Dissipated Energy at a Bimaterial Crack Tip Under Cyclic Loading

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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Abstract

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A new theory of fatigue crack growth in ductile solids has recently been proposed based on the total plastic energy dissipation per cycle ahead of the crack. This, and previous energy-based approaches in the literature, suggest that the total plastic dissipation per cycle can be closely correlated with fatigue crack growth rates under mode I loading. The goal of the current research is to extend the dissipated energy approach to steady-state crack growth under mixed-mode I/II loading conditions, with application to cyclic delamination of ductile bimaterial interfaces. Such systems can occur in brazing, soldering, welding, and a variety of layered manufacturing applications, where high-temperature material deposition can result in a mismatch in mechanical properties between the deposited material and the substrate. The total plastic dissipation per cycle is obtained by 2-D elastic-plastic finite element analysis of a stationary crack in a general mixed-mode specimen geometry under constant amplitude loading. Numerical results for a dimensionless plastic dissipation per cycle are presented over the full range of relevant material combinations and mixed-mode loading conditions. Results suggest that while applied mode-mix ratio is the dominant parameter, mismatches in yield strength and hardening modulus can have a significant effect on the total plastic dissipation per cycle, which is dominated by the weaker/softer material. Results extended to general elastic-plastic mismatches provide substantial insight into the effects of crack-tip constraint, material hardening behavior and applied mode-mix ratio on the dissipated energy during fatigue crack growth. A consistent definition of the mode mix ratio is presented based on minimizing the elastic strain energy at a crack tip. Next, application of the current theory is demonstrated for thermomechanical fatigue of bonded bimaterials. Finally, the plastic dissipation computations are performed in a probabilistic framework in an attempt to assess the variability of the fatigue crack growth rate based on variation in bulk properties.
# Contents

## 1 Introduction

1.1 Motivation ....................................................... 1  
1.2 Background ..................................................... 3  
   1.2.1 Dissipated Energy Approach ................................. 3  
   1.2.2 Fatigue Crack Growth in Mixed-Mode ......................... 5  
   1.2.3 Fracture Mechanics of Bimaterial Interfaces .................. 6  
1.3 Scope and Limitations ........................................... 7  
1.4 Overview and Contributions .................................... 8

## 2 Plastic Dissipation in Homogeneous Materials Under Mixed-Mode Loading ........................................... 10

2.1 Modeling Approach ............................................... 10  
   2.1.1 Dissipated Energy Theory .................................... 10  
   2.1.2 Stationary Crack Modeling ................................. 11  
2.2 Mixed-Mode Layered Specimen Geometry ......................... 12  
2.3 Numerical Modeling Procedures ................................ 14  
   2.3.1 Finite Element Modeling .................................... 14  
   2.3.2 Bi-Linear Kinematic Hardening Model ...................... 15  
   2.3.3 Crack Tip Plasticity ....................................... 17  
   2.3.4 Plastic Dissipation Per Cycle .............................. 21  
   2.3.5 Nondimensionalization ..................................... 23  
2.4 Numerical Results and Discussion ............................... 24  
   2.4.1 Effect of Mode-Mix ........................................ 24  
   2.4.2 Effect of Plastic Constraint ............................... 25
3 Plastic Dissipation in Mixed-Mode Along Plastically Mismatched Interfaces

3.1 Problem Considered
3.1.1 Global Problem Geometry
3.1.2 Elastic Solution
3.1.3 Elastic-Plastic Response
3.1.4 Plastic Mismatches Considered

3.2 Numerical Modeling Procedures
3.2.1 Nondimensionalization

3.3 Numerical Results and Discussion
3.3.1 Effect of Yield Strength Mismatch
3.3.2 Effect of Hardening Modulus Mismatch
3.3.3 Effect of Combined Strength and Hardening Mismatch
3.3.4 Implications of Results

3.4 Conclusion

4 Determination of the Mode Mix in the Presence of an Elastic Mismatch

4.1 Introduction
4.1.1 Bimaterial Example

4.2 Elastic Strain Energy Near the Crack Tip
4.2.1 The Homogeneous Case
4.2.2 Curve Fit to Plastic Dissipation of Chapter 2
4.2.3 The Case of a Yield Strength Mismatch
4.2.4 The Bimaterial Case
4.2.5 Determine the Extrema of the Elastic Strain Energy
4.2.6 Gradients and Sensitivity of the Elastic Strain Energy
4.2.7 Limitations of Using the Elastic Strain Energy Within the Yield Zone

4.3 Strain Energy Contour Integral
4.3.1 Homogeneous Case ............................................. 74
4.3.2 The Bimaterial Case ........................................... 77
4.4 Definition of Mode ............................................... 79
  4.4.1 Spanning the Complete Range of Energy ...................... 79
  4.4.2 Determining the Mode Given $K_1$ and $K_2$ ................. 88
  4.4.3 Determining the Stress Intensity Factors given $G$ and $\psi$ .... 90
4.5 Conclusion ......................................................... 91

5 Plastic Dissipation from Cyclic Loading of a Bimaterial Interface Crack 93
  5.1 The Interface Crack Problem .................................... 94
    5.1.1 Equivalent Loading and Superposition ....................... 94
    5.1.2 Strain Energy Release Rate .................................. 96
    5.1.3 Interface Stress Intensity Factors .......................... 97
  5.2 Numerical Determination of $w(a, b, h)$ ........................ 98
    5.2.1 Determining the Values of the Applied Moments ............ 99
    5.2.2 Discussion of the Results for $w$ ............................ 101
  5.3 Numerical Determination of Plastic Dissipation Energy .......... 103
    5.3.1 Specimen Geometry .......................................... 103
    5.3.2 Loading Conditions .......................................... 103
    5.3.3 Determining the Applied Moments ........................... 104
    5.3.4 Normalization .............................................. 106
  5.4 Results and Discussion ........................................... 106
    5.4.1 Setting $\beta = 0$ in Plane Strain .......................... 106
    5.4.2 Elastic Modulus Mismatches without Oscillation ............ 108
    5.4.3 Effect of Plastic Constraint ................................ 108
    5.4.4 Elastic Modulus Mismatches with Non-zero Oscillation Index .... 113
    5.4.5 Effect of Elastic and Plastic Mismatches ..................... 114
    5.4.6 Representations of the Plastic Zones ....................... 117
  5.5 Application to Thermomechanical Fatigue ......................... 117
  5.6 Conclusion ........................................................ 122
6 Predicting the Variation in Fatigue Crack Growth Rate 124

6.1 Introduction ................................................................. 124

6.2 Statistical Analysis of Material Properties ............................ 125

6.2.1 Deterministic quantities ................................................. 126

6.2.2 Stochastic Quantities .................................................. 126

6.2.2.1 Yield Strength ....................................................... 127

6.2.2.2 Strain Hardening Modulus ....................................... 127

6.2.2.3 Correlation .......................................................... 128

6.2.3 Fracture Toughness .................................................... 129

6.2.3.1 Coefficient of variation .......................................... 130

6.2.3.2 Parametric Distribution of Fracture Toughness ............... 130

6.3 Simulation Procedure and Results .................................. 131

6.3.1 Probabilistic Integration ............................................... 131

6.3.2 Plastic Dissipation Results .......................................... 132

6.4 Discussion and Conclusions ........................................... 133

7 Conclusions and Contributions 137

A Calculating the J-Integral 140

A.1 Mathematical Definition of the J Integral ......................... 140

A.2 Physical Interpretation .................................................. 141

A.3 Homogeneous Beam Analysis ........................................ 142

A.4 Composite Beam Analysis .............................................. 145

A.5 Strain Energy Density .................................................... 147

A.5.1 Homogeneous Beam Section ....................................... 147

A.5.2 Composite Beam Section ........................................... 148

A.6 J-Integral evaluations .................................................... 148

A.6.1 Special Cases .......................................................... 149

A.6.2 A Numerical Example ............................................... 150

A.6.3 Relating the Stress Intensity Factors to the J-Integral ........... 153

A.6.4 Crack Tip Opening Displacements ............................... 154
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>A bonded interface between a fin and internal wall of a heat exchanger pipe.</td>
<td>2</td>
</tr>
<tr>
<td>1.2</td>
<td>The dissipated energy theory requires that the same amount of energy per unit crack extension is dissipated from either monotonic or fatigue loading. (figure taken from [1])</td>
<td>4</td>
</tr>
<tr>
<td>1.3</td>
<td>Validation of fatigue crack growth rate prediction based on the dissipated energy approach for mode I loading. (figure taken from [1])</td>
<td>4</td>
</tr>
<tr>
<td>1.4</td>
<td>Different modes of loading of a crack tip.</td>
<td>5</td>
</tr>
<tr>
<td>2.1</td>
<td>Mixed mode specimen geometry with a symmetry condition on the right side.</td>
<td>12</td>
</tr>
<tr>
<td>2.2</td>
<td>Finite element mesh, loading, and boundary conditions.</td>
<td>16</td>
</tr>
<tr>
<td>2.3</td>
<td>Bi-linear kinematic hardening material model.</td>
<td>17</td>
</tr>
<tr>
<td>2.4</td>
<td>Forward and reversed plastic zones in pure mode I with $\Delta G = 200 J/m^2$, $E = 73.1$ GPa, $\nu = 0.3$, and $\sigma_y = 300$ MPa.</td>
<td>18</td>
</tr>
<tr>
<td>2.5</td>
<td>Forward and reversed plastic zone when $\psi = 41^\circ$ with $\Delta G = 200 J/m^2$, $E = 73.1$ GPa, $\nu = 0.3$, and $\sigma_y = 300$ MPa.</td>
<td>19</td>
</tr>
<tr>
<td>2.6</td>
<td>Forward and reversed plastic zones in pure mode II with $\Delta G = 200 J/m^2$, $E = 73.1$ GPa, $\nu = 0.3$, and $\sigma_y = 300$ MPa.</td>
<td>20</td>
</tr>
<tr>
<td>2.7</td>
<td>Effect of Mode Mix on $\frac{dW}{dN}$ in plane strain when $E_i/E = 0$, $E = 73.1$ GPa, $\nu = 0.3$, and $\sigma_y = 300$ MPa.</td>
<td>22</td>
</tr>
<tr>
<td>2.8</td>
<td>Dimensionless plastic dissipation $\frac{dW}{dN}$ vs. mode mix ratio $\psi$ for $E_i/E = 0$ and $\nu = 0.3$ in plane strain.</td>
<td>24</td>
</tr>
<tr>
<td>2.9</td>
<td>Dimensionless plastic dissipation energy $\frac{dW}{dN}$ vs. mode mix $\psi$ in plane stress and plane strain.</td>
<td>25</td>
</tr>
</tbody>
</table>
2.10 Effect of the tangent modulus ratio on $\frac{dW^*}{dN}$ vs. $\psi$. ........................................... 27
2.11 Comparison of current specimen (DCB) results in mode I and C(T) results from [1]. 28

3.1 Plastic mismatches considered. ................................................................. 33
3.2 Dimensionless plastic dissipation energy as a function of yield strength mismatch $\hat{\sigma}$ for elastic-perfectly plastic materials ($E_{t1}/E = E_{t2}/E = 0$) over the full range of applied mode mix $\psi$. ................................................................. 36
3.3 Effect of strength mismatch on reversed plastic zones in pure mode I, for elastic-perfectly plastic materials ($E_{t1}/E = E_{t2}/E = 0$) with $E = 73.1$ GPa, $\nu = 1/3$, $\sigma_{y2} = 300$ MPa and $\Delta G = 200$ J/m$^2$. ................................................................. 38
3.4 Effect of strength mismatch on reversed plastic zones for $\psi = 41^\circ$, for elastic-perfectly plastic materials ($E_{t1}/E = E_{t2}/E = 0$) with $E = 73.1$ GPa, $\nu = 1/3$, $\sigma_{y2} = 300$ MPa and $\Delta G = 200$ J/m$^2$. ................................................................. 39
3.5 Effect of strength mismatch on reversed plastic zones in pure mode II, for elastic-perfectly plastic materials ($E_{t1}/E = E_{t2}/E = 0$) with $E = 73.1$ GPa, $\nu = 1/3$, $\sigma_{y2} = 300$ MPa and $\Delta G = 200$ J/m$^2$. ................................................................. 40
3.6 Dimensionless plastic dissipation $\frac{dW^*}{dN}$ as a function of mode mix $\psi$ for elastic-perfectly plastic materials ($E_{t1}/E = E_{t2}/E = 0$) over the full range of strength mismatches. ................................................................. 42
3.7 Family of plots showing the effect of the hardening modulus ratios $E_{t1}/E$ and $E_{t2}/E$ for the case of no strength mismatch ($\hat{\sigma} = 0$). The response is dominated by the layer with the least hardening. ................................................................. 44
3.8 Family of plots showing the effect of the hardening modulus ratios $E_{t1}/E$ and $E_{t2}/E$ for the case of $\hat{\sigma} = 0.1$. ................................................................. 45
3.9 Family of plots showing the effect of the hardening modulus ratios $E_{t1}/E$ and $E_{t2}/E$ for the case of $\hat{\sigma} = 0.25$. ................................................................. 46
3.10 Family of plots showing the effect of the hardening modulus ratios $E_{t1}/E$ and $E_{t2}/E$ for the case of $\hat{\sigma} = 0.5$. The layer with the higher yield strength is not contributing anything to the plastic dissipation. ................................................................. 47
4.1 A stress-strain diagram of an elastic mismatch with equal plastic properties (both strain hardening rate and yield strength). ................................................................. 52

4.2 A schematic of an interfacial crack. ................................................................. 53

4.3 A plot of the dimensionless elastic strain energy for different values of $\nu$ in plane stress and plane strain. ................................................................. 62

4.4 A curve fit using Eq. (4.34) and the coefficients listed in Eq. (4.35) for the plastic dissipation of a homogeneous material when $\nu = 1/3$. ................................................................. 64

4.5 The influence of a yield strength mismatch on the normalized elastic strain energy in plane strain. These plots are the graphs of Eq. (4.40) in plane strain. ................................. 67

4.6 A contour map of the elastic strain energy within the yield contour of a the brass/-solder interface ($\alpha = 0.549$, $\beta = 0.1496$) for different values of the characteristic length and mode mix. The strain energy release rate ($G$) was set to unity. The square is located near the minimum value of $U$. The maximum is located within the concentric contours near $\phi = 1.5$. Notice that two logarithmic periods of the strain energy is shown. ................................................................. 69

4.7 A contour map of the elastic strain energy within the yield contour of a the brass/-solder interface for different values of the loading and mode mix. The characteristic length ($l$) was set to unity. Notice that two logarithmic periods of the strain energy is shown. ................................................................. 70

4.8 The effect of choosing a characteristic length proportional to $G$. ................................. 70

4.9 Plots of Eq. (4.69) for different values of Poisson’s ratio in plane strain. ......................... 77

4.10 Response of the strain energy contour integral as a function of the characteristic length $l$ and mode $\phi$ for a brass-solder interface. The radius of the contour is 1 and $G = 1$. ................................................................. 78

4.11 Response of the strain energy contour integral as a function of load $G$ and mode $\phi$ when the characteristic length is equal the inverse of the load. The effects of $\varepsilon$ (through $\beta = \alpha/4$) on the contour plots of $U/G$ when the characteristic length was set to $r_G$. ................................. 80
4.12 The location of the maximum and minimum values of \( U / G \) with respect to \( G \) for different values of \( \alpha \). The maximum values may reflect the one logarithmic period difference from the maximum values in Fig. 4.11. .......................... 81

4.13 The location of the maximum and minimum values of \( U / G \) with respect to \( \phi \) for different values of \( \alpha \). .................................................. 82

4.14 The maximum values of \( U / G \) different values of \( \alpha \). ................................. 83

4.15 The minimum values of \( U / G \) different values of \( \alpha \). ................................. 84

4.16 Contour plot for a brass solder interface showing the line segment connecting the extrema and a sampling of across \( \phi \) for \( G^* = 100 \). ............................... 85

4.17 The extraction of \( U / G \) along the line segments from Fig. 4.16 results in the plot shown here. The arrows indicate the transformation from \( \phi \) to \( \psi \) for the portion of the curve with a negative slope. The curve with the circles show the energy extracted from the different values of \( \phi \) when \( G = 100 \). ................................. 86

4.18 The transformation of the bottom axis from \( \phi \) to \( \psi \) results in the plot shown here. The value of the mode mix is known based on the contour energy \( U / G \). The circles the results of the interpolation for different values of \( \phi \) when \( G = 100 \). ................. 87

4.19 The top plot has the same values of \( U / G \) and \( \psi \) as shown in Fig. 4.18 plotted as circles. From those values, the characteristic length was calculated for each point and shown in the bottom plot. Finally, the \( \times \) symbols were plotted on the top plot using the classic definition of \( \psi \) to verify the new definition of mode mix. ...... 89

5.1 A generalized mixed mode specimen and corresponding equivalent loading obtained by superposition. .................................................. 95

5.2 Loading conditions to generate pure mode I and pure mode II loading conditions for different values of \( \alpha \) according to the definition of mode presented in Ref. [51]. The pure mode II conditions are the negative ratios. ..................... 100

5.3 Possible values for Poisson’s ratio when \( \beta = 0 \) in plane strain for different values of \( \alpha \). Only half the solutions are provided because the line \( \nu_2 = \nu_1 \) is a line of symmetry. 107
5.4 Results of the normalized plastic dissipation energy for positive values of \( \alpha \) and all ranges of mode mix when \( \beta = 0 \). The variation between all the levels of elastic mismatch is attributed different values of Poisson’s ratio. .......................................................... 109

5.5 The effect of the plastic constraint on the plastic dissipation energy when normalized by the properties of the top layer and loaded in pure mode 1. ................................. 110

5.6 The effect of the plastic constraint on the plastic dissipation energy when normalized by the properties of the top layer and loaded in in a middle mode when \( M_2 = 0 \). . . 111

5.7 The effect of the plastic constraint on the plastic dissipation energy when normalized by the properties of the top layer and loaded in pure mode II. ............................... 112

5.8 Plot of the effect of elastic mismatch with nonzero values of \( \epsilon \). For this plot, \( \beta = \alpha/4 \) which corresponded to \( \nu_1 = \nu_2 = 1/3 \). The characteristic length for each result was linearly interpolated from the values in Table 5.2. ......................................................... 114

5.9 Plot of the effect of elastic mismatch with nonzero values of \( \epsilon \) for different extremes of yield strength mismatch. For these plots, \( \beta = \alpha/4 \) which corresponds to \( \nu_1 = \nu_2 = 1/3 \). ................................................................. 116

5.10 Plastic zones of representative loading cases and material properties when \( J = 1000 \) N/m, \( \nu_1 = \nu_2 = 1/3, \) and \( E_1 = 100 \) MPa. ......................................................... 118

5.11 Plastic zones of representative loading cases and material properties when \( J = 1000 \) N/m, \( \nu_1 = \nu_2 = 1/3, \) and \( E_1 = 100 \) MPa. ......................................................... 119

5.12 Plastic zones of representative loading cases and material properties when \( J = 1000 \) N/m, \( \nu_1 = \nu_2 = 1/3, \) and \( E_1 = 100 \) MPa. ......................................................... 120

5.13 Plastic zones of representative loading cases and material properties when \( J = 1000 \) N/m, \( \nu_1 = \nu_2 = 1/3, \) and \( E_1 = 100 \) MPa. ......................................................... 121

6.1 Yield Strength Distribution ................................................................. 127

6.2 Strain hardening modulus distribution for the sample of Ti-6Al-4V. .................... 128

6.3 Fracture Toughness Distribution .......................................................... 131

6.4 Histogram of the dimensionless plastic dissipation based on the distributions of yield strength and hardening modulus. .......................................................... 132
6.5 Comparison of the predicted results of the fatigue crack growth rate to the collapsed experimental results for $\Delta K = 75 \text{ MPa}\sqrt{\text{m}}$. .......... 134

6.6 A plot of the experimental results of the fatigue crack growth rate compared to the results from sampling and expanding the empirical distribution of $da/dN$ shown in the bottom of Fig. 6.5. .......... 135

A.1 An arbitrary contour, $\Gamma$, around a crack propagating along a bimaterial interface. ... 141

A.2 A rectangular composite beam with an interfacial crack. The dashed line represents a contour around the crack tip to be used in our $J$-integral hand calculations. ... 142

A.3 The beam from Figure A.2 shown cut along the path $\Gamma$. The arrows represent the kinematic requirement of a linear deflection of a beam in pure bending. The dashed lines represent the location of the neutral axes. ................. 143

A.4 The profile of the stress of a composite beam in pure bending. The stress is dependent on the elastic modulus. ................. 146

A.5 A contour plot of the bending stress ($\sigma_x$) from the finite element analysis. The values used in this plane strain analysis are given in Table A.1. .......... 151

A.6 Crack opening displacements. The distance from the crack tip of the original, undeformed model is taken as $r$. .......... 154

A.7 A plot showing two solutions to the example problem. The numerical solution uses the finite element analysis to determine the nodal displacements. The analytical solution shown in Eq. (A.17) is plotted as lines on top of the nodal solutions from the FEA. The presence of an $x$ component indicates mode II loading (shear along the interface). .......... 155
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1 Introduction

This dissertation presents concepts and results that can be used in predicting the fatigue crack growth rate of ductile materials. The treatment is sufficient to cover cracks growing along bimaterial interfaces, where a mismatch in both elastic and plastic properties can exist across the interface. The reason this research is exciting and novel is the fact that Paris regime fatigue crack growth rates can be predicted based entirely on monotonic material properties and physics based models of the plastic dissipation without calibration by crack growth measurements. This opens the door for more advanced life prediction models and the possibility for rapid introduction of new aerospace materials.

1.1 Motivation

Advancements in technology have led to an increase in the use of materials that are bonded together in such a fashion that the interface may be more susceptible to fracture than either of the two bulk materials. This phenomena is found in layered manufacturing, electronics packaging, soldering, brazing, welding, or any other material system comprising a substrate bonded to another layer. An example of a bonded interface is shown in Fig. 1.1 which shows a section of a heat exchanger pipe with vacuum brazed fins.

In a recent paper, Klingbeil [1] proposed a new theory of fatigue crack growth in ductile solids based on the total plastic energy dissipation per cycle ahead of the crack. The results of this and previous energy-based approaches in the literature suggest that the total plastic dissipation per cycle is a driving force for fatigue crack growth in ductile solids, and can be closely correlated with fatigue crack growth rates under mode I loading. The goal of this dissertation is to extend the dissipated energy approach to steady-state crack growth under mixed-mode loading conditions, with ultimate
application to fatigue delamination of ductile interfaces in layered materials.

Klingbeil’s research proposed a surprisingly simple concept: The fatigue crack growth rate is directly proportional to the dissipated plastic energy. The proportionality constant is the inverse of the monotonic fracture toughness. In equation form:

\[
\frac{da}{dN} = \frac{1}{G_c} \frac{dW}{dN}
\]  

(1.1)

This equation implies there is no distinction between the amount of energy dissipated per unit crack extension, whether it be in fatigue loading or monotonic loading.

Equation (1.1) provides sufficient motivation for computing the plastic dissipation; however, the dissipated energy from cyclic loading at a crack is fundamentally important in its own right. Therefore, the results presented herein are useful whether or not Eq. (1.1) is valid.\(^*\) Since Klingbeil’s work was limited to mode I loading, the research in this dissertation extended the computational efforts of determining the plastic dissipation into mixed mode I/II of general bimaterial systems. This extension resulted in two journal papers, Refs. [2, 3], with others in progress.

The focus of the research herein is on determining the amount of plastic strain energy dissipated per cycle, \(dW/dN\). The term “plastic dissipation energy” is sometimes referred to as the “hysteretic” energy or the “plastic work”. This quantity is important in its own right because it represents the

\(^*\)Klingbeil showed the validity of Eq. (1.1) for mode I loading of homogeneous materials, but the experimental validation of Eq. (1.1) for mixed mode and bimaterial interfaces is the subject of ongoing research.
energy dissipated in a hysteresis loop. The subject of continuum damage mechanics uses dissipated energy as criteria for predicting failure with a promising proposition found in Ref. [4]. Therefore, the computation of \( dW/dN \) provides a significant contribution to the field whether or not Eq. (1.1) holds true for all material systems considered. On page 79 of his monograph [5], Ellyin provides further motivation, “The significance of the energy approach is in its ability to unify microscopic and macroscopic testing data, and to formulate a multiaxial failure criteria.”

Fatigue crack growth rate data is known to have substantial of variation [6, 7]. Similarly, fracture toughness data follows a fairly wide distribution [8–14], and, according to Eq. (1.1), may influence the scatter found in the fatigue crack growth rate. The fracture toughness is a mechanical property of the material system which is obtained experimentally using the ASTM standard E399. However, significant statistical correlation exists between the material properties (yield strength, hardening modulus, and fracture toughness) which needs to be resolved in order to predict the scatter of the fatigue crack growth rate. Since significant computational efforts are required to compute the plastic dissipation, efficient sampling methods, as described in [15], are desired. Understanding the variation of fracture toughness is important when applying the results of the research herein to actual fatigue crack growth data. Knowledge and predictive capabilities of probability density functions describing both fatigue crack growth rate, as well as fracture toughness, can be used in component and system reliability assessments, with ultimate cost savings and life extension of aerospace components.

1.2 Background

1.2.1 Dissipated Energy Approach

A critical plastic dissipation criterion for fatigue crack extension in ductile solids was first suggested by Rice [16]. Dissipated energy approaches to fatigue crack growth prediction have since been the subject of numerous analytical [17–27] and experimental [28–36] investigations. The basic premise of the dissipated energy theory as described in [1] is shown in Fig. 1.2 where the energy balance in fatigue uses a chain rule for differentiation to develop the relationship of Eq. (1.1).

The dissipated energy approach considers the total plastic dissipation per cycle occurring throughout the reversed plastic zone ahead of the crack, which is a quantity of both theoretical and practical
Figure 1.2: The dissipated energy theory requires that the same amount of energy per unit crack extension is dissipated from either monotonic or fatigue loading. (figure taken from [1])

Figure 1.3: Validation of fatigue crack growth rate prediction based on the dissipated energy approach for mode I loading. (figure taken from [1])
interest. As shown herein, the total plastic dissipation per cycle is directly related to the range of applied energy release rate, which is typically used to correlate fatigue crack growth rates under mixed-mode loading [37]. Moreover, as opposed to the crack tip stresses and strains, the total plastic dissipation per cycle is a bounded quantity, which allows for straightforward interpretation of numerical results. Finally, numerical results for the total plastic dissipation per cycle can be directly compared to measurements of dissipated energy during fatigue crack growth, which have been reported in the literature by a number of researchers [28–36]. Measurements of fatigue crack growth rate have typically been restricted to mode I loading, as plotted in Fig. 1.3. Notice the dissipated energy approach predicts the mean behavior of the fatigue crack growth rate. Since most fatigue data is from mode I, the mixed-mode results of this work can be compared with subsequent measurements of dissipated energy during mixed-mode fatigue crack growth along bimaterial interfaces.

1.2.2 Fatigue Crack Growth in Mixed-Mode

In this work, the term “mixed-mode” in this work refers to a combination of mode I and mode II loading conditions at the crack tip. As shown in Fig. 1.4, the in-plane loading consists of a symmetric loading component (mode I) and an anti-symmetric loading component (mode II). Out-of-plane shearing (mode III) is not considered in plane problems and is not considered in this dissertation.
As discussed in the review paper by Qian and Fatemi [37], surface flaws and short cracks in homogeneous materials are typically subject to mixed-mode loading conditions, yet ultimately orient themselves such that Paris-regime crack growth occurs primarily in mode I. As such, the majority of the fatigue crack growth literature has focused on mode I loading. Recent studies of fatigue crack growth under mixed-mode loading have typically been concerned with the growth of short cracks [38], fatigue crack threshold behavior [38–40], and the effect of mode-mix on crack growth direction [41–43]. A noteworthy investigation of fatigue crack growth in a homogeneous material under sustained mixed-mode loading has been conducted by Magill and Zwerneman [44].

The current dissertation considers the plastic energy dissipation associated with steady-state fatigue crack growth under mixed-mode loading, which can occur during cyclic delamination of ductile interfaces in layered materials. Fatigue delamination is a potential mode of failure in a variety of applications involving bonded layers of material, where mixed-mode crack growth along the bonded interface can be energetically favorable to mode I crack growth within either bonded layer. While little comprehensive experimental data is available, researchers have begun to investigate fatigue crack growth along solder joints and other bonded interfaces [45–50], where mixed-mode delamination can be a predominant mode of failure.

1.2.3 Fracture Mechanics of Bimaterial Interfaces

Bonded layers of dissimilar materials occur in a variety of applications. They are the basis for numerous solid free-form fabrication and layered manufacturing applications, and they occur in welding, soldering, coating, and electronic packaging applications [51–57]. In general, a mismatch in both elastic and plastic properties can exist across the interface; however, special cases exist in which the material combinations have identical elastic properties, but have mismatches in plastic properties. A direct application of such interfaces can be found in [58], where commercially pure titanium was bonded to Ti 6Al-4V. Other example applications are found in under matched or over matched welds, and in heat affected zones where there are distinct layer boundaries [59,60]. In many of the above applications, delamination of the bimaterial interface can be a predominant mode of failure.

Delamination of bimaterial interfaces generally involves both mode I and II loading. Elastic fracture mechanics concepts for interface cracks under mixed-mode loading have been summarized by
Hutchinson and Suo [51, 52], and have been extended by Shih et al to elastic-plastic analysis of interface cracks under monotonic [61, 62] and cyclic [63] loading. A special case of such systems occurs when one material is relatively brittle, such as ceramic-metal interfaces, where crack tip plasticity is restricted to the ductile (metal) layer [63–65]. Although little comprehensive experimental data is available, researchers have begun to investigate fatigue crack growth along solder joints and other bimaterial interfaces [45–50]. While a number of researchers have characterized the stress fields and driving forces for cracks along plastically mismatched interfaces [66–68], none have provided comprehensive results for the total cyclic plastic dissipation ahead of the crack.

1.3 Scope and Limitations

The focus of the current dissertation is numerical calculation of the total plastic dissipation per cycle ahead of the crack, which is a quantity of both theoretical and practical interest. As shown in [2], the total plastic dissipation per cycle is directly related to the range of applied energy release rate, which is typically used to correlate fatigue crack growth rates under mixed-mode loading. This research pertains to cases where at least one layer of the bimaterial has the capability of dissipating energy through hysteretic losses. In metals, the hysteretic losses occur from dislocation motion which manifests itself as plastic deformation [69, 70].

It is presumed that away from the crack tip, the component or specimen under consideration is not loaded beyond its proportional limit. This precludes large scale yielding analysis; however, in the vicinity of the crack tip, the elasticity solution shows that the stresses become singular and a plastic zone will develop. This plastic zone is governed by the applied loads and geometry, which can be represented by Linear Elastic Fracture Mechanics (LEFM) parameters. Therefore, as long as the assumptions for LEFM are maintained, then the results presented herein are valid. A method of verifying this assumption is presented in subsequent chapters.

Since the total plastic dissipation energy is desired, resolving the plastic zone details and showing the nature of the plastic singularity fields is not required. A qualitative statement of sufficient mesh resolution only requires that the reversed plastic zone be fully resolved. Within the plastic zones, the materials are assumed isotropic, which negates the direct application to a functionally graded material system. This, however, could be a natural extension of the current work.
In this study, the total plastic dissipation per cycle is determined by 2-D elastic-plastic finite element modeling of a stationary crack in a general mixed-mode layered specimen geometry. The merits and limitations of stationary (as opposed to growing) crack modeling are discussed in [1], and are reiterated in an upcoming section. The results are presented for stationary cracks and do not consider the energy of an advancing crack. This is not as limiting as it may seem due to the small amount of energy released in creating a new surface (in comparison to the plastic dissipation). Also, the model used in this research assumes a mathematically perfect crack. This assumption becomes more valid as the scale of the asperities in the bond become smaller compared to the plastic zone size.

The material properties and geometry are time independent. Therefore, creep effects, temperature changes, and corrosion are not considered in this analysis. This precludes a useful discussion of thermomechanical fatigue (TMF) of thermal barrier coatings (TBCs) on turbine blades because thermal effects, the effect of a thermally grown oxide (TGO) layer, and metal creep or “ratcheting” is not considered. For a more detailed discussion of TMF, see Refs. [71–73].

1.4 Overview and Contributions

After the introductory remarks, this dissertation is written in four nearly separate chapters. Chapter 2 maps out the response of the plastic dissipation energy as a function of mode mix in a homogeneous material. This work was a result of the research done in fulfillment of a master’s degree [74], which introduced some of the topics covered in this dissertation. All the results from Chapter 3 and beyond are results from research conducted after Ref. [74] was published. As such, Chapter 2 should be viewed as background for the research presented herein. Comprehensive numerical results for a dimensionless plastic dissipation per cycle are presented over the full range of mixed-mode loading for both elastic-perfectly plastic and bi-linear kinematic hardening materials. The numerical results provide significant insight into the role of crack tip constraint, material hardening behavior and mode-mix ratio on the energy dissipated during fatigue crack growth.

Chapter 3 extends the dissipated energy results for mixed mode to interface cracks between layers with similar elastic properties and dissimilar plastic properties. The general case of a material interface is considered in Chapter 5, where a mismatch exists in both elastic and plastic properties.
The technical challenge of interpreting the results for the case of elastic mismatches is how to consistently define the mode mix. The reason for this challenge is the fact that the ratio of shear to normal stresses at a point depends on the magnitude of the loading. In other words, mode of loading and the magnitude of loading are coupled. The mapping of the plastic dissipation energy for all possible bimaterial systems is the most significant contribution of this research, and comprises Chapter 5.

Once the plastic dissipation energy has been mapped for the general bimaterial system, the results can be applied to the problem of thermomechanical fatigue. Of particular interest are the dissipated energy from a steady-state crack growing in a solder joint where the steady state assumptions of the previous section are not violated. The details of this problem are included in an appendix.

Finally, a treatment of the variability inherent to fatigue testing and fatigue crack growth data is presented based on the dissipated energy theory of Eq. (1.1). A method of quantifying the variation of fatigue crack growth rate (FCGR) is based on the variation of basic material properties: yield strength, strain hardening, elastic modulus, and fracture toughness. A recently introduced confidence interval minimization scheme [15] is employed to reduce the required number of simulations used to assess the variation of the fatigue crack growth rate. These results provide some insight and explanation for the variability found in fatigue crack growth data.
2 Plastic Dissipation in Homogeneous Materials Under Mixed-Mode Loading

2.1 Modeling Approach

2.1.1 Dissipated Energy Theory

Following the work of Bodner et al. [20], Klingbeil [1] proposed the crack growth law shown in Eq. (1.1) which is repeated here:

\[
\frac{da}{dN} = \frac{1}{G_c} \frac{dW}{dN},
\]

(2.1)

where \( \frac{da}{dN} \) is the fatigue crack growth rate, \( G_c \) is the critical strain energy release rate under monotonic loading (i.e., the fracture toughness), and \( \frac{dW}{dN} \) is the total plastic dissipation per cycle occurring throughout the reversed plastic zone ahead of the crack tip.\(^*\) The proposed crack growth law assumes that the total energy required to propagate a crack a unit distance in a given material is independent of the manner in which the energy is dissipated, be it monotonic or fatigue loading conditions. As outlined in [1], the proposed crack growth law results in a \((\Delta K)^4\) dependence of the fatigue crack growth rate, and has been shown to collapse the measured Paris-regime crack growth data for several ductile metals under constant amplitude, mode I loading conditions. Moreover, numerical results for the plastic dissipation per cycle were shown to be consistent with a variety of dissipated energy measurements reported in the literature.

In theory, the crack growth law of Eq. (2.1) is applicable to fatigue crack growth under general mixed-mode loading conditions, where both \( G_c \) and \( \frac{dW}{dN} \) depend on the mode-mix ratio. Hence, application of the crack growth law requires numerical calculation of the quantity \( \frac{dW}{dN} \), which is simply the total plastic dissipation per cycle integrated over the reverse plastic zone ahead of the crack.

\(^*\)The plastic dissipation \( W \) is per unit width, as required by the units of Eq. (2.1).
crack:
\[ \frac{dW}{dN} = \int_{r_p} \left\{ \oint \sigma_{ij} \, d\varepsilon_{ij} \right\} \, dA. \]  \hspace{1cm} (2.2)

2.1.2 Stationary Crack Modeling

In the current study, the total plastic dissipation per cycle of Eq. (2.2) is obtained by 2-D elastic-plastic finite element analysis of a stationary crack in a general mixed-mode layered specimen geometry. As discussed in [1], a stationary (as opposed to growing) crack modeling approach neglects the contribution of the actual crack extension to the total plastic dissipation occurring during any given load cycle. However, for Paris-regime crack growth in ductile solids, both the plastic work and surface energy contributions associated with the actual crack extension in any given cycle are negligible compared to the total plastic dissipation occurring throughout the reversed plastic zone ahead of the crack. As such, modeling the actual crack extension is unnecessary.

That said, it is important to note that stationary crack modeling is unable to capture the transient evolution of the cyclic constitutive behavior as the fatigue crack extends through previously yielded material [75], and neglects the possibility of plasticity-induced crack closure. In the current study, only elastic-perfectly plastic and bilinear kinematic hardening constitutive behaviors are considered, each of which predicts plastic shakedown after only a single cycle. As such, the results of this work should be viewed as a first approximation to the stabilized cyclic response under constant amplitude loading, and do not attempt to account for load ratio effects typically associated with fatigue crack closure.†

It should finally be noted that numerical results presented herein can be interpreted from a number of standpoints. First, in the context of the fatigue crack growth law of Eq. (2.1), the results are applicable to stabilized, self-similar crack extension under mixed-mode loading conditions. As previously outlined, such results are most applicable to layered material systems, where sustained mixed-mode crack growth is a potential mode of failure. However, the results may also be taken at face value, i.e., as simply the plastic dissipation associated with a single load cycle applied to a stationary crack tip under mixed-mode loading. In this context, the results of this work may be useful in the development of energy-based approaches for predicting crack growth direction or

†In the absence of crack closure, the applied load ratio \( R = K_{\text{min}}/K_{\text{max}} \) was shown in [1] to have only a negligible effect on the total plastic dissipation per cycle, and is not considered further herein.
mixed-mode fatigue crack threshold behavior. Finally, the trends in plastic dissipation with mode-mix ratio presented herein may provide insight into discrepancies between mode I model results and dissipated energy measurements reported in the literature, which have been attributed in part to a mix of crack extension modes at the crack tip [1, 36].

2.2 Mixed-Mode Layered Specimen Geometry

The mixed-mode layered specimen geometry considered herein is shown in Figure 2.1. The specimen is composed of two bonded layers of identical elastic-plastic, isotropic materials having equal thickness \( h \) and length \( L \), and with a crack of length \( a \) along the interface. The loading consists of pure bending moments per unit width \( M_1 \) and \( M_2 \) applied to the top and bottom layers, which are equilibrated by a symmetry condition on the right hand side. As discussed shortly, variation in the bending moments \( M_1 \) and \( M_2 \) allows consideration of the full range of mode-mix values, from pure mode I to pure mode II.

![Figure 2.1: Mixed mode specimen geometry with a symmetry condition on the right side.](image)

Both the length \( L \) and the crack length \( a \) are sufficiently long to allow for steady-state conditions at the crack tip, so that the energy release rate is independent of crack length. As discussed in [54, 55, 57], steady-state conditions prevail once the crack is sufficiently long so that the specimen exhibits beam-type behavior far behind and ahead of the crack tip. The dimensions used in all numerical analysis discussed in the next section were \( L = 50 \ mm \), \( h = 5 \ mm \), and \( a = 25 \ mm \), which are sufficient for steady-state conditions at the crack tip.

A semi-analytical solution for steady-state cracking along the interface of a general bimaterial
specimen configuration, having mismatches in both layer thickness and elastic properties, has been provided by Suo and Hutchinson [51]. The results of [51] can be reduced to provide an exact closed-form solution for the specimen configuration considered herein, in which there is no mismatch in either elastic properties or layer thickness. The resulting mode I and mode II stress intensity factors for the problem of Fig. 2.1 are

\[ K_I = \frac{\sqrt{3}(M_1 + M_2)}{h^{3/2}} \]  

(2.3)

and

\[ K_{II} = \frac{-3(M_1 - M_2)}{2h^{3/2}}. \]  

(2.4)

As previously noted, variations in the bending moments \( M_1 \) and \( M_2 \) can be used to span the entire range of mode-mix values, which are defined herein in terms of the phase angle

\[ \psi = \tan^{-1} \left( \frac{K_{II}}{K_I} \right). \]  

(2.5)

Inspection of Eqs. (2.3) and (2.4) reveals that when \( M_1 = M_2 \), the \( K_{II} \) component vanishes leaving pure mode I loading (\( \psi = 0^\circ \)). Also, when \( M_1 = -M_2 \), the \( K_I \) component vanishes leaving a pure mode II condition (\( \psi = 90^\circ \)). Another simplification of Eqs. (2.3) and (2.4) occurs when \( M_1 = 0 \) (or \( \psi \approx 41^\circ \)), which is a special case of the four-point bend test specimen geometry commonly used for interfacial fracture testing of layered materials [53, 54].

The mixed-mode stress intensity factors are related to the strain energy release rate by the well-known fracture mechanics relation

\[ G = \frac{|K|^2}{ar{E}}. \]  

(2.6)

where \( \bar{E} = E/(1 - v^2) \) for plane strain, \( E \) for plane stress and \( |K| = \sqrt{K_I^2 + K_{II}^2} \). Substitution of Eqs. (2.3) and (2.4) into Eq. (2.6) gives the steady-state energy release rate for the problem of Fig. 2.1 as

\[ G = \frac{3(7M_1^2 + 2M_1M_2 + 7M_2^2)}{4Eh^3}. \]  

(2.7)

The above result can also be determined directly from the difference in strain energy per unit crack area far behind and ahead of the crack tip, which is the hallmark of steady-state delamination problems.
2.3 Numerical Modeling Procedures

2.3.1 Finite Element Modeling

The total plastic dissipation per cycle is obtained herein from a plane strain finite element model of the geometry of Fig. 2.1 under constant amplitude, mixed-mode loading. The finite element mesh, applied loads and boundary conditions are illustrated in Fig. 2.2a. For ease of implementation, the moments $M_1$ and $M_2$ are applied in the form of equal and opposite uniform stress distributions. The loading illustrated in Fig. 2.2a results in equal and opposite bending moments, which corresponds to the case of pure mode I. A pure mode II loading would have the applied moments in the same direction. Throughout this study, the full range of mixed-mode loading has been considered by first holding $M_1$ constant and varying $M_2$ in the range $-M_1 \leq M_2 \leq M_1$ (or $-90^\circ \leq \psi \leq 0^\circ$), and then holding $M_2$ constant and varying $M_1$ in the range $-M_2 \leq M_1 \leq M_2$ (or $0^\circ \leq \psi \leq 90^\circ$).

The finite element model uses 8-node bi-quadratic reduced integration elements provided by the commercial software package ABAQUS. The analysis employs incremental small-strain elastoplasticity with Von Mises yield criterion, which is generally appropriate for metals and other ductile solids. Reduced integration elements are chosen for their accuracy during nearly incompressible material response, which results from the pressure-independent yielding assumed in the elastoplasticity formulation. The elements are highly biased toward the crack tip, with the smallest element measuring only 0.5 μm. As discussed in [1, 2], such fine mesh resolution is needed to accurately resolve the reversed plastic zone, and to ensure convergence of the continuum theory solution.‡

The total plastic dissipation per cycle is insensitive to the choice of crack-tip elements, so standard (as opposed to quarter-point) elements are used at the crack tip. A rigorous convergence study was performed in both time and space by successively halving both the element edge length and the maximum time step used in ABAQUS’ automatic time-stepping algorithm.

As described in [1, 2], the total plastic dissipation is automatically calculated by ABAQUS, and is readily extracted from the finite element output. The quantity $\frac{dW}{dN}$ is determined as the change in total plastic dissipation per unit width during the second of two complete load cycles (where $M_1$ and $M_2$ are proportionally varied from zero to their maximum values). Two load cycles are necessary.

‡It should be noted that convergence of the continuum solution does not police its applicability. As such, care should be taken in applying the results of this work for cases in which the reversed plastic zone is on the order of the grain size of the material.
to allow plastic shakedown of the bi-linear hardening model, after which the plastic dissipation per cycle remains constant. While the results presented herein correspond to a load ratio of \( R = \frac{K_{\text{min}}}{K_{\text{max}}} = 0 \), the applied load ratio was shown in [1, 2] to have only a negligible effect on the plastic dissipation per cycle, and is not considered further herein.

For all cases presented in this chapter, the applied energy release rate was \( G = 200 \text{ J/m}^2 \). Each layer had identical elastic properties of \( E = 73.1 \text{ GPa} \) and \( v = 1/3 \), with a yield strength of \( \sigma_y = 300 \text{ MPa} \). These values were shown in [2] to provide a sufficiently resolved reversed plastic zone, while still maintaining small-scale yielding over the full range of applied mode-mix ratios.

The small scale yielding criteria was independently verified for each analysis using the contour integral approach implemented in ABAQUS. The contour integral is capable of giving \( J \)-integral estimates based on a series of expanding rings of elements around the crack tip. The values of the \( J \)-integral have been calculated at maximum load and directly compared to Eq. (2.7) (\( J = G \) for linear elastic fracture). While crack tip plasticity invalidates \( J \)-integral estimates within the plastic zone, those taken from contours outside the plastic zone have been found to agree with Eq. (2.7) to five significant digits. Such agreement can only be obtained in the presence of small-scale yielding when the elastic stress fields are governing the plasticity near the crack tip.

### 2.3.2 Bi-Linear Kinematic Hardening Model

In this study, the effects of material hardening behavior have been included by means of the bi-linear kinematic hardening model available in ABAQUS. As shown in Fig. 2.3, the uniaxial response for bi-linear kinematic hardening is completely described by the elastic modulus \( E \), the yield strength \( \sigma_y \) and the hardening (or tangent) modulus \( E_t \). For a given yield strength, the hardening behavior can be characterized by the ratio \( E_t/E \), which varies in the range \( 0 \leq E_t/E \leq 1 \) (i.e. from elastic-perfectly plastic to purely elastic response). As shown in Fig. 2.3, bi-linear kinematic hardening provides for a reduced yield strength upon reversal (the Bauschinger effect), and predicts plastic shakedown after only a single cycle. In the context of classical small-strain elastoplasticity, the bi-linear kinematic hardening model can be used to approximate the stabilized cyclic response during constant amplitude loading.

\[ \frac{\text{The plane strain plastic dissipation has been shown to be independent of Poisson's ratio for typical ductile metals, where } v \geq 0.3. \text{ A detailed discussion on the effects of Poisson's ratio and plastic constraint is given in [1, 2], and is not reiterated here.} \]
Figure 2.2: Finite element mesh, loading, and boundary conditions.

(a) Undeformed finite element model with boundary and loading conditions

(b) Finite element model with a deformation factor of 1000
2.3.3 Crack Tip Plasticity

The effect of mode-mix ratio on the evolution of forward and reversed plastic zones during a complete load cycle ($R = G_{\text{min}}/G_{\text{max}} = 0$) are illustrated for both plane stress and plain strain in Figs. 2.4-2.6. The material considered is elastic-perfectly plastic ($E_t/E = 0$) with elastic modulus $E = 73.1$ GPa, yield strength $\sigma_y = 300$ MPa, and Poisson’s ratio $\nu = 0.3$. For ease of comparison, the applied range of energy release rate is held constant at $\Delta G = 200$ J/m$^2$.

As shown in Figs. 2.4a and 2.4b, both the shape and size of the forward plastic zones under pure mode I loading are in keeping with expectations from classical fracture mechanics analyses, as well as with previous results in the literature [76]. In particular, unconstrained yielding results in a much larger plastic zone in plane stress (Fig. 2.4b) than in plane strain (Fig. 2.4a). Moreover, while the forward plastic zones scale with $(\Delta K/\sigma_y)^2$, the reversed plastic zones scale with $(\Delta K/2\sigma_y)^2$, which is in keeping with the plastic superposition argument first put forth by Rice [16]. As such, the sizes of the reversed plastic zones of Figs. 2.4c and 2.4d are roughly 1/4 those of the forward plastic zones of Figs. 2.4a and 2.4b.

The asymmetry of crack tip plasticity during mixed-mode loading is evident from Fig. 2.5, where the phase angle of $\psi = 41^\circ$ represents a nearly equal mix of mode I and II loading. More importantly, a comparison of the scale factors in Figs. 2.4 and 2.5 reveals that an increase in mode II
Figure 2.4: Forward and reversed plastic zones in pure mode I with $\Delta G = 200 J/m^2$, $E = 73.1$ GPa, $\nu = 0.3$, and $\sigma_y = 300$ MPa.
Figure 2.5: Forward and reversed plastic zone when $\psi = 41^\circ$ with $\Delta\gamma = 200 J/m^2$, $E = 73.1$ GPa, $\nu = 0.3$, and $\sigma_y = 300$ MPa.
Figure 2.6: Forward and reversed plastic zones in pure mode II with $\Delta G = 200 \text{J/m}^2$, $E = 73.1 \text{ GPa}$, $\nu = 0.3$, and $\sigma_y = 300 \text{ MPa}$. 
component significantly increases the extent of crack tip plasticity in both plane stress and plane strain. Indeed, the sizes of the forward and reversed plastic zones under pure mode II loading (Fig. 2.6) are several times those of pure mode I, which is also in keeping with classical fracture mechanics analyses [76]. However, the increase in plastic zone size with mode-mix ratio is greater in plane strain than in plane stress, so that the difference between the two decreases with increasing mode-mix. This result might be explained in terms of a decrease in plastic constraint with increasing shear component, which is a direct result of the Von Mises yield criterion. In the limit of pure shear, the principal stresses directly ahead of the crack are equal and opposite, so that the out-of-plane stress at the onset of yielding is zero even in plane strain. Hence, in the limit of pure mode II loading, the plastic constraint against first yielding along the crack plane vanishes entirely. That said, the incompressibility assumed in the classical plasticity formulation still provides for a constraint against subsequent yielding within the reversed plastic zone in plane strain.

It should finally be noted that the plastic zone sizes of Figs. 2.4-2.6 are well within the range of small-scale yielding, which has been independently verified for all cases considered herein. First, $J$-integral estimates available in ABAQUS have been calculated at maximum load and directly compared to Eq. (2.7) ($J = \mathcal{G}$ for linear elastic fracture). While crack tip plasticity invalidates $J$-integral estimates within the plastic zone, those taken from contours outside the plastic zone have been found to agree with Eq. (2.7) to five significant digits. Such agreement can only be obtained in the presence of small-scale yielding. In addition, interaction integral estimates for the stress intensity factors have been obtained from elastic finite element runs of the specimen geometry. The results have been in excellent agreement with the the closed-form solutions of Eqs. (2.3) and (2.4), as well as with the $J$-integral estimates obtained from the elastic-plastic analysis.

### 2.3.4 Plastic Dissipation Per Cycle

Representative plane strain finite element results for the total plastic dissipation per cycle $\frac{dW}{dN}$ as a function of applied range of energy release rate $\Delta\mathcal{G}$ are plotted over the full range of mode-mix values in Fig. 2.7, where the material considered is that of Figs. 2.4-2.6. Since ABAQUS automatically calculates the total plastic dissipation per unit width $W$ during each load step, extraction of the finite element results is trivial. As shown in Fig. 2.7, the quantity $\frac{dW}{dN}$ is calculated as the change in plastic dissipation per unit width during the second cycle, denoted here as $\frac{dW}{dN} = \Delta W_{24}$. Two cycles
Figure 2.7: Effect of Mode Mix on \( \frac{dW}{dN} \) in plane strain when \( E_t/E = 0 \), \( E = 73.1 \) GPa, \( v = 0.3 \), and \( \sigma_y = 300 \) MPa.

are necessary because the plastic deformation during the first load cycle occurs throughout the forward plastic zone, while plastic deformation in subsequent cycles is restricted to the reversed plastic zone. Moreover, for both elastically-perfectly plastic and bi-linear kinematic hardening materials, the plastic dissipation remains constant after the second cycle. As such, the quantity \( \frac{dW}{dN} = \Delta W_{24} \) represents a steady-state value of \( \frac{dW}{dN} \) in all subsequent cycles. It should finally be noted that a rigorous convergence study was performed in both time and space by successively halving both the element edge length and the maximum time step used in ABAQUS’ automatic time-stepping algorithm. In so doing, the value of \( \frac{dW}{dN} \) from the production mesh of Fig. 2.2 varied less than 1 percent from the value of \( \frac{dW}{dN} \) obtained from the finest mesh.

As evident from the log-log plot of Fig. 2.7, the numerical data follows a power-law relation of the form
\[
\frac{dW}{dN} = C (\Delta \mathcal{G})^m.
\] (2.8)

Least square curve fits of the numerical data showed that the exponent of the power law relation for all cases considered was in the range \(1.99 \leq m \leq 2.04\). Thus, to within numerical error, the exponent of the power law relation is \(m = 2\), and is independent of the mode-mix ratio. In light of Eq. (2.1), the predicted fatigue crack growth rate is proportional to \((\Delta \mathcal{G})^2\), which is within the range of observations of mixed-mode fatigue crack growth on ductile interfaces [45–50]. It should also be noted that for an applied load ratio \(R = 0\), the quantity \((\Delta \mathcal{G})^2\) corresponds directly to \((\Delta |K|)^4\), or for the case of mode I loading, \((\Delta K)^4\). This is in keeping with previous energy-based theories of fatigue crack growth under mode I loading, and is consistent with fatigue crack growth data for a variety of ductile metals [1].

### 2.3.5 Nondimensionalization

In order to facilitate a general presentation of results, the plastic dissipation per cycle can be nondimensionalized in terms of the yield strength, applied energy release rate and elastic modulus as

\[
\frac{dW^*}{dN} = \frac{\sigma_y^2}{E \Delta \mathcal{G}^2} \frac{dW}{dN}.
\] (2.9)

In light of Eq. (2.1), the fatigue crack growth rate can be written in terms of the dimensionless plastic dissipation per cycle as

\[
\frac{da}{dN} = \frac{E \Delta \mathcal{G}^2}{\sigma_y^2 \mathcal{G}_c^2} \frac{dW^*}{dN}.
\] (2.10)

In general, the dimensionless plastic dissipation \(\frac{dW^*}{dN}\) depends on the applied mode-mix ratio \(\psi\), Poisson’s ratio \(v\) and the hardening ratio \(E_t/E\). However, as defined in Eq. (3.3), \(\frac{dW^*}{dN}\) is independent of the ratio \(\sigma_y/E\) [1]. An analytical justification of the nondimensionalization can be found in Section 4.2.
Normalized Plastic Dissipation

\[ \frac{dW^*}{dN} = \tan^{-1}(K_{II}/K_I) \]

**Figure 2.8:** Dimensionless plastic dissipation \( \frac{dW^*}{dN} \) vs. mode mix ratio \( \psi \) for \( E_t/E = 0 \) and \( \nu = 0.3 \) in plane strain.

### 2.4 Numerical Results and Discussion

In this section, results are presented for the dimensionless plastic dissipation per cycle \( \frac{dW^*}{dN} \) over the full range of applied mode-mix ratios and relevant constitutive model parameters. Results provide significant insight into the effect of mode-mix ratio, crack tip constraint and material hardening behavior on the energy dissipated during fatigue crack growth.

#### 2.4.1 Effect of Mode-Mix

When normalized according to Eq. (3.3), the data plotted in Fig. 2.7 are collapsed onto the master plot of Fig. 2.8. This plot contains all ninety points in Fig. 2.7, which validates the normalization of the data with Eq. (3.3). Clearly, the plastic dissipation increases significantly with mode mix, and is between one and two orders of magnitude greater in mode II than in mode I. This result might be
2.4.2 Effect of Plastic Constraint

A family of curves showing the dimensionless plane strain plastic dissipation as a function of mode-mix over the full range of Poisson’s ratio and for $E_i/E = 0$ is shown in Fig. 2.9. Results are also shown for plane stress, although these are independent of $v$. As shown in Fig. 2.9, the plastic dissipation is greatest in plane stress, and decreases with increasing plastic constraint (i.e., increasing $\theta$).

---

Figure 2.9: Dimensionless plastic dissipation energy $\frac{dW^*}{dN}$ vs. mode mix $\psi$ in plane stress and plane strain.

---

Note that for the case of $v = 0$, plane stress and plane strain are equivalent only in the elastic regime; for the case of plane strain, the incompressibility of plastic deformation results in constrained yielding in the elastic-plastic regime.
Poisson’s ratio in plane strain). This result might be expected based on the relative plastic zone sizes in plane stress and plane strain, and is in keeping with the mode I results of [1].

The plot of Fig. 2.9 also contains the master curve of Fig. 2.8, which corresponds to \( v = 0.3 \) in plane strain. Evidently, changes in Poisson’s ratio (i.e., changes in plastic constraint) result in roughly uniform shifts of the master curve, although the magnitude of such shifts decreases with increasing plastic constraint. An important result is that for \( v \geq 0.3 \), values of \( \frac{dW^*}{dN} \) vary by less than 0.5%. Thus, for all values of the mode-mix ratio, the effect of Poisson’s ratio on \( \frac{dW^*}{dN} \) is negligible for typical ductile metals, where \( v \geq 0.3 \). It should finally be noted that since the plastic dissipation increases with mode-mix, the uniform shifts in the master curve suggest that the relative effect of plastic constraint decreases with increasing shear component. This supports the previous observation regarding the similarity of plane stress and plane strain plastic zone sizes for pure mode II.

2.4.3 Effect of Hardening Modulus

Plane strain numerical results for \( \frac{dW^*}{dN} \) vs. \( \psi \) are plotted in Fig. 2.10 over the full range of \( E_t/E \) and for \( v = 0.3 \). The case of \( E_t/E = 0 \) (elastic-perfectly plastic response) coincides with the master curve of Fig. 2.8, and represents an upper bound on the plastic work per cycle in plane strain. As should be expected, \( \frac{dW^*}{dN} \) decreases with increasing hardening modulus, and approaches zero for the case of \( E_t/E = 1 \) (purely elastic response). Thus, for all values of mode-mix, the effect of increasing material hardening is to decrease the plastic work. In an absolute sense, the results of Fig. 2.10 indicate that \( \frac{dW^*}{dN} \) is more sensitive to hardening ratio for high values of \( \psi \). On the other hand, the effect of mode-mix is substantially more pronounced for low values of \( E_t/E \), which is typical of ductile metals.

The effects of hardening modulus on the dimensionless plastic dissipation have been considered for the case of mode I loading (C(T) specimen geometry) in [1]. However, different specimen geometries and loading typically exhibit slight differences in both the shape and size of the crack tip plastic zones, which is typically attributed to differences in “T-stress” at the crack tip [76]. In order to investigate the sensitivity of \( \frac{dW^*}{dN} \) to specimen geometry, both the current results for mode I loading and those of [1] are plotted verses \( E_t/E \) for both plane stress and plain strain (\( v = 0.3 \)) in Fig. 2.11. The most measurable difference is for the case of \( E_t/E = 0 \) in plane stress; however,
Figure 2.10: Effect of the tangent modulus ratio on $\frac{dW^*}{dN}$ vs. $\psi$. 

$\psi = \tan^{-1}(\frac{K_{II}}{K_I})$
Effect of Specimen Geometry

Figure 2.11: Comparison of current specimen (DCB) results in mode I and C(T) results from [1].
this difference decreases with increasing hardening modulus, and appears to be negligible in plane strain. Hence, results suggest that specimen geometry has only a limited effect on the total plastic dissipation during plane strain fatigue crack growth in ductile solids.

### 2.4.4 Implications of Results

The numerical results suggest that the mode-mix ratio is a dominant factor in the total plastic dissipation per cycle, which increases significantly with mode II component. In light of Eq. (2.1), this might suggest that fatigue crack growth rates should also increase with mode-mix ratio, which is in contrast to experimental observations [37]. However, the fatigue crack growth rate of Eq. (2.1) is inversely proportional to the critical energy release rate $G_c$, which is also known to increase sharply with mode-mix ratio. Hence, validation of Eq. (2.1) requires both mixed-mode fracture toughness data and Paris-regime crack growth data under sustained mixed-mode loading.

In light of the difficulty in achieving self-similar crack propagation in a homogeneous material under sustained mixed-mode loading, such data is not readily available. Moreover, work by Magill and Zwerneman [44] suggests that propagation of cracks in homogeneous materials under constrained mixed-mode loading conditions can be three-dimensional in nature, with the crack orienting itself in favor of mode I conditions along the crack front. It is currently unclear to what extent this occurs during steady-state delamination of layered material systems. Subsequent experimental studies of steady-state delamination of layered materials under both monotonic and fatigue loading conditions would provide further insight into this matter, and could be used to assess the validity of the crack growth law of Eq. (2.1).

Even in the absence of a comparison with sustained mixed-mode crack growth data, the results of this work are useful for comparison with dissipated energy measurements during fatigue crack growth. In particular, Ranganathan [36] has reported dissipated energy measurements under mode I loading which are substantially higher than those predicted by finite element models. Such discrepancies had been attributed in part to a mix of crack extension modes associated with the deformation mechanism at the crack tip. In light of the current work, the presence of a mode II component can significantly increase the dissipated energy at a fatigue crack tip, which tends to support the observation in [36]. Perhaps subsequent experimental studies of dissipated energy under sustained mixed-mode crack growth will allow a more thorough comparison with the results of this work.
2.5 Conclusion

Previous results in the literature suggest that fatigue crack growth rates under mode I loading can be closely correlated with the total plastic dissipation per cycle ahead of the crack. The current chapter has extended the dissipated energy approach to steady-state crack growth under mixed-mode loading conditions. The total plastic dissipation per cycle has been extracted from 2-D elastic-plastic finite element analyses of a stationary crack in a general mixed-mode specimen geometry under constant amplitude loading. Both elastic-perfectly plastic and bilinear kinematic hardening constitutive behaviors have been considered, and numerical results for a dimensionless plastic dissipation per cycle have been presented over the full range of relevant mechanical properties and mixed-mode loading conditions. Numerical results reveal that the total plastic dissipation per cycle decreases with increases in both material hardening behavior and crack tip constraint, which is in keeping with previous results for mode I loading. Results further indicate that the total plastic dissipation is a strong function of applied mode-mix ratio, and increases sharply with an increase in mode II component.
3 Plastic Dissipation in Mixed-Mode Along Plastically Mismatched Interfaces

In this chapter, the total plastic dissipation per cycle is determined by plane strain elastic-plastic finite element modeling of a stationary crack in a general mixed-mode layered specimen geometry, with mismatches in both yield strength and hardening modulus across the interface. Comprehensive numerical results for a dimensionless plastic dissipation per cycle are presented over the full range of relevant material combinations and mixed-mode loading conditions. The numerical results provide substantial insight into the relative effects of applied mode-mix ratio, strength mismatch and hardening mismatch on the plastic dissipation during fatigue crack growth, and may ultimately contribute to the design of fatigue-resistant interfaces.

3.1 Problem Considered

3.1.1 Global Problem Geometry

The mixed-mode layered specimen geometry considered herein is shown in Fig. 2.1, and follows that used in the previous chapter and in Ref. [2]. The specimen is composed of two bonded layers of plastically mismatched materials #1 and #2 having equal thickness $h$ and length $L$, and with a crack of length $a$ along the interface. The loading consists of pure bending moments per unit width $M_1$ and $M_2$ applied to the top and bottom layers, which are equilibrated by a symmetry condition on the right hand side. As discussed shortly, variation in the bending moments $M_1$ and $M_2$ allows consideration of the full range of mode-mix values, from pure mode I to pure mode II. Both the length $L$ and the crack length $a$ are sufficiently long to allow for steady-state conditions at the crack tip, so that the energy release rate is independent of crack length. As discussed in [51, 53–55, 57],
steady-state conditions prevail once the crack is sufficiently long such that the specimen exhibits beam-type behavior far behind and ahead of the crack tip. As a reminder, the dimensions used in the numerical analyses in this Chapter were \( L = 50 \text{ mm} \), \( h = 5 \text{ mm} \), and \( a = 25 \text{ mm} \), which are sufficient for steady-state conditions at the crack tip.

### 3.1.2 Elastic Solution

Since the elastic problem is the same as in the previous chapter, the results for the stress intensity factors, mode mix and strain energy release rate are unchanged.

### 3.1.3 Elastic-Plastic Response

In this study, the elastic-plastic constitutive response of each material is assumed to obey a bi-linear kinematic hardening model. As shown in Fig. 2.3, the uniaxial response for bi-linear kinematic hardening is completely described by the elastic modulus \( E \), the yield strength \( \sigma_y \) and the hardening (or tangent) modulus \( E_t \). For a given yield strength, the hardening behavior can be characterized by the ratio \( E_t/E \), which varies in the range \( 0 \leq E_t/E \leq 1 \) (i.e., from elastic-perfectly plastic to purely elastic response). As shown in Fig. 2.3, bi-linear kinematic hardening provides for a reduced yield strength upon load reversal (the Bauschinger effect), and predicts plastic shakedown after only a single cycle. In the context of classical small-strain elastoplasticity, the bi-linear kinematic hardening model represents a first approximation to the stabilized cyclic response during constant amplitude loading.

### 3.1.4 Plastic Mismatches Considered

As shown in Fig. 3.1, the material combinations considered herein include a mismatch in yield strength (Fig 3.1a), a mismatch in hardening modulus (Fig. 3.1b), or a mismatch in both yield strength and hardening modulus (Fig. 3.1c). Throughout this dissertation, mismatches in hardening modulus are investigated by varying the ratios \( E_{t1}/E \) and \( E_{t2}/E \), while mismatches in yield strength are characterized in terms of the dimensionless parameter

\[
\hat{\sigma} = \frac{\sigma_{y1} - \sigma_{y2}}{\sigma_{y1} + \sigma_{y2}}.
\] (3.1)
(a) Different yield strengths.

(b) Different hardening moduli.

(c) General case of plastic mismatch.

Figure 3.1: Plastic mismatches considered.
As defined in Eq. (3.1), values of $\hat{\sigma}$ are bounded in the range $-1 \leq \hat{\sigma} \leq 1$. A positive value of $\hat{\sigma}$ signifies a top layer which is stronger than the bottom layer. As the relative strength of the top layer increases, then $\hat{\sigma} \to 1$. Likewise, a negative value of $\hat{\sigma}$ signifies a bottom layer which is stronger than the top layer. As the relative strength of the bottom layer increases, then $\hat{\sigma} \to -1$. The case when $\hat{\sigma} = 0$ means there is no strength mismatch across the interface. For a given value of $\hat{\sigma}$, the ratio of yield strengths is given by

$$\frac{\sigma_y_1}{\sigma_y_2} = \frac{\hat{\sigma} + 1}{1 - \hat{\sigma}},$$

(3.2)

and lies in the range $0 \leq \frac{\sigma_y_1}{\sigma_y_2} \leq \infty$.

### 3.2 Numerical Modeling Procedures

The finite element modeling used for plastically mismatched interfaces was identical to the procedure outlined in Section 2.3.1 on page 14 with the exception that the plastic material properties are different for each layer. In this chapter, values of strength mismatch in the range $0 \leq \hat{\sigma} \leq 1$ were investigated by fixing the strength of the bottom layer at $\sigma_y_2 = 300$ MPa and increasing the strength of the top layer. Similarly, values of strength mismatch in the range $-1 \leq \hat{\sigma} \leq 0$ were investigated by fixing the strength of the top layer at $\sigma_y_1 = 300$ MPa and increasing the strength of the bottom layer. Since the extent of crack-tip plasticity is controlled by the weaker material, this ensured small-scale yielding for all material combinations considered.

#### 3.2.1 Nondimensionalization

As described in [2], the total plastic dissipation per cycle at the tip of a mixed-mode crack in a homogeneous material scales with the square of the applied energy release rate, and can be normalized in terms of the elastic properties and the yield strength of the material. Throughout the current study, where a mismatch in plastic properties exists across the interface, the plastic dissipation is normalized in terms of the yield strength of the bottom layer as

$$\frac{dW^*}{dN} = \frac{\sigma_y_2^2}{E\Delta G^2} \frac{dW}{dN},$$

(3.3)
As defined in Eq. (3.3), the dimensionless plastic dissipation \( \frac{dW^*}{dN} \) depends on the applied mode-mix ratio \( \psi \), the strength mismatch parameter \( \hat{\sigma} \), and the hardening modulus ratios \( E_{t1}/E \) and \( E_{t2}/E \). In the context of the dissipated energy theory of Eq. (1.1), the fatigue crack growth rate can be written in terms of the dimensionless plastic dissipation as

\[
\frac{da}{dN} = \frac{E_D G_c^2}{\hat{\sigma}^2 G_c^2 dW^* dN},
\]

where numerical results for \( \frac{dW^*}{dN} \) are provided in the next section. As previously mentioned, \( G_c \) is the critical strain energy release rate which represents the monotonic fracture toughness of the interface. As such, \( G_c \) is a mechanical property of the interface. In the context of Eq. (3.4), the energy released in monotonic loading is the same as the energy released in cyclic loading as described in reference [1].

### 3.3 Numerical Results and Discussion

In this section, numerical results are presented for the dimensionless plastic dissipation per cycle \( \frac{dW^*}{dN} \) over the full range of applied mode-mix ratios and plastic mismatches. The results provide substantial insight into the relative effects of applied mode-mix ratio, strength mismatch and hardening mismatch on the plastic dissipation during fatigue crack growth.

#### 3.3.1 Effect of Yield Strength Mismatch

This section considers the material combinations illustrated Fig. 3.1a, where each material is elastic-perfectly plastic, but where a mismatch in yield strength exists across the interface. Without loss of generality, comprehensive results the geometry of Fig. 2.1 can be presented by considering both positive and negative values of the strength mismatch parameter \( \hat{\sigma} \) over only half the range of mode mix values (e.g., \( 0^\circ \leq \psi \leq 90^\circ \)), or by considering both positive and negative values of the mode-mix parameter \( \psi \) over only half the range of strength mismatches (e.g., \( 0 \leq \hat{\sigma} \leq 1 \)). In this study, the latter approach is adopted, so that attention is restricted to strength mismatches in the range \( 0 \leq \hat{\sigma} \leq 1 \). Hence, given the normalization of Eq. (3.3) in terms of the bottom layer, increasing the strength mismatch in the range \( 0 \leq \hat{\sigma} \leq 1 \) corresponds to increasing the strength of the top layer.
Figure 3.2: Dimensionless plastic dissipation energy as a function of yield strength mismatch $\hat{\sigma}$ for elastic-perfectly plastic materials ($E_{11}/E = E_{12}/E = 0$) over the full range of applied mode mix $\psi$. 
A plot of the dimensionless plastic dissipation $\frac{\partial W^*}{\partial N}$ as a function of the strength mismatch parameter $\hat{\sigma}$ is shown in Fig. 3.2 over the full range of applied mode-mix ratio $\psi$. In general, there is a slight difference in $\frac{\partial W^*}{\partial N}$ for positive and negative values of the same mode-mix ratio, except for the case of pure mode II ($\psi = \pm 90^\circ$). This can be expected based on the asymmetry of the problem for general mixed-mode loading. However, for all applied mode-mix ratios, the plastic dissipation is greatest for the case of no mismatch ($\hat{\sigma} = 0$), and decreases with increasing values of $\hat{\sigma}$. Interestingly, there is an asymptotic effect of the yield strength mismatch for values of $|\hat{\sigma}| \geq 0.25$, which corresponds to a yield strength ratio of $\sigma_{y1}/\sigma_{y2} = 5/3$. Hence, with respect to the plastic dissipation at the crack tip, an interface between two ductile metals with a strength ratio of only $5/3$ is no different than an interface with a strength ratio of infinity, where one material exhibits no yielding at all!

The above results can best be explained by considering the plastic zones at the crack tip. Figures 3.3, 3.4 and 3.5 show the effect of strength mismatch on the reversed crack-tip plastic zones for $\psi = 0^\circ$, $\psi = 41^\circ$ and $\psi = 90^\circ$, respectively. In each figure, results are presented for values of $\hat{\sigma} = 0, 0.1, 0.25$ and 1.0. For the case of no mismatch ($\hat{\sigma} = 0$), the crack tip plastic zones are symmetric about the crack plane for both pure mode I (Fig. 3.3a) and pure mode II (Fig. 3.5a), while Fig. 3.4a illustrates the asymmetry of crack tip plasticity for general mixed-mode loading. For each value of mode-mix, increasing the plastic mismatch parameter acts to increase the strength of the top layer, which results in a decrease in plastic zone size. The crack tip plastic zone in the top layer becomes negligible when $\hat{\sigma} = 0.25$, in which case the crack tip plasticity closely resembles that for $\hat{\sigma} = 1.0$. Similar asymptotic behavior in crack tip plastic zones for plastically-mismatched interfaces has been reported by Lee [66,67], and tends to support the results of Fig. 3.2.

At first thought, the symmetry of the problem might suggest that the plastic dissipation in pure modes I and II would be reduced by half for the case of an infinite strength mismatch ($\hat{\sigma} = 1.0$), when one layer does not yield. However, according to Fig. 3.2, the decrease in plastic dissipation between $\hat{\sigma} = 0$ and $\hat{\sigma} = 1.0$ is much less than 50%. A factor of two difference in the plastic dissipation energy would require that the problem could be modeled with a symmetry condition along the crack plane, so that the plastic zone size in the bottom layer would be identical for $\hat{\sigma} = 0$ and $\hat{\sigma} = 1.0$. Inspection of the plastic zones in Figs. 3.3 and 3.5 reveals that this is clearly not the case. In particular, increasing the strength mismatch results in a measurable increase in the plastic
Figure 3.3: Effect of strength mismatch on reversed plastic zones in pure mode I, for elastic-perfectly plastic materials \( (E_{t1}/E = E_{t2}/E = 0) \) with \( E = 73.1 \) GPa, \( \nu = 1/3, \sigma_{y2} = 300 \) MPa and \( \Delta \mathcal{G} = 200 \) J/m\(^2\).
Figure 3.4: Effect of strength mismatch on reversed plastic zones for $\psi = 41^\circ$, for elastic-perfectly plastic materials ($E_{1}/E = E_{2}/E = 0$) with $E = 73.1$ GPa, $\nu = 1/3$, $\sigma_{y2} = 300$ MPa and $\Delta G = 200 J/m^2$. 

(a) $\dot{\sigma} = 0$
(b) $\dot{\sigma} = 0.1$
(c) $\dot{\sigma} = 0.25$
(d) $\dot{\sigma} = 1.0$
Figure 3.5: Effect of strength mismatch on reversed plastic zones in pure mode II, for elastic-perfectly plastic materials \((E_{11}/E = E_{12}/E = 0)\) with \(E = 73.1\) GPa, \(\nu = 1/3\), \(\sigma_{y2} = 300\) MPa and \(\Delta G = 200\) J/m\(^2\).
zone size of the bottom layer. As the strength mismatch increases, the increased load carrying capacity of the top layer is unmatched by the bottom layer, which consequently experiences an increase in crack tip plasticity.

It should finally be noted that compared to strength mismatch, the applied mode-mix ratio has a dominant effect on the plastic dissipation per cycle for elastic-perfectly plastic materials. This is further illustrated by Fig. 3.6, in which the normalized plastic dissipation $\frac{dW^*}{dn}$ is plotted as a function of the mode mix ratio $\psi$ over the full range of strength mismatches. The “master curve” corresponding to $\hat{\sigma} = 0$ is identical to that reported in [2] for values of $0^\circ \leq \psi \leq 90^\circ$, and is symmetric about $\psi = 0^\circ$. A slight asymmetry about $\psi = 0^\circ$ exists for the remaining values of $\hat{\sigma}$, except for the case of pure mode II. Based on the results in Fig. 3.6, changes in applied mode-mix ratio can affect the plastic dissipation by more than an order of magnitude, while the effects of strength mismatch are comparatively small. The dominant effect of mode-mix ratio can be attributed to the plastic zone sizes in Figs. 3.3-3.5, which increase significantly with an increase in mode II component.

### 3.3.2 Effect of Hardening Modulus Mismatch

This section considers the material combinations illustrated Fig. 3.1b, where the two materials have the same yield strength, but where a mismatch in hardening modulus exists across the interface. Since the shape and size of the crack tip plastic zones are primarily controlled by the yield strength, they are generally similar to those illustrated in Figs. 3.3-3.5 for the case of $\hat{\sigma} = 0$. However, the plastic dissipation occurring within each plastic zone is controlled by the area bounded by the bi-linear kinematic hardening response (Fig. 2.3), which is heavily dependent on the hardening modulus in each material.

A family of plots for the plastic dissipation $\frac{dW^*}{dn}$ as a function of the mode mix ratio $\psi$ is provided in Fig. 3.7 over the full range of mismatches in hardening modulus. Each of the four plots shows the effect of increasing $E_{t1}/E$ in the range $0 < E_{t1}/E < 1$ for a fixed value of $E_{t2}/E$. The plot of Fig. 3.7a is for the case of $E_{t2}/E = 0$, which corresponds to elastic-perfectly plastic response in the bottom layer. The curve for $E_{t1}/E = 0$ corresponds to the master curve of Fig. 3.6, where both materials are elastic-perfectly plastic with no mismatch in yield strength across the interface. As might be expected, increasing the hardening ratio $E_{t1}/E$ acts to decrease the plastic dissipation per cycle. The case $E_{t1}/E = 1.0$ results in zero plastic dissipation in the top layer, and is identical to the
Figure 3.6: Dimensionless plastic dissipation $\frac{\partial W}{\partial N^*}$ as a function of mode mix $\psi$ for elastic-perfectly plastic materials ($E_1/E = E_{ij}/E = 0$) over the full range of strength mismatches.
case of an infinite strength mismatch ($\hat{\sigma} = 1.0$) plotted in Fig. 3.6. In general, increasing the ratio $E_{t1}/E$ for the case $E_{t2}/E = 0$ has only a limited effect on the plastic dissipation per cycle, which is dominated by the elastic-perfectly plastic response of the bottom layer.

The effect of increasing the ratio $E_{t2}/E$ is illustrated in Figs. 3.7b-d. As the hardening of the bottom layer increases, changes in $E_{t1}/E$ have a much larger effect on the plastic dissipation per cycle, particularly for high values of the mode-mix ratio $\psi$. It should be noted that the hardening modulus for typical ductile metals falls in the range $0 < E_t/E < 0.1$. Hence, the practical effect of mismatches in hardening modulus for ductile metal interfaces is bounded by the plots of Figs. 3.7a and 3.7b.

The limiting case of $E_{t2}/E = 1.0$ is plotted in Fig. 3.7d, and yields the greatest effect of changes in $E_{t1}/E$. The curve corresponding to $E_{t1}/E = 0$ is the upper bound in Fig. 3.7d, and is the mirror image about $\psi = 0$ of the lower bound curve corresponding to $E_{t1}/E = 1$ in Fig. 3.7a. This is a result of the problem symmetry, where switching the materials is equivalent to changing the sign of the applied mode-mix ratio. As must be the case, the lower bound in Fig. 3.7d corresponds to both $E_{t1}/E = E_{t2}/E = 1.0$, in which case the plastic dissipation is zero.

In general, the results of Fig. 3.7 suggest that the plastic dissipation per cycle is dominated by the softer material. If one material is elastic-perfectly plastic (Fig. 3.7a), then changing the hardening modulus of the other material has only a limited effect. However, if one material is purely elastic (Fig. 3.7d), then changing the hardening modulus of the ductile layer can have a significant effect, even if variations are restricted to the practical range $0 < E_{t1}/E < 0.1$. Neglecting any difference in elastic properties, the latter result could have substantial implications for the design of ceramic/metal or other brittle/ductile interfaces, where yielding is restricted to the ductile layer. In other words, it is possible to make a more fatigue resistant interface, assuming fracture toughness is maintained, by increasing the yield strength of one of the layers. This mismatch has the effect of reducing the amount of plastic dissipation, even if just a small amount.

### 3.3.3 Effect of Combined Strength and Hardening Mismatch

The combined effect of mismatches in yield strength and hardening modulus (Fig. 3.1c) can be illustrated by considering families of plots similar to Fig. 3.7, but for increasing values of the strength mismatch parameter $\hat{\sigma}$. Such results are plotted in Figs. 3.8, 3.9, and 3.10 for values of $\hat{\sigma} = 0.1$, 0.43,
Figure 3.7: Family of plots showing the effect of the hardening modulus ratios $E_{i1}/E$ and $E_{i2}/E$ for the case of no strength mismatch ($\hat{\sigma} = 0$). The response is dominated by the layer with the least hardening.

Increasing the strength mismatch parameter $\hat{\sigma}$ acts to mitigate the effects of the hardening modulus mismatch, and tends to collapse the numerical results toward the lower bound curves in Figs. 3.7a-d. This is because the lower-bound curves correspond to $E_{i1}/E = 1.0$, which with respect to the crack tip plasticity is no different than an infinite strength mismatch $\hat{\sigma} = 1.0$. As such, the results for the limiting case of $\hat{\sigma} = 1.0$ are simply the bottom most curves in Figs. 3.7a-d, which are unchanged in Figs. 3.8-3.10.

The asymptotic effect of strength mismatch is illustrated for the case $E_{i2}/E = 0$ by comparing the results Figs. 3.8a and 3.9a, in which the curves are nearly collapsed for $\hat{\sigma} = 0.1$, and fully collapsed for $\hat{\sigma} = 0.25$. In the latter case, the plastic dissipation is completely controlled by the
Figure 3.8: Family of plots showing the effect of the hardening modulus ratios $E_{i1}/E$ and $E_{i2}/E$ for the case of $\hat{\sigma} = 0.1$. 

\(\psi = \tan^{-1}(K_i/K_c)\)
Figure 3.9: Family of plots showing the effect of the hardening modulus ratios $E_{t1}/E$ and $E_{t2}/E$ for the case of $\bar{\sigma} = 0.25$. 
Figure 3.10: Family of plots showing the effect of the hardening modulus ratios $E_{i1}/E$ and $E_{i2}/E$ for the case of $\sigma = 0.5$. The layer with the higher yield strength is not contributing anything to the plastic dissipation.
elastic-perfectly plastic response of the bottom layer, which is both the softer and weaker material. Hence, for an elastic-perfectly plastic bottom layer and a relatively strong top layer ($\hat{\sigma} > 0.25$), there is no effect of changing the hardening modulus of the top layer. That said, there is an increasing effect of hardening modulus mismatch for increasing values of $E_{t2}/E$, for which a higher value of strength mismatch is required for asymptotic response. For example, the data for $E_{t2}/E = 0.1$ in Fig. 3.10b are not quite collapsed even for $\hat{\sigma} = 0.5$, which is well into the asymptotic range for the case of $E_{t2}/E = 0$. Only in the limit of $\hat{\sigma} = 1.0$ are the data collapsed for all values of $E_{t2}/E$, in which case varying the hardening modulus of the top layer has no effect.

### 3.3.4 Implications of Results

Application of the current results to the fatigue crack growth law of Eq. (3.4) requires the critical energy release rate (i.e., the mixed-mode fracture toughness) for the given interface, which must be determined experimentally. However, assuming the fracture resistance is maintained, the results of this work suggest that intentional mismatches in strength and/or hardening modulus could potentially contribute to the design of a fatigue-resistant interface. In particular, increasing the relative strength or hardening modulus of one layer acts to decrease the plastic dissipation per cycle, which suggests a reduction in the fatigue crack growth rate as well. Such results are especially relevant for ceramic/metal and other brittle/ductile interfaces, where the plastic dissipation is sensitive to both the yield strength and hardening modulus of the ductile layer.

### 3.4 Conclusion

Numerical results for a dimensionless plastic dissipation per cycle have been presented over the full range of applied mode mix ratios and bi-linear kinematic hardening parameters, including mismatches in both yield strength and hardening modulus. Results suggest that the applied mode-mix ratio is a dominant factor, and can increase the plastic dissipation by more than an order of magnitude compared to mode I loading. For matching layers, elastic-perfectly plastic behavior is an upper bound on the dimensionless plastic dissipation per cycle, which decreases with increases in material hardening behavior. For a mismatch in plastic properties, increasing either the yield strength or hardening modulus of one layer acts to decrease the plastic dissipation per cycle, which is domi-
nated by the weaker or softer material. There is an asymptotic effect of yield strength mismatch for strength ratios beyond approximately 5/3, so long as at least one material exhibits elastic-perfectly plastic response. However, increasing hardening in the softer material acts to increase the strength ratio required for asymptotic response. In the limit where one material is fully elastic, the hardening modulus of the ductile layer can have a significant effect of the plastic dissipation per cycle. Ultimately, the results of this chapter suggest that intentional mismatches in plastic properties could potentially contribute to the design of fatigue resistant interfaces.
4 Determination of the Mode Mix in the Presence of an Elastic Mismatch

In previous chapters, the definition of the mode is straightforward since the elastic properties are homogeneous. However, if a crack is growing along an interface between layers with a mismatch in elastic properties, the mode of loading is not well understood. This chapter addresses this issue and proposes a technique of determining the mode mix based on an elastic energy criterion. Two candidate quantities were considered, the first being the total elastic strain energy within the contour defined by the yield strength and the second was a contour integral of the elastic strain energy density for a circular contour. The strain energy density was computed in closed form and numerical integration provided the energy quantity desired. Analytical solutions for the strain energy quantities of the homogeneous case provide insight for normalization as well as verification of the numerical procedures. The energy criterion based on the contour integral was chosen because it only relies on elastic properties and it is easier to compute.

The mode is defined by interpolating the range of dimensionless energy values when the minimum energy was set to pure mode I ($\psi = 0$) and the maximum energy was set to pure mode II ($\psi = \pm \pi/2$). Spanning the whole range of mode mix values requires varying both the ratio of the stress intensity factors as well as their magnitude. An example of a brass/solder interface illustrates this concept.

The proposed procedure for determining mode-mix is consistent with the classic definition of mode and requires choosing an appropriate characteristic length. The characteristic length can be given physical meaning based on an energetic definition of the mode.
4.1 Introduction

Applications of bonded interface mechanics arise from layered manufacturing, structural coatings, electronic packaging and other processes where a material is deposited on a dissimilar substrate. It is often energetically favorable for a crack to grow along an interface which may result in mixed mode cracking, both from the mode of the externally applied loads and a possible mismatch in elastic properties across the interface. In-plane mixed mode loading considers combinations of in-plane symmetric loading (mode I) and in-plane antisymmetric loading (mode II). The third mode of loading is for out of plane shear (tearing). These modes of loading stem from the solution to the boundary value problem governed by the theory of elasticity.

Williams [77] is credited with an eigenfunction expansion solution to the crack problem that shows both opening (mode I) and shearing (mode II) behavior is present at an interfacial crack, even if the externally applied loading is mode I. Also, Williams revealed the oscillating stress singularity near the crack tip. The singular stress fields oscillate in a logarithmic fashion which requires the stresses to change sign an infinite number of times as the distance from the crack tip approaches zero as demonstrated by the following equation:

$$\sigma_{i,j} = \frac{1}{\sqrt{2\pi r}} r^j f_{i,j}(\theta)$$ (4.1)

For the analysis of previous researchers and those contained herein to be valid, a small scale yielding assumption is necessary such that the perturbation from crack tip yielding does not significantly violate the assumptions of a mathematically perfect, traction free crack.

Further advances of the solution of the field equations of elasticity based on Mushkelishvili’s complex potential technique were performed by Cherepanov [78], England [79], Ergodan [80], and Rice and Sih [81] in which a complex stress intensity factor,

$$K = K_1 + iK_2$$ (4.2)

was introduced to characterize the stress fields. The magnitude of the complex stress intensity factor is determined from the relationship:

$$|K|^2 = K_1^2 + K_2^2$$ (4.3)
Figure 4.1: A stress-strain diagram of an elastic mismatch with equal plastic properties (both strain hardening rate and yield strength).

and the ratio of the stress intensity factors is defined in this chapter as

\[ \phi = \tan^{-1} \frac{K_2}{K_1} \]  
(4.4)

A detailed discussion of the development of interfacial fracture mechanics can be found in Broberg’s text [82]. The explicit form of the asymptotic near tip displacement and stress expressions are provided by Sun and Jih [83] as well as Nishioka [84]. These expressions are shown in Appendix B.

Alternative characterizations of the crack tip stress fields are based on the strain energy release rate. Rice [85] showed the equivalence of the \( J \)-integral to the strain energy release rate for elastic problems. While the strain energy release rate, \( \mathcal{G} \), governs the intensity of the crack tip singularities, it fails to represent the nature of the mode mix. As such, complete characterization of the crack tip stress fields requires either the complex stress intensity factor (in which the real part of \( K \) is analogous to the classic \( K_I \) and the imaginary part of \( K \) is analogous to \( K_{II} \)) or the strain energy release rate \( \mathcal{G} \) and the mode mix, \( \psi \).

Rice [86] explains in detail the issue of interpreting the mode mix in the context of a complex stress intensity factor. For the mode mix to be dimensionally meaningful, the mode mix must be determined as

\[ \psi = \tan^{-1} \frac{\Im \{K_I^{ie}\}}{\Re \{K_I^{ie}\}} \]  
(4.5)

where \( \Im \) refers to the imaginary part and \( \Re \) refers to the real part of a complex number. The quantity \( l \) is an arbitrary characteristic length required for dimensional reasons. Note that \( \phi = \psi \) when \( l = 1 \).
Figure 4.2: A schematic of an interfacial crack.

or \( \varepsilon = 0 \). Hutchinson and Suo [51, 52] have championed this definition of the mode in their work discussing the fracture mechanics of layered systems and have set the arbitrary length equal to the layer thickness of the top layer \( l = h \). Many recent researchers [48, 54–57] have also used this definition of the mode.

Equation (4.5) contains the term \( \varepsilon \), known as the oscillation index, which stems from Dundurs’ [87] two parameters

\[
\alpha = \frac{\mu_1 (\kappa_2 + 1) - \mu_2 (\kappa_1 + 1)}{\mu_1 (\kappa_2 + 1) + \mu_2 (\kappa_1 + 1)},
\]

\[
\beta = \frac{\mu_1 (\kappa_2 - 1) - \mu_2 (\kappa_1 - 1)}{\mu_1 (\kappa_2 + 1) + \mu_2 (\kappa_1 + 1)},
\]

where \( \mu_i \) is the shear modulus (modulus of rigidity) of the \( i \)th layer from Fig. 4.2, \( \kappa_i = 3 - 4v_i \) for plane strain and \( \kappa_i = (3 - v_i)/(1 + v_i) \) for plane stress. A more insightful form of Dundurs’ first parameter has the form

\[
\alpha = \frac{\bar{E}_1 - \bar{E}_2}{\bar{E}_1 - \bar{E}_2},
\]

where

\[
\bar{E}_i = \frac{E_i}{1 - v_i^2} \quad \text{for plane strain and}
\]

\[
\bar{E}_i = E_i \quad \text{for plane stress}.
\]

The value of \( \alpha \) is bounded between \(-1\) and \( 1 \) and \(-0.5 < \beta < 0.5 \) for solids \( 0 \leq v \leq 0.5 \). The oscillation index, or the bimaterial constant, is defined as

\[
\varepsilon = \frac{1}{2\pi} \ln \left( \frac{1 - \beta}{1 + \beta} \right).
\]
Substituting the bounds for $\beta$ gives approximate permissible values of $\varepsilon$ between $-0.175$ and $0.175$. Since the value of $\varepsilon$ is typically small (e.g. $\varepsilon = 0.039$ for Ti/Al$_2$O$_3$), researchers [48] often let $\beta = 0$ and proceed with the analysis. While this practice may lead to acceptable engineering results, it excludes the possibility of capturing the effects of nonzero $\beta$ when analyzing trends. Furthermore, there are an infinite number of material property combinations that satisfy $\beta = 0$ for a given value of $\alpha$.

The consequences of nonzero values of $\beta$ are that the mode and the magnitude of the load are coupled. In other words, both the mode $\psi$ and the strain energy release rate $G$ are required to characterize the crack tip stress fields, but $\psi$ is also a function of $G$. Also, given a value of the mode $\psi_1$ and a corresponding characteristic length $l_1$, a new mode can be obtained by choosing a new characteristic length using the transformation

$$\psi_2 = \psi_1 + \varepsilon \ln \frac{l_2}{l_1}. \quad (4.10)$$

Since the choice of the characteristic length is arbitrary, the value of the mode is necessarily arbitrary.

The consequence of this is not dire in that the results from experiments or analysis simply require the reporting of the characteristic length. However, when examining trends as a function of mode mix, the mode mix itself is meaningless as it depends on an arbitrary characteristic length. A recent example from Daily and Klingbeil [2] show the trends of the plastic dissipation as a function of mode mix for homogeneous materials. Since $\beta = \varepsilon = 0$ for homogeneous materials, $l^{\varepsilon} = 1$ and the definition of mode is well defined. However, extending the results of [2] to the general bimaterial interface becomes problematic due to the lack of a consistent definition of a mode.

The goal of this chapter is to define the mode mix in such a way as to eliminate the arbitrary aspect of the characteristic length. A procedure based on determining the extrema of the elastic strain energy contained in the region defined by the Von Mises stress contour corresponding to the uniaxial yield strength leads to a physically based determination of the characteristic length. Similarly, a procedure for determining the complex stress intensity factor from the strain energy release rate and mode mix is shown. The properties of the design space are presented and concepts are illustrated using the properties of a brass solder interface presented by Nayeb-Hashemi and

54
Table 4.1: Material properties for a brass/solder interface.

Yang [48].

4.1.1 Bimaterial Example

Consider the scenario where a crack along a brass solder interface exists. The pertinent fracture mechanics parameters ($K_1 + iK_2$) and $\mathcal{G}$ and $\psi$ for a given load can be calculated. Physically meaningful values can be assigned to all the material properties, loads and geometry in the SI system. Also, the same specimen can be analyzed in the US system. Let the characteristic length be $l = 1\text{mm}$.

SI Units

The material properties of the brass solder interface are shown in Table 4.1.

Consider a load on the crack such that $\mathcal{G} = 200\text{ J/m}^2$ and $\psi = \pi/4\text{ radians}$. A relationship between the strain energy release rate and the stress intensity factors was given by Malyshev and
Salganik [88] as
\[ G = \frac{(K_1^2 + K_2^2)(1 - \beta^2)}{E^*}, \] (4.11)
where the equivalent elastic modulus \( E^* \) is defined with the relationship
\[ \frac{1}{E^*} = \frac{1}{2} \left( \frac{1}{E_1} + \frac{1}{E_2} \right). \] (4.12)

It should also be noted that \( 1 - \beta^2 = \frac{1}{\cosh^2(\pi \varepsilon)} \). From Eqs. (4.3) and (4.11) we can determine the magnitude of the stress intensity factor,
\[ |K| = \cosh(\pi \varepsilon) \sqrt{G E^*}. \] (4.13)

For this example, \( |K| = 3.259 \text{ MPa}\sqrt{\text{m}} \). The next question is to determine the values of \( K_1 \) and \( K_2 \) based on the given mode mix. Since \( l = 1 \text{ mm} \), \( i \varepsilon = 1 \), the imaginary part of \( K \) is \( K_2 \), and the real part of \( K \) is \( K_1 \). Therefore, from Eq. (4.5) we obtain the ratio:
\[ \tan \frac{\pi}{4} = 1 = \frac{3 |K| \varepsilon}{|K| \varepsilon} = \frac{K_2}{K_1} \]

Since \( K_2 = K_1 \) and \( K_1^2 + K_2^2 = |K|^2 \), the values for \( K_1 \) and \( K_2 \) are 2.304 MPa\sqrt{\text{m}}.

**US Units**

The analysis in US units is begun by converting the applied stress intensity factors previously obtained to US units:
\[ K_1 = K_2 = 2.097 \text{kpsi}\sqrt{\text{in}}. \]

Converting these stress intensity factors to an equivalent strain energy release rate gives
\[ G = \frac{2(2.097)^2(1 - 0.1496^2)}{7530} = 1.142 \text{ lb}\sqrt{\text{in}}. \]

Performing a dimensional analysis will reveal that \( G \) is the same physical quantity. Evaluating the stress fields at the same location from the crack tip (i.e. 1 mm = 0.03937 in) gives the mode mix
according to Eq. (4.5) as

\[
\psi = \tan^{-1} \left( \frac{\Im[K^i]}{\Re[K^i]} \right) \\
= \tan^{-1} \left( \frac{\Im[(2.097 + i2.097)(0.03937)^{-0.0480}]}{\Re[(2.097 + i2.097)(0.03937)^{-0.0480}]} \right) \\
= \tan^{-1} \left( \frac{2.3961}{1.7475} \right) = 0.9407 \text{ radians}
\]

This calculated mode is not \( \pi/4 \) as originally given. To obtain a value of the characteristic length required for changing the mode back to \( \pi/4 \) we must manipulate Eq. (4.10) to solve for \( l_2 \):

\[
l_2 = l_1 \exp \left( \frac{\psi_2 - \psi_1}{\varepsilon} \right) . \quad (4.14)
\]

The length \( l_1 = 0.03937 \) inches corresponds to \( \psi_1 = 0.94 \) radians which gives \( l_2 = 1.00 \) inches when \( \psi_2 = \pi/4 \). This shows the importance of knowing the characteristic length when analyzing interface crack problems.

Further study would reveal that if we had started from \( G = 1.142 \text{ lb}\text{in} \), \( \psi = \pi/4 \), and \( l = 0.03937 \) in, then the resulting stress intensity factors would not be equal as expected from the analysis in SI. However, transforming the characteristic length can give any value of mode mix we desire. As such, the meaning of the mode is not explicit. The different results for the mode mix arising simply from changing unit systems demands a better method of defining mode mix! The proposition presented herein based on quantifying energy near a crack tip works toward this end.

### 4.2 Elastic Strain Energy Near the Crack Tip

In the development of the solution of the asymptotic stress fields, the strain energy stored near a crack tip must be bounded. The total strain energy is determined as

\[
U = \int_{-\pi}^{\pi} \int_{0}^{R(\theta)} U_o(r, \theta) r dr d\theta ,
\]

where \( R(\theta) \) is an arbitrary contour surrounding the crack tip, \( U \) is the total strain energy, and \( U_o \) is the strain energy density at a point. Notice this requirement does not preclude the possibility of
points of unbounded strain energy density (e.g., when \( r = 0, U_0 \) is unbounded). The arbitrary contour \( R(\theta) \) is chosen based on the yield strength of the material. This eliminates the arbitrary aspect of choosing a contour to evaluate the elastic strain energy near the crack tip, \( U \). The disadvantage of this choice is that the definition of the mode inherently depends on the plastic properties of the material system, even if an elastic-plastic analysis is not performed.

Using the distortion energy (Von Mises) failure criterion for yield enables a solution for an approximate contour of the plastic zone. This contour is only approximate because equilibrium may not be satisfied if yielding actually occurs within the plastic zone. The yield criterion is satisfied when

\[
2 (\sigma')^2 = (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6 (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2),
\]

where \( \sigma' \) is the equivalent (Mises) stress at a point. The evaluation of the contour \( R(\theta) \) is determined by solving for the locus of points where \( \sigma_y = \sigma' \).

### 4.2.1 The Homogeneous Case

Determining the amount of elastic strain energy within the approximate plastic zone can be done analytically for homogeneous materials. Consider a crack tip loaded with \( K_1 \) and \( K_2 \). The singular stress fields around the crack tip have the form [76]:

\[
\sigma_{xx} = \frac{K_1}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[ 1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] - \frac{K_2}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) \left[ 2 \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \right] 
\]

\[
\sigma_{yy} = \frac{K_1}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[ 1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] + \frac{K_2}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) \left[ \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \right] 
\]

\[
\tau_{xy} = \frac{K_1}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) + \frac{K_2}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[ 1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] 
\]

For plane strain conditions,

\[
\sigma_{zz} = v(\sigma_{xx} + \sigma_{yy}) = \frac{2K_1}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) + \frac{2K_2}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right).
\]
For plane stress conditions,
\[ \sigma_{zz} = 0. \]  

(4.20)

Substituting Eqs. (4.16)-(4.19) into Eq. (4.15) gives an expression in terms of \( \sigma', r, \theta, K_1, \) and \( K_2. \) When the Von Mises stress is equal to the yield strength, \( \sigma_{ys}, \) then the radial distance at angle \( \theta \) is defined. The equation defining the yield radius in plane strain is

\[
 r_y(\theta) = \frac{1}{16\pi \sigma_y^2} \left( K_1^2 [7 + 16\nu (\nu - 1)] + K_2^2 [19 + 16\nu (\nu - 1)] + 4(K_1^2 - K_2^2)(1 - 2\nu)^2 \cos(\theta) 
- 3(K_1^2 - 3K_2^2) \cos(2\theta) + 8K_1K_2[3\cos(\theta) - (1 - 2\nu)^2] \sin(\theta) \right) \]  

(4.21)

and the equation defining the yield radius in plane stress is

\[
 r_y(\theta) = \frac{1}{16\pi \sigma_y^2} \left( 7K_1^2 + 19K_2^2 + 4(K_1^2 - K_2^2) \cos(\theta) 
- 3(K_1^2 - 3K_2^2) \cos(2\theta) + 8K_1K_2[3\cos(\theta) - 1] \sin(\theta) \right). \]  

(4.22)

The total elastic strain energy within the yield radius zone is computed as

\[
 U = \int_{-\pi}^{\pi} \int_{0}^{r_y} U_o r \, dr \, d\theta, \]  

(4.23)

where \( U_o \) is the strain energy density. The elastic strain energy density \( U_o \) can be calculated as

\[
 U_o = \frac{1}{2E} \left( \sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 \right) - \frac{\nu}{E} \left( \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} \right) + \frac{1 + \nu}{E} \left( \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \right). \]  

(4.24)

Substitution of Eqs. (4.16)-(4.19) into Eq. (4.24) gives:

\[
 U_o = \frac{1}{16E \pi r} \left( (1 + \nu)(5K_1^2 + 9K_2^2 - 8\nu(K_1^2 + K_2^2)) - 4(K_1^2 - K_2^2)(2\nu - 1) \cos(\theta) 
- (K_1^2 - 3K_2^2) \cos(2\theta) + 8K_1K_2[\cos(\theta) - 1 + 2\nu] \sin(\theta) \right) \]  

(4.25)
in plane strain and

\[
U_o = \frac{1}{16E\pi r} \left( K_1^2 (5 - 3v) + K_2^2 (9 + v) - 4(K_1^2 - K_2^2)(v - 1)\cos(\theta) \right.
\]

\[
- (K_1^2 - 3K_2^2)(1 + v)\cos(2\theta) + 8K_1K_2[(1 + v)\cos(\theta) - 1 + v]\sin(\theta) \right)
\]  

(4.26)

in plane stress. Notice the \(r\) terms cancel in when performing the integration of Eq. (4.23) giving the result for

\[
U = \frac{K_1^4 f_1(v) + K_1^2 K_2^2 f_2(v) + K_2^4 f_3(v)}{256\pi E \sigma_y^2}
\]  

(4.27)

where

\[
f_1 = -(1 + v)(-89 + 16v[23 - 38v + 24v^2])
\]

\[
f_2 = -6(1 + v)(-63 + 16v[11 - 14v + 8v^2])
\]

\[
f_3 = 385 + v(-303 + 16v[3 + 22v - 24v^2])
\]

for plane strain and

\[
f_1 = 89 - 55v
\]

\[
f_2 = 378 - 102v
\]

\[
f_3 = 385 + 49v
\]

for plane stress.

Dividing both sides of Eq. (4.27) by \(K_1^4\) gives

\[
\frac{U}{K_1^4} = \frac{f_1(v) + \frac{K_1^2}{K_1^4} f_2(v) + \frac{K_2^4}{K_1^4} f_3(v)}{256\pi E \sigma_y^2}
\]  

(4.28)

Recognizing that \(\tan \phi = K_2/K_1\) yields

\[
\frac{256\pi E \sigma_y^2 U}{K_1^4} = f_1(v) + f_2(v)\tan^2 \phi + f_3(v)\tan^4 \phi
\]  

(4.29)
Squaring both sides of Eq. (4.11), letting $1 - \beta^2 = 1$, and recognizing that $E^* = E$ for a homogeneous material gives

$$\sigma^2 = \frac{(K_1^2 + K_2^2)^2}{E^2}.$$  \hfill (4.30)

Expanding and solving for $K_1^4$ gives an expression for normalizing the energy

$$K_1^4 = \frac{\sigma^2 E^2}{1 + 2\tan^2 \phi + \tan^4 \phi}.$$  \hfill (4.31)

Substituting Eq. (4.31) into Eq. (4.29) gives

$$\frac{\sigma^2 U}{\sigma^2 E} = f_1(v) + f_2(v)\tan^2 \phi + f_3(v)\tan^4 \phi \quad \frac{256}{256(1 + 2\tan^2 \phi + \tan^4 \phi)}.$$  \hfill (4.32)

The right hand side of Eq. (4.32) has no dimension as it only depends on Poisson’s ratio and the ratio of the stress intensity factors. The quantity on the right hand side is the dimensionless elastic strain energy denoted as $U^*$:

$$U^* = \frac{\sigma^2 U}{\sigma^2 E}.$$  \hfill (4.33)

This dimensionless quantity verifies the normalization of the plastic dissipation results in [2]. A plot of Eq. (4.32) is shown in Fig. 4.3 for different values of $v$, which mimics the curves for plastic dissipation in Chapter 3.

### 4.2.2 Curve Fit to Plastic Dissipation of Chapter 2

The results of the previous section show that the functional form of the dissipated energy curve is a rational fraction of the form

$$\frac{dW^*}{dN} = \frac{a_2x^2 + a_1x + a_0}{x^2 + b_1x + b_0}.$$  \hfill (4.34)

where $x = \tan^2 \phi$ and $a_i$ and $b_i$ are unknown coefficients that can be determined based on a linear least squares fit. From the results of Chapter 2, the value of $dW/dN^*$ is given for different values of
Figure 4.3: A plot of the dimensionless elastic strain energy for different values of $\nu$ in plane stress and plane strain.
Writing Eq. (4.34) in matrix form:

\[
\left\{ \frac{x^2 dW^*}{dN} \right\} = \begin{bmatrix} x^2 \times 1 - x \frac{dW^*}{dN} - \frac{dW^*}{dN} \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \\ b_1 \\ b_0 \end{bmatrix}
\]

The solution for the coefficients is:

\[
\begin{bmatrix} a_2 \\ a_1 \\ a_0 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} x^2 \times 1 - x \frac{dW^*}{dN} - \frac{dW^*}{dN} \end{bmatrix}^+ \begin{bmatrix} a_2 \\ a_1 \\ a_0 \\ b_1 \\ b_0 \end{bmatrix}
\]

where \([\cdot]^+\) indicates the Moore-Penrose pseudo inverse. Inclusion of the \(\phi = 90^\circ\) points in the pseudo-inverse renders an ill-conditioned matrix, and ignoring that point makes the solution stable.

For the homogeneous case, the coefficients are:

\[
\begin{bmatrix} a_2 \\ a_1 \\ a_0 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 0.1179 \\ 0.0918 \\ 0.0099 \\ 2.3206 \\ 1.5945 \end{bmatrix}
\]

These results are important because they provide a technique of rapidly determining the dissipated energy per cycle for different mode mix values. The interpolation scheme is physically based on the elastic strain energy and can easily be implemented in a computer code that predicts fatigue crack growth rate.
Figure 4.4: A curve fit using Eq. (4.34) and the coefficients listed in Eq. (4.35) for the plastic dissipation of a homogeneous material when $v = 1/3$.

### 4.2.3 The Case of a Yield Strength Mismatch

When dealing with the case of a yield strength mismatch, the elastic portion of the solution remains the same as a homogeneous solution. The difference arises when the yield contour is calculated. Therefore, we can follow the same analysis as in the previous section except the integration with respect to $\theta$ must be done in two parts due to the different radii of the yield zone.

$$U = \int_0^{\pi} \int_0^{R_1} U_o \ r \ dr \ d\theta + \int_0^{\pi} \int_0^{R_2} U_o \ r \ dr \ d\theta$$  \hspace{1cm} (4.36)

The results of the integration are:

$$U = \frac{(\sigma'_{y1} + \sigma'_{y2}) \left[ K_1^4 g_1(v) + \sigma'_{y1} K_1^3 K_2 g_2(v) + K_1^2 K_2^2 g_3(v) + \sigma'_{y1} K_1 K_2^2 g_4(v) + K_2^4 g_5(v) \right]}{1536 \pi^2 E \sigma_{y1}^2 \sigma_{y2}^2}$$  \hspace{1cm} (4.37)

where

$$\sigma'_{y} = \frac{\sigma'_{y1} - \sigma'_{y2}}{\sigma_{y1}^2 + \sigma_{y2}^2}$$  \hspace{1cm} (4.38)
and

\[
\begin{align*}
g_1(v) &= -3\pi(55v - 89) \\
g_2(v) &= -768(v - 1) \\
g_3(v) &= -18\pi(17v - 63) \\
g_4(v) &= -256(7v - 11) \\
g_5(v) &= 21\pi(7v + 55)
\end{align*}
\]

in plane stress and

\[
\begin{align*}
g_1(v) &= -3\pi(-89 + 16v(23 - 38v + 24v^2))(1 + v) \\
g_2(v) &= -768(2v - 1)^3(1 + v) \\
g_3(v) &= -18\pi(-63 + 16v(11 - 14v + 8v^2))(1 + v) \\
g_4(v) &= -256(2v - 1)(11 - 16v + 12v^2)(1 + v) \\
g_5(v) &= -3\pi(-385 + 16v(43 - 46v + 24v^2))(1 + v)
\end{align*}
\]

for plane strain. The value of \( \sigma'_y \) from Eq. (4.38) is a measure of the relative yield strength mismatch. It has a value bounded by \(-1\) and \(1\) where positive values indicate the top layer (#1) is stronger than the bottom layer (#2) and negative values indicate the bottom layer (#2) is stronger than the top. The case for no mismatch corresponds to \( \sigma'_y = 0 \). Also, the yield strength mismatch influences only the asymmetric terms in Eq. (4.37). These asymmetric terms disappear when there is no yield strength mismatch or when the material exhibits incompressible behavior (\( v = 0.5 \)).

Eq. (4.37) can be simplified by dividing through by \( K_1^4 \):

\[
\frac{E \sigma'^2 \sigma''_y U}{K_1^4(\sigma'_{\parallel}^2 + \sigma'_{\perp}^2)} = \frac{g_1(v) + \sigma'_y \tan(\phi)g_2(v) + \tan^2(\phi)g_3(v) + \sigma'_y \tan^3(\phi)g_4(v) + \tan^4(\phi)g_5(v)}{1536\pi^2}
\]

(4.39)
Substituting Eq. (4.31) into Eq. (4.39) to get an expression in terms of \( \mathcal{G} \) gives

\[
\frac{2\sigma_1^2 \sigma_2^2 U}{\mathcal{G}^2 E (\sigma_{y1}^2 + \sigma_{y2}^2)} = \frac{g_1(v) + \sigma_1' \tan(\phi)g_2(v) + \tan^2(\phi)g_3(v) + \sigma_1' \tan^3(\phi)g_4(v) + \tan^4(\phi)g_5(v)}{768\pi^2(1 + 2\tan^2\phi + \tan^4\phi)}.
\]

(4.40)

Furthermore, letting

\[
\sigma_y^* = \frac{2\sigma_1^2 \sigma_2^2}{\sigma_{y1}^2 + \sigma_{y2}^2},
\]

(4.41)

then the dimensionless elastic strain energy can be written as

\[
U^* = \frac{\sigma_y^* U}{\mathcal{G}^2 E},
\]

(4.42)

where \( U^* \) is a function of Poisson’s ratio and the relative mismatch in yield strength \( \sigma_y^* \). The plots of the effect of a yield strength mismatch are shown in Fig. 4.5 on the next page for three different values of Poisson’s ratio. The case for equal yield strengths gives \( \sigma_y^* = \sigma_y^2 \) and matches the definition in Eq. (4.33). The value of \( \sigma_y^* \) is bounded on the lower end by zero which is not physically probable because most solid materials have some yield strength. The value of \( \sigma_y^* \) is bounded on the upper end by twice the square of the smaller yield strength. The bounds are expressed mathematically as

\[
0 \leq \sigma_y^* \leq 2 \min(\sigma_{y1}^2, \sigma_{y2}^2).
\]

### 4.2.4 The Bimaterial Case

In the previous section, a closed form solution exists for elastic strain energy for the homogeneous case. When an interface comprises two different layers with different elastic moduli, then the closed form solution may not exist and a numerical technique is needed to compute the integral. A two dimensional integration scheme was developed in Matlab to numerically determine the elastic strain energy.

An important consideration is understanding the effect of the characteristic length on the elastic strain energy. As shown in Fig. 4.6, the pattern repeats itself (on a log scale) based on \( \varepsilon \). The oscillation has a period of

\[
T = \log_{10}\left(\exp\left[\frac{n\pi}{\varepsilon}\right]\right)
\]

where \( n \) is an integer and \( T \) is the period of the exponent \( (10^T) \). Since the energy is periodic with
Figure 4.5: The influence of a yield strength mismatch on the normalized elastic strain energy in plane strain. These plots are the graphs of Eq. (4.40) in plane strain.
respect to the logarithm of \( l \), there exists many equivalent characteristic lengths. They are all related by

\[
l = l_o \exp \left[ \frac{n\pi}{\varepsilon} \right],
\]

(4.43)

where \( n \) is an integer. Therefore, any value of \( l \) can be cast into the space illustrated in Fig. 4.6 by choosing an appropriate value of \( n \).

As expected, the design space is also periodic with respect to \( \phi \). Since the elastic strain energy is determined as the square of the mode mix, the periodicity is \( \pi \) rather than \( 2\pi \).

An alternative approach is to set the characteristic length to unity and vary the load through changing \( G \). Since increasing the load will obviously increase the strain energy, the elastic strain energy must be normalized. In keeping with the homogeneous case, a dimensionless parameter can be obtained as

\[
U^* = \frac{\cosh^4(\pi\varepsilon)\sigma_0^\ast U}{G^2E^\ast},
\]

(4.44)

where \( E^\ast \) is determined from Eq. (4.12) and the \( \cosh^4(\pi\varepsilon) \) term is an artifact from the relationship between \( \mathcal{G} \) and \( |K| \). For a given material system, the yield strength, equivalent elastic modulus and oscillation index are all constant which implies the elastic strain energy is proportional to \( \mathcal{G}^2 \). However, as illustrated by Fig. 4.7, \( U^* \) still exhibits a dependence on \( \mathcal{G} \). The logarithmic periodicity of the dependence on \( \mathcal{G} \) is

\[
T = \log_{10} \left( \exp \left[ \frac{n}{\pi\varepsilon} \right] \right),
\]

so transforming one value of \( \mathcal{G} \) to another equivalent value uses the transformation

\[
\mathcal{G} = \mathcal{G}_0 \exp \left[ \frac{n}{\pi\varepsilon} \right].
\]

(4.45)

Notice the periodicity of \( \mathcal{G} \) is faster than \( l \) by a factor of \( \pi^2 \).

If the characteristic length is proportional to \( \mathcal{G} \), then the variation of the elastic energy is only dependent on \( \phi \) (the ratio of \( K_2 \) to \( K_1 \)), as shown in Fig. 4.8! Furthermore, the value of the proportionality constant determines where the curve is located with respect to the \( \phi \) axis. For example, if Fig. 4.8, the dimensionless energy was determined by setting the characteristic length equal to \( \mathcal{G} \). This resulted in a surface that only varied with \( \phi \). Shifting the curve along the \( \phi \) axis requires a change of \( l \) and the definition of \( \psi \) from Eq. (4.5). To obtain the contour shown in Fig. 4.8b
Figure 4.6: A contour map of the elastic strain energy within the yield contour of a the brass/solder interface ($\alpha = 0.549, \beta = 0.1496$) for different values of the characteristic length and mode mix. The strain energy release rate ($G$) was set to unity. The square is located near the minimum value of $U$. The maximum is located within the concentric contours near $\phi = 1.5$. Notice that two logarithmic periods of the strain energy is shown.

where the minimum corresponds to $\psi = 0$, we must determine a new characteristic length to use in Eq. (4.5):

$$l = \exp \left[ -\frac{\Phi_{\text{min}}}{\epsilon} \right].$$

(4.46)

It is also possible to set the maximum of the $G$-independent curve to $\psi = \pi/2$ by selecting the characteristic length

$$l = \exp \left[ \frac{\pi/2 - \Phi_{\text{max}}}{\epsilon} \right].$$

(4.47)

This means it is possible to choose a proportionality constant that will ensure the minimum is always at $\phi = 0$ or the maximum is always at $\phi = \pi/2$. Unfortunately, the proportionality constants determined from Eqs. (4.46) and (4.47) are not the same; therefore, a definition of mode based
Figure 4.7: A contour map of the elastic strain energy within the yield contour of a brass/solder interface for different values of the loading and mode mix. The characteristic length ($l$) was set to unity. Notice that two logarithmic periods of the strain energy is shown.

Figure 4.8: The effect of choosing a characteristic length proportional to $\mathcal{J}$. 

(a) Character length is unity

(b) Character length Proportional to $\mathcal{J}$
solely on this criteria will only work if the proportionality constants are the same.

4.2.5 Determine the Extrema of the Elastic Strain Energy

Consider a single period of the design space of $U^*$ with respect to $G$ and $\phi$ while the characteristic length set to unity. Within that period, there exists a maximum value of the normalized elastic strain energy ($U_{\text{max}}^*$) and a minimum value of the normalized elastic strain energy ($U_{\text{min}}^*$). An optimization algorithm is capable of determining the locations of those values within the period. An implementation of the popular Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm [89] in Matlab located those points for various values of $\alpha$ and $\beta = \alpha/4$. Unfortunately, the results depend on the yield strength of the material. As such, a purely elastic argument for the mode mix is not possible based on these results.

In a similar fashion, the location of the extrema can be found with respect to $l$ and $\phi$ while setting the applied load $G = 1$.

4.2.6 Gradients and Sensitivity of the Elastic Strain Energy

Beginning with the expression for the elastic strain energy from Eq. (4.23), it is possible to perform a sensitivity analysis of $U$ with respect to $G$ and $\phi$. Since integration and differentiation are linear operations and the limits of integration are functions of $G$ and $\phi$, we must use Leibniz’s rule for differentiation:

$$\frac{\partial U}{\partial G} = \int_{-\pi}^{\pi} \left[ r_y \frac{\partial r_y}{\partial G} U_o(r_y, \theta) + \int_0^{r_y} \frac{\partial U_o}{\partial G} r dr \right] d\theta$$

$$\frac{\partial U}{\partial \phi} = \int_{-\pi}^{\pi} \left[ r_y \frac{\partial r_y}{\partial \phi} U_o(r_y, \theta) + \int_0^{r_y} \frac{\partial U_o}{\partial \phi} r dr \right] d\theta.$$ 

The first terms require the derivative of the approximate yield zone radius with respect to the variables. Since the yield zone is computed numerically for the general case of a bimaterial, the derivatives of the yield zone radius must also be computed numerically. However, an analytical derivative
of the strain energy density can be found using the chain rule and product rule:

\[
\frac{\partial U_o}{\partial \phi} = \frac{1}{E} \left( \sigma_{xx} \frac{\partial \sigma_{xx}}{\partial \phi} + \sigma_{yy} \frac{\partial \sigma_{yy}}{\partial \phi} + \sigma_{zz} \frac{\partial \sigma_{zz}}{\partial \phi} \right) - \frac{\nu}{E} \left( \sigma_{xx} \frac{\partial \sigma_{yy}}{\partial \phi} + \frac{\partial \sigma_{xx}}{\partial \phi} \sigma_{yy} \right) + \left( \sigma_{yy} \frac{\partial \sigma_{zz}}{\partial \phi} + \frac{\partial \sigma_{yy}}{\partial \phi} \sigma_{zz} \right) + \frac{2(1+\nu)}{E} \left( \tau_{xy} \frac{\partial \tau_{xy}}{\partial \phi} + \tau_{yz} \frac{\partial \tau_{yz}}{\partial \phi} + \tau_{xx} \frac{\partial \tau_{xx}}{\partial \phi} \right), \tag{4.48}
\]

\[
\frac{\partial U_o}{\partial \phi} = \frac{1}{E} \left( \sigma_{xx} \frac{\partial \sigma_{xx}}{\partial \phi} + \sigma_{yy} \frac{\partial \sigma_{yy}}{\partial \phi} + \sigma_{zz} \frac{\partial \sigma_{zz}}{\partial \phi} \right) - \frac{\nu}{E} \left( \sigma_{xx} \frac{\partial \sigma_{yy}}{\partial \phi} + \frac{\partial \sigma_{xx}}{\partial \phi} \sigma_{yy} \right) + \left( \sigma_{yy} \frac{\partial \sigma_{zz}}{\partial \phi} + \frac{\partial \sigma_{yy}}{\partial \phi} \sigma_{zz} \right) + \frac{2(1+\nu)}{E} \left( \tau_{xy} \frac{\partial \tau_{xy}}{\partial \phi} + \tau_{yz} \frac{\partial \tau_{yz}}{\partial \phi} + \tau_{xx} \frac{\partial \tau_{xx}}{\partial \phi} \right). \tag{4.49}
\]

For the in-plane problems considered herein, the out-of-plane shearing terms are zero. Moreover, the out of plane axial stress is zero for plane stress and

\[
\frac{\partial \sigma_{zz}}{\partial \phi} = \nu \left( \frac{\partial \sigma_{xx}}{\partial \phi} + \frac{\partial \sigma_{yy}}{\partial \phi} \right) \tag{4.50}
\]

\[
\frac{\partial \sigma_{zz}}{\partial \phi} = \nu \left( \frac{\partial \sigma_{xx}}{\partial \phi} + \frac{\partial \sigma_{yy}}{\partial \phi} \right). \tag{4.51}
\]

The field equations given by Nishioka [84] have the form

\[
\sigma_{ij} = \frac{K_1 f_{1,ij}(\theta, r, \epsilon, l) + K_2 f_{2,ij}(\theta, r, \epsilon, l)}{2\sqrt{2\pi r \cosh(\pi \epsilon)}}, \tag{4.52}
\]

where \(ij = xx, yy, xy\). The individual stress intensity factors can be expressed with the relationship

\[
K_1 = |K| \cos \phi
\]

\[
K_2 = |K| \sin \phi,
\]
where $|K|$ is a function of $G$ through Eq. (4.13). Recasting Eq. (4.52) in terms of $G$ and $\phi$ gives the following expression for the field equations:

$$
\sigma_{ij} = \frac{\cosh(\pi \varepsilon) \sqrt{G E^*} (\cos(\phi) f_{1,ij}(\theta, r \varepsilon, l) + \sin(\phi) f_{2,ij}(\theta, r \varepsilon, l))}{2 \sqrt{2} \pi r \cosh(\pi E^*)} \tag{4.53}
$$

From Eq. (4.53) the derivatives are

$$
\frac{\partial \sigma_{ij}}{\partial G} = \frac{E^* (\cos(\phi) f_{1,ij}(\theta, r \varepsilon, l) + \sin(\phi) f_{2,ij}(\theta, r \varepsilon, l))}{4 \sqrt{2} \pi r \sqrt{G E^*}} \tag{4.54}
$$

and

$$
\frac{\partial \sigma_{ij}}{\partial \phi} = \frac{\sqrt{G E^*} (-\sin(\phi) f_{1,ij}(\theta, r \varepsilon, l) + \cos(\phi) f_{2,ij}(\theta, r \varepsilon, l))}{2 \sqrt{2} \pi r}. \tag{4.55}
$$

An implementation of the equations in MATLAB showed the semi-analytical derivative agreed with a converged finite difference scheme to five digits. However, the derivative of $U$ was more computationally demanding than a simpler finite differencing scheme because it requires a numerical derivative for the yield radius calculation in addition to the original numerical integration. This added computational burden renders the results of this section a novelty with little practical use, although they are included here for completeness.

4.2.7 Limitations of Using the Elastic Strain Energy Within the Yield Zone

The elastic strain energy within the yield strength contour is useful in understanding trends and interpreting the dimensional aspects of energy at a crack tip. However, using this quantity to define a meaningful value of mode-mix is problematic because it requires plastic properties (i.e., the yield strength). Furthermore, the inclusion of the plastic properties in an energy-based argument renders the classic definition of the mode meaningless due to the effect of the yield strength mismatch shifting the point of maximum and minimum away from the classic mode II and mode I marks, respectively.
4.3 Strain Energy Contour Integral

Consider a symmetric contour around a bimaterial crack tip. This symmetric contour could be the same contour used to evaluate the $J$-integral. Now consider a quantity of energy computed by

$$U = \oint U_0 ds,$$  \hspace{1cm} (4.56)

where $U_0$ is the strain energy density at a point and $\Gamma$ is a symmetric contour. For the remaining analysis $\Gamma$ will be a circular contour with an arbitrary radius, $r_\Gamma$. Notice that no plastic properties (i.e., yield strength) are used in computing the contour integral of Eq. (4.56).

4.3.1 Homogeneous Case

Determining the amount of elastic strain energy density along a contour can be done analytically for homogeneous materials. Consider a crack tip loaded with $K_1$ and $K_2$. The singular stress fields around the crack tip have the form [76]:

$$\sigma_{xx} = \frac{K_1}{\sqrt{2\pi r}} \cos \left( \frac{\theta}{2} \right) \left[ 1 - \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{3\theta}{2} \right) \right] - \frac{K_2}{\sqrt{2\pi r}} \sin \left( \frac{\theta}{2} \right) \left[ 2 + \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{3\theta}{2} \right) \right] \hspace{1cm} (4.57)$$

$$\sigma_{yy} = \frac{K_1}{\sqrt{2\pi r}} \cos \left( \frac{\theta}{2} \right) \left[ 1 + \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{3\theta}{2} \right) \right] + \frac{K_2}{\sqrt{2\pi r}} \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{3\theta}{2} \right) \hspace{1cm} (4.58)$$

$$\tau_{xy} = \frac{K_1}{\sqrt{2\pi r}} \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{3\theta}{2} \right) + \frac{K_2}{\sqrt{2\pi r}} \cos \left( \frac{\theta}{2} \right) \left[ 1 - \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{3\theta}{2} \right) \right] \hspace{1cm} (4.59)$$

For plane stress conditions:

$$\sigma_{zz} = 0. \hspace{1cm} (4.60)$$

For plane strain conditions:

$$\sigma_{zz} = \nu (\sigma_{xx} + \sigma_{yy})$$

$$= \frac{2K_1 \nu}{\sqrt{2\pi r}} \cos \left( \frac{\theta}{2} \right) + \frac{2K_2 \nu}{\sqrt{2\pi r}} \sin \left( \frac{\theta}{2} \right). \hspace{1cm} (4.61)$$
The elastic strain energy density $U_o$ can be calculated as

$$
U_o = \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{V}{E}(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \frac{1+v}{E}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2). \quad (4.62)
$$

Substitution of Eqs. (4.16)-(4.19) into Eq. (4.24) gives

$$
U_o = \frac{1}{16E\pi r} \left( K_1^2(5 - 3v) + K_2^2(9+v) - 4(K_1^2 - K_2^2)(v - 1)\cos(\theta)
\right.

\left. - (K_1^2 - 3K_2^2)(1 + v)\cos(2\theta) + 8K_1K_2[(1 + v)\cos(\theta) - 1 + v]\sin(\theta) \right) \quad (4.63)
$$

in plane stress and

$$
U_o = \frac{1}{16E\pi r} \left( (1 + v)(5K_1^2 + 9K_2^2 - 8v(K_1^2 + K_2^2)) - 4(K_1^2 - K_2^2)(2v - 1)\cos(\theta)
\right.

\left. - (K_1^2 - 3K_2^2) \cos(2\theta) + 8K_1K_2[(1 + v)\cos(\theta) - 1 + 2v]\sin(\theta) \right) \quad (4.64)
$$

in plane strain. The integration of the strain energy density $U_o$ using Eq. (4.56) when $ds = r d\theta$ gives

$$
U = \frac{K_1^2f_1(v) + K_2^2f_2(v)}{8E} \quad (4.65)
$$

where

$$
f_1(v) = 5 - 3v
$$

$$
f_2(v) = 9 + v
$$

in plane stress and

$$
f_1(v) = (1 + v)(5 - 8v)
$$

$$
f_2(v) = (1 + v)(9 - 8v)
$$
in plane strain. Dividing Eq. (4.65) through by $K_1^2$ and substituting the definition for mode mix as $\tan \phi = (K_2/K_1)$ gives:

$$\frac{UE}{K_1^2} = \frac{f_1(v) + f_2(v)\tan^2 \phi}{8} \quad (4.66)$$

We can let $1 - \beta^2 = 1$ and recognize that $E^* = E$ for a homogeneous material to simplify Eq. (4.11) as:

$$J = \frac{(K_1^2 + K_2^2)}{E} \quad (4.67)$$

Rearranging Eq. (4.67) gives form in terms of $K_1^2$:

$$K_1^2 = \frac{JE}{1 + \tan^2 \phi} \quad (4.68)$$

Substitution of Eq. (4.68) into Eq. (4.66) gives:

$$\frac{U}{J} = \frac{f_1(v) + f_2(v)\tan^2 \phi}{8(1 + \tan^2 \phi)} \quad (4.69)$$

and the dimensionless plot of $U/J$ is shown in Fig. 4.9. It is important to note that the quantity $U/J$ is independent of the contour radius, $r_G$. 

76
4.3.2 The Bimaterial Case

The field equations presented in [84] were used to develop a numerical integration scheme in Matlab that solves for the integral of Eq. (4.56). In the bimaterial case, when $\varepsilon \neq 0$, the response of $U/G$ is dependent on the ratio of the applied stress intensity factors as well as the characteristic length and the radius of the contour, as shown in Fig. 4.10. Only one logarithmic period in $l$ and one period in $f$ is shown in Fig. 4.10. Within this period, it can be seen that one minimum exists (square) and one maximum exists. These minimum and maximum are repeated every period. When $\varepsilon = 0$, the period with respect to the characteristic length extends to infinity and the minimum occurs when $f = 0$ and the maximum occurs when $f = \pi/2$.

Interestingly, changing the radius of the path of integration ($r_T$) only effects the location of the contour plot of Fig. 4.10 in relation to the bottom axis ($l$). However, if we multiply the characteristic length by $r_T$, then changes in $r_T$ no longer influence the location of the contour plot.
Dimensionless Results from the Elastic Strain Energy Contour Integral

$f = \text{atan}(k_2/k_1)$

Figure 4.10: Response of the strain energy contour integral as a function of the characteristic length $l$ and mode $\phi$ for a brass-solder interface. The radius of the contour is 1 and $\gamma = 1$. 
Consider a characteristic length such that \( l = r_1 \mathcal{G}^* \), where

\[
\mathcal{G}^* = \frac{\mathcal{G}}{E^* L}
\]

is the normalized strain energy release rate. The value of \( L \) corresponds to some physical length scale (e.g., the total thickness of a sandwich specimen). With this definition of the characteristic length, the whole range of \( U/\mathcal{G} \) can be spanned by varying \( \mathcal{G} \) and \( f \). The logarithmic periodicity of \( U \) with respect to \( l \) (now \( \mathcal{G}^* \)) is

\[
l = l_0 \exp \left[ \frac{1}{\pi \varepsilon} \right],
\]

which is different than the periodicity of having equivalent characteristic lengths for the field equations. The difference is a factor of \( \pi^2 \).

### 4.4 Definition of Mode

The in-plane mode mix should be defined such that mode I \( (\psi = 0) \) corresponds to the minimum normalized strain energy \( (U/\mathcal{G}) \) computed from the contour integral of Eq. (4.56) when the characteristic length is defined as \( l = r_1 \mathcal{G} \). Similarly, the maximum normalized strain energy should be equivalent to pure mode II \( (\psi = \pi/2 \text{ or } \psi = -\pi/2) \). The remaining modes will be determined from an energy interpolation scheme outlined in this section. From this technique of defining the mode, the characteristic length used in Eq. (4.5) will be assigned and no longer be arbitrary. As such, a significant contribution of this work is providing a physical basis for choosing the characteristic length. The full range of mode mix values will be capable of mapping the complete response of energy based quantities with respect to both the ratio of the applied stress intensity factors as well as the magnitude.

#### 4.4.1 Spanning the Complete Range of Energy

In this section, a technique of choosing a function that spans the complete range of energy is shown. Begin with locating the energy extrema, \((\mathcal{G}_{\min}, \phi_{\min}, U_{\min}) \text{ and } (\mathcal{G}_{\max}, \phi_{\max}, U_{\max})\). These points can be determined using an optimization scheme or by interpolating the plots of Figs. 4.12-4.15.

Once the locations of the extrema are known, line segments can connect the maximum to the
Figure 4.11: Response of the strain energy contour integral as a function of load $G$ and mode $f$ when the characteristic length is equal to the inverse of the load. The effects of $e$ (through $\beta = \alpha/4$) on the contour plots of $U/G$ when the characteristic length was set to $1/G$. 
The Effect of $a$ on the Location of the Extrema in Plane Strain when $b = a/4$

Figure 4.12: The location of the maximum and minimum values of $U/G$ with respect to $G$ for different values of $\alpha$. The maximum values may reflect the one logarithmic period difference from the maximum values in Fig. 4.11.
The Effect of $\alpha$ on the Location of the Extrema in Plane Strain when $\beta = \alpha/4$

Figure 4.13: The location of the maximum and minimum values of $U/G$ with respect to $\phi$ for different values of $\alpha$. 

\[
\phi = \tan^{-1}\left(\frac{K_2}{K_1}\right)
\]
The Effect of $\alpha$ on Maximum Energy in Plane Strain when $\beta=\alpha/4$

![Graph showing the effect of $\alpha$ on maximum energy in plane strain with $\beta=\alpha/4$.](image)

Figure 4.14: The maximum values of $U/\mathcal{G}$ different values of $\alpha$. 

83
The Effect of $a$ on Minimum Energy in Plane Strain when $\beta=a/4$

Figure 4.15: The minimum values of $U/G$ different values of $\alpha$. 
Figure 4.16: Contour plot for a brass solder interface showing the line segment connecting the extrema and a sampling of across $\phi$ for $\mathcal{G}^* = 100$.

minimum. The first segment is contained within one period and the second segment is plotted across the plot boundary, as shown in Fig. 4.16. The equations for the line segments are

$$\phi - \phi_{\text{min}} = \frac{\phi_{\text{max}} - \phi_{\text{min}}}{\log \mathcal{G}_{\text{max}} - \log \mathcal{G}_{\text{min}}} (\log \mathcal{G} - \log \mathcal{G}_{\text{min}}) \quad \phi_{\text{max}} \leq \phi \leq \phi_{\text{min}} \quad (4.70)$$

$$\phi - \phi_{\text{min}} = \frac{\pi + \phi_{\text{max}} - \phi_{\text{min}}}{\log \mathcal{G}_{\text{max}} - \log \mathcal{G}_{\text{min}}} (\log \mathcal{G} - \log \mathcal{G}_{\text{min}}) \quad \phi_{\text{min}} < \phi \leq (\phi_{\text{max}} + \pi) \quad (4.71)$$

The value of $U/\mathcal{G}$ is computed for every point along the lines and those values of $U/\mathcal{G}$ comprise a function of $\phi$, as shown in Fig. 4.17. Since the new definition of mode is energy dependent and the energy computed for a specific value of $\mathcal{G}$ can be changed by multiplying the characteristic length by some value, we can find a characteristic length for each value of $\mathcal{G}$ and $\phi$ that will match the energy determined from the interpolation function. Furthermore, one can adjust the value of the mode mix defined by Eq. (4.5) by selecting a new characteristic length. This transforms the $\phi$ axis
Figure 4.17: The extraction of $U/G$ along the line segments from Fig. 4.16 results in the plot shown here. The arrows indicate the transformation from $\phi$ to $\psi$ for the portion of the curve with a negative slope. The curve with the circles show the energy extracted from the different values of $\phi$ when $\mathcal{G} = 100$.

to the $\psi$ axis. Since the value of $l$ required to make this transformation is unknown, consider the following technique.

For each line segment, assign the minimum values of $U/G$ to $\psi = 0$. This is shown in Fig. 4.17 by the arrow indicating the transformation of $\phi_{min}$ to zero. The maximum of the segment whose values are entirely less than $\phi_{min}$ (+ symbols) shall be assigned $\psi = -\pi/2$. Similarly, the maximum of the segment whose values are predominately greater than than $\phi_{min}$ (dot symbols) shall be assigned $\psi = +\pi/2$. Every point in between shall scale linearly to create the curve shown in Fig. 4.18. The circles shown in Fig. 4.18 show the results of the transformation indicated by the arrows in Fig. 4.17.

The samples of $U/G$ when $\mathcal{G} = 100$ correspond to some location on the interpolated curve. These points are mapped to the interpolated curve according to their value and the sign of their slope. This ensures that the same values of $U/G$ are mapped to different interpolation lines. If this does not
Figure 4.18: The transformation of the bottom axis from $\phi$ to $\psi$ results in the plot shown here. The value of the mode mix is known based on the contour energy $U/G$. The circles the results of the interpolation for different values of $\phi$ when $G = 100$. 

\[ y = \text{atan}(\text{Im}[K]) / \text{Re}[K] \]
happen, then the final characteristic length computed for the mode will not be unique.

The determination of the characteristic length is possible based on Eq. (4.10) where \( \psi_1 = \phi \), \( l_1 = 1 \), and \( \psi_2 = \psi \). The value of \( l \) corresponding to \( \psi \) based on interpolating the quantity \( U/G \) is computed as:

\[
l = \exp \left[ \frac{\psi - \phi}{\varepsilon} \right].
\]

Keep in mind that this value of \( l \) periodic and other equivalent characteristic lengths can be determined by the transformation of Eq. (4.43) as

\[
l = l_0 \exp \left[ \frac{n\pi}{\varepsilon} \right],
\]

where \( n \) is an integer. For small values of \( \varepsilon \), values of \( l \) become quite large or small compared to physical length scales.

The characteristic length has been calculated in such a fashion as to tie it to a physical parameter, namely the amount of energy contained in a circular contour. This means the characteristic length is no longer arbitrary and it does not need to be reported with results since this technique is general in nature. A couple of examples may illustrate this concept.

### 4.4.2 Determining the Mode Given \( K_1 \) and \( K_2 \)

Consider the brass/solder interface loaded with \( K_1 = 200 \text{ MPa} \sqrt{\text{m}} \) and \( K_2 = 100 \text{ MPa} \sqrt{\text{m}} \). Using Eqs. (4.11) and (4.4) gives \( G = 941.6 \text{ N/m} \) and \( \phi = 0.4636 \). Normalizing \( G \) by a length \( L = 1 \text{ m} \) and \( E^* = 51.91 \text{ GPa} \) gives

\[
G^* = \frac{G}{E^*L} = \frac{941.6}{(51.91 \times 10^9)(1)} = 18.14 \times 10^{-6}.
\]

This value of \( G^* \) can be transformed using Eq. (4.45) since it is logarithmically periodic. If \( n = -2 \), then

\[
G^* = (18.14 \times 10^{-6}) \exp \left[ \frac{-2}{\pi(-0.0480)} \right] = 10.4926.
\]

With values of \( G^* \) and \( \phi \), we can numerically determine the value of \( U/G \) using the integral of Eq. (4.56) when the field equations use the characteristic length of \( l = r_1G \). This results in \( U/G = \ldots \)
Figure 4.19: The top plot has the same values of $U/G$ and $\psi$ as shown in Fig. 4.18 plotted as circles. From those values, the characteristic length was calculated for each point and shown in the bottom plot. Finally, the $\times$ symbols were plotted on the top plot using the classic definition of $\psi$ to verify the new definition of mode mix.
If we perturb the value of $\phi$, we can obtain a numerical estimation of $dU/d\phi$. This results in a positive slope, therefore, the value of $U/\mathcal{G}$ must be interpolated off the positive sloping portion of the graph in Fig. 4.18 (positive values of $\psi$). Interpolating off the graph in Fig. 4.18 gives the value of the mode to be $\psi = 0.7559 \text{ radians}$ and the corresponding characteristic length is

$$l = \exp \left[ \frac{\psi - \phi}{\varepsilon} \right] = \exp \left[ \frac{0.7559 - 0.4636}{-0.0480} \right] = 0.002266.$$ 

Verification of this computation can be made by using the classic definition of $\psi$:

$$\psi = \tan^{-1} \frac{\Im[K^{i\varepsilon}]}{\Re[K^{i\varepsilon}]}.$$ 

Noting that $K^{i\varepsilon} = K_1 \cos(\varepsilon \ln l) - K_2 \sin(\varepsilon \ln l) + i[K_1 \sin(\varepsilon \ln l) + K_2 \cos(\varepsilon \ln l)],$

$$\psi = \tan^{-1} \frac{K_1 \sin(\varepsilon \ln l) + K_2 \cos(\varepsilon \ln l)}{K_1 \cos(\varepsilon \ln l) - K_2 \sin(\varepsilon \ln l)}$$

$$= \tan^{-1} \frac{200 \sin(-0.048 \ln[0.002266]) + 100 \cos(-0.048 \ln[0.002266])}{200 \cos(-0.048 \ln[0.002266]) - 100 \sin(-0.048 \ln[0.002266])}$$

$$= \tan^{-1} \frac{57.63 + 95.76}{191.5 - 28.81}$$

$$= 0.7559 \text{ rad.}$$

Note that the normalization of $\mathcal{G}$ was based on $l = 1 \text{ m}$. If this length changed, then the results may change. Therefore, care must be taken to normalize $\mathcal{G}$ with respect to 1 meter, regardless of measurement systems. A similar analysis in US units (when $\mathcal{G}$ is normalized by 39.37 inches) results in the same characteristic length and mode mix. This suggests the characteristic length determined with this method is not unit dependent.

### 4.4.3 Determining the Stress Intensity Factors given $\mathcal{G}$ and $\psi$

It has been shown how to determine the mode given $K_1$ and $K_2$ or $\mathcal{G}$ and $\phi$. However, to determine $K_1$ and $K_2$ based on $\mathcal{G}$ and $\psi$ without knowing a characteristic length is possible by using the procedure outlined in the previous section in reverse.
Consider the example using the brass/solder interface where $J = 200$ N/m and $\psi = 0.5$ radians. The normalized strain energy release rate is $J^* = 1.6949$ (after being transformed with $n = -3$). Also, if we know $\psi$, then $U/J$ can be found from Fig. 4.18. Given the value of $J^*$ and $U/J$, the value of $\phi$ is determined by matching the values of $U/J$, one from looking up in Fig. 4.18 and the other from calculation technique used to generate the contour plots. The interpolated value of $U/J$ for this example is 0.4587 which gives $\phi = 0.5647$ radians.

The two stress intensity factors are determined as

$$K_1 = |K| \cos \phi$$

and

$$K_2 = |K| \sin \phi,$$

where $|K|$ is determined from Eq. (4.13). For this example,

$$|K| = \sqrt{J^* E^* \cosh(\pi \varepsilon)}$$

$$= \sqrt{200(51.91 \times 10^9) \cosh(\pi(-0.048))}$$

$$= 3.2588 \text{ MPa} \sqrt{\text{m}}$$

so $K_1 = 2.7529 \text{ MPa} \sqrt{\text{m}}$ and $K_2 = 1.7439 \text{ MPa} \sqrt{\text{m}}$.

### 4.5 Conclusion

A technique of determining the mode mix based on an energy criterion was presented. Two candidate quantities were considered, the first being the total elastic strain energy within the contour defined by the yield strength and the second was a contour integral of the elastic strain energy density for a circular contour. The strain energy density was computed in closed form and the numerical integration proved much faster for the contour integral. An analytical solution for the homogeneous case provided insight for normalization as well as verification for the numerical procedures.

The mode was defined by interpolating the range of dimensionless energy values when the minimum was set to pure mode I ($\psi = 0$) and the maximum was pure mode II ($\psi = \pm \pi/2$). An example
of a brass/solder interface illustrated the procedure and consistency with the classic definition of mode was maintained by choosing an appropriate characteristic length. The characteristic length can be given physical meaning based on this definition of the mode. Furthermore, spanning the whole range of mode mix values requires varying both the ratio of the stress intensity factors as well as their magnitude.
5 Plastic Dissipation from Cyclic Loading of a Bimaterial Interface Crack

The previous chapters have shown the significance of mode-mix on the cyclic plastic dissipation energy at a crack tip. To extend the research into the general case of a bimaterial interface, where there are mismatches in both elastic and plastic properties, requires an unambiguous definition for the mode. The elasticity solution to the bimaterial crack problem gives oscillating stresses along the interfaces and the definition of the mode in a classic sense is not dimensionally feasible. To ensure the dimensions are correct, the complex stress intensity factor is multiplied by an arbitrary length raised to a complex power. This arbitrary length subsequently makes the definition of the mode mix arbitrary.

Since the plastic dissipation energy is at a minimum for mode I in a homogeneous material, it is plausible to define mode I as the state of loading that yields the minimum plastic dissipation. Similarly, the maximum plastic dissipation should occur in mode II as it does for a homogeneous material.

In light of Chapter 3, the changes in plastic properties do not change the mode for the minimum plastic dissipation. A goal of this chapter is to set forth a criteria for selecting the characteristic length for bimaterial interface problems based on minimizing the plastic dissipation energy in the plastic zone around the crack tip. Once the mode-mix is unambiguously defined through choosing an appropriate characteristic length, a complete mapping of the plastic dissipation as a function of mode-mix for a general bimaterial interface is possible.
5.1 The Interface Crack Problem

Suo and Hutchinson [51] published a paper of an elastic analysis of a layered specimen with an interface crack under general loading. This specimen, shown in Fig. 5.1a, has enjoyed many applications throughout the literature concerning layered materials, surface coatings, and systems where steady state cracking exists. This section provides an overview of the analysis in [51]. Those results are then specialized for the particular problem at hand.

5.1.1 Equivalent Loading and Superposition

Consider the geometry of Fig. 5.1b as an equivalent loading on the same specimen used in the finite element runs (Fig. 2.1). Superposition arguments can be used to simplify the loading into two load parameters \( P \) and \( M \) as shown in Eqs. (5.1) and (5.2):

\[
P = P_1 - C_1 P_3 - C_2 \frac{M_3}{h} \tag{5.1}
\]

\[
M = M_1 - C_3 M_3 \tag{5.2}
\]

where,

\[
C_1 = \frac{\Sigma}{A_o} \tag{5.3}
\]

\[
C_2 = \frac{\Sigma}{I_o} \left( \frac{1}{\eta} - \Delta + \frac{1}{2} \right) \tag{5.4}
\]

\[
C_3 = \frac{\Sigma}{12I_o}. \tag{5.5}
\]

Here \( \Sigma \) is an elastic modulus mismatch \( \Sigma = c_2/c_1 \), where \( c_i = (\kappa_i + 1)/\mu_i \). Also, \( \kappa_i = 3 - 4\nu_i \) for plane strain and \( \kappa = (3 - \nu)/(1 + \nu) \) for plane stress. It is also noted that

\[
\Sigma = \frac{1 + \alpha}{1 - \alpha} \tag{5.6}
\]

where \( \alpha \) is Dundurs’ first parameter. The following terms are derived from elementary beam theory for a composite beam:
neutral axis

(a) General loading conditions for an interface crack.

(b) Generalized self equilibrating specimen shown with equal layer thickness.

Figure 5.1: A generalized mixed mode specimen and corresponding equivalent loading obtained by superposition.

\[
A_o = \frac{1}{\eta} + \Sigma \quad (5.7)
\]

\[
I_o = \frac{1}{3} \left\{ \left( \Delta - \frac{1}{\eta} \right)^2 - 3 \left( \Delta - \frac{1}{\eta} \right) + 1 \right\} + \frac{3}{\eta} \left( \Delta - \frac{1}{\eta} \right) + \frac{1}{\eta^3} \quad (5.8)
\]

\[
\Delta = \frac{1 + 2 \Sigma \eta + \Sigma \eta^2}{2 \eta (1 + \Sigma \eta)} = \frac{\delta}{h} \quad (5.9)
\]

where \( \eta = \frac{h}{H} = 1 \) for the case of equal layer thickness and \( \delta \) refers to the offset of the neutral axis in a composite beam. Using Eqs. (5.3)-(5.7) in Eqs. (5.1) and (5.2) will give an equivalent loading for any generalized specimen.
5.1.2 Strain Energy Release Rate

The strain energy release rate is a measure of the amount of energy required to advance a crack for some given area. The definition of steady state crack extension is a crack whose strain energy release rate \( G \) is independent of the actual crack length. For steady state crack extension, the strain energy release rate can be determined by taking the difference of the strain energy ahead of the crack and behind the crack.

The strain energy release rate reported in [51] for the geometry of Figure 5.1b is

\[
G = \frac{c_1}{16} \left[ \frac{p^2}{Ah} + \frac{M^2}{In^3} + 2 \frac{PM}{\sqrt{Ah}} \sin \gamma \right],
\]

where

\[
\sin \gamma = 6 \Sigma \eta^2 (1 + \eta) \sqrt{AI}
\]

\[
A = 1/[1 + \Sigma(4\eta + 6\eta^2 + 3\eta^3)]
\]

\[
I = 1/[12(1 + \Sigma \eta^3)].
\]

For a geometry with matching layer thicknesses (\( \eta = 1 \)),

\[
\sin \gamma = \Sigma \sqrt{\frac{12}{1 + 14\Sigma + 13\Sigma^2}}.
\]

In terms of Dundurs parameter \( \alpha \), the elastic mismatch constant \( \Sigma \) can be expressed as

\[
\Sigma = \frac{E_2}{E_1} = \frac{1 - \alpha}{1 + \alpha}.
\]

The external loads and applied moments in Eq. (5.10) can be determined from an equivalent loading by the superposition argument from Section 5.1.1. If only moments \( M_1 \) and \( M_2 \) are applied and the
layers are of equal thickness, then the results for $P$ and $M$ are

$$P = \frac{M_1(6\alpha^2 + \alpha - 7) - M_2(\alpha + 1)}{6\alpha^2 - 8}$$  \hspace{1cm} (5.16)$$

and

$$M = \frac{3(M_2 - M_1)(\alpha^2 - 1)}{5(3\alpha^2 - 4)}. \hspace{1cm} (5.17)$$

Substituting Eqs. (5.15-5.17) into Eq. (5.10) and letting $\eta = 1$ yields

$$G = \frac{3[M_1^2(6\alpha + 7)(\alpha - 1)^2 + 2M_1M_2(1 - \alpha^2) + M_2^2(7 - 6\alpha)(\alpha^2 - 1)]}{(3\alpha^2 - 4)(\alpha - 1)E_1h^3}, \hspace{1cm} (5.19)$$

which is a result for the strain energy release rate that is specific to the geometry considered in this research. In the case of no elastic mismatch, $(\alpha = 0)$, the value of the strain energy release rate reduces to

$$G = \frac{3(7M_1^2 + 2M_1M_2 + 7M_2^2)}{4Eh^3} \hspace{1cm} (5.20)$$

which is reported in [2].

### 5.1.3 Interface Stress Intensity Factors

The interface crack tip stress field have the form

$$\left. (\sigma_{yy} + i\sigma_{xy}) \right|_{y=0} = \frac{(K_1 + iK_2)r_\epsilon^\psi}{\sqrt{2\pi r_x}}. \hspace{1cm} (5.21)$$

The oscillations in the stress field occur because the asymptotic solution for the stress fields yield a complex solution in the form $K = K_1 + iK_2 = |K|e^{i\psi}$, where $\psi$ is a measure of the mode-mix. The stress fields oscillate for nonzero values of $\epsilon$, therefore, the ratio $K_2/K_1$ does not correspond directly to the ratio $\tau_{xy}/\sigma_{yy}$. In order to define a dimensionally acceptable measure of mode-mix, it is necessary to consider the quantity $Kl^{\psi\epsilon}$, where

$$Kl^{\psi\epsilon} = K_1\cos(\epsilon \ln l) - K_2\sin(\epsilon \ln l) + i[K_2\cos(\epsilon \ln l) + K_1\sin(\epsilon \ln l)]. \hspace{1cm} (5.22)$$

Here, $l$ is any characteristic length which is arbitrary.
For the problem of Figure 5.1, Suo and Hutchinson selected the arbitrary characteristic length to be the thickness of the top layer. Therefore, \( l = h \) and the real and imaginary parts of \( Kh^i \) are [51, 54]

\[
\Re[Kh^i] = \frac{p}{\sqrt{2}} \left[ \frac{P}{\sqrt{Ah}} \cos \omega + \frac{M}{\sqrt{Ih^3}} \sin(\omega + \gamma) \right]
\]

\( (5.23) \)

\[
\Im[Kh^i] = \frac{p}{\sqrt{2}} \left[ \frac{P}{\sqrt{Ah}} \sin \omega - \frac{M}{\sqrt{Ih^3}} \cos(\omega + \gamma) \right],
\]

\( (5.24) \)

where \( p = \sqrt{(1 - \alpha)/(1 - \beta^2)} \), \( A = 1/[1 + \Sigma(4\eta + 6\eta^2 + 3\eta^3)] \), and \( I = 1/[12(1 + \Sigma\eta^3)] \). Equations (5.23) and (5.24) contain a function \( \omega(\alpha, \beta, \eta) \) that is tabulated in [51] and ranges from \( 37^\circ \leq \omega \leq 65^\circ \). Another contribution of this dissertation is an alternative method of determining the value of \( \omega \), as shown in Section 5.2.

A numerical technique involving energy domain integrals, which is known to be stable and accurate for relatively coarse meshes, is used to determine the stress intensity factors. The interaction integral method for extracting stress intensity factors was formulated by Shih and Asaro, and it is implemented in the ABAQUS FEA software as the keyword *CONTOUR INTEGRAL, CONTOURS=n, Type=K Factors [90].

### 5.2 Numerical Determination of \( \omega(\alpha, \beta, \eta) \)

The tabulated solution for \( \omega \) presented by Suo and Hutchinson [51] is based upon the characteristic length being set to the thickness of the top layer (i.e., \( l = h \)). Once the characteristic length is defined, the results from [51] can be used to determine the angle \( \omega \). Both mode I and mode II loading condition can be used to determine \( \omega \). In the case of the layered geometry shown in Fig. 5.1:

\[
\frac{P}{\sqrt{Ah}} \sin \omega = \frac{M}{\sqrt{Ih^3}} \cos(\omega + \gamma)
\]

\( (5.25) \)

when the imaginary part of the complex stress intensity factor vanishes (see Eq. (5.24)). According to Eq. (4.5), when \( \Im[Kh^i] = 0 \) a pure mode I condition exists. Using the interaction integral (which defines a characteristic length of unity), we can search for a loading condition that produces \( \Re[Kh^i] = |K| \) and \( \Im[Kh^i] = 0 \). Once the loading conditions (\( P \) and \( M \)) are known, Eq. (5.25) can be solved for \( \omega \).
In a similar fashion,
\[
\frac{P}{\sqrt{Ah}} \cos \omega = -\frac{M}{\sqrt{Ih^3}} \sin(\omega + \gamma) \tag{5.26}
\]
when the real part of the complex stress intensity factor vanishes. When \(\Re[Khi] = 0\) a pure mode II condition exists and one can numerically search for the loading conditions that satisfy this criteria. Once the loading conditions \((P \text{ and } M)\) are known, Eq. (5.26) can be solved for \(\omega\). The above are two techniques to determine the value of \(\omega\).

Suo and Hutchinson present results of their numerical solution for \(\omega\) in Ref. [51], and a subset of those results are repeated in Table 5.1 to compare to the values obtained through the finite element method presented in this section.

### 5.2.1 Determining the Values of the Applied Moments

The ABAQUS finite element analysis package has a feature called the interaction integral which is capable of numerically estimating the stress intensity factors at a crack tip. The interaction integrals are only available for a purely elastic analysis so plastic material behavior must be reserved for different analysis. The output of the interaction integral is given as \(K_1\) and \(K_2\) which are the real and imaginary parts of the complex stress intensity factor for the general case of a bimaterial interface crack.

To determine a ratio of the applied bending moments, a search algorithm was used to determine the moments for each value of \(\alpha\). Only half the values of \(\alpha\) had to be determined because values of \(f\) are simply inverted when \(\alpha\) changes sign (i.e. \(f(\alpha) = f(-\alpha)^{-1}\)). The results are interpreted as a ratio of the bending moments (i.e. \(f(\alpha) = \frac{M_1}{M_2}\)) and shown in Figure 5.2.

This technique is illustrated in Listing C in the Appendix which uses the secant method to search for the zeros of the real or imaginary part of the complex stress intensity factor which is determined from the results of the interaction integral. The value of \(\Re[Khi]\) should be zero when looking for the pure mode II ratio and the value of \(\Im[Khi]\) should be zero when looking for pure mode I. The independent variable in this case is the ratio of \(M_1\) to \(M_2\). Once the ratio of \(M_1\) to \(M_2\) is known, the value of \(\omega\) can be determined.
Figure 5.2: Loading conditions to generate pure mode I and pure mode II loading conditions for different values of \( \alpha \) according to the definition of mode presented in Ref. [51]. The pure mode II conditions are the negative ratios.
5.2.2 Discussion of the Results for $\omega$

For a pure mode I loading condition Eq. (5.24) should be equal to zero. Since the values of the loads required to generate a pure mode were determined in the previous section, the only way to enforce $\Im[Kh^{i\omega}]$ to be equal to zero is by finding the appropriate value of $\omega$. The value of $\gamma$ is determined from elastic and geometric properties. A simple root finding algorithm is all that is required to search for the value of $\omega$ that makes $\Im[Kh^{i\omega}] = 0$. For this case (mode I), the value of $\omega$ can be numerically determined from

$$0 = \lambda \sin(\omega) - \cos(\omega + \gamma),$$

where $\lambda = \frac{P h}{\pi T} \sqrt{\frac{l}{A}}$. The values of $P$ and $M$ are determined from the superposition equations and, for the case of only an applied moment, $\lambda$ is only dependent on the ratio of $M_1$ to $M_2$. The same procedure is used for the pure mode II case except the real part of the complex stress intensity factor is set to zero. For the case where $\Re[Kh^{i\omega}] = 0$ (mode II), the value of $\omega$ can be numerically determined from

$$0 = \lambda \cos(\omega) + \sin(\omega + \gamma).$$

It is only necessary to solve for $\omega$ using either mode I or mode II. The results of $\omega$ from both modes are reported in Table 5.1.

Notice on the right table of results of Table 5.1 that the values of $\beta = \alpha/4$. This may seem peculiar since the nonzero $\beta$ values introduce the oscillatory elastic response. However, since the loading conditions were solved for using a known reference length, the formulation remains valid. Therefore, the same reference length $l = h$ must be used for both the computation of the load in section 5.2.1 as well as the root finding procedure contained in this section.

As far as numerical methods are concerned, the process of repeating highly involved finite element analysis is not computationally efficient. The actual algorithm may not be optimized for speed but the results shown in Table 5.1 validate the solutions found in the literature as well as provide more confidence in the model.
<table>
<thead>
<tr>
<th>$(\alpha, \beta)$</th>
<th>$\omega_{\text{Mode I}}$</th>
<th>$\omega_{\text{Mode II}}$</th>
<th>$\omega_{\text{Suo}}$</th>
</tr>
</thead>
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<tr>
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<td>47.549</td>
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<td>49.107</td>
<td>49.110</td>
</tr>
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<table>
<thead>
<tr>
<th>$(\alpha, \beta)$</th>
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<th>$\omega_{\text{Mode II}}$</th>
<th>$\omega_{\text{Suo}}$</th>
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<tr>
<td>(0.9, 0.225)</td>
<td>48.179</td>
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</tbody>
</table>

Table 5.1: Comparison of the values of $\omega(\alpha, \beta, \eta)$ (in degrees) obtained from the interaction integrals compared to the values reported by Suo and Hutchinson in [51]. The value of $\eta = 1$ is used for all cases.
5.3 Numerical Determination of Plastic Dissipation Energy

5.3.1 Specimen Geometry

The geometry used for analysis in this chapter is the same as the geometry in previous chapters and is shown in Fig. 2.1. However, in this chapter the mixed-mode layered specimen geometry of Figure 2.1 is composed of two bonded layers of isotropic materials #1 and #2, which can have equal thicknesses (\(h_1\) and \(h_2\)), different elastic properties (\(E_1\), \(E_2\), \(\nu_1\), and \(\nu_2\)) and different plastic properties (\(\sigma_y_1\), \(\sigma_y_2\), \(E_t_1\), and \(E_t_2\)).

5.3.2 Loading Conditions

The loads need to be chosen as to span the range of plastic dissipation energy while ensuring the minimum plastic dissipation is at \(\psi = 0\) and the maximum plastic dissipation is at \(\psi = \pm \pi/2\) radians. For this criteria to be satisfied, the characteristic length will be a function of the mode mix ratio. We have seen in Chapter 4 that the mode and the load are coupled; however, the effect of changing the load is negligible when computing the result for the normalized plastic dissipation.

For mapping out the plastic dissipation, the magnitude of the load shall be fixed at \(G = 1000\) N/m and the ratio of the loads shall be evenly spaced with the loading condition. Each applied load ratio has its own characteristic length. There is a linear relationship between the applied load and the log of the characteristic length to be used in the analysis. This relationship depends on the location of the extrema for different \(\epsilon\) and follows the linear form of Eqs. (5.27)-(5.29).

\[
\phi - \phi_{\text{min}} = \frac{\phi_{\text{max}} - \phi_{\text{min}}}{\log l_{\text{max}} - \log l_{\text{min}}} (\log l - \log l_{\text{min}}) \quad \phi_{\text{min}} \leq \phi \leq \phi_{\text{max}} \quad (5.27)
\]

\[
\phi - \phi_{\text{min}} = \frac{(\phi_{\text{max}} - 180^\circ) - \phi_{\text{min}}}{\log l_{\text{max}} - \log l_{\text{min}}} (\log l - \log l_{\text{min}}) \quad \phi < \phi_{\text{min}} \quad (5.28)
\]

\[
\phi - \phi_{\text{min}} = \frac{(\phi_{\text{max}} + 180^\circ) - \phi_{\text{min}}}{\log l_{\text{max}} - \log l_{\text{min}}} (\log l - \log l_{\text{min}}) \quad \phi > \phi_{\text{max}} \quad (5.29)
\]

These equations represent a similar interpolation scheme seen in Fig. 4.16 on page 85. The values of the extrema used in the analysis are tabulated in Table 5.2. These results represent the first physically meaningful definition of mode mix for bimaterial crack problems which is a primary contribution of this dissertation.
\[
\begin{array}{cccccccc}
\alpha & \beta & \epsilon & \phi_{\text{min}} \text{ (deg)} & \phi_{\text{max}} \text{ (deg)} & l_{\text{min}} & l_{\text{max}} \\
0.0 & 0.0 & 0.0 & 0 & 90 & \text{N/A} & \text{N/A} \\
0.1 & 0.025 & -0.00796 & -1.86 & 89.2 & 0.0168 & 0.182 \\
0.2 & 0.05 & -0.01593 & -3.90 & 88.2 & 0.0140 & 0.1009 \\
0.3 & 0.075 & -0.02392 & -6.19 & 86.9 & 0.0109 & 0.1055 \\
0.4 & 0.1 & -0.03194 & -8.90 & 84.8 & 0.00770 & 0.0583 \\
0.5 & 0.125 & -0.04000 & -12.0 & 81.9 & 0.00529 & 0.0296 \\
0.6 & 0.15 & -0.04811 & -16.1 & 78.7 & 0.00291 & 0.0164 \\
0.7 & 0.175 & -0.05628 & -20.7 & 74.8 & 0.00161 & 0.00895 \\
0.8 & 0.2 & -0.06453 & -25.5 & 71.3 & 0.00101 & 0.00589 \\
\end{array}
\]

Table 5.2: Characteristic lengths and stress intensity factor ratios to make minimum plastic dissipation at \( \psi = 0 \) and maximum plastic dissipation at \( \psi = 90^\circ \) when \( v_1 = v_2 = 1/3 \). The results are \( \epsilon \) driven so tabulation of \( \alpha \) is unnecessary but shown for illustration.

### 5.3.3 Determining the Applied Moments

The characteristic length for the solution of the layered specimen was chosen by Suo and Hutchinson to be the thickness of the top layer. This choice is certainly valid and provides consistent results between analysis and experiment. Its physical significance, however, is separated from the stress intensity factors themselves. Also, the choice of the using the top layer thickness did not render a closed form solution for the crack tip stress intensity factors; the solution depends on the numerical determination of \( \omega \), as shown in Section 5.2.

Since the specialized solution presented by Suo and Hutchinson uses characteristic length of the top layer, using any other characteristic length to define the mode mix requires a conversion. Consider two characteristic lengths: \( h \) (the top layer thickness) and \( l \) (the length determined based on energy considerations). A specification of the ratio of the applied complex stress intensity factors is made such that

\[
\phi = \tan^{-1} \frac{K_2}{K_1},
\]

where \( \phi, l \) and \( |K| \) are specified. The question is how to apply the moments to the model of Fig. 5.1b to generate the expected near tip fields. One can begin with the trigonometric relations:

\[
K_1 = |K| \cos \phi \tag{5.31}
\]

\[
K_2 = |K| \sin \phi \tag{5.32}
\]
The quantity $K_{h}^{ie}$ (where $h$ is the top layer thickness) must be split into real and imaginary parts:

\[
\Re[K_{h}^{ie}] = K_1 \cos(\varepsilon \ln h) - K_2 \sin(\varepsilon \ln h) \tag{5.33}
\]

\[
\Im[K_{h}^{ie}] = K_1 \sin(\varepsilon \ln h) + K_2 \cos(\varepsilon \ln h) \tag{5.34}
\]

Next, the quantities $\gamma$ and $\omega$ are needed. A specific formula for $\gamma$ for this geometry is

\[
\gamma = \sin^{-1} \left( \frac{1 + \alpha}{2\sqrt{\alpha - 1}} \right),
\]

and the value of $\omega$ can be determined from Section 5.2. It should be noted that for no elastic mismatch and equal layer thicknesses, $\gamma = \sin^{-1} \sqrt{\frac{3}{7}}$ and $\omega = \cos^{-1} \sqrt{\frac{3}{7}}$. The equivalent loading can be determined as:

\[
P = \sqrt{\frac{h-h_{a}}{\pi 6a}} \left( \Re[K_{h}^{ie}] \cos(\omega + \gamma) + \Im[K_{h}^{ie}] \sin(\omega) \right) \cos(\omega) \sqrt{\frac{\alpha - 1}{\beta - 1}},
\]

and

\[
M = \frac{\sqrt{-h^{4}(\alpha - 1)}}{2\sqrt{3} \cos(\omega) \sqrt{\frac{\alpha - 1}{\beta - 1}}} \left( \Im[K_{h}^{ie}] \sin(\omega) - \Re[K_{h}^{ie}] \cos(\omega) \right).
\tag{5.36}
\]

Once the equivalent loading is known, the applied moments can be determined as

\[
M_1 = M + \frac{hP}{6\alpha - 6} \tag{5.37}
\]

and

\[
M_2 = M + hP \left( 1 + \frac{1}{6\alpha + 6} \right). \tag{5.38}
\]

This provides the loading required to give the crack tip fields a mode defined by $|K|$ and

\[
\psi = \tan^{-1} \frac{\Im[K_{h}^{ie}]}{\Re[K_{h}^{ie}]}. \tag{5.39}
\]

Verification of this process was performed using the interaction integrals implemented in ABAQUS. Notice that the mode defined using $h$ is generally different than the mode defined by $l$, which
corresponds to more physically based energy considerations.

5.3.4 Normalization

When a mismatch exists across an interface, an equivalent elastic modulus is desired for normalization. One such modulus is defined as $E^*$ in Eq. (4.12). Since the plastic dissipation goes as $\Delta \mathcal{G}^2$ when $\beta = 0$, one can use a similar definition of the normalization as the previous chapters:

$$
\frac{dW^*}{dN} = \frac{\sigma^*}{\mathcal{G}^2 E^*} \frac{dW}{dN},
$$

where

$$
\frac{1}{E^*} = \frac{1}{2} \left( \frac{1}{E_1} + \frac{1}{E_2} \right),
$$

and

$$
\sigma^* = \min(\sigma_{y1}^2, \sigma_{y2}^2).
$$

It can also be noted that $\tilde{E}_2(1 + \alpha) = E^*$. Upon normalizing the plastic dissipation, the dimensionless plastic dissipation is dependent on the mode-mix and the Poisson’s ratios of each material:

$$
\frac{dW^*}{dN} = \frac{dW^*}{dN}(\psi, \nu_1, \nu_2).
$$

5.4 Results and Discussion

5.4.1 Setting $\beta = 0$ in Plane Strain

When $\beta = 0$, from either a lack of elastic mismatch or the proper choice of Poisson’s ratios, the asymptotic stress fields do not display oscillatory behavior. This can be shown by substituting $\beta = 0$ into Eq. (4.9) and simplifying Eq. (5.22), which gives $K = K_1 + iK_2$, and the mode mix is defined as

$$
\psi = \tan^{-1} \left( \frac{K_2}{K_1} \right).
$$

Given a value for $\alpha$, the values of $\nu_1$ and $\nu_2$ that make $\beta = 0$ are not unique. However, one can develop a relationship between $\nu_1$ and $\nu_2$. Begin by letting $\kappa = 3 - 4\nu$ for plane strain and solving
Relationship Between the Poisson’s Ratios for $\beta = 0$

![Graph showing the relationship between Poisson’s ratios for different values of $\alpha$.](image)

Figure 5.3: Possible values for Poisson’s ratio when $\beta = 0$ in plane strain for different values of $\alpha$. Only half the solutions are provided because the line $v_2 = v_1$ is a line of symmetry.

Eq. (4.6) for $\mu_1$:

$$\mu_1 = \frac{\mu_2 - \mu_2 v_1 + \alpha \mu_2 - \alpha \mu_2 v_1}{1 - \alpha - v_2 + \alpha v_2}.$$  (5.45)

Substituting this expression for $\mu_1$ into the Eq. (4.7) and simplifying gives:

$$\beta = \frac{v_1 - v_2 + 2\alpha - 3v_1 \alpha - 3v_2 \alpha + v_1 + 4v_1 v_2 \alpha}{4 - 4v_1 - 4v_2 + 4v_1 v_2}.$$  (5.46)

Solving for $v_2$ gives the relationship:

$$v_2 = \frac{-v_1 - 2\alpha + 3v_1 \alpha + 4\beta - 4v_1 \beta}{-1 - 3\alpha + 4v_1 \alpha + 4\beta - 4v_1 \beta}.$$  (5.47)

When $\beta = 0$, Eq. (5.47) simplifies to

$$v_2 = \frac{-v_1 - 2\alpha + 3v_1 \alpha}{-1 - 3\alpha + 4v_1 \alpha}.$$  (5.48)

which is plotted in Figure 5.3.
While forcing $\beta = 0$ in a real material system may be mathematically unsatisfying, it has very little consequences when making engineering decisions. Commonly the decision to make $\beta = 0$ is justified by eliminating the ambiguity of the definition of the mode \cite{54, 55, 86}. In this chapter, the effects of $\beta$ on the plastic dissipation is further expanded.

5.4.2 Elastic Modulus Mismatches without Oscillation

To assess the case where only an elastic mismatch exists, one can set the value of $\hat{s} = 0$ and the hardening modulus in both materials to zero (elastic perfectly plastic). The plot in Fig. 5.4 shows the dimensionless plastic dissipation $dW/dN^*$ as a function of the mode-mix, $\psi$, for the case when $\beta = 0$ and the bottom layer was held at $v = 1/3$. Keep in mind that the mode mix is well defined because no oscillations exist. The significant influence of the mode-mix overshadows the variations induced from using different values of $v_1$ and $v_2$. Furthermore, all curves closely resemble the “master” curve in Fig. 2.8 on page 24.

As the absolute value of $\alpha$ increases, the stiffness of one layer becomes larger than the other. According to Eq. (5.40), the effect of the elastic mismatch should be symmetric about the line $\psi = 0$ since the sum of the inverses of the elastic moduli is the same regardless of the order designation of the layers. However, the plot in Fig. 5.4 shows that the normalized results are not symmetric. This is due to the fact that the Poisson’s ratios are not the same across the layers and the plastic constraint effect is discussed in the next section.

5.4.3 Effect of Plastic Constraint

Both the computation and interpretation of results is simplified when $\beta = 0$. The plastic dissipation, however, is not purely a function of $\beta$, but also a function of both Poisson’s ratios, $v_1$ and $v_2$. To determine the variation of plastic dissipation energy for different combinations of plastic constraint within $\beta = 0$, a numerical experiment was conducted by varying the top layer between 0.0 and 0.495.

The values for Poisson’s ratio for the numerical experiment are determined using Eq. (5.48). Notice how as the elastic modulus mismatch increases (higher $\alpha$), Poisson’s ratio for layer #2 is tending toward 0.5 (incompressible). This result, combined with the author’s previous work \cite{1, 2},

108
Figure 5.4: Results of the normalized plastic dissipation energy for positive values of $\alpha$ and all ranges of mode mix when $\beta = 0$. The variation between all the levels of elastic mismatch is attributed different values of Poisson’s ratio.

suggests that the dissipated energy will be less influenced by the changes in plastic constraint for higher values of $\alpha$. Another point is that there would be a larger influence on the plastic dissipation energy for low values of $\alpha$ because each material is contributing its effect from the plastic constraint; whereas, with higher values of $\alpha$ the effect from the plastic constraint is limited to variation in only one layer.

The results of the numerical experiment are shown in Figs. 5.5-5.7. The top line in Fig. 5.5 shows the most influence of changing the value of $\nu_1$ occurs when there is no elastic mismatch. As the level of the elastic modulus mismatch increases, the value of $\nu_2$ becomes closer to 0.5 to satisfy the $\beta = 0$ criterion. When this incompressible material behavior is approached, the effect of the plastic constraint of the other layer is marginalized as shown in Figs. 5.5-5.7. This is true for all modes and the influence of the plastic constraint appears to be independent of the mode. The normalization of the results in Figs. 5.5-5.7 are performed using the normalization in Chapter 2. The results could be normalized using Eq. (5.40) which would result in the curves falling on top of each other so that the trends would be more difficult to interpret.
Figure 5.5: The effect of the plastic constraint on the plastic dissipation energy when normalized by the properties of the top layer and loaded in pure mode 1.
Figure 5.6: The effect of the plastic constraint on the plastic dissipation energy when normalized by the properties of the top layer and loaded in in a middle mode when $M_2 = 0$. 
Figure 5.7: The effect of the plastic constraint on the plastic dissipation energy when normalized by the properties of the top layer and loaded in pure mode II.
5.4.4 Elastic Modulus Mismatches with Non-zero Oscillation Index

Upon normalizing the results of the plastic dissipation according to Eq. (5.40) and applying the appropriate characteristic length according to Table 5.2, one can see from Fig. 5.8 that the mode mix has the most influence of any non-normalizing variable. The most interesting observation is that the deviation of the normalized plastic dissipation from the so-called master curve from Chapter 2 is relatively small compared to the response changed from the mode mix. This means the normalizing equation coupled with the appropriate characteristic length renders normalized plastic dissipation results for cases where an oscillation index exists similar to the results for no oscillation. The differences in the results between those shown in Figs. 5.4 and 5.8 is due to differences in the plastic constraint (Poisson’s ratio effects).

Keep in mind that the results shown in Fig. 5.8 are for elastic mismatches with no plastic mismatch. More meaningful results would be given if a mismatch in the plastic properties also exists, as discussed in the next section.
5.4.5 Effect of Elastic and Plastic Mismatches

The general case of a bimaterial system contains mismatches in both elastic and plastic properties. A plastic property mismatch can be due to a difference in yield strengths as well as a difference in hardening modulus. As seen in Chapter 3, the effects of the hardening modulus are similar to that of the yield strength. Moreover, many ductile metals have similar hardening moduli that are at or below 10% of the elastic moduli and are well represented by elastic-perfectly plastic response. As such, a mismatch in yield strength is sufficient to understand the effect of a plastic mismatch coupled with an elastic mismatch.

Since the effect of an elastic mismatch can be significantly normalized out of the response, the effect of the plastic mismatch on the normalized plastic dissipation for material systems with an
elastic mismatch follows the same trends outlined in Chapter 3. Those trends show that the normalized plastic dissipation follows the trend of the elastic mismatch. In other words, the normalized plastic dissipation increases in relation to the case of no elastic mismatch ($\alpha = 0$) when both the elastic mismatch and the yield strength mismatch increase (Fig. 5.9a). Similarly, the normalized plastic dissipation decreases in relation to no elastic mismatch when $\hat{s}$ is decreased. When the elastic mismatch becomes great, the stiffer layer dissipates little energy in relation to the more compliant layer. Therefore, when the stiffer layer does not yield at all, as for the case of a positive $\alpha$ and a positive $\hat{s}$, then the energy is completely dissipated by the more compliant layer. Since the stiff layer is not contributing significantly to the plasticity to begin with, the effect is fairly negligible. This can be seen by comparing the curve in Fig. 5.9a for $\alpha = 0.8$ with the curve in Fig. 5.8 for $\alpha = 0.8$.

The most obvious difference between Figs. 5.9a and 5.9b is the larger spread of the curves for the negative modes for a negative value of $\hat{s}$. This situation occurs when the bottom (more compliant) layer is stronger than the top layer. This means that all of the energy is being dissipated through plasticity in the stiffer layer in Fig. 5.9b. Not only does the normalized dissipated energy decrease when plasticity is forced to remain in the stiffer layer, but also the effect of that stiffness becomes more pronounced. Also, the minimum and maximum dissipated energy may no longer coincide with pure mode I and II, respectively. This is consistent with the elastic strain energy observations from Section 4.2.3 for a homogeneous material with a yield strength mismatch, However, caution should be used when interpreting these results since $\alpha$ is part of the normalization.
Figure 5.9: Plot of the effect of elastic mismatch with nonzero values of $\varepsilon$ for different extremes of yield strength mismatch. For these plots, $\beta = \alpha/4$ which corresponds to $v_1 = v_2 = 1/3$. 

(a) Top layer is much stronger than the bottom layer, $\sigma = 1$

(b) Bottom layer is much stronger than the top layer, $\sigma = -1$
5.4.6 Representations of the Plastic Zones

To better understand the nature of the mode mix at a bimaterial crack, the plastic zones for representative cases of an elastic mismatch are presented in Figs. 5.10-5.13. These plots contain the plastic zones at a crack tip for three different material combinations: homogeneous, \( \alpha = 0.4 \), and \( \alpha = 0.8 \). Eight different modes of loading are shown as columns in each figure defined using the value of \( l \) given in Table 5.2. The crack tip is located at the origin and the crack is growing in a positive \( x \) direction.

Examination of the figures reveals that the shapes (not necessarily the sizes) of the plastic zones are similar in each column. Since each column represents one mode, the definition of the mode based on minimizing energy produces plastic zones with similar shapes! This provides more physical basis for choosing a mode based on energy and eliminating the arbitrary aspect of the characteristic length. It should be noted that the magnitude of the stress intensity factor determines the size of the plastic zone. As such, the different sizes of plastic zone for different values of \( \alpha \) are primarily due to the different composite moduli (\( E^* \)).

Each plastic zone shown is from a material where there is no mismatch in plastic constraint. A mismatch in the plastic constraint changes the plastic zone length along the interface. This change, however, does not greatly influence the plastic dissipation.

As the mode changes from –90 to 0 and back to +90, the plastic zones tend to rotate in a clockwise direction. At mode I, the major axis of the plastic zone is lined with the \( y \)-axis. Similarly, the major axis of the plastic zone is lined with the \( x \)-axis in mode II. Incidentally, the same criteria holds true for the estimation of the plastic zone for a homogeneous material based on only the elastic fields.

5.5 Application to Thermomechanical Fatigue

Consider a steady state crack between a layer of brass and a layer of solder. The interface is assumed to be stress free at \( T_o \) and the temperature of the system is increased to \( T \). Since the materials have different coefficients of thermal expansion, different strains are produced in the layers, thus driving the crack. The analysis presented in [51] for a “misfit” stress across an interface gives

\[
\sigma_T = \frac{8 \mu_1 \Delta \alpha \Delta T}{\kappa_1 + 1}. \tag{5.49}
\]
Figure 5.10: Plastic zones of representative loading cases and material properties when $G = 1000$ N/m, $v_1 = v_2 = 1/3$, and $E_1 = 100$ MPa.
Figure 5.11: Plastic zones of representative loading cases and material properties when $G = 1000$ N/m, $v_1 = v_2 = 1/3$, and $E_1 = 100$ MPa.
Figure 5.12: Plastic zones of representative loading cases and material properties when $G = 1000$ N/m, $v_1 = v_2 = 1/3$, and $E_1 = 100$ MPa.
Figure 5.13: Plastic zones of representative loading cases and material properties when $G = 1000$ N/m, $v_1 = v_2 = 1/3$, and $E_1 = 100$ MPa.
where $\mu_1$ is the shear modulus of the top layer, $\Delta \alpha = (\alpha_1 - \alpha_2)$ is the difference between the thermal expansion of the top layer and bottom layer, $\Delta T = (T - T_0)$ is the change in temperature, and $\kappa_1 = 3 - 4v_1$ for plane strain and $\kappa_1 = (3 - v_1)/(1 + v_1)$ for plane stress.

Reference [51] continues by providing a solution for $P$ and $M$ based on superposition as

$$P = \sigma_T h \left[ 1 - C_1 - C_2 \left( \frac{1}{\eta} - \Delta + \frac{1}{2} \right) \right]$$

$$M = -\sigma_T h^2 C_3 \left( \frac{1}{\eta} - \Delta + \frac{1}{2} \right)$$

where $\delta$ is shown in Fig. 5.1 on page 95 and the remaining terms are defined on page 94. Notice that the loading is only dependent on the geometry and the equivalent thermal stress. The equivalent loads from Eqs. (5.50) and (5.51) can be used in Eqs. (5.23) and (5.24) to get the stress intensity factor.

Once the stress intensity factors are determined, the strain energy release rate can also be found. Since the characteristic length in Eqs. (5.23) and (5.24) is the layer thickness, a transformation of the characteristic length must be performed to use the results in this chapter. Once the dimensionless plastic dissipation is determined and the loading is known, the actual plastic dissipation can be found. Furthermore, the results of the plastic dissipation can be divided by the interface fracture toughness to get the predicted fatigue crack growth rate.

### 5.6 Conclusion

This chapter provides a detailed analysis and summary of the general bimaterial interface crack problem. The analysis of the layered specimen was reviewed and specific cases to the problem at hand were formulated for the elastic solution of the case of a crack along a bimaterial interface. An independent technique was developed to determine the numerical results for Suo and Hutchinson’s $\omega(\alpha, \beta, \eta)$ function using the interaction integral. These results were used to determine the loading conditions to span the complete mode of a bimaterial crack. A characteristic length is required to
define the mode of a crack along a bimaterial interface and the choices presented in this Chapter rendered the plastic dissipation a minimum for mode I and a maximum for mode II. Given the characteristic lengths as a function of mode and mismatch, the results of the dimensionless plastic dissipation were plotted against mode. The normalized plastic dissipation is highly mode dependent with minor effects from the elastic mismatch. Also, the results for general bimaterial systems were plotted and the effect of increasing the yield strength of the stiffer layer had little effect as the stiffness of that layer increased. However, increasing the yield strength of the more compliant layer tended to decrease the dimensionless plastic dissipation as the elastic modulus mismatch was increased. Finally, an example (in the appendix) of a brass solder interface demonstrated the utility of the equations and dimensionless plots presented herein.
6 Predicting the Variation in Fatigue Crack Growth Rate

A technique to predict the variability of the Paris regime fatigue crack growth rates in ductile materials based on variation in bulk monotonic properties (yield strength, hardening modulus, and fracture toughness) is presented. The prediction, based on the plastic dissipation in the reversed plastic zone ahead of the crack tip, is carried out for Ti-6Al-4V. The empirical distributions of the bulk properties of Ti-6Al-4V are characterized and directly used in the probabilistic assessment of the fatigue crack growth rate. Since computing the plastic dissipation is a computationally intensive procedure, a previously developed sampling scheme based on confidence interval minimization was used to generate the empirical distribution of fatigue crack growth rate.

6.1 Introduction

It is well known that both the fracture toughness, $K_{Ic}$, and the fatigue crack growth rate, $da/dN$, are highly variable properties—much more so than the yield strength. As a result, probabilistic assessment (i.e., choosing the correct distribution) requires a significant amount of tests for the material properties. These tests can be expensive and time consuming. Therefore, if one series of tests can be eliminated by the ability to relate the results of the other tests, then a potential for savings exists when assessing the properties of the material. To this end, a technique is developed to predict the statistical distribution of the fatigue crack growth rate based on the results of monotonic testing parameters and the dissipated energy theory of Eq. (1.1).

A probabilistic lifing system can only be as good as the data used to describe the phenomena being modeled. There are two major aspects of structural analysis, the first is assessing the variable
loading conditions and the second is assessing the variation in material properties. Highly controlled tests in a laboratory setting are used to assess the material properties, which still show inherent variability due to variations in the micro-structure, random distributions of lattice defects, crystal sizes and shapes, grain boundary parameters, and macro defects like cracks, bubbles, and casting defects [14, 91].

In aerospace components, the material quality, (i.e. cleanliness) tends to be superior, which has the combined effect of improving the fracture toughness of a material as well as increasing its variability. This concept seems counter-intuitive and may be explained by realizing that there are fewer locations to consistently initiate a failure. In other words, poor quality materials consistently fail at defects in the micro-structure which entail a small volume fraction of the bulk material. However, if these defects are not present, then the volume of material dictating failure becomes larger which inherently increases the scatter in the fracture toughness.

Given the inherent variability in material properties, a robust estimation of these properties is required to incorporate them into a probabilistic lifing system.

These material properties can be combined with a numerical prediction of the plastic work per cycle \((dW/dN)\) to give the fatigue crack growth rate \((da/dN)\) for ductile metals [1]. This represents a departure from the conventional stochastic approach where the fatigue crack growth rate is modeled phenomenologically. If the combinations of the distributions and the numerical predictions match the distributions from fatigue crack growth testing, then the numerical prediction of the fatigue crack growth rate will show significant promise as a technique for the rapid introduction of new materials into service.

The goal of this chapter is to provide a parametric distribution for the fatigue crack growth rate \((da/dN)\) based on the distributions of three material properties: fracture toughness \((K_{IC})\), yield strength \((\sigma_y)\), and the strain hardening modulus \((E_T)\).

### 6.2 Statistical Analysis of Material Properties

The material data used was generated under the US Air Force’s High Cycle Fatigue program by the Air Force Research Laboratory’s Materials and Manufacturing Directorate [92]. The titanium specimen data had variations in heat treatment of the specimens and consisted of 68 tests. The
introduction or variation from testing is limited since a single lab generated the entire data set. Data provided by J. Tiley of the Air Force Research Laboratory [92] provided multiple test results for yield strength, ultimate strength, elongation and fracture toughness. There was sufficient data to estimate the shape of the probability function for each variable. Those distributions were then sampled to generate a series of input data for analysis of Mode I plastic work per cycle which, when combined with samples of the fracture toughness, provide a predicted distribution of the fatigue crack growth rate for specific loading. The distributions of the material properties are shown in this section.

6.2.1 Deterministic quantities

The elastic modulus (Young’s modulus) and Poisson’s ratio have small variations compared to other material properties such as the yield strength and fracture toughness. As a result, these values were fixed using typical data for Ti-6Al-4V such that

\[ E = 114.5 \text{ GPa} \]

\[ \nu = 0.32. \]

6.2.2 Stochastic Quantities

The analysis depends on first characterizing the variation of the Ti-6-4 bulk properties. The input parameters that vary are the yield strength, strain hardening modulus, and fracture toughness. Those distributions are the probabilistic input to the simulation used to predict the variation in crack growth behavior. The integration technique used to estimate the crack growth rate variation is a variance reduction sampling technique.

The general approach used to characterize the distributions is to test the empirical distribution against an assumed parametric distribution. The empirical distribution from the data set defines the initial assumed distribution. When an assumed distribution matches the data through a statistical test that distribution is used as the input to the probabilistic analysis.

Each of the tests reports a P-value that reflects the necessary type-II error to reject the assumed distribution. Type-II error means that the null hypothesis is rejected when it is actually true. A
higher P-value relates to a more exact match between the data and the assumed distribution.

6.2.2.1 Yield Strength

The yield stress is well-modeled by a normal distribution. The empirical distribution, as shown by both a histogram and an empirical cumulative distribution function (CDF), compares well to the normal distribution in Fig. 6.1. The normal distribution has the following form for its density function:

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right], \tag{6.1} \]

where \( \mu \) is the mean value and \( \sigma \) is the standard deviation. When tested with the Lilliefors test for normality implemented in Matlab, the null hypothesis that the distribution is normal cannot be rejected with a confidence of 95% (\( \alpha = 0.05 \)). The P-value for the data set is 0.09, implying if \( \alpha \) were greater than the P-value, then the null hypothesis would be rejected. The seemingly close call for the normality assumption comes from the two “weak” links around the 720 MPa region of the data plots in Fig. 6.1.
6.2.2.2 Strain Hardening Modulus

The strain hardening modulus is clearly not normally distributed as seen in Fig. 6.2. The beta distribution seemed to fit well when using $\alpha = 1$ and $\beta = 3.15$ with shifted lower limit of 620 MPa and upper limit at 4200 MPa. This corresponds to a range of hardening modulus ratios of between $0.0054 < E_t/E < 0.037$. The probability density function of the Beta distribution has the functional form:

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} (1-x)^{\beta-1} x^{\alpha-1}$$  \hspace{1cm} (6.2)

where $x \in [0, 1]$ and can be determined as $x = \frac{E_t - 620}{4200 - 620}$ MPa.

A Kolmogorov-Smirnov test cannot reject the null hypothesis when comparing the data to the Beta distribution for $\alpha$ set to 0.05. The P-value of the test is 0.63, meaning there is a significant margin before the hypothesis that the Beta distribution correctly models the data can be rejected.

6.2.2.3 Correlation

It should not be a surprise that the yield strength and hardening modulus are statistically correlated. The plastic hardening modulus was estimated by determining the slope of the line that passed through the yield point and the point of ultimate failure on a stress strain curve. Therefore, as the yield strength decreases, the hardening modulus increases, which gives a negative correlation.
Yield Stress    | Hardening Modulus | Ultimate Stress | Fracture Toughness
---|---|---|---
Yield Stress    | 1   | -0.343  | 0.624  | -0.008
Hardening Modulus| 1   | 0.439   | -0.007 |
Ultimate Stress  | 1   | 1       | -0.016 |
Fracture Toughness| 1   | 1       | 1       |

Table 6.1: Correlation coefficient results among the input variables.

larly, as the ultimate strength decreases, the hardening modulus decreases (positive correlation). The results of the correlation coefficient are summarized in Table 6.1. Notice there is little correlation of the strength and hardening with the fracture toughness.

6.2.3 Fracture Toughness

Damage tolerant designs have been tauted as a method of design that allows for lighter structures at the cost of non-destructive inspection. These damage tolerant designs use fracture toughness values as design criteria.

At issue, however, is the fact that fracture toughness values reported in texts and handbooks [93] are “typical” and a sense of the inherent variation is lost [12]. It was reported in Ref. [94], that “it is well recognized that the [fracture toughness] \( K_{IC} \) is a random variable, and should be treated as such when significant statistical data are available.” In 1993, Hoysan and Sinclair published a paper emphasizing the fact that there is significant variation in the values of fracture toughness reported in the literature [8]. For example, mill annealed Ti-6Al-4V has a range of fracture toughness from 39.2-124.0 ksi from 29 data points. This critical discussion of the fracture toughness motivates the need for a robust estimation of the distribution of fracture toughness values for use in probabilistic design assessment tools.

As manufacturing techniques improve to increase the purity of an alloy and remove tramp elements, the fracture toughness expectantly increases. However, these higher purity materials will exhibit more variation in the fracture toughness as the mechanism of fracture transitions from having definite microvoid-producing particles that affect a smaller volume in an impure material to having higher volume dictating the toughness of the material. For turbine engine applications, a high fracture toughness, thus a high purity alloy is desired. As a consequence, the possibility exists for high variations in the plane strain fracture toughness values.
6.2.3.1 Coefficient of variation

The coefficient of variation is defined as the ratio of the standard deviation to the mean. For a sample, the estimated coefficient of variation is

\[ C = \frac{s}{\bar{x}} \]  

(6.3)

where \( s \) is the sample standard deviation and \( \bar{x} \) is the sample mean. The problem with this approach is that the estimators have inherent uncertainty due to the small number of samples. Therefore, the uncertainty associated with the estimator itself must be taken into consideration. This can be done using a Student-t distribution and a \( \chi^2 \) distribution.

Previous researches have indicated a need for understanding the coefficient of variation and have even tried to quantify its maximum value. The values listed in [12] are as follows:

- 0.14 for aluminum alloys
- 0.27 for titanium alloys
- 0.22 for steels

These values were the most conservative estimates based on an empirical distribution of the coefficient of variation.

6.2.3.2 Parametric Distribution of Fracture Toughness

Researchers have investigated distributions used to model the variation of fracture toughness [14, 95]. Claims of the quality of fit of certain distributions were made without any justifying quantitative arguments. A normal distribution was suggested for fracture toughness in [12].

In this work, the distribution of the data was driven by the data available at hand. These data showed a normal distribution with a mean of 100.6 MPa\(\sqrt{m}\) and a standard deviation of 6.658 MPa\(\sqrt{m}\). Notice the 6.6% coefficient of variation is significantly lower than the COV reported in [12] for titanium alloys. The results of 500 samples from this distribution are shown by the distributions in Fig. 6.3.
The fracture toughness, like the yield stress, seems to be normally distributed. As with the other parameters, the distributions are shown in Fig. 6.3. This data is strikingly normal as is evidenced by the Lilliefors test. The null hypothesis cannot be rejected for $\alpha$ of 0.05 and the P-value is greater than 0.20. Through this testing, there is significant confidence that the fracture toughness is normally distributed.

6.3 Simulation Procedure and Results

6.3.1 Probabilistic Integration

The probabilistic integration technique is a low-discrepancy, variance reduction technique based on Latin Hypercube Sampling (LHS). The process [15] combines centroidal Voronoi tessellation with the univariate LHS constraints to quickly converge to the correct output distribution.

The analysis process is similar to other sampling methods such as Monte Carlo. To determine the distribution, a set of analysis points are evaluated by the model and the results are tabulated. The resultant distribution is the most likely estimate of the distribution of response of the system being modeled.

The system will not account for error introduced by incorrect input distributions or bias error for an inaccurate system model. Since the output distribution also only accounts for the variation from
4.4

4.6

4.8

5

5.2

5.4

5.6

5.8

6

$10^{-3}$

$\times$

Figure 6.4: Histogram of the dimensionless plastic dissipation based on the distributions of yield strength and hardening modulus.

the uncertain parameters, it will not reflect the error seen in the validation data set. The reason is that test variation is not modeled in either the input distributions or the crack growth model.

6.3.2 Plastic Dissipation Results

The results of the probabilistic integration (500 samples) is shown in the histogram of Fig. 6.4. These results are intermediate as they only include the variation associated with the yield strength and hardening modulus. The distribution is highly skewed toward higher values of the dimensionless plastic dissipation due to the nature of the beta distribution of the hardening modulus. The hardening modulus was never zero so the maximum normalized mode I plastic dissipation is lower than the elastic-perfectly plastic results from Chapter 2.

The intermediate results can be combined with the fracture toughness and normalizing parameters to get an estimated distribution of the actual fatigue crack growth, as shown in the bottom of
The experimental data was provided by Southwest Research Institute and was taken from Refs. [96, 97]. Since the details of the actual material being tested was unknown, the total data set was used in this analysis. To reduce the fatigue crack growth rate (FCGR) data for all $\Delta K$ values, the experimental $da/dN$ data were reduced to a single loading parameter of $\Delta K = 75 \text{ MPa} \sqrt{\text{m}}$ by transforming each data point according to the following rule:

$$\left. \frac{da}{dN} \right|_{\Delta K=75} = \exp \left[ 4 \ln \left( \frac{75}{\Delta K_{\text{exp}}} \right) + \ln \left( \frac{da}{dN} \right) \right]. \quad (6.4)$$

The above relation effectively collapsed all of the data to its equivalent location for $\Delta K = 75 \text{MPa} \sqrt{\text{m}}$ as shown in the top density function of Fig. 6.5. It is derived from a the power law

$$\frac{da}{dN} = C(\Delta K)^4.$$

Likewise, the estimated density function obtained from the probabilistic integration can also be expanded to account for different loading scenarios according to a similar expansion transformation as shown in Fig. 6.6. A random sample from the estimated density function was generated for each experimental data point and then transformed according to the following rule:

$$\left. \frac{da}{dN} \right|_{\text{predict}} = \exp \left[ 4 \ln \left( \frac{\Delta K_{\text{exp}}}{75 \text{ MPa} \sqrt{\text{m}}} \right) + \ln \left( \frac{da}{dN} \right) \right], \quad (6.5)$$

It should be noted that the slope of the trend in Fig. 6.6 is 4 as expected from Eq. (6.5). Also, the spurious predicted data points below the scatter band were generated from sampling the lower tail of the estimated distribution shown in the bottom part of Fig. 6.5. These points are an artifact of the uncertainty associated with the tail regions and bring no relevance to the discussion of the results.

### 6.4 Discussion and Conclusions

As seen in both Figs. 6.5 and 6.6, the mean trend of the predicted fatigue crack growth rate matches the mean trends of the actual fatigue crack growth rate data closely. This agreement gives credence to the dissipated energy theory as it demonstrated the first order effects driving the fatigue crack growth rate. Interestingly, the experimental data used for comparison was completely unrelated
Figure 6.5: Comparison of the predicted results of the fatigue crack growth rate to the collapsed experimental results for $\Delta K = 75$ MPa$\sqrt{m}$.
Figure 6.6: A plot of the experimental results of the fatigue crack growth rate compared to the results from sampling and expanding the empirical distribution of $da/dN$ shown in the bottom of Fig. 6.5.
to data used for estimating the variation associated with the input data other than the fact that the material was Ti-6Al-4V.

The experimental data shows significantly more scatter than the input. There are three main reasons for this:

1. No mean stress (R-ratio) effects were considered when reporting the actual fatigue crack growth rate data. Since the dissipated energy does not address the issue of crack closure or other mean stress effects, the effect of applied R-ratio is also not included in prediction.

2. The measurement process and experimental procedures always increase the uncertainty associated with a phenomena– fatigue crack growth rate is no exception.

3. The material scatter obtained from the input data may be less than reality. In other words, the variability of the material properties of the specimens being measured for FCGR are potentially higher than the sample provided by the Air Force Research Laboratory.

The shape of both the predicted and empirical distribution in Fig. 6.5 are similar in that they have the same skew direction. The current model does not show any deviation of the slope of the scatter band in log-log space whereas the actual data does not follow a pure power law relationship. This indicates that there are some second order effects that influence the shape of the FCGR curve. The location of the curve, however, is predicted nicely using the dissipated energy theory.
7 Conclusions and Contributions

This dissertation presented a thorough analysis of the energy dissipated due to cyclic plasticity at a crack tip oriented along a bimaterial interface. These bimaterial interfaces occur in soldering, welding, layered manufacturing, or any other system where a material is deposited upon a substrate and an energetically favorable conditions exist to grow a crack along the interface. The results of quantifying the plastic dissipation were presented in three main parts: the case of mixed mode cracking in a homogeneous material, the case of a crack along an interface with a mismatch in plastic properties, and the general case of an elastic and plastic mismatch. Furthermore, an analysis of the elastic strain energy near a crack tip was insightful in that it provided for a functional form of the dissipated energy curves and verified the normalization of the plastic dissipation.

The analysis of a homogeneous material revealed that the effect of the load changed the plastic dissipation according to a power law with the same exponent for all modes of loading. After multiplying the dissipation by a normalizing parameter it was seen that the mode of loading has the largest effect on the dimensionless plastic dissipation \( \frac{dw}{dN^*} \). It was also noted that increasing Poisson’s ratio reduced \( \frac{dw}{dN^*} \) with little effect after \( \nu > 0.3 \). Chapter 2 revealed the effect of mode mixity on the plastic dissipation.

Chapter 3 logically extended the results for the plastic dissipation to the scenario where there is a mismatch in plastic properties, both hardening modulus and yield strength, across the interface. A dimensionless measure of the yield strength mismatch was provided as \( \bar{\sigma} \) and the dimensionless measures of hardening moduli were with the ratios \( E_{t1}/E \) and \( E_{t2}/E \). Plots were presented showing the effects of varying all three of these dimensionless mismatch quantities. A mismatch in the yield strength only effects the dimensionless plastic dissipation for minor mismatches (i.e. \( \bar{\sigma} > 0.25 \)), after which there is no difference in the dimensionless plastic dissipation for an interface between some ductile material and a material that does not yield and an interface between the same ductile
material and a material that is about 5/3 as strong. The implication is if fracture toughness could be maintained, then a more fatigue resistant interface could be made by increasing the yield strength of one of the layers.

The definition of the mode for an elastically homogeneous material is well understood and easily interpreted. This definition can also be cast in terms of an energy criteria where the elastic strain energy within a contour defined by the yield strength can be determined based on the field equations of elasticity. This analysis rendered a closed form solution for the elastic strain energy and justified the normalization parameters used in Chapter 2. The functional form of the dimensionless elastic strain energy is a quadratic rational fraction. This form can be applied to the dimensionless plastic dissipation for efficient prediction of the dissipated energy in mixed mode.

A closed form solution for the bimaterial interface crack does not exist. Also, defining the mode requires the introduction of an arbitrary characteristic length which subsequently renders the definition of the mode for a bimaterial interface arbitrary. The numerical examination of the elastic strain energy near a crack tip provided insight on the nature of the elasticity problem and provided further insight into the logarithmic oscillating singularity. A new definition of the mode was proposed by choosing a characteristic length that would render the mode to be $\psi = \pm 90^\circ$ when the energy quantity is maximum and $\psi = 0^\circ$ when the energy quantity is minimum.

The determination of the plastic dissipation for a bimaterial is shown in Chapter 5. Initially, the classic definition of the mode was explained using the layer thickness as the characteristic length. However, the results were presented by using a characteristic length that depended upon the ratio of the stress intensity factors and the extrema of the plastic dissipation. Using this new energetic criteria for the characteristic length, the results for the dimensionless plastic dissipation were presented for the case of only an elastic mismatch as well as cases of both an elastic and plastic mismatches. The pictures of the plastic zones for different elastic mismatches revealed that the energetic definition of the mode rendered similar shaped plastic zones for each mode, thus reaffirming a physical basis (energy) for the characteristic length for bimaterial elasticity problems. Finally, an application of the results is presented for thermomechanical cycling of a bimaterial.

The dissipated energy theory can also be used in the context of probabilistic analysis. Determining the variability of fatigue crack growth rate (FCGR) is possible by using the dissipated energy theory and the variability associated with the monotonic properties used in determining the plastic
dissipation and the fracture toughness. The data used to assess the variability in the monotonic properties were completely independent of the data used to compare the fatigue crack growth rate. Assessing the variability employed an optimized sampling scheme since the computational cost of a full scale Monte Carlo analysis is prohibitive. Results show the mean trends of the predicted FCGR and the experimental data coincide. The experimental data showed higher amounts of variation attributed to mean stress effects and experimental noise. The probabilistic assessment revealed the need for understanding the mean stress effects while showing the validity of the dissipated energy theory for predicting mean crack growth behavior in mode I.

Future work can include more detailed probabilistic analysis using the same specimen to generate both FCGR data and monotonic property data. Also, an experimental verification of the dissipated energy criteria is needed for mixed mode loading conditions. Including second order effects such as creep, frequency, crack-closure and mean stress should be considered. Finally, as computational power increases, plastic dissipation from 3-D models should be investigated.
A Calculating the $J$-Integral

A.1 Mathematical Definition of the $J$ Integral

The $J$-integral was proposed by Rice [85] in efforts to develop a quantification of the strength of the concentrated strain fields found at the root of a notch or crack. The path-independent integral defining $J$ is written as:

$$J = \int_{\Gamma} \left( U_o dy - T \frac{\partial u}{\partial x} ds \right)$$  \hspace{1cm} (A.1)

In matrix form:

$$J = \int_{\Gamma} U_o dy - \int_{\Gamma} \begin{bmatrix} \sigma_n \\ \tau_n \end{bmatrix} \begin{bmatrix} \frac{\partial u_n}{\partial x} \\ \frac{\partial v_n}{\partial x} \end{bmatrix} ds$$  \hspace{1cm} (A.2)

where $U_o = \frac{E}{2} - \frac{E}{2}$ is the strain energy density

$I_1 = \sigma_x + \sigma_y + \sigma_z$ is the first stress invariant

$I_2 = (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) - (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)$ is the second stress invariant

$E$ is the elastic modulus

$\mu$ is the shear modulus

$\sigma_n$ is the normal component of the traction vector

$\tau_n$ is the shear component of the traction vector

$u_n$ is the normal component of the displacement in the $x$-direction

$v_n$ is the normal component of the displacement in the $y$-direction.

Though the $J$-Integral can be used in three dimensions, we will restrict our discussion to planar problems, more specifically, plane strain. The definition of plane stress says $\sigma_z = 0$ and by the definition of plane strain gives $\sigma_z = v(\sigma_x + \sigma_y)$. Also, the shear modulus is related to the elastic
modulus using Poisson’s ratio:
\[ \mu = \frac{E}{2(1 + v)} \]  

The \( J \)-integral has gained favor in the study of fracture mechanics because of its ability to apply in elastic-plastic analysis. Also, in the limiting case of Linear Elastic Fracture Mechanics (LEFM), the \( J \)-integral is equivalent to the strain energy release rate, \( G \).

### A.2 Physical Interpretation

The value of \( J \) is also considered as a general strain energy release rate because it represents the change in potential energy per change in area for nonlinear materials.

\[ J = \frac{-d\Pi}{dA} \]

where \( \Pi \) is the energy stored in the body of the cracked part less the work done by external forces.

When the material behaves in a linear fashion, the release of potential energy for a small change in crack area is defined as \( G \). Furthermore, there is a direct relationship to the mode I stress intensity factor:

\[ J = G = \frac{K_I^2}{E} \]

where \( E \) for plane stress and \( E = E/(1 - v^2) \) for plane strain. When all three modes of loading
are present in a linear elastic material under small scale yielding,

\[ J = \mathcal{J} = \frac{K_1^2}{E} + \frac{K_{II}^2}{E} + \frac{K_{III}^2}{2\mu} \]

The nonlinear release rate is interpreted as the area between the load-displacement curves when the first curve is for a crack length \( a \) and the second curve is a crack length \( a + da \). This physical basis for the \( J \)-integral lends itself to laboratory testing and many elastic-plastic materials have \( J_{IC} \) values reported. A more detailed discussion of experimental determinations of \( J \) can be found in many textbooks.

Let us examine an example that is tractable for hand calculation. Consider Figure A.2 which shows a rectangular sandwich beam growing along its interface. The arbitrary path for this example was chosen for its mathematical simplicity. The beam is loaded by moments \( M_1 \) and \( M_2 \) which are equilibrated by \( M_3 \). Since path integration is a linear operator, the \( J \)-integral can be broken into six sections:

\[ J = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \]

corresponding to the six different sections of the path. The \( J \)-integral vanishes for Sections 2 and 5 because \( dy \) and \( T \) are both zero along the outer surface.

**A.3 Homogeneous Beam Analysis**

Let us define our coordinate system such that positive \( x \) is in the direction of crack propagation along the interface. The \( y \)-axis intersects the \( x \)-axis at the root of the crack. If we apply pure
moments to the left end of a composite beam as shown in Fig. A.2 and assume that plane section remain plane during deformation (according the Euler-Bernoulli beam theory), then the deflection of a plane section follows that of a straight line with no deflection at the neutral axis.

Consider analyzing a section cut from the beam from Fig. A.2 along the first contour section, \( \Gamma_1 \). This section is sufficiently far away from both the applied load and the crack to be in a state of pure bending (Saint Venant’s principle). Since the material in this section is homogeneous and the cross section is rectangular, the neutral axis is located at the mid-plane of this section. Therefore, the equation for differential deflection over a differential length of beam is

\[
\frac{du_1(y)}{dx} = e_1(y) = -k \left( y + \frac{h_2}{2} \right)
\]

which gives the kinematic definition of strain as a function of \( y \) along the first section of the path \( \Gamma_1 \). This relationship is shown pictorially in Fig. A.3. According to Hooke’s law,

\[
\sigma_1(y) = E_2 e_1(y)
\]
For the rest of this analysis functional dependence on $y$ will be assumed so for brevity, $\sigma = \sigma(y)$.

From equilibrium arguments, the sum of the forces must be zero:

$$0 = \int_{-h_2}^{0} \sigma \, dy = - \int_{-h_2}^{0} E_2 k \left( y + \frac{h_2}{2} \right) \, dy$$

$$0 = -E_2 k \left[ \frac{y^2}{2} + \frac{h_2 y}{2} \right]_{-h_2}^{0} = 0$$

The equation from the equilibrium of forces is somewhat uninformative; however, the sum of moments lead to an expression for $k$:

$$M_2 = \int_{-h_2}^{0} y \sigma \, dy = -E_2 k \int_{-h_2}^{0} y \left( y + \frac{h_2}{2} \right) \, dy$$

$$M_2 = -E_2 k \left[ \frac{y^3}{3} + \frac{h_2 y^2}{4} \right]_{-h_2}^{0} = 0$$

$$M_2 = -E_2 k \left( \frac{-h_3^3}{3} + \frac{h_3^3}{4} \right) = E_2 k \left( \frac{h_3^3}{12} \right)$$

$$k = \frac{12M_2}{E_2 h_3^3} \quad (A.5)$$

Substituting the expression for $k$ into Eq. (A.4) gives:

$$\sigma_1 = E_2 \left[ - \left( \frac{12M_2}{E_2 h_3^3} \right) \left( y + \frac{h_2}{2} \right) \right]$$

$$\sigma_1 = -\frac{M_2}{l_2} \left( y + \frac{h_2}{2} \right) \quad (A.6)$$

where $l_2 = \frac{1}{12} h_3^3$ is the area moment of inertia for a rectangular cross section of unit width. Computing the strain gives:

$$\frac{du}{dx} = \varepsilon_1 = - \left( \frac{12M_2}{E_2 h_3^3} \right) \left( y + \frac{h_2}{2} \right) \quad (A.7)$$

A similar analysis of section 6 of the path starts with the observation that the kinematic expression for strain is

$$\varepsilon_6 = k \left( y - \frac{h_1}{2} \right) \quad (A.8)$$
which results in the stress as

\[ \sigma_6 = \frac{M_1}{I_1} \left( y - \frac{h_1}{2} \right) \]  

(A.9)

and

\[ \frac{du}{dx} = \varepsilon_6 = \left( \frac{12M_1}{E_1h_1^3} \right) \left( y - \frac{h_1}{2} \right) \]  

(A.10)

### A.4 Composite Beam Analysis

The section of the path defined by \( \Gamma_3 \) and \( \Gamma_4 \) make up a composite beam where the elastic modulus and Poisson’s ratio of one section may be different than the other. Even though the moduli may be different, the kinematic and compatibility requirement of a continuum must still hold. In light of Fig. A.3 we can see that the strain follows a straight line but the location of the neutral axis may no longer be in the middle of the section. Thus,

\[ \varepsilon = k(y - \bar{y}) \]  

(A.11)

From Hooke’s law we can write down the expressions for stress in the two different materials:

\[ \sigma_3(y) = E_2 \varepsilon(y) \quad -h_2 < y < 0 \]  

(A.12)

\[ \sigma_4(y) = E_1 \varepsilon(y) \quad 0 < y < h_1 \]  

(A.13)

A graphical representation of the stress is shown in Fig. A.4.

Some texts [98] use the ratio of the elastic moduli as a parameter \( n = \frac{E_2}{E_1} \) in an area moment analysis. However, we will explicitly use \( E_1 \) and \( E_2 \).

Equation (A.11) has two unknowns that can be solved using the equilibrium of both forces and moments at the face under consideration. The equilibrium of forces gives the following relationship:

\[ 0 = \int_{-h_2}^{0} \sigma_3 \, dy + \int_{0}^{h_1} \sigma_4 \, dy \]

\[ 0 = E_2 k \int_{-h_2}^{0} (y - \bar{y}) \, dy + E_1 k \int_{0}^{h_1} (y - \bar{y}) \, dy \]

\[ * \]  

*Throughout this discussion the thickness of each member is considered unity.*
Evaluating and simplifying gives:

\[ E_1 k \int_0^{h_1} (y - \bar{y}) \, dy = -E_2 k \int_{-h_2}^0 (y - \bar{y}) \, dy \]

\[ E_1 k \left[ \frac{y^2}{2} - \bar{y}y \right]_{0}^{h_1} = E_2 k \left[ \frac{y^2}{2} - \bar{y}y \right]_{-h_2}^{0} \]

\[ \frac{1}{2} E_1 k (h_1^2 - 2h_1 \bar{y}) = \frac{1}{2} E_2 k (h_2^2 + 2h_2 \bar{y}) \]

Dividing through by \( \frac{1}{2} E_1 k \) and solving for \( \bar{y} \) gives:

\[ \bar{y} = \frac{h_1^2 - nh_2^2}{2(h_1 + h_2)} \]

where \( n = E_2/E_1 \). The simple case of \( h_1 = h_2 \) and \( n = 1 \) puts the neutral axis at zero as expected. This is also a special case of the formula for the neutral axis location presented in [98].

Once the location of the neutral axis is determined, we can find the slope, \( k \), based on the equi-
librium of moments:

\[ M_3 = - \int_{-h_2}^0 y\sigma_3 \, dy - \int_0^{h_1} y\sigma_4 \, dy \]

\[ M_3 = - \frac{1}{6} E_2 k(2h_2^3 + 3h_2^2\bar{y}) - \frac{1}{6} E_1 k(2h_1^3 - 3h_1^2\bar{y}) \]

Solving for \( k \):

\[ k = \frac{-6M_3}{E_2(2h_2^3 + 3h_2^2\bar{y}) + E_1(2h_1^3 - 3h_1^2\bar{y})} \]

Now our stress equations, Eqs. (A.12)-(A.13), are functions of \( y \), \( M \), geometry and moduli; all of which are known.

### A.5 Strain Energy Density

The strain energy density is given in [99] as:

\[ U_o = \frac{I_1^2}{2E} - \frac{I_2}{2\mu} \]

where \( I_1 \) and \( I_2 \) are the stress invariants given on page 140.

#### A.5.1 Homogeneous Beam Section

The strain energy density for the case of pure bending in plane strain of a homogeneous section:

\[ U_o = \frac{(\sigma_x + v\sigma_z)^2}{2E} - \frac{2(1 + v)[\sigma_x\sigma_z]}{2E} \]

\[ = \frac{\sigma_x^2(1 + v)^2 - 2(1 + v)[v\sigma_x^2]}{2E} \]

\[ = \frac{\sigma_x^2[(1 + v^2) - (2v + 2v^2)]}{2E} \]

\[ = \frac{\sigma_x^2[1 + 2v + v^2 - 2v - 2v^2]}{2E} \]

\[ = \frac{\sigma_x^2[1 - v^2]}{2E} \]
If plane stress conditions exist for a beam in bending:

\[ U_o = \frac{\sigma^2}{2E} \]

which leads us to use the notation of placing a bar over the elastic modulus, that is \( \bar{E} = E \) for plane stress and \( \bar{E} = E/(1 - v^2) \) for plane stress. If we use the results of stress, we can write expressions for the strain energy density of different paths in terms of the load and geometry:

\[ U_{o1} = \frac{\left( \frac{M_2}{I_2} \left( y + h_2 \right) \right)^2}{2E_2} \]

\[ U_{o6} = \frac{72M_2^2(y + h_2)^2}{E_2h_2^6} \]

and for section 6:

\[ U_{o6} = \frac{\left( \frac{M_1}{I_1} \left( y - h_1 \right) \right)^2}{2E_1} \]

\[ U_{o6} = \frac{72M_1^2(y - h_1)^2}{E_1h_1^6} \]

### A.5.2 Composite Beam Section

The direct formula for the strain energy density in terms of the loads, geometry, and elastic properties is too large to write. However, we can take the results of the previous section and substitute them into the strain energy density equation. This will need to be done twice, once for path 3 and the other for path 4. The algebra is tedious to write long form, but can easily be implemented in a computer.

### A.6 J-Integral evaluations

The final term needing determination from Eq. A.2) is the rate of change of the normal displacements with respect to \( x \). For vertical paths, the normal vector is horizontal and the derivative is simply the strain, i.e. \( \frac{du}{dx} = \varepsilon_x \). Therefore, the piece wise contributions to the J-integral are as
follows:

\[ J_1 = \int_0^{-h_2} U_{o1} - \sigma_1 \frac{du_1}{dx} \, dy \]

\[ J_1 = \frac{6M_2^2}{E_2h_2^3} \]

and

\[ J_6 = \int_{h_1}^0 U_{o6} - \sigma_6 \frac{du_6}{dx} \, dy \]

\[ J_6 = \frac{6M_1^2}{E_1h_1^3} \]

The integration of paths 3 and 4 are a bit tedious and messy. As such, a computer algebra system was used (Mathematica 5.0) to determine the following values:

\[ J_3 = \frac{-3E_2(M_1 - M_2)^2 \left( [E_2h_2^2 - E_1h_1^2]^3 + [E_2h_2^2 + E_1h_1(h_1 + 2h_2)]^3 \right)}{(E_2h_2 + E_1h_1) \left( E_2^2h_2^4 + 2E_1E_2h_1h_2[2h_1^2 + 3h_1h_2 + 2h_2^2] + E_1^2h_1^4 \right)^2} \]

\[ J_4 = \frac{-3E_1(M_1 - M_2)^2 \left( [E_2h_2(2h_1 + h_2) + E_1h_1^2]^3 - [E_2h_2^2 - E_1h_1^2]^3 \right)}{(E_2h_2 + E_1h_1) \left( E_2^2h_2^4 + 2E_1E_2h_1h_2[2h_1^2 + 3h_1h_2 + 2h_2^2] + E_1^2h_1^4 \right)^2} \]

Finally, the contributions of the horizontal paths to the J-Integral is zero \((J_2 = J_5 = 0)\). Once all the individual contributions to the J-Integral from the sections of the path are determined, the overall value of \(J\) can be found:

\[ J = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \quad (A.14) \]

**A.6.1 Special Cases**

There are two special cases of this evaluation of \(J\). the first is from a case reported in [2] where \(h_1 = h_2 = h\) and \(E_1 = E_2 = E\). Making these simplification of a homogeneous material with equal thicknesses and a steady state crack give the formula:

\[ J = \frac{3(7M_1^2 + 2M_1M_2 + 7M_2^2)}{4Eh^3} \quad (A.15) \]

which, in the case of linear elastic material response, is the same as the strain energy release rate.
If the thickness of each layer are equal, then the formula for $J$ is

$$J = \frac{6E_2^2(13E_1 + E_2)M_1^2 + 2E_1E_2(E_1 + E_2)M_1M_2 + E_1^2(E_1 + 13E_2)M_2^2}{E_1E_2(E_1 + 14E_1E_2 + E_2^2)h^3}$$  \hspace{1cm} (A.16)

It should be noted that the relative thickness between the layers changes the ratio of shear to normal tractions at the crack tip. This can also be achieved by changing the applied moments. This motivates the use of Eq. (A.16) because a crack tip can be loaded across the full range of mode (I/II) mix ratios by solely changing the ratio of the applied loads (moments).

### A.6.2 A Numerical Example

Given the geometry shown in Figure A.2 and the values listed in Table A.1, a numerical value of $J$ in plane strain can be determined.

- $h_1 = 0.5 \text{ mm}$  \hspace{0.5cm} $M_1 = 100 \text{ N-m}$  \hspace{0.5cm} $E_1 = 10,000 \text{ Pa}$  \hspace{0.5cm} $v_1 = 0.3$
- $h_2 = 1 \text{ mm}$  \hspace{0.5cm} $M_2 = 800 \text{ N-m}$  \hspace{0.5cm} $E_2 = 50,000 \text{ Pa}$  \hspace{0.5cm} $v_2 = 0.35$

Table A.1: Exemplar values for the computation of the $J$-integral.

The first step in determining the contributions to $J$ is to find the values of $E_1$ and $E_2$ in plane strain:

$$E_1 = \frac{E_1}{1 - v_1^2} = \frac{10,000}{1 - 0.3^2} \approx 10989$$

$$E_2 = \frac{E_2}{1 - v_2^2} = \frac{50,000}{1 - 0.35^2} \approx 56980$$

Simply substituting these values and the values in Table A.1 into the components of $J$ give the following results:

$$J = J_1 + J_2 + J_3 + J_4 + J_5 + J_6$$

$$= 67.392 + 0 + (-20.744) + (-11.150) + 0 + 43.68$$

$$= 79.178$$

The isoparametric quadratic plane strain element with reduced integration was used to determine the response of the example problem. The analysis, performed in ABAQUS v6.4, gave the stress
Figure A.5: A contour plot of the bending stress ($\sigma_x$) from the finite element analysis. The values used in this plane strain analysis are given in Table A.1.

ABAQUS has an implementation of the $J$-integral calculation that is invoked using the keyword: *CONTOUR INTEGRAL, CONTOURS=n, Type=J. The implementation is based on the divergence theorem and calculates the area integral defined from the path. This has been shown to be accurate, even for relatively coarse meshes [76]. For this example, 20 contours were calculated and the results from the data file are shown below.

The value of $J$ is estimated for different contours, each on slightly larger than the previous. This is done by evaluating successive rings of elements. This means the first contour is a ring of elements in the immediate vicinity of the crack tip and successive contours encompass more area. As a result, multiple values of $J$ are estimated in the data file and the higher contours contain more area. Since the numerical solution of the stresses in the elements next to the crack tip are not stable, the estimated value of the $J$ integral should be viewed with caution. However, the estimation of $J$ becomes more accurate as the number of contours increases until there are no more rings available. In the presence of elastic-plastic material response, the $J$ integral has spatial dependence. Once the contour is outside the plastic zone, however, the value of $J$ converges to the strain energy release.
Table A.2: ABAQUS output of the results of the contour integral.

<table>
<thead>
<tr>
<th>CRACK NUMBER</th>
<th>CRACKFRONT NODE SET</th>
<th>J - INTEGRAL ESTIMATES CONTOURS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 3 4 5</td>
<td>78.97 79.07 79.18 79.19 79.19</td>
</tr>
<tr>
<td>6</td>
<td>7 8 9 10</td>
<td>79.19 79.19 79.18 79.18 79.18</td>
</tr>
<tr>
<td>11</td>
<td>12 13 14 15</td>
<td>79.18 79.18 79.18 79.17 79.17</td>
</tr>
<tr>
<td>16</td>
<td>17 18 19 20</td>
<td>79.17 79.17 79.13 79.16 79.16</td>
</tr>
</tbody>
</table>

LABELS REFERENCED IN THE ABOVE TABLE
-6- ASSEMBLY_TIP

Table A.2: ABAQUS output of the results of the contour integral.

<table>
<thead>
<tr>
<th>CRACK NUMBER</th>
<th>CRACKFRONT NODE SET</th>
<th>K FACTOR ESTIMATES CONTOURS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 3 4 5</td>
<td>K1: 1233. 1231. 1232. 1232. 1232.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>K2: 63.58 59.33 59.63 59.64 59.64</td>
</tr>
<tr>
<td></td>
<td></td>
<td>J from Ks:79.40 79.01 79.17 79.16 79.16</td>
</tr>
<tr>
<td>6</td>
<td>7 8 9 10</td>
<td>K1: 1232. 1232. 1232. 1232. 1232.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>K2: 59.64 59.64 59.64 59.63 59.63</td>
</tr>
<tr>
<td></td>
<td></td>
<td>J from Ks:79.16 79.16 79.16 79.16 79.16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>K2: 59.62 59.61 59.60 59.59 59.58</td>
</tr>
<tr>
<td></td>
<td></td>
<td>J from Ks:79.15 79.15 79.15 79.15 79.15</td>
</tr>
<tr>
<td>16</td>
<td>17 18 19 20</td>
<td>K1: 1232. 1232. 1232. 1232. 1232.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>K2: 59.57 59.55 59.53 59.51 59.48</td>
</tr>
<tr>
<td></td>
<td></td>
<td>J from Ks:79.14 79.14 79.14 79.13 79.13</td>
</tr>
</tbody>
</table>

***NOTE: THIS IS A CRACK LYING ON THE INTERFACE BETWEEN TWO DIFFERENT MATERIALS

Table A.3: ABAQUS output of the results of the interaction integral that is used to determine the stress intensity factors.
rate, \( \mathcal{J} \), provided the yielding at the crack tip is governed by the far field elastic fields (small scale yielding).

**A.6.3 Relating the Stress Intensity Factors to the \( J \)-Integral**

Computing the second Dundurs parameter, \( \beta \), requires the shear moduli,

\[
\mu = \frac{E}{2(1 + \nu)}
\]

\[
\mu_1 = \frac{E_1}{2(1 + \nu_1)} = \frac{10000}{2(1 + 0.3)} \approx 3846.15
\]

\[
\mu_2 = \frac{E_2}{2(1 + \nu_2)} = \frac{50000}{2(1 + 0.35)} \approx 18518.51
\]

and the values for \( \kappa \) in plane strain:

\[
\kappa = 3 - 4\nu
\]

\[
\kappa_1 = 3 - 4(0.3) = 1.8
\]

\[
\kappa_2 = 3 - 4(0.35) = 1.6
\]

The second Dundurs parameter is

\[
\beta = \frac{\mu_1(\kappa_2 - 1) - \mu_2(\kappa_1 - 1)}{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)}
\]

\[
= \frac{3846.15(1.6 - 1) - 18518.51(1.8 - 1)}{3846.15(1.6 + 1) + 18518.51(1.8 + 1)}
\]

\[
\approx -0.2022
\]

From the second Dundurs parameter, we can determine the oscillation index:

\[
\varepsilon = \frac{1 - \beta}{2\pi \ln 1 + \beta}
\]

\[
= \frac{1 - (-0.2022)}{2\pi \ln 1 + (-0.2022)}
\]

\[
\approx 0.06527
\]
Figure A.6: Crack opening displacements. The distance from the crack tip of the original, undeformed model is taken as \( r \).

From the FEA results in Table A.3, \( K_1 = 1232 \) and \( K_2 = 59.6 \) \((K_{II} = 0)\). The value of \( E^* = \left[ \frac{1}{2} \left( \frac{1}{10989} + \frac{1}{56980.1} \right) \right]^{-1} = 18424.67 \). Combining these values according to Eq. (4.11) gives:

\[
J = \mathcal{G} = \frac{1 - B^2}{E^*} (K_1^2 + K_2^2) \\
\approx \frac{1 - (-0.2022)^2}{18424.67} (1232^2 + 59.6^2) \\
= 79.196
\]

This value of \( J \) is close to the analytical value of \( J \) computed from the beam theory. The previous computation shows how the number reported as \( J \) from \( K \) in Table A.3 is determined.

**A.6.4 Crack Tip Opening Displacements**

The crack tip opening displacements a distance \( r \) behind the crack are computed as:

\[
\delta_x + i\delta_y = \frac{(K_1 + iK_2)\sqrt{ri}e^{i\pi}}{2\sqrt{2\pi(1+2i\varepsilon)}\cosh(\pi\varepsilon)} \left( \frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2} \right) \quad (A.17)
\]

Figure A.6 gives a schematic defining the crack opening displacements. This equation is only valid in the \( K \) dominated region. In other words, as the radius becomes larger, the influence of the stress intensity factor becomes less. A technique of extracting stress intensity factors from crack tip opening displacements is outlined in [84]; however, the energy domain techniques implemented in
ABAQUS through the interaction integral are used for numerically determining the stress intensity factors.

Equation (A.17) was analyzed for this example and plotted in Fig. A.7.

Figure A.7: A plot showing two solutions to the example problem. The numerical solution uses the finite element analysis to determine the nodal displacements. The analytical solution shown in Eq. (A.17) is plotted as lines on top of the nodal solutions from the FEA. The presence of an \( x \) component indicates mode II loading (shear along the interface).
B Stress and Displacement Field Equations
Near a Crack Tip

The field equations for stress and displacements near a crack tip along a bimaterial interface were published by Sun and Jih in [83], and again by Nishioka, Syano, and Fujimoto in [84]. The following equations are for in-plane stresses and displacements. The superscript \( (m) \) refers to the material (layer) number.

\[
\sigma_{xx}^{(m)} = \frac{K_1}{2\sqrt{2\pi r} \cosh(\pi \epsilon)} \left[ e^{(\epsilon \theta - (3-2m)\epsilon \pi)} \left\{ 3 \cos \left( \frac{\theta}{2} + \epsilon \ln \frac{r}{l} \right) + 2\epsilon \sin \theta \cos \left( \frac{3\theta}{2} + \epsilon \ln \frac{r}{l} \right) - \sin \theta \sin \left( \frac{3\theta}{2} + \epsilon \ln \frac{r}{l} \right) \right\} \right. \\
\left. - e^{(2\epsilon \theta + (3-2m)\epsilon \pi)} \cos \left( \frac{\theta}{2} - \epsilon \ln \frac{r}{l} \right) \right]
\]

\[
\sigma_{xx}^{(m)} = \frac{K_2}{2\sqrt{2\pi r} \cosh(\pi \epsilon)} \left[ e^{(\epsilon \theta - (3-2m)\epsilon \pi)} \left\{ 3 \sin \left( \frac{\theta}{2} + \epsilon \ln \frac{r}{l} \right) + 2\epsilon \sin \theta \sin \left( \frac{3\theta}{2} + \epsilon \ln \frac{r}{l} \right) + \sin \theta \cos \left( \frac{3\theta}{2} + \epsilon \ln \frac{r}{l} \right) \right\} \\
+ e^{(2\epsilon \theta + (3-2m)\epsilon \pi)} \sin \left( \frac{\theta}{2} - \epsilon \ln \frac{r}{l} \right) \right]
\]  
(B.1)

156
\[ \sigma_{xy}^{(m)} = \frac{K_1}{2\sqrt{2\pi r} \cosh(\pi \varepsilon)} \left[ e^{i\varepsilon \theta} - e^{-i\varepsilon \theta} \right] \left \{ 3 \cos \left( \frac{i \theta}{2} + \varepsilon \ln \frac{r}{r_0} \right) - 2\varepsilon \sin \cos \left( \frac{3\theta}{2} + \varepsilon \ln \frac{r}{r_0} \right) \right \} \]

\[ - \frac{K_2}{2\sqrt{2\pi r} \cosh(\pi \varepsilon)} \left[ e^{i\varepsilon \theta} - e^{-i\varepsilon \theta} \right] \left \{ \sin \left( \frac{i \theta}{2} + \varepsilon \ln \frac{r}{r_0} \right) - 2\varepsilon \sin \left( \frac{3\theta}{2} + \varepsilon \ln \frac{r}{r_0} \right) \right \} \]

The displacements in the \( xy \) plane are given as:

\[ u_x^{(m)} = \frac{K_1 \sqrt{2\pi r}}{4\pi \mu^{(m)} \cosh(\pi \varepsilon)} \left[ e^{i\varepsilon \theta} - e^{-i\varepsilon \theta} \right] \left \{ 1 + 4\varepsilon^2 \right \} \left \{ 3 \cos \left( \frac{i \theta}{2} - \varepsilon \ln \frac{r}{r_0} \right) - 2\varepsilon \sin \cos \left( \frac{3\theta}{2} - \varepsilon \ln \frac{r}{r_0} \right) \right \} \]

\[ - e^{i\varepsilon \theta} \left \{ 1 + 4\varepsilon^2 \right \} \left \{ \sin \left( \frac{i \theta}{2} - \varepsilon \ln \frac{r}{r_0} \right) + 2\varepsilon \cos \left( \frac{3\theta}{2} - \varepsilon \ln \frac{r}{r_0} \right) \right \} \]

\[ - e^{-i\varepsilon \theta} \left \{ 1 + 4\varepsilon^2 \right \} \left \{ \sin \left( \frac{i \theta}{2} + \varepsilon \ln \frac{r}{r_0} \right) + 2\varepsilon \cos \left( \frac{3\theta}{2} + \varepsilon \ln \frac{r}{r_0} \right) \right \} \]

\[ + \frac{K_2 \sqrt{2\pi r}}{4\pi \mu^{(m)} \cosh(\pi \varepsilon)} \left[ e^{i\varepsilon \theta} - e^{-i\varepsilon \theta} \right] \left \{ 1 + 4\varepsilon^2 \right \} \left \{ \sin \left( \frac{i \theta}{2} - \varepsilon \ln \frac{r}{r_0} \right) + 2\varepsilon \cos \left( \frac{3\theta}{2} - \varepsilon \ln \frac{r}{r_0} \right) \right \} \]

\[ - e^{i\varepsilon \theta} \left \{ 1 + 4\varepsilon^2 \right \} \left \{ \sin \left( \frac{i \theta}{2} + \varepsilon \ln \frac{r}{r_0} \right) + 2\varepsilon \cos \left( \frac{3\theta}{2} + \varepsilon \ln \frac{r}{r_0} \right) \right \} \]

\[ - e^{-i\varepsilon \theta} \left \{ 1 + 4\varepsilon^2 \right \} \left \{ \sin \left( \frac{i \theta}{2} + \varepsilon \ln \frac{r}{r_0} \right) + 2\varepsilon \cos \left( \frac{3\theta}{2} + \varepsilon \ln \frac{r}{r_0} \right) \right \} \]
\[ u_r^{(m)} = \frac{K_1 \sqrt{2\pi r}}{4\pi \mu^{(m)} \cosh(\pi \varepsilon)} \]

\[ \begin{aligned}
&\left[ e^{(\varepsilon \theta - (3-2m)\varepsilon \pi)} \frac{k^{(m)}}{1 + 4\varepsilon^2} \left\{ \sin \left( \frac{\theta}{2} - \varepsilon \ln \frac{r}{l} \right) - 2\varepsilon \cos \left( \frac{\theta}{2} - \varepsilon \ln \frac{r}{l} \right) \right\} \\
&\quad - e^{(-\varepsilon \theta + (3-2m)\varepsilon \pi)} \frac{1}{1 + 4\varepsilon^2} \left\{ - \sin \left( \frac{\theta}{2} + \varepsilon \ln \frac{r}{l} \right) + 2\varepsilon \cos \left( \frac{\theta}{2} + \varepsilon \ln \frac{r}{l} \right) \right\} \\
&\quad - e^{(+\varepsilon \theta - (3-2m)\varepsilon \pi)} \sin \theta \cos \left( \frac{\theta}{2} + \varepsilon \ln \frac{r}{l} \right) \right] \\
&\quad + \frac{K_2 \sqrt{2\pi r}}{4\pi \mu^{(m)} \cosh(\pi \varepsilon)} \left[ e^{(\varepsilon \theta - (3-2m)\varepsilon \pi)} \frac{k^{(m)}}{1 + 4\varepsilon^2} \left\{ \cos \left( \frac{\theta}{2} - \varepsilon \ln \frac{r}{l} \right) - 2\varepsilon \sin \left( \frac{\theta}{2} - \varepsilon \ln \frac{r}{l} \right) \right\} \\
&\quad + e^{(-\varepsilon \theta + (3-2m)\varepsilon \pi)} \frac{1}{1 + 4\varepsilon^2} \left\{ \cos \left( \frac{\theta}{2} + \varepsilon \ln \frac{r}{l} \right) + 2\varepsilon \sin \left( \frac{\theta}{2} + \varepsilon \ln \frac{r}{l} \right) \right\} \\
&\quad + e^{(+\varepsilon \theta - (3-2m)\varepsilon \pi)} \sin \theta \sin \left( \frac{\theta}{2} + \varepsilon \ln \frac{r}{l} \right) \right] \quad (B.5)
\end{aligned} \]

where \( m = 1, 2 \) which refers to the material number,

\( K_1 \) and \( K_2 \) are the stress intensity factors

\( r \) is the radius from the crack tip,

\( l \) is the characteristic length that normalizes the singular oscillation,

\( \theta \) is the angle measured from the interface ahead of the crack,

\( k = 3 - 4\nu \) for plane strain or \( k = \frac{3-\nu}{1+\nu} \) for plane stress,

\( \mu \) is the shear modulus,

\( \nu \) is Poisson’s ratio, and

\( \varepsilon \) is the oscillation index.
C Mode Searching Algorithm

The secant method used to search for the pure mode I and pure mode II loading factors is outlined below:

for mode = I, II

• guess initial \( f \), set \( err = 100 \), set \( tol = 0.001 \)
• Set \( M_1 = f \), \( M_2 = 1 \)
• perturb \( f \) to generate \( f' \)
• Set \( M'_1 = f' \), and \( M'_2 = 1 \)
• Use FEA with \( M_1 \) and \( M_2 \) to solve for \( K_1 \) and \( K_2 \)
• Use FEA with \( M'_1 \) and \( M'_2 \) to solve for \( K'_1 \) and \( K'_2 \)

while \( err > tol \)

if mode = I,

\[ \text{Set } f^{(new)} = f' - K_2 f' - \frac{f}{K'_2 - K_2} \]

if mode = II,

\[ \text{Set } f^{(new)} = f' - K_2 f' - \frac{f}{K'_2 - K_2} \]

- Set \( f = f' \), \( M_1 = M'_1 \), \( M_2 = M'_2 \)
- Set \( M'_1 = f^{(new)} \) and \( M'_2 = 1 \)
– Use FEA with $M'_1$ and $M'_2$ to solve for $K'_1$ and $K'_2$

if mode = I,

– Set $err = K'_2$

if mode = II,

– Set $err = K'_1$

return $f^{(new)}$
D Elastic Strain Energy Matlab Code Listings

Listing D.1: examplebetazero.m

```matlab
close all
clear all
clc
format short
E=[101e9 30e9]
v=[.35 .324]
yield=[70E6 70E6]
N=200;
l=1;

mu=E./(2*(1+v))
kappa=3-4*v
Ebar=E./(1-v.^2) %plane strain

% Adjust v1 and v2 to ensure beta =0
beta=0
tol=1e-8;
diff=1;
while abs(diff) > tol
    alpha=(Ebar(1)-Ebar(2))/(Ebar(1)+Ebar(2));
    v1 =v(1);
    v2= (-v1 - 2*alpha + 3*v1*alpha + 4*beta - 4*v1*beta) / ...
        (-1 - 3*alpha + 4*v1*alpha + 4*beta - 4*v1*beta);
    v=[v1 v2];
    mu=E./(2*(1+v));
kappa=3-4*v;
Ebar=E./(1-v.^2) ; %plane strain
diff=(Ebar(1)-Ebar(2))/(Ebar(1)+Ebar(2)) - alpha;
end

E =
v
alpha=(Ebar(1)-Ebar(2))/(Ebar(1)+Ebar(2))

beta=(mu(1)*(kappa(2)-1) - mu(2)*(kappa(1)-1)) / ...
     (mu(1)*(kappa(2)+1) + mu(2)*(kappa(1)+1))

epsilon=(1/2*pi)*log((1-beta)/(1+beta))

Estar =1/(0.5*(1/Ebar(1)+1/Ebar(2))) ;
titlestr=['Load= '];
F=figure
G=logspace(-3,3,51);
```

161
for j=1:length(G)
    Constant=G*Estar/(1-beta^2);
    K1= sqrt(Constant);
    K2=0;
    U(j)=StrainEnergy(yield,G(j),K2,E,v,1);
end

for j=1:length(G)
    Constant=G*Estar/(1-beta^2);
    K1= sqrt(Constant);
    K2=0;
    theta=linspace(0,2*pi,N);
    r=yieldRadius(theta,yield,G(j),K2,E,v,1);
    l=find(sin(theta)<0);
    xbot=[ r(l).*cos(theta(l)) ];
    ybot=[ r(l).*sin(theta(l)) ];
    l=find(sin(theta)>=0);
    xtop=r(l).*cos(theta(l));
    ytop=r(l).*sin(theta(l));
    subplot(2,1,1)
    plot(xbot,ybot,'r',xtop,ytop)
    axis equal
    if rem(j,5)==0
        titlestr=[titlestr ' | '];
    end
    title(titlestr);
end

subplot(2,1,2)
semilogx(G,U./G.^2,G(j),U(j)/G(j)^2,'o')
axis([10e-4 10e2 507.1 507.3])
xlabel('G')
ylabel('U/G^2')

M(j) = getframe(gcf);
saveas(F,[ 'betazero_ ' num2str(j) '.fig'])
end

%movie(M)
movie2avi(M, 'betazero.avi', 'Compression', 'cinepak')

Listing D.2: strainenergy.m
Listing D.3: yieldradius.m

```matlab
function r = yieldRadius (theta, yield, G, K2, E, v, l)

for i = 1:length (theta)
    if sin (theta(i)) < 0
        m = 2;
    else
        m = 1;
    end

    r(i) = fzero (@(x) (yield(m) - misesStress (x, theta(i), G, K2, E, v, 1)), [1e-99 1e19]);
end
```

Listing D.4: globalstrainenergydensity.m

```matlab
function Uo = GlobalStrainEnergyDensity (r, theta, K2, G, v, E, l)

% Determine the elastic strain energy density
% r and theta denote the location of the point under consideration
% K2 is the second stress intensity factor
% G is the strain energy release rate
% v is a 2 element list describing Poisson's Ratio
% E is a 2 element list of Young's moduli
% l is the characteristic length

% Calculate the shear modulus:
mu = E ./ (2 * (1 + v));

% determine if the point under consideration is above or below the interface
if sin (theta) < 0
    m = 2;
else
    m = 1;
end

% Call the function that calculates the stress and displacement fields
[s11, s22, s12, u1, u2] = newfields (r, theta, G, K2, E, v, l);

% Compute the strain energy density in plane strain.
Uo = (1/(2*E(m))) * (s11^2 + s22^2) ...
    - (v(m)/E(m)) * (s11*s22) ...
    + 1/(2*mu(m)) * (s12^2);
```
Listing D.5: Elasticity field equations

```matlab
function [s11, s22, s12, u1, u2] = newfields(r, theta, G, K2, E, v, l)
%condition r and theta
x = r * cos(theta);
y = r * sin(theta);
r = sqrt(x^2 + y^2);
theta = atan2(y, x);
mu = E / (2 * (1 + v));
kappa = 3 - 4 * v;
Ebar = E / (1 - v^2); %plane strain
alpha = (Ebar(1) - Ebar(2)) / (Ebar(1) + Ebar(2));
beta = (mu(1) * (kappa(2) - 1) - mu(2) * (kappa(1) - 1)) / ...
      (mu(1) * (kappa(2) + 1) + mu(2) * (kappa(1) + 1));
epsilon = (1 / 2 * pi) * log((1 - beta) / (1 + beta));
Estar = 1 / (0.5 * (1 / Ebar(1) + 1 / Ebar(2)));
C2 = G * Estar / (1 - beta^2);
K1 = sqrt(C2 - K2^2);
if r <= 0
  s11 = 0;
s22 = 0;
s12 = 0;
u1 = 0;
u2 = 0;
return
end
if sin(theta) < 0
  m = 2;
else
  m = 1;
end
mu = mu(m);
kappa = kappa(m);
C1 = 1 / (2 * sqrt(2 * pi * r) * cosh(epsilon * pi));
f1_11 = exp(epsilon * theta - (3 - 2*m) * epsilon * pi) * ...
        (3 * cos(theta/2 + epsilon * log(r/1)) + ...
         2 * epsilon * sin(theta) * cos(3 * theta/2 + epsilon * log(r/1)) - ...
         sin(theta) * sin(3 * theta/2 + epsilon * log(r/1)) - ...
         exp(-epsilon * theta + (3 - 2*m) * epsilon * pi) * ...
         cos(theta/2 - epsilon * log(r/1));
f2_11 = exp(epsilon * theta - (3 - 2*m) * epsilon * pi) * ...
        (3 * sin(theta/2 + epsilon * log(r/1)) + ...
         2 * epsilon * sin(theta) * sin(3 * theta/2 + epsilon * log(r/1)) + ...
         sin(theta) * cos(3 * theta/2 + epsilon * log(r/1)) + ...
         exp(-epsilon * theta + (3 - 2*m) * epsilon * pi) * ...
         sin(theta/2 - epsilon * log(r/1));
f1_22 = exp(epsilon * theta - (3 - 2*m) * epsilon * pi) * ...
```

164
\[
\begin{align*}
\text{f}_{2,22} &= \exp (\text{epsilon} \cdot \theta - (3 - 2m) \cdot \text{epsilon} \cdot \pi) \cdot \ldots \\
(\sin (\theta / 2 + \epsilon \cdot \log (r/l)) - \ldots \\
2 \cdot \epsilon \cdot \cos (3 \cdot \theta / 2 + \epsilon \cdot \log (r/l)) + \ldots \\
\sin (\theta) \cdot \cos (3 \cdot \theta / 2 + \epsilon \cdot \log (r/l)) + \ldots \\
\exp (- \epsilon \cdot \cos (\theta + (3 - 2m) \cdot \epsilon \cdot \pi) \cdot \ldots \\
\cos (\theta / 2 - \epsilon \cdot \log (r/l)); \\
\text{f}_{1,12} &= \exp (\text{epsilon} \cdot \theta - (3 - 2m) \cdot \text{epsilon} \cdot \pi) \cdot \ldots \\
(\sin (\theta / 2 + \epsilon \cdot \log (r/l)) + \ldots \\
2 \cdot \epsilon \cdot \cos (3 \cdot \theta / 2 + \epsilon \cdot \log (r/l)) + \ldots \\
\sin (\theta) \cdot \cos (3 \cdot \theta / 2 + \epsilon \cdot \log (r/l)) + \ldots \\
\exp (- \epsilon \cdot \cos (\theta + (3 - 2m) \cdot \epsilon \cdot \pi) \cdot \ldots \\
\cos (\theta / 2 - \epsilon \cdot \log (r/l)); \\
\text{f}_{2,12} &= \exp (\text{epsilon} \cdot \theta - (3 - 2m) \cdot \text{epsilon} \cdot \pi) \cdot \ldots \\
(- \cos (\theta / 2 + \epsilon \cdot \log (r/l)) - \ldots \\
2 \cdot \epsilon \cdot \cos (3 \cdot \theta / 2 + \epsilon \cdot \log (r/l)) - \ldots \\
\sin (\theta) \cdot \cos (3 \cdot \theta / 2 + \epsilon \cdot \log (r/l)) - \ldots \\
\exp (- \epsilon \cdot \cos (\theta + (3 - 2m) \cdot \epsilon \cdot \pi) \cdot \ldots \\
\cos (\theta / 2 - \epsilon \cdot \log (r/l)); \\
\text{s}_1 &= \text{Cl} \cdot (K_1 \cdot f_{1,12} - K_2 \cdot f_{2,11}); \\
\text{s}_{22} &= \text{Cl} \cdot (K_1 \cdot f_{1,22} - K_2 \cdot f_{2,22}); \\
\text{s}_{12} &= \text{Cl} \cdot (K_1 \cdot f_{1,12} - K_2 \cdot f_{2,12}); \\
\text{u}_1 &= \text{K}_1 \cdot \sqrt{2 \cdot \pi \cdot r} / (4 \cdot \pi \cdot \mu \cdot \cosh (\pi \cdot \epsilon \cdot \pi)) \cdot \ldots \\
(\exp (\epsilon \cdot \cos (\theta - (3 - 2m) \cdot \epsilon \cdot (\pi \cdot \epsilon \cdot \pi)) \cdot \kappa / (1 + \epsilon \cdot \epsilon \cdot \pi \cdot \epsilon \cdot \pi) \cdot \ldots \\
(\cos (\theta / 2 - \epsilon \cdot \log (r/l)) - \ldots \\
2 \cdot \epsilon \cdot \cos (3 \cdot \theta / 2 - \epsilon \cdot \log (r/l)) - \ldots \\
\exp (- \epsilon \cdot \cos (\theta + (3 - 2m) \cdot \epsilon \cdot \pi) \cdot \ldots \\
(\cos (\theta / 2 \cdot \epsilon \cdot \log (r/l)) + \ldots \\
2 \cdot \epsilon \cdot \cos (3 \cdot \theta / 2 \cdot \epsilon \cdot \log (r/l)) + \ldots \\
\exp (\cos (\epsilon \cdot \theta + (3 - 2m) \cdot \epsilon \cdot \pi) \cdot \ldots \\
(\sin (\epsilon \cdot \theta / 2 \cdot \epsilon \cdot \log (r/l)) + \ldots \\
\text{K}_2 \cdot \sqrt{2 \cdot \pi \cdot r} / (4 \cdot \pi \cdot \mu \cdot \cosh (\pi \cdot \epsilon \cdot \pi)) \cdot \ldots \\
(\exp (\epsilon \cdot \cos (\theta - (3 - 2m) \cdot \epsilon \cdot (\pi \cdot \epsilon \cdot \pi)) \cdot \kappa / (1 + \epsilon \cdot \epsilon \cdot \pi \cdot \epsilon \cdot \pi) \cdot \ldots \\
(\sin (\theta / 2 - \epsilon \cdot \log (r/l)) + \ldots \\
2 \cdot \epsilon \cdot \cos (3 \cdot \theta / 2 - \epsilon \cdot \log (r/l)) - \ldots \\
\exp (- \epsilon \cdot \cos (\theta + (3 - 2m) \cdot \epsilon \cdot \pi) \cdot \ldots \\
(- \sin (\epsilon \cdot \theta / 2 \cdot \epsilon \cdot \log (r/l)) + \ldots \\
2 \cdot \epsilon \cdot \cos (3 \cdot \theta / 2 \cdot \epsilon \cdot \log (r/l)) + \ldots \\
\exp (\epsilon \cdot \theta - (3 - 2m) \cdot \epsilon \cdot \pi) \cdot \ldots \\
\sin (\epsilon \cdot \theta \cdot \epsilon \cdot \log (r/l)) + \ldots \\
\text{u}_2 &= \text{K}_1 \cdot \sqrt{2 \cdot \pi \cdot r} / (4 \cdot \pi \cdot \mu \cdot \cosh (\pi \cdot \epsilon \cdot \pi)) \cdot \ldots \\
(\exp (\epsilon \cdot \cos (\theta - (3 - 2m) \cdot \epsilon \cdot (\pi \cdot \epsilon \cdot \pi)) \cdot \kappa / (1 + \epsilon \cdot \epsilon \cdot \pi \cdot \epsilon \cdot \pi) \cdot \ldots \\
(\sin (\theta / 2 - \epsilon \cdot \log (r/l)) + \ldots \\
2 \cdot \epsilon \cdot \cos (3 \cdot \theta / 2 - \epsilon \cdot \log (r/l)) - \ldots \\
\exp (- \epsilon \cdot \cos (\theta + (3 - 2m) \cdot \epsilon \cdot \pi) \cdot \ldots \\
(- \sin (\epsilon \cdot \theta / 2 \cdot \epsilon \cdot \log (r/l)) + \ldots \\
2 \cdot \epsilon \cdot \cos (3 \cdot \theta / 2 \cdot \epsilon \cdot \log (r/l)) + \ldots \\
\exp (\epsilon \cdot \theta - (3 - 2m) \cdot \epsilon \cdot \pi) \cdot \ldots \\
\sin (\epsilon \cdot \theta \cdot \epsilon \cdot \log (r/l)) + \ldots
\end{align*}
\]
\[ K2 \sqrt{2\pi r}/(4\pi \mu \cosh(\pi \epsilon)) \ast \dot{\ldots} \]
\[ (\exp(\epsilon \theta -(3-2m)\epsilon \pi) - \kappa)/(1+4\epsilon^2) \ast \dot{\ldots} \]
\[ \cos(\theta/2-\epsilon \log(r/l)) + \ldots \]
\[ 2\epsilon \sin(\theta-\epsilon \log(r/l)) + \ldots \]
\[ \exp(-\epsilon \theta + (3-2m)\epsilon \pi) * 1/(1+4\epsilon^2) \ast \ldots \]
\[ \cos(\theta/2+\epsilon \log(r/l)) + \ldots \]
\[ 2\epsilon \sin(\theta+\epsilon \log(r/l)) + \ldots \]
\[ \exp(\epsilon \theta -(3-2m)\epsilon \pi) \ast \ldots \]
\[ \sin(\theta) \ast \sin(\theta/2+\epsilon \log(r/l)) ; \]
Listing E.1: Main Function

```python
# This must be run within ABAQUS
from abaqus import *
from abaqusConstants import *
from sketch import *
from part import *
from material import *
from section import *
from assembly import *
from load import *
from visualization import *
from interaction import *
from step import *
from mesh import *
from job import *
from odbAccess import *
from shutil import *

import assembly
import regionToolset
import displayGroupMdbToolset as dgm
import part
import step
import interaction
import load
import mesh
import job
import visualization
import xyPlot
import displayGroupOdbToolset as dgo
import material
import section
import sys
import os

# Initialize the output file
file = open('modelData.csv', 'w')
file.write('# number,G,magK,alpha,beta,epsilon,E1,E2,v1,v2,Sy1,Sy2,Et1,Et2,K1,K2,' +
           'CharLen,phi,psi(h),psi(1),Ebar1,Ebar2,+' +
           'P,M,M2,omega,h,sighat,psi,prime,Estar,Systar,topLoad,botLoad,' +
           'topBendStress,botBendStress2,+' +
           'mode1ratio,mode2ratio,J,JfromK,K1(int),K2(int),phicheck,psicheck,' +
           'Jplastic,dwn,dwdsnstar\n')
file.close()
```
# Initialize the LaTeX output file (Needs latex to run)
graphics = open('plasticzones.tex', 'w')

# Set up the document class
documentclass[english]{article}
usepackage[T1]{fontenc}
usepackage[latin1]{inputenc}
usepackage{geometry}
setlength\parskip{\smallskipamount}
setlength\parindent{0pt}

\geometry{verbose, letterpaper, tmargin=1in, bmargin=−0.5in, lmargin=1in, rmargin=0.7in}
pagestyle{empty}
usepackage{amsmath}
usepackage{graphicx}
usepackage{amssymb}
makeatletter
usepackage{pslatex}
psset{unit=1in}
usepackage{ babel}
makeatother
\begin{document}

# Initialize the options dictionary
options={}
name='\textbf{strength}'
options[\textit{name}]=name

# Initialize the model parameters
model={}

# What is the height and length of the model?
model[\textit{h1}]=5. \#mm
model[\textit{h2}]=5. \#mm
model[\textit{L}]=50. \#mm

# What should the smallest element be in milimeters?
model[\textit{elm}]=0.00001

# Is the model in plane stress or plane strain?
model[\textit{plane}]='\textbf{strain}'
model[\textit{E1}]=100.0e3
model[\textit{E2}]=25.0e3
model[\textit{v1}]=1/3.
model[\textit{v2}]=1/3.

if model[\textit{plane}]=='\textbf{stress}':
  model[\textit{E1bar}]=model[\textit{E1}]
  model[\textit{E2bar}]=model[\textit{E2}]
else:
  model[\textit{E1bar}]=model[\textit{E1}]/(1−model[\textit{v1}]**2)
  model[\textit{E2bar}]=model[\textit{E2}]/(1−model[\textit{v2}]**2)

Estar = 1/(0.5*(1/model[\textit{E1bar}]+1/model[\textit{E2bar}]))
print \textbf{Estar = \textbf{\%g}} \%Estar
model[\textit{Estar}]=Estar
alpha, beta = getDundurs(model['E1'], model['E2'], model['v1'], model['v2'], model['plane'])

epsilon = 1/(2*pi)*log((1-beta)/(1+beta))
print 'epsilon = %.6g' %epsilon

model['alpha'] = alpha
model['beta'] = beta
model['epsilon'] = epsilon

# what are the tangent moduli?
model['et1'] = 1.e-25
model['et2'] = 1.e-25

# what are the strengths?
sy1 = 120.0
sy2 = 120.0
model['sy1'] = sy1
model['sy2'] = sy2

Systar = (sy1**2*sy2**2)/(sy1**2+sy2**2)
syprime = (sy1**2-sy2**2)/(sy1**2+sy2**2)
sighat = (sy1-sy2)/(sy1+sy2)

model['Systar'] = Systar
model['syprime'] = syprime
model['sighat'] = sighat

# what are the strain energy release rates applied?
model['G'] = []
model['magK'] = []
for i in [0, -.25, -.5, -.75, -1.]:
    G = exp(i/abs(pi*epsilon))
    model['G'].append(G)
    magK = sqrt(G*Estar/(1-beta**2))
    magK = cosh(pi*epsilon)*(sqrt(G*Estar))
    model['magK'].append(magK)

n=5 # n must be or more

# what is the arctangent of (K2/K2)?
# what are the characteristic lengths applied?
model['phi'] = []
model['l'] = []
for i in range(n):
    model['phi'].append(-pi/2+i/float((n-1))*pi)
    model['l'].append(10.)

# What should the mesh resolution be? (coarse, medium, fine)
options['mesh'] = 'medium'

# What should the time resolution be? (coarse, medium, fine)
options['timestep'] = 'medium'

options['name'] = name + ' - plastic'
inputs, plasticresults = plasticanalysis(model, options)

options['name'] = name + ' - elastic'
elasticresults = elasticanalysis(model, options)

# write results to a file
```python
file = open('modelData.csv', 'a')
for input, elasticresults, plasticresults in zip(inputs, elasticresults, plasticresults):
    for item in input:
        file.write('%g,' %item)
    for item in elasticresults:
        file.write('%g,' %item)
    for item in plasticresults:
        file.write('%g,' %item)
    file.write('
')
file.close()

# finish the plastic zones using LaTeX and PSTricks
graphics = open('plasticzones.tex', 'a')
graphics.write('\\end{document}
')
graphics.close()

os.system('latex plasticzones.tex')
os.system('dvips -o plasticzones.ps plasticzones.dvi')
```

Listing E.2: Plastic Analysis

```python
def plasticanalysis(model, options):
    name = options['name']

    # Set the time as mesh resolution
    if options['mesh'] == 'medium':
        meshsize = 2
    elif options['mesh'] == 'fine':
        meshsize = 4
    else:
        meshsize = 1

    if options['timestep'] == 'medium':
        timestep = 1.
    elif options['timestep'] == 'fine':
        timestep = 2.
    else:
        timestep = 0.5

    h1 = model['h1']
    h2 = model['h2']
    L = model['L']
    elm = model['elm']

    jobnumber = 0

    plane = model['plane']
    buildModel(h1, L, meshsize, timestep, elm, plane, name)
    alpha = model['alpha']

    inputs = []
    results = []
    for G, magK in zip(model['G'], model['magK']):
        for phi, l in zip(model['phi'], model['l']):
            jobnumber = jobnumber + 1
            number = '%g' % (jobnumber)
            M, P, M1, M2, gamma, h, psi, psil, K1, K2, reK, imK, omega = createLoads(magK, phi, l, model)
```

170
bendstress1 = abs(6*M1/h**2)
bendstress2 = abs(6*M2/h**2)

# what are the strengths?
sy1 = max(abs(M1/h**2), abs(M2/h**2))*30
sy2 = sy1
model["sy1"] = sy1
model["sy2"] = sy2

Systar = (sy1**2 + sy2**2) / (sy1**2 + sy2**2)
syprime = (sy1**2 + sy2**2) / (sy1**2 - sy2**2)
sighat = (sy1 - sy2) / (sy1 + sy2)
model["Systar"] = Systar
model["syprime"] = syprime
model["sighat"] = sighat

assignMaterials(model)

G, K, tc, bc, Elbar, E2bar, E1, E2, v1, v2 = setLoads(M1, M2, number)
inputs. append([jobnumber, G, magK, model["alpha"], model["beta"], model["epsilon"], E1, E2, v1, v2, model["sy1"], model["sy2"],
model["et1"], model["et2"], K1, K2, l, phi, psi, psil, Elbar,
E2bar, P, M1, M2, omega, gamma, h, model["sighat"],
model["syprime"], model["Estar"], model["Systar"], tc,
bc, bendstress1, bendstress2])

if jobnumber > 0:
submitJobs(name, number)
dwdn, dWdNStar, Jplastic = extractWork(name+"−"+number, model["Estar"],
model["Systar"], model["epsilon"], G)
results. append([Jplastic, dwdn, dWdNStar])

if jobnumber > 0:
    printf2(name+"−"+number, alpha)
    graphics = open("plasticzones.txt", 'a')
    graphics . write("Job" Name is %s where $\mathcal{G}=%g, \nu_1=%g, \nu_2=%g, l=m^{2}\$\$
\psi=%0.4f, \phi=%0.4f, \alpha=%g, \nu_1[1]=%g, \nu_2[2]=%g, \psi[1]frac{dW}{dn}=%g, \psi[2]frac{dW}{dn}=%g, \psi[1]frac{dW}{dn}=%g, \psi[2]frac{dW}{dn}=%g
$\$ integral=%g, \%((name+"−"+number), G*1000, psi, phi, alphabet, v1, v2, dwdn, dWdNStar, Jplastic*1000))

graphics.close()

return inputs, results

def submitJobs(name, number):
    """ This submits jobs """
mymodel= mdb. model["ModeMix"]
a = mdb. model["ModeMix"]. rootAssembly
myjob = mdb. Job(name=name+"−"+number, model=mymodel.name, type=ANALYSIS,
    explicitPrecision=SINGLE, nodalOutputPrecision=SINGLE,
    description='', userSubroutine='',
    numCpus=1, preMemory=256, standardMemory=256,
    standardMemoryPolicy=MODERATE, scratch='', echoPrint=OFF,
    modelPrint=OFF, contactPrint=OFF, historyPrint=OFF)
a. regenerate()
myjob.submit()}
def elastic_analysis(model, options):
    name = options['name']

    # Delete the extra load steps
    del mdb.models['ModeMix'].steps['Unload']
    del mdb.models['ModeMix'].steps['Reload']
    del mdb.models['ModeMix'].steps['Reunload']

    mdb.models['ModeMix'].keywordBlock.insert(83, 'Contour Integral, contours = 50, type = K factors Tip, 0')
    mdb.models['ModeMix'].keywordBlock.replace(93, '')
    mdb.models['ModeMix'].keywordBlock.replace(94, '')
    mdb.models['ModeMix'].keywordBlock.replace(95, '')

    # Delete the plastic material properties
    del mdb.models['ModeMix'].materials['Bot'].plastic
    del mdb.models['ModeMix'].materials['Top'].plastic

    # Reduce the number of required steps
    mdb.models['ModeMix'].steps['Load'].setValues(initialInc=1.0, minInc=1e-05, maxInc=1.0)

    modelRatio, mode2Ratio = findFactor(model['epsilon'], model['alpha'])
    omega1, omega2 = determineOmega(modelRatio, mode2Ratio, model['alpha'], 'strain')
    omega = (omega1 + omega2) / 2. * pi / 180.

    jobNumber = 0
    results = []
    for G, magK in zip(model['G'], model['magK']):
        for phi, l in zip(model['phi'], model['l']):
            jobNumber = jobNumber + 1
            number = '%g %g' % (jobNumber, l)

            M, P, M1, M2, gamma, h, psi, psil, K1, K2, reK, imK, omega = createLoads(magK, phi, l, model)

            G, K, tc, bc, E1bar, E2bar, E1, E2, v1, v2 = setLoads(M1, M2, number, 'elastic')

            if jobNumber > 0:
                submitJobs(name, number)

            J, JfromK, K1int, K2int, phiInt, psiInt = extractMode(name, number, l, model['epsilon'], G)
            results.append([modelRatio, mode2Ratio, J, JfromK, K1int, K2int, phiInt, psiInt])

    return results
Listing E.4: Scripts for Assigning Loads and Materials

```python
def setLoads(M1, M2, number, plastic='yes'):
    print("Setting Loads...")

    mymodel = mdb.model['ModeMix']
a = mdb.model['ModeMix'].rootAssembly

# extract y-coordinates and measure height
y_top = a.sets['top-right-corner'].vertices[0].pointOn[0][1]
y_bot = a.sets['bottom-right-corner'].vertices[0].pointOn[0][1]
y_tip = a.sets['Tip'].vertices[0].pointOn[0][1]

h1 = y_top - y_tip
h2 = y_tip - y_bot
h = h1

eta = h1 / h2

# extract elastic material properties
top_material = mdb.model['ModeMix'].material['Top']
bottom_material = mdb.model['ModeMix'].material['Bottom']
E1 = top_material.elastic.table[0][0]
E2 = bottom_material.elastic.table[0][0]
v1 = top_material.elastic.table[0][1]
v2 = bottom_material.elastic.table[0][1]

if plastic == 'yes':
    # extract plastic material properties
    y1 = top_material.plastic.table[0][0]
y2 = bottom_material.plastic.table[0][0]
y1a = top_material.plastic.table[1][0]
y2a = bottom_material.plastic.table[1][0]
strain1 = top_material.plastic.table[1][1] - y1 / E1
strain2 = bottom_material.plastic.table[1][1] - y2 / E2

t1 = (y1a - y1) / strain1 / E1
t2 = (y2a - y2) / strain2 / E2

    print('Top material: %g %g' % (y1, t1))
    print('Bot material: %g %g' % (y2, t2))

alpha, beta = getDundurs(E1, E2, v1, v2)
epsilon = 1 / (2 * pi) * log((1 - beta) / (1 + beta))
p = sqrt((1 - alpha) / (1 - beta ** 2))

# define the shear modulus
shear1 = E1 / (2 * (1 + v1))
shear2 = E2 / (2 * (1 + v2))

# kappa for plane strain
k1 = 3 - 4 * v1
k2 = 3 - 4 * v2

# Modulus from plane stress to plane strain
E1bar = E1 / (1 - v1 ** 2)
E2bar = E2 / (1 - v2 ** 2)
c1 = (k1 + 1) / shear1
c2 = (k2 + 1) / shear2
```

173
SIGMA = c2 / c1

A = 1 / ( 1 + SIGMA * ( 4 * eta + 6 * eta**2 + 3 * eta**3 ) )
I = 1 / ( 12 * ( 1 + SIGMA * eta**3 ) )
delta = h1 * ( 1 + 2 * SIGMA * eta + SIGMA * eta**2 ) / ( 2 * eta * ( 1 + SIGMA * eta ) )
DELTA = delta / h1

P1 = 0
P2 = 0
P3 = P1 - P2 # eqn (1.1)
M3 = ( M1 - M2 + P1 * ( h1 / 2. + h2 - delta ) + P2 * ( delta - h2 / 2. ) ) # eqn (1.1)
Ao = 1 / ( eta + SIGMA)
Io = ( 1 / 3. ) * ( SIGMA * ( 3 * ( DELTA - 1 / eta )**2 - 3 * ( DELTA - 1 / eta ) + 1 ) + 3 * DELTA / ( eta * ( DELTA - 1 / eta ) + 1 / eta**3 ) )

C1 = SIGMA / Ao
C2 = ( SIGMA / Io ) * ( 1 / eta - DELTA + 1 / 2 )
C3 = SIGMA / ( 1 / 2 * Io )
P = P1 - C1 * P3 - C2 * M3 / h1 # eqn (1.2)
M = M1 - C3 * M3
print 'Pcheck = %g' % P
print 'Mcheck = %g' % M

siny = 6 * SIGMA * eta**2 * ( 1 + eta ) * ( A + 1 )**0.5
gamma = asin( siny )
print 'gamma_check = %g' % gamma
omega = lookupomega( alpha, beta )
print 'omega = %g' % omega

# eq (2.16)
ReKh = p / sqrt( 2. ) * ( P / sqrt( A + h1 ) * cos( omega ) + M / sqrt( I + h1**3 ) * sin( omega + gamma ) )
print 'ReKh_check = %g' % ReKh

# eq (2.16)
ImKh = p / sqrt( 2. ) * ( P / sqrt( A + h1 ) * sin( omega ) - M / sqrt( I + h1**3 ) * cos( omega + gamma ) )
print 'ImKh_check = %g' % ImKh

# eqn (2.7)
G = ( c1 / 16.0 ) * ( P**2 / ( A + h1 ) + M**2 / ( I + h1**3 ) + 2 * P * M * siny / (( A + 1 )**0.5 * h1**2 ) )
magK = ( Elbar * ( 1 - alpha ) + G / ( 1 - beta**2 ) )**.5
print 'Gcheck = %g' % G
print 'magKcheck = %g' % magK
G = ( c1 + c2 ) / ( 16.0 * cosh( pi * epsilon )**2 ) * ( magK**2 )
magK = ( P**2 / 2. ) * ( P**2 / ( A + h1 ) + M**2 / ( I + h1**3 ) + 2 * P * M * siny / (( A + 1 )**0.5 * h1**2 ) )**.5
print 'Gcheck = %g' % G
print 'magKcheck = %g' % magK

topcouple = 4 * M1 / h1**2
bottomcouple = 4 * M2 / h2**2

mdb.model[ 'ModeMix' ].load[ 'Load-1' ].setValues( magnitude=topcouple )
mdb.model[ 'ModeMix' ].load[ 'Load-2' ].setValues( magnitude=-topcouple )
mdb.model['ModeMix'].load['Load-3'].setValues(magnitude=bottomcouple)
mdb.model['ModeMix'].load['Load-4'].setValues(magnitude=bottomcouple)

if plastic=='yes':
    mdb.model['ModeMix'].load['Reload-1'].setValues(magnitude=topcouple)
    mdb.model['ModeMix'].load['Reload-2'].setValues(magnitude=topcouple)
    mdb.model['ModeMix'].load['Reload-3'].setValues(magnitude=bottomcouple)
    mdb.model['ModeMix'].load['Reload-4'].setValues(magnitude=bottomcouple)
    print 'Reload_values_set...
return G, mag, topcouple, bottomcouple, E1bar, E2bar, E1, E2, v1, v2

def assignMaterials(model):
    "This_assigns_the_materials_according_to_the_properties"
    print ('Assigning_Materials...
Etop = model['E1']
Ebot = model['E2']
    vtop = model['v1']
    vbot = model['v2']

    #delete materials in model
    mymodel=mdb.model['ModeMix']
    mymodel.material['Top'].Elastic(table=((Etop, vtop), ))
    mymodel.material['Bot'].Elastic(table=((Ebot, vbot), ))

    #Plastic Properties
    EttopoverE=model['et1']
    EtbotoverE=model['et2']
    yieldstr=float(model['sy1'])
    print 'yieldstr=%g' %yieldstr
    stress = yieldstr*1.0002
    topplasticstable = [(yieldstr,0.0)]
    plasticstrain = ((stress-yieldstr)/EttopoverE-stress+yieldstr)/Etop
totalstrain=stress/Etop+plasticstrain
topplasticstable.append((stress,plasticstrain))
    print topplasticstable
    top_material=mymodel.Material(name='Top')
    mymodel.material['Top'].Elastic(table=((Etop,vtop), ))
    mymodel.material['Top'].Plastic(table = topplasticstable)
    mymodel.material['Top'].plastic.setValue(hardening=KINEMATIC)

    yieldstr=float(model['sy2'])
    print 'yieldstr=%g' %yieldstr
    stress = yieldstr*1.0002
    botplasticstable = [(yieldstr,0.0)]
    plasticstrain = ((stress-yieldstr)/EtbotoverE-stress+yieldstr)/Ebot
totalsstrain=stress/Ebot+plasticstrain
    botplasticstable.append((stress,plasticstrain))
    print botplasticstable
    bottom_material = mymodel.Material(name='Bot')
    mymodel.material['Bot'].Elastic(table=((Ebot,vbot), ))

    # delete materials in model
    mymodel=mdb.model['ModeMix']
    mymodel.material['Top'].Elastic(table=((Etop,vtop), ))
    mymodel.material['Bot'].Elastic(table=((Ebot,vbot), ))
mymodel.material[Bot].Plastic(table=botplasticstable)
mymodel.material[Bot].plastic.setValues(hardening=KINEMATIC)

#assign material to section
section1=mymodel.section[Top]
section2=mymodel.section[Bottom]
section1.setValue(material=top_material.name, thickness=1.0)
section2.setValue(material=bottom_material.name, thickness=1.0)

def createLoads(magK, phi, l, model):
  print('Creating Loads...')
mymodel= mdb.model['ModeMix']
a = mdb.model['ModeMix'].rootAssembly

  print('h1= y_top - y_tip')
h1 = y_top - y_tip
  print('h2 = y_tip - y_bot')
h2 = y_tip - y_bot

h=h1

  alpha = model['alpha']
  beta = model['beta']
  epsilon = model['epsilon']

  print('l = %g' % l)
  print('epsilon*log(l) = %g *(epsilon*log(1))

K1=magK*cos(phi)
K2=magK*sin(phi)

  print('K1 = %g' % K1)
  print('K2 = %g' % K2)

reKl=K1*cos(epsilon*log(1))-K2*sin(epsilon*log(1))
imKl=K1*sin(epsilon*log(1))+K2*cos(epsilon*log(1))

  print('reKl = %g' % reKl)
  print('imKl = %g' % imKl)

psil=atan(imKl/reKl)

  print('psi_l = %g' % psil)

reK=K1*cos(epsilon*log(h))-K2*sin(epsilon*log(h))
imK=K1*sin(epsilon*log(h))+K2*cos(epsilon*log(h))

  print('reK = %g' % reK)
  print('imK = %g' % imK)

psi=atan(imK/reK)

  print('psi = %g' % psi)

#NOT From Suo and Hutchinson
p=sqrt((alpha-1)/(beta**2-1))

#From Mathematica Result
  print('gamma = %g' % gamma)

omega=lookupomega(alpha,beta)
print 'omega = %.2g' %omega
M = (sqrt(-h**3*(alpha - 1))*(-imK*cos(omega)+reK*sin(omega)))/(cos(gamma)*2*sqrt(3)*p)
P = (sqrt(((h-h+alpha)/(7+6*alpha))(*(reK*cos(omega+gamma)+imK*sin(omega+gamma))))/(cos(gamma)*p))
print 'P = %.2g' %P
print 'M = %.2g' %M
M1 = M + h*P/(6*alpha - 6)
M2 = M + h*P*(1+1/(6+6*alpha))
return M, P, M1, M2, gamma, h, psi, psi1, K1, K2, reK, imK, omega

Listing E.5: Plastic Zone Creation

def printfig(NAME, alpha):
    print 'Printing Figure ...
    alpha = '+alpha+' 
    name = '+NAME
    import displayGroupOdbToolset as dgo
    o1 = session.openOdb(name=NAME+'.odb')
    session.viewports['Viewport:1'].setValues(displayedObject=o1)
    legend=OFF, title=OFF, state=OFF)
    session.viewports['Viewport:1'].odbDisplay.setPlotMode(CONTOUR)
    session.viewports['Viewport:1'].odbDisplay.setFrame(step=3, frame=1)
    session.viewports['Viewport:1'].odbDisplay.setPrimaryVariable(
        variableLabel='AC_YIELD', outputPosition=INTEGRATION_POINT)
    session.viewports['Viewport:1'].odbDisplay.contourOptions.setValues(
        numIntervals=2, contourMethod=TESSELLATED, spectrumType=WHITE_TO_BLACK,
        modelShape=UNDEFORMED, maxValue=.5, minValue=.4, maxAutoCompute=OFF,
        minAutoCompute=OFF)
    session.viewports['Viewport:1'].odbDisplay.contourOptions.setValues(
        outsideLimitsBelowColor='White')
    session.viewports['Viewport:1'].odbDisplay.contourOptions.setValues(
        outsideLimitsAboveColor='Grey60')
    session.viewports['Viewport:1'].odbDisplay.contourOptions.setValues(  
        #elementShrinkFactor = 0.05)
    session.viewports['Viewport:1'].setValues( origin=(0.0, 0), width=180, height=150)
    session.epsOptions.setValues(imageSize=(3.00, 2.5), resolution=DPI_450,
        fontType=PS_ALWAYS)
    session.printOptions.setValues( rendition=GREYSCALE, vpDecorations=OFF,
        vpBackground=OFF)
    session.viewports['Viewport:1'].restore()
    session.viewports['Viewport:1'].view.fitView()

k=1
zoomlist=[80,320,1280,5120,20480,81920 ]
for zoomfactor in zoomlist:
    for stepnumber in [3]:
        imageFile=NAME+'-zoomfactor '+zoomfactor +'.eps'
        graphics=open('plasticzones.tex', 'a')
        graphics.write(\includegraphics[width=3in, height=2.5in, bb=0_0_216_180][%s ]\psframe(0,0)(-3.2,5)\nnode(-3.0){LC}\nnode(0,0){RC}\npsline [linewidth=.5pt]{->}(0,1.25)(.25,1.25)\nuput[u](.25,1.25){$x$}\npsline [linewidth=.5pt]{->}(-1.5,2.5)(-1.5,2.65)\nuput[r]{(-1.5,2.65){$y$}\n}
Listing E.6: Extract Various Quantities

def extractWork(name, Estar, Systar, epsilon, G):
    myodb = session.openOdb(name+'.odb')
    odb = session.odbs[name+'.odb']
    for stepname in myodb.steps.keys():
        history = myodb.steps[stepname].historyOutput['Assembly'].data
        pdata = history.historyOutput['ALLPD'].data
        if stepname == myodb.steps.keys()[1]:
            w2 = pdata[-1][-1]
        if stepname == myodb.steps.keys()[3]:
            w4 = pdata[-1][-1]
    dwdn=(w4-w2)
    dWdNStar = dwdn*Systar*cosh(pi*epsilon)**4 / float(G**2 * Estar)
    k=30
    session.XYDataFromHistory(name='J'+str(k), odb=odb, outputVariableName='J-Integral of contour at onset crack front node set', steps=('Load', 'Unload', 'Reload', 'Reunload',))
    J = session.xyData['J'+str(k)].data
    J1=J[0][1]
myodb.close()
return dwdn, dWdNStar, J1

def getDundurs(E1, E2, v1, v2, plane='strain'):
    shear1 = E1 / (2*(1+v1))
    shear2 = E2 / (2*(1+v2))
    if plane=='stress':
        k1 = (3−v1)/(1+v1)
        k2 = (3−v2)/(1+v2)
    else:
        k1 = 3−4*v1
        k2 = 3−4*v2
        E1=E1/(1−v1**2)
        E2=E2/(1−v2**2)
    GAMMA = shear1/shear2
    alpha = ( GAMMA*(k2 + 1) − (k1 + 1) ) / ( GAMMA*(k2 + 1) + (k1 + 1) )
    beta = ( GAMMA*(k2 − 1) − (k1 − 1) ) / ( GAMMA*(k2 + 1) + (k1 + 1) )
    alpha = (E1−E2)/(E1+E2)
    beta = ( shear1*(k2 − 1) − shear2*(k1 − 1) ) / ( shear1*(k2 + 1) + shear2*(k1 + 1) )
    return alpha, beta

def extractMode(originame, number, h, epsilon, G):
    name=originame+'-'+number
    o3 = session.openOdb(name=name+'.'odb')
    session.viewports['Viewport:1'].setValues(displayedObject=o3)
    odb = session.odbs[name+'.'odb']

    # J Integrals
    session.XYDataFromHistory(name='J integral', odb=odb, outputVariableName='J integral at of contour 10 on crackfront node set L5:J integral for Whole Model', steps=('Load',))
    session.XYDataFromHistory(name='J from K', odb=odb, outputVariableName='J integral estimated from K at of contour 10 on crackfront node set L5:J from K for Whole Model', steps=('Load',))
    session.XYDataFromHistory(name='K1', odb=odb, outputVariableName='Stress intensity factor K1 at of contour 10 on crackfront node set L5:K1 for Whole Model', steps=('Load',))
    session.XYDataFromHistory(name='K2', odb=odb, outputVariableName='Stress intensity factor K2 at of contour 10 on crackfront node set L5:K2 for Whole Model', steps=('Load',))
    session.XYDataFromHistory(name='K2−25', odb=odb, outputVariableName='Stress intensity factor K2 at of contour 25 on crackfront node set L5:K2 for Whole Model', steps=('Load',))
    session.XYDataFromHistory(name='K1−25', odb=odb, outputVariableName='Stress intensity factor K1 at of contour 25 on crackfront node set L5:K1 for Whole Model', steps=('Load',))
    session.XYDataFromHistory(name='J from K−25', odb=odb,
outputVariableName='J-integral estimated from Ks at of contour 25 on crackfront node set L5: Jf for Whole Model'.
steps=('Load',))
session.XYDataFromHistory(name='J−25',odb=odb,
outputVariableName='J-integral at of contour 25 on crackfront node set L5: Jf for Whole Model'.
steps=('Load',))
JfromK = session.xyDataObjects['JfromK'].data[0][1]
J = session.xyDataObjects['J integral'].data[0][1]
K1 = session.xyDataObjects['K1'].data[0][1]
K2 = session.xyDataObjects['K2'].data[0][1]
JfromK25 = session.xyDataObjects['JfromK−25'].data[0][1]
J25 = session.xyDataObjects['J−25'].data[0][1]
K125 = session.xyDataObjects['K1−25'].data[0][1]
K225 = session.xyDataObjects['K2−25'].data[0][1]
tanpsi = (K2 / K1)
phi = atan(K2 / K1)
realKh = K1 * cos(epsilon * log(h)) - K2 * sin(epsilon * log(h))
imagKh = K2 * cos(epsilon * log(h)) + K1 * sin(epsilon * log(h))
psi = atan(imagKh / realKh)
adjustedmode = phi + epsilon * log(1)
adjustedmodel = psi + epsilon * log(h / l)
odb.close()
return J, JfromK, K1, K2, phi, psi

def lookupomega(alpha, beta, eta=1):
    # epsilon = 1/(2*pi) * log((l−beta)/(l+beta))
    # modelratio, mode2ratio = findfactor(epsilon, alpha)
    # omegai, omegai2 = determineOmega(modelratio, mode2ratio, alpha, plane)
    #
    if beta == 0:
        omega = [44.26, 44.2, 44.4, 44.9, 45.46, 46.2, 46.81, 47.6, 48.2, acos(sqrt(3/7)) * 180/pi, 49.931, 50.8, 51.686, 52.632, 53.635, 55.0, 55.882, 57.2, 58.712]
    else:
        omega = [54.795, 53.9, 52.701, 52.016, 51.946, 51.504, 50.403, 49.942, 49.510, acos(sqrt(3/7)) * 180/pi, 48.733, 48.39, 48.098, 47.852, 47.671, 47.573, 47.587, 47.761, 48.179]
    for i in range(len(omega)):
        alpha = -0.9 + i * 1
        if alpha <= alpha:
            y = (omega[i] − omega[i − 1]) / (alpha − alpha) + omega[i]
        return y/pi/180.

def determineOmega(modelratio, mode2ratio, alpha, plane):
    mymodel = mdb.model['ModeMix']
a = mdb.model['ModeMix'].rootAssembly
    # extract y-coordinates and measure height
    y_top = a.sets['top−right−corner'].vertices[0].pointOn[0][1]
y_bot = a.sets['bottom−right−corner'].vertices[0].pointOn[0][1]
y_tip = a.sets['Tip'].vertices[0].pointOn[0][1]
h1 = y_top - y_tip
h2 = y_tip - y_bot
l = h1

eta = h1 / h2

# extract elastic material properties
top_material = mdb.model['ModeMix'].material['Top']
bottom_material = mdb.model['ModeMix'].material['Bot']
E1 = top_material.elastic.table[0][0]
E2 = bottom_material.elastic.table[0][0]
v1 = top_material.elastic.table[0][1]
v2 = bottom_material.elastic.table[0][1]

# define the shear modulus
shear1 = E1 / (2*(1+v1))
shear2 = E2 / (2*(1+v2))

if plane == 'stress':
k1 = (3-v1)/(1+v1)
k2 = (3-v2)/(1+v2)
else:
k1 = 3-4*v1
k2 = 3-4*v2
E1 = E1 / (1-v1**2)
E2 = E2 / (1-v2**2)

c1 = (k1 + 1)/shear1
c2 = (k2 + 1)/shear2

SIGMA = c2/c1
A = 1. / (1. + SIGMA*(4*eta + 6*eta**2 + 3*eta**3))
I = 1. / (12*(1. + SIGMA*eta**3))
delta = h1*((1 + 2*SIGMA*eta + SIGMA*eta**2) / (2*eta*(1 + SIGMA*eta))
DELTA = delta/h1
Ao = 1. / eta + SIGMA
Io = (1/3.)*(SIGMA*(3*(DELTA - 1/eta)**2 - 3*(DELTA - 1/eta) + 1) + 3*DELTA/
     eta*(DELTA - 1/eta) + 1/eta**3)
C1 = SIGMA / Ao
C2 = (SIGMA / Io)*(1/eta - DELTA + 1./2.)
C3 = SIGMA / (12*Io)
siny = 6*SIGMA*eta**2*(1 + eta)*(A*1)**0.5
gamma = asin(siny)
P1 = 0
P2 = 0

M2 = 1.
for Mi in [mode2ratio, mode1ratio]:
P3 = P1 - P2 #eqn (1.1)
M3 = (Mi - M2 + P1*(h1/2. + h2 - delta) + P2*(delta - h2/2.)) #eqn (1.1)
P = P1 - C1*P3 - C2*M3/h1 # eqn (1.2)
\[ M = M_1 - C_3 M_3 \]
\[ \lambda = \left( I / A \right)^{0.5} P h_1 / M \]
\[ \text{err} = 100 \]
\[ \text{tol} = 0.0001 \]
\[ \omega = [49.0 \pi / 180, 50.0 \pi / 180] \]
\[ \text{lhs} = [] \]

for \( k \) in range(2):
    if \( M_1 \) == mode2ratio:
        \text{lhs}.append(\( \lambda \cdot \cos(\omega[k]) + \sin(\omega[k] + \gamma) \))
        \text{err} = abs(lhs[k])
    if \( M_1 \) == mode1ratio:
        \text{lhs}.append(\( \lambda \cdot \sin(\omega[k]) - \cos(\omega[k] + \gamma) \))
        \text{err} = abs(lhs[k])
    if \text{err} < \text{tol}:
        break

while \text{err} > \text{tol}:
    \omega.append(\( \omega[k] - (\text{lhs}[k] \cdot (\omega[k] - \omega[k-1]) / (\text{lhs}[k] - \text{lhs}[k-1]) \))
    \( k = k + 1 \)

if \( M_1 \) == mode2ratio:
    \text{lhs}.append(\( \lambda \cdot \cos(\omega[k]) + \sin(\omega[k] + \gamma) \))
    print 'mode2'
if \( M_1 \) == mode1ratio:
    \text{lhs}.append(\( \lambda \cdot \sin(\omega[k]) - \cos(\omega[k] + \gamma) \))
    print 'mode1'

\text{err} = abs(lhs[k])

if \( M_1 \) == mode2ratio:
    \omega2 = \omega[k] \cdot 180 / \pi
    print ('\omega2 = ' + 'omega2 ')
    print ('\text{lhs} = ' + 'lhs ')
    \omega.append(\( \omega[k] - (\text{lhs}[k] \cdot (\omega[k] - \omega[k-1]) / (\text{lhs}[k] - \text{lhs}[k-1]) \))
    \( k = k + 1 \)

if \( M_1 \) == mode2ratio:
    \omega2 = \omega[k] \cdot 180 / \pi
    print ('\omega2 = ' + 'omega2 ')
print
if \( M_1 \) == mode1ratio:
    \omega1 = \omega[k] \cdot 180 / \pi
    print ('\omega1 = ' + 'omega1 ')
print

return \omega1, \omega2

---

Listing E.7: Determine Loading Factors

def findfactor(epsilon, alpha):
    print "Finding Pure Moment Ratios Using the Secant Method . . . "

mymodel = mdb.model['ModeMix ']
a = mdb.model['ModeMix '].rootAssembly

# extract y-coordinates and measure height
y_top = a.sets['top-right-corner '].vertices[0].pointOn[0][1]
y_bot = a.sets['bottom-right-corner '].vertices[0].pointOn[0][1]
y_tip = a.sets['Tip '].vertices[0].pointOn[0][1]
h1 = y_top - y_tip
h2 = y_tip - y_bot
l = h1

# generate Loading conditions
m2 = 1.e3

for mode in [1, 2]:
    err = 100.
    tol = .001
    if mode == 1:
        point1 = ((1.0 - 0.831 * alpha + 0.00187 * alpha**2 - 3.01 * alpha**3 + 5.03 * alpha**4 + 8.32 * alpha**5 - 12.7 * alpha**6 - 10.1 * alpha**7 + 12.7 * alpha**8))
    else:
        point1 = ((-1.0 + 1.06 * alpha - 0.143 * alpha**2 + 3.92 * alpha**3 - 6.70 * alpha**4 - 11.1 * alpha**5 + 17 * alpha**6 + 13.4 * alpha**7 - 17.0 * alpha**8))
    point = [point1, point1 * .99]

MyLoads = [[m2 * point[0], m2], [m2 * point[1], m2]]
K1 = []
K2 = []
realKh = []
imagKh = []
k = 0

for load in MyLoads:  # vary loads to change modes
    M1 = load[0]
    M2 = load[1]
    factor = M1 / M2
    print('alpha = ' + 'alpha ')
    print('k = ' + 'k ')
    print('M1 = ' + 'M1 ')
    print('M2 = ' + 'M2 ')
    print('M1 factor = ' + 'factor ')
    if M2 == 0.:
        M2 = 1e-15

    # Begin Applying forces to model
    topcouple = 4*M1/h1**2
    bottomcouple = 4*M2/h2**2

    mymodel.load['Load-1'].setValues(magnitude=topcouple)
    mymodel.load['Load-2'].setValues(magnitude=-topcouple)
    mymodel.load['Load-3'].setValues(magnitude=bottomcouple)
    mymodel.load['Load-4'].setValues(magnitude=bottomcouple)

    # Submit the job and wait for completion
    myjob = mdb.Job(name='findFactor', model=mymodel.name, type=ANALYSIS, explicitPrecision=SINGLE, nodalOutputPrecision=SINGLE,
                    description='G=f'+mode2factor+'factor', userSubroutine=''
                    numCpus=1, preMemory=512.0, standardMemory=512.0,
                    standardMemoryPolicy=MODERATE, scratch='', echoPrint=OFF,
                    modelPrint=OFF, contactPrint=OFF, historyPrint=OFF)
name = myjob.name
a. regenerate()
myjob.submit()
myjob.waitForCompletion()

o3 = session.openOdb(name='findfactor.odb')
    session.viewports['Viewport:1'].setValues(displayedObject=o3)

odb = session.odb[0].odb

# J Integrals
session.XYDataFromHistory(name='Jintegral', odb=odb, outputVariableName='J
    integral at of contour 10 on crackfront node set 5:J for Whole
    Model', steps=('Load', ))
session.XYDataFromHistory(name='JfromK', odb=odb,
    outputVariableName='J integral estimated from Ks at of contour 10 on
    crackfront node set 5:JfK for Whole Model', steps=('Load', ))
session.XYDataFromHistory(name='K1', odb=odb,
    outputVariableName='Stress intensity factor K1 at of contour 10 on
    crackfront node set 5:K1 for Whole Model', steps=('Load', ))
session.XYDataFromHistory(name='K2', odb=odb,
    outputVariableName='Stress intensity factor K2 at of contour 10 on
    crackfront node set 5:K2 for Whole Model', steps=('Load', ))
JfromK = session.xyDataObjects['JfromK'].data[0][1]
J = session.xyDataObjects['Jintegral'].data[0][1]
K1.append(session.xyDataObjects['K1'].data[0][1])
K2.append(session.xyDataObjects['K2'].data[0][1])

odb.close()

print('JIntegral = ' + 'J')
print('JfromK = ' + 'JfromK')
print('K1 = ' + 'K1[k]')
print('K2 = ' + 'K2[k]')
tanpsi = (K2[k] / K1[k])
modemix = atan(K2[k] / K1[k]) * 180 / pi

print('mode_mix = ' + 'modemix')
realKh.append(K1[k] * cos(epsilon * log(1)) - K2[k] * sin(epsilon * log(1)))
imagKh.append(K2[k] * cos(epsilon * log(1)) + K1[k] * sin(epsilon * log(1)))
phase = atan(imagKh[k] / realKh[k]) * 180 / pi
adjustedmode = modemix + epsilon * log(1) * 180 / pi

print('realKh = ' + 'realKh[k]')
print('imagKh = ' + 'imagKh[k]')
print('phase = ' + 'phase')
print('adjustedmode = ' + 'adjustedmode')

print('err = ' + 'err')
print('tol = ' + 'tol')
if err < tol:
    print('this should break !!!!!!!!!!!!!!!!')
    break
if len(realKh) < 2:
123 \( k = k + 1 \)

124 \textbf{while} \( \text{err} > \text{tol} \):

125 \texttt{print(‘alpha = ‘ + ‘alpha ’)}

126 \texttt{print(‘k = ‘ + ‘k ‘)}

127 \textbf{if} \( \text{mode} == 2 \):

128 \hspace{1em} \texttt{point.append(point[k] - (realKh[k] - point[k]) / (realKh[k] - realKh[k - 1])})

129 \textbf{else}:

130 \hspace{1em} \texttt{point.append(point[k] - (imagKh[k] - point[k]) / (imagKh[k] - imagKh[k - 1])})

131 \( k = k + 1 \)

132 \( M1 = m2 \times \texttt{point}[k] \)

133 \( M2 = m2 \)

134 \texttt{factor} = \( M1 / M2 \)

135 \texttt{print(‘\(M1\) ‘ + ‘M1’)}

136 \texttt{print(‘\(M2\) ‘ + ‘M2’)}

137 \texttt{print(‘\(M1\) factor ‘ + ‘factor ‘)}

138 \textbf{if} \( \text{M2} == 0. : \)

139 \hspace{1em} \texttt{M2 = 1e-15}

140 # Begin Applying forces to model

141 \( \text{topcouple} = 4 \times M1 / h1 ** 2 \)

142 \( \text{bottomcouple} = 4 \times M2 / h2 ** 2 \)

143 \texttt{mymodel.load[‘Load–1’].setValue(\texttt{magnitude=topcouple})}

144 \texttt{mymodel.load[‘Load–2’].setValue(\texttt{magnitude=topcouple})}

145 \texttt{mymodel.load[‘Load–3’].setValue(\texttt{magnitude=bottomcouple})}

146 \texttt{mymodel.load[‘Load–4’].setValue(\texttt{magnitude=bottomcouple})}

147 # submit the job and wait for completion

148 \texttt{myjob = mdb.Job(name=’findfactor’, model=mymodel.name, type=ANALYSIS,}
149 \hspace{1em} \texttt{explicitPrecision=SINGLE, nodalOutputPrecision=SINGLE,}
150 \hspace{1em} \texttt{description=’G ‘ + ‘G ‘ + ‘mode2factor ‘ + ‘factor ‘, userSubroutine=’’,}
151 \hspace{1em} \texttt{numCpus=1, preMemory=512.0, standardMemory=512.0,}
152 \hspace{1em} \texttt{standardMemoryPolicy=MODERATE, scratch=’’, echoPrint=OFF,}
153 \hspace{1em} \texttt{modelPrint=OFF, contactPrint=OFF, historyPrint=OFF})

154 \texttt{name} = \texttt{myjob.name}

155 \texttt{a.regenerate()}\texttt{myjob.submit()}\texttt{myjob.waitForCompletion()}

156 \texttt{print ”FindFactor \_ Job \_ Finished \_ \_ \_ \_ \_ \_ ”}

157 \texttt{o3 = session.openOdb(name=’findfactor.odb’)}

158 \hspace{1em} \texttt{session.viewports[‘Viewport : 1’].setValue(\texttt{displayedObject=o3})}

159 \texttt{odb = session.odb[name+’’.odb’]}

160 # Integrals

161 \texttt{session.XYDataFromHistory(name=’J integral’, odb=odb, outputVariableName=’J ‘ + \texttt{‘integral\_at\_of\_contour\_10\_on\_crackfront\_node\_set\_5\_for\_Whole\_Model’, steps=(‘Load’,))}

162 \texttt{session.XYDataFromHistory(name=’JfromK’, odb=odb,}
outputVariableName='J_integral_estimated_from_Ks_at_of_contour_10_on_crackfront_node_set_15:JKf_for_Whole_Model',
steps=('Load',))
session.XYDataFromHistory(name='K1',odb=odb,
outputVariableName='Stress_intensity_factor_K1_at_of_contour_10_on_crackfront_node_set_15:K1_for_Whole_Model',
steps=('Load',))
JfromK = session.xyDataObjects['JfromK'].data[0][1]
J = session.xyDataObjects['J_integral'].data[0][1]
K1.append(session.xyDataObjects['K1'].data[0][1])
K2.append(session.xyDataObjects['K2'].data[0][1])
odb.close()

print('JIntegral = '+str(J))
print('JfromK = '+str(JfromK))
print('K1 = '+str(K1[k]))
print('K2 = '+str(K2[k]))
tanpsi=(K2[k]/K1[k])
modemix = atan(K2[k]/K1[k])*180/pi
print('mode_mix = '+str(modemix))
realKh.append(K1[k]*cos(epsilon*log(1))-K2[k]*sin(epsilon*log(1)))
imagKh.append(K2[k]*cos(epsilon*log(1))+K1[k]*sin(epsilon*log(1)))
phase=atan(imagKh[k]/realKh[k])*180/pi
adjustedmode = modemix + epsilon*log(1)*180/pi
print('realKh = '+str(realKh[k]))
print('imagKh = '+str(imagKh[k]))
print('phase = '+str(phase))
print('adjustedmode = '+str(adjustedmode))
if mode == 1:
    err = (abs(imagKh[k]))
else:
    err = (abs(realKh[k]))

print('points = '+str(points))
print('RealKh = '+str(realKh))
print('ImagKh = '+str(imagKh))
if mode == 1:
    ModelRatio = point[k]
    print('ModelRatio = '+str(ModelRatio))
else:
    Mode2Ratio = point[k]
    print('Mode2Ratio = '+str(Mode2Ratio))
print('Done!')
#clean files
types = ('.stt','.023','.res','.sta','.prt','.inp','.ipm','.mdl','.com')
for extension in filetypes:
    file = open(name + extension, 'w')
    file.close()
    os.remove(name + extension)
morefiletypes = ('.diary', '.msg', '.dat')
for extension in morefiletypes:
    file = open(name + extension, 'w')
    file.close()
    os.remove(name + extension)

if mode == 1:
    err = (abs(imagKh[k]))
else:
    err = (abs(realKh[k]))
return Mode1Ratio, Mode2Ratio

Listing E.8: Build the Model (ABAQUS v6.4)
def buildModel(h, L, meshsize, timestep, elm, plane, name):
    "This function builds a model"

    # create the model database
    Mdb

    # create the model
    mymodel = mdb.Model('ModeMix')
    if mdb.model.keys()[0] == 'Model-1':
        del mdb.model['Model-1']

    # create the sketch profile
    s = mymodel.Sketch(name='__profile__', sheetSize=L)
    g, v, d = s.geometry, s.vertex, s.dimension
    s.setPrimaryObject(option=STANDALONE)
    s.rectangle(point1=(-L/2., 0.0), point2=(L/2., h))

    # create the top part
    p = mymodel.Part(name='Top', dimensionality=TWO_D_PLANAR, type=DEFORMABLE_BODY)
    p.BaseShell(sketch=s)
    s.unsetPrimaryObject()

    # delete the sketch profile
    del mdb.model['ModeMix'].sketch['__profile__']

    # begin defining partitions
    p0 = mdb.model['ModeMix'].part['Top']
    f, e, d0 = p0.face, p0.edge, p0.datum
    t = p0.MakeSketchTransform(sketchPlane=f[0], sketchPlaneSide=SIDE1)
    s0 = mdb.model['ModeMix'].Sketch(name='__profile__', sheetSize=L, gridSpacing=10.0, transform=t)
    g, v, d = s0.geometry, s0.vertex, s0.dimension
    s0.setPrimaryObject(option=SUPERIMPOSE)
p0 = mdb.model['ModeMix'].part['Top']
p0.projectReferencesOntoSketch(sketch=s0, filter=COPLANAR_EDGES)

r, r0 = s0.referenceGeometry, s0.referenceVertex

#draw the horizontal partition
s0.Line(point1=(-L/2., 0.0), point2=(L/2.0, 0.0))

#draw the vertical partition
s0.Line(point1=(0.0, h/2.), point2=(0.0, -h/2.))

#draw the biasing box
s0.rectangle(point1=(-h/2., h/2.), point2=(h/2., -h/2.))

#draw the crack tip box
s0.rectangle(point1=(-4*elm, -h/2.+4*elm), point2=(4*elm, -h/2.))

#draw the radials
s0.Line(point1=(4*elm, -h/2.+4*elm), point2=(h/2., 0.0))
s0.Line(point1=(-h/2., 0.0), point2=(-4*elm, -h/2.+4*elm))

f, e, d0 = p0.face, p0.edge, p0.datum
faces = (f[0],)
p0.PartitionFaceBySketch(faces=faces, sketch=s0)
s0.unsetPrimaryObject()

del mdb.model['ModeMix'].sketch['__profile__']

#Copy top to bottom
mdb.model['ModeMix'].Part('Bottom', mdb.model['ModeMix'].part['Top'])

#create -materials
mdb.model['ModeMix'].Material('Top')
mdb.model['ModeMix'].Material('Bot')
mdb.model['ModeMix'].HomogeneousSolidSection(name='Top', material='Top',
thickness=1.0)
mdb.model['ModeMix'].HomogeneousSolidSection(name='Bottom', material='Bot',
thickness=1.0)

#create assembly
a = mdb.model['ModeMix'].rootAssembly
a.DatumCsysByDefault(CARTESIAN)
p = mdb.model['ModeMix'].part['Bottom']
a.Instance(name='Bottom-1', part=p)
p2 = a.instance['Bottom-1']
p2.rotateAboutAxis(axisPoint=(0.0, 0.0, 0.0), axisDirection=(0.0, 0.0, 1.0),
angle=180.0)
p = mdb.model['ModeMix'].part['Top']
a.Instance(name='Top-1', part=p)
p1 = mdb.model['ModeMix'].part['Top']
f = p1.face
faces = f[0:12]
region =(faces ,)
p0 = mdb.model['ModeMix'].part['Top']
p0.assignSection(region=region, sectionName='Top')
pl = mdb.model['ModeMix'].part['Bottom']
f = plface
faces = f[0:12]
region = (faces,)
p0 = mdb.model['ModeMix'].part['Bottom']
p0.assignSection(region=region, sectionName='Bottom')

# defines surface sets
e1 = a.instances['Bottom-1'].edges
d1 = d1[16:17]+d1[19:21]
a.Surface(side1Edges=d1, name='right-bottom-interface')
e1 = a.instances['Bottom-1'].edges
a.Surface(side1Edges=edges1, name='left-bottom-interface')
e1 = a.instances['Top-1'].edges
a.Surface(side1Edges=edges1, name='right-top-interface')
e1 = a.instances['Top-1'].edges
d1 = d1[16:17]+d1[19:21]
a.Surface(side1Edges=d1, name='left-top-interface')
e1 = a.instances['Top-1'].edges
d1 = d1[7:9]
e2 = a.instances['Bottom-1'].edges
a.Set(edges=edges1+edges2, name='right-side')
v1 = a.instances['Bottom-1'].verticess
verts1 = v1[11:12]
a.Set(verticess=verts1, name='bottom-right-corner')
v1 = a.instances['Top-1'].verticess
verts1 = v1[8:9]
a.Set(verticess=verts1, name='top-right-corner')
v1 = a.instances['Top-1'].verticess
verts1 = v1[3:4]
a.Set(verticess=verts1, name='top-middle-point')
e1 = a.instances['Top-1'].edges
edges1 = e1[13:14]
a.Surface(side1Edges=edges1, name='left-edge-1')
e1 = a.instances['Top-1'].edges
edges1 = e1[15:16]
a.Surface(side1Edges=edges1, name='left-edge-2')
e1 = a.instances['Bottom-1'].edges
edges1 = e1[7:8]
a.Surface(side1Edges=edges1, name='left-edge-3')
e1 = a.instances['Bottom-1'].edges
edges1 = e1[8:9]

189
a.Surface(side1Edges=edges1, name='left-edge-4')

vl = a.instance['Top-1'].vertex
verts1 = vl[16:17]
v2 = a.instance['Bottom-1'].vertex
verts2 = v2[16:17]
a.GeometrySet(vertexSeq=(verts1, verts2), name='Tip')

# extract y-coordinates and measure height
y_top = a.sets['top-right-corner'].vertices[0].pointOn[0][1]
y_bot = a.sets['bottom-right-corner'].vertices[0].pointOn[0][1]
y_tip = a.sets['Tip'].vertices[0].pointOn[0][1]

h1 = y_top - y_tip
h2 = y_tip - y_bot

print('h1 = ' + str(h1))
print('h2 = ' + str(h2))

logfile = open(name+'.diary', 'a')
logfile.write('The heights extracted from the buildModel routine are:
h1 = %g and h2 = %g
(h1, h2))
logfile.close()
region = region, distribution=UNIFORM, magnitude=−1000.0,
                amplitude=UNSET)
region = a.surfaces['left−edge−3']
mdb.model['ModeMix'].Pressure(name='Load−3', createStepName='Load',
                region=region, distribution=UNIFORM, magnitude=−1000.0,
                amplitude=UNSET)
region = a.surfaces['left−edge−4']
mdb.model['ModeMix'].Pressure(name='Load−4', createStepName='Load',
                                region=region, distribution=UNIFORM, magnitude=−1000.0,
                                amplitude=UNSET)

mdb.model['ModeMix'].Load('Reload−4', mdb.model['ModeMix'].load['Load−4'])
mdb.model['ModeMix'].Load('Reload−1', mdb.model['ModeMix'].load['Load−1'])
mdb.model['ModeMix'].Load('Reload−3', mdb.model['ModeMix'].load['Load−3'])
mdb.model['ModeMix'].Load('Reload−2', mdb.model['ModeMix'].load['Load−2'])

mdb.model['ModeMix'].load['Load−1'].deactivate('Unload')
mdb.model['ModeMix'].load['Load−2'].deactivate('Unload')
mdb.model['ModeMix'].load['Load−3'].deactivate('Unload')
mdb.model['ModeMix'].load['Load−4'].deactivate('Unload')

mdb.model['ModeMix'].load['Reload−1'].move('Load', 'Unload')
mdb.model['ModeMix'].load['Reload−1'].move('Unload', 'Reload')
mdb.model['ModeMix'].load['Reload−2'].move('Load', 'Unload')
mdb.model['ModeMix'].load['Reload−2'].move('Unload', 'Reload')
mdb.model['ModeMix'].load['Reload−3'].move('Load', 'Unload')
mdb.model['ModeMix'].load['Reload−3'].move('Unload', 'Reload')
mdb.model['ModeMix'].load['Reload−4'].move('Load', 'Unload')
mdb.model['ModeMix'].load['Reload−4'].move('Unload', 'Reload')

mdb.model['ModeMix'].load['Reload−1'].deactivate('Reunload')
mdb.model['ModeMix'].load['Reload−2'].deactivate('Reunload')
mdb.model['ModeMix'].load['Reload−3'].deactivate('Reunload')
mdb.model['ModeMix'].load['Reload−4'].deactivate('Reunload')

# create mesh

a0 = mdb.model['ModeMix'].rootAssembly
f01 = a0.instance['Bottom−1'].face
f02 = a0.instance['Top−1'].face
regions = [f01[0], f01[1], f01[2], f01[3], f01[4], f01[5], f01[6], f01[7],
            f01[8], f01[9], f01[10], f01[11], f02[0], f02[1], f02[2], f02[3],
            f02[4], f02[5], f02[6], f02[7], f02[8], f02[9], f02[10], f02[11])
a0.setMeshControls(regions=regions, technique=FREE)

if plane=='stress':
    elemType1 = ElemType(elemCode=CPS8R)
    elemType2 = ElemType(elemCode=CPS8R)
else:
    elemType1 = ElemType(elemCode=CPE8R)
    elemType2 = ElemType(elemCode=CPE8R)

print('Element type = ' + 'elemType1 ')
logfile = open(name + '.diary', 'a')
logfile.write('nThe elements are: ' + 'elemType1 ' + '
')
logfile.close()

f1 = a0.instance['Bottom-1'].face
faces1 = f1[0:12]
f2 = a0.instance['Top-1'].face
faces2 = f2[0:12]
regions = ((faces1, faces2, ), )
a0.setElementType(regions=regions, elemTypes=(elemType1, elemType2))
a0 = mdb.model['ModeMix'].rootAssembly
e01 = a0.instance['Top-1'].edge
e02 = a0.instance['Bottom-1'].edge
edges = (e01[0], e01[2], e01[8], e01[10], e01[13], e01[14], e01[15], e01[5],
e01[7], e02[14], e02[15], e02[0], e02[2], e02[8], e02[10], e02[13],
e02[5], e02[7], e02[29], e02[30], e02[1], e02[3], e01[1], e01[3],
e01[29], e01[30])
a0.seedEdgeByNumber(edges=edges, number=2*meshsize, constraint=FIXED)
e11 = a0.instance['Bottom-1'].edge
e12 = a0.instance['Top-1'].edge
end1Edges = (e11[4], e11[6], e12[12], e11[12], e12[4], e12[6])
end2Edges = (e11[9], e12[11], e12[16], e11[11], e11[16], e12[9])
edges = ((end1Edges, END1), (end2Edges, END2))
a0.seedEdgeByBias(edges=edges, ratio=5.0, number=6*meshsize, constraint=FIXED)
e01 = a0.instance['Top-1'].edge
e02 = a0.instance['Bottom-1'].edge
end1Edges = (e01[25], e01[24], e02[25], e02[24])
end2Edges = (e01[18], e01[28], e02[19], e02[18], e02[28], e01[19])
edges = ((end1Edges, END1), (end2Edges, END2))
a0.seedEdgeByBias(edges=edges, ratio=1.0, number=20*meshsize, constraint=FIXED)
e11 = a0.instance['Bottom-1'].edge
e12 = a0.instance['Top-1'].edge
edges = (e11[26], e11[27], e12[20], e12[22], e12[17], e12[21], e12[23],
e11[20], e11[22], e12[26], e12[27], e11[17], e11[21], e11[23])
a0.seedEdgeByNumber(edges=edges, number=2*meshsize, constraint=FIXED)
e01 = a0.instance['Top-1'].edge
e02 = a0.instance['Bottom-1'].edge
end1Edges = (e01[25], e01[24], e02[25], e02[24])
end2Edges = (e01[18], e01[28], e02[19], e02[18], e02[28], e01[19])
edges = ((end1Edges, END1), (end2Edges, END2))
a0.seedEdgeByBias(edges=edges, ratio=h/(elm*10.), number=20*meshsize)
f01 = a0.instances['Bottom-1'].faces
f02 = a0.instances['Top-1'].faces
regions = (f01[1], f01[2], f01[3], f01[4], f02[1], f02[2], f02[3], f02[4])
a0.setMeshControls(regions=regions, technique=STRUCTURED)
partInstances = (a0.instance['Bottom-1'], a0.instance['Top-1'], )
a0.generateMesh(regions=partInstances)
mdb.model['ModeMix'].fieldOutputRequest['F-Output-1'].setValues(variables=('S', 'E', 'PE', 'PEEQ', 'U', 'RF', 'CF', 'COORD'), frequency=LAST_INCREMENT)

mdb.model['ModeMix'].historyOutputRequest['H-Output-1'].setValues(variables=('',), frequency=LAST_INCREMENT)

session.viewport['Viewport:1'].assemblyDisplayOptions.setValues(datumPoints=OFF, datumAxes=OFF, datumPlanes=OFF, datumCoordSystems=OFF)

session.viewport['Viewport:1'].setValues(displayedObject=a)

session.viewport['Viewport:1'].view.fitView()

mdb.model['ModeMix'].keywordBlock.synchVersions()

mdb.model['ModeMix'].keywordBlock.insert(78, "Contour_Integral, contours=50, type=J, Tip, 1, 0"")

mdb.model['ModeMix'].keywordBlock.insert(103, "Contour_Integral, contours=50, type=J, Tip, 1, 0"")

mdb.model['ModeMix'].keywordBlock.insert(124, "Contour_Integral, contours=50, type=J, Tip, 1, 0"")

mdb.model['ModeMix'].keywordBlock.insert(146, "Contour_Integral, contours=50, type=J, Tip, 1, 0"")
Bibliography


