waveform’s PSD as a function of usable bandwidth, where each frequency has probability \( p \) of being usable. This is equivalent to saying the PSD has a spectral efficiency \( p \), defined in [1] as the ratio of usable frequencies, \( M_p \), to complete PSD samples, \( M \):

\[
p = \frac{M_p}{M}.
\]

(4.4)

Most radar systems desire waveforms with an ACS possessing a narrow mainlobe and low sidelobes. All lags from lag zero out to the first local minimum of the ACS comprise the mainlobe, while lags outside the first local minimum define the sidelobes. In practice, we frequently seek to control ISLR and PSLR. A radar system would desire a low integrated sidelobe ratio to prevent masking weak targets. Our definition of ISLR is modified with an expectation, to adopt our definition of the PSD:

\[
    \text{ISLR} \approx \frac{\mathbb{E}\left\{\sum_{\Theta} |r(k)|^2\right\}}{\mathbb{E}\left\{\sum_{\Lambda} |r(k)|^2\right\}}
\]

(4.5)

A radar system trying to minimize false alarms would desire an ACS with a low peak-to-sidelobe ratio. To quantify PSLR for our assumed PSD model, further examination of the autocorrelation sequence is required.
To characterize the ACS, we examine the first and second moments of the ACS defined in (4.2) with a PSD defined by (4.3). We can compute the mean of the ACS, $\bar{r}(k)$, as

$$\bar{r}(k) = E \{r(k)\}$$

$$= \frac{1}{M} \sum_{m=0}^{M-1} E \{\phi(m)\} \exp \left\{ j\frac{2\pi mk}{M} \right\}$$

$$= \begin{cases} 
  p & k = 0 \\
  0 & k \neq 0
\end{cases} \quad (4.6)$$

and the variance of the ACS, $\sigma^2_r(k)$, as

$$\sigma^2_r(k) = E \{|r(k) - \bar{r}(k)|^2\}$$

$$= E \left\{ \left| \frac{1}{M} \sum_{m=0}^{M-1} (\phi(m) - E \{\phi(m)\}) \exp \left\{ j\frac{2\pi km}{M} \right\} \right|^2 \right\}$$

$$= \frac{1}{M^2} \sum_{m=0}^{M-1} \sum_{l=0}^{M-1} E \left\{ [\phi(m) - \bar{r}(k)] [\phi(l) - \bar{r}(k)] \right\} \exp \left\{ j\frac{2\pi k(m - l)}{M} \right\}$$

$$= \frac{1}{M^2} \sum_{m=0}^{M-1} \sum_{l=0}^{M-1} p(1 - p) \delta_{m,l}$$

$$= \frac{1}{M^2} \sum_{m=0}^{M-1} p(1 - p)$$

$$= \frac{p(1 - p)}{M} \quad (4.7)$$

To validate the variance prediction, we generated 500 Bernoulli sequences, each of length $M = 511$, for varying values of $p$. For each, we recorded the variance and averaged, to get an average variance for each value of $p$. Figure 4.1 shows the average variances in black and our predicted variance in purple.
Figure 4.1: Eqn. (4.7) variance prediction for \( p = [0.2, 0.9] \) and \( M = 511 \). The black curve represents the averaged variance for each spectral efficiency for 500 Monte Carlo trials of generating a PSD from a Bernoulli distribution.

We can predict the ISLR for a waveform with a PSD as in (4.3), by utilizing the first and second moments. Breaking down our definition of ISLR shows that integrating the sidelobe variance and normalizing by the ACS peak squared is approximately equivalent to ISLR:

\[
\text{ISLR}(p) \approx \frac{\mathbb{E}\left\{ \sum_{\Theta} |r(k)|^2 \right\}}{\mathbb{E}\left\{ \sum_{\Lambda} |r(k)|^2 \right\}} = \frac{(M - 1) \sigma_r(k)^2}{r(0)^2} = \frac{(M - 1) (1 - p)}{Mp} \tag{4.8}
\]

This is within a factor of \((M - 1)/M\) of the generalized integrated sidelobe level bound derived in [1] for the high average pass-to-stopband power ratio (APSPR) case. To verify this we ran the same Bernoulli experiment described for Figure 4.1, but recording the ISLR of the ACS rather than the variance. Figure 4.2 shows that the average ISLR of a spectrally-disjoint waveform with a PSD like (4.3), follows our predicted ISLR (4.5), and concurs with the GISL bound from [1].
Figure 4.2: ISLR predictions for $p = [0.2, 0.9]$ and $M = 511$. The green curve is the GISL bound [1] where APSPR = 1000. The black curve represents the averaged ISLR for each spectral efficiency for 500 Monte Carlo trials of generating a PSD from a Bernoulli distribution.

When $M$ is large, the statistics of the sidelobes of the ACS are well represented by a Gaussian distribution, $N(\bar{r}(k), \sigma^2_r(k))$, according to the Central Limit Theorem [21], as shown in Figure 4.3 and 4.4. These figures were generated by fixing $p$, generating 500 Bernoulli sequences, and storing the ACS of each Bernoulli sequence. The mainlobe region was removed from each ACS, then the sidelobes were concatenated to form one large vector of sidelobes for a particular value of $p$. Histograms were computed using the real and imaginary parts of the sidelobes, and are shown in blue. A Gaussian PDF was estimated and fitted using the sidelobes, and plotted in red.
Figure 4.3: Real part of ACS sidelobes for 500 Monte Carlo trials compared to a Gaussian PDF when $p = 0.6$ and $M = 511$.

Figure 4.4: Imaginary part of ACS sidelobes for 500 Monte Carlo trials compared to a Gaussian PDF when $p = 0.6$ and $M = 511$. 
The squared magnitude of the sidelobes may then be well represented by an exponential distribution,

\[ f(x; \lambda) = \begin{cases} 
\lambda \exp \{-\lambda x\}, & x \geq 0 \\
0, & x < 0 
\end{cases} \]  

with \( \lambda = 1/2\sigma^2(k) \) [21]. This is verified by computing the histogram of the squared magnitude of the sidelobes from Figs. 4.3 and 4.4, and is shown in blue in Figure 4.5. An exponential PDF is also fitted and shown in red.

![Figure 4.5: Magnitude squared of ACS sidelobes for 500 Monte Carlo trials compared to a Exponential PDF, \( E(\lambda) \), when \( p = 0.6 \) and \( M = 511 \).](image)

The maximum of a sequence of standard (\( \lambda = 1 \)) exponentially distributed random variables is asymptotically (\( N \rightarrow \infty \)) represented with the Gumbel distribution (Fisher-Tippett Type I) [22–24],

\[ f(x; \mu, \beta) = \frac{1}{\beta} \exp \left\{ -\left( \frac{x - \mu}{\beta} \right) - \exp \left\{ -\left( \frac{x - \mu}{\beta} \right) \right\} \right\}, \]
with location parameter $\mu = 0$ and scale parameter $\beta = 1$. For finite $M$, we observe that the maximum of a sequence of exponentially distributed random variables follow a Gumbel distribution with $\mu = 1/M$ and $\beta = 1/\lambda = 2\sigma_r^2(k)$. Therefore, we hypothesize the maximum of the magnitude of the sidelobes follows the Gumbel distribution, $f(x; 1/M, 2\sigma_r^2(k))$, and we can predict the PSLR by using the mean of the Gumbel distribution [23], $\mu + \gamma\beta$, and normalizing by the ACS peak squared,

$$
\text{PSLR} \approx \left[ \mu + \gamma\beta \right] \frac{1}{p^2} \\
= \left[ \frac{1}{M} + \gamma 2\sigma_r^2(k) \right] \frac{1}{p^2} \\
= \left[ \frac{1}{M} + 2\gamma p(1 - p) \right] \frac{1}{p^2} \\
= \frac{1}{M} \frac{1}{p^2} + 2\gamma \frac{1 - p}{Mp} \quad (4.10)
$$

where $\gamma = .5772$ (Euler’s Constant).

To verify this, we ran the same Bernoulli experiment described for Figure 4.1, but recording the PSLR of the ACS rather than the variance. Figure 4.6 shows the PSLR prediction, and the average PSLR of a Bernoulli sequence modeled PSD for 500 Monte Carlo trials. Figure 4.6 shows the PSLR prediction to work well for all values of $p$.

\footnote{We arrived at this $\mu$ and $\beta$ by examining the asymptotic relationship between the standard exponential distribution and the standard Gumbel distribution in [22] and verifying each empirically.}
Figure 4.6: Eqn. 4.10, PSL predictions for $p = [0.2, 0.9]$ and $M = 511$. The black curve represents the averaged PSL for each spectral efficiency for 500 Monte Carlo trials of generating a PSD from a Bernoulli distribution.

4.2 Sidelobe Prediction Comparison

We perform a Monte Carlo simulation of the alternating projections algorithm in section 2.2.2 and a modified version of the gradient descent-based waveform/filter design in section 2.2.1, to compare to the PSLR and ISLR predictions.

Previously we investigated the algorithm in [2], to examine the bounds of unconstrained radar waveform design. We modified $R$ to dictate interference and noise statistics. To adopt our PSD model from 4.3, we define $R$ as

$$
R = \frac{1}{r^{(1)}} \begin{bmatrix}
  r(1) & r(2) & \ldots & r(N) \\
  r^*(2) & r(1) & \ldots & r(N-1) \\
  \vdots & \vdots & \vdots & \vdots \\
  r^*(N) & r^*(N-1) & \ldots & r(1)
\end{bmatrix}.
$$

(4.11)
where,

\[ r(k) = \frac{1}{2N-1} \sum_{k=-N+1}^{N-1} (1 - \phi(m)) \exp \left\{ -j \frac{2\pi mk}{2N-1} \right\} = \mathcal{F}^{-1} \{ \nu - \phi \}, \]  \hspace{1cm} (4.12)

and \( \nu \) is a length \( M \) vector of ones. By subtracting the Bernoulli sequence from a vector of ones, we model the unusable frequency bands in the statistics of \( R \). We then use the rest of the algorithm in \[2\] to design a spectrally-disjoint waveform and receive filter.

Each Monte Carlo trial is a new length \( M \) random Bernoulli sequence, \( \phi \), with spectral efficiency \( p \). The Bernoulli sequence represents the desired spectrum for each design.

### 4.2.1 Simulation Results

To exhaustively characterize the average PSLR and ISLR of the waveforms, we vary spectral efficiency, \( p \), and for each spectral efficiency randomly generate a Bernoulli sequence. We compute the PSLR and ISLR of this Bernoulli sequence, design each waveform/filter, and then compute the PSLR and ISLR of each designed correlation sequence. This is done for 500 Monte Carlo trials and the recorded PSLRs and ISLRs are averaged and compared to our PSLR and ISLR predictions. Figure 4.7 shows the average ISLR of the alternating projections design in blue, the average ISLR of the modified design from \[2\] in green, and the ISLR prediction in purple.
Figure 4.7: ISLR predictions for $p = [0.2, 0.9]$, $M = 511$, and the average ISLR of waveforms with varying bands to avoid.

As the spectral efficiency approaches either extreme, zero or one, the variance of the sidelobes is decreasing. As the variance of the sidelobes decreases, the sidelobes are not well represented with a Normal distribution. As expected, the ISLR of the modified algorithm from section 2.2.1 becomes more difficult to predict. It is not understood why the average ISLR dips at low spectral efficiencies for this algorithm.

Figure 4.8 shows the average PSLR of the alternating projections design in blue, the average PSLR of the modified design from section 2.2.1 in green, and the PSLR prediction in purple.
Figure 4.8: PSL predictions for $p = [0.2, 0.9]$, $M = 511$ and the average PSLR of waveforms with varying bands to avoid.

Again we see the best PSLR predictions for spectral efficiencies near $p = 0.5$ and a dip in the average PSLR of the design from [2] for low spectral efficiencies.

We can then examine the average power levels of each waveform in the usable and unusable frequency bands. In Figure 4.9, we show the ratio of the average power in the usable bands to the average power in the unusable bands. A contiguous bandwidth waveform, one whose spectrum is relatively flat over the bandwidth of the waveform, would have a ratio equal to 1. If a spectrally-disjoint waveform has a ratio of 2, the waveform puts twice as much energy in the usable bands as the unusable bands on average.
Figure 4.9: Ratio of average power in usable bands to average power in unusable bands for $p = [0.2, 0.9]$, $M = 511$. 
Chapter 5

Experimental Validation of Hardware Effects on Spectrally-Disjoint Waveforms

As waveform designs become more exotic, it is critical that future spectrally-disjoint waveforms consider constraints imposed by hardware, such as modulus and phase restrictions, while maintaining reasonable ACS properties. To obtain a general understanding of how spectrally-disjoint waveforms will be affected by hardware, we have selected four algorithms with different features and system constraints. We investigate the waveform designs described in sections 2.2.1, 2.2.2, 2.2.3, and 2.2.4, using both computer simulations and hardware demonstrations. By subjecting each design to a hardware-in-the-loop test, we can evaluate the underlying constraints of each design. This will provide insight to future spectrally-disjoint waveform designs, and shed light on the most important constraints for maintaining performance. Waveform/filter correlation sequences and frequency-domain characteristics are investigated for each design for three particular scenarios.
5.1 Simulation and Experimental Results

Three simulation scenarios are used to characterize the performance of each design method. First, a single band is chosen for each spectrally-disjoint waveform to avoid. Each design method will return a waveform/filter pair where the time-domain, correlation function, and frequency response will be studied. The second simulation will dictate one wide band and one narrow band for each waveform to avoid. The third scenario dictates five extremely narrow frequency bands as unusable. After each simulation scenario, the waveforms were saved and loaded into a Tektronix AWG5014C arbitrary waveform generator (AWG). One waveform at a time was transmitted from the AWG into a Tektronix TDS5054B-NV oscilloscope, with markers at each end of the waveform. Each waveform was sampled at 1.2GHz and modulated to a carrier frequency of 250MHz. To investigate the control of out-of-band energy and somewhat mimic a spectral mask, the first and last 37.5% of the baseband spectrum (where the baseband spectrum is defined to be between $-f_s/2$ and $f_s/2$) are dictated as unusable bands. This is a strict requirement, because at best a contiguous bandwidth waveform would only have 25% of the available spectrum for transmission. A block diagram depicting the experiment and processing is shown in Figure 5.1.

![Figure 5.1: Diagram and post-processing outline of the hardware-in-the-loop experiment.](image-url)
Although our objective is not to perform a one-to-one comparison of the designs, it is important to set each design as close to a common baseline as possible. We first note that the alternating projections [14] and piecewise-linear phase designs [11] have no parameters to tune, so they will be fixed constant modulus designs. The gradient descent-based waveform design only has relative weights of importance for the avoided frequency bands, so it is also fixed for our purposes. The receive filter for the gradient descent-based design has a convex weight to trade the cost between reducing range sidelobes and filtering energy in the avoided bands. We found that raising \( \beta \) did not have any consistent effect of lowering sidelobes, so we set \( \beta = 0.05 \) as the author notes in [2].

The chirp-like design with notches [12] can be tuned by varying the number and multiplicity of nulls in the frequency spectrum. The more nulls and the higher the multiplicity of each null, the greater the spectral suppression. In turn, this raises the sidelobes of the ACS. One can imagine that there is a range of the number of nulls to consider. You need a certain number of nulls to even begin suppressing the frequencies in the unusable bands. This would be the point at which the waveform has the worst frequency suppression, but lowest range sidelobe levels. There is also another extreme in which adding any more nulls has little effect on the spectrum. This would be the waveform with the greatest frequency suppression, but highest range sidelobe levels. We chose the latter extreme because we found that the minimum number of nulls required to suppress frequencies in the unusable band, resulted in an ACS with a PSLR still greater than the other three designs. Therefore, baselining all four designs to PSLR would be difficult. The chirp-like design was set with 1000 nulls, equally spaced, each with multiplicity four.

### 5.1.1 Scenario 1

Figures 5.2a, 5.3a, 5.4a, and 5.5a show the baseband spectrums of the transmitted waveforms in green and the received waveform spectrums, demodulated to baseband, in blue. The spectrums are median filtered using 10 taps for display purposes. The usable frequency bands, -150Mhz to -120MHz and -70MHz to 150MHz (at baseband), or permitted bands for transmission, are shown in the dashed black curves.

The autocorrelation sequences of the transmitted waveforms are shown in Figures 5.2b, 5.3b,
5.4b, and 5.5b in green, where the cross correlation sequences of the transmitted waveforms and the received waveforms are shown in blue. The only exception to this is the gradient descent-based approach, where the algorithm designs a mismatched filter. In that case, the green curve represents the XCS of the designed filter and the transmitted waveform, while the blue curve represents the XCS between the designed filter and the received waveform.

![Baseband spectrums and correlation sequences](image)

(a) Baseband spectrums
(b) Correlation sequences

Figure 5.2: Spectrum and correlation sequences of the transmitted and received waveforms for the gradient descent based approach where $\beta = 0.05$.

![Baseband spectrums and correlation sequences](image)

(a) Baseband spectrums
(b) Correlation sequences

Figure 5.3: Spectrum and correlation sequences of the transmitted and received waveforms for the alternating projections design.
Each waveform experiences some spectral tapering toward the edges of the waveform’s band. This is likely a common attenuation after passing through several pieces of hardware and several stages of processing. It can also be seen that each cross correlation sequence exhibits an approximate 2.0dB increase in PSLR. Even with recorded waveform markers, varying cable lengths and timing inconsistencies can result in higher responses at non-zero lags. The piecewise linear-phase waveform exhibits a one sided taper in the cross correlation sequence with the received waveform.
This is a result of the hardware tapering higher frequencies, and the chirp-like waveform sweeping through the higher frequencies toward the end of the pulse.

### 5.1.2 Scenario 2

The second simulation was tuned in the same manner as the first, but each algorithm was forced to avoid two frequency bands, leaving -150MHz to -112.5MHz, -27MHz to 30MHz, and 36MHz to 150MHz (at baseband) as the usable bands for transmission. We can see in the following figures that each algorithm has more difficulty avoiding the narrow unusable bands, and in exchange for avoiding more frequencies, the range sidelobes rise.

Figure 5.6: Spectrum and correlation sequences of the transmitted and received waveforms for the gradient descent based approach.
Figure 5.7: Spectrum and correlation sequences of the transmitted and received waveforms for the alternating projections design.

Figure 5.8: Spectrum and correlation sequences of the transmitted and received waveforms for the piecewise linear phase design.
Similar spectral and cross-correlation observations can be made as in the first scenario. We continue to see the spectral tapering and rise in sidelobes. It can be seen in the first algorithm that there is also a rise in the power of the narrow unusable frequency band. The piecewise linear phase algorithm was able to maintain its spectrum fairly well, but unable to avoid transmitting in the narrow unusable frequency band.

### 5.1.3 Scenario 3

The final simulation was also tuned in the same manner, but each algorithm was forced to avoid seven frequency bands. The six usable bands were -150MHz to -120MHz, -114MHz to -84MHz, -72MHz to -12MHz, 6MHz to 78MHz, 84MHz to 120MHz, and 126MHz to 150MHz (at baseband).
Figure 5.10: Spectrum and correlation sequences of the transmitted and received waveforms for the gradient descent based approach.

Figure 5.11: Spectrum and correlation sequences of the transmitted and received waveforms for the alternating projections design.
Several observations can be made in the final scenario. By fixing the waveform design length we can see how difficult it is for each algorithm to maintain transmission in the narrow usable bands. This could be remedied by increasing the waveform design length. We can also see how the strict frequency requirements force the XCS sidelobes to rise dramatically in comparison to the other scenarios. Just as in the previous two scenarios, there is also a slight increase in the correlation sequence sidelobes when correlating with the received waveforms.
Chapter 6

Conclusion

This performance comparison illustrates the benefits of waveform design in colored interference. With increasing levels of complexity, the matched filter design is outperformed by the disjoint mismatched design, which is in turn outperformed by the joint mismatched design, when subjected to an interference dominant environment. When presented with a noise dominant environment, little performance is to be gained by waveform diversity, design, and optimization. These performance gains could be critical to a radar system operating in a congestion spectrum.

We have derived the autocorrelation sequence mean and variance of a spectrally-disjoint waveform by modeling the waveform’s usable and unusable frequencies as a sequence of Bernoulli random variables. This allowed us to develop closed-form predictions for ISLR and PSLR, for any percentage of usable bandwidth. Our predictions were verified against two spectrally-disjoint waveform designs for varying spectral efficiencies.

We can make a few final conclusions about the effects of a congested spectrum on radar waveforms. As general intuition would suggest, as the waveform is required to avoid more frequencies and more frequencies in narrower bands, the range sidelobes of the correlation sequence rise. This is in agreement with the PSLR and ISLR predictions derived in Chapter 4. We can also see how each design has to be evaluated for its ability to avoid narrow frequency bands. Hardware will also have an affect on the frequency content of these waveforms. Each design investigated here delivered constant modulus waveforms, to maximize the energy on target. This need for a constant modulus
waveform restricts the waveforms to phase only designs, and in turn restricts the ability to achieve low sidelobe levels while also limiting transmit energy in the unusable frequency bands. If a mismatch loss on receive is considered acceptable, the mismatch filter design, such as in [2], could prove most useful for radar waveform designs in congested spectrums.

6.1 Future Work

As waveform diversity and design for congested spectrums matures, performance predictions and bounds will need to become more accurate. A straightforward extension to the ISLR and PSLR predictions in Chapter 4 would include a Markov chain model for the usable and unusable frequencies, rather than a Bernoulli sequence. This would provide the ability to model probabilities of bands of frequencies being usable or unusable, rather than single independent frequencies. One should also investigate arbitrary magnitudes for each usable frequency, rather than simply zero or one.

For spectrally-disjoint radar waveform designs to make significant advancements toward real-world operations, hardware effects must continue to be exploited. Spectral masks, amplifier models, and doppler effects should all be considered during waveform optimization. Decision making logic for adaptive radars, as alluded to in Chapter 1, could be critical to spectral cooperation and maximizing performance.
Bibliography


